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# NORMS IN PRODUCT SPACES WHICH PRESERVE APPROXIMATION PROPERTIES

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Let E be a normed linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and L be a subset of E.

A point $y\in L$ is said to be <u>best approximation</u> to $x\in E$ from L, $y\in P_L(x)$, if $\|x-y\|\leq\|x-z\|$, for every $z\in L$.

A point $y\in L$ is said to be <u>better approximation</u> to $x\in E$ from L than other point $z\in L$, $y\prec_L^x z$, if $\|x-y\|\leq\|x-z\|$.

A point $x\in E$ is said to be <u>Birkhoff-orthogonal</u> to other point $y\in E$, $x\perp y$, if $\|x\|\leq\|x+ty\|$, for every $t\in\mathbb{K}$, |2|, |3|.

Let $E_1$ and $E_2$ be normed linear spaces over $\mathbb{K}$. We shall say that a norm $\|\ \|$ in $E_1xE_2$ is of <u>type A</u>, |1|, if

$$x_1\perp y_1-x_1,\ x_2\perp y_2-x_2 \Rightarrow \|(x_1,x_2)\|\leq\|(y_1,y_2)\|$$

and we shall say that $\|\ \|$ is of <u>type M</u>, |1|, if

$$\|x_1\|_1\leq\|y_1\|_1,\ \|x_2\|_2\leq\|y_2\|_2 \Rightarrow \|(x_1,x_2)\|\leq\|(y_1,y_2)\|$$

The importance in Approximation Theory of the A and M-norms is due to the following facts.

THEOREM 1. A norm in $E_1xE_2$ is of type M if and only if for every $x_k\in E_k$ and every $L_k$ subset of $E_k$, (k=1,2), it verifies

$$y_1\in P_{L_1}(x_1),\ y_2\in P_{L_2}(x_2)\Rightarrow(y_1,y_2)\in P_{L_1xL_2}(x_1,x_2) \qquad (1)$$

In other words, the M-norms are the only norms in $E_1xE_2$ satisfying the minimum requirement of compatibility given by (1). In this sense we propose the M-norms as the widest class of suitable norms for the approximation in normed product spaces.

THEOREM 2. |1|. A norm in $E_1xE_2$ is of type A if and only

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if for every $x_k \in E_k$ and every $L_k$ linear subspace of $E_k$, $(k=1,2)$ it verifies (1).

THEOREM 3. |1|. A norm in $E_1 x E_2$ is of type M if and only if for every $x_k \in E_k$ and every $L_k$ linear subspace of $E_k$, $(k=1,2)$ it verifies

$$y_1 \overset{x_1}{\underset{L_1}{<}} z_1, \quad y_2 \overset{x_2}{\underset{L_2}{<}} z_2 \Longrightarrow (y_1, y_2) \overset{(x_1, x_2)}{\underset{L_1 x L_2}{<}} (z_1, z_2) \qquad (2)$$

We can paraphrase the above theorems by saying that the A-norms (M-norms) are the only norms which preserve best (better) linear approximations in the passage to the product.

If is obvious that every M-norm is A-norm and it is easy to see that if $E_1 = E_2 = \mathbb{R}$ , the norm in $E_1 x E_2$ defined by

$$||(x_1, x_2)|| = \begin{cases} |x_1| + |x_2| & , \text{ if } x_1 x_2 \geq 0 \\ \sup(|x_1|, |x_2|), & \text{if } x_1 x_2 < 0 \end{cases}$$

is of type A but not of type M.

However, aside the trivial case in which $E_1$ and $E_2$ are real and of dimension 1, it is conjectured in |1| that every A-norm is M-norm, and this paper is essentially devoted to the proof that such conjecture is true in the case $\mathbb{K} = \mathbb{R}$.

THEOREM 4. If $E_1$ and $E_2$ are real normed linear spaces and if the dimension of any of them is $\geq 2$, then every A-norm in $E_1 x E_2$ is an M-norm.

The proof is based in the following.

**LEMMA**. Let E be the real linear space $\mathbb{R}^2$ endowed with any norm. If $x, y \in E$ are such that $0 < ||x|| < ||y||$, then there exist a finite number of points $x_1, \ldots, x_m \in E$ and a real number $0 < \theta \leq 1$ such that

$$x \perp x_1 - x, \ x_1 \perp x_2 - x_1, \ldots, x_m \perp \theta y - x_m$$

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