DESINGULARIZATION OF THREE-DIMENSIONAL VECTOR FIELDS

F. Cano

Departamento de Geometría
Facultad de Ciencias. 47005-Valladolid. SPAIN

Here we present some results obtained in [6, 7, 8, 9]. If F is a singular foliation over a variety X, dim X = 2, making a finite number of quadratic blowing-ups of X, one obtains a (saturated) foliation F' given at each singular point by a vector field D = aX/aX + bY/5Y whose linear part has an eigenvalue λ ≠ 0 and the other root μ of the characteristic polynomial verifies μ/λ ∈ Q*. There are two problems: first to show that if v = min (v(a), v(b)) then it decreases after a finite number of blowing-ups if v ≥ 2, second to obtain the above special feature if v = 1; [6, 7] deal with the first problem and [8, 9] with the second one in the case dim X = 3.

1. Logarithmic viewpoint. An unidimensional distribution D over X is an invertible OX-submodule of the tangent sheaf TX. It defines an unidimensional singular foliation FD over X, D is multiplicatively irreducible iff D = a(D), where a(D) is the double orthogonal. The saturation of FD is sat(FD) = Fq(D). P is a singular point of D iff vp(D) ≥ 1,
vp(D) = order of DP as a submodule of OX,P. The behaviour of vp(D) is not very good under quadratic blowing-ups. Moreover it is an important question to study the leaves of FD through a singular point ([1], [2], [3] if dim X = 2). The strict transforms of these leaves are the leaves of F' not contained in the exceptional divisor E. This allows us to introduce E. D is adapted to E iff D ⊂ OX[log E] = dual sheaf of the logarithmic forms OX[log E]. D is m.i. and adapted to E iff D = aE(D), where aE(D) is obtained as above. The adapted order ν(D, E, P) is the order of DP
as a submodule of OX,P[log E]. One has

Theorem 1. ([6]. I. 3.1.4); [4]. II. 4.3.3). If π: X' → X is a quadratic blowing-up and (X', E', D') is the strict transform of (X, E, D) (E' = π⁻¹(E ∪ {center})), then ν(D', E', P') ≤ ν(D, E, π(P')) for each P'. The blowing-ups to be considered have center Y with normal crossings
with \( E \) in order that \( E' \) would be a normal crossings divisor. If \( Y \) verifies that \( \nu(D', E', P') \leq \nu(D, E, \pi(P')) \) and \( \dim Y \leq \dim X - 2 \), then \( Y \) is weakly permissible.

2. Reduction games. Let us begin with \((X, E, D, P)\) with \( \nu(D, E, P) = r \geq 2 \). We are player A. Choose a weakly permissible center \( Y \) at \( P \) and let \( \pi: X' \to X \) be the corresponding blowing-up. Assume that the player B chooses a point \( P' \) over \( P \). If \( \nu(D', E', P') \leq 1 \), we have won. Otherwise the game begins at the status \((X', E', D', P')\). \([6]\) is devoted to the proof of

\[ \text{Theorem 2.} \quad ([6] \text{ I. 4.2.9}). \] If \( \dim X = 3 \), then there is a strategy for the player A in order to win in a finite number of steps.

3. Global results. \([7]\) is devoted to prove that for a special kind of singularities one can reduce the adapted order in a global manner. Let \( J(f) = \text{ideal of the strict tangent space of } cl^r(f) \) \((\nu(f) \geq r, cl^r(f) = \text{image in } m_p^r / m_p^{r+1})\). Let \( W_0 = G \cap R \) and \( W_1 = J^r(D(\pi(W_{l-1}) \cap G_n(R))) \) and set \( W(D, E, P) = \bigcap W_i \). \( P \) is of the type zero iff \( W(D, E, P) \neq 0 \). The \( W \)-directrix \( \text{Dir}_W(D, E, P) \) is the subvariety of \( T_P X \) given by \( W(D, E, P) \). A weakly permissible curve \( Y \) is permissible iff \( \nu_Y(D(I(Y))) > r \), \( \nu_Y(D(I(P))) > r - 1 \), where \( r = \min \{ r + 1, \nu_D(D(I(Y))) \} \). The main results of \([7]\) are the following ones.

**Proposition 1.** If \( Y \) is permissible, tangent to the \( W \)-directrix, \( \pi: X' \to X \) is the blowing-up centered at \( Y \) and \( \nu(D', E', P') = \nu(D, E, \pi(P')) \), then \( P' \in \text{Proj}(\text{Dir}_W(D(E, P)) / T_{\pi(P')} Y) \) and \( \dim \text{Dir}_W(D', E', P') \leq \dim \text{Dir}_W(D, E, \pi(P')) \). \([7]. \text{ (2.4)}\).

**Theorem 3.** ([7] 3.1). Assume that \( r = \text{biggest adapted order}, r \geq 2, \) \( \dim X = 3 \) and if \( \nu(D, E, P) = r \) then \( P \) is of the type zero. Then there is a finite sequence of permissible blowing-ups such that the strict transform \((X', D', P')\) verifies \( \nu(D', E', P') < r \) for each \( P' \).

4. Final forms. One has also the analogous of \( \nu / \lambda \notin \mathbb{Q}_+ \) for \( \dim X = 3 \). Assume that \( \nu(D, E, P) = 0, P \) is regular iff \( \nu(D, E, P) = 0, P \) is a simple point iff it is not regular, \( \lambda = e(E, P) = \#(\text{components of } E) \) and the characteristic polynomial of the linear part of a generator of \( D \) has roots \((1, \lambda, \mu)\) where \( \lambda \notin \mathbb{Q}_+, \mu \notin \mathbb{Q}_+ \). P is a simple corner iff it is not regular, \( e(E, P) > 2 \) and one can take a generator \( D \) of \( D \) and
\( Y = E_1 \cap E_2 \) (\( E_1 = \text{irr. comp. of } E \)) verifying that the linear map 
\( f \mapsto \text{cl}^1(D(f)) \) in \( \text{In}^1(E(Y)) \) has roots \( 1, \lambda, 1 \not\in \mathbb{Q}_+ \) for its characteristic polynomial. The main results of [9] are the following ones.

**Theorem 4.** ([9], 4.10). Let \( \dim X = 3 \) and assume that \( \nu(D,E,P) = 0 \) for each \( P \). Then after a finite number of blowing-ups, the strict transform (\( X', E', D' \)) verifies that each closed point is a regular point, a simple point or a simple corner.

**Corollary.** ([9], 7.5, also [5], [8]). Let \( \pi: X' \to X \) be the above morphism. Then there is a bijection \( \Psi \) between the integral branches of \( D \) through a singular point \( P \) of \( D \) not contained in \( E \) and the simple points \( P' \) over \( P \), such that \( \Psi(P) = \) infinitely near point of \( \Gamma \) over \( E' \).

**REFERENCES**


(*) A.M.S. Subject Classification (1980): 14-05, 32-C-40, 32-C-45