EXTRACTA MATHEMATICAE 1, 32-34 (1986)

THE INTERNAL STRUCTURE OF $\mathcal{G}$-SPACES

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A mi madre, sine qua non

We have considered in a unified form the following three general problems:

1. How the topological structure of a l.c.s. reflects the fact of being an A-space (A an operator ideal)?. This question appears proposed in [A.Pietsch, Opeartor Ideals. North-Holland, p 402] and has been quite extensively studied in the case of Schwartz and nuclear spaces, but, besides this, not much seems to be known. We undertake the study of the class of $\mathcal{G}$-spaces, defined by the ideal $\mathcal{G}$ of approximable operators, and thus intermediate between the classess of Schwartz and nuclear spaces.

2. It is known that "almost all" Schwartz spaces possess the A.P. (Approximation Property), while "few" Schwartz spaces do enjoy the B.A.P.(Bounded Approximation Property). We introduce two "local" versions of the B.A.P. and identify the Schwartz spaces with those properties precisely as the $\mathcal{G}$ and the dual $\mathcal{G}$ spaces.

3. M.S.Ramanujan proposed [Proceedings of the International Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics. Teubner Texte zür Mathematics. Leizpig 1978, p 128] : Is every Schwartz space with A.P. a $\mathcal{G}$-space?. We give a partial answer which generalices the previous known results [3].

Let U(E) and B(E) fundamental systems of 0-neighbourhoods -resp. bounded sets- in E. We will say that a net (A_i) of operators of L(E,E) is equicontinuous with respect to $U \in U(E)$ when $\bigcap_i A_i^{-1}(U)$ is a 0-nbhd in E; and it is equibounded with respect to a $B \in B(E)$ when $\bigcup_i A_i(B)$ is a bounded set of E.

AMS (1980) clas. 46A05-46A12

Definition 1. A l.c.s. E is a G-space when for each $U \in U(E)$ there exists a net (A_i) of finite rank operators of L(E,E), equicontinuous with respect to U, and such that for each $x \in E$, the net $(A_i x)$ converges to x in the seminorm p_U.

Roughly, property G means that the space possesses the B.A.P. with respect to any finite set of seminorms.

Definition 2. A l.c.s. E is an L-space when for each $B \in B(E)$ there exists a net (A_i) of finite rank operators of L(E,E), equibounded with respect to B, and such that for each $x \in B$, the net $(A_i x)$ converges to x in the topology of E.

Roughly, property L means that the space possesses the B.A.P. on bounded sets.

Properties G and L can be regarded as generalizations of the B.A.P.: it is clear that in Banach spaces, B.A.P., G and L are equivalent properties; in Fréchet spaces, B.A.P. $\Rightarrow$ G and B.A.P. $\Rightarrow$ L , though the other implications do not hold.

We have proved:

Theorem 1. *Let E be a l.c.s.*
(a) E is a $\mathcal{G}$-space iff E is Schwartz and a G-space.
(b) E'_b is a $\mathcal{G}$-space iff E is co-Schwartz and an L-space.

The tech niques developped led to obtain and compare some notions related to the A.P. (mixed-$\mathcal{G}$ and a local version of property L). It is worth to notice, as a consequence of theorem 1, that **though a Fréchet or DF nuclear space does not need to have the B.A.P., it does have it "locally" , that is: both, on bounded sets or with respect to any finite set of seminorms.**

An internal description of a $\mathcal{G}$-space, in the sense of 1, can be obtained as follows:

Definition 3. A l.c.s. E is said to have the Uniform Approximation Property (U.A.P.) when for each $U \in U(E)$ there exist a $V \in U(E)$ and a sequence (T_n) of finite rank operators of L(E,E) such that

$$p_U(x-T_n x) \leqslant n^{-1} p_V(x) \qquad \forall x \in E$$

Theorem 2. *A l.c.s. is a 𝒢-space iff it possesses the U.A.P.*

The dual version is also true: E'_b is a 𝒢-space iff E possesses the co-U.A.P. , where this property is defined as follows:

Definition 4. A l.c.s. is said to have the co-U.A.P. iff for each $A \in B(E)$ there exist a $B \in B(E)$ and a sequence (T_n) of finite rank operators of L(E,E), such that

$$p_B(x-T_n x) \leqslant n^{-1} \qquad \forall x \in A$$

These results apply to problem 3. The part (a) of theorem 1. can itself be considered as a partial answer. In particular:

A Schwartz space with B.A.P. is a 𝒢-space.

This result was proved in [3] when E is a Fréchet space. Our method of proving provides an affirmative answer to 3. in some favorable cases: when E is a Schwartz reflexive space with A.P., (T_i) and (R_j) being the nets which define the A.P. in E and E'_b respectively, and one of the following conditions is satisfied:

-for each $U \in U(E)$ a subnet of (T_i) can be picked, equicontinuous with respect to U.

-for each $B \in B(E'_b)$ a subnet of (R_j) can be picked, equibounded with respect to B or such that $\bigcup_j R_j(B)$ is metrizable.

then E is a 𝒢-space.

This is the case of DFM spaces with A.P (which can also be found in [3] with a different proof).

REFERENCES

|1| **Jesús M. Fdez Castillo.** An internal characterization of 𝒢-spaces (to appear in Portugaliae Math)

|2| **Jesús M. Fdez Castillo.** The internal structure of 𝒢-spaces. (Preprint 1985)

|3| **E. Nelimarkka.** The Approximation Property and Locally Convex Spaces defined by the Ideal of Approximable Operators. Math Nachr 107 (1982) 349-356