

A quantum-like model of decision: the Allais paradox

Oliva Gonzalo, Eduardo (eduardo.oliva@upm.es)

Departamento de Ingeniería Energética

Escuela Técnica Superior de Ingenieros Industriales—UPM

Álvarez-López, Alberto A. (aalvarez@cee.uned.es)

Departamento de Teoría Económica y Economía Matemática

Facultad de CC. Económicas y Empresariales—UNED

ABSTRACT

En este artículo presentamos un modelo de decisión inspirado en el formalismo matemático de la Mecánica Cuántica. El modelo incorpora los resultados conocidos de la utilidad esperada clásica y los amplía. Tras detallar los supuestos, algunas aplicaciones y otras peculiaridades del modelo, se ilustran sus posibilidades mediante una explicación de la paradoja de Allais dentro del nuevo marco.

Palabras clave: [Formalismo cuántico Utilidad esperada Paradoja de Allais]

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ABSTRACT

In this paper we present a decision model inspired by the mathematical formalism of Quantum Mechanics. The model encompasses the well known results of classical expected utility, and enhances them. After detailing the assumptions, some applications, and further peculiarities of the model, its possibilities are illustrated by giving an explanation of the Allais paradox within the new framework.

1 Introduction

The expected utility hypotheses, first enunciated by Daniel Bernoulli in 1730 in a paper (Bernoulli, 1954) he wrote while a member of the newly established St. Petersburg Sciences Academy, was formalised by John von Neumann and Oskar Morgenstern in their famous book of 1944 (von Neumann & Morgenstern, 1944), with a relevant second edition in 1947. Since then, the expected utility theory is at the heart of Economic Theory, with very important applications to fields such as Decision Theory, Game Theory, or Finance.

However, in some experiments, the behaviour of agents when facing some particular choices has challenged the validity of the expected utility hypotheses. The oldest and most famous of these challenges is probably the Allais paradox (Allais, 1953). Other relevant puzzles are the Ellsberg paradox (Ellsberg, 1961) and Machina's paradox (Mas-Colell et al., 1995). In essence, all of these puzzles describe situations when the *actual* decisions of the agents are different (sometimes, very different) from those *predicted* by the theory. This fact has led to propose different refinements of the expected utility theory, and even to new theories. Perhaps the most relevant of them is Prospect Theory, proposed by Daniel Kahneman and Amos Tversky in 1979 (Kahneman & Tversky, 1979; Wakker, 2010). The goal has always been to explain these paradoxes while maintaining both the manageability and the

normative and predictive power of expected-utility-based models (Machina, 1982, 1987).

Among those models that try to enhance—or even replace—the expected utility paradigm, some based on the mathematical formalism of Quantum Mechanics have been considered (see Haven & Khrennikov, 2013; Busemeyer & Bruza, 2012, and the wide list of references therein). In this paper we present a simple model of this kind that will allow us to shed light on the Allais paradox. As we shall see, our model also includes the classical utility model as a particular case.

An outline of the paper is the following: Section 2 describes the model—a quantum-like model of decision—; Section 3 shows the diagonal operators—which are the simplest and allow us to include the classical utility in the model—; Section 4 is devoted to non-diagonal operators—which may lead to a reversal of a classical utility—; Section 5 studies the Allais paradox in this new context; and, finally, Section 6 summarises the paper.

2 A quantum-like model of decision

In this section we present our proposal for a model of decision. We describe the decision process by considering five *postulates*, in turn inspired by the postulates of the Copenhagen interpretation of Quantum Mechanics (see, for instance, Alvarellos Bermejo et al., 2007).

We consider an agent who faces a decision among a finite number of *alternatives*, $a_1, a_2, \dots, a_N$.

- Postulate 1. For each $1 \leq i \leq N$, the alternative a_i is represented by a vector $\mathbf{a}_i$ in a *complex* Hilbert space $\mathcal{H}$, so that the system of vectors $\mathbf{B} \equiv (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N)$ is an *orthonormal* basis of $\mathcal{H}$ —the *basis of alternatives*. Thus the space $\mathcal{H}$ is spanned by the vectors of alternatives and its dimension equals

the number of options available to the agent.¹

- Postulate 2. The *state of mind* of the agent when faced with this decision is modelled by a vector $\boldsymbol{\psi} \in \mathcal{H}$ such that $\|\boldsymbol{\psi}\| = 1$ —that is, a *normalised* vector of the Hilbert space. We have:

$$\boldsymbol{\psi} = \sum_{i=1}^N \psi_i \mathbf{a}_i \quad \text{for some } \psi_i \in \mathbb{C}, \quad 1 \leq i \leq N, \quad \text{with } \sum_{i=1}^N |\psi_i|^2 = 1.$$

- Postulate 3. To try to make a decision, the agent applies some *criterion*. A criterion is modelled by a linear operator defined on the space $\mathcal{H}$ —that is, an endomorphism of $\mathcal{H}$ —that is diagonalisable.
- Postulate 4. Given an agent's state of mind $\boldsymbol{\psi} \in \mathcal{H}$ and a criterion U , the agent applies the following procedure. Let $V_1, V_2, \dots, V_p$ the eigenspaces of the operator U . For each $1 \leq j \leq p$, the squared norm of the projection of $\boldsymbol{\psi}$ onto V_k is calculated: $\|\pi|_{V_j}(\boldsymbol{\psi})\|^2$. Taking the projection with the *greatest* squared norm among all projections onto the eigenspaces, the agent's state of mind is *updated* to the result of normalising this projection. That is, the new state of mind is:

$$\boldsymbol{\psi}' = \frac{\pi|_{V_k}(\boldsymbol{\psi})}{\|\pi|_{V_k}(\boldsymbol{\psi})\|} \quad \text{such that} \quad \max_{1 \leq j \leq p} \|\pi|_{V_j}(\boldsymbol{\psi})\|^2 = \|\pi|_{V_k}(\boldsymbol{\psi})\|^2.$$

In case there are more than one projection with the greatest squared norm—that is, if this maximum is attained for more than one index $1 \leq i \leq N$ —, the state of mind is postulated to update to the sum of all those projections.

- Postulate 5. The new state of mind $\boldsymbol{\psi}'$ can be expanded in the basis $\mathbf{B}$: $\boldsymbol{\psi}' = \sum_{i=1}^N \psi'_i \mathbf{a}_i$ for some $\psi'_i \in \mathbb{C}$. Let D be the set of those alternatives whose corresponding coefficient in this expansion is non-null: $D = \{a_i \mid \psi'_i \neq 0\}$. We

¹For instance, we could simply take the Hilbert space $\mathcal{H} = \mathbb{C}^N$ with the usual scalar product: $\langle \mathbf{u} | \mathbf{v} \rangle = \sum_{i=1}^N \bar{u}_i v_i$ for any $\mathbf{u} = (u_1, u_2, \dots, u_N)$ and $\mathbf{v} = (v_1, v_2, \dots, v_N)$ in $\mathbb{C}^N$.

say that *the alternatives of the agent has been reduced* to those belonging to D . If the set D has only one element, that is, for some $1 \leq i \leq N$, $D = \{a_i\}$ —or equivalently: $\psi' = \psi'_i \mathbf{a}_i$ —, then *the agent has made a decision*, namely the alternative a_i .

Given a state of the mind and a criterion, the set D may have more than one element. In this case, as Postulate 2 states, the *actual* alternatives for the agent after applying the decision process are those in D , but it is not possible to choose among them without a further analysis: we are in a situation of *indecision*. If the cardinal of D is (strictly) lower than N , applying the criterion has indeed reduced the number of alternatives, but the agent would need to consider another criterion, to be applied to the new state of mind, with which to try to achieve a single choice.

Those readers familiar with the formalism of Quantum Mechanics could recognise some elements from this theory: the state of mind would correspond to the *wavefunction*, the criterion would be an *observable*, and the updating of the state would be similar to the *collapse* of the wavefunction due to a mesure represented by the observable. There is a significant difference, however, concerning the role of the eigenvalues: in Quantum Mechanics, they are very relevant—just the possible values that the measure represented by the observable may take—, but in our theory their specific value is in principle less important.

Just as in Quantum Mechanics—where the wavefunction “collects” all the physical information of the corresponding state, to be obtained by the observable—, when a vector represents the state of mind of the agent facing those alternatives, this vector embodies all the information about the agent related to that decision, such as tastes, preferences, impressions, thoughts, etc., and the objective of a criterion is to extract that information, or at least part of it, and materialise it in a form relevant to the decision-making process.

In the following sections, we will study different particular cases that will help

us better understand the role of the different elements of the model.

3 Diagonal operators

In this section we focus on criteria that are *diagonal* in the basis of alternatives. We start with a simple example.

Consider an agent trying to decide where to spend the holidays. There are two alternatives (i.e. $N = 2$): **beach** and **mountain**. Write² $\mathbf{B} = (|\mathbf{beach}\rangle, |\mathbf{mountain}\rangle)$ to stand for an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$ representing these alternatives (the system $\mathbf{B}$ could be, for instance, the usual canonical basis of $\mathbb{C}^2$). Let $\boldsymbol{\psi} \in \mathbb{C}^2$ be the agent's state of mind; we have:

$$\boldsymbol{\psi} = \psi_b |\mathbf{beach}\rangle + \psi_m |\mathbf{mountain}\rangle \quad \text{for some } \psi_b \in \mathbb{C} \text{ and } \psi_m \in \mathbb{C} \text{ with } |\psi_b|^2 + |\psi_m|^2 = 1.$$

To make a decision, the agent needs a criterion. Let U be a linear endomorphism of $\mathbb{C}^2$ represented in the basis $\mathbf{B}$ by the diagonal matrix³

$$U = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where a and b are real numbers.

The basis $\mathbf{B}$ is in fact a basis of eigenvectors of the matrix U . Assume that $a \neq b$, that is: *different* eigenvalues. The corresponding eigenspaces have dimension equal to 1: $V_b = \text{Lin}(|\mathbf{beach}\rangle)$ and $V_m = \text{Lin}(|\mathbf{mountain}\rangle)$, and the respective projections of the state $\boldsymbol{\psi}$ onto these eigenspaces are

$$\pi|_{V_b}(\boldsymbol{\psi}) = \psi_b |\mathbf{beach}\rangle \quad \text{and} \quad \pi|_{V_m}(\boldsymbol{\psi}) = \psi_m |\mathbf{mountain}\rangle.$$

²In this example (and in others later), we find useful the Dirac *bra-ket* notation (see, for instance, Dirac, 1958, or Hall, 2013), which is usual in Quantum Mechanics.

³With a slight abuse of notation, we use the same letter for both the endomorphism and the matrix that represents it in the basis $\mathbf{B}$.

The squared norm of these projections are equal to $|\psi_b|^2$ and $|\psi_m|^2$, respectively. If, for instance, $|\psi_b|^2 > |\psi_m|^2$, the agent's state of mind updates to $\psi' = (\psi_b/|\psi_b|)|\text{beach}\rangle$, and $D = \{\text{beach}\}$: the agent will go on vacation to the beach.

If we have that $|\psi_b|^2 = |\psi_m|^2$, the agent's state of mind is not actually updated, and the alternatives are not reduced: a situation of *indecision*.

What would have happened if both eigenvalues coincide ($a = b$)? In this case, there is a unique eigenspace: the entire $\mathbb{C}^2$, and a unique projection of ψ . The state of mind is not really updated either, and the alternatives are the same: again, there is indecision.

Come back to the case of different eigenvalues: $a \neq b$. Assume that the agent has defined a (classical) utility u for these alternatives. The decision according to this utility would be **beach** if and only if $u(\text{beach}) > u(\text{mountain})$. The decision process described *replicates* this utility-maximising behaviour if the numbers $|\psi_b|^2$ and $|\psi_m|^2$ are respectively proportional to the utilities $u(\text{beach})$ and $u(\text{mountain})$.

As we can notice with this example, when a criterion that is diagonal in the basis of alternatives is applied, the existence of equal eigenvalues may make us lose information from the agent's state of mind, leading to indecision. In this paper, from now on, we will make use only of criteria with different eigenvalues. We have the following result, which generalises what we have just seen in the example with a criterion of this type.

Theorem 1 *Given the (orthonormal) basis of alternatives $\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N)$, consider a state of mind $\psi = \sum_{i=1}^N \psi_i \mathbf{a}_i$, and define: $J = \arg \max_{1 \leq i \leq N} |\psi_i|$. Let U be a criterion that is diagonal in the basis $\mathbf{B}$ and whose eigenvalues are distinct from each other. The decision process, applying the criterion U to the state ψ , reduces the actual agent's alternatives to those in the set $\{\mathbf{a}_k \mid k \in J\}$.*

Proof. The basis of alternatives is a basis of eigenvectors of the matrix U , and each eigenspace of U has dimension equal to 1. For each $1 \leq i \leq N$, we have

that $V_i = \text{Lin}(\mathbf{a}_i)$, and the corresponding projection of $\boldsymbol{\psi}$ takes the form:

$$\pi|_{V_i}(\boldsymbol{\psi}) = \psi_i \mathbf{a}_i.$$

Now we have:

$$\max_{1 \leq i \leq N} \|\pi|_{V_i}(\boldsymbol{\psi})\|^2 = \max_{1 \leq i \leq N} \|\psi_i \mathbf{a}_i\|^2 = \max_{1 \leq i \leq N} |\psi_i|^2,$$

and this maximum is attained at *any* $k \in J$. Therefore, the updated state is

$$\boldsymbol{\psi}' = \frac{1}{\sqrt{\sum_{k \in J} |\psi_k|^2}} \sum_{k \in J} \psi_k \mathbf{a}_k,$$

and hence $D = \{a_k \mid k \in J\}$. □

If the set J has only one element—that is, the maximum of the numbers $|\psi_i|$, $1 \leq i \leq N$, is *strict*—, then the decision process leads the agent to a single choice. Otherwise, there is indecision.

As in the previous example, if a utility function u is defined for each alternative, we retrieve the classical selection of the utility-maximising alternative if the numbers $|\psi_1|^2, |\psi_2|^2, \dots, |\psi_N|^2$ are respectively proportional to the utilities $u(a_1), u(a_2), \dots, u(a_N)$. Therefore, according to Theorem 1, *the classical utility theory can be included as a particular case of our framework, via any operator with different eigenvalues that is diagonal in the basis of alternatives.*

4 Non-diagonal operators

In this section, we study the effect of a criterion that is *non-diagonal* in the basis of alternatives. As we could see in Section 3, looking at the expansion in the basis of alternatives of the agent's state of mind, in the diagonal case only the modulus of its complex coefficients is relevant. We shall see that, with non-diagonal criteria, also the arguments of those coefficients play a role.

For the sake of simplicity, we consider only $N = 2$ alternatives (as in the example in Section 3); thus the basis of alternatives is $\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_2)$, and a state of mind takes the form $\boldsymbol{\psi} = \psi_1 \mathbf{a}_1 + \psi_2 \mathbf{a}_2$ for some complex numbers ψ_1 and ψ_2 with $|\psi_1|^2 + |\psi_2|^2 = 1$. According to Theorem 1, if $|\psi_1| > |\psi_2|$, a diagonal criterion (in the basis $\mathbf{B}$) with distinct eigenvalues would extract from the state of mind $\boldsymbol{\psi}$ the choice of the alternative a_1 . In the following theorem, we see that a criterion “almost” diagonal could lead the agent to the other alternative.

Theorem 2 *Consider the basis of alternatives $\mathbf{B} = (\mathbf{a}_1, \mathbf{a}_2)$ and the state of mind $\boldsymbol{\psi} = \psi_1 \mathbf{a}_1 + \psi_2 \mathbf{a}_2$, with $\psi_j = |\psi_j| e^{i\theta_j}$, $1 \leq j \leq 2$. Let V be the criterion whose matrix with respect to the basis $\mathbf{B}$ is given by:*

$$V = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix},$$

where a , b and c are real numbers and $a \neq b$. If the following conditions are held:

$$2|\psi_1| |\psi_2| |c| |\cos(\theta_1 - \theta_2)| > \left| |\psi_1|^2 - |\psi_2|^2 \right| |a - b| \quad \text{and} \quad (a - b)c \cos(\theta_1 - \theta_2) < 0, \quad (1)$$

then the decision process, applying the criterion V to the state $\boldsymbol{\psi}$, leads the agent to the alternative a_2 . If the conditions written in (1) are not held, there is indecision.

Proof. The following are normalised eigenvectors of the matrix V respectively associated to the eigenvalues a and b :

$$\mathbf{a} \equiv \frac{1}{\sqrt{(a-b)^2 + c^2}} ((a-b)\mathbf{a}_1 + c\mathbf{a}_2) \quad \text{and} \quad \mathbf{b} \equiv \mathbf{a}_2.$$

The corresponding eigenspaces are $V_1 = \text{Lin}(\mathbf{a})$ and $V_2 = \text{Lin}(\mathbf{b})$, and the projections of the state $\boldsymbol{\psi}$ onto these eigenspaces take the form:

$$\pi|_{V_1}(\boldsymbol{\psi}) = \frac{\psi_1(a-b) + \psi_2 c}{\sqrt{(a-b)^2 + c^2}} \mathbf{a} \quad \text{and} \quad \pi|_{V_2}(\boldsymbol{\psi}) = \psi_2 \mathbf{b}.$$

On one hand, we have:

$$\begin{aligned}\|\pi|_{V_1}(\boldsymbol{\psi})\|^2 &= \frac{|\psi_1(a-b) + \psi_2 c|^2}{(a-b)^2 + c^2} = \frac{|\psi_1|^2 (a-b)^2 + |\psi_2|^2 c^2 + (a-b)c(\psi_1 \bar{\psi}_2 + \bar{\psi}_1 \psi_2)}{(a-b)^2 + c^2} \\ &= \frac{|\psi_1|^2 (a-b)^2 + |\psi_2|^2 c^2 + 2(a-b)c |\psi_1| |\psi_2| \cos(\theta_1 - \theta_2)}{(a-b)^2 + c^2},\end{aligned}$$

on the other hand: $\|\pi|_{V_2}(\boldsymbol{\psi})\|^2 = |\psi_2|^2$. The inequality $\|\pi|_{V_1}(\boldsymbol{\psi})\|^2 < \|\pi|_{V_2}(\boldsymbol{\psi})\|^2$ —which would indicate that the alternative a_2 is chosen—is equivalent to:

$$2(a-b)c |\psi_1| |\psi_2| \cos(\theta_1 - \theta_2) < -\left||\psi_1|^2 - |\psi_2|^2\right| (a-b)^2,$$

which is in turn equivalent to the inequalities written in (1). Finally, if $\|\pi|_{V_1}(\boldsymbol{\psi})\|$ is equal to $\|\pi|_{V_2}(\boldsymbol{\psi})\|$, then the state of mind $\boldsymbol{\psi}$ remains unchanged; and if $\|\pi|_{V_1}(\boldsymbol{\psi})\|$ is greater than $\|\pi|_{V_2}(\boldsymbol{\psi})\|$, the state is updated to:

$$\boldsymbol{\psi}' = \frac{\pi|_{V_1}(\boldsymbol{\psi})}{\|\pi|_{V_1}(\boldsymbol{\psi})\|} = \frac{\psi_1(a-b) + \psi_2 c}{|\psi_1(a-b) + \psi_2 c|} \mathbf{a} = K(a-b)\mathbf{a}_1 + Kc\mathbf{a}_2,$$

for some number $K \in \mathbb{C}$. In both cases, the final state of mind implies indecision. $\square$

If a utility u is defined for these alternatives, with the numbers $|\psi_1|^2$ and $|\psi_2|^2$ respectively proportional to the utilities $u(a_1)$ and $u(a_2)$, Theorem 2 gives a condition for choosing the alternative a_2 *even if* $u(a_1) > u(a_2)$. Notice that the difference $\left||\psi_1|^2 - |\psi_2|^2\right|$ is in turn proportional to the difference of utilities $|u(a_1) - u(a_2)|$. If these utilities are close, the first condition in (1) could be easily held. On the other hand, the second condition in (1) can be satisfied if $a > b$, $c > 0$, and the complex numbers ψ_1 and ψ_2 form (in the complex plane) an angle greater than $\pi/2$. Therefore, a non-diagonal operator, “playing” with *close* utilities, could yield a sort of *reversal of utility*.

5 Applying the model: the Allais paradox

The Allais paradox is the oldest and most famous example of an experimental failure of the expected utility model (Allais, 1953). In this paradox, as presented in

Mas-Colell et al. (1995), an agent faces two different choices:

$$L_1 = \left\{ \begin{array}{lll} 1.00 & \text{chance of} & \$500,000 \end{array} \right. \text{ versus } L_2 = \left\{ \begin{array}{lll} 0.10 & \text{chance of} & \$2,500,000 \\ 0.89 & \text{chance of} & \$500,000 \\ 0.01 & \text{chance of} & \$0, \end{array} \right.$$

and

$$L_3 = \left\{ \begin{array}{lll} 0.11 & \text{chance of} & \$500,000 \\ 0.89 & \text{chance of} & \$0 \end{array} \right. \text{ versus } L_4 = \left\{ \begin{array}{lll} 0.10 & \text{chance of} & \$2,500,000 \\ 0.90 & \text{chance of} & \$0. \end{array} \right.$$

More formally, the agent has to choose between lotteries corresponding to three different outcomes (which are monetary prizes, of \$2,500,000, \$500,000, \$0): on one hand, $L_1 = (0, 1, 0)$ and $L_2 = (0.1, 0.89, 0.01)$; on the other hand, $L_3 = (0, 0.11, 0.89)$ and $L_4 = (0.1, 0, 0.9)$.

The expected utility model predicts that if L_4 is preferred to L_3 , then L_2 should be preferred to L_1 . However, it is experimentally observed that many agents choose L_4 and L_1 . We shall show how this paradox can fit into our model, also unveiling some experimentally testable predictions about the agent's behaviour.

Consider the diagonal criterion

$$U = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

for some real numbers a and b , with $a \neq b$. For the choice between L_4 and L_3 , the state of mind of the agent can be written as $\psi_{43} = \psi_4 |L_4\rangle + \psi_3 |L_3\rangle$. We assume that the application of the criterion U to the state ψ_{43} yields the alternative L_4 , what is equivalent (cf. Theorem 1) to $|\psi_4| > |\psi_3|$.

On the other hand, for the choice between L_2 and L_1 , the state of mind of the agent would take the form: $\psi_{21} = \psi_2 |L_2\rangle + \psi_1 |L_1\rangle$, so that the criterion U would be to be applied to this state. But the lottery L_1 represents a *sure* gain, while L_2 includes a probability, however very low, of gaining nothing. This fact may perturb

the perception of the agent, in the sense that there might be some *regret* of having chosen L_2 instead of L_1 in case of a result of \$0. We assume that this perturbation makes the agent apply the diagonal criterion U , not in the basis $\mathbf{B} = (|L_2\rangle, |L_1\rangle)$, but in a “slightly perturbed” basis $\mathbf{B}' = (|L'_2\rangle, |L_1\rangle)$, with $|L'_2\rangle = \alpha|L_2\rangle + \beta|L_1\rangle$ for some positive real numbers α and β such that $\alpha \gg \beta$ and $\alpha^2 + \beta^2 = 1$.⁴ The operator U is represented in the basis of alternatives $\mathbf{B}$ as a *non-diagonal* matrix, namely:

$$V = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \quad \text{with } c = \frac{\beta}{\alpha}(a - b).$$

According to Theorem 2, applying the operator V to the state of mind ψ_{21} will yield the choice of L_1 if the following conditions are held:

$$2|\psi_1||\psi_2||c|\cos(\theta_1 - \theta_2) > \left| |\psi_1|^2 - |\psi_2|^2 \right| |a - b| \quad \text{and} \quad (a - b)c \cos(\theta_1 - \theta_2) < 0, \quad (2)$$

where $\psi_j = |\psi_j|e^{i\theta_j}$, $1 \leq j \leq 2$. Note that $(a - b)c > 0$, so that the second condition is verified for $\theta_1 - \theta_2 > \pi/2$. In addition, as we saw in Section 4, the first condition is easily satisfied when the numbers $|\psi_1|^2$ and $|\psi_2|^2$ are close.

Now, consider a utility u defined on all these alternatives verifying the assumptions of the expected utility theory. It is well known that $u(L_4) - u(L_3) = u(L_2) - u(L_1)$ (what in turn justifies that preferring L_4 to L_3 implies preferring L_2 to L_1), and it is also well known that this equality of differences of utilities remains true if we change the amount of \$500,000 for another one, say \$1. However, those differences are greater for \$1 than for \$500,000. Indeed, the difference $u(L_4) - u(L_3)$,

⁴We are considering that, due to the mentioned possible regret, the possibility of not gaining any reward, albeit with low probability, induces the agent to *entangle* the sure gain and the uncertain alternative.

or $u(L_2) - u(L_1)$, would take the form:

$$0.1u(\$2,500,000) + 0.01u(0) - 0.11u(\$500,000) < 0.1u(\$2,500,000) + 0.01u(0) - 0.11u(\$1),$$

taking into account that the function u is increasing.

The Allais paradox can be translated to our model when the coefficients of the states of mind take the following form:

$$\psi_1 = \frac{\sqrt{u_1}}{\sqrt{u_1 + u_2}} e^{i\theta_1}, \quad \psi_2 = \frac{\sqrt{u_2}}{\sqrt{u_1 + u_2}} e^{i\theta_2}, \quad \psi_3 = \frac{\sqrt{u_3}}{\sqrt{u_3 + u_4}} e^{i\theta_3}, \quad \psi_4 = \frac{\sqrt{u_4}}{\sqrt{u_3 + u_4}} e^{i\theta_4},$$

where $u_i \equiv u(L_i)$, $1 \leq i \leq 4$. One can easily check that $|\psi_1|^2 + |\psi_2|^2 = |\psi_3|^2 + |\psi_4|^2 = 1$, and also that:

$$|\psi_1|^2 - |\psi_2|^2 = k_{12}(u(L_1) - u(L_2)) \quad \text{and} \quad |\psi_3|^2 - |\psi_4|^2 = k_{34}(u(L_3) - u(L_4)),$$

where $k_{ij} = 1/(U_i + U_j)$, for $i \in \{1, 3\}$ and $j \in \{2, 4\}$. Therefore we see that the difference $||\psi_1|^2 - |\psi_2|^2|$ is greater (possibly much greater) considering the amount of \$1 than considering the original amount of \$500,000. In the latter case, the numbers $|\psi_1|^2$ and $|\psi_2|^2$ are closer, and the first inequality in (2) can be held for a low value of c —and in turn for a low value of the coefficient β in the perturbed basis $\mathbf{B}'$. However, with the amount of \$1, this inequality needs a much greater value of the perturbation β to be held.

As we can see, in our model, the amount that is a sure gain in the lottery L_1 has a clear effect on an eventual reversal of utility. If this amount is low, the regret component—materialised in the perturbation β —reduces its influence, and the agent's behaviour is that prescribed by the classical theory. If the amount is high, a very small perturbation can lead the agent to reverse the choice. This dependence of the agent's behaviour on the sure amount that defines the lottery might be tested experimentally, to possibly confirm the validity of our model.

6 Summary

In this paper we present a model for describing decision processes. This model is built upon ideas borrowed from Quantum Mechanics and quantum models of agents: the state of mind of the agent is described by a (normalised) vector in a Hilbert space spanned by the vectors corresponding to the alternatives among which the agent makes the choice. The criterion the agent applies to choose an alternative are represented by linear operators acting in the Hilbert space. To make a choice, the agent projects the state of mind onto the eigenspaces of the operator, and updates this state of mind as the projection that exhibits the maximum squared norm. If the resulting state of mind is proportional to one and only one vector of the basis of alternatives, the agent has made a choice, consisting of the corresponding alternative.

We also study both diagonal and non-diagonal operators. The former allows to include the classical utility theory in the new framework, simply by considering that the coefficients of the expansion of the state of mind in the basis of alternatives are proportional to the corresponding utilities. On the contrary, non-diagonal operators can lead to a reversal of utility under some conditions.

Finally, this model allows to explain the Allais paradox. The criterion used by the agent is represented by a non diagonal operator that results in choices that are paradoxical for the classical utility model. These non-diagonal terms arise from cross effects within alternatives, as the regret of having chosen one alternative that is revealed not to be optimal. Moreover, our model predicts that these perturbations are important when the expected utilities of the alternatives are similar, and that the behaviour of the agents will change depending on this difference, thus making testable predictions.

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