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Categoricity and amalgamation for AEC, and κ measurable

Categoricidad y amalgamación para clases elementales abstractas, y κ medible

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Abstract. In the original version of this paper, we assume a theory T in the logic $\mathbb{L}_{\kappa,\aleph_0}$ is categorical in a cardinal $\lambda > \kappa$, and κ is a measurable cardinal. There we prove that the class of models of T of cardinality $< \lambda$ but $\geq |T| + \kappa$ has the amalgamation property under a natural order; this is a step toward understanding the character of such classes of models.

In this revised version we replaced the class of models of T by $\mathfrak{k}$, an AEC (abstract elementary class) which has LST-number $< \kappa$, or at least which behaves nicely for ultra-powers by $\mathbf{D}$, some normal ultra-filter on κ or just $\mathrm{LST}^+_{\mathfrak{k}}$-complete non-principal ultra-filters on κ.

Presently sub-section §2A deals with $T \subseteq \mathbb{L}_{\kappa^+,\aleph_0}$ (and so does a large part of the introduction and little in the rest of §2), but otherwise, all is done in the context of AEC.

The reader may in the first reading for transparency fix $\mathbf{D}$, a normal ultrafilter on the measurable cardinal κ and either fix the $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$ or fix an AEC $\mathfrak{k}$ with $\mathrm{LST}_{\mathfrak{k}} < \kappa$.

We leave the original introduction adding a few comments at the end, after the three stars.

Keywords: Model theory, abstract elementary classes, AEC, categoricity, infinitary logic, amalgamation.

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Resumen. En la versión original de este artículo, suponemos que una teoría T en la lógica $\mathbb{L}_{\kappa,\aleph_0}$ es categórica en un cardinal $\lambda > \kappa$, y que κ es un cardinal medible. Allí demostramos que la clase de modelos de T de cardinalidad $< \lambda$ con $\geq |T| + \kappa$ tiene la propiedad de amalgamación bajo un orden natural; este es un paso hacia la comprensión del carácter de dichas clases de modelos.

En esta versión revisada, reemplazamos la clase de modelos de T por $\mathfrak{k}$, una AEC (clase elemental abstracta) que tiene número de Löwenheim-Skolem-Tarski $< \kappa$, o al menos que se comporta bien para ultrapotencias por $\mathbf{D}$, algún ultrafiltro normal sobre κ o simplemente ultrafiltros no principales $LST_{\mathfrak{k}}^{+}$-completos sobre κ.

Actualmente, la subsección §2A trata con $T \subseteq \mathbb{L}_{\kappa^+,aleph_0}$ (y también una gran parte de la introducción y poco en el resto del §2), pero por lo demás, todo se hace en el contexto de las AEC.

Para una primera lectura, y en aras de la transparencia, el lector puede fijar $\mathbb{D}$ un ultrafiltro normal sobre el cardinal medible κ, y bien fijar $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$ o bien fijar una AEC $\mathfrak{k}$ con $LST_{\mathfrak{k}} < \kappa$

Dejamos la introducción original añadiendo algunos comentarios al final, después de los tres asteriscos.

Palabras claves: teoría de modelos, clases elementales abstractas, AEC, categoricidad, lógica infinitaria, amalgamación.

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Annotated Content

[In §2A we review materials on fragments $\mathcal{F}$ of $\mathbb{L}_{\kappa,\aleph_0}$ (including the theory T) and basic model-theoretic properties (Tarski-Vaught property and L.S.T.), and we define amalgamation. In §2B we move to AEC $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ which is our main framework now, and spell out the connection. In §3C, D we deal with indiscernibles and E.M. models, then we deal with limit ultra-powers which are suitable (for $\mathbb{L}_{\kappa,\aleph_0}$ and for our AECs) and in particular ultra-limits. Next, we introduce a notion basic for this paper: $M \underset{\text{nice}}{\leq} N$ if there is a $\leq_{\mathfrak{k}}$-embedding of N into suitable ultra-limit of M extending the canonical embbeding.]

[We get amalgamation in $(K_\lambda, \leq_{\mathfrak{k}})$ when one of the extensions is nice, see Claim 2.1. We prove that if $\mathfrak{k}$ is categorical in the regular $\lambda > \mathrm{LST}_{\mathfrak{k}} + \kappa$, then $(K_{<\lambda}, \leq_{\mathfrak{k}})$ has the amalgamation property. For this, we show that nice extension (in $K_{<\lambda}$) preserves being a non-amalgamation basis. We also start investigating (in Theorem 2.5) the connection between extending the linear order I and the model $\mathrm{EM}(I)$: $I \underset{\text{nice}}{\subseteq} J \Rightarrow \mathrm{EM}(I) \underset{\text{nice}}{\leq} \mathrm{EM}(J)$; and give sufficient condition for $I \underset{\text{nice}}{\subseteq} J$ (in Criterion 2.6). From this, we get in K_λ a model such that any sub-model of a suitable expansion is a $\underset{\text{nice}}{\leq}$-sub-model (in Fact 2.8, Theorem 2.11(2)), and conclude the amalgamation property in $(K_{<\lambda}, \leq_{\mathfrak{k}})$ when λ is regular (in Theorem 2.10) and something for singulars in Theorem 2.11.]

§3 Toward removing the assumption of regularity from the existence of universal extensions, pg. 26.

[The problem is that $\mathrm{EM}(\lambda)$ has many sub-models which "sit" well in it and we can prove that there are many amalgamation bases but we need to get this simultaneously. First in Theorem 3.1 we show that, if $\langle M_i \colon i < \theta^+\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous sequence of models from K_θ, then for a club of $i < \theta^+$ we have $M_i \underset{\text{nice}}{\leq} \bigcup\{M_j \colon j < \theta^+\}$. In Definition 3.6, we define nice models (essentially, every reasonable extension is nice). Next (in Theorem 3.4) we show that nice models are dense in K_θ. Also (by Theorem 3.5) many embeddings are nice and (in Corollary 3.6) we show that being nice implies being amalgamation base. Then we define a universal extension of $M \in K_\theta$ in K_∂ (Definition 3.7), we prove existence over a model in Lemma 3.10 and after preparation prove the existence (Corollary 3.13, Corollary 3.14).]

§4 (θ, ∂)-saturated models, pg. 32.

[If $M_i \in K_\theta$ for $i \leq \partial$ is increasing continuous, M_{i+1} universal over M_i, and each M_i is nice, then[1] we say M_∂ is (θ, ∂)-*saturated* over M_0. We show existence (and uniqueness). We connect this to more usual saturation and prove that (θ, ∂)-saturation implies niceness (in Theorem 4.11).]

§5 The amalgamation property for $K_{<\lambda}$, pg. 38.

[After preliminaries we prove that for $\theta \leq \lambda$ (and $\theta \geq \mathrm{LST}(\mathfrak{k}) + \kappa$ of course) every member of K_θ can be extended to one with many nice sub-models, this is done by induction on θ using the niceness of (θ_1, ∂_1)-saturated models. Lastly, we conclude that every $M \in K_{<\lambda}$ is nice hence $K_{<\lambda}$ has the amalgamation property.]

[1] In [33] we say M_j is (θ, ∂)-brimmed.

0. Introduction

The main result[2] of this paper is a proof of the following theorem:

Theorem 0.1. *Suppose that T is a theory in a fragment of $\mathbb{L}_{\kappa,\aleph_0}$ where κ is a measurable cardinal. If T is categorical in the cardinal $\lambda > \kappa + |T|$, then $\mathcal{K}_{<\lambda}$, the class of models of T of power strictly less than λ (but $\geq \chi = \kappa + |T|$), has the amalgamation property (see Definition 1.12 (1)(2)).*

The interest in this theorem stems in part from its connection with the study of categoricity spectra. For a theory T in a logic $\mathscr{L}$ let us define $\mathrm{Cat}(T)$, the *categoricity spectrum of* T, to be the collection of those cardinals λ in which T is categorical. In the 1950's Łos conjectured that if T is a countable theory in first-order logic, then $\mathrm{Cat}(T)$ contains every uncountable cardinal or no uncountable cardinal. This conjecture, based on the example of algebraically closed fields of fixed characteristic, was verified by Morley [12], who proved that if a countable first-order theory is categorical in some uncountable cardinal, then it is categorical in every uncountable cardinal. Following advances made by Rowbottom [15], Ressayre [14] and Shelah [17], Shelah [18] proved the Łos conjecture for uncountable first-order theories: if T is a first-order theory categorical in some cardinal $\lambda > |T|+\aleph_0$, then T is categorical in every cardinal $\lambda > |T| + \aleph_0$. It is natural to ask whether analogous results hold for theories in logics other than first-order logic. Perhaps the best-known extensions of first-order logic are the infinitary logics $\mathbb{L}_{\lambda,\kappa}$. As regards theories in $\mathbb{L}_{\kappa,\aleph_0}$, Shelah (see [21] and [22]) continuing work begun in [19] introduced the concept of excellent classes: these have models in all cardinalities, have the amalgamation property and satisfy the Łos conjecture. In particular, if φ is an excellent sentence of $\mathbb{L}_{\aleph_1,\aleph_0}$, then the Łos conjecture holds for φ. Furthermore, under some set-theoretic assumptions (weaker than the Generalized Continuum Hypothesis) if φ is a sentence in $\mathbb{L}_{\aleph_1,\aleph_0}$ which is categorical in $\aleph_n$ for every natural number n (or even just if φ is a sentence in $\mathbb{L}_{\aleph_1,\aleph_0}$ with at least one uncountable model not having too many models in each $\aleph_n$), then φ is excellent. Now, [25], [34] try to develop classification theory in some non-elementary classes. We cannot expect much for $\mathbb{L}_{\lambda,\kappa}$ for $\kappa > \aleph_0$. The first author conjectured that if φ is a sentence in $\mathbb{L}_{\aleph_1,\aleph_0}$ categorical in some $\lambda > \beth_{\omega_1}$, then φ is categorical in every $\lambda > \beth_{\omega_1}$. (Recall that the Hanf number of $\mathbb{L}_{\aleph_1,\aleph_0}$ is $\beth_{\omega_1}$, so if ψ is a sentence in $\mathbb{L}_{\aleph_1,\aleph_0}$ and ψ has a model of power $\lambda \geq \beth_{\omega_1}$, then ψ has a model in every power $\lambda \geq \beth_{\omega_1}$, see [8]) . There were some who asked why so tardy the beginning. Recent work of Hart and Shelah [5] showed that for every natural number k greater than 1 there is a sentence ψ_k in $\mathbb{L}_{\aleph_1,\aleph_0}$ which is categorical in the cardinals $\aleph_0, \ldots, \aleph_{k-1}$, but which has many models of power λ for every cardinal $\lambda \geq 2^{\aleph_{k-1}}$. The general conjecture for $\mathbb{L}_{\aleph_1,\aleph_0}$ remains open nevertheless. As regards theories in $\mathbb{L}_{\kappa,\aleph_0}$, progress has been recorded under the assumption that κ is a strongly compact cardinal. Under this assumption

[2] In the old version.

Shelah and Makkai [11] have established the following results for a λ-categorical theory T in a fragment $\mathcal{F}$ of $\mathbb{L}_{\kappa,\aleph_0}$:

1) if λ is a successor cardinal and $\lambda > ((\kappa')^{\kappa})^+$ where $\kappa' = \max(\kappa, |\mathcal{F}|)$, then T is categorical in every cardinal greater than or equal to $\min(\lambda, \beth_{(2^{\kappa'})^+})$,

2) if $\lambda > \beth_{\kappa+1}(\kappa')$, then T is categorical in every cardinal of the form $\beth_\delta$ with δ divisible by $(2^{\kappa'})^+$ (i.e. for some ordinal $\alpha > 0$, $\delta = (2^{\kappa'})^+ \cdot \alpha$ (ordinal multiplication)).

In proving theorems of this kind, one has recourse to the amalgamation property which makes possible the construction of analogs of saturated models. In turn, these are of major importance in categoricity arguments. The amalgamation property holds for theories in first-order logic [2] and in $\mathbb{L}_{\kappa,\kappa}$ when κ is a strongly compact cardinal (see e.g. [11]: although $\prec_{\mathbb{L}_{\kappa,\kappa}}$ fails the Tarski-Vaught property for unions of chains of length $<\kappa$ (whereas $\prec_{\mathbb{L}_{\kappa,\aleph_0}}$ satisfies it), under a categoricity assumption it can be shown that $\prec_{\mathbb{L}_{\kappa,\aleph_0}}$ and $\prec_{\mathbb{L}_{\kappa,\kappa}}$ coincide). However, it is not known in general for theories in $\mathbb{L}_{\kappa,\aleph_0}$ or $\mathbb{L}_{\kappa,\kappa}$ when one weakens the assumption on κ, in particular when κ is just a measurable cardinal. Nevertheless, categoricity does imply the existence of reasonably saturated models in an appropriate sense, and it is possible to begin classification theory. This is why the main theorem of the present paper is of relevance regarding the categoricity spectra of theories in $\mathbb{L}_{\kappa,\aleph_0}$ when κ is measurable.

A sequel to this paper under preparation (which is now [31]) tries to provide a characterization of $\mathrm{Cat}(T)$ at least parallel to that in [11] and we hope to deal with the corresponding classification theory later. This division of labor both respects historical precedent and is suggested by the increasing complexity of the material. Another sequel deals with abstract elementary classes (in the sense of [23]) (see [31], [29] respectively). On more work see [30], [33].

The paper is divided into five sections. Section 2 is preliminary and notational. In section 3 it is shown that if the theory $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$ or just suitable AEC $\mathfrak{K}$ is categorical in the regular cardinal $\lambda > \kappa + |T|$, then $K_{<\lambda}$ has the amalgamation property. Section 4 deals with weakly universal models, section 5 with (θ, ∂)-saturated and $\bar{\theta}$-saturated models. In section 6 the amalgamation property for $K_{<\lambda}$ is established.

All the results in this paper (other than those explicitly credited) are due to Saharon Shelah.

* * *

On a more recent survey see [35] and a recent one see [36], in particular on the history of κ-compact AEC.

We had stated that clearly, the proof of [9] works for AEC, but the referee of [36] asked to do it explicitly. Here we justify [36, 4.7]. Note that, [9, 1.1, 1.2] essentially proves that $(\mathrm{Mod}(T), \prec_T)$ is an AEC ignoring Ax. V of AEC (see Definition 1.17), so Fact 1.11(2) was added.

We thank Shimoni Garti for his help in proofreading and the referee for pointing out some obscure points.

1. Preliminaries

To start things off in this section, let us fix notation, provide basic definitions and well-known facts, and formulate our working assumptions.

The working assumptions in force throughout the paper are these.

***Assumption* 1.1.** κ is an uncountable cardinal, and $\mathbf{D}$ is an uniform non-principal ultra-filter on κ.

***Assumption* 1.2.**

(1) The theory T is a theory in the infinitary logic $\mathbb{L}_{\kappa,\aleph_0}$, $\chi = \kappa + |T|$ and vocabulary $\tau = \tau_T$, and κ is a measurable cardinal, $\mathbf{D}$ is a κ-complete non-principal ultra-filter on κ, or

(2) $\mathfrak{k}$ is an AEC which is $\mathbf{D}$-compact (see Definition 1.17 and Definition 1.40 respectively) and $\chi = \kappa + \mathrm{LST}(\mathfrak{k})$.

Our main theorem for the logic $\mathbb{L}_{\kappa,\aleph_0}$ is:

Theorem 1.3. *If $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$ is categorical in $\lambda > \kappa + |\tau_T|$ then the class of models of T of cardinality $< \lambda$ but $\geq \kappa + |\tau_T|$ (under the so called $\prec_{\mathcal{F}}$, see Notation 1.8(2)(7), (8)) has the amalgamation property.*

Proof. Use Theorem 1.19 on AEC which is applicable by Conclusion 1.23 and recalling Definition 1.21 and Claim 1.22. □

From these assumptions follow certain facts, of which the most important are these.

Fact 1.4. For each model M of T, κ-complete ultra-filter D over I and suitable set G of equivalence relations on $I \times I$ (see Definition 1.32) the limit ultra-power $\mathrm{Op}(M) = \mathrm{Op}(M, I, D, G)$ is a model of T.

Fact 1.5. For each linear order $I = (I, \leq)$ there exists an Ehrenfeucht-Mostowski model $\mathrm{EM}(I)$ of T (see Definition 1.26(6)).

This section is divided into several subsections: in §1A we deal with a theory T in $\mathbb{L}_{\kappa,\aleph_0}$, in §1B we move to AEC $\mathfrak{k}$ showing that the context in §1A is a special case. Then in §1C we deal with EM models. Finally, in §1D we deal with ultra-powers, ultra-limits, and nice sub-models.

1(A). Frame for $\mathbb{L}_{\kappa,\aleph_0}$

Relevant set-theoretic and model-theoretic information on measurable cardinals can be found in [7], [2], and [4].

Notation 1.6. Let τ denote[3] a vocabulary, i.e. a set of finitary relation and function symbols, including equality (i.e. the arity of the symbols in τ_{sk} is always finite). So $|\tau|$ is the cardinality of the vocabulary τ.

[3] In the old version it was called "language" and denoted by L.

Definition 1.7.

(1) For cardinals $\kappa \leq \lambda$, $\mathbb{L}_{\lambda,\kappa}$ is the logic such that for any vocabulary τ, $\mathbb{L}_{\lambda,\kappa}(\tau)$ is the smallest set of (possibly infinitary) formulas in the vocabulary τ which contains all first-order formulas and which is closed under:

(A) the formation of conjunctions (disjunctions) of any set of formulas of power less than λ, provided that the set of free variables in the conjunctions (disjunctions) has power less than κ,

(B) the formation of $\forall \bar{x}\varphi, \exists \bar{x}\varphi$, where $\bar{x} = \langle x_\alpha : \alpha < \alpha_* \rangle$ is a sequence with no repetitions of variables of length $\alpha_* < \kappa$.

(2) Whenever we use the notation $\varphi(\bar{x})$ to denote a formula in $\mathbb{L}_{\lambda,\kappa}$, we mean that $\bar{x}$ is a sequence $\langle x_\alpha : \alpha < \alpha_* \rangle$ as above. So if $\varphi(\bar{x})$ is a formula in $\mathbb{L}_{\kappa,\aleph_0}$, then $\bar{x}$ is a finite sequence of variables.

(3) So $\mathbb{L} = \mathbb{L}_{\aleph_0,\aleph_0}$ is a first order logic.

Notation 1.8.

(1) $\mathcal{F}$ denotes a fragment of $\mathbb{L}_{\kappa,\aleph_0}(\tau)$, i.e. a set of formulas of $\mathbb{L}_{\kappa,\aleph_0}(\tau)$ which contains all atomic formulas of τ, and which is closed also under negations, finite conjunctions (finite disjunctions), and the formation of subformulas. An $\mathcal{F}$-formula is just an element of $\mathcal{F}$.

(2) T is a theory in $\mathbb{L}_{\kappa,\aleph_0}(\tau)$, so there is a fragment $\mathcal{F}$ of $\mathbb{L}_{\kappa,\aleph_0}$ such that $T \subseteq \mathcal{F}$ and $|\mathcal{F}| < |T|^+ + \kappa$. Let $\mathcal{F}_T$ be the minimal such $\mathcal{F}$. If not said otherwise, T and $\mathcal{F} = \mathcal{F}_T$ are fixed.

(3) Models of T (invariably referred to as models) are τ-structures which satisfy the sentences of T. They are generally denoted $M, N, \ldots$, and $|M|$ is the universe of the τ-structure M; $\|M\|$ is the cardinality of $|M|$.

(4) For a set A, $|A|$ is the cardinality of A and ${}^{<\omega}A$ is the set of finite sequences in A and for $\bar{a} = \langle a_0 \ldots a_{n-1} \rangle \in {}^{<\omega}A$, $\lg(\bar{a}) = \mathrm{n}$ is the length of $\bar{a}$. Similarly, if $\bar{a} = \langle a_\zeta : \zeta < \delta \rangle$, we write $\lg(\bar{a}) = \delta$, where δ is an ordinal.

(5) For an element R of τ and a τ-model M, let $\mathrm{val}(M, R)$, or R^M, be the interpretation of R in the τ-structure M. Similary for a function symbol $F \in \tau$.

(6) We ignore models of power less than κ. K is the class of all models of T;

$$K_\lambda = \{M \in K : ||M|| = \lambda\}, K_{<\lambda} = \bigcup_{\mu<\lambda} K_\mu, K_{\leq\lambda} = \bigcup_{\mu\leq\lambda} K_\mu, K_{[\mu,\lambda)} = \bigcup_{\mu\leq\chi<\lambda} K_\chi.$$

(7) We write $f \colon M \underset{\mathcal{F}}{\to} N$ (may be abbreviated $f : M \to N$) to mean that f is an $\mathcal{F}$-elementary embedding (briefly, an embedding) of M into N, i.e. f is a function with domain $|M|$ into $|N|$ such that for every $\mathcal{F}$-formula $\varphi(\bar{x})$, and $\bar{a} \in {}^{<\omega}|M|$ with $\lg(\bar{a}) = \lg(\bar{x}), M \models \varphi[\bar{a}]$ iff $N \models \varphi[f(\bar{a})]$, where if $\bar{a} = \langle a_i : i < n \rangle$, then $f(\bar{a}) := \langle f(a_i) : i < n \rangle$.

(8) In the special case where an embedding f is a set-inclusion (so that $|M| \subseteq |N|$), we write $M \prec_{\mathcal{F}} N$ (briefly $M \prec N$), instead of $f : M \underset{\mathcal{F}}{\to} N$. We

may say that M is an $\mathcal{F}$-elementary sub-model of N, or N is an $\mathcal{F}$-elementary extension of M.

Notation 1.9.

(1) $(I, \leq_I), (J, \leq_J)$ are partial orders; we will not bother to subscript the order relation unless really necessary; we may write I for $(I, \leq)$. We say $(I, \leq)$ is directed iff for every i_1 and i_2 in I, there is $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$. $(I, <)^*$ is the (reverse) partial order $(I^*, <^*)$ where $I^* = I$ and $s <^* t$ iff $t < s$.

(2) A sequence $\langle M_i : i \in I \rangle$ of models indexed by I is a $\prec_{\mathcal{F}}$-directed system iff $(I, \leq)$ is a directed partial order and for $i \leq j$ in $I, M_i \prec_{\mathcal{F}} M_j$.

Note that, the union $\bigcup_{i \in I} M_i$ of a $\prec_{\mathcal{F}}$-directed system $\langle M_i \colon i \in I \rangle$ of τ-structures is an τ-structure. In fact, more is true.

Fact 1.10.

(1) (Tarski-Vaught property) The union of a $\prec_{\mathcal{F}}$-directed system $\langle M_i : i \in I \rangle$ of models of T is a model of T, and for every $j \in I, M_j \prec_{\mathcal{F}} \bigcup_{i \in I} M_i$.

(2) For $\bar{M}$ as above, if M is a fixed model of T such that for every $i \in I$ there is $f_i \colon M_i \underset{\mathcal{F}}{\to} M$, I is directed, and for all $i \leq j$ in $I, f_i \subseteq f_j$, then $\bigcup_{i \in I} f_i : \bigcup_{i \in I} M_i \underset{\mathcal{F}}{\to} M$. In particular, if $M_i \prec_{\mathcal{F}} M$ and f_i is the identity function on M_i for every $i \in I$, then $\bigcup_{i \in I} M_i \prec_{\mathcal{F}} M$. Let α be an ordinal. A $\prec_{\mathcal{F}}$-chain of models of length α is a sequence $\langle M_\beta : \beta < \alpha \rangle$ of models such that if $\beta < \gamma < \alpha$, then $M_\beta \prec_{\mathcal{F}} M_\gamma$. The chain is continuous if for every limit ordinal $\beta < \alpha, M_\beta = \bigcup_{\gamma < \beta} M_\gamma$.

Fact 1.11.

(1) (Downward Löwenheim-Skolem Property): Suppose that M is a model of T, $A \subseteq |M|$ and $\max(\kappa + |T|, |A|) \leq \lambda \leq ||M||$. Then there is a model N such that $A \subseteq |N|, ||N|| = \lambda$ and $N \prec_{\mathcal{F}} M$.

(2) If N and $M_1 \subseteq M_2$ are τ-models, $\mathcal{F}$ is a fragment of $\mathbb{L}_{\kappa, \aleph_0}$, and $M_\ell \prec_{\mathcal{F}} N$ for $\ell = 1, 2$ then $M_1 \prec_{\mathcal{F}} M_2$.

Now we turn from the rather standard model-theoretic background to the more specific concepts which are central in our investigation.

Definition 1.12.

(1) Suppose that $<$ is a binary relation on a class K of models (mainly $(K, <) = (K_{\mathfrak{k}}, <_{\mathfrak{k}})$, see below). We say $\mathcal{K} = \langle K, < \rangle$ has the *amalgamation property* (AP) iff for every $M, M_1, M_2 \in K$, if f_i is an isomorphism from M onto $\mathrm{rng}(f_i)$ and $\mathrm{rng}(f_i) < M_i$ for $i = 1, 2$, then there exist $N \in K$ and isomorphisms g_i from M_i onto $\mathrm{rng}(g_i)$ for $i = 1, 2$ such that $\mathrm{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$. The model N is called an *amalgam* of M_1, M_2 over M with respect to f_1, f_2.

(2) An τ-structure M is an *amalgamation base* (a.b.) for $\mathcal{K} = \langle K, < \rangle$ iff $M \in K$ and whenever for $i = 1, 2$, $M_i \in K$ and f_i is an isomorphism from

M onto $\mathrm{rng}(f_i)$, $\mathrm{rng}(f_i) < M_i$, then there exist $N \in K$ and isomorphisms g_i $(i = 1, 2)$ from M_i onto $\mathrm{rng}(g_i)$ such that $\mathrm{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$.

(3) We say $\mathcal{K} = \langle K, < \rangle$ has AP iff every model in K is an a.b. for $\mathcal{K}$.

Example 1.13. Suppose that T is a theory in first-order logic having an infinite model. Define, for M, N in the class $K_{\leq |T| + \aleph_0}$ of models of T of power at most $|T| + \aleph_0$, $M < N$ iff the indentity on $|M|$ is an embedding of M onto an elementary sub-model of N. Then $\mathcal{K}_{\leq |T|+\aleph_0} = \langle K_{\leq |T|+\aleph_0}, < \rangle$ has AP, (see [2]).

Example 1.14. Suppose that T is a theory in $\mathbb{L}_{\kappa,\aleph_0}$ and $\mathcal{F}$ is a fragment of $\mathbb{L}_{\kappa,\aleph_0}$ containing T with $|\mathcal{F}| < |T|^+ + \kappa$. Let $<$ be the binary relation $\prec_{\mathcal{F}}$ defined on the class K of all models of T. $M \in K$ is an a.b. for $\mathcal{K}$ iff whenever for $i = 1, 2, M_i \in K$ and f_i is an $\prec_{\mathcal{F}}$-elementary embedding of M into M_i, there exist $N \in K$ and $\mathcal{F}$-elementary embeddings $g_i (i = 1, 2)$ of M_i into N such that $g_1 f_1 = g_2 f_2$.

Definition 1.15. Suppose that $<$ is a binary relation on a class K of models. Let μ be a cardinal. $M \in K_{\leq \mu}$ is a *μ-counter amalgamation basis* (μ-c.a.b.) of $\mathcal{K} = \langle K, < \rangle$ iff there are $M_1, M_2 \in K_{\leq \mu}$ and isomorphisms f_i from M into M_i such that:

(a) $\mathrm{rng}(f_i) < M_i (i = 1, 2)$,

(b) there is no amalgam $N \in K_{\leq \mu}$ of M_1, M_2 over M with respect to f_1, f_2.

Observation 1.16. Suppose that $T, \mathcal{F}$ and $<$ are as in Example 1.14 and $\kappa + |T| \leq \mu < \lambda$. Note that if there is an amalgam N' of M_1, M_2 over M (for M_1, M_2, M in $K_{\leq \mu}$), then by Fact 1.11(1) there is an amalgam $N \in K_{\leq \mu}$ of M_1, M_2 over M.

1(B). Replacing T by AEC

On AEC see [23], [32] or [1], recall:

Definition 1.17. We say $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$ is an a.e.c. with L.S.T. number $\lambda(\mathfrak{k}) = \mathrm{LST}_{\mathfrak{k}} = \mathrm{LST}(\mathfrak{k})$, we may write K for $K_{\mathfrak{k}}$, when K is a class of $\tau_{\mathfrak{k}}$-models, $\leq_{\mathfrak{k}}$ a two-place relation on K and

- Ax 0: The holding of $M \in K, N \leq_{\mathfrak{k}} M$ depend on N, M only up to isomorphism, i.e. $[M \in K, M \cong N \Rightarrow N \in K]$ and [if $N \leq_{\mathfrak{k}} M$ and f is an isomorphism from M onto the τ-model M' and $f \restriction N$ is an isomorphism from N onto N' then $N' \leq_{\mathfrak{k}} M'$.]

- Ax I: if $M \leq_{\mathfrak{k}} N$ then $M \subseteq N$ (i.e. M is a sub-model of N).

- Ax II: $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_2$ implies $M_0 \leq_{\mathfrak{k}} M_2$ and $M \leq_{\mathfrak{k}} M$ for $M \in K$.

- Ax III: If λ is a regular cardinal, M_i $(i < \lambda)$ is a $\leq_{\mathfrak{k}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{k}} M_j$) and continuous (i.e. for every limit ordinal $\delta < \lambda, M_\delta = \bigcup_{i<\delta} M_i$) then $M_0 \leq_{\mathfrak{k}} \bigcup_{i<\lambda} M_i$. Hence $M_j \leq_\kappa \bigcup_{i<\lambda} M_i$ for every $j < \lambda$.

- Ax IV: If λ is a regular cardinal and M_i (for $i < \lambda$) is $\leq_{\mathfrak{k}}$-increasing continuous and $M_i \leq_{\mathfrak{k}} N$ for $i < \lambda$ then $\bigcup_{i<\lambda} M_i \leq_{\mathfrak{k}} N$.

- Ax V: If $N_0 \subseteq N_1 \leq_{\mathfrak{k}} M$ and $N_0 \leq_{\mathfrak{k}} M$ then $N_0 \leq_{\mathfrak{k}} N_1$.

- Ax VI: If $A \subseteq N \in K$ and $|A| \leq$ LST$(\mathfrak{k})$ then for some $M \leq_{\mathfrak{k}} N, A \subseteq |M|$ and $\|M\| \leq$ LST$(\mathfrak{k})$ (and LST$(\mathfrak{k})$ is the minimal infinite cardinal satisfying this axiom which is $\geq |\tau|$; the $\geq |\tau|$ is for notational simplicity).

Definition 1.18.
(1) We define "$\mathfrak{k}$ categorical in λ", $\mathfrak{k}_{<\lambda}$, "$\mathfrak{k}$ has amalgamation" "$M \in K_{\mathfrak{k}}$ is a.b.", "M is c.a.b." naturally (see Definitions 1.12 and 1.15).
(2) Let $\mathfrak{k}_\lambda = (K_\lambda, \leq_{\mathfrak{k}}\restriction K_\lambda)$, where $K_\lambda = \{M \in K_{\mathfrak{k}} \colon \|M\| = \lambda\}$.
(3) For $\chi < \lambda$, let $\mathfrak{k}_{[\chi,\lambda)} = (K_{[\chi,\lambda)}, \leq_{\mathfrak{k}}\restriction K_{[\chi,<\lambda)]})$, where $K_{[\chi,\lambda)} = \bigcup\{K_\mu \colon \mu \in [\chi, \lambda)\}$.

So our main theorem is:

Theorem 1.19. *Assume κ is a measurable cardinal, $\mathfrak{k}$ is an AEC, and $\chi =$ LST$_{\mathfrak{k}} + \kappa < \lambda$, and* LST$_{\mathfrak{k}} < \kappa$ *or just $\mathfrak{k}$ is* **D**-*compact (see Definition 1.2(1) and Assumption 1.1). If $\mathfrak{k}$ is categorical in λ then $\mathfrak{k}_{[\chi,\lambda)}$ has amalgamation, see Definition 1.12.*

Proof. First, without loss of generality, assume that Hypothesis 1.45 holds.
[Why? If LST$_{\mathfrak{k}} < \kappa$ then by Claim 1.28(0), without loss of generality $|\tau_{\mathfrak{k}}| \leq 2^{\mathrm{LST}(\mathfrak{k})}$, hence $|\tau_{\mathfrak{k}}| < \chi$ and by Claim 1.41(1), $\mathfrak{k}$ is **D**-compact (see Assumption 1.1). So in any case $\mathfrak{k}$ is **D**-compact and by Claim 1.42, Hypothesis 1.45(1) holds.
By Claim 1.28(1), (2) also Hypothesis 1.45(2) holds. So Hypothesis 1.45 holds indeed.]
Recall that, in §2-§5 we assume Hypothesis 1.45.
Second, if λ is regular, then the desired conclusion holds by §2, that is, by Theorem 2.10.
Third, if λ is singular, then the desired conclusion holds by §5, that is, by Corollary 5.6. $\square$

Claim 1.20. Assume $\mathfrak{k}$ is an AEC and $\tau = \tau_{\mathfrak{k}}$. Then;
There are $\tau_1 = \tau_{\mathfrak{k},1} \supseteq \tau_{\mathfrak{k}}$ of cardinality $|\tau| +$ LST$_{\mathfrak{k}}$ and a set $\mathscr{P}$ of q.f. (quantifier free) 1-types in $\mathbb{L}(\tau_1)$ such that:

(A) a τ-structure M belongs to $K_{\mathfrak{k}}$ iff it can be expanded to a τ_1-model M^+ from K_+, where:

- $K_+ = K^+_{\mathfrak{k}} = \{N\colon N$ a τ_1-structure omitting every $p \in \mathscr{P}\}$.

(B) If $M^+ \in K_+$ and $M^+ \restriction \tau \leq_{\mathfrak{k}} N$ then there is a model $N^+ \in K_+$ expanding N such that $M^+ \subseteq N^+$. Also, for $M, N \in K$, we have $M \leq_{\mathfrak{k}} N$ <u>iff</u> there are expansions $M^+, N^+ \in K_+$ of M, N respectively such that $M^+ \subseteq N^+$.

(C) $(K_+, \subseteq)$ is an AEC with $\mathrm{LST}(K^+, \subseteq) = \mathrm{LST}(\mathfrak{k})$.

(D) There is a set $\tau'_1 \subseteq \tau_1$ of cardinality $\mathrm{LST}_{\mathfrak{k}}$ such that $A \subseteq M^+ \in K_+ \Rightarrow \mathrm{cl}_{\tau'_1}(\mathrm{A}, \mathrm{M}^+) \subseteq \mathrm{M}^+$.

(E) Some $\psi \in \mathbb{L}_{(2^\lambda)^+, \aleph_0}$ defines $K_{\mathfrak{k},1}$ where $\lambda = \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$.

Proof. 1.20 By [32, 1.7]. $\square$

Definition 1.21.
Assume T is a theory in $\mathbb{L}_{\kappa,\aleph_0}(\tau_T)$, τ_T determined by T (so $|T| \leq (|\tau_T| + \kappa)^{<\kappa}$) and recall $\mathcal{F}_T$ is the set of formulas $\varphi(\bar{x})$ such that $\varphi(\bar{x})$ is a sub-formula of some sentence $\psi \in T$. We define $\mathfrak{k} = \mathfrak{k}_T$ as follows:

(A) $K_{\mathfrak{k}}$ is the class of τ_T-models of T of cardinality $\geq \kappa + |T|$.

(B) $M \leq_{\mathfrak{k}} N$ <u>iff</u>:

  (a) $M, N \in K_{\mathfrak{k}}$,
  (b) $M \subseteq N$,
  (c) $M \preceq_{\mathcal{F}} N$ i.e., if $\varphi(\bar{x}) \in \mathcal{F}_T$ (see below, so $\lg(\bar{x})$ is finite and $\bar{a} \in {}^{\lg(\bar{x})}M$) then $M \models \varphi[\bar{a}]$ <u>iff</u> $N \models \varphi[\bar{a}]$.

Claim 1.22. If T is a theory in $\mathbb{L}_{\kappa,\aleph_0}(\tau_T)$, then:

(A) $\mathfrak{k}_T$ is an AEC.

(B) $\mathrm{LST}_{\mathfrak{k}_T} = \mathrm{LST}(\mathfrak{k}_T) \leq |T| + \kappa$.

(C) If $T \subseteq \mathbb{L}_{\lambda^+,\aleph_0}(\tau_T)$ then $\mathrm{LST}_{\mathfrak{k}} \leq |T| + \lambda$.

(D) $\mathfrak{k}_T$ has no model of cardinality $< |\tau| + \kappa$ but for any $\tau(T)$-model M of cardinality $\geq |T| + \kappa$, $M \in K_{\mathfrak{k}_T} \Leftrightarrow M \models T$.

(E) If D is a κ-complete non-principal ultra-filter on κ, <u>then</u> the AEC $\mathfrak{k}$ is D-compact (By Łos's theorem for $\mathbb{L}_{\kappa,\aleph_0}$, even $\mathbb{L}_{\kappa,\kappa}$) proved by Hanf (see Definition 1.40).

Proof. Mainly, this holds by Fact 1.10 and Fact 1.11, but see fully in the proof of Claim 1.25, except clause (E) which is proved in 1.41. $\square$

Conclusion 1.23. To prove our results for $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$ it suffices to prove them for the AEC $\mathfrak{k}_T$(see Definition 1.21).

Proof. By Claim 1.22 just check the definitions and assumptions. □

Definition 1.24. We say the AEC $\mathfrak{k}$ *is* (μ, λ, κ)*-representable* <u>when</u> there are (τ_1, T_1, Γ) such that:

(a) $\tau_1 \supseteq \tau_{\mathfrak{k}}$ has cardinality $\leq \lambda$,

(b) $T_1 \subseteq \mathbb{L}(\tau_1)$ is a first order logic universal theory, so $|T_1| \leq \lambda$,

(c) Γ is a set of $\leq \mu$ qf-types in $\mathbb{L}(\tau_1)$, each of cardinality $< \kappa$,

(d) $M \in K_{\mathfrak{k}}$ <u>iff</u> M is the $\tau_{\mathfrak{k}}$-reduct of some $M_2 \in \mathrm{EC}(T_1, \Gamma)$, where

$$\mathrm{EC}(T_1, \Gamma) = \{N \colon N \text{ a } \tau_1\text{-model of } T_1 \text{ omitting every } p(x) \in \Gamma\},$$

(e) $M \leq_{\mathfrak{k}} N$ <u>iff</u> for every $M_1 \in \mathrm{EC}(T_1, \Gamma)$ expanding M, there is $N_1 \in \mathrm{EC}(T_1, \Gamma)$ expanding N and extending M_1.

Claim 1.25.
(1) Let $\mathfrak{k}$ be an AEC. If $\lambda \geq \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$, then $\mathfrak{k}$ is $(2^\lambda, \lambda, \lambda^+)$-representable.
(2) If $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$ is a theory then $\mathfrak{k}_T$ is $(|\tau| + \kappa, |\tau| + \kappa, \kappa)$-representable. If in addition κ is a limit regular cardinal and $|T| < \kappa$ hence is $\subseteq \mathbb{L}_{\theta,\aleph_0}$ for some $\theta < \kappa$, then it is $(|\mathcal{F}_T|, |\mathcal{F}_T|, \theta)$-representable.

Proof.
(1) By Claim 1.20, that is, by [32] and classical theorems, see e.g. [28, Ch. VII].
(2) Just consider Definition 1.24 and the proof of Claim 1.22. □

1(C). Indiscernibles and Ehrenfeucht-Mostowski structures

The basic results on generalized Ehrenfeucht-Mostowski models can be found in [20] or [26, VII].

Definition 1.26.
(1) We recall here some notation. Let $\mathbf{I}$ be a class of models which we call the *index models.* Denote the members of $\mathbf{I}$ by $I, J \ldots$, etc.
(2) For $I \in \mathbf{I}$ we say that $\langle a_s \colon s \in I \rangle$ is *indiscernible in* M iff the a_s-s are pairwise distinct and for every $\bar{s}, \bar{t} \in {}^{<\omega}I$ realizing the same atomic type in I, $\bar{a}_{\bar{s}}$ and $\bar{a}_{\bar{t}}$ realize the same quatifier free type in M (where $\bar{a}_{\langle s_0,\ldots,s_n \rangle} = \langle a_{s_0}, \ldots, a_{s_n} \rangle$).
(3) Assume $\tau \subseteq \tau'$ are vocabularies and Φ is a function with domain including

$$\{\mathrm{tp}_{\mathrm{at}}(\bar{s}, \emptyset, I) \colon \bar{s} \in {}^{<\omega}I \text{ for some } I \in \mathbf{I}\}$$

and if $\bar{s} \in {}^nI$ then $\Phi(\mathrm{tp}(\bar{s}, \emptyset, I))$ is a complete quantifier free n-type in $\mathbb{L}(\tau')$, let $\tau_\Phi = \tau'$. Moreover, if $I \in \mathbf{I}$, we let $\mathrm{GEM}'(I, \Phi)$ be an τ'-model generated by $\{a_s : s \in I\}$ such[4] that $\mathrm{tp}_{\mathrm{at}}(\bar{a}_{\bar{s}}, \emptyset, M) = \Phi\big(\mathrm{tp}_{\mathrm{at}}(\bar{s}, \emptyset, I)\big)$; $\langle a_s \colon s \in I\rangle$ is called the skeleton.

(4) We say that Φ is *proper for* $\mathbf{I}$ if for every $I \in \mathbf{I}$, $\mathrm{GEM}'(I, \Phi)$ is well-defined.

(5) Let $\mathrm{GEM}(I, \Phi)$ be the τ-reduct of $\mathrm{GEM}'(I, \Phi)$.

Pedantically, we should write $\mathrm{GEM}_\tau(I, \Phi)$ but τ is constant.

(6) For the purposes of this paper we'll let $\mathbf{I}$ be the class $\mathbf{LO}$ of linear orders and Φ will be proper for $\mathbf{LO}$ and then write EM (instead GEM). For $I \in \mathbf{LO}$ we may abbreviate $\mathrm{EM}'(I, \Phi)$ by $\mathrm{EM}'(I)$ and $\mathrm{EM}(I, \Phi)$ by $\mathrm{EM}(I)$, when Φ is clear from the context.

We first deal with pairs $(T, \mathcal{F})$.

Claim 1.27. If $T \subseteq \mathbb{L}_{\kappa,\aleph_0}(\tau)$ is a theory which has a model of cardinality $\geq \kappa$, then there are τ, Φ as in Definition 1.26 such that, for each linear order $I = (I, \leq)$ there exists a Ehrenfeucht-Mostowski model $\mathrm{EM}(I, \Phi)$ is a model of T.

Proof. See Nadel [13] and Dickmann [3] or [26, VII, §5] <u>or</u> see the limit ultrapower below. □

But now we use the AEC framework.

Claim 1.28.

(0) If $\mathfrak{k}$ is an AEC then without loss of generality $\tau_{\mathfrak{k}}$ has cardinality $\leq 2^{\mathrm{LST}(\mathfrak{k})}$. Fully we have $\tau_{\mathfrak{k}}/E_{\mathfrak{k}}$ has $\leq 2^{\mathrm{LST}(\mathfrak{k})}$ equivalent classes when $E_{\mathfrak{k}} = \{(R_1, R_2) \colon R_1, R_2$ are both predicates or both function symbols and are of the same arity and $M \in K_{\mathfrak{k}} \Rightarrow R_1^M = R_2^M\}$.

(1) Assume $\mathfrak{k}$ is an AEC, $\mu = 2^{\mathrm{LST}(\mathfrak{k}) + |\tau(\mathfrak{k})|}$. If $\mathfrak{k}$ has a model of cardinality $\geq \beth_{\mu^+}$ (or just model of cardinality $\geq \beth_\alpha$ for every $\alpha < \mu^+$) <u>then</u> there is Φ such that:

- (a) Φ is as in Definition 1.26,
- (b) $\tau_\Phi = \tau_{\mathfrak{k},1}$, where $\tau_{\mathfrak{k},1}$ is from Claim 1.20 or Definition 1.24,
- (c) $\mathrm{EM}(I) \in K_{\mathfrak{k}}$ has cardinality $\mathrm{LST}_{\mathfrak{k}} + |I|$,
- (d) for (τ_1, T_1, Γ) as in Definition 1.24, every model of the form $\mathrm{EM}'(I)$ is in $\mathrm{EC}(\Gamma, T_1)$ and $\tau_\Phi = \tau_1$.

(2) In particular,

- (a) $\mathrm{EM}'(I)$ is a τ_1-model,
- (b) $\mathrm{EM}(I) = \mathrm{EM}'(I) \restriction \tau$ belongs to K,
- (c) (follows) if $I \subseteq J$ then $\mathrm{EM}(I) \leq_{\mathfrak{k}} \mathrm{EM}(J)$, both models from K of cardinality $|I| + \mathrm{LST}(\mathfrak{k})$.

Proof. As in [32, 1.13], [26, Ch. VII]. □

[4] Equivalently, we can use $\mathrm{tp}_{\mathrm{qf}}$, the quantifier free type.

1(D). Limit ultra-powers, iterated ultra-powers and nice extensions

An important technique we shall use in studying the categoricity spectrum of a theory in $\mathbb{L}_{\kappa,\aleph_0}$ or suitable AECs is the limit ultra-power. It is convenient to record here the well-known definitions and properties of limit and iterated ultra-powers (see Chang and Keisler [2], Hodges-Shelah [6]) and then to examine nice extensions of models.

Definition 1.29. Suppose that M is an τ-structure, I is a non-empty set, D is an ultra-filter on I (but see Definition 1.30(5)), and G is a filter on $I \times I$.

(1) For each $g \in {}^I|M|$, let

(a) $\mathrm{eq}(g) \coloneqq \{\langle i,j\rangle \in I \times I \colon g(i) = g(j)\}$, and

(b) $g/D \coloneqq \{f \in {}^I|M| \colon g = f \text{ Mod } D\}$ where,

$$g = f \text{ Mod } D \text{ iff } \{i \in I \colon g(i) = f(i)\} \in D.$$

(2) Let $\prod\limits_{D/G}|M| \coloneqq \{g/D \colon g \in {}^I|M| \text{ and } \mathrm{eq}(g) \in G\}$. Note that $\prod\limits_{D/G}|M|$ is a non-empty subset of $\Pi_D|M| = \{g/D \colon g \in {}^I|M|\}$ and is closed under the constants and functions of the ultra-power $\Pi_D M$ of M modulo D.

(3) The limit ultra-power $\prod\limits_{D/G} M$ of the τ-structure M (with respect to (I,D,G)) is the substructure of $\Pi_D M$ whose universe is the set $\prod\limits_{D/G}|M|$. The canonical map d from M into $\prod\limits_{D/G} M$ is defined by $d(a) = \langle a_i : i \in I\rangle/D$, where $a_i = a$ for every $i \in I$.

(4) Note that the limit ultra-power $\prod\limits_{D/G} M$ depends only on the equivalence relations which are in G, i.e. if $\mathbf{E}$ is the set of all equivalence relations on I and $G \cap \mathbf{E} = G' \cap \mathbf{E}$, where G' is a filter on $I \times I$, then $\prod\limits_{D/G} M = \prod\limits_{D/G'} M$.

Definition 1.30. Assume,

(a) M be an τ-structure, $\langle Y, <\rangle = \langle Y, <_Y\rangle$ a linear order,

(b) for each $y \in Y$, let D_y be an ultra-filter on a non-empty set I_y,

(c) $\bar{I} = \langle I_y \colon y \in Y\rangle$,

(d) $\bar{D} = \langle D_y \colon y \in Y\rangle$,

(e) $I = \prod\limits_{y \in Y} I_y$.

Then,

(1) Let $E = \prod\limits_{y \in Y} D_y$ be the set of $s \subseteq I$ such that there are $y_1 < \cdots < y_n$ in Y satisfying:

(α) for all $i, j \in I$, if $i \restriction \{y_1, \dots, y_n\} = j \restriction \{y_1, \dots, y_n\}$ then $i \in s$ iff $j \in s$,

(β) $\{\langle i(y_1), \dots, i(y_n)\rangle : i \in s\} \in D_{y_1} \times \cdots \times D_{y_n}$.

(2) The iterated ultra-power $\prod_{\bar{D}} |M|$ or $\prod_E |M|$ of the set $|M|$, noting E is a filter on I, is the set $\{f/E \colon f \in {}^I|M|$ and for some finite $Z_f \subseteq Y$ for all $i, j \in I$, if $i \restriction Z_f = j \restriction Z_f$, then $f(i) = f(j)\}$.

(2A) Note that $\langle Y, <\rangle, \bar{I}, \bar{D}, E, I$ and E can be defined from E and can be defined from $\bar{D}$, so we may indeed write Π_E, $\Pi_{\bar{D}}$ above.

(3) The iterated ultra-power $\prod_E M$ of the τ-structure M with respect to $\langle D_y : y \in Y\rangle$ is the τ-structure whose universe is the set $\Pi_E|M|$; for each n-ary predicate symbol R of $L, R^{\Pi_E M}(f_1/E, \dots, f_n/E)$ iff $\{i \in I \colon R^M(f_1(i), \dots, f_n(i))\} \in E$; for each n-ary function symbol F of L, $F^{\Pi_E M}(f_1/E, \dots, f_n/E) = \langle F^M(f_1(i), \dots, f_n(i)) \colon i \in I\rangle/E$.

(4) The canonical map $d \colon M \to \Pi_E M$ is defined as usual by:

$$d(a) = \langle a \colon i \in H\rangle/E.$$

(5) In Definition 1.29, we do not need "D is an ultra-filter on I", just "D is a filter on I such that, if $e \in G$ is an equivalence relation on I, then D/e is an ultra-filter on I/e".

(6) We say $\mathbf{u}$ is an iterated ultra-powers parameter when it consists of $\langle Y, <\rangle$, $\bar{I} = \langle I_y \colon y \in Y\rangle$, $\bar{D} = \langle D_y \colon y \in Y\rangle$ and I as in the beginning of Definition 1.30, E as in Definition 1.30(1) and

- $G = \{e\colon$ is an equivalence relation on I such that, for some finite subset Z of Y, we have $f, g \in I \wedge f \restriction Z = g \restriction Z \Rightarrow f \; e \; g\}$.

(6A) So $\mathbf{u} = (Y_{\mathbf{u}}, \dots,)$ definable from $\bar{D}_{\mathbf{u}}$ and from $E_{\mathbf{u}}$ and we may write $\prod_{\mathbf{u}}$.

Remark 1.31.

(1) Every ultra-power is a limit ultra-power: take $G = \mathscr{P}(I \times I)$ and note that $\Pi_D M = \underset{D/G}{\Pi} M$.

(2) Every iterated ultra-power is a limit ultra-power, hence in Definition 1.30 we may write $\mathrm{Op}_{\bar{D}}$, Op_E or $\mathrm{Op}_{\mathbf{u}}$.

[Why? let the iterated ultra-power be defined by $\langle Y, <\rangle$ and $\langle (I_y, D_y) \colon y \in Y\rangle$ (see Definition 1.30). For $Z \in [Y]^{<\omega}$, let $A_Z = \{(i, j) \in I \times I \colon i \restriction Z = j \restriction Z\}$. Note that $\{A_Z \colon Z \in [Y]^{<\omega}\}$ has the finite intersection property and hence can be extended to a filter G on $I \times I$. Now for any model M we have $\Pi_E M \cong \underset{D/G}{\Pi} M$ for every filter D over I extending E under the map $f/E \to f/D$.]

Definition 1.32.

(1) We say that (I, D, G) is *suitable* when:

(a) D is an ultra-filter on a non-empty set I (or just a filter, see Definition 1.30(5)),

(b) G is a suitable, pedantically a D-suitable filter on $I \times I$ or just a set of equivalence relations on I, which means:

(i) if $e \in G$ and e' is an equivalence relation on I coarser than e, then $e' \in G$,

(ii) G is closed under finite intersections,

(iii) (I, D, G) is κ-complete, which means that, if $e \in G$, then $D/e = \{A \subseteq I/e : \bigcup_{x \in A} x \in D\}$ is a κ-complete ultra-filter on I/e which, for simplicity, has cardinality κ.

(2) A an iterated ultra-power parameter $\mathbf{u}$ is *suitable* when $(I_{\mathbf{u}}, E_{\mathbf{u}}, G_{\mathbf{u}})$ is.

(3) Suppose that M is an τ-structure and (I, D, G) is suitable. Then $\mathrm{Op}(M, I, D, G) = \mathrm{Op}_{I,D,G}(M)$ is the limit ultra-power $\prod_{D/\hat{G}} M$ where $\hat{G}$ is the filter on $I \times I$ generated by G. When clear from the context one abbreviates $\mathrm{Op}(M, I, D, G)$ by $\mathrm{Op}(M)$, pedantically Op stand for $\mathrm{Op}_{I,M,G}$ and one writes $f_{\mathrm{Op}} = f_{\mathrm{Op},M}$ for the canonical map $d \colon M \to \mathrm{Op}(M)$; so we may write f_{Op} or f^M_{Op} instead $f_{\mathrm{Op},M}$ when M is clear from the context.

Recall that,

***Observation / Convention* 1.33.**

(1) For any τ-structure $N, f_{\mathrm{Op}} = f_{\mathrm{Op},N}$ is an elementary embedding of N into $\mathrm{Op}(N)$ and if $N \in K_{\mathfrak{k}}$ then $f_{\mathrm{Op}} : N \underset{\mathfrak{k}}{\to} \mathrm{Op}(N)$.

(2) Since f_{Op} is canonical, one very often identifies N with the τ-structure $\mathrm{rng}(f_{\mathrm{Op}})$ which is an $\mathfrak{k}$-elementary substructure of $\mathrm{Op}(N)$, and one writes $N \leq_{\mathfrak{k}} \mathrm{Op}(N)$. In particular for any model $M \in K$ and $\mathrm{Op}, f_{\mathrm{Op}} \colon M \to_{\mathfrak{k}} \mathrm{Op}(M)$ (briefly written $M \leq_{\mathfrak{k}} \mathrm{Op}(M)$) so that $\mathrm{Op}(M)$ is a model from K too.

(3) Remark that if D is a κ-complete ultra-filter on I and G is a filter on $I \times I$, then $\mathrm{Op}(M, I, D, G)$ is well defined.

(4) Suitable limit ultra-power means one using a suitable triple, for such Op in Observation/Convention 1.33(2) we get a $\mathbb{L}_{\kappa,\aleph_0}$-elementary embedding.

More information on limit and iterated ultra-powers can be found in [2] and [6].

Observation 1.34. (1) Given κ-complete ultra-filters D_1 on I_1, D_2 on I_2 and suitable filters G_1 on $I_1 \times I_1, G_2$ on $I_2 \times I_2$ respectively, there exist a κ-complete ultra-filter D on a set I and a filter G on $I \times I$ such that:

$$\mathrm{Op}(M, I, D, G) = \mathrm{Op}(\mathrm{Op}(M, I_1, D_1, G_1), I_2, D_2, G_2)$$

and (D, G, I) is κ-complete.

(2) Also iterated ultra-power (along any linear order) with each iterand being ultra-power by κ-complete ultra-filter, gives a suitable triple (in fact, even iteration of suitable limit ultra-powers is a suitable ultra-power).

Definition 1.35. Suppose that K is a class of τ-structures and $<=<_{\mathcal{K}}$ is a binary relation on K (usually $(K,<) = (K_{\mathfrak{k}}, <_{\mathfrak{k}})$). For $M, N \in K$, write $f\colon M \leq_{\mathcal{K}}^{\text{nice}} N$ to mean (if $<$ is clear from the context we may write $f\colon M \underset{\text{nice}}{\rightarrow} N$ and, if $f = \text{id}_M$ we may write $M \underset{\text{nice}}{\leq} N$):

(a) f is an isomorphism from M onto $\text{rang}(f) = N \restriction \text{rang}(f)$ and $\text{rng}(f) < N$. Which means $f(M) < N$, where $f(M)$ is the model M' with universe $\text{rng}(f)$ such that f is an isomorphism from M into M',

(b) for some[5] ultra-limit parameter $\mathbf{u} = (Y, <_Y, \bar{I}, \bar{D}, I, E, G)$, so G is a suitable set of equivalence relations on I (so Definition 1.32 clause (i), (ii), (iii) holds) each[6] D_i is isomorphic to $\mathbf{D}$, and an isomorphism g from N onto $\text{rang}(g) = \text{Op}(M, I, E, G) \restriction \text{rang}(g)$, such that $\text{rng}(g) < \text{Op}(M, I, E, G)$ and $gf = f_{\text{Op}}$, where f_{Op} is the canonical embedding of M into $\text{Op}(M, I, E, G)$. Then f is called a $<$-nice embedding of M into N. Of course, one writes $f\colon M \underset{\text{nice}}{\rightarrow} N$ and says that f is a nice embedding of M into N when $<$ is clear from the context.

Example 1.36. Consider $T, \mathcal{F}$ and $\mathcal{K} = \langle K, < \rangle = (K, <_{\mathcal{K}})$ as set up in Example 1.14. In this case $f\colon M \underset{\text{nice}}{\rightarrow} N$ holds iff $f\colon M \underset{\mathcal{F}}{\rightarrow} N$ and for some suitable ultra-limit parameter $\mathbf{u}$ and some $g\colon N \underset{\mathcal{F}}{\rightarrow} \text{Op}_{\mathbf{u}}(M)$ we have $gf = f_{\text{Op}}$.

Abusing notation one may writes $M \underset{\text{nice}}{\rightarrow} N$ to mean that there are f, g and Op such that $f\colon M \underset{\text{nice}}{\rightarrow} N$ using g and Op. IF NOT SAID OTHERWISE, $<$ is $<_{\mathfrak{k}}$. We may also write $M \underset{\text{nice}}{\leq} N$, and for linear orders we use $I \underset{\text{nice}}{\subseteq} J$.

Example 1.37. Let **LO** be the class of linear orders and let $(I, \leq_I) < (J, \leq_J)$ mean that $(I, \leq_I) \subseteq (J, \leq_J)$, i.e. $(I, \leq_I)$ is a suborder of $(J, \leq_J)$. If $f : (I, \leq_I) \underset{\text{nice}}{\rightarrow} (J, \leq_J)$, then identifying isomorphic orders, one has $(I, \leq_I) \subseteq (J, \leq_J) \subseteq \text{Op}(I, \leq_I)$ and we may write $(I, \leq_I) \underset{\text{nice}}{\subseteq} (I, \leq_J)$.

Observation 1.38. Assume that $\mathcal{K} = (K, <_K)$ is as in Def. 1.35. Suppose further $M \underset{\text{nice}}{\leq} N$ and $M \subseteq M' \leq_{\mathfrak{k}} N$ where $M, M', N \in K$. Then $M \underset{\text{nice}}{\leq} M'$.

---

[5]We could use here and Theorem 2.5 suitable tuples (I, D, G). However, then we have to add to the definition of "$\mathfrak{k}$ is (I, D, G)-compact" a clause saying:

(∗) if $M \in K_{\mathfrak{k}}$, $e_1 \supseteq e_2$ are from G and $M_\ell = \prod_{D/G} M \restriction \{f \in {}^I M\colon \text{eq}(f) \supseteq e_\ell\}$ for $\ell = 1, 2$, then $M_1 \leq_{\mathfrak{k}} M_2$.

In [9] this issue does not arise.

[6]Can fix a family of filters.

Proof. For some f, g and Op, $f\colon M \underset{\mathfrak{k}}{\to} N$, $g\colon N \underset{\mathfrak{k}}{\to} \mathrm{Op}(M)$ and $gf = f_{\mathrm{Op}}$. Now $g\colon M' \underset{\mathfrak{k}}{\to} \mathrm{Op}(M)$ (since $M' \leq_{\mathfrak{k}} N$) and $gf = f_{\mathrm{Op}}$ so that $M \underset{\mathrm{nice}}{\leq} M'$. □

Observation 1.39. Suppose that δ is any ordinal, $\langle M_i \colon i \leq \delta\rangle$ is a continuous increasing chain and for each $i < \delta$, $M_i \underset{\mathrm{nice}}{\leq} M_{i+1}$. Then for every $i < \delta$, $M_i \underset{\mathrm{nice}}{\leq} M_\delta$.

Proof. Like the proof of Remark 1.31(2). For each $i < \delta$, there is a $\mathbf{u}_i$ as in Definition 1.32 which witnesses $M_i \underset{\mathrm{nice}}{\leq} M_{i+1}$ and and let $Y_i = Y_{\mathbf{u}_i}$ for $i < \delta$. Without loss of generality, $\langle Y_i \colon i < \delta\rangle$ are pairwise disjoint. We define $\mathbf{u}$ by:

(a) $Y = \bigcup\{Y_i \colon i < \delta\}$,

(b) $s <_y t$ iff $\bigvee_{i<\delta} s <_i t$ or $s \in Y_{\mathbf{u}_i} \wedge t \in Y_{\mathbf{u}_j} \wedge i < j$,

(c) $D_j = D_{\mathbf{u}_i, s}$ when $s \in Y_{\mathbf{u}_i}$ for $i < \delta$.

This is enough and the rest should be clear. □

Definition 1.40.

(1) Assume D is an ultra-filter on κ. For an AEC $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$ we say $\mathfrak{k}$ is *D-compact* <u>when</u>:

(a) if $M \in K_{\mathfrak{k}}$ then the ultra-power M^κ/D belongs to $K_{\mathfrak{k}}$,

(b) moreover, the canonical embedding of M into M^κ/D is a $\leq_{\mathfrak{k}}$-embedding,

(c) if $M \leq_{\mathfrak{k}} N$ then the canonical embedding of M^κ/D into N^κ/D is a $\leq_{\mathfrak{k}}$-embedding,

(d) $\mathfrak{k}$ has a model of cardinality $\geq \kappa$ (or at least of cardinality $\geq \theta$ where D is not θ-complete).

(2) If $\mathbf{u} = (Y, \bar{I}, \bar{D}, I, E, G)$ is as in Definition 1.30, then for an AEC $\mathfrak{k}$ we say $\mathfrak{k}$ is $\mathbf{u}$-compact and E-compact <u>when</u>:

(a) if $M \in K_{\mathfrak{k}}$ and $\prod_E M \in K_{\mathfrak{k}}$,

(b) moreover, the canonical embedding of M into $\prod_E M$ is a $\leq_{\mathfrak{k}}$-embedding,

(c) if $M \leq_{\mathfrak{k}} N$ then the canonical embedding of $\prod_E M$ into $\prod_E N$ is a $\leq_{\mathfrak{k}}$-embedding.

Claim 1.41. Assume D is a non-principal κ-complete ultra-filter (usually on κ).

(1) If $\mathfrak{k}$ is an AEC and $|\tau_{\mathfrak{k}}| + \mathrm{LST}(\mathfrak{k}) < \kappa$ <u>then</u> $\mathfrak{k}$ is D-compact.

(2) If $\mathfrak{k}$ is (μ, λ, κ)-representable, <u>then</u> $\mathfrak{k}$ is D-compact.

(3) Also the claim on Op generalizes, that is, if $\langle Y, <\rangle, \bar{I}, \bar{D}, E, I$ is as in Definition 1.30 and $\mathfrak{k}_s$ is D_s-compact for every $s \in Y$ then in (1) and (2), $\mathfrak{k}$ is E-compact.

(4) So if there is one ultra-filter D on κ which is normal or just non-principal κ-complete ultra-filter on κ, then for every linear order $\langle Y, <\rangle$ then we can find $\bar{I}, \bar{D}, E, I$ such that they together are as in 1.41(3)

Proof.

(1) By 1.25 and part (c).

(2), (3), (4) Easy. □

Claim 1.42. Assume D is a non-principal κ-complete ultra-filter on κ and $\mathfrak{k}_1$ is a D-compact AEC, $\chi \geq \mathrm{LST}_{\mathfrak{k}_1}$ and let $\mathfrak{k}_2 = (\mathfrak{k}_1)_{[\chi,\infty)}$, see Definition 1.18(3).

(1) If $\chi \geq \kappa$ and $\mathfrak{k}_1$ is D-compact then $\mathfrak{k}_2$ is D-compact.

(2) If $\lambda \geq \chi$, then $\mathfrak{k}_1$ is categorical in λ iff $\mathfrak{k}_1$ is categorical in λ.

(3) If $\lambda \geq \chi$, then $(\mathfrak{k}_2)_{[\chi,<\lambda)}$ has amalgamation iff $(\mathfrak{k}_1)_{[\chi,<\lambda)}$ has amalgamation.

Proof. Straightforward. □

Remark 1.43.

(1) Claim 1.41 justifies the assumption $\mathrm{LST}_{\mathfrak{k}} \geq \chi$ in Hypothesis 1.45 below (e.g. to prove 1.19).

(2) Usually λ denotes a power in which $\mathfrak{k}$ is categorical.

Claim 1.44. For every model M of cardinality $\geq \kappa$ and $\lambda \geq \kappa + \mathrm{LST}_{\mathfrak{k}} + \|M\|$ there is N such that $M \underset{\text{nice}}{\leq} N$, $M \neq N$ and $\|N\| = \lambda$.

Proof. As $\mathfrak{k}$ is $\mathbf{D}$-compact, by Assumption 1.2(2) no $M \in K_{\geq\kappa}$ is $\leq_{\mathfrak{k}}$-maximal, so by Definition 1.17 we are done. □

For the rest of this work,

Hypothesis 1.45. Assume $\chi \geq \kappa$.

(1) $\mathfrak{k}$ is a $\mathbf{D}$-compact AEC with $\mathrm{LST}_{\mathfrak{k}} = \chi$, no $M \in K_{\mathfrak{k}}$ has cardinality $< \chi$, $\mathbf{D}$ a κ-complete non-principal ultra-filter on κ, $K = K_{\mathfrak{k}}$ and similarly for any $\langle Y, <\rangle, \bar{I}, \bar{D}, I$ or E derived from $\mathbf{D}$ as in Definition 1.30.

(2) Φ, $\mathbf{a} = \langle a_s \colon s \in I\rangle$ are as in Definition 1.26 for $\mathfrak{k}$ with τ_Φ of cardinality $\leq \chi$, hence $\lambda \geq \chi \Rightarrow (K_{\mathfrak{k}})_\lambda \neq \emptyset$.

2. The amalgamation property for regular categoricity

The main aim of this section is to show that if K is categorical in the regular cardinal $\lambda > \mathrm{LST}_{\mathfrak{k}}$, then $\mathfrak{k}_{<\lambda} = \langle K_{<\lambda}, \leq_{\mathfrak{k}}\rangle$ has the amalgamation property (AP) (Definition 1.12 (1)). Categoricity is not presumed if not required.

Recall Hypothesis 1.45 is assumed.

Lemma 2.1. *Suppose that $\chi \leq \mu \leq \lambda, M, M_1, M_2 \in K_{\leq\mu}, f_1 \colon M \underset{nice}{\to} M_1, f_2 : M \underset{\mathfrak{k}}{\to} M_2$. Then there is an amalgam $N \in K_{\leq\mu}$ of M_1, M_2 over M with respect to f_1, f_2.*

Moreover, there are N and $g_\ell \colon M_\ell \underset{\mathfrak{k}}{\to} N$ for $\ell = 1,2$ such that $g_1 f_1 = g_2 f_2$ hence $\mathrm{rng}(g_2 f_2) = \mathrm{rng}(g_1 f_1)$ and $g_1 \colon M_1 \underset{\mathrm{nice}}{\to} N$.

Proof. There are g and Op such that $g \colon M_1 \underset{\mathfrak{k}}{\to} \mathrm{Op}(M)$, and $g f_1 = f_{\mathrm{Op},M}$. Now, f_2 induces an $\leq_{\mathfrak{k}}$-elementary embedding f_2^* of $\mathrm{Op}(M)$ into $\mathrm{Op}(M_2)$ such that $f_2^* f_{\mathrm{Op}}^M = f_{\mathrm{Op}}^{M_2} f_2$. Let $g_1 = f_2^* g$ and $g_2 = f_{\mathrm{Op},M_2}$. By Fact 1.11 one finds $N \in K_{\leq_\mu}$ such that $\mathrm{rng}(g_1) \cup \mathrm{rng}(g_2) \subseteq N \leq_{\mathfrak{k}} \mathrm{Op}(M_2)$. Now N is an amalgam of M_1, M_2 over M with respect to f_1, f_2 since $g_1 f_1 = f_2^* g f_1 = f_2^* f_{0_p,M} = f_{\mathrm{Op},M_2} f_2 = g_2 f_2$. The last phrase in the lemma is easy by properties of Op. $\square$

Lemma 2.2. *Suppose that $M \in K_{\leq\mu}$ is a μ-c.a.b., $\chi \leq \mu < \lambda$. Then $N \in K_{<\lambda}$ is a $\|N\|$-c.a.b. whenever $f \colon M \underset{nice}{\to} N$.*

Proof. By the assumption, there is $g \colon N \underset{\mathfrak{k}}{\to} \mathrm{Op}(M)$ such that $gf = f_{\mathrm{Op},M}$. Recall M is a μ- c.a.b., so for some $M_i \in K_{\leq\mu}$ and $f_i \colon M \underset{\mathfrak{k}}{\to} M_i$ (for $i = 1,2$) there is no amalgam of M_1, M_2 over M w.r.t. f_1, f_2. Let f_i^* be the $\leq_{\mathfrak{k}}$-elementary embedding from $\mathrm{Op}(M)$ into $\mathrm{Op}(M_i)$ induced by f_i (note that $f_i^* f_{\mathrm{Op},M} = f_{\mathrm{Op},M_i} f_i, i = 1,2$). Choose N_i of power $||N||$ such that $M_i \cup \mathrm{rng}(f_i^* g) \subseteq N_i \leq_{\mathfrak{k}} \mathrm{Op}(M_i)$. Note that $f_i^* g \colon N \underset{\mathfrak{k}}{\to} N_i$. It suffices to show that there is no amalgam of N_1, N_2 over N w.r.t. $f_1^* g, f_2^* g$.

Well, suppose that one could find an amalgam N^* and $h_i \colon N_i \underset{\mathfrak{k}}{\to} N^*, i = 1,2$, with $h_1(f_1^* g) = h_2(f_2^* g)$. Using Fact 1.11 choose $M^*, ||M^*|| \leq \mu, M^* \leq_{\mathfrak{k}} N^*, \mathrm{rng}(h_1 f_{\mathrm{Op}} \restriction M_1) \cup \mathrm{rng}(h_2 f_{\mathrm{Op}} \restriction M_2) \subseteq |M^*|$. Set $g_i = h_i f_{\mathrm{Op}} \restriction M_i$, for $i = 1,2$, and note that:

$$\begin{aligned} g_1 f_1 &= h_1 f_{\mathrm{Op}} f_1 = h_1 f_1^* f_{\mathrm{Op}} = h_1 f_1^* g f = h_2 f_2^* g f = h_2 f_2^* f_{\mathrm{Op}} \\ &= h_2 f_{\mathrm{Op}} f_2 = g_2 f_2. \end{aligned}$$

In other words, M^* is an amalgam of M_1, M_2 over M w.r.t. f_1, f_2-contradiction. It follows that N is a $\|N\|$-c.a.b. $\square$

Corollary 2.3. *Suppose that μ, λ satisfy $\chi \leq \mu < \lambda$. If $M \in K_\mu$ is a μ-c.a.b., then there exists $M^* \in K_\lambda$ such that:*

(∗) *$M \leq_{\mathfrak{k}} M^*$ and for every $M' \in K_{<\lambda}$, if $M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M^*$, then M' is a $||M'||$-c.a.b.*

Proof. As $\|M\| \geq \kappa$, for some appropriate Op one has $||\mathrm{Op}(M)|| \geq \lambda$, and by Fact 1.11 one finds $M^* \in K_\lambda$ such that $M \subseteq M^* \leq_{\mathfrak{k}} \mathrm{Op}(M)$, hence $M \leq_{\mathfrak{k}} M^*$. Let us check that M^* works in (∗). Take $M' \in K_{<\lambda}, M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M^*$; so $M \underset{\mathrm{nice}}{\leq} M'$ since $M^* \leq_{\mathfrak{k}} \mathrm{Op}(M)$, see Observation 1.38; hence by Lemma 2.2, M' is a $||M'||$-c.a.b. $\square$

Theorem 2.4. *Suppose that $\mathfrak{k}$ is λ-categorical, $\lambda = \mathrm{cf}(\lambda) > \chi$. If $K_{<\lambda}$ fails AP, <u>then</u> there is $N^* \in K_\lambda$ such that for some continuous increasing $\leq_{\mathfrak{k}}$-chain $\langle N_i \in K_{<\lambda} : i < \lambda\rangle$ of models,*

(1) $N^* = \bigcup\limits_{i<\lambda} N_i$,

(2) for every $i < \lambda, N_i \underset{nice}{\not\leq} N_{i+1}$ (and so $N_i \underset{nice}{\not\leq} N^*$).

***Proof*.** By an assumption $\mathfrak{k}_{<\lambda}$ fails AP, so for some $\mu \in [\chi, \lambda)$ and $M \in K_{\leq\mu}, M$ is a μ-c.a.b. recalling Definition 1.15. By Lemma 2.2 and Claim 1.44 without loss of generality $M \in K_\mu$. Choose by induction a continuous strictly increasing $\leq_{\mathfrak{k}}$-chain $\langle N_i \in \mathrm{K}_{<\lambda} : i < \lambda\rangle$ as follows:

$N_0 = M$; at a limit ordinal i, take the union; at a successor ordinal $i = j+1$, if there is $N \in K_{<\lambda}$ such that $N_j \leq_{\mathfrak{k}} N$ and $N_j \underset{\text{nice}}{\not\leq} N$ (so necessarily $N_j <_{\mathfrak{k}} N$), choose $N_i = N$, otherwise choose for N_i any non-trivial $\leq_{\mathfrak{k}}$-elementary extension of N_j of power less than λ. Next, we prove:

$\boxplus$ $(\exists j_0 < \lambda)(\forall j \in (j_0, \lambda))(N_j$ is a $||N_j||$-c.a.b.).

Why $\boxplus$ holds? Suppose not. So one has a strictly increasing sequence $\langle j_i : i < \lambda\rangle$ such that for each $i < \lambda, N_{j_i}$ is not a $\|N_{j_i}\|$-c.a.b. Let $N_* = \bigcup\limits_{i<\lambda} N_{j_i}$. So $\|N_*\| = \lambda$. Applying 2.3 one can find $M^* \in K_\lambda$ such that $M \in K_{<\lambda}$ and $M \leq_{\mathfrak{k}} M^*$ and whenever $M' \in K_{<\lambda}$ and $M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M^*$, then M' is a $\|M'\|$-c.a.b.

Since $\mathfrak{k}$ is λ-categorical, there is an isomorphism g of N_* onto M^*. Let $N = g^{-1}(M)$ and $M_i = g^{-1}(N_i)$ for $i < \lambda$. Now, $||N|| = \mu < \mathrm{cf}(\lambda) = \lambda$, so there is $i_0 < \lambda$ such that $N \subseteq N_{j_{i_0}}$, hence $N \leq_{\mathfrak{k}} N_{j_{i_0}}$.

In fact $N_{j_{i_0}}$ is a $||N_{j_{i_0}}||$-c.a.b. [Otherwise, consider $N_{j_{i_0}}$. Since $M \leq_{\mathfrak{k}} f^{-1}(N_{j_{i_0}}) \leq_{\mathfrak{k}} M^*$ and $\|M_{j_{i_0}}\| < \lambda, M_{j_{i_0}}$ is a $\|M_{j_{i_0}}\|$-c.a.b., so there are $f_\ell \colon M_{j_{i_0}} \underset{\mathcal{F}}{\to} M'_\ell, (\ell = 1, 2)$, with no amalgam of M'_1, M'_2 over $M_{j_{i_0}}$ w.r.t. f_1, f_2. If $N_{j_{i_0}}$ is *not* a $||N_{j_{i_0}}||$-c.a.b., then one can find an amalgam $N^+ \in K_{\leq ||N_{j_{i_0}}||}$ of M'_1, M'_2 over $N_{j_{i_0}}$ w.r.t. $f_1 g, f_2 g$ such that $h_\ell \colon M'_\ell \underset{\mathfrak{k}}{\to} N^+$ and $h_1(f_1 g) = h_2(f_2 g)$; so $h_1 f_1 = h_2 f_2$ and N^+ is thus an amalgam of M'_1, M'_2 over $M_{j_{i_0}}$ w.r.t. $f_1, f_2, \|N^+\| \leq \|N_{j_{i_0}}\| = ||M_{j_{i_0}}||$-contradiction.] This contradicts the choice of $N_{j_{i_0}}$. So the statement $\boxplus$ is correct.

It follows that for each $j \in (j_0, \lambda)$ there are N_j^1, N_j^2 in $\mathrm{K}_{<\lambda}$ and $f_\ell \colon N_j \to_{\mathfrak{k}} N_j^\ell$ such that no amalgam of N_j^1, N_j^2 over N_j w.r.t. f_1, f_2 exists. By Lemma 2.1 for both $\ell \in \{1, 2\}, N_j \underset{\text{nice}}{\not\leq} N_{j+1}^\ell$. So by the inductive choice of $\langle N_{j+1} : j < \lambda\rangle, \forall j \in (j_0, \lambda)(N_j \underset{\text{nice}}{\not\leq} N_{j+1})$. Taking $N^* = \bigcup\limits_{j_0<j<\lambda} N_j$, one completes the proof (of course for $j_0 < j < \lambda, N_j \underset{\text{nice}}{\not\leq} N^*$: if $N_j \underset{\text{nice}}{\leq} N^* \leq_{\mathfrak{k}} \mathrm{Op}(N_j)$, then by Observation 1.38 $N_j \underset{\text{nice}}{\leq} N_{j+1}$-contradiction). $\square$

Theorem 2.5. *Suppose that* $I = (I, <_I)$, $J = (J, <_J)$ *are linear orders and* I *is a suborder of* J. *Let* $\mathrm{EM}'(I, \Phi)$ *be as in Definition 1.26, so let* $\langle a^1_s \colon s \in I\rangle$ *be a skeleton of* $M'_1 = \mathrm{EM}'(I) = \mathrm{EM}'(I, \Phi)$, *a* τ_Φ*-model,* $\langle a^1_s \colon s \in I\rangle$ *is an indiscernible sequence in* $\mathrm{EM}'(I)$ *which generates it. Similarly,* $M'_2 = \mathrm{EM}'(J, \Phi)$, $\langle a_s \in s \in J\rangle$ *and as standard, we assume* $M'_1 \subseteq M'_2$, $s \in I \Rightarrow a^1_s = a_s$, *let* $M_\ell = \mathrm{EM}(I) = M'_\ell \restriction \tau_{\mathfrak{k}}$. *If* $(I, <_I) \underset{nice}{\subseteq} (J, <_J)$, *then* $\mathrm{EM}(I) \underset{nice}{\leq} \mathrm{EM}(J)$.

Proof. So there is a suitable ultra-limit[7] parameter $\mathbf{u} = (Y, <_Y, \bar{I}, \bar{D}, I_{\mathbf{u}}, E, G)$ witnessing $(I, <_I) \underset{\text{nice}}{\subseteq} (J, <_J)$, that is, we have $(I, \leq_I) \subseteq \mathrm{Op}_{I,D,G}((I, <_I))$ and $(J, <)$ is isomorphic over $(I, \leq)$ to some $(J', <)$ such that $(I, <_I) \subseteq (J', <) \subseteq \mathrm{Op}_{I,D,G}((I, <))$ and let π be such isomorphism.

So for each $t \in J$, there exists $f_t \in {}^{I(\mathbf{u})}I$ such that $\pi(t) = f_t/D$. Note that if $t \in I$, then $f_t/D = f_{\mathrm{Op}}(t)$ so that without loss of generality for all $i \in I_{\mathbf{u}}, f_t(i) = t$. Define a map h from $\mathrm{EM}(J)$ into $\mathrm{Op}(\mathrm{EM}(I))$ as follows. An element of $\mathrm{EM}(J)$ has the form

$$\sigma^{\mathrm{EM}'(J)}(a_{t_1}, \ldots, a_{t_n}),$$

where $t_1, \ldots, t_n \in J$, σ an τ_Φ-term. Define, for $t \in J, g_t \in {}^I\,\mathrm{EM}(I)$ by $g_t(i) = a_{f_t(i)}$.

Note that $f_t(i) \in I$, so that $a_{f_t(i)} \in \mathrm{EM}(I)$ and so $g_t/D \in \mathrm{Op}(\mathrm{EM}(I))$. Let $h(\sigma^{\mathrm{EM}'(J)}(a_{t_1}, \ldots, a_{t_n})) = \sigma^{\mathrm{Op}(\mathrm{EM}'(I))}(g_{t_1}/D, \ldots, g_{t_n}/D)$ which is an element in $\mathrm{Op}(\mathrm{EM}(I))$. The reader is invited to check that h is an $\leq_{\mathfrak{k}}$-elementary embedding of $\mathrm{EM}(J)$ into $\mathrm{Op}(\mathrm{EM}(I))$, and consequently $\mathrm{EM}(I) \leq_{\mathfrak{k}} \mathrm{EM}(J)$, but we elaborate. Prove by induction on $n < \omega$ that:

- $\oplus$ if $\bar{s} = \langle s_i \colon i < n\rangle$ is $<_Y$-increasing then let $m \leq n$ and $N_{\bar{s}} := M_2 \restriction \{f \in {}^H M \colon \ \mathrm{eq}(f)$ is refined by $\mathrm{eq}_{\bar{s}\restriction m} = \{(h_1, h_2) \colon h_1, h_2 \in \prod_{s \in I} I_s$ and $\ell < m \Rightarrow h_1(s_\ell) = h_2(s_\ell)\}\}$,

- $\boxplus$ for $\bar{s} = \langle s_\ell \colon \ell < n\rangle$ as above, $N_{\bar{s}\restriction m} \leq_{\mathfrak{k}} N_{\bar{s}}$.

[Why? Prove by induction on n that it suffices to conclude that $m = n - 1$ and now read the Definition.]

- $\boxplus$ if $\bar{s}$ is as above and $\bar{t}$ is a sub-sequence of $\bar{s}$ then $N_{\bar{t}} \leq_{\mathfrak{k}} N_{\bar{s}}$.

Why? By Ax. V of AEC (see Definition 1.17): The rest should be clear.

Finally note that if $b = \sigma^{\mathrm{EM}'(I)}(a_{t_1}, \ldots, a_{t_n}) \in \mathrm{EM}(I), t_1, \ldots, t_n \in I$, then $h(a) = \sigma^{\mathrm{Op}(\mathrm{EM}'(I))}(g_{t_1}/D, \ldots, g_{t_n}/D) = \sigma^{\mathrm{Op}(\mathrm{EM}'(I))}(\langle a_{f_{t_i(i)}} \colon i < \mu\rangle/D, \ldots, \langle a_{f_{t_n(i)}} \colon i < \mu\rangle/D) = f_{\mathrm{Op}}(\sigma^{\mathrm{EM}'(I)}(a_{t_1}, \ldots, a_{t_n})) = f_{\mathrm{Op}}(b)$. Thus $\mathrm{EM}(I) \underset{\text{nice}}{\leq} \mathrm{EM}(J)$. $\square$

[7]We write $I_{\mathbf{u}} = I(\mathbf{u})$ to distinguish it from $(I, <_I)$.

Criterion 2.6. Suppose that $(I,<)$ is a suborder of the linear order $(J,<)$. We have $(I,\le) \underset{\text{nice}}{\subseteq} (J,<)$ when:

- $(*)$ for every $t \in J \setminus I$,
- $(\aleph)$ $\operatorname{cf}((I,<) \restriction \{s \in I : (J,<) \models s < t\}) = \kappa$,

  or

- $(\beth)$ $\operatorname{cf}((I,<)^* \restriction \{s \in I : (J,<)^* \models s <^* t\}) = \kappa$.

Notation 2.7. $(I,<)^*$ is the (reverse) linear order $(I^*,<^*)$ where $I^* = I$ and $(I^*,<^*) \models s <^* t$ iff $(I,<) \models t < s$.

Proof. We shall use freely Assumption 1.1, that is, "**D** is a uniform ultra-filter on κ". Let us list some general facts which facilitate the proof.

Fact (A): Let $\underline{\kappa}$ denote the linear order $(\kappa,<)$ where $<$ is the usual order $\in\restriction \kappa \times \kappa$. If $J_1 = \underline{\kappa} + J_0$, then $\underline{\kappa} \underset{\text{nice}}{\subseteq} J_1$ ($+$ is the addition of linear orders in which all elements in the first order precede those in the second).

Fact (B): If $\underline{\kappa} \subseteq (I,<)$, $\underline{\kappa}$ is unbounded in $(I,<)$ and $J_1 = I + J_0$, then $I \underset{\text{nice}}{\subseteq} J_1$.

Fact (C): If $I \underset{\text{nice}}{\subseteq} J$, then $I + J_1 \underset{\text{nice}}{\subseteq} J + J_1$.

Fact (D): $I \underset{\text{nice}}{\subseteq} J$ iff $(J <)^* \underset{\text{nice}}{\subseteq} (I,<)^*$.

Fact (E): If $\langle I_\alpha : \alpha \le \delta\rangle$ is a continuous increasing sequence of linear orders and for $\alpha < \delta$, $I_\alpha \underset{\text{nice}}{\subseteq} I_{\alpha+1}$, then $I_\alpha \underset{\text{nice}}{\subseteq} I_\delta$.

Now using these facts, let us prove the criterion. Define an equivalence relation E on $J \setminus I$ as follows: tEs iff t and s define the same Dedekind cut in $(I,<)$. Let $\{t_\alpha : \alpha < \delta\}$ be a set of representatives of the E-equivalence classes. For each $\beta \le \delta$, define

$$I_\beta = J \restriction \left\{ t : t \in I \vee \bigvee_{\alpha<\beta} tEt_\alpha \right\}$$

so $I_0 = I$, $I_\delta = J$ and $\langle I_\alpha : \alpha \le \delta\rangle$ is a continuous increasing sequence of linear orders. By Fact (E), to show that $I \underset{\text{nice}}{\subseteq} J$, it suffices to show that $I_\alpha \underset{\text{nice}}{\subseteq} I_{\alpha+1}$ for each $\alpha < \delta$.

Fix $\alpha < \delta$. Now t_α belongs to $J \setminus I$, so by $(*)$, $(\aleph)$ or $(\beth)$ holds. By Fact (D), it is enough to treat the case $(\aleph)$. So without loss of generality $\operatorname{cf}\big((I,<) \restriction \{s \in I : (J,<) \models s < t_\alpha\}\big) = \kappa$.

Let

$$I^a_\alpha = \{t \in I_\alpha : t < t_\alpha\},$$
$$I^b_\alpha = \{t \in I_{\alpha+1} : t \in I^a_\alpha \vee tEt_\alpha\},$$

$$I^c_\alpha = \{t \in I_\alpha : t > t_\alpha\}.$$

Note that $I_\alpha = I^a_\alpha + I^c_\alpha$ and $I_{\alpha+1} = I^b_\alpha + I^c_\alpha$. Recalling Fact (C), it is now enough to show that $I^a_\alpha \underset{\text{nice}}{\subseteq} I^b_\alpha$. Identifying isomorphic orders and using $(\aleph)$, one has that $\underline{\kappa}$ is unbounded in I^a_α and $I^b_\alpha = I^a_\alpha + (I^b_\alpha \setminus I^a_\alpha)$ so by Fact (B), $I^a_\alpha \underset{\text{nice}}{\subseteq} I^b_\alpha$ as required.

We still owe the five facts, we prove (A), (B), and (E) as (C) and (D) are obvious.

Proof of Fact (A): Recall that $\mathbf{D}$ is a uniform ultra-filter on κ. For every linear order J_0 (or J^*_0) there is $\mathrm{Op}_{I,\mathbf{D}}(-)$, the iteration of I ultra-powers $(-)^\kappa/\mathbf{D}$, ordered in the order J_0 (or J^*_0), giving the required embedding (use Observation 1.34).

Proof of Fact (B) Since $\underline{\kappa} \subseteq I$ and using Fact (A), we know that

- $\bullet_1$ let d_0 be the identity map from $\underline{\kappa}$ into I,
- $\bullet_2$ let d_1 be the canonical map from $\underline{\kappa}$ into $\mathrm{Op}(\underline{\kappa})$, which exists by properties of Op,
- $\bullet_3$ let d_2 be the embedding of J_0 into $\mathrm{Op}(\underline{\kappa})$ as in the choice of Op, so $\mathrm{rang}(d_1)$ is below $\mathrm{rang}(d_2)$,
- $\bullet_4$ let d_3 be the canonical embedding from $\mathrm{Op}(\underline{\kappa})$ into $\mathrm{Op}(I)$ by lifting (really is the identity),
- $\bullet_5$ let d_4 be the canonical embedding of I into $\mathrm{Op}(I)$, which exists by properties of Op extending d_1 by the properties of Op,
- $\bullet_6$ So $\mathrm{rang}(d_4 \restriction \underline{\kappa})$ is unbounded in $\mathrm{rang}(d_4)$ in the order $\mathrm{Op}(I)$,
- $\bullet_7$ $\mathrm{rang}(d_4) \subseteq \mathrm{Op}(I)$ is below $d_3 \circ d''_2 \in J_0$.

Chasing through the diagram, we obtain the required embedding. So we are done.

Proof of Fact (E). Apply Observation 1.39 to the chain $\langle I_\alpha : \alpha \leq \delta\rangle$. ☐

So we are done proving Criterion 2.6. ☐

Fact 2.8. Suppose that $\lambda \geq \kappa$. There exist a linear order $(I, <_I)$ of power λ and a sequence $\langle A_i \subseteq I : i \leq \lambda\rangle$ of pairwise disjoint subsets of I, each of power κ such that $I = \underset{i\leq\lambda}{\cup} A_i$ and,

$(*)$ if $\lambda \in X \subseteq \lambda \dotplus 1$, then $I \restriction \underset{i\in X}{\cup} A_i \underset{\text{nice}}{\subseteq} I$.

Proof. Let $I = (\lambda \dotplus 1) \times \kappa$ and define $<_I$ on I: $(i_1, \alpha_1) <_I (i_2, \alpha_2)$ iff $i_1 < i_2$ or ($i_1 = i_2$ and $\alpha_1 > \alpha_2$). For each $i \leq \lambda$, let $A_i = \{i\} \times \kappa$. Let us check $(*)$ of Criterion 2.6: suppose that $\lambda \in X \subseteq \lambda + 1$. Write $I_X = I \restriction \left(\bigcup_{i\in X} A_i\right)$. To show that $I_X \underset{\text{nice}}{\subseteq} I$, we can assume without loss of generality that $I_X \neq I$ and then one employs Criterion 2.6. Consider $t \in I - I_X$, say $t = (i, \alpha)$ (note that $\alpha < \kappa$ and $i < \lambda$, since $\lambda \in X$) and $i \notin X$. Let $j = \min(X - i)$; note that j is well-defined, since $\lambda \in X - i$, and $j \neq i$. Now, if $s \in I_X$ and $s \leq_I t$, then for every $\beta < \kappa$, one has $s <_I (j, \beta)$ and $(j, \beta) \in I_X$. Also if $s \in I_X$ and $t <_I s$, then for some $\beta < \kappa$, we have $(j, \beta) <_I s$. Thus $\langle (j, \beta) \colon \beta < \kappa \rangle$ is a cofinal sequence in $(I_X \restriction \{s \in I \colon t <_I s\})^*$. By the criterion, $I_X \underset{\text{nice}}{\subseteq} I$. □

Theorem 2.9. *Suppose that $\kappa = \mathrm{cf}(\delta) \leq \delta < \lambda$. Then $\mathrm{EM}(\delta) \underset{nice}{\leq} \mathrm{EM}(\lambda)$.*

Proof. By Fact (B) of Criterion 2.6, one has that $\delta \underset{\text{nice}}{\subseteq} \lambda$; so by Theorem 2.5, $\mathrm{EM}(\delta) \underset{\text{nice}}{\leq} \mathrm{EM}(\lambda)$. □

Now let us turn to the main theorem of this section.

Theorem 2.10. *Suppose that $\mathfrak{k}$ is categorical in the regular cardinal $\lambda > \chi$. Then $\mathfrak{k}_{<\lambda}$ has the amalgamation property.*

Proof. 2.10 Suppose that $\mathfrak{k}_{<\lambda}$ fails AP. Note that $\|\mathrm{EM}(\lambda)\| = \lambda$. Apply Theorem 2.4 to find $M^* \in K_\lambda$ and $\langle M_i \colon i < \lambda \rangle$ satisfying Theorem 2.4(1) and Theorem 2.4(2). Since $\mathfrak{k}$ is λ-categorical, $M^* \cong EM(\lambda)$, so without loss of generality $\mathrm{EM}(\lambda) = \bigcup_{i<\lambda} M_i$ and so $C = \{i < \lambda \colon M_i = \mathrm{EM}(i)\}$ is a club of λ. Choose $\delta \in C, \mathrm{cf}(\delta) = \kappa$. By Theorem 2.9, $\mathrm{EM}(\delta) \underset{\text{nice}}{\leq} \mathrm{EM}(\lambda)$, so $M_\delta \underset{\text{nice}}{\leq} M^*$. But of course by Theorem 2.4(2) $M_\delta \underset{\text{nice}}{\nleq} M^*$-contradiction. □

The last theorem of this section applies to singular cardinals as well.

Theorem 2.11. *Suppose that K is categorical in $\lambda > \chi$ (notice that λ is not necessarily regular). Then:*

(1) K has a model M of power λ such that if $N \leq_{\mathfrak{k}} M$ and $\|N\| < \lambda$, then there exists N' such that:

(α) $N \leq_{\mathfrak{k}} N' \leq_{\mathfrak{k}} M$,

(β) $\|N'\| = \|N\| + \chi$,

(γ) $N' \underset{nice}{\leq} M$.

(2) K has a model M of power λ and an expansion M^+ of M by at most χ functions such that if $N^+ \subseteq M^+$, then $N^+ \restriction \tau \underset{nice}{\leq} M$.

Proof. (1) Let $\langle I, \langle A_i \colon i \leq \lambda \rangle \rangle$ be as in Fact 2.8. Let $M = \mathrm{EM}(I)$. Suppose that $N \leq_{\mathfrak{k}} M, \|N\| < \lambda$. Then there exists $J \subseteq I, |J| < \lambda$ such that $N \subseteq \mathrm{EM}(J)$ so by Fact 2.8 there exists $X \subseteq \lambda+1$ such that $\lambda \in X$, $|X| < \lambda$ and $J \subseteq \bigcup_{i \in X} A_i$.

Note that $\left|\bigcup_{i\in X} A_i\right| \leq |X| \cdot \kappa < \lambda$. Now $N' = \mathrm{EM}(I \restriction \bigcup_{i\in X} A_i)$ is as required, since $I \restriction \bigcup_{i\in X} A_i \underset{\text{nice}}{\leq} I$ and so by Theorem 2.5 $\mathrm{EM}(I \restriction (\bigcup_{i\in X} A_i)) \underset{\text{nice}}{\leq} \mathrm{EM}(I)$. This proves (1).

(2) We expand $M = \mathrm{EM}(I)$ with skeleton $\langle a_s \colon s \in I\rangle$ as follows:

(a) by all functions of $\mathrm{EM}'(I)$,

(b) by the unary functions $f_\ell(\ell < n)$ which are chosen as follows: we know that for each $b \in M$ there is σ_b an τ_1-term (τ_1-the vocabulary of $\mathrm{EM}'(I)$) and $t(b,0) < t(b,1) < \ldots < t(b, n_{\sigma_b} - 1)$ from I such that

$$b = \sigma_b(a_{t(b,0)}, a_{t(b,1)}, \ldots, a_{t(b,n_{\sigma_b}-1)})$$

(it is not unique, but we can choose one; really if we choose it with n_b minimal it is almost unique). We let

$$f_\ell(b) = \begin{cases} a_{t((b,\ell))}, & \text{if } \ell < n_{\sigma_b}, \\ b, & \text{if } \ell \geq n_{\sigma_b}. \end{cases}$$

(c) by unary functions g_α, g^α for $\alpha < \kappa$ such that <u>if</u> $t < s$ are in I, $\alpha = \mathrm{otp}[(t,s)^*_I]$ then $g^\alpha(a_t) = a_s$, $\bigvee_{\beta<\kappa} g_\beta(a_s) = a_t$ (more formally $g^\alpha(a_{(i,\beta)}) = a_{(i,\beta+\alpha)}$ and $g_\alpha(a_{(i,\beta)}) = a_{(i,\alpha)}$) in the other cases $g^\alpha(b) = b$, $g_\alpha(b) = b$.

(d) by individual constants $c_\alpha = a_{(\lambda,\alpha)}$ for $\alpha < \kappa$.

Call the expanded model M^+. Now suppose N^+ is a sub-model of M^+ and N its τ-reduct. Let $J \overset{\text{def}}{=} \{t \in I \colon a_t \in N\}$, now J is a subset of I of cardinality $\leq ||N||$ as for $t \neq s$ from J, $a_t \neq a_s$. Also if $b \in N$ by clause (b), $a_{t(b,\ell)} \in N$ hence $b \in \mathrm{EM}(J)$; on the other hand if $b \in \mathrm{EM}(J)$ then by clause (a) we have $b \in N$; so we can conclude $N = \mathrm{EM}(J)$. So far this holds for any linear suborder of I.

By clause (c) $J = \bigcup_{i\in X} A_i$ for some $X \subseteq \lambda + 1$, and by clause (d), $\lambda \in X$.

Now $\mathrm{EM}(J) \underset{\text{nice}}{\leq} \mathrm{EM}(I) = M$ by Fact 2.8. $\square$

3. Towards removing the assumption of regularity from the existence of universal extensions

In §2 we showed that $\mathfrak{k}_{<\lambda}$ has the amalgamation property when $\mathfrak{k}$ is categorical in the regular cardinal $\lambda > \chi$. We now study the situation in which λ is not assumed to be regular.

Our problem is that while we know that most sub-models of $N \in K_\lambda$ sit well in N (see Theorem 2.11(2)) and that there are quite many $N \in K_{<\lambda}$ which are amalgamation bases, our difficulty is to get those things together: constructing $N \in K_\lambda$ as $\bigcup_{i<\lambda} N_i$, $N_i \in K_{<\lambda}$ means N has $\leq_{\mathfrak{k}}$-sub-models not included in any N_i.

Recall we are assuming Hypothesis 1.45.

Theorem 3.1. *Suppose that $\mathfrak{k}$ is categorical in λ and $\chi \leq \theta < \lambda$. If $\langle M_i \in K_\theta : i < \theta^+ \rangle$ is an increasing continuous $\leq_{\mathfrak{k}}$-chain, then:*

$$\left\{ i < \theta^+ : M_i \underset{nice}{\leq} (\cup_{j<\theta^+} M_j) \right\} \in D_{\theta^+}.$$

Remark 3.2.

(1) We cannot use Theorem 2.11(1) as possibly λ has cofinality $< \chi$.

(2) Recall that D_{θ^+} is the closed unbounded filter on θ^+.

Proof. Write $M_{\theta^+} = \underset{i<\theta^+}{\cup} M_i$. Choose an operation Op such that for all $i < \theta^+$, $\| \mathrm{Op}(M_i) \| \geq \lambda$. Let $M_i^* = \mathrm{Op}(M_i)$, hence $M_i \underset{\text{nice}}{\leq} M_i^*$. Applying Fact 1.11 for non-limit ordinals, Fact 1.10 for limit ordinals, one finds inductively an increasing continuous $\leq_{\mathfrak{k}}$-chain $\langle N_i : i \leq \theta^+ \rangle$ such that for $i < \theta^+, M_i \leq_{\mathfrak{k}} N_i \leq_{\mathfrak{k}} M_i^*, \|N_i\| = \lambda$, so $M_i \underset{\text{nice}}{\leq} N_i$ and $N_{\theta^+} = \underset{i<\theta^+}{\cup} N_i$. Note that $\|N_{\theta^+}\| = \theta^+ \cdot \lambda = \lambda$.

Since $\mathfrak{k}$ is λ-categorical, $N_{\theta^+} \cong \mathrm{EM}(I)$ where Fact 2.8 furnishes I of power λ. By Theorem 2.11(2), there is an expansion $N^+_{\theta^+}$ of N_{θ^+} by at most $\kappa + |\tau_{\mathfrak{k}}|$ functions such that if $A \subseteq |N^+_{\theta^+}|$ is closed under the functions of $N^+_{\theta^+}$, then $(N^+_{\theta^+} \restriction \tau_{\mathfrak{k}}) \restriction A \underset{\text{nice}}{\leq} N_{\theta^+}$.

Choose a set A_i and an ordinal j_i, by induction on $i < \theta^+$, satisfying:

(1) $A_i \subseteq |N_{\theta^+}|, |A_i| \leq \theta; \langle A_i : i < \theta^+ \rangle$ is continuous increasing,

(2) $\langle j_i : i < \theta^+ \rangle$ is continuous increasing,

(3) A_i is closed under the functions of $N^+_{\theta^+}$,

(4) $A_i \subseteq |N_{j_{i+1}}|$,

(5) $|M_i| \subseteq A_{i+1}$.

This is possible: for zero, let $A_0 \coloneqq \emptyset$, $j_0 \coloneqq 0$ and for limit ordinals unions work; for $i+1$ choose j_{i+1} to satisfy (2) and (4), and A_{i+1} to satisfy (1), (3) and (5).

By (2), $C = \{i < \theta^+ : i$ is a limit ordinal and $j_i = i\}$ is a club of θ^+ i.e. $C \in D_{\theta^+}$.

Fix $i \in C$. Note that $|M_i| \subseteq A_i$ and $A_i \subseteq |N_i|$ (since $|M_i| = \underset{j<i}{\cup} |M_j| \subseteq \underset{j<i}{\cup} A_{j+1} = A_i = \underset{i'<i}{\cup} A_{i'} \subseteq \underset{i'<i}{\cup} |N_{j_{i'+1}}| = N_{j_i} = N_i$ (using (5), (1), (4), (2) and $j_i = i$)) and recalling (3), $M_i \leq_{\mathfrak{k}} (N^+_{\theta^+} \restriction \tau_{\mathfrak{k}}) \restriction A_i \leq_{\mathfrak{k}} N_i \leq_{\mathfrak{k}} M_i^* = \mathrm{Op}(M_i)$, so that $M_i \underset{\text{nice}}{\leq} (N^+_{\theta^+} \restriction \tau_{\mathfrak{k}}) \restriction A_i$. However by (3) and the choice of N_{θ^+} and $N^+_{\theta^+}$ one has also that $(N^+_{\theta^+} \restriction \tau_{\mathfrak{k}}) \restriction A_i \underset{\text{nice}}{\leq} N_{\theta^+}$. So by transitivity of $\underset{\text{nice}}{\leq}$, one obtains $M_i \underset{\text{nice}}{\leq} N_{\theta^+}$.

Finally remark that $M_{\theta^+} \leq_{\mathfrak{k}} N_{\theta^+}$ since $M_i \underset{\text{nice}}{\leq} N_i \leq_{\mathfrak{k}} N_{\theta^+}$ for every $i < \theta^+$. Hence $i \in C \Rightarrow M_i \leq_{\mathfrak{k}} M_{\theta^+} \leq_{\mathfrak{k}} N_{\theta^+}$, so recalling $i \in C \Rightarrow M_i \underset{\text{nice}}{\leq} N_{\theta^+}$ we have $i \in C \Rightarrow M_i \underset{\text{nice}}{\leq} M_{\theta^+}$; so $C \subseteq \left\{ i < \theta^+ \colon M_i \underset{\text{nice}}{\leq} M_{\theta^+} \right\} \in D_{\theta^+}$. □

Definition 3.3. Suppose that $\theta \in [\chi, \lambda)$ and $M \in K_\theta$. M is nice iff whenever $M \leq_{\mathfrak{k}} N \in K_\theta$, then $M \underset{\text{nice}}{\leq} N$. (The analogous $\leq_{\mathfrak{k}}$-elementary embedding definition runs: M is nice iff whenever $f \colon M \underset{\mathfrak{k}}{\to} N \in K_\theta$ then $f \colon M \underset{\text{nice}}{\to} N$).

Theorem 3.4. *Suppose that $\mathfrak{k}$ is categorical in λ and $M \in K_\theta, \theta \in [\chi, \lambda)$. Then there exists $N \in K_\theta$ such that $M \leq_{\mathfrak{k}} N$ and N is nice.*

Proof. Suppose otherwise. We'll define a continuous increasing $\leq_{\mathfrak{k}}$-chain $\langle M_i \in K_\theta \colon i < \theta^+ \rangle$ such that for $j < \theta^+$:

$(*)_j$ $M_j \underset{\text{nice}}{\not\leq} M_{j+1}$.

For $i = 0$, put $M_0 = M$; if i is a limit ordinal, put $M_i = \bigcup_{j<i} M_j$; if $i = j+1$, then since Theorem 3.4 is assumed to fail, M_{j+1} exists as required in $(*)_j$ (otherwise M_j works as N in Theorem 3.4). But now $\langle M_i \colon i < \theta^+ \rangle$ yields a contradiction to Theorem 3.1, since $C = \{ i < \theta^+ \colon M_i \underset{\text{nice}}{\leq} \bigcup_{j<\theta^+} M_j \} \in D_{\theta^+}$ by Theorem 3.1 so that choosing j from C one has $M_j \underset{\text{nice}}{\leq} M_{j+1}$ by Observation 1.38, contradicting $(*)_j$. □

Theorem 3.5. *Suppose that $\mathfrak{k}$ is categorical in λ and $\theta \in [\chi, \lambda)$. If $M \in K_\theta$ is nice and $f \colon M \underset{\mathfrak{k}}{\to} N \in K_{\leq\lambda}$, then $f \colon M \underset{nice}{\to} N$.*

Proof. Choosing an appropriate Op and using Fact 1.11 one finds N_1 such that $N \leq_{\mathfrak{k}} N_1$ and $\|N_1\| = \lambda$. Find $M_1' \underset{\text{nice}}{\leq} N_1$ by Theorem 2.11(2) such that $\operatorname{rng}(f) \subseteq |M_1'|, \|M_1'\| = \theta$. So $M_1' \leq_{\mathfrak{k}} N_1$ and therefore $N_1 \restriction \operatorname{rng}(f) \leq_{\mathfrak{k}} M_1'$. Recall M is nice, so $f \colon M \underset{\text{nice}}{\to} M_1'$. Now $M_1' \underset{\text{nice}}{\leq} N_1$, therefore $f \colon M \underset{\text{nice}}{\to} N_1$. So there are Op and $g \colon N_1 \underset{\mathfrak{k}}{\to} \operatorname{Op}(M)$ satisfying $gf = f_{\operatorname{Op}}$. Since $N \leq_{\mathfrak{k}} N_1$ it follows that $f \colon M \underset{\text{nice}}{\to} N$ as required. □

Corollary 3.6. *Suppose that $M \in K_\theta$ is nice, $\theta \in [\chi, \lambda)$. Then M is an a.b. in $\mathfrak{k}_{\leq\lambda}$ i.e. if $f_i \colon M \underset{\mathfrak{k}}{\to} M_i, M_i \in K_{\leq\lambda} (i = 1, 2)$, then there exists an amalgam $N \in K_{\leq\lambda}$ of M_1, M_2 over M w.r.t. f_1, f_2.*

Proof. By Definition 3.5 $f_i \colon M \underset{\text{nice}}{\to} M_i (i = 1, 2)$. Hence by Lemma 2.1 there is an amalgam $N \in K_{\leq\lambda}$ of M_1, M_2 over M w.r.t. f_1, f_2. □

Definition 3.7. Suppose that $\theta \in [\chi, \lambda)$ and ∂ is a cardinal.

(1) A model $M \in K_\theta$ is ∂-universal iff for every $N \in K_\partial$, there exists an $\leq_{\mathfrak{k}}$-elementary embedding $f\colon N \underset{\mathfrak{k}}{\to} M$. We say M is universal iff M is $\|M\|$-universal.

(2) A model $M_2 \in K_\theta$ is ∂-universal over the model M_1 (and one writes $M_1 \underset{\partial-\mathrm{univ}}{\preceq} M_2$) <u>iff</u> $M_1 \leq_{\mathfrak{k}} M_2$ and whenever $M_1 \leq_{\mathfrak{k}} M_2' \in K_\partial$, then there exists an $\leq_{\mathfrak{k}}$-elementary embedding $f\colon M_2' \underset{\mathfrak{k}}{\to} M_2$ such that $f \restriction M_1$ is the identity. (The embedding version runs: there exists $h\colon M_1 \underset{\mathfrak{k}}{\to} M_2$ and whenever $g\colon M_1 \underset{\mathfrak{k}}{\to} M_2' \in K_\partial$, then there exists $f\colon M_2' \underset{\mathfrak{k}}{\to} M_2$ such that $fg = h$.) M_2 is universal over M_1 ($M_1 \underset{\mathrm{univ}}{\preceq} M_2$) iff M_2 is $\|M_2\|$-universal over M_1.

(3) A model M_2 is ∂-universal over M_1 in M <u>iff</u> $M_1 \leq_{\mathfrak{k}} M_2 \leq_{\mathfrak{k}} M$, $||M_1|| \leq \partial$ and whenever $M_2' \in K_\partial$ and $M_1 \leq_{\mathfrak{k}} M_2' \leq_{\mathfrak{k}} M$, then there exists an $\leq_{\mathfrak{k}}$-elementary embedding $f\colon M_2' \underset{\mathfrak{k}}{\to} M_2$ such that $f \restriction M_1$ is the identity. M_2 is universal over M_1 in M iff M_2 is $\|M_2\|$-universal over M_1 in M.

(4) M_2 is weakly ∂-universal over M_1 (written $M_1 \underset{\partial-\mathrm{wu}}{\prec} M_2$) <u>iff</u> $M_1 \leq_{\mathfrak{k}} M_2 \in K_\partial$ and whenever $M_2 \leq_{\mathfrak{k}} M_2' \in K_\partial$, then there exists an $\leq_{\mathfrak{k}}$-elementary embedding $f\colon M_2' \underset{\mathfrak{k}}{\to} M_2$ such that $f \restriction M_1$ is the identity. (The embedding version is: there exists $h\colon M_1 \underset{\mathfrak{k}}{\to} M_2$ and whenever $g\colon M_2 \underset{\mathfrak{k}}{\to} M_2' \in K_\partial$, then there exists $f\colon M_2' \underset{\mathfrak{k}}{\to} M_2$ such that $h = fgh$ (written $h\colon M_1 \underset{\partial-\mathrm{wu}}{\to} M_2$)). We say M_2 is weakly universal over $M_1 (M_1 \underset{\mathrm{wu}}{\preceq} M_2)$ iff M_2 is $\|M_2\|$-weakly universal over M_1.

Remark 3.8. In $\mathfrak{k}_{<\lambda}$, if M_1 is an a.b., then weak universality over M_1 is equivalent to universality over M_1.

Proof. Suppose that $h\colon M_1 \underset{\mathrm{wu}}{\to} M_2$ and $g\colon M_1 \underset{\mathfrak{k}}{\to} M_2' \in K_{\|M_2\|}$. Since M_1 is an a.b. there exist a model N and $h'\colon M_2 \underset{\mathfrak{k}}{\to} N$, $g'\colon M_2' \underset{\mathfrak{k}}{\to} N$ satisfying $h'h = g'g$. By Fact 1.11 without loss of generality $\|N\| = \|M_2\|$. Since M_2 is weakly universal over M_1, there exists $h''\colon N \underset{\mathfrak{k}}{\to} M_2$, $h = h''h'h$. Let $f = h''g'\colon M_2' \to M_2$, and note that $fg \restriction M_1 = h''g'g = h''h'h = h$, so that M_2 is universal over M_1. □

Remark 3.9. For any model M, universality over M implies weak universality over M.

Lemma 3.10. *Suppose that $\mathfrak{k}$ is categorical in $\lambda, \theta \in [\chi, \lambda)$. If $M \in K_\theta$ and $M \leq_{\mathfrak{k}} N \in K_\lambda$, then there exists $M^+ \in K_\theta$ such that:*

(a) $M \leq_{\mathfrak{k}} M^+ \leq_{\mathfrak{k}} N$,

(b) M^+ *is universal over* M *in* N.

Proof. We choose I such that:

$(*)$ (a) I is a linear order of cardinality λ,

(b) if $\partial \in [\aleph_0, \lambda)$, $J_0 \subseteq I$, $|J_0| = \partial$ then there is J_1 satisfying $J_0 \subseteq J_1 \subseteq I$, $|J_1| = \partial$, and for every $J^* \subseteq I$ of cardinality $\leq \partial$ there is an order-preserving (one to one) mapping from $J_0 \cup J^*$ into $J_0 \cup J_1$ which is the identity on J_0.

Essentially the construction follows Laver [10] and [24, Appendix], see more in §2 of [16]; but for our present purpose let $I = ({}^{\omega>}\lambda, <_{\ell\text{ex}})$; given θ and J_0 we can increase J_0 so without loss of generality $J_0 = {}^{\omega>}A$, $A \subseteq \lambda$, $|A| = \theta$. Define an equivalence relation E on $I \setminus J_0$: $\eta E \nu \Leftrightarrow (\forall \rho \in J_0)(\rho <_{\ell\text{ex}} \eta \equiv \rho <_{\ell\text{ex}} \nu)$, easily it has $\leq \theta$ equivalence classes, so let $\{\eta^*_i \colon i < i^* \leq \theta\}$ be a set of representatives each of minimal length, so $\eta^*_i \restriction (\lg \eta^*_i - 1) \in J_0$, $\eta^*_i(\lg \eta^*_i - 1) \in \lambda \setminus A$.

Let $J_1 = I \cup \{\eta^*_i {}^\frown \nu \colon \nu \in {}^{\omega>}\theta \text{ and } i < i^*\}$, so clearly $J_0 \subseteq J_1 \subseteq I$, $|J_1| = \theta$. Suppose $J_0 \subseteq J \subseteq I$, $|J| \leq \theta$, and we should find the required embedding h. As before without loss of generality $J = {}^{\omega>}B$, $|B| = \theta$ and $A \subseteq B$. Now $h \restriction J_0 = \mathrm{id}_{J_0}$ so it is enough to define $h \restriction (J_1 \cap (\eta^*_i/E))$, hence it is enough to embed $J_1 \cap (\eta^*_i/E)$ into $\{\eta^*_i {}^\frown \nu \colon \nu \in {}^{\omega>}\theta\}$ (under $<_{\ell\text{ex}}$).

Let $\gamma = \mathrm{otp}(B)$, so it is enough to show $({}^{<\omega}\gamma, <_{\ell\text{ex}})$ can be embedded into ${}^{\omega>}\theta$, where of course $|\gamma| \leq \theta$. This is proved by induction on γ.

Since $\mathfrak{k}$ is λ-categorical and $\mathrm{EM}(I)$ is a model of $\mathfrak{k}$ of power λ, there is an isomorphism g from $\mathrm{EM}(I)$ onto N. It follows from $(*)$ that $M^+ = g''(\mathrm{EM}(J)) \in K_\theta$ satisfies (1) and (2). (Analogues of (1) and (2) are checked in more detail in the course of the proof of Corollary 3.14.) □

Lemma 3.11. *Suppose that $\mathfrak{k}$ is categorical in $\lambda, \theta \in [\chi, \lambda)$, and $\langle M_i \in K_\theta \colon i < \theta^+\rangle$, $\langle N_i \in K_\lambda \colon i < \theta^+\rangle$ are continuous $\leq_{\mathfrak{k}}$-chains such that for every $i < \theta^+$ we have $M_i \leq_{\mathfrak{k}} N_i$. Then there exists $i(*) < \theta^+$ such that $(i(*),\ \theta^+) \subseteq C :=$ $\{i < \theta^+ \colon M_{i+1}$ can be $\leq_{\mathfrak{k}}$-elementarily embedded into N_i over $M_0\}$.*

Proof. Apply Lemma 3.10 for $M_0 \in K_\theta$ and $N_{\theta^+} = \bigcup_{i<\theta^+} N_i \in K_\lambda$ (noting that $M_0 \leq_{\mathfrak{k}} N_0 \leq_{\mathfrak{k}} N_{\theta^+}$) to find $M^+ \in K_\theta$ such that $M_0 \leq_{\mathfrak{k}} M^+ \leq_{\mathfrak{k}} N_{\theta^+}$ and M^+ is universal over M_0 in N_{θ^+}.

For some $i(*) < \theta^+, M^+ \subseteq N_{i(*)}$ and so $M^+ \leq_{\mathfrak{k}} N_{i(*)}$. If $i \in (i(*), \theta^+)$, then $M_{i+1} \in K_\theta$ and $M_0 \leq_{\mathfrak{k}} M_{i+1} \leq_{\mathfrak{k}} N_{i+1} \leq_{\mathfrak{k}} N_{\theta^+}$, so there is an $\leq_{\mathfrak{k}}$-elementary embedding $f \colon M_{i+1} \underset{\mathfrak{k}}{\to} M^+$ and $f \restriction M_0$ is the identity. Now $M^+ \leq_{\mathfrak{k}} N_{i(*)} \leq_{\mathfrak{k}} N_i$, so $f \colon M_{i+1} \underset{\mathfrak{k}}{\to} N_i$. Hence $(i(*),\ \theta^+) \subseteq C$ as required. □

Theorem 3.12. *Suppose that $\mathfrak{k}$ is categorical in $\lambda, \theta \in [\chi, \lambda), M \in K_\theta$. <u>Then</u> there exists $M^+ \in K_\theta$ such that:*

(ℵ) *$M \leq_{\mathfrak{k}} M^+$ and M^+ is nice,*

(ℶ) *M^+ is weakly universal over M.*

***Proof*.** Define by induction on $i < \theta^+$ continuous $\leq_{\mathfrak{k}}$-chains $\langle M_i \in K_\theta : i < \theta^+ \rangle$, $\langle N_i \in K_\lambda : i < \theta^+ \rangle$ such that:

(0) $M_0 = M$,

(1) $M_i \leq_{\mathfrak{k}} N_i$,

(2) if $(*)_i$ holds, then M_{i+1} cannot be $\leq_{\mathfrak{k}}$-elementarily embedded into N_i over M_0, where $(*)_i$ is the statement:

$(*)_i$ there are $M' \in K_\theta$ and $N' \in K_\lambda$ such that $M_i \leq_{\mathfrak{k}} M'$, $N_i \leq_{\mathfrak{k}} N'$, $M' \leq_{\mathfrak{k}} N'$ and M' cannot be $\leq_{\mathfrak{k}}$-elementarily embedded into N_i over M_0,

(3) $M_{i+1} \underset{\text{nice}}{\leq} N_{i+1}$.

This is possible. N_0 is obtained by an application of Fact 1.11 to an appropriate $\mathrm{Op}(M_0)$ of power at least λ. At limit stages, continuity dictates that one take unions. Suppose that M_i, N_i have been defined. If $(*)_i$ does not hold, by Theorem 2.11(2) there is $M'' \in K_\theta$, $M_i \leq_{\mathfrak{k}} M'' \underset{\text{nice}}{\leq} N_i$. Let $M_{i+1} = M''$, $N_{i+1} = N_i$. If $(*)_i$ does hold for M', N', let $N_{i+1} = N'$; note that by Theorem 2.11(2) there exists $M^{''} \in K_\theta, M' \leq_{\mathfrak{k}} M^{''} \underset{\text{nice}}{\leq} N'$; now let $M_{i+1} = M^{''}$. Note that in each case, (3) is satisfied.

Find $i(*) < \theta^+$ and C as in Lemma 3.11 and choose $i \in C$. By (1), we have $M_{i+1} \leq_{\mathfrak{k}} N_{i+1}$ so by Lemma 3.10 there exists $M^- \in K_\theta$ such that $M_{i+1} \leq_{\mathfrak{k}} M^- \leq_{\mathfrak{k}} N_{i+1}$ and M^- is weakly universal over M_{i+1} in N_{i+1}. By Theorem 3.4 one can find $M^+ \in K_\theta$ such that $M^- \leq_{\mathfrak{k}} M^+$ and M^+ is nice. So M^+ satisfies $(\aleph)$. It remains to show that M^+ is weakly universal over M. Suppose not and let $g \colon M^+ \underset{\mathfrak{k}}{\rightarrow} M^* \in K_\theta$ where M^* cannot be $\leq_{\mathfrak{k}}$-elementarily embedded in M^+ over M hence cannot be $\leq_{\mathfrak{k}}$-elementarily embedable in M^- over M, hence in N_{i+1} over M. Now, $M_{i+1} \leq_{\mathfrak{k}} M^* \in K_\theta$ and by (3) $M_{i+1} \underset{\text{nice}}{\leq} N_{i+1} \in K_\lambda$, so by 2.1 there is an amalgam $N^* \in K_\lambda$ of M^*, N_{i+1}. The existence of M^*, N^* implies that $(*)_{i+1}$ holds since M^* cannot be $\leq_{\mathfrak{k}}$-elementarily embedded into N_{i+1} over M_0, hence M_{i+2} cannot be $\leq_{\mathfrak{k}}$-elementarily embedded into N_{i+1} in contradiction to the choice of i as by Lemma 3.10 $i+1$ is in C. □

Corollary 3.13. *If $\mathfrak{k}$ is categorical in $\lambda, \theta \in [\chi, \lambda)$ and $M \in K_\theta$ is an a.b. (e.g. M is nice, see 2.1), then there exists $M^+ \in K_\theta$ such that:*

$(\aleph)$ *$M \leq_{\mathfrak{k}} M^+$ and M^+ is nice,*

$(\beth)$ *M^+ is universal over M.*

***Proof*.** By Theorem 3.12 and Remark 3.8. □

Corollary 3.14. *Suppose that $\mathfrak{k}$ is categorical in λ and $\theta \in [\chi, \lambda)$. Then there is a nice universal model $M \in K_\theta$.*

Proof. By 3.4 it suffices to find a universal model of power θ, noting that universality is preserved under $\leq_{\mathfrak{k}}$-elementary extensions in the same power. As in the proof of 3.10, there is a linear order $(I, <_I)$ of power λ and $J \subseteq I, |J| = \theta$, such that:

(∗) $(\forall J' \subseteq I)$ (if $|J'| \leq \theta$, then there is an order-preserving injective map g from J' into J).

To finish the proof it suffices to prove:

⊞ $\mathrm{EM}(J) \in K_\theta$ is universal.

Why ⊞ holds? $\mathrm{EM}(J)$ is a model of power θ since $\max(|J|, \chi) \leq \theta$ and $\theta = |J| \leq \|\mathrm{EM}(J)\|$. Let us show that $\mathrm{EM}(J)$ is universal. Suppose that $N \in K_\theta$. Applying Fact 1.11 to a suitably large $\mathrm{Op}(N)$ find $M \in K_\lambda, N \leq_{\mathfrak{k}} M$, so that by λ-categoricity of $\mathfrak{k}, M \cong \mathrm{EM}(I)$. There is a surjective $\leq_{\mathfrak{k}}$-elementary embedding $h\colon N \underset{\mathfrak{k}}{\rightarrow} N' \leq_{\mathfrak{k}} \mathrm{EM}(I)$ and there exists $J' \subseteq I$, $|J'| \leq \|N'\| + \chi = \theta$, such that $N' \subseteq \mathrm{EM}(J')$. So by (∗) there is an order preserving injective map g from J' into J. Now g induces an $\leq_{\mathfrak{k}}$-elementary embedding $\hat{g}$ from $\mathrm{EM}(J')$ into $\mathrm{EM}(J)$. Let $f = \hat{g}h$, then $f\colon N \underset{\mathfrak{k}}{\rightarrow} \mathrm{EM}(J)$ is as required. □

Theorem 3.15. *Suppose that $\mathfrak{k}$ is categorical in λ, $\theta \in [\kappa + |T|, \lambda)$, $N \in K_{<\lambda}$ is nice, $M \in K_\theta$ and $M \underset{nice}{\leq} N$. Then M is nice.*

Proof. Let $B \in K_\theta$, $M \leq_{\mathfrak{k}} B$ and we show that $M \underset{\text{nice}}{\leq} B$. Well, since $M \underset{\text{nice}}{\leq} N$ and $M \leq_{\mathfrak{k}} B$, by Lemma 3.1 there exists an amalgam $M^* \in K_{<\lambda}$ of N, B over M. Without loss of generality by 1.16 $\|M^*\| = \|N\|$. But N is nice, hence $N \underset{\text{nice}}{\leq} M^*$. Since $M \underset{\text{nice}}{\leq} N$, it follows by Observation 1.34 that $M \underset{\text{nice}}{\leq} M^*$. Since $M \leq_{\mathfrak{k}} B \leq_{\mathfrak{k}} M^*$, it follows by Observation 1.39 that $M \underset{\text{nice}}{\leq} B$. □

4. (θ, ∂)-saturated models

In this section, we define notions of saturation which will be of use in proving amalgamation for $\mathfrak{k}_\lambda$.

Definition 4.1. Suppose that ∂ is an ordinal, $\aleph_0 \leq \partial \leq \theta \in [\chi, \lambda)$.

(1) An τ-structure M is (θ, ∂)-saturated[8] iff:

(a) $\|M\| = \theta$,

(b) there exists a continuous $\leq_{\mathfrak{k}}$-chain $\langle M_i \in K_\theta \colon i < \partial\rangle$ witnessing it, which means:

  (i) M_0 is nice and universal,

[8]Called (θ, ∂)-trimmed in [33].

(ii) M_{i+1} is universal over M_i,

(iii) M_i is nice, and,

(iv) $M = \bigcup_{i<\partial} M_i$.

(2) M is θ-saturated iff M is $(\theta, \mathrm{cf}(\theta))$-saturated.

(3) M is (θ, ∂)-saturated over N iff M is (θ, ∂)-saturated as witnessed by a chain $\langle M_i \colon i < \partial\rangle$ such that $N \leq_{\mathfrak{k}} M_0$.

The principal facts established in this section connect the existence, uniqueness and niceness of (θ, ∂)-saturated models.

Theorem 4.2. *Suppose that $\mathfrak{k}$ is categorical in λ and $\partial \leq \theta \in [\chi, \lambda)$. Then:*

(1) there exists a (θ, ∂)-saturated model M,

(2) for ∂ a limit ordinal, M is unique up to isomorphism,

(3) M is nice.

Proof. One proves (1), (2), and (3) simultaneously by induction on ∂.

Ad (1). Choose a continuous $\leq_{\mathfrak{k}}$-chain $\langle M_i \in K_\theta \colon i < \partial\rangle$ of nice models by induction on i as follows. For $i = 0$, apply 3.14 to find a nice universal model $M_0 \in K_\theta$. For $i = j+1$, note that M_j is an a.b. by 3.6 (since M_j is nice), hence by 3.13 there exists a nice model $M_i \in K_\theta, M_j \leq_{\mathfrak{k}} M_i, M_i$ universal over M_j. For limit i, let $M_i = \bigcup_{j<i} M_j$. Note that by the inductive hypothesis (3) on ∂ for $i < \partial$, since M_i is (θ, i)-saturated, M_i is nice. Thus $M = \bigcup_{i<\partial} M_i$ is (θ, ∂)-saturated (witnessed by $\langle M_i \colon i < \partial\rangle$). Note that M is universal since $\langle M_i \colon i < \partial\rangle$ is continuous and M_0 is universal.

Ad (2). Recall that each M_i is an amalgamation base by Lemma 2.1. As ∂ is a limit ordinal standard back-and-forth argument shows that if M and N are (θ, ∂)-saturated models, then M and N are isomorphic.

Ad (3). By the uniqueness (i.e. by Ad(2)) it suffices to prove that some (θ, ∂)-saturated model is nice. Suppose that M is (θ, ∂)-saturated. We'll show that M is nice.

If $\mathrm{cf}(\partial) < \partial$, then M is also $(\theta, \mathrm{cf}(\partial))$-saturated and hence by the inductive hypothesis (3) on ∂ for $\mathrm{cf}(\partial), M$ is nice. So we'll assume that $\mathrm{cf}(\partial) = \partial$. Choose a continuous $\leq_{\mathfrak{k}}$-chain $\langle M_i \in K_\theta \colon i < \theta^+\rangle$ such that: M_0 is nice and universal (possible by 3.14); if M_i is nice, then $M_{i+1} \in K_\theta$ is nice and universal over M_i (possible by 3.6 and 3.13); if M_i is not nice (so necessarily i is a limit ordinal), then $M_{i+1} \in K_\theta, M_i \leq_{\mathfrak{k}} M_{i+1}$ and $M_i \underset{\text{nice}}{\not\leq} M_{i+1}$. By Theorem 3.1 and Fact 1.38 there is a club C of θ^+ such that if $i \in C$, then $M_i \underset{\text{nice}}{\leq} M_{i+1}$. So by the choice of $\langle M_i \colon i < \theta^+\rangle$, if $i \in C$, then M_i is nice. Choose $i \in C, i = \sup(i \cap C)$, $\mathrm{cf}(i) = \partial$. It suffices to show that M_i is (θ, ∂)-saturated (for then by (2) M_i is isomorphic to M and so M is nice). Choose a continuous increasing sequence $\langle \alpha_\zeta \colon \zeta < \partial\rangle \subseteq C$ such that $i = \bigcup\{\alpha_\zeta \colon \zeta < \partial\}$ (recall that $i = \sup(i \cap C)$, $\mathrm{cf}(i) = \partial$). Now $M_i = \bigcup_{\zeta<\partial} M_{\alpha_\zeta}$. Of course M_{α_0} is universal (since M_0 is universal and $M_0 \leq_{\mathfrak{k}} M_{\alpha_0}$), $M_{\alpha_{\zeta+1}}$ is universal over M_{α_ζ} since $M_{\alpha_\zeta+1}$ is universal over M_{α_ζ}

and $M_{\alpha_\zeta} \leq_{\mathfrak{k}} M_{\alpha_\zeta+1} \leq_{\mathfrak{k}} M_{\alpha_{\zeta+1}}$. Also M_{α_ζ} is nice for each $\zeta < \partial$ since $\alpha_\zeta \in C$. Hence M_i is (θ, ∂)-saturated, recall that M_i is nice because $i \in C$, so we are done. □

Remark 4.3. Remember that by Theorem 3.15, if $\mathfrak{k}$ is categorical in λ, $\theta \in [\chi, \lambda)$, $N \in K_{<\lambda}$ is nice, $M \in K_\theta$ and $M \underset{\text{nice}}{\leq} N$, then M is nice.

Theorem 4.4. *Suppose that $\mathfrak{k}$ is categorical in λ, $\chi \leq \theta < \theta^+ < \lambda$. If $\langle M_i \in K_\theta : i < \theta^+ \rangle$ is a continuous $\leq_{\mathfrak{k}}$-chain of nice models such that M_{i+1} is universal over M_i for $i < \theta^+$, <u>then</u> $\bigcup_{i<\theta^+} M_i$ is (θ^+, θ^+)-saturated.*

Remark 4.5. Why this is not trivial? Because here M_i is of cardinality θ whereas in Definition 4.1 the M_i are of cardinality θ^+.

Proof. Write $M = \bigcup_{i<\theta^+} M_i$. Note that if $\langle M'_i \in K_\theta : i < \theta^+ \rangle$ is any other continuous $\leq_{\mathfrak{k}}$-chain of nice models such that M'_{i+1} is universal over M'_i then $\bigcup_{i<\theta^+} M'_i \cong M$ (use again the back and forth argument recalling that M_0, is an a.b., so as M_j is universal over M_0, it is universal).

By Theorem 4.2 there exists a (θ^+, θ^+)-saturated model N which is unique and nice. In particular $||N|| = \theta^+$ and there exists a continuous $\leq_{\mathfrak{k}}$-chain $\langle N_i \in K_{\theta^+} : i < \theta^+ \rangle$ such that:

(i) N_0 is nice and universal,

(ii) N_{i+1} is universal over N_i,

(iii) N_i is nice,

(iv) $N = \bigcup_{i<\theta^+} N_i$.

It suffices to prove that M and N are isomorphic models.

Without loss of generality $|N| = \theta^+$. By Fact 1.11, the set $C_1 = \{\delta < \theta^+ : N \restriction \delta \leq_{\mathfrak{k}} N\}$ contains a club of θ^+. By Theorem 3.1 there exists a club $C_2 \subseteq C_1$ of θ^+ such that for every $\delta \in C_2$, $N \restriction \delta \underset{\text{nice}}{\leq} N$. Since $\{|N_i| : i < \theta^+\}$ is a continuous increasing sequence of subsets of θ^+, it follows that $C_3 = \{\delta < \theta^+ : \delta \subseteq |N_\delta|\}$ is a club of θ^+. Hence there is a club C_4 of θ^+ such that $C_4 \subseteq C_1 \cap C_2 \cap C_3 \cap [\theta, \theta^+)$. Note that for $\delta \in C_4$ one has $N \restriction \delta \underset{\text{nice}}{\leq} N$, $|N \restriction \delta| = \delta \subseteq |N_\delta|$ and $N_\delta \leq_{\mathfrak{k}} N$, so that $N \restriction \delta \leq_{\mathfrak{k}} N_\delta \leq_{\mathfrak{k}} N$ and so by 1.38 $N \restriction \delta \underset{\text{nice}}{\leq} N_\delta$. Also, $\langle N_\delta : \delta \in C_4 \rangle$ is a continuous increasing $\leq_{\mathfrak{k}}$-chain, $N_\delta \in K_{\theta^+}$ and $N \restriction \delta \in K_\theta$.

By Theorem 3.15 $N \restriction \delta$ is nice since N_δ is nice (by (iii)). So by Corollary 3.13 $N \restriction \delta$ has a nice $\leq_{\mathfrak{k}}$-extension $B_\delta \in K_\theta$ which is universal over $N \restriction \delta$, without loss of generality $N \restriction \delta \leq_{\mathfrak{k}} B_\delta \leq_{\mathfrak{k}} N$.

[Why? since $N \restriction \delta \leq_{\mathfrak{k}} B_\delta$ (in fact $N \restriction \delta \underset{\text{nice}}{\leq} B_\delta$) and $N \restriction \delta \underset{\text{nice}}{\leq} N_\delta$, by Lemma 2.1 there exists an amalgam $A_\delta \in K_{\leq\theta^+}$ of B_δ, N_δ over $N \restriction \delta$. Let $f_\delta : B_\delta \underset{\mathfrak{k}}{\to} A_\delta$ be a witness. But $N_{\delta+1}$ is universal over N_δ (by (ii)), so A_δ can be $\leq_{\mathfrak{k}}$-elementarily embedded into $N_{\delta+1}$ over N_δ (say by g_δ), hence B_δ can be $\leq_{\mathfrak{k}}$-elementarily embedded into N (using $g_\delta f_\delta$).]

Let $C_5 = \{\delta \in C_4 \colon \text{if } \alpha \in C_4 \cap \delta, \text{ then } |B_\alpha| \subseteq \delta\}$. Note that C_5 is a club of θ^+ since $||B_\alpha|| = \theta$. [Why? For $\alpha \in C_4$, let $E_\alpha = (\sup|B_\alpha|, \theta^+) \cap C_4$, let $E_\alpha = \theta^+$ for $\alpha \notin C_4$ and let E be the diagonal intersection of $\langle E_\alpha \colon \alpha < \theta^+\rangle$, i.e. $E = \{\delta < \theta^+ \colon (\forall \alpha < \delta)(\delta \in E_\alpha)\}$. Note that E is a club of θ^+ and $C_5 \supseteq E \cap C_4$ which is a club of θ^+.]

Thus $\langle N \restriction \delta \colon \delta \in C_5\rangle$ is a continuous $\leq_{\mathfrak{k}}$-chain of nice models, each of power θ. If $\delta_1 \in C_5$ and $\delta_2 = \min(C_5 \setminus (\delta_1 + 1))$, then $N \restriction \delta_1 \leq_{\mathfrak{k}} B_{\delta_1} \leq_{\mathfrak{k}} N \restriction \delta_2$. Hence $N \restriction \delta_2$ is universal over $N \restriction \delta_1$ (since B_{δ_1} is universal over $N \restriction \delta_1$). Let $\{\delta_i \colon i < \theta^+\}$ enumerate C_5 and set $M'_i = N \restriction \delta_i$. Note that $N = \bigcup_{i<\theta^+} M'_i$. Then $\langle M'_i \in K_\theta \colon i < \theta^+\rangle$ is a continuous $\leq_{\mathfrak{k}}$-chain of nice models, M'_{i+1} is universal over M'_i. Therefore N and M are isomorphic (as said at the beginning of the proof), as required. $\square$

Notation 4.6. $\Theta = \{\bar\theta \colon \bar\theta = \langle\theta_i \colon i < \delta\rangle$ is a (strictly) continuous increasing sequence of cardinals, $\chi < \theta_0$, $\delta < \theta_0$ (a limit ordinal), $\bigcup_{i\leq\delta} \theta_i \leq \lambda\}$ and $\Theta^- = \{\bar\theta \in \Theta \colon \sup\theta_i < \lambda\}$.

Remark 4.7. Let $\theta = \sup(\bar\theta) = \sup\{\theta_i \colon i < \lg(\bar\theta)\}$ for $\bar\theta \in \Theta$. Then θ is singular, since $\mathrm{cf}(\theta) \leq \delta < \theta_0 \leq \theta$.

Definition 4.8. Let $\bar\theta \in \Theta$. A model M is $\bar\theta$-saturated iff there is a continuous $\leq_{\mathfrak{k}}$-chain $\langle M_i \in K_{\theta_i} \colon i < \delta\rangle$ such that $M = \bigcup_{i<\delta} M_i$, M_i is nice and M_{i+1} is θ_{i+1}-universal over M_i.

Definition 4.9. Suppose that $\bar\theta \in \Theta$. $\Pr(\bar\theta)$ holds iff every $\bar\theta$-saturated model is nice.

Remark 4.10. (1) If $\bar\theta_1, \bar\theta_2 \in \Theta$, $\mathrm{rng}(\bar\theta_1) \subseteq \mathrm{rng}(\bar\theta_2)$, $\sup\mathrm{rng}(\bar\theta_1) = \sup\ \mathrm{rng}(\bar\theta_2)$, and M is $\bar\theta_2$-saturated, then M is $\bar\theta_1$-saturated.

(2) For $\bar\theta \in \Theta^-$ and $\Pr(\bar\theta')$ whenever $\bar\theta' \in \Theta$ is a proper initial segment of $\bar\theta$, there is a $\bar\theta$-saturated model and it is unique.

Theorem 4.11. *Suppose that $\bar\theta \in \Theta^-$ and for every limit ordinal $\alpha < \lg(\bar\theta)$, $\Pr(\bar\theta \restriction \alpha)$. Then $\Pr(\bar\theta)$.*

Proof. Let $\theta = \sup(\bar\theta)$. By Remark 4.10(1) and the uniqueness of $\bar\theta$-saturated models 4.10(2), without loss of generality one may assume that $\lg(\bar\theta) = \mathrm{cf}(\sup(\bar\theta)) = \mathrm{cf}(\theta)$. Now, by Remark 4.7, we know $(\mathrm{cf}(\theta))^+ < \theta$ $(= \sup(\bar\theta))$, so by [27, §1] there exists $\langle S, \langle C_\alpha \colon \alpha \in S\rangle\rangle$ such that:

(α) $S \subseteq \theta^+$ is a set of ordinals; $0 \notin S$,

(β) $S_1 = \{\alpha \in S\colon \mathrm{cf}(\alpha) = \mathrm{cf}(\theta)\}$ is a stationary subset of θ^+,

(γ) if $\alpha \in S$ then $\alpha = \sup(C_\alpha)$ and, if $\alpha \in S \setminus S_1$ then $\mathrm{otp}(\mathrm{C}_\alpha) < \mathrm{cf}(\theta)$,

(δ) if $\beta \in C_\alpha$, then $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$,

(ϵ) C_α is a set of successor ordinals.

[Note that the existence of $\langle S, \langle C_\alpha\colon \alpha \in S\rangle\rangle$ is provable in ZFC.]

Without loss of generality $S \setminus S_1 = \cup\{C_\alpha\colon \alpha \in S_1\}$. We shall construct the required model by induction, using $\langle C_\alpha\colon \alpha \in S\rangle$. Remember $\bar{\theta} = \langle \theta_\zeta\colon \zeta < \mathrm{cf}(\theta)\rangle$. Let us start by defining by induction on $\alpha < \theta^+$ the following entities: M_α, $M_{\alpha,\xi}$ (for $\alpha < \theta^+$, $\xi < \mathrm{cf}(\theta)$), and N_α (only when $\alpha \in \bigcup\limits_{\beta \in S} C_\beta$) such that:

(A1) $M_\alpha \in K_\theta$,

(A2) $\langle M_\alpha\colon \alpha < \theta^+\rangle$ is a continuous increasing $\leq_{\mathfrak{k}}$-chain of models,

(A3) $M_{\alpha+1}$ is nice, and if M_α is not nice, then $M_\alpha \underset{\text{nice}}{\not\leq} M_{\alpha+1}$,

(A4) $M_\alpha \neq M_{\alpha+1}$,

(A5) $M_{\alpha+1}$ is weakly universal over M_α,

(B1) $M_\alpha = \bigcup\limits_{\xi < \mathrm{cf}(\theta)} M_{\alpha,\xi}$, $||M_{\alpha,\xi}|| = \theta_\xi$,

(B2) if $\alpha \in S_1$, $\beta \in C_\alpha$, $\gamma \in C_\alpha$, $\beta < \gamma$, then:

(a) $N_\beta \leq_{\mathfrak{k}} M_\beta$,

(b) $||N_\beta|| = \theta_{\mathrm{otp}(C_\beta)}$,

(c) $(\forall \xi < \mathrm{otp}(C_\beta))(M_{\beta,\xi} \leq_{\mathfrak{k}} N_\gamma)$,

(d) N_β is nice,

(e) N_γ is $\theta_{\mathrm{otp}(C_\gamma)}$-universal over N_β.

There are now two tasks at hand. First of all, we shall explain how to construct these entities (THE CONSTRUCTION, below). Then we shall use them to build a nice $\bar{\theta}$-saturated model (PROVING $\mathrm{Pr}(\bar{\theta})$, below). From the uniqueness of $\bar{\theta}$-saturated models it will thus follow that $\mathrm{Pr}(\bar{\theta})$ holds.

<u>THE CONSTRUCTION</u>: we consider several cases:

<u>Case (i)</u>: $\beta = 0$. Choose $M_0 \in K_\theta$ and $\langle M_{0,\xi} \in K_\theta\colon \xi < \mathrm{cf}(\theta)\rangle$ with $M_0 = \bigcup\limits_{\xi < \mathrm{cf}(\theta)} M_{0,\xi}$ using Fact 1.11. There is no need to define N_0 since $0 \notin C_\alpha$.

Case (ii): β is a limit ordinal. Let $M_\beta = \bigcup\limits_{\gamma < \beta} M_\gamma$ and choose $\langle M_{\beta,\xi}\colon \xi < \mathrm{cf}(\theta)\rangle$ using Fact 1.11. Again there's no call to define N_β since C_α is always a set of successor ordinals.

Case (iii): β is a successor ordinal, $\beta = \gamma + 1$. Choose $M'_\gamma \in K_\theta$ such that $M_\gamma \leq_{\mathfrak{k}} M'_\gamma$ and if possible $M_\gamma \underset{\text{nice}}{\not\leq} M'_\gamma$; without loss of generality M'_γ is weakly universal over M_γ. If $\beta \notin S$, then define things as above, taking into account (A3). The definitions of M_β, $M_{\beta,\xi}$ present no special difficulties. Now suppose that $\beta \in S$. The problematic entity to define is N_β.

If $C_\beta = \emptyset$, choose for N_β any nice sub-model (of power θ_0) of M_γ.

If $C_\beta \neq \emptyset$, then first define $N^-_\beta = \bigcup_{\gamma \in C_\beta} N_\gamma$. Note that N^-_β is nice. [If C_β has a last element β', then $N^-_\beta = N_{\beta'}$ which is nice; if C_β has no last element, then $N^-_\beta = \bigcup_{\gamma \in C_\beta} N_\gamma$ is $\overline{\theta} \restriction \mathrm{otp}(C_\beta)$-saturated, and, by the hypothesis of the theorem, $\Pr(\overline{\theta} \restriction \mathrm{otp}(C_\beta))$, so N^-_β is nice.] Also $N^-_\beta \preceq_{\mathfrak{k}} M_\gamma$. If otp($\mathrm{C}_\beta$) is a limit ordinal we let $N_\beta = N^-_\beta$ and $M_\beta = M'_\gamma$, so we have finished, so assume $\mathrm{otp}(C_\beta)$ is a successor ordinal. To complete the definition of N_β, one requires a Lemma (the proof of which is similar to Corollary 3.12, Theorem 3.13):

(∗) if $A \subseteq M \in K_\theta$, $|A| \leq \theta_j < \theta$, then there exist a nice $M^+ \in K_\theta$, $M \leq_{\mathfrak{k}} M^+$, and nice models N^*, $N^+ \in K_{\theta_j}$, $A \subseteq N^* \leq_{\mathfrak{k}} N^+ \leq_{\mathfrak{k}} M^+$ and N^+ is universal over N^*.

Why is this enough? Use the Lemma with $M = M'_\beta$ and $A = N^-_\beta \cup \bigcup_{\substack{\xi < \mathrm{otp}(C_\beta) \\ \gamma \in C_\beta}} M_{\gamma,\xi}$ to find N^*, N^+, M^+ and choose N^+, M^+ as N_β, M_β respectively.

Now, why (∗) holds? The proof of (∗) is easy as M'_β is nice.

PROVING $\Pr(\overline{\theta})$:

For $\alpha \in S_1$, consider $\langle N_\beta \colon \beta \in C_\alpha \rangle$. For $\beta, \gamma \in C_\alpha, \beta < \gamma$, one has by (B2)(c) $\bigcup_{\xi < \mathrm{otp}(C_\beta)} M_{\beta,\xi} \subseteq N_\gamma$. Therefore $M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma$. (Recalling $M_\beta = \bigcup_{\xi < \mathrm{cf}(\theta)} M_{\beta,\xi} = \bigcup_{\xi < \mathrm{cf}(\alpha)} M_{\beta,\xi}$ (since $\alpha \in S_1$); for $\xi < \mathrm{cf}(\alpha)$, choose $\gamma \in C_\alpha$, $\xi < \gamma, \beta < \gamma$; so $M_{\beta,\xi} \subseteq N_\gamma$ and $M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma$).

Thus for every $\beta \in C_\alpha$, $M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma$ hence $M_\alpha = \bigcup_{\beta \in C_\alpha} M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma$ (remember $\alpha = \sup(C_\alpha)$ as $\alpha \in S_1$). If $\gamma \in C_\alpha$, then $N_\gamma \leq_{\mathfrak{k}} M_\gamma$ (by (B2)(a)), and so $\bigcup_{\gamma \in C_\alpha} N_\gamma \subseteq \bigcup_{\beta \in C_\alpha} M_\beta = M_\alpha$ by continuity. So $M_\alpha = \bigcup_{\beta \in C_\alpha} N_\beta$ hence $\langle N_\beta \colon \beta \in C_\alpha \rangle$ exemplifies M_α is $\bar{\theta}$-saturated (remember $\Pr(\bar{\theta} \restriction \delta)$ for every limit $\delta < \lg(\overline{\theta})$). So M_α is $\overline{\theta}$-saturated for every $\alpha \in S_1$. In other words $\{\alpha < \theta^+ \colon M_\alpha \text{ is } \overline{\theta}\text{-saturated}\} \supseteq S_1$ and is stationary, so, applying 3.1, there exists $\alpha < \theta^+$ such that M_α is $\overline{\theta}$-saturated and $M_\alpha \underset{\text{nice}}{\leq} \bigcup_{\beta < \theta^+} M_\beta$. Hence by 1.38 $M_\alpha \underset{\text{nice}}{\leq} M_{\alpha+1}$ and so, since $M_{\alpha+1}$ is nice (A3), M_α is nice (by Theorem 3.15).

We conclude that $\Pr(\overline{\theta})$ holds. □

To round off this section of the paper, let us make the connection between $\overline{\theta}$-saturation and $(\theta, \mathrm{cf}(\theta))$-saturation (Notation follows 4.6–Remark 5.10).

Theorem 4.12. *Let $\bar{\theta} \in \Theta^-$ and $\theta = \sup_i(\theta_i)$. Every $\bar{\theta}$-saturated model is $(\theta, \mathrm{cf}(\theta))$-saturated.*

Proof. Let $\langle M_\alpha \colon \alpha < \theta^+ \rangle$ be as in the proof of Theorem 4.11. By Theorem 3.1 there exists a club C of θ^+ such that for every $\alpha \in C$, $M_\alpha \underset{\text{nice}}{\leq} \bigcup_{\beta<\theta^+} M_\beta$ hence by the construction M_α is nice. So if $\alpha, \beta \in C$ and $\alpha < \beta$, then M_β is a universal extension of M_α and for $\gamma = \sup(\gamma \cap C)$, $\gamma \in C$, one has that M_γ is $(\theta, \mathrm{cf}(\gamma))$-saturated. Choose $\gamma \in S_1 \cap C$ and $\sup(\gamma \cap C) = \gamma$. So M_γ is $(\theta, \mathrm{cf}(\theta))$-saturated and also $\bar{\theta}$-saturated (see proof of 4.11). Together we finish. □

5. The amalgamation property for $\mathfrak{k}_{<\lambda}$

Corollaries 5.5 and 5.6 are the goal of this section, showing that if K is categorical in λ then every element of $\mathfrak{k}_{<\lambda}$ is nice (see 5.5) and $\mathfrak{k}_{<\lambda}$ has the amalgamation property (see 5.6).

Lemma 5.1. *Suppose that μ is singular, $\langle \mu_i \colon i < \mathrm{cf}(\mu)\rangle$ is a continuous strictly increasing sequence of cardinals, $\mu = \sup_{i<\mathrm{cf}(\mu)} \mu_i$, and $\chi \leq \mu_0 < \mu \leq \lambda$. Then there exist a linear order I of power μ and a continuous increasing sequence $\langle I_i \colon i < \mathrm{cf}(\mu)\rangle$ of linear orders such that:*

(a) $\chi \leq |I_i| \leq \mu_i$ and $|I_i| < |I_{i+1}|$ for each i,

(b) $\bigcup_{i<\mathrm{cf}(\mu)} I_i = I$,

(c) every $t \in I_{i+1} \setminus I_i$ defines a Dedekind cut of I_i in which (at least) one side of the cut has cofinality κ.

Proof. Let $I = (\{0\} \times \mu) \cup (\{1\} \times \kappa)$, $I_i = (\{0\} \times \mu_i) \cup (\{1\} \times \kappa)$ ordered by:

$$(i, \alpha)_{<_I}(j, \beta) \text{ iff i < j or } (0 = \text{i} = \text{j and } \alpha < \beta) \text{ or } (1 = \text{i} = \text{j and } \alpha > \beta).$$

□

Lemma 5.2. *Suppose that $\mathfrak{k}$ is categorical in $\lambda > \mathrm{cf}(\lambda)$, $\kappa + \mathrm{LST}_{\mathfrak{k}} < \mu \leq \lambda$. If $M \in K_\lambda$, <u>then</u> there exists a continuous increasing $\leq_{\mathfrak{k}}$-chain $\langle M_i \colon i < \mathrm{cf}(\lambda)\rangle$ of models such that:*

(a) $M \leq_{\mathfrak{k}} \bigcup_{i<\mathrm{cf}(\lambda)} M_i$,

(b) $\| \bigcup_{i<\mathrm{cf}(\lambda)} M_i \| = \lambda$,

(c) $\kappa + |T| \leq \|M_i\| < \|M_{i+1}\| < \lambda$,

(d) *for each* $i < \mathrm{cf}(\lambda), M_i \underset{nice}{\leq} \left(\underset{j<\mathrm{cf}(\lambda)}{\cup} M_j \right)$.

Proof. As λ is a limit cardinal, choose a continuous increasing sequence $\langle \mu_i : i < \mathrm{cf}(\lambda) \rangle$, $\lambda = \sup_{i<\mathrm{cf}(\lambda)} \mu_i$, $\kappa + |T| \leq \mu_0 < \lambda$. Let $\langle I, \langle I_i : i < \mathrm{cf}(\lambda) \rangle \rangle$ be as in 5.1. By λ-categoricity of $\mathfrak{k}$ without loss of generality $M = \mathrm{EM}(\lambda)$. Let $M_i = \mathrm{EM}(I_i)$ for $i < \mathrm{cf}(\lambda)$. Clearly (a), (b), and (c) hold. To obtain (d), observe that by 2.6 and Corollary 3.6 it suffices to remark that by demand (c) from Lemma 5.1 on $\langle I_i : i < \mathrm{cf}(\lambda) \rangle$ clauses ($\aleph$) or ($\beth$) in 2.6 holds for each $t \in I \setminus I_i$. $\square$

Theorem 5.3. *For every* $\mu \in [\chi, \lambda]$ *and* $M \in K_\mu$, *there exists* $M' \in K_\mu, M \leq_{\mathfrak{k}} M'$ *such that:*

$(*)_{M'}$ *for every* $A \subseteq |M'|$, $|A| < \lambda \ \wedge \ |A| \leq \mu$, *there is* $N \in K_{\chi+|A|}$ *such that* $A \subseteq N \leq_{\mathfrak{k}} M'$ *and* N *is nice.*

Proof. The proof is by induction on μ.

Case 1: $\mu = \chi$. By Theorem 3.4 there is $M' \in K_\mu$, $M \leq_{\mathfrak{k}} M'$ and M' is nice. Given $A \subseteq |M'|$ let $N = M'$ and note that N is as required in $(*)_{M'}$.

Case 2: $\chi < \mu$. Without loss of generality, one can replace M by any $\leq_{\mathfrak{k}}$-extension in K_μ. Choose a continuous increasing sequence $\langle \mu_i : i < \mathrm{cf}(\mu) \rangle$ such that if μ is a limit cardinal it is a strictly increasing sequence with limit μ; if μ is a successor, use $\mu_i^+ = \mu$ for every $i < \mathrm{cf}(\mu)$, and in both cases $\chi \leq \mu_i < \mu$. Find $\bar{M} = \langle M_i : i < \mathrm{cf}(\mu) \rangle$ such that:

(a) $M \leq_{\mathfrak{k}} \bigcup_{i<\mathrm{cf}(\mu)} M_i$,

(b) $|| \bigcup_{i<\mathrm{cf}(\mu)} M_i || = \mu$,

(c) $||M_i|| = \mu_i$,

(d) $M_i \underset{\text{nice}}{\leq} \bigcup_{j<\mathrm{cf}(\mu)} M_j$.

Why does $\bar{M}$ exist? If $\mu = \lambda$ by Lemma 5.2, otherwise by Theorem 4.4 (μ regular) and Theorem 4.12 (μ singular).

Choose by induction on $i < \mathrm{cf}(\mu)$ models L_i^0, L_i^1, L_i^2 in that order such that:

(α) $M_i \leq_{\mathfrak{k}} L_i^0 \leq_{\mathfrak{k}} L_i^1 \leq_{\mathfrak{k}} L_i^2 \in K_{\mu_i}$,

(β) $j < i \Rightarrow L_j^2 \leq_{\mathfrak{k}} L_i^0$,

(γ) $(*)_{L_i^1}$ holds , i.e. for each $A \subseteq |L_i^1|$, there is $N \in K_{\leq \kappa+|T|+|A|}$ such that $A \subseteq N \leq_{\mathfrak{k}} L_i^1$ and N is nice (so in particular L_i^1 is nice, letting $A = |L_i^1|$),

(δ) L_i^2 is nice and μ_i-universal over L_i^1,

(ε) L_i^0 is $\leq_{\mathfrak{k}}$-increasing continuous,

(ζ) $L_i^\ell \cap \bigcup_{j<\mathrm{cf}(\mu)} M_j = M_i$ (or use system of $\leq_{\mathfrak{k}}$-embeddings).

For $i = 0$, let $L_i^0 = M_0$. For $i = j+1$, note that by Criterion 2.1 there is an amalgam $L_i^0 \in K_{\mu_i}$ of M_i, L_j^2 over M_j since $M_j \underset{\text{nice}}{\leq} M_i$ and $M_j \leq_{\mathfrak{k}} L_j^2$ (use last phrase of Fact 1.11 for clause (ζ)); actually not really needed. For limit i, continuity necessitates choosing $L_i^0 = \bigcup_{j<i} L_j^0$ (note that in this case $L_i^0 = \bigcup_{j<i} L_j^2$). To choose L_i^1 apply the inductive hypothesis with respect to μ_i, L_i^0 to find L_i^1 so that $L_i^0 \leq_{\mathfrak{k}} L_i^1$ and $(\gamma)(*)_{(L_i^1)}$ holds. To choose L_i^2 apply Lemma 3.10 to $L_i^1 \in K_{\mu_i}$ giving $L_i^1 \leq_{\mathfrak{k}} L_i^2$, L_i^2 is nice and μ_i-universal over L_i^1 (so (δ) holds).

Let $L = \bigcup_{i<\mathrm{cf}(\mu)} L_i^0 = \bigcup_{i<\mathrm{cf}(\mu)} L_i^1 = \bigcup_{i<\mathrm{cf}(\mu)} L_i^2$, and let $L_i = L_i^0$ if i is a limit, L_i^1 otherwise. Now show by induction on $i < \mathrm{cf}(\mu)$ that L_i is nice.

[Why? show by induction on i for $i = 0$ or i successor that $L_i = L_i^1$ hence use clause (γ), if i is limit then L_i is $(\bar{\theta} \restriction i)$-saturated, hence L_i is nice by Theorem 4.9, 4.11.]

Now $\langle L_i \colon i < \mathrm{cf}(\mu)\rangle$ witnesses that <u>if</u> μ is regular, L is (μ,μ)-saturated by Theorem 4.4, <u>if</u> μ is singular, L is $\bar{\mu}$-saturated; in all cases L is $\bar{\mu}$-saturated of power μ, hence by the results of section 5 (Theorem 4.9, 4.11) if $\mu < \lambda$ then L is nice. Claim 5.4 below provides the desired model M', so we are done. □

Claim 5.4. $M' = L$ is as required (in 5.3).

Proof. $M \leq_{\mathfrak{k}} \underset{i<\mathrm{cf}(\mu)}{\cup} M_i \leq_{\mathfrak{k}} \underset{i<\mathrm{cf}(\mu)}{\cup} L_i^0 = L \in K_\mu$. Suppose that $A \subseteq |L|$. If $|A| = \mu$, then necessarily $\mu < \lambda$ and we take $N = L$. So without loss of generality, $|A| < \mu$. If $\mu = \mathrm{cf}(\mu)$ or $|A| < \mathrm{cf}(\mu)$, then there is $i < \mathrm{cf}(\mu)$ such that $A \subseteq L_i^1$ and, by $(\gamma), (*)_{L_i^1}$ holds, so there is $N \in K_{\kappa+\mathrm{LST}(\mathfrak{k})+|A|}, A \subseteq N \leq_{\mathfrak{k}} L_i^1, N$ is nice and $N \leq_{\mathfrak{k}} L$ as required. So suppose that $\mathrm{cf}(\mu) \leq |A| < \mu$. Choose by induction on $i < \mathrm{cf}(\mu)$ models N_i^0, N_i^1, N_i^2 in that order such that:

(α) $N_i^0 \leq_{\mathfrak{k}} N_i^1 \leq_{\mathfrak{k}} N_i^2$,

(β) $N_i^2 \leq_{\mathfrak{k}} N_{i+1}^0$,

(γ) $A \cap L_i^0 \subseteq N_i^0 \leq_{\mathfrak{k}} L_i^0$,

(δ) $N_i^1 \leq_{\mathfrak{k}} L_i^1$ and N_i^1 is nice,

(ε) $N_i^2 \leq_{\mathfrak{k}} L_i^2, N_i^2$ is nice and universal over N_i^1,

(ζ) N_i^0, N_i^1, N_i^2 have power at most $\min\{\chi + |A|, \mu_i\}$.

For $i = 0$, apply Fact 1.11 for $A \cap L_0^0, L_0^0$; for $i = j + 1$, apply Fact 1.11 to find $N_i^0 \in K_{\mu_i}, (A \cap L_i^0) \cup N_j^2 \subset N_i^0 \leq_{\mathfrak{k}} L_i^0$ (in particular $N_j^2 \leq_{\mathfrak{k}} N_i^0$); for limit $i, N_i^0 = \bigcup_{j<i} N_j^0$. To choose N_i^1, use $(*)_{L_i^1}$ for the set $A_i = N_i^0$ to find a nice $N_i^1 \in K_{\leq \chi+|A|}, N_i^0 \leq_{\mathfrak{k}} N_i^1 \leq_{\mathfrak{k}} L_i^1$. Note that $\|N_i^1\| \leq \mu_i$. Finally to choose N_i^2 note that by Lemma 3.12 the model N_i^1 has a nice extension N_i^+ (of power $\|N_i^1\|$) weakly universal over N_i^1. Now N_i^1 is nice, hence N_i^+ is universal over N_i^1 (by Definition 3.7(5)) and by Lemma 2.1 there is an amalgam N_i of N_i^+, L_i^1 over N_i^1 such that $\|N_i\| \leq \mu_i$. Since L_i^2 is universal over L_i^1 one can find an $\leq_{\mathfrak{k}}$-elementary sub-model N_i^2 of L_i^2 isomorphic to N_i. Let N_i be N_i^0 if i is a limit, N_i^1 otherwise; prove by induction on i that N_i is nice (by Theorem 4.2).

Now $\bigcup_{i<\mathrm{cf}(\mu)} N_i^0$ is an $\leq_{\mathfrak{k}}$-elementary sub-model of L of power at most $\kappa + |T| + |A|$, including A (by (γ)) and $\bigcup_{i<\mathrm{cf}(\mu)} N_i^0$ is $(\chi + |A|, \mathrm{cf}(\mu))$-saturated, hence (by Theorem 4.2) nice, as required. □

Corollary 5.5. *If K is categorical in λ then every element of $K_{<\lambda}$ is nice.*

Proof. 5.5 Suppose otherwise and let $N_0 \in K_{<\lambda}$ be a model which is not nice. Choose a suitable Op such that $\|\mathrm{Op}(N_0)\| \geq \lambda$ and by Fact 1.11 find $M_0 \in K_\lambda, N_0 \leq_{\mathfrak{k}} M_0 \leq_{\mathfrak{k}} \mathrm{Op}(N_0)$ i.e. $N_0 \underset{\text{nice}}{\leq} M_0$. It follows that:

⊞ if $N_0 \leq_{\mathfrak{k}} N \leq_{\mathfrak{k}} M_0$ and $N \in K_{<\lambda}$ then N is not nice.

[Why? By 4.3; alternatively, suppose contrariwise that N is nice. So there is $N_1 \in K_{<\lambda}, N_0 \leq_{\mathfrak{k}} N_1, N_0 \underset{\text{nice}}{\leq} N_1$, so $N_0 \underset{\text{nice}}{\leq} N$ since $N_0 \underset{\text{nice}}{\leq} M_0$ and $N \leq_{\mathfrak{k}} M_0$, hence there is an amalgam $N' \in K_{<\lambda}$ of N_1, N over N_0. But N is nice, so $N \underset{\text{nice}}{\leq} N'; N_0 \underset{\text{nice}}{\leq} N$, so $N_0 \underset{\text{nice}}{\leq} N'$ and so $N_0 \underset{\text{nice}}{\leq} N_1$ contradiction.]

On the other hand, applying Theorem 5.3 for $\mu = \lambda$ there exists $M' \in K_\lambda$ satisfying $(*)_{M'}$. By λ-categoricity of $\mathfrak{k}$ without loss of generality, $(*)_{M_0}$ holds (see Theorem 5.3) and $A = |N_0|$ yields a nice model $N \in K_{\kappa+|T|+\|N_0\|}$ such that $N_0 \leq_{\mathfrak{k}} N \leq_{\mathfrak{k}} M_0$ contradicting ⊞. □

Corollary 5.6. *If K is categorical in λ, then $\mathfrak{k}_{<\lambda}$ has the amalgamation property.*

Proof. 5.6 As every nice $M \in K_{<\lambda}$ is an amalgamation base (by Corollary 3.6) we are done by the previous corollary. □

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