Forcing techniques for Cichoń's Maximum: Lecture notes for the mini-course at the University of Vienna

Técnicas de forcing para Cichoń Maximal: Notas de un mini-curso en la Universidad de Viena

Diego A. Mejia[1,a]

Abstract. Cichoń's diagram describes the connections between combinatorial notions related to measure, category, and compactness of sets of irrational numbers. In the second part of the 2010's, Goldstern, Kellner and Shelah constructed a forcing model of Cichoń's Maximum (meaning that all non-dependent cardinal characteristics are pairwise different) by using large cardinals. Some years later, we eliminated this large cardinal assumption. In this mini-course, we explore the forcing techniques to construct the Cichoń's Maximum model and much more.

Keywords: cardinal characteristics of the continuum, Cichoń's diagram, Tukey connections, forcing iteration theory, forcing iterations with ultrafilters and measures, Boolean ultrapowers, forcing intersection with submodels.

Resumen. El diagrama de Cichoń describe las conexiones entre nociones combinatorias relacionadas con medida, categoría y compacidad de conjuntos de irracionales. En la segunda mitad de la década del 2010, Goldstern, Kellner y Shelah construyeron un modelo de forcing de Cichoń Maximal (i.e. los valores no dependientes de los cardinales característicos del diagrama de Cichoń son diferentes dos a dos) usando grandes cardinales. Algunos años más tarde, eliminamos la hipótesis de grandes cardinales. En este mini-curso, exploramos las técnicas de forcing para construir el modelo de Cichoń Maximal y mucho más.

Palabras claves: cardinales característicos del continuo, diagrama de Cichoń, conexiones de Tukey, teoría de forcing iterado, forcing iterado con ultrafiltros y medidas, ultrapotencias booleanas, intersección de forcing con submodelos.

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1. Tukey connections and cardinal characteristics of the continuum

A large part of the contents of this section are taken almost verbatim from: Section 1, up to Figure 3, of [14]; and Section 1, up to Fact 1.2, of [27].

Many cardinal characteristics of the continuum and their relations can be represented by relational systems as follows. This presentation is based on [51, 6, 8].

Definition 1.1. We say that $R = \langle X, Y, \sqsubset \rangle$ is a *relational system* if it consists of two non-empty sets X and Y and a relation $\sqsubset$.

(1) A set $F \subseteq X$ is *R-bounded* if $\exists\, y \in Y \ \forall\, x \in F \colon x \sqsubset y$.

(2) A set $E \subseteq Y$ is *R-dominating* if $\forall\, x \in X \ \exists\, y \in E \colon x \sqsubset y$.

We associate two cardinal characteristics with this relational system R:

$\mathfrak{b}(R) := \min\{|F| \colon F \subseteq X \text{ is } R\text{-unbounded}\}$ the *unbounding number of R*, and

$\mathfrak{d}(R) := \min\{|D| \colon D \subseteq Y \text{ is } R\text{-dominating}\}$ the *dominating number of R*.

Note that $\mathfrak{d}(R) = 1$ iff $\mathfrak{b}(R)$ is undefined (i.e. there are no R-unbounded sets, which is the same as saying that X is R-bounded). Dually, $\mathfrak{b}(R) = 1$ iff $\mathfrak{d}(R)$ is undefined (i.e. there are no R-dominating families).

A very representative general example of relational systems is given by directed preorders.

Definition 1.2. We say that $\langle S, \leq_S \rangle$ is a *directed preorder* if it is a preorder (i.e. $\leq_S$ is a reflexive and transitive relation on S) such that

$$\forall\, x, y \in S \ \exists\, z \in S \colon x \leq_S z \text{ and } y \leq_S z.$$

A directed preorder $\langle S, \leq_S \rangle$ is seen as the relational system $S = \langle S, S, \leq_S \rangle$, and its associated cardinal characteristics are denoted by $\mathfrak{b}(S)$ and $\mathfrak{d}(S)$. The cardinal $\mathfrak{d}(S)$ is actually the *cofinality of S*, typically denoted by $\mathrm{cof}(S)$ or $\mathrm{cf}(S)$.

Fact 1.3. *If a directed preorder S has no maximum element then $\mathfrak{b}(S)$ is infinite and regular, and $\mathfrak{b}(S) \leq \mathrm{cf}(\mathfrak{d}(S)) \leq \mathfrak{d}(S) \leq |S|$. Even more, if L is a linear order without maximum then $\mathfrak{b}(L) = \mathfrak{d}(L) = \mathrm{cof}(L)$.*

Proof. First notice that $\mathfrak{d}(S)$ is infinite, otherwise, by directedness, $\mathfrak{d}(S) = 1$ and S would have a top element.

We prove the less obvious $\mathfrak{b}(S) \leq \mathrm{cf}(\mathfrak{d}(S))$. Assume that $\lambda < \mathfrak{b}(S)$ is a cardinal and $\langle A_\alpha \colon \alpha < \lambda \rangle$ is a sequence of subsets of S of size $<\mathfrak{d}(S)$. It is enough to show that $A := \bigcup_{\alpha < \lambda} A_\alpha$ is not cofinal in S. For each $\alpha < \lambda$, since $|A_\alpha| < \mathfrak{d}(S)$, A_α is not cofinal in S, so there is some $x_\alpha \in S$ such that $x_\alpha \not\leq_S y$

for all $y \in A_\alpha$. Now, $|\{x_\alpha \colon \alpha < \lambda\}| \leq \lambda < \mathfrak{b}(S)$, so there is some $x^* \in S$ such that $x_\alpha \leq_S x^*$ for all $\alpha < \lambda$. Then, $x^* \not\leq_S y$ for all $y \in A$, i.e. A is not cofinal in S.

A similar argument shows that $\mathfrak{b}(S)$ is regular. □

Example 1.4. Consider $\omega^\omega = \langle \omega^\omega, \leq^* \rangle$, which is a directed preorder. The cardinal characteristics $\mathfrak{b} := \mathfrak{b}(\omega^\omega)$ and $\mathfrak{d} := \mathfrak{d}(\omega^\omega)$ are the well-known *bounding number* and *dominating number*, respectively.

Example 1.5. For any ideal $\mathcal{I}$ on X, we consider the following relational systems.

(1) $\mathcal{I} := \langle \mathcal{I}, \subseteq \rangle$ is a directed partial order. Note that

$$\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I}) := \min \left\{ |\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{I}, \ \bigcup \mathcal{F} \notin \mathcal{I} \right\} \text{ the } \textit{additivity of } \mathcal{I},$$

$$\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}).$$

(2) $\mathbf{C}_\mathcal{I} := \langle X, \mathcal{I}, \in \rangle$. When $\bigcup \mathcal{I} = X$,

$$\mathfrak{b}(\mathbf{C}_\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \min\{|F| \colon F \subseteq X, \ F \notin \mathcal{I}\} \text{ the } \textit{uniformity of } \mathcal{I},$$

$$\mathfrak{d}(\mathbf{C}_\mathcal{I}) = \mathrm{cov}(\mathcal{I}) = \min \left\{ |\mathcal{C}| \colon \mathcal{C} \subseteq \mathcal{I}, \ \bigcup \mathcal{C} = X \right\} \text{ the } \textit{covering of } \mathcal{I}.$$

Definition 1.6. Let $\Xi \colon \mathbb{B} \to [0, \infty]$ be a fam (finitely additive measure) on a Boolean algebra $\mathbb{B}$. We define the Ξ-*null ideal* by

$$\mathcal{N}(\Xi) := \{a \in \mathbb{B} \colon \Xi(a) = 0\}.$$

When $\mathbb{B}$ is a field of sets over X, we extend the definition to

$$\mathcal{N}(\Xi) := \{a \subseteq X \colon \exists b \in \mathbb{B} \colon a \subseteq b \text{ and } \Xi(b) = 0\}.$$

This is clearly an ideal on X. When $\bigcup \mathcal{N}(\Xi) = X$, i.e. every singleton has measure zero, we say that the fam Ξ is *free*.

Denote by Lb the Lebesgue measure on $\mathbb{R}$, and let $\mathcal{N} := \mathcal{N}(\mathrm{Lb})$.

Definition 1.7. Let X be a topological space. We say that $F \subseteq X$ is *nowhere dense (nwd)* if, for any non-empty open $U \subseteq X$, there is some non-empty open $U' \subseteq U$ disjoint from F. We say that $A \subseteq X$ is *meager (or of first category)* if $A = \bigcup_{n<\omega} F_n$ for some nwd F_n ($n < \omega$).

Denote by $\mathcal{M}(X)$ the collection of all meager subsets of X, and let $\mathcal{M} := \mathcal{M}(\mathbb{R})$.

Definition 1.8. Define by $\mathcal{E}$ the ideal generated by the F_σ measure zero subsets of $\mathbb{R}$.

It is clear that $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$, even more, $\mathcal{E} \subsetneq \mathcal{M} \cap \mathcal{N}$ ([5, Lem. 2.6.1], see also [23, Thm. 3.7]).

Definition 1.9.

(1) For $a, b \in [\omega]^{\aleph_0}$, we define $a \subseteq^* b$ iff $a \smallsetminus b$ is finite;

(2) and we say that a *splits* b if both $a \cap b$ and $b \smallsetminus a$ are infinite, that is, $a \not\supseteq^* b$ and $\omega \smallsetminus a \not\supseteq^* b$.

(3) $F \subseteq [\omega]^{\aleph_0}$ is a *splitting family* if every $y \in [\omega]^{\aleph_0}$ is split by some $x \in F$. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family.

(4) $D \subseteq [\omega]^{\aleph_0}$ is an *unreaping family* if no $x \in [\omega]^{\aleph_0}$ splits every member of D. The *reaping number* $\mathfrak{r}$ is the smallest size of an unreaping family.

(5) Define the relational system $\mathrm{Spl} := \langle [\omega]^{\aleph_0}, [\omega]^{\aleph_0}, \sqsubset^{\mathrm{nsp}} \rangle$ by

$$a \sqsubset^{\mathrm{nsp}} b \text{ iff either } a \supseteq^* b \text{ or } \omega \smallsetminus a \supseteq^* b.$$

Note that $a \not\sqsubset^{\mathrm{nsp}} b$ iff a splits b, so $\mathfrak{b}(\mathrm{Spl}) = \mathfrak{s}$ and $\mathfrak{d}(\mathrm{Spl}) = \mathfrak{r}$.

Inequalities between cardinal characteristics associated with relational systems can be determined by the dual of a relational system and also via Tukey connections, which we introduce below.

Definition 1.10. If $R = \langle X, Y, \sqsubset \rangle$ is a relational system, then its *dual relational system* is defined by $R^\perp := \langle Y, X, \sqsubset^\perp \rangle$ where $y \sqsubset^\perp x$ if $\neg(x \sqsubset y)$.

Fact 1.11. *Let $R = \langle X, Y, \sqsubset \rangle$ be a relational system.*

(a) $(R^\perp)^\perp = R$.

(b) The notions of $R^\perp$-dominating set and R-unbounded set are equivalent.

(c) The notions of $R^\perp$-unbounded set and R-dominating set are equivalent.

(d) $\mathfrak{d}(R^\perp) = \mathfrak{b}(R)$ *and* $\mathfrak{b}(R^\perp) = \mathfrak{d}(R)$.

Definition 1.12. Let $R = \langle X, Y, \sqsubset \rangle$ and $R' = \langle X', Y', \sqsubset' \rangle$ be relational systems. We say that $(\Psi_-, \Psi_+) \colon R \to R'$ is a *Tukey connection from R into R'* if $\Psi_- \colon X \to X'$ and $\Psi_+ \colon Y' \to Y$ are functions such that

$$\forall\, x \in X \ \forall\, y' \in Y' \colon \Psi_-(x) \sqsubset' y' \Rightarrow x \sqsubset \Psi_+(y').$$

The *Tukey order* between relational systems is defined by $R \preceq_\mathrm{T} R'$ iff there is a Tukey connection from R into R'. *Tukey equivalence* is defined by $R \cong_\mathrm{T} R'$ iff $R \preceq_\mathrm{T} R'$ and $R' \preceq_\mathrm{T} R$

Fact 1.13. *Assume that $R = \langle X, Y, \sqsubset \rangle$ and $R' = \langle X', Y', \sqsubset' \rangle$ are relational systems and that $(\Psi_-, \Psi_+) \colon R \to R'$ is a Tukey connection.*

(a) If $D' \subseteq Y'$ is R'-dominating, then $\Psi_+[D']$ is R-dominating.

(b) $(\Psi_+, \Psi_-) \colon (R')^\perp \to R^\perp$ *is a Tukey connection.*

(c) If $E \subseteq X$ is R-unbounded then $\Psi_-[E]$ is R'-unbounded.

Corollary 1.14. *(a) $R \preceq_T R'$ implies $(R')^\perp \preceq_T R^\perp$.*

(b) $R \preceq_T R'$ implies $\mathfrak{b}(R') \leq \mathfrak{b}(R)$ and $\mathfrak{d}(R) \leq \mathfrak{d}(R')$.

(c) $R \cong_T R'$ implies $\mathfrak{b}(R') = \mathfrak{b}(R)$ and $\mathfrak{d}(R) = \mathfrak{d}(R')$.

Example 1.15. The diagram in Figure 1 can be expressed in terms of the Tukey order since $\mathbf{C}_\mathcal{I} \preceq_T \mathcal{I}$ and $\mathbf{C}_\mathcal{I}^\perp \preceq_T \mathcal{I}$ when $\mathcal{I}$ is an ideal on X such that $\bigcup \mathcal{I} = X$. The first inequality is obtained via the Tukey connection $x \in X \mapsto \{x\} \in \mathcal{I}$ and $A \in \mathcal{I} \mapsto A \in \mathcal{I}$, and the second is obtained via $A \in \mathcal{I} \mapsto A \in \mathcal{I}$ and $B \in \mathcal{I} \mapsto y_B \in X$ such that $y_B \notin B$.
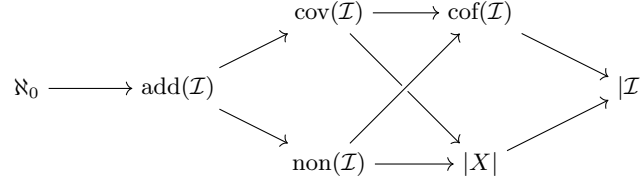

Figure 1: Diagram of the cardinal characteristics associated with $\mathcal{I}$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$.

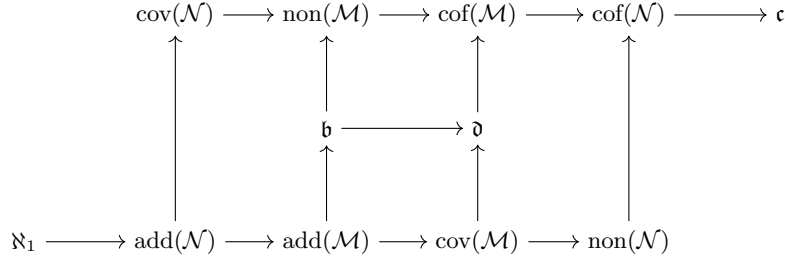Figure 2: Cichoń's diagram. The arrows mean $\leq$ and dotted arrows represent $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$, which we call the *dependent values*.

Cichoń's diagram (Figure 2) illustrates the inequalities between the cardinal characteristics associated with measure and category of the real numbers. The initial study of this diagram was completed between 1981 and 1993. Inequalities were proved by Bartoszyński, Fremlin, Miller, Rothberger and Truss. The name "Cichoń's diagram" was given by Fremlin [22]. On the other hand, the diagram is complete in the sense that no more arrows can be added. Moreover, for any $\aleph_1$-$\aleph_2$ assignment to the cardinals in Cichoń's diagram that does not contradict

the arrows (and the dependent values), there is a forcing poset that forces the corresponding model. This part of the study was completed by Bartoszyński, Judah, Miller and Shelah. In fact, the inequalities in Cichoń's diagram can be obtained via the Tukey connections as illustrated in Figure 3. See e.g. [5, 8] for all the details.
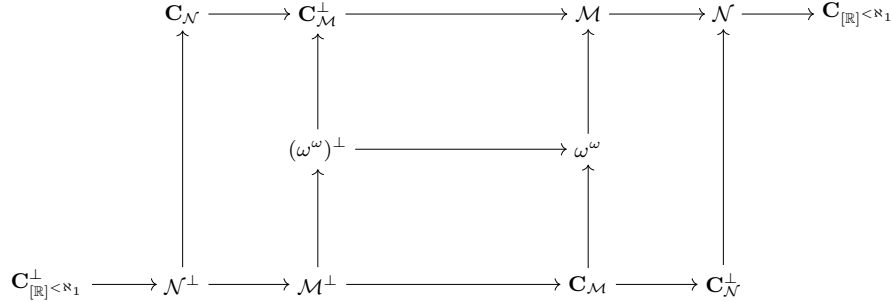

Figure 3: Cichoń's diagram via Tukey connections. Any arrow represents a Tukey connection in the given direction.

We look at more classical cardinal characteristics. Concerning those associated with $\mathcal{E}$:

Lemma 1.16 ([5, Lem. 7.4.3])**.** $\mathbf{C}_{\mathcal{E}} \preceq_{\mathrm{T}} \mathrm{Spl}$.

Theorem 1.17 ([4], see also [5, Sec. 2.6])**.**

(a) $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} \leq \mathrm{non}(\mathcal{E}) \leq \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$.

(b) $\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} \leq \mathrm{cov}(\mathcal{E}) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$.

(c) $\mathrm{add}(\mathcal{E}) = \mathrm{add}(\mathcal{M})$ *and* $\mathrm{cof}(\mathcal{E}) = \mathrm{cof}(\mathcal{M})$.

Definition 1.18.

(1) $D \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* when:

    (i) if $a \in [\omega]^{\aleph_0}$, $b \in D$ and $a \subseteq^* b$, then $a \in D$,

    (ii) if $\langle I_n \colon n < \omega \rangle$ is an interval partition of ω then $\bigcup_{n \in a} I_n \in D$ for some $a \in [\omega]^{\aleph_0}$.

The *groupwise density number* $\mathfrak{g}$ is the smallest size of a collection of groupwise dense sets whose intersection is empty.

(2) The *distributivity number* $\mathfrak{h}$ is the smallest size of a collection of dense subsets of $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ whose intersection is empty.

(3) Say that $a \in [\omega]^{\aleph_0}$ is a *pseudo-intersection of* $F \subseteq [\omega]^{\aleph_0}$ if $a \subseteq^* b$ for all $b \in F$.

(4) The *pseudo-intersection number* $\mathfrak{p}$ is the smallest size of a filter base of subsets of $[\omega]^{\aleph_0}$ without pseudo-intersection.

(5) The *tower number* $\mathfrak{t}$ is the smallest length of a (transfinite) $\subseteq^*$-decreasing sequence in $[\omega]^{\aleph_0}$ without pseudo-intersection.

(6) Given a class $\mathcal{P}$ of forcing notions, $\mathfrak{m}(\mathcal{P})$ denotes the minimal cardinal κ such that, for some $Q \in \mathcal{P}$, there is some collection $\mathcal{D}$ of size κ of dense subsets of Q without a filter in Q intersecting every member of $\mathcal{D}$.

(7) Let $\mathbb{P}$ be a poset. A set $A \subseteq \mathbb{P}$ is *k-linked (in $\mathbb{P}$)* if every k-element subset of A has a lower bound in $\mathbb{P}$. A is *centered* if it is k-linked for all $k \in \omega$.

(8) A poset $\mathbb{P}$ is *k-Knaster*, if for each uncountable $A \subseteq \mathbb{P}$ there is a k-linked uncountable $B \subseteq A$. And $\mathbb{P}$ *has precaliber $\aleph_1$* if such a B can be chosen centered. For notational convenience, *1-Knaster* means ccc, and *ω-Knaster* means precaliber $\aleph_1$.

(9) For $1 \leq k \leq \omega$ denote $\mathfrak{m}_k := \mathfrak{m}(k\text{-Knaster})$ and $\mathfrak{m} := \mathfrak{m}_1$. We also set $\mathfrak{m}_0 := \aleph_1$.

(10) Define the relational system $\mathrm{Pred} = \langle \omega^\omega, \mathrm{Pr}, \sqsubset^{\mathrm{pr}} \rangle$ where Pr is the set of functions π (called *predictors*) into ω with domain $\bigcup_{n \in D_\pi} \omega^n$ for some $D_\pi \in [\omega]^{\aleph_0}$, and

$$x \sqsubset^{\mathrm{pr}} \pi \text{ iff } \exists\, m < \omega \ \forall\, n \geq m \colon n \in D_\pi \Rightarrow x(n) = \pi(x{\upharpoonright}n),$$

in which case we say that π *predicts* x. We define $\mathfrak{e} := \mathfrak{b}(\mathrm{Pred})$ the *evasion number*.

(11) Two sets a and b are *almost disjoint* if $a \cap b$ is finite. A family A of sets is an *almost disjoint (a.d.) family* if any pair of members of A are almost disjoint. We say that $A \subseteq [\omega]^{\aleph_0}$ is a *maximal almost disjoint (mad) family* if A is $\subseteq$-maximal among the a.d. families contained in $[\omega]^{\aleph_0}$. The *almost disjointness number* $\mathfrak{a}$ is the smallest size of an infinite mad family in $[\omega]^{\aleph_0}$.

(12) The *ultrafilter number* $\mathfrak{u}$ is the smallest size of a filter base generating an ultrafilter contained in $[\omega]^{\aleph_0}$, i.e. a *non-principal ultrafilter*.

(13) $\mathrm{Fn}(A, 2)$ denotes the set of finite partial functions from A into $2 = \{0, 1\}$ (see Definition 2.22). When $A \subseteq \mathcal{P}(\omega)$, for $s \in \mathrm{Fn}(A, 2)$ denote

$$a^s := \bigcap_{x \in s^{-1}[\{0\}]} x \cap \bigcap_{x \in s^{-1}[\{1\}]} (\omega \smallsetminus x).$$

A family $A \subseteq \mathcal{P}(\omega)$ is said to be *independent* if a^s is infinite for all $s \in \mathrm{Fn}(A, 2)$, and we say that it is a *maximal independent family* if it is $\subseteq$-maximal among the independent families in $\mathcal{P}(\omega)$. The *independence number* $\mathfrak{i}$ is defined as the smallest size of a maximal independent family.
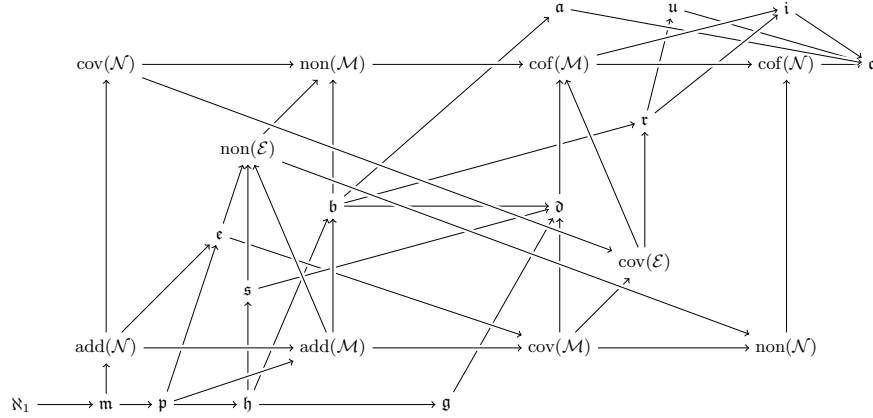
Figure 4: Diagram of inequalities between classical cardinal characteristics

The inequalities between the cardinal characteristics presented so far are summarized in Figure 4. See [8, 5] for the definitions and the proofs for the inequalities (with the exception of $\mathrm{cof}(\mathcal{M}) \leq \mathfrak{i}$, which was proved in [1]). Below we list some additional properties of these cardinals. Unless noted otherwise, proofs can be found in [8].

Fact 1.19.

(1) *[40]* $\mathfrak{p} = \mathfrak{t}$.[1]

(2) *The cardinals* $\mathrm{add}(\mathcal{N})$, $\mathrm{add}(\mathcal{M})$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{h}$ *and* $\mathfrak{g}$ *are regular.*

(3) $\mathrm{cf}(\mathfrak{s}) \geq \mathfrak{t}$ *(see [20]).*

(4) $2^{<\mathfrak{t}} = \mathfrak{c}$.

(5) $\mathrm{cf}(\mathfrak{c}) \geq \mathfrak{g}$.

(6) *For* $1 \leq k \leq k' \leq \omega$, $\mathfrak{m}_k \leq \mathfrak{m}_{k'}$.

(7) *For* $1 \leq k \leq \omega$, $\mathfrak{m}_k > \aleph_1$ *implies* $\mathfrak{m}_k = \mathfrak{m}_\omega$ *(well-known, but see e.g. [26, Lemma 4.2]).*

Concerning cofinalities:

Fact 1.20. *Let* $\mathcal{I}$ *be an ideal on* X *such that* $\bigcup \mathcal{I} = X$.

(1) $\mathrm{add}(\mathcal{I})$ *is regular,* $\mathrm{cf}(\mathrm{cof}(\mathcal{I})) \geq \mathrm{add}(\mathcal{I})$ *and* $\mathrm{cf}(\mathrm{non}(\mathcal{I})) \geq \mathrm{add}(\mathcal{I})$.

(2) $\mathrm{cf}(\mathrm{cov}(\mathcal{M})) \geq \mathrm{add}(\mathcal{N})$ *(Bartoszyński and Judah 1989, [5, Thm. 5.1.5]).*

(3) *(Bartoszyński 1988, [5, Thm. 5.1.17])* $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}$ *implies* $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) > \omega$.

(4) *If* $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{d}$ *then* $\mathrm{cf}(\mathrm{cov}(\mathcal{E})) > \omega$ *(Miller, see [5, Thm. 5.1.18]).*

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[1]Only the trivial inequality $\mathfrak{p} \leq \mathfrak{t}$ is used in this mini-course.

The problem of the cofinality of $\mathrm{cov}(\mathcal{N})$ was settled with the following result.

Theorem 1.21 (Shelah [49]). *It is consistent with* ZFC *that* $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \omega$.

The following question is still unsolved.

Question 1.22. *Is it consistent with* ZFC *that* $\mathrm{cf}(\mathrm{cov}(\mathcal{E})) = \omega$?

To solve this problem in the positive, it is necessary to force $\mathfrak{d} < \mathrm{cov}(\mathcal{E})$, which implies $\mathrm{cov}(\mathcal{E}) = \mathrm{cov}(\mathcal{N})$ (see Theorem 1.17), so it would be necessary to force $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \omega$ via an ω^ω-bounding forcing.

Figure 4 is quite complete, but the following is still unknown.

Question 1.23. *Is* $\mathfrak{a} \leq \mathfrak{i}$?

It is not even known how to solve:

Question 1.24. *Does* $\mathfrak{i} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$?

A positive answer to this problem is implied by a positive answer to the following famous problem in set theory.

Question 1.25 (Roitman's problem). *Does* $\mathfrak{d} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$?

The following strengthening of Roitman's problem was formulated by Brendle and Raghavan [11].

Question 1.26. *Does* $\mathfrak{b} = \mathfrak{s} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$?

2. Finite Support iterations

2.1. Generic reals

We first look at the types of generic reals we intend to add by forcing. Recall that a *Polish space* is a separable completely metrizable space. The real line $\mathbb{R}$ and any product $\prod_{n<\omega} b(n)$ of countable discrete spaces, such as the Cantor space 2^ω and the Baire space ω^ω, are canonical examples. Polish spaces share many of the combinatorial and descriptive set-theoretic properties of the real line.

For a Polish space Z, denote by $\bar{\mathbf{\Sigma}}(Z)$ the field of sets generated by the analytic subsets of Z.

Definition 2.1. We say that $R = \langle X, Y, \sqsubset \rangle$ is a *relational system of the reals* if

(i) $X \in \bar{\mathbf{\Sigma}}(Z_1)$ and $Y \in \bar{\mathbf{\Sigma}}(Z_2)$ for some Polish spaces Z_1 and Z_2, and

(ii) $\sqsubset \in \bar{\mathbf{\Sigma}}(Z_1 \times Z_2)$.

In most of the cases, $X = Z_1$ is a perfect Polish space and, for any $y \in Y$, $\{x \in X : x \sqsubset y\}$ is meager in X. More precisely, we are interested in the following notion: We say that $R = \langle X, Y, \sqsubset \rangle$ is a *Polish relational system (Prs)* if it satisfies the requirements below:

(i) X is a perfect Polish space,

(ii) Y is a non-empty analytic subspace of some Polish space Z and

(iii) $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ where $\langle \sqsubset_n \rangle_{n<\omega}$ is some increasing sequence of closed subsets of $X \times Z$ such that $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is closed nowhere dense for any $n < \omega$ and $y \in Y$.

By (iii), the maps $x \mapsto x$ and $y \mapsto \{x \in X : x \sqsubset y\} \in \mathcal{M}(X)$ form a Tukey connection for $\mathbf{C}_{\mathcal{M}(X)} \preceq_{\mathrm{T}} R$.

Although it is enough to look at Prs's for the applications, the results of this section are valid for relational systems of the reals (and sometimes with extra assumptions). The strength of the notion of Prs is necessary for the iteration theorems in section 3.

The reason we use $\bar{\boldsymbol{\Sigma}}$ in the definition of relational system of the reals is to have absoluteness of the statements "$x \in X$", "$y \in Y$" and "$x \sqsubset y$". In general, we can just use definable sets X, Y and $\sqsubset$ such that the previous statements are absolute for the arguments we are carrying out.

For the rest of this section, we fix a relational system of the reals $R = \langle X, Y, \sqsubset \rangle$. We introduce the following type of (generic) reals related to R.

Definition 2.2. Let M be a (transitive) model of ZFC.[2]

(1) A point $y^* \in Y$ is *R-dominating over M* if $\forall x \in X \cap M : x \sqsubset y^*$.

(2) A point $x^* \in X$ is *R-unbounded over M* if $\forall y \in Y \cap M : x^* \not\sqsubset y$.

We look at many Prs related to the cardinals in Cichoń's diagram.

Definition 2.3 (Localization). For $h \in \omega^\omega$ and $H \subseteq \omega^\omega$, define

$$\mathcal{S}(\omega, h) := \prod_{i<\omega} [\omega]^{\leq h(i)},$$

$$\mathcal{S}(\omega, H) := \bigcup_{h \in H} \mathcal{S}(\omega, h).$$

Objects in these sets are usually called *slaloms*.

For functions x and y with domain ω, we define the relation "y *localizes* x" by

$$x \in^* y \text{ iff } \exists\, m < \omega \,\forall\, i \geq m : x(i) \in y(i).$$

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[2]Since such set models cannot exist, most of the time this expression means that M satisfies a large enough fragment of ZFC to perform the arguments at hand.

Define the following *localization relational systems*:

$$\mathrm{Lc}(\omega, h) := \langle \omega^{\omega}, \mathcal{S}(\omega, h), \in^* \rangle,$$
$$\mathrm{Lc}(\omega, H) := \langle \omega^{\omega}, \mathcal{S}(\omega, H), \in^* \rangle.$$

It is easy to check that these are Prs when H is countable. We often work with $H_* := \{\mathrm{id}^{k+1} \colon k < \omega\}$ (powers of the identity function id on ω)

The localization relational systems work to easily characterize the cardinal characteristics associated with $\mathcal{N}$.

Theorem 2.4 (Bartoszyński [2](1984), see also [15, Sec. 4]).
If $h \to \infty$ and $H \subseteq \omega^{\omega}$ is a countable set containing some function diverging to infinity, then
$$\mathrm{Lc}(\omega, h) \cong_{\mathrm{T}} \mathrm{Lc}(\omega, H) \cong_{\mathrm{T}} \mathcal{N}.$$

In particular,

$$\mathfrak{b}(\mathrm{Lc}(\omega, h)) = \mathfrak{b}(\mathrm{Lc}(\omega, H)) = \mathrm{add}(\mathcal{N}) \ and$$
$$\mathfrak{d}(\mathrm{Lc}(\omega, h)) = \mathfrak{d}(\mathrm{Lc}(\omega, H)) = \mathrm{cof}(\mathcal{N}).$$

We now introduce a forcing to modify $\mathrm{Lc}(\omega, h)$. In the context of forcing, V always refers to the *ground model*.

Definition 2.5 (Localization forcing). For $h \in \omega^{\omega}$, define the poset[3]

$$\mathbb{L}\mathrm{c}_h := \{(n, \varphi) \in \omega \times \mathcal{S}(\omega, h) \colon \exists\, m < \omega \colon \varphi \in \mathcal{S}(\omega, m)\}$$

ordered by

$$(n', \varphi') \leq (n, \varphi) \text{ iff } n \leq n', \ \varphi' {\upharpoonright} n = \varphi {\upharpoonright} n \text{ and } \forall\, i < \omega \colon \varphi(i) \subseteq \varphi'(i).$$

When $h \to \infty$ we have that $\mathbb{L}\mathrm{c}_h$ is ccc (even σ-k-linked for any $k < \omega$) and it adds a generic slalom $\varphi_* \in \mathcal{S}(\omega, h)$ which localizes all functions in the ground model, i.e. it is $\mathrm{Lc}(\omega, h)$-dominating over the ground model. If G is $\mathrm{Lc}(\omega, h)$-generic over V, the generic slalom is defined by $\varphi_*(i) := \varphi(i)$ when $(n, \varphi) \in G$ and $i < n$ (this value is the same for any such (n, φ)).

We present a Prs that represents the relational system $\mathbf{C}_{\mathcal{N}}$ (more precisely, its dual). For this purpose, we code measure zero sets as follows.

Definition 2.6. For any topological space X, denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X. Let Lb_2 be the measure on $\mathcal{B}(2^{\omega})$ defined as the product measure of the uniform measure on $2 = \{0, 1\}$.[4] Recall that $\{[s] \colon s \in 2^{<\omega}\}$ forms a base of 2^{ω} and that each $[s]$ is clopen in 2^{ω}. Then, Lb_2 is the unique measure on $\mathcal{B}(2^{\omega})$ such that $\mathrm{Lb}_2([s]) = 2^{-|s|}$ for any $s \in 2^{<\omega}$.[5]

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[3]The m in $\mathcal{S}(\omega, m)$ refers to the constant function with value m.
[4]The uniform measure on a finite non-empty set b assigns probability $\frac{1}{|b|}$ to each point.
[5]For $s \in 2^{<\omega}$, $|s| = |\mathrm{dom}\, s|$ is the length of s as a sequence.

We abuse notation and denote $[F] := \bigcup_{s \in F}[s]$ for $F \subseteq 2^{<\omega}$. Since 2^ω is compact, we have that the clopen sets are precisely of the form $[c]$ for $c \subseteq 2^{<\omega}$ finite.

We code measure zero subsets of 2^ω in the following way. Fix a sequence $\bar{\varepsilon} = \langle \varepsilon_n : n < \omega \rangle$ of positive real numbers such that $\sum_{n<\omega} \varepsilon_n < \infty$. Define

$$\Omega_{\bar{\varepsilon}} := \left\{ \bar{c} = \langle c_n : n < \omega \rangle : \forall n < \omega : c_n \in [2^{<\omega}]^{<\aleph_0} \text{ and } \mathrm{Lb}_2([c_n]) < \varepsilon_n \right\}.$$

For any sequence $\bar{c} = \langle c_n : n < \omega \rangle$ of finite subsets of $2^{<\omega}$, denote

$$N(\bar{c}) = \bigcap_{m<\omega} \bigcup_{n \geq m} [c_n],$$

i.e. for $x \in 2^\omega$, $x \in N(\bar{c})$ iff $x \in [c_n]$ for infinitely many n.

Define the relational system $\mathrm{Cn} := \langle \Omega_{\bar{\varepsilon}}, 2^\omega, \not\blacktriangleright \rangle$ such that $\bar{c} \not\blacktriangleright y$ iff $y \notin N(\bar{c})$. This is a Prs.

The sequences in $\Omega_{\bar{\varepsilon}}$ are simple codes of (a base of) measure zero sets in 2^ω.

Fact 2.7 (See e.g. [5, Lemma 2.3.10]). *If $\bar{c} \in \Omega_{\bar{\varepsilon}}$, then $N(\bar{c}) \in \mathcal{N}(\mathrm{Lb}_2)$ and, for any $A \in \mathcal{N}(\mathrm{Lb}_2)$, there exists $\bar{c} \in \Omega_{\bar{\varepsilon}}$ such that $A \subseteq N(\bar{c})$.*

In combinatorics of the reals, working in the Cantor space is the same as working in $\mathbb{R}$ because functions in 2^ω represent the numbers in $[0, 1]$ when expressed in base 2. For this reason, the measure theory of 2^ω is equivalent to the one of $[0, 1]$ (with the Lebesgue measure), so $\mathcal{N}(\mathbb{R}) \cong_{\mathrm{T}} \mathcal{N}(2^\omega)$ and $\mathbf{C}_{\mathcal{N}(\mathbb{R})} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{N}(2^\omega)}$. See details in [39, Ch. VII, §3].

As a direct consequence of Fact 2.7, we obtain:

Fact 2.8. $\mathrm{Cn} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{N}(2^\omega)}^{\perp}$, *so $\mathfrak{b}(\mathrm{Cn}) = \mathrm{cov}(\mathcal{N})$ and $\mathfrak{d}(\mathrm{Cn}) = \mathrm{non}(\mathcal{N})$.*

Definition 2.9. Random forcing is $\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega)$ ordered by $\subseteq$. If G is a generic set over V, then we can define $r \in 2^\omega$ by $r := \bigcup\{s \in 2^{<\omega} : [s] \in G\}$. Such r is called a *random real (over V)*.

Random forcing is ccc (even σ-k-linked for any $k < \omega$).

Fact 2.10. *If r is a random real over V, then $r \notin N(\bar{c})$ for any $\bar{c} \in \Omega_{\bar{\varepsilon}} \cap V$, i.e. any random real over V is Cn-dominating over V.*

This says that any random real over V evades the Borel measure zero sets coded in the ground model.

Concerning the directed preorder $\langle \omega^\omega, \leq^* \rangle$:

Fact 2.11. $\langle \omega^\omega, \leq^* \rangle$ *is a Prs.*

Definition 2.12. *Hechler forcing* is the poset $\mathbb{D} := \omega^{<\omega} \times \omega^\omega$ ordered by

$$(t, g) \leq (s, f) \text{ iff } s \subseteq t, \ \forall i < \omega : f(i) \leq g(i), \text{ and } \forall i \in |t| \smallsetminus |s| : t(i) \geq f(i).$$

This poset is ccc (even σ-centered).

If G is $\mathbb{D}$-generic over V, then $d := \bigcup\{s : \exists f : (s, f) \in G\}$ is $\langle \omega^\omega, \leq^* \rangle$-dominating over V.

We now turn to $\mathbf{C}_{\mathcal{M}}$. First, we introduce a useful characterization of its cardinal characteristics.

Definition 2.13. Define the relational system $\mathrm{Ed} := \langle \omega^\omega, \omega^\omega, \neq^* \rangle$ where

$$x \neq^* y \text{ iff } \exists\, m < \omega \; \forall\, i \geq m \colon x(i) \neq y(i).$$

This is a Prs.

Theorem 2.14 (Miller [46], Bartoszyński [3], see also [15, Thm. 5.3]). $\mathfrak{b}(\mathrm{Ed}) = \mathrm{non}(\mathcal{M})$ *and* $\mathfrak{d}(\mathrm{Ed}) = \mathrm{cov}(\mathcal{M})$.

Definition 2.15. Define the *eventually different real forcing* by

$$\mathbb{E} := \left\{ (s, \varphi) \colon s \in \omega^{<\omega}, \ \varphi \in \bigcup_{m < \omega} \mathcal{S}(\omega, m) \right\}$$

ordered by

$$(t, \psi) \leq (s, \varphi) \text{ iff } s \subseteq t, \ \forall\, i < \omega \colon \varphi(i) \subseteq \psi(i), \text{ and } \forall\, i \in |t| \smallsetminus |s| \colon t(i) \notin \varphi(i).$$

This forcing is ccc (even σ-centered).

If G is $\mathbb{E}$-generic over V then $e := \bigcup \{ s \colon \exists\, \varphi \colon (s, \varphi) \in G \}$ is Ed-dominating over V.

Therefore, by Theorem 2.14, $\mathbb{E}$ can be used to increase $\mathrm{non}(\mathcal{M})$. But it actually does more:

Theorem 2.16 (Cardona & Mejía [17, Clm. 4.11]). $\mathbb{E}$ *adds a* $\mathbf{C}_{\mathcal{E}}$*-dominating real over* V*, i.e. an* F_σ *null subset of* $\mathbb{R}$ *that covers* $\mathbb{R} \cap V$*.*

As for measure, we have that $\mathcal{M}(\mathbb{R}) \cong_{\mathrm{T}} \mathcal{M}(2^\omega)$ and $\mathbf{C}_{\mathcal{M}(\mathbb{R})} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}(2^\omega)}$, so we obtain the same cardinal characteristics for the meager ideal using the Cantor space instead of $\mathbb{R}$. More generally, as a consequence of [34, Ex. 8.32 & Thm. 15.10]:

Theorem 2.17. *For any perfect Polish space* X*,*

$$\mathcal{M}(X) \cong_{\mathrm{T}} \mathcal{M}(\mathbb{R}) \text{ and } \mathbf{C}_{\mathcal{M}(X)} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}(\mathbb{R})}.$$

In particular:

Fact 2.18. *Let* $R = \langle X, Y, \sqsubset \rangle$ *be a Prs in* V*. Then, any Cohen real* $x \in X$ *over* V *is* R*-unbounded over* V*.*

We now look at the effect of Cohen forcing on meager sets. As we did with measure zero, we introduce a coding of (a base of) meager subsets of 2^ω.

Definition 2.19. Let $\mathbb{I}$ be the set of interval partitions of ω. Define the relational system $\mathrm{M} := \langle 2^\omega, 2^\omega \times \mathbb{I}, \sqsubset^{\mathrm{m}} \rangle$ where

$$x \sqsubset^{\mathrm{m}} (y, \bar{I}) \text{ iff } \exists\, m < \omega\ \forall\, n \geq m \colon x{\upharpoonright} I_n \neq y{\upharpoonright} I_n.$$

This is a Prs. The members of $2^\omega \times \mathbb{I}$ are usually called *matching reals*. For any matching real $(y, \bar{I})$, define $M(y, \bar{I}) := \{x \in 2^\omega \colon x \sqsubset^{\mathrm{m}} (y, \bar{I})\}$.

Fact 2.20 (See e.g. [8]). *For any matching real $(y, \bar{I})$, $M(y, \bar{I})$ is meager in 2^ω. And, for any $A \in \mathcal{M}(2^\omega)$, there is some matching real $(y, \bar{I})$ such that $A \subseteq M(y, \bar{I})$.*

Corollary 2.21. $\mathrm{M} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}(2^\omega)}$. *In particular,* $\mathfrak{b}(\mathrm{M}) = \mathrm{non}(\mathcal{M})$ *and* $\mathfrak{d}(\mathrm{M}) = \mathrm{cov}(\mathcal{M})$.

Definition 2.22. Let I be a set and $\bar{b} = \langle b(i) \colon i \in I \rangle$ a sequence of non-empty sets. Define the poset

$$\mathrm{Fn}(\bar{b}) := \{p \colon p \text{ is a finite function, } \mathrm{dom}\, p \subseteq I \text{ and } \forall\, i \in \mathrm{dom}\, p \colon p(i) \in b(i)\}$$

ordered by $\supseteq$. The generic real added by this poset is $g := \bigcup G \in \prod \bar{b} := \prod_{i \in I} b(i)$ whenever G is $\mathrm{Fn}(\bar{b})$-generic over V.

We use this forcing to add Cohen reals, not just over 2^ω or ω^ω, but over any perfect space of the form $\prod_{n < \omega} b(n)$, endowed with the product topology for countable discrete spaces $b(n)$ $(n < \omega)$.

Fix a countable sequence $\bar{b} := \langle b(n) \colon n < \omega \rangle$ of countable non-empty sets. Note that $\prod \bar{b}$ is a perfect Polish space iff $|b(n)| \geq 2$ for infinitely many $n < \omega$. In this case, we call $\mathrm{Fn}(\bar{b})$ the *forcing adding a Cohen real in* $\prod \bar{b}$, usually referred to as *Cohen forcing*. We use c to denote the generic real in $\prod \bar{b}$ added by this poset, which we often call *Cohen real*. For example, $\Omega_{\bar{\varepsilon}}$ is such a space, and a Cohen real in $\Omega_{\bar{\varepsilon}}$ over V codes a measure zero set that covers $2^\omega \cap V$. The letter $\mathbb{C}$ is reserved for any version of Cohen forcing.

For any set I, denote $\mathbb{C}_I := \mathrm{Fn}(\hat{b})$ where $\hat{b} := \langle b(i, n) \colon i \in I,\ n < \omega \rangle$ is defined by $b(i, n) := b(n)$. This poset adds a sequence $\langle c_i \colon i \in I \rangle$ where each $c_i \in \prod_{n < \omega} b(n)$ is a Cohen real over V (and even over $V^{\mathbb{C}_{I \smallsetminus \{i\}}}$).

All the versions of Cohen forcing are forcing equivalent:

Theorem 2.23. *Any countable atomless forcing notion is forcing equivalent with $\mathbb{C}$.*

In general, for any perfect Polish space X, it is possible to define a countable atomless forcing that adds a generic real $c \in X$.[6] The main property of this generic real is that it evades all the Borel meager subsets of X coded in the ground model. In particular,

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[6]Using finite fragments of Cauchy sequences coming from a countable dense subset of X.

Theorem 2.24. *If $c \in 2^\omega$ is a Cohen real over V, then $c \notin M(y, \bar{I})$ for any matching real $(y, \bar{I}) \in V$. In particular, any Cohen real is* M-*unbounded over V.*

Remark 2.25. Both M and Ed have the same cardinal characteristics as $\mathbf{C}_\mathcal{M}$, M $\cong_\mathrm{T} \mathbf{C}_\mathcal{M}$ and $\mathbf{C}_\mathcal{M} \preceq_\mathrm{T}$ Ed. The converse Tukey-inequality is not provable in ZFC. This follows by Zapletal's result stating that there is a proper poset adding an Ed-unbounded real but not adding Cohen reals [53].

2.2. FS iterations

We now turn to FS (finite support) iterations. Any FS iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \pi, \ \beta < \pi \rangle$ of length π is defined by recursion as follows:

(I) $\mathbb{P}_0 := \{\langle \, \rangle\}$ is the poset containing the empty sequence $\langle \, \rangle$, usually called the *trivial poset*.

(II) When $\mathbb{P}_\alpha$ has been defined, we pick a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}_\alpha$ of a poset and define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

(III) For limit $\gamma \leq \pi$, $\mathbb{P}_\gamma := \mathrm{limdir}_{\alpha<\gamma} \, \mathbb{P}_\alpha = \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$ ordered by

$$q \leq_\gamma p \text{ iff } \exists\, \alpha < \gamma \colon p, q \in \mathbb{P}_\alpha \text{ and } q \leq_\alpha p.$$

Here, $\leq_\alpha$ denotes the preorder of $\mathbb{P}_\alpha$. It can be proved by induction that $\mathbb{P}_\alpha \lessdot \mathbb{P}_\beta$ whenever $\alpha \leq \beta \leq \pi$, where $\lessdot$ denotes the *complete-subposet* relation.[7]

If G is $\mathbb{P}_\pi$-generic over V and $\alpha \leq \pi$, then $G_\alpha := \mathbb{P}_\alpha \cap G$ is $\mathbb{P}_\alpha$-generic over V, so $G_\pi = G$. In the context of FS iterations, we denote $V_\alpha := V[G_\alpha]$, so $V_0 = V$. The relation $\lessdot$ indicates that $V_\alpha \subseteq V_\beta$ whenever $\alpha \leq \beta \leq \pi$. So, when $\alpha < \pi$, we call V_α an *intermediate generic extension*, and V_π the *final generic extension*.

In this context, we abbreviate the forcing relation $\Vdash_{\mathbb{P}_\alpha}$ by $\Vdash_\alpha$.

We review some basic facts about FS iterations of ccc posets.

Lemma 2.26. *Any FS iteration of ccc posets is ccc, i.e. if $\Vdash_\beta \dot{\mathbb{Q}}_\beta$ is ccc for all $\beta < \pi$, then $\mathbb{P}_\alpha$ is ccc for all $\alpha \leq \pi$.*

Lemma 2.27. *In any FS iteration of ccc posets of length π: if $\mathrm{cf}(\pi) > \omega$ then $\mathbb{R} \cap V_\pi = \bigcup_{\alpha<\pi} \mathbb{R} \cap V_\alpha$.*

Lemma 2.28. *Any FS iteration of non-trivial[8] posets adds Cohen reals at limit stages. Concretely, $\mathbb{P}_{\alpha+\omega}$ adds a Cohen real over V_α.*

The Cohen reals added by a FS iteration determine a Tukey connection for $\mathbf{C}_\mathcal{M}$ as follows.

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[7] $\mathbb{P} \lessdot \mathbb{Q}$ iff $\mathbb{P}$ is a suborder of $\mathbb{Q}$, the incompatibility relation is preserved, and any predense subset of $\mathbb{P}$ is predense in $\mathbb{Q}$.

[8] A poset is *trivial* if all its conditions are pairwise compatible. This is equivalent to saying that the poset is forcing equivalent with the trivial poset.

Corollary 2.29. *Any FS iteration of ccc posets of length π with uncountable cofinality forces* $\mathrm{non}(\mathcal{M}) \leq \mathrm{cf}(\pi) \leq \mathrm{cov}(\mathcal{M})$, *even more,* $\pi \preceq_{\mathrm{T}} \mathbf{C}_{\mathcal{M}}$ *and* $\mathfrak{g} \leq \mathrm{cf}(\pi)$.

Proof. Work in V_π. For any matching real $(y, \bar{I})$, by Lemma 2.27 there is some $\alpha_{y,\bar{I}} < \pi$ such that $(y, \bar{I}) \in V_{\alpha_{y,\bar{I}}}$. On the other hand, by Lemma 2.28, there is some Cohen real $c_\alpha \in 2^\omega \cap V_{\alpha+\omega}$ over V_α. Then by Theorem 2.24, $c_\alpha \not\sqsubseteq^{\mathrm{m}} (y, \bar{I})$ whenever $(y, \bar{I}) \in V_\alpha$, which happens when $\alpha_{y,\bar{I}} \leq \alpha$. This indicates that the maps $\alpha \mapsto c_\alpha$ and $(y, \bar{I}) \mapsto \alpha_{y,\bar{I}}$ form a Tukey connection for $\pi \preceq_{\mathrm{T}} \mathrm{M}$.

We now show that $\mathfrak{g} \leq \mathrm{cov}(\mathcal{M})$ in V_π. Let $\langle \alpha_\xi : \xi < \mathrm{cf}(\pi) \rangle$ be an increasing sequence of limit ordinals with limit $\nu := \mathrm{cf}(\pi)$. Set $V'_\xi := V_{\alpha_\xi}$ for $\xi < \nu$ and $V'_\nu := V_\pi$. Then $\langle \omega^\omega \cap V'_\xi : \xi < \nu \rangle \in V'_\nu$ is strictly increasing and $\bigcup_{\xi < \nu} \omega^\omega \cap V'_\xi = \omega^\omega \cap V'_\nu$. Hence, by a result of Blass [7, Thm. 2] (see also Brendle's proof [10, Lem. 1.17]), it follows that $\mathfrak{g} \leq \nu$ in V'_ν. $\qquad\square$

The previous result puts a restriction on the models of Cichoń's diagram that can be obtained via FS iterations of ccc posets (of uncountable cofinality), since they force the inequalities $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$ and $\mathfrak{g} \leq \mathrm{cov}(\mathcal{M})$. Therefore, in such models, the diagram of cardinal characteristics presented in Figure 4 takes the form as in Figure 5.

Figure 5: Cichoń's diagram with other classical cardinal characteristics after a FS iteration of ccc (non-trivial) posets of length with uncountable cofinality, as an effect of the forced inequalities $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$ and $\mathfrak{g} \leq \mathrm{cov}(\mathcal{M})$.

Below, we summarize the effect of the forcings introduced in this section to modify the cardinals in Cichoń's diagram:

(1) When $h \to \infty$, $\mathbb{L}c_h$ adds $\mathrm{Lc}(\omega, h)$-dominating reals (so it affects $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$).

(2) Random forcing adds Cn-dominating reals (affecting $\mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{N})$).

(3) Hechler forcing adds $\langle \omega^\omega, \leq^* \rangle$-dominating reals (affecting $\mathfrak{b}$ and $\mathfrak{d}$).

(4) The forcing $\mathbb{E}$ adds Ed-dominating reals, and also $\mathbf{C}_{\mathcal{E}}$-dominating reals (affecting $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{M})$, $\mathrm{cov}(\mathcal{E})$).

(5) Cohen forcing adds M-unbounded reals (affecting $\mathrm{cov}(\mathcal{M})$ and $\mathrm{non}(\mathcal{M})$).

We are going to use these forcings to modify the cardinals in Cichoń's diagram. However, we cannot just simply add dominating reals without any particular restriction, as indicated in the following result.

Lemma 2.30. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \pi, \ \beta < \pi \rangle$ be a FS iteration of ccc posets. Assume that $\mathrm{cf}(\pi) > \omega$, $K \subseteq \pi$ is cofinal, $R = \langle X, Y, \sqsubset \rangle$ is a relational system of the reals (see Definition 2.1), and assume that, for $\alpha \in K$, $\dot{\mathbb{Q}}_\alpha$ adds an R-dominating real over V_α.*

Then $\mathbb{P}_\pi$ forces $\mathfrak{d}(R) \leq \mathrm{cf}(\pi) \leq \mathfrak{b}(R)$, even $R \preceq_{\mathrm{T}} \pi$.

Moreover, if ZFC proves $\mathbf{C}_{\mathcal{M}} \preceq_{\mathrm{T}} R$, then $\mathbb{P}_\pi$ forces $R \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}} \cong_{\mathrm{T}} \pi$, so $\mathfrak{b}(R) = \mathfrak{d}(R) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{cf}(\pi)$.

Proof. Work in V_π. If $x \in X$, by Lemma 2.27 there is some $\alpha_x < \pi$ such that $x \in V_{\alpha_x}$. On the other hand, for any $\alpha < \pi$, there is some $\beta_\alpha \in K$ above α, so $\dot{\mathbb{Q}}_{\beta_\alpha}$ adds an R-dominating real $y_\alpha \in Y$ over V_{β_α}. Then the maps $x \mapsto \alpha_x$ and $\alpha \mapsto y_\alpha$ form the Tukey connection for $R \preceq_{\mathrm{T}} \pi$.

The rest is a consequence of Corollary 2.29. $\qquad\square$

When aiming to force many different values to cardinal characteristics, we cannot add *full* dominating reals as in the previous lemma. However, there is a way to add *restricted* dominating reals, allowing better control of the cardinal characteristics. We develop this technique in the following part.

2.3. Book-keeping arguments

Fix, for the rest of this section:

(1) A relational system $R = \langle X, Y, \sqsubset \rangle$ of the reals (see Definition 2.1) such that $|X| = \mathfrak{c} = 2^{\aleph_0}$

(2) A very definable (i.e. Suslin) ccc poset $\mathbb{Q}_R$ adding R-dominating reals over the ground model, such that $|\mathbb{Q}_R| \leq \mathfrak{c}$. Note that $\mathbb{L}c_h$, $\mathbb{D}$, $\mathbb{E}$, random forcing and Cohen forcing satisfy these conditions (for certain R as in the previous subsection).

(3) An infinite cardinal θ.

We aim to force $\mathfrak{b}(R) = \theta$. For the rest of this section we deal with $\mathfrak{b}(R) \geq \theta$, and from the following section onwards we deal with the converse inequality.

Forcing $\mathfrak{b}(R) \geq \theta$ means to force that $\forall F \in [X]^{<\theta} \ \exists y \in Y \ \forall x \in F \colon x \sqsubset y$. One way is to deal with one F at a time along a FS iteration. Concretely, if we are at step α of a FS iteration, we pick some $F_\alpha \in [X]^{<\theta} \cap V_\alpha$ and aim to add a $y_\alpha \in Y \cap V_{\alpha+1}$ that R-dominates all members of F_α. A very effective idea to do this comes from Brendle [9]: in V_α, pick a transitive model N_α of ZFC such that $F_\alpha \subseteq N_\alpha$ and $|N_\alpha| = \max\{\aleph_0, |F_\alpha|\} < \theta$.[9] So forcing with $\mathbb{Q}_\alpha = \mathbb{Q}_R^{N_\alpha}$ (which is ccc) does the job: it adds an R-dominating y_α over N_α, hence it dominates all members of F_α. The hope is that this y_α does not dominate much larger fragments of X.

Now, assume that π is an ordinal of uncountable cofinality, and that we perform a FS iteration of ccc posets of length π as explained before. To force $\mathfrak{b}(R) \geq \theta$, it is enough to guarantee that, in V_π, $\{F_\alpha \colon \alpha < \pi\}$ is *cofinal in* $[X]^{<\theta}$. Indeed, if $F \in [X]^{<\theta}$ then $F \subseteq F_\alpha$ for some $\alpha < \pi$, so y_α dominates all members of F_α, and then all members of F.

In the practice, we do not use all steps $\alpha < \pi$ to take care of $\mathfrak{b}(R) \geq \theta$, but only steps $\alpha \in K$ for some (cofinal) $K \subseteq \pi$, while other steps can be used to take care of something else. So we explain how to construct an iteration as above ensuring that some choice of $\{F_\alpha \colon \alpha \in K\}$ is cofinal in $[X]^{<\theta}$.

To do this, we first have to look at what happens to $|X|$ in the final extension. Recall that $|X| = \mathfrak{c}$. Assume $\theta \leq \lambda = \lambda^{\aleph_0}$. Then, in a FS iteration of length $\pi < \lambda^+$ of ccc posets, we can ensure that $|\mathbb{P}_\alpha| \leq \lambda$ and $\Vdash_\alpha \mathfrak{c} \leq \lambda$ as long as we have $\Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| \leq \lambda$ for all $\alpha < \pi$. This is fine in this context because all forcings we use to iterate have size $\leq\mathfrak{c}$.

If $\lambda \leq \pi < \lambda^+$ then, in V_π, $|X| = \mathfrak{c} = \lambda$, so $\mathrm{cof}([X]^{<\theta}) = \mathrm{cof}([\lambda]^{<\theta})$. Now, the existence of a collection $\{F_\alpha \colon \alpha \in K\}$ cofinal in $[X]^{<\theta}$ for some $K \subseteq \pi$, $K \in V$, implies that $\mathrm{cof}([\lambda]^{<\theta}) = \mathrm{cof}([X]^{<\theta}) \leq |K| \leq |\pi| \leq \lambda$. Hence, a requirement to obtain such a cofinal family is that $\mathrm{cof}([\lambda]^{<\theta}) = \lambda$ and $|K| = \lambda$.

We now show that the assumptions $\theta \leq \mathrm{cf}(\lambda) \leq \lambda = \lambda^{\aleph_0}$ and $\mathrm{cof}([\lambda]^{<\theta}) = \lambda$ are enough to construct such an iteration via a *book-keeping argument*. We only explain the construction for $\pi = \lambda$, but it works when $\pi = \lambda\delta$ (ordinal product) for some $0 < \delta < \lambda^+$ and $\mathrm{cf}(\pi) \geq \theta$. Let $K \subseteq \lambda$ of size λ and fix a bijection $h \colon K \to \lambda \times \lambda$ such that $h(\alpha) = (\xi, \eta)$ implies $\xi \leq \alpha$. Now, perform a FS iteration of ccc posets and assume we have reached the stage $\alpha < \lambda$. Since $\mathrm{cov}([\lambda]^{<\theta})$ is not modified by ccc forcing[10], in V_α we can pick a cofinal $\{F_{\alpha,\eta} \colon \eta < \lambda\}$ in $[X]^{<\theta} \cap V_\alpha$ (because $|X| = \mathfrak{c} \leq \lambda$). In the previous steps $\xi \leq \alpha$, in the same way we had picked in V_ξ a cofinal $\{F_{\xi,\eta} \colon \eta < \lambda\}$ in $[X]^{<\theta} \cap V_\xi$. If $\alpha \notin K$ then we can force with any ccc poset, but when $\alpha \in K$, the book-keeping function h makes the choice: letting $h(\alpha) = (\xi, \eta)$, pick $F_{h(\alpha)} = F_{\xi,\eta}$ (which exists because $\xi \leq \alpha$). As before, let N_α be a transitive model of ZFC such that $F_{h(\alpha)} \subseteq N_\alpha$ and $|N_\alpha| < \theta$, and we force with $\mathbb{Q}_\alpha := \mathbb{Q}_R^{N_\alpha}$ to go to the step $\alpha + 1$.

At the end of the iteration, in V_λ, we have ensured that each member of

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[9]This is possible because the members of X are "reals".

[10]Because, when θ is uncountable, in any ccc generic extension, any set of ordinals of size $<\theta$ is covered by a set in the ground model of size $<\theta$.

$\{F_{h(\alpha)}\colon \alpha \in K\} = \{F_{\xi,\eta}\colon \xi, \eta < \lambda\}$ is R-bounded. It remains to ensure that this family is cofinal in $[X]^{<\theta}$: If $F \in [X]^{<\theta}$ then, since $\mathrm{cf}(\lambda) \geq \theta$, we have that $F \in V_\xi$ for some $\xi < \lambda$, so $F \subseteq F_{\xi,\eta}$ for some $\eta < \lambda$.

Concerning $\mathrm{cof}([\lambda]^{<\theta})$, by Fact 1.3 we have $\mathrm{cf}(\mathrm{cof}([\lambda]^{<\theta})) \geq \mathrm{add}([\lambda]^{<\theta}) = \mathrm{cf}(\theta)$, so $\mathrm{cof}([\lambda]^{<\theta}) = \lambda$ implies $\mathrm{cf}(\lambda) \geq \mathrm{cf}(\theta)$, which is $\mathrm{cf}(\lambda) \geq \theta$ in the case when θ is regular.

Using the book-keeping argument presented above, we are now ready to present the first important construction of models with several pairwise different cardinal characteristics.

Theorem 2.31. *Let $\aleph_1 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ be regular cardinals, and assume λ is a cardinal such that $\lambda = \lambda^{\aleph_0}$ and $\mathrm{cf}([\lambda]^{<\theta_i}) = \lambda$ for $i = 1, \ldots, 4$. Then, we can construct a FS iteration of length (and size) λ of ccc posets forcing $\mathrm{add}(\mathcal{N}) = \theta_1$, $\mathrm{cov}(\mathcal{N}) = \theta_2$, $\mathfrak{b} \geq \theta_3$, $\mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{M}) = \theta_4$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$ (see Figure 6).*

The argument of the proof only gives $\mathfrak{b} \geq \theta_3$. In section 4 we are going to show how to obtain $\mathfrak{b} = \theta_3$, in addition.

In the first part of the proof we only deal with equalities of the form $\mathfrak{b}(R) \geq \theta_i$ and $\mathfrak{c} = \lambda$. In the next section, we deal with the rest of the proof.

Proof of Theorem 2.31, part 1. Partition $\lambda = K_1 \cup K_2 \cup K_3 \cup K_4$ with $|K_i| = \lambda$. Proceed in two steps:

Step 1. Force with $\mathbb{C}_\lambda$ (i.e. add λ-many Cohen reals).

Step 2. In $V_0 := V^{\mathbb{C}_\lambda}$, using book-keeping as before at each K_i, iterate with length λ and at:

$\alpha \in K_1$: force with $\mathbb{L}\mathrm{c}_{\mathrm{id}}^{N_\alpha}$, $|N_\alpha| < \theta_1$, which guarantees $\mathrm{add}(\mathcal{N}) \geq \theta_1$ in the final extension;

$\alpha \in K_2$: force with $(\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega))^{N_\alpha}$, $|N_\alpha| < \theta_2$, which guarantees $\mathrm{cov}(\mathcal{N}) \geq \theta_2$ in the final extension;

$\alpha \in K_3$: force with $\mathbb{D}^{N_\alpha}$, $|N_\alpha| < \theta_3$, which guarantees $\mathfrak{b} \geq \theta_3$ in the final extension;

$\alpha \in K_4$: force with $\mathbb{E}^{N_\alpha}$, $|N_\alpha| < \theta_4$, which guarantees $\mathrm{non}(\mathcal{M}) \geq \theta_4$, and even $\mathrm{non}(\mathcal{E}) \geq \theta_4$ (by Theorem 2.16), in the final extension.

It is clear by the construction that, in V_λ, $\mathfrak{c} = \lambda$. $\qquad\square$

Note that we have not used the Cohen reals from step 1. These will be used to prove the converse inequalities in the next section.
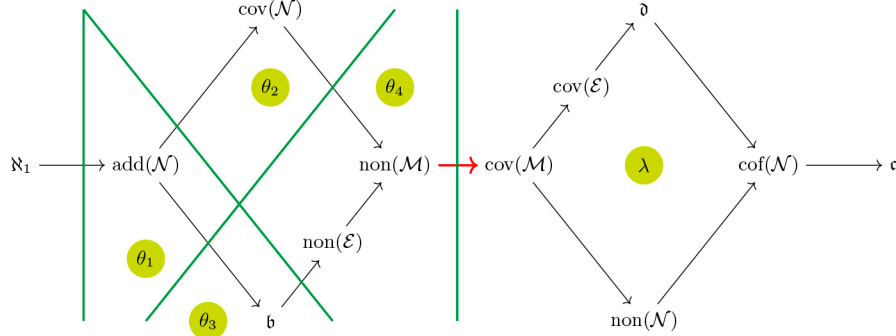
Figure 6: The constellation of Cichoń's diagram forced in Theorem 2.31. For $\mathfrak{b}$ we can only guarantee that $\mathfrak{b} \leq \theta_3$, but the converse inequality is settled in Theorem 4.20.

3. Preservation theory for cardinal characteristics

We deal with the problem of forcing $\mathfrak{b}(R) \leq \theta$ (and much more) to conclude the proof of Theorem 2.31. We proceed in two steps: we first add a strong type of R-unbounded family (using Cohen reals), and then show that this strong unbounded family is not destroyed in the remaining part of the iteration.

The strong type of unbounded family is defined as follows.

Definition 3.1. Let $R = \langle X, Y, \sqsubset \rangle$ be a relational system, and θ an infinite cardinal. We say that $\{x_i \colon i \in I\} \subseteq X$ is a *θ-R-unbounded family* if $|I| \geq \theta$ and $|\{i \in I \colon x_i \sqsubset y\}| < \theta$ for all $y \in Y$.

Although a θ-R-unbounded family is quite large, it has the property that any subset of size θ is R-unbounded, which guarantees $\mathfrak{b}(R) \leq \theta$. But we get much more, as indicated in the following result.

Lemma 3.2. *Assume that $|I| \geq \theta$. Then there exists a θ-R-unbounded family $\{x_i \colon i \in I\}$ iff $\mathbf{C}_{[I]^{<\theta}} \preceq_\mathrm{T} R$. In particular, $\mathfrak{b}(R) \leq \mathrm{non}([I]^{<\theta}) = \theta$ and $\mathrm{cov}([I]^{<\theta}) \leq \mathfrak{d}(R)$.*

Proof. If $\{x_i \colon i \in I\}$ is a θ-R-unbounded family then the maps

$$i \mapsto x_i \text{ and } y \mapsto \{i \in I \colon x_i \sqsubset y\}$$

yield the desired Tukey connection.

Conversely, assume that $\mathbf{C}_{[I]^{<\theta}} \preceq_\mathrm{T} R$ is witnessed by the Tukey connection $i \mapsto x_i$ and $y \mapsto A_y \in [I]^{<\theta}$, i.e. $x_i \sqsubset y$ implies $i \in A_y$. Therefore $\{i \in I \colon x_i \sqsubset y\} \subseteq A_y$, so it has size $<\theta$. Hence, $\{x_i \colon i \in I\}$ is a θ-R-unbounded family. $\square$

Concerning $\mathrm{cov}([I]^{<\theta})$, we have

$$\mathrm{cov}([I]^{<\theta}) = \begin{cases} |I| & \text{if } \theta < |I|, \\ \mathrm{cf}(\theta) & \text{if } \theta = |I|. \end{cases}$$

Hence $\mathrm{cov}([I]^{<\theta}) = |I|$ when θ is regular.

Recall from the previous section the Polish relational systems describing the cardinal characteristics in Cichoń's diagram, as illustrated in Figure 7.
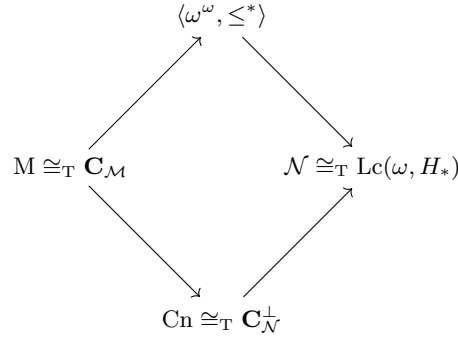

$$\langle \omega^\omega, \leq^* \rangle$$

$$\mathrm{M} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}} \qquad\qquad \mathcal{N} \cong_{\mathrm{T}} \mathrm{Lc}(\omega, H_*)$$

$$\mathrm{Cn} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{N}}^{\perp}$$

Figure 7: Tukey connections between the relational systems determining the non-dependant values in Cichoń's diagram, along with their equivalent Polish relational systems.

For the rest of this section, we fix a Polish relational system $R = \langle X, Y, \sqsubset \rangle$. In this case, θ-R-unbounded families can easily be added using Cohen reals.

Lemma 3.3. *Let λ be an uncountable cardinal. Then the Cohen reals $\{c_\alpha : \alpha < \lambda\} \subseteq X$ added by $\mathbb{C}_\lambda$ form an $\aleph_1$-R-unbounded family in $V^{\mathbb{C}_\lambda}$.*

Proof. Working in $V^{\mathbb{C}_\lambda}$, let $y \in Y$. Since y is a real, it only depends on countably many maximal antichains, so there is some $C \in [\lambda]^{<\aleph_1} \cap V$ such that $y \in V^{\mathbb{C}_C}$. For any $\alpha \in \lambda \smallsetminus C$, c_α is Cohen over $V^{\mathbb{C}_C}$, hence R-unbounded over $V^{\mathbb{C}_C}$ by Fact 2.18, so $c_\alpha \not\sqsubset y$. Therefore, $\{\alpha < \lambda : c_\alpha \sqsubset y\} \subseteq C$, which is countable. $\square$

Note that $\theta \leq \theta'$ implies that any θ-R-unbounded family $\{x_i : i \in I\}$ is θ'-R-unbounded, as long as $\theta' \leq |I|$. Therefore, the Cohen reals added by $\mathbb{C}_\lambda$ form a θ-R-unbounded family for all $\aleph_1 \leq \theta \leq \lambda$.

The reason we start with $\mathbb{C}_\lambda$ in Step 1 of the proof of Theorem 2.31 is to add θ_i-unbounded families. Now we aim to show how to preserve them in the iteration of Step 2. For this purpose, we introduce the preservation theory from Judah and Shelah [31] and Brendle [9].

Definition 3.4. Let κ be an infinite cardinal. A poset $\mathbb{P}$ is κ-R-*good* if, for any $\mathbb{P}$-name $\dot{y}$ for a member of Y, there is a non-empty set $H \subseteq Y$ (in the ground

model) of size $<\kappa$ such that, for any $x \in X$, if x is R-unbounded over H then $\Vdash x \not\sqsubset \dot{h}$.

We say that $\mathbb{P}$ is R-*good* if it is $\aleph_1$-R-good.

Note that $\kappa \leq \kappa'$ implies that any κ-R-good poset is κ'-R-good.

Goodness guarantees the preservation of strong unbounded families as follows.

Lemma 3.5. *If κ and θ are infinite cardinals, and $\kappa \leq \mathrm{cf}(\theta)$, then any κ-R-good poset preserves all the θ-R-unbounded families from the ground model.*

Proof. Let $\mathbb{P}$ be a κ-R-good poset. Assume that $\{x_i \colon i \in I\} \subseteq X$ is a θ-R-unbounded family. Let $\dot{y}$ be a $\mathbb{P}$-name of a member of Y. Find $H \in [Y]^{<\kappa}$ non-empty as in Definition 3.4. For each $y \in H$ let $A^y := \{i \in I \colon x_i \sqsubset y\}$ and $A := \bigcup_{y \in H} A^y$. Then $|A^y| < \theta$ and $|A| < \theta$, the latter because $\mathrm{cf}(\theta) \geq \kappa$.

We claim that $\Vdash \{i \in I \colon x_i \sqsubset \dot{y}\} \subseteq A$. Indeed, if $i \in I \smallsetminus A$, $x_i \not\sqsubset y$ for all $y \in H$, so $\Vdash x_i \not\sqsubset \dot{y}$. $\square$

Now, goodness is preserved along FS iterations.

Theorem 3.6. *Let κ be an uncountable regular cardinal. Then, any FS iteration of κ-cc κ-R-good posets is again κ-R-good.*

Proof. See e.g. [13, Thm. 4.15]. $\square$

This result can be weakened as follows.

Theorem 3.7. *Let κ and θ be uncountable cardinals such that κ is regular and $\mathrm{cf}(\theta) \geq \kappa$. Then, any FS iteration of κ-cc posets preserving θ-R-unbounded families, preserves θ-R-unbounded families.*

Proof. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \pi, \ \beta < \pi \rangle$ be a FS iteration of κ-cc posets preserving θ-R-unbounded families, and let $\{x_i \colon i \in I\}$ be a θ-R-unbounded family (in V). We show by recursion on $\alpha \leq \pi$ that $\mathbb{P}_\alpha$ forces that $\{x_i \colon i \in I\}$ is θ-R-unbounded. This is clear for $\alpha = 0$ and for the successor step.

The limit step α with $\mathrm{cf}(\alpha) \geq \kappa$ is easy: if $y \in Y \cap V_\alpha$ then $y \in V_\xi$ for some $\xi < \alpha$ (becase a nice name of the real y depends on $<\kappa$-many conditions, cf. Lemma 2.27), so $V_\xi \models |\{i \in I \colon x_i \sqsubset y\}| < \theta$ by inductive hypothesis. The same is satisfied in V_α.

We have to work more when $\mathrm{cf}(\alpha) < \kappa$. Fix an increasing cofinal sequence $\{\alpha_\xi \colon \xi < \mathrm{cf}(\alpha)\}$ in α. Since Y is analytic, there is some continuous surjection $f \colon \omega^\omega \to Y$. Let $\dot{y}$ be a $\mathbb{P}_\alpha$-name of a member of Y and pick some $\mathbb{P}_\alpha$-name $\dot{z}$ of a real in ω^ω such that $\Vdash_\alpha f(\dot{z}) = \dot{y}$. For each $\xi < \mathrm{cf}(\alpha)$ pick $\mathbb{P}_{\alpha_\xi}$-names $\dot{z}^\xi$ and $\langle \dot{p}_n^\xi \colon n < \omega \rangle$ such that $\mathbb{P}_{\alpha_\xi}$ forces that $\langle \dot{p}_n^\xi \colon n < \omega \rangle$ is a decreasing sequence in $\mathbb{P}_\alpha/\mathbb{P}_{\alpha_\xi}$, $\dot{z}^\xi \in \omega^\omega$, and $\dot{p}_n^\xi \Vdash_{\mathbb{P}_\alpha/\mathbb{P}_{\alpha_\xi}} \dot{z}{\upharpoonright} n = \dot{z}^\xi {\upharpoonright} n$ for all $n < \omega$. By induction hypothesis, $\Vdash_{\alpha_\xi} |\{i \in I \colon x_i \sqsubset f(\dot{z}^\xi)\}| < \theta$.

To finish the proof, if is enough to show that, for any $i \in I$, $\mathbb{P}_\alpha$ forces that $x_i \not\sqsubset f(\dot{z})$ whenever x_i is R-unbounded over $\{f(\dot{z}^\xi) \colon \xi < \mathrm{cf}(\alpha)\}$ (because

$\mathrm{cf}(\alpha) < \kappa \leq \mathrm{cf}(\theta)$). Fix $p \in \mathbb{P}_\alpha$, $\ell < \omega$ and assume that $p \Vdash x_i \sqsubset_\ell f(\dot{z})$. Find some $\xi < \mathrm{cf}(\alpha)$ such that $p \in \mathbb{P}_{\alpha_\xi}$. Let G_{α_ξ} be a $\mathbb{P}_{\alpha_\xi}$-generic set containing p and work in $V_{\alpha_\xi} = V[G_{\alpha_\xi}]$. Since $\Vdash_{\mathbb{P}_\alpha / \mathbb{P}_{\alpha_\xi}} x_i \sqsubset_\ell f(\dot{z})$, we have that $x \sqsubset_\ell f(\dot{z}^\xi[G_{\alpha_\xi}])$ (otherwise, by [13, Lem. 4.9], there would be some $n < \omega$ such that $\dot{p}_n^\xi[G_{\alpha_\xi}]$ forces $x_i \not\sqsubset_\ell f(\dot{z})$). Therefore, in V, $p \Vdash_{\alpha_\xi} x_i \sqsubset_\ell f(\dot{z}^\xi)$. $\qquad\square$

We now turn to particular cases. One very useful fact is that small posets are good.

Lemma 3.8. *Any poset $\mathbb{P}$ is κ-R-good for any infinite $\kappa > |\mathbb{P}|$.*

In particular, Cohen forcing is κ-R-good for all uncountable κ.

Proof. See [41, Lem. 4], also [13, Lem. 4.10]. $\qquad\square$

More concrete examples of R-good posets come from the connection between the combinatorics of a forcing and R. We formalize this with the following notions.

Definition 3.9 ([42])**.** We say that Γ is a *linkedness property* if $\Gamma(\mathbb{P}) \subseteq \mathcal{P}(\mathbb{P})$ for any poset $\mathbb{P}$.

Let μ and κ be infinite cardinals.

(1) A poset $\mathbb{P}$ is *μ-Γ-linked* if it can be covered by $\leq\mu$-many subsets in $\Gamma(\mathbb{P})$.

When $\mu = \aleph_0$, we write *σ-Γ-linked*.

(2) A poset $\mathbb{P}$ is *κ-Γ-Knaster* if $\forall B \in [\mathbb{P}]^\kappa \, \exists A \in [B]^\kappa \colon A \in \Gamma(\mathbb{P})$.

When $\kappa = \aleph_1$, we just write *Γ-Knaster*.

If Γ satisfies that $Q' \subseteq Q \in \Gamma(\mathbb{P})$ implies $Q' \in \Gamma(\mathbb{P})$, then any μ-Γ-linked poset is μ^+-Γ-Knaster. A more concrete discussion about linkedness properties and iterations can be found in [42, Sec. 5].

Example 3.10. The following are examples of linkedness properties. Here, $\mathbb{P}$ denotes an arbitrary poset.

(1) $\Lambda_{<\omega}$: *Centered.* $Q \in \Lambda_{<\omega}(\mathbb{P})$ iff Q is a centered subset of $\mathbb{P}$, i.e. for any finite $F \subseteq Q$, there is a $q \in \mathbb{P}$ stronger than all members of F.

Then, μ-$\Lambda_{<\omega}$-linked means μ-centered, and κ-$\Lambda_{<\omega}$-Knaster means pre-caliber κ.

(2) Λ_{int}: *Positive intersection number.* For $n < \omega$ non-zero and $s \in \mathbb{P}^n$, define

$$\iota_*(s) := \max\left\{ |e| \colon e \subseteq n \text{ and } \{s_i \colon i \in e\} \text{ has a lower bound in } \mathbb{P} \right\}.$$

For $Q \subseteq \mathbb{P}$, define *the intersection number of Q in $\mathbb{P}$* by

$$\mathrm{int}_{\mathbb{P}}(Q) := \inf\left\{ \frac{\iota_*(s)}{n} \colon s \in Q^n, \ 0 < n < \omega \right\}.$$

We say that $Q \in \Lambda_{\mathrm{int}}(\mathbb{P})$ iff $\mathrm{int}_{\mathbb{P}}(Q) > 0$.

Notice that $\Lambda_{<\omega}(\mathbb{P}) \subseteq \Lambda_{\mathrm{int}}(\mathbb{P})$ because any centered poset has intersection number 1.

According to the following result, $\Lambda_{<\omega}$ *is good for* Cn:

Theorem 3.11 (Brendle [9]). *Any μ-centered poset is μ^+-Cn-good. In particular, any σ-centered poset is* Cn-*good.*

Inspired by a result of Kamburelis [33], we have that Λ_{int} *is good for* $\mathrm{Lc}(\omega, H_*)$. Recall that $H_* = \{\mathrm{id}^{k+1} \colon k < \omega\}$.

Theorem 3.12. *Any μ-Λ_{int}-linked poset is μ^+-$\mathrm{Lc}(\omega, H_*)$-good.*

Proof. Let $\mathbb{P}$ be a μ-Λ_{int}-linked poset witnessed by $\langle Q_\alpha \colon \alpha < \mu \rangle$ and let $\dot\varphi$ be a $\mathbb{P}$-name of a member of $\mathcal{S}(\omega, H_*)$. Notice that $\mathbb{P}$ is μ^+-cc since no member of $\Lambda_{\mathrm{int}}(\mathbb{P})$ contains infinite antichains. So find a maximal antichain $A \subseteq \mathbb{P}$ and a sequence $\langle k_p \colon p \in A \rangle$ of natural numbers such that $p \Vdash \dot\varphi \in \mathcal{S}(\omega, \mathrm{id}^{k_p+1})$ for all $p \in A$.

Fix $\alpha < \mu$ and $p \in A$. Since $\mathrm{int}_{\mathbb{P}}(Q_\alpha) > 0$, there is some $m_\alpha < \omega$ satisfying $\frac{1}{m_\alpha} < \mathrm{int}_{\mathbb{P}}(Q_\alpha)$. Define

$$\varphi_{\alpha,p}(n) := \begin{cases} \{i < \omega \colon \exists\, q \in Q_\alpha \colon q \leq p,\ q \Vdash i \in \dot\varphi(n)\} & \text{if } n \geq m_\alpha, \\ \emptyset & \text{if } n < m_\alpha. \end{cases}$$

We show that $|\varphi_{\alpha,p}(n)| \leq n^{k_p+2}$ for all $n < \omega$, i.e. $\varphi_{\alpha,p} \in \mathcal{S}(\omega, H_*)$. This is clear for $n < m_\alpha$. In the case $n \geq m_\alpha$, for each $i \in \varphi_{\alpha,p}(n)$ pick some $q_i \in Q_\alpha$, stronger than p, forcing $i \in \dot\varphi(n)$, and let $\bar{q} := \langle q_i \colon i \in \varphi_{\alpha,p}(n) \rangle$. If $|\varphi_{\alpha,p}(n)| \geq n^{k_p+2}$ then

$$\iota_*(\bar{q}) \geq \mathrm{int}_{\mathbb{P}}(Q)|\varphi_{\alpha,p}(n)| \geq \mathrm{int}_{\mathbb{P}}(Q)n^{k_p+2} > n^{k_p+1},$$

so there is some $q' \in \mathbb{P}$ such that $a := \{i \in \varphi_{\alpha,p}(n) \colon q' \leq q_i\}$ has size $> n^{k_p+1}$. Then, $q' \Vdash a \subseteq \dot\varphi(n)$, which contradicts that $q' \Vdash |\dot\varphi(n)| \leq n^{k_p+1}$ (because $q' \leq p$). Therefore, $|\varphi_{\alpha,p}(n)| < n^{k_p+2}$.

We show that $S := \{\varphi_{\alpha,p} \colon \alpha < \mu,\ p \in A\}$ (which has size $\leq \mu$) witnesses goodness. Let $x \in \omega^\omega$ be $\mathrm{Lc}(\omega, H_*)$-unbounded over S, $n_0 < \omega$ and $p_0 \in \mathbb{P}$. Then p_0 is compatible with some $p \in A$, so there is a common stronger condition $q \in \mathbb{P}$. Now, pick an $\alpha < \mu$ such that $q \in Q_\alpha$. Then, there is some $n \geq \max\{m_\alpha, n_0\}$ such that $x(n) \notin \varphi_{\alpha,p}(n)$, so $q \nVdash x(n) \in \dot\varphi(n)$, i.e. there is some $q' \leq q$ forcing $x(n) \notin \dot\varphi(n)$. This shows that $\Vdash x \notin^* \dot\varphi$. $\qquad\square$

Corollary 3.13. *Any μ-centered poset is μ^+-$\mathrm{Lc}(\omega, H_*)$-good.*

Other examples are obtained using Boolean algebras with finitely additive measures.

Theorem 3.14 (Kelley [35]). *Let $\mathbb{B}$ be a Boolean algebra. Then $\mathbb{B} \smallsetminus \{0_{\mathbb{B}}\}$ is σ-Λ_{int}-linked iff there is a strictly positive fam $\Xi \colon \mathbb{B} \to [0,1]$ (i.e. $\Xi(b) = 0$ iff $b = 0$).*

In combination with Theorem 3.12, we obtain

Corollary 3.15. *If N is a transitive model of* ZFC, *then* $(\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega))^N$ *is* $\mathrm{Lc}(\omega, H_*)$-*good.*

In the next section, we will present a good linkedness property for $\langle \omega^\omega, \leq^* \rangle$. For the moment, we present the following examples.

Theorem 3.16 (Miller [45]). $\mathbb{E}$ *is* $\langle \omega^\omega, \leq^* \rangle$-*good.*

Theorem 3.17. *Random forcing is* $\langle \omega^\omega, \leq^* \rangle$-*good.*[11]

We are finally ready to conclude the proof of Theorem 2.31.

Proof of Theorem 2.31, part 2. It remains to show that, in V_λ, $\mathrm{add}(\mathcal{N}) \leq \theta_1$, $\mathrm{cov}(\mathcal{N}) \leq \theta_2$, $\mathrm{non}(\mathcal{M}) \leq \theta_4$ and $\lambda \leq \mathrm{cov}(\mathcal{M})$.

By Lemma 3.3, the Cohen reals added at step 1 give us $\aleph_1$-R-unbounded families of size λ for any Polish relational system R, in particular, we obtain in $V^{\mathbb{C}_\lambda}$ an $\aleph_1$-$\mathrm{Lc}(\omega, H_*)$-unbounded $\{c_\alpha^1 \colon \alpha < \lambda\}$, an $\aleph_1$-Cn-unbounded $\{c_\alpha^2 \colon \alpha < \lambda\}$, and an $\aleph_1$-M-unbounded $\{c_\alpha^4 \colon \alpha < \lambda\}$. Now, if we prove that the iteration of step 2 is θ_1-$\mathrm{Lc}(\omega, H_*)$-good, θ_2-Cn-good and θ_4-M-good, we obtain by Lemma 3.5 that the previous families are, in the final extension, θ_1-$\mathrm{Lc}(\omega, H_*)$-unbounded, θ_2-Cn-unbounded, and θ_4-M-unbounded, respectively. Therefore, by Lemma 3.2, $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\mathrm{Lc}(\omega, H_*)) \leq \theta_1$, $\mathrm{cof}(\mathcal{N}) = \mathfrak{b}(\mathrm{Cn}) \leq \theta_2$, $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\mathrm{M}) \leq \theta_4$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(\mathrm{M}) \geq \mathrm{cov}([\lambda]^{<\theta_4}) = \lambda$.

By virtue of Theorem 3.6, it is enough to prove that all the iterands used in step 2 are θ_1-$\mathrm{Lc}(\omega, H_*)$-good, θ_2-Cn-good and θ_4-M-good. Indeed, for:

$\alpha \in K_1$: $\mathbb{Q}_\alpha = \mathbb{L}\mathrm{c}_{\mathrm{id}}^{N_\alpha}$ has size $< \theta_1$ because $|N_\alpha| < \theta_1$, so it is θ_1-R-good (and κ-R-good for any $\kappa \geq \theta_1$) for any Polish relational system R (by Lemma 3.8).

$\alpha \in K_2$: $\mathbb{Q}_\alpha = (\mathcal{B}(2^\omega)/\mathcal{N}(2^\omega))^{N_\alpha}$ has size $< \theta_2$, so it is θ_2-R-good for any Polish relational system R. On the other hand, by Corollary 3.15, $\mathbb{Q}_\alpha$ is $\mathrm{Lc}(\omega, H_*)$-good.

$\alpha \in K_3$: $\mathbb{Q}_\alpha = \mathbb{D}^{N_\alpha}$ has size $< \theta_3$, so it is θ_3-R-good for any Polish relational system R. On the other hand, $\mathbb{Q}_\alpha$ is $\mathrm{Lc}(\omega, H_*)$-good and Cn-good by Theorem 3.11 and Corollary 3.13, respectively.

$\alpha \in K_4$: $\mathbb{Q}_\alpha = \mathbb{E}^{N_\alpha}$ has size $< \theta_4$, so it is θ_4-R-good for any Polish relational system R. On the other hand, $\mathbb{Q}_\alpha$ is $\mathrm{Lc}(\omega, H_*)$-good and Cn-good by Theorem 3.11 and Corollary 3.13, respectively. $\qquad\square$

In the previous proof, we have that $\mathbb{Q}_\alpha$ is θ_3-$\langle \omega^\omega, \leq^* \rangle$-good for $\alpha \in K_1 \cup K_2 \cup K_3$. However, although $\mathbb{E}$ is $\langle \omega^\omega, \leq^* \rangle$-good, there are examples of restrictions of the form $\mathbb{E}^N$ for transitive models N of ZFC that are not $\langle \omega^\omega, \leq^* \rangle$-good.

---

[11]This easily follows from the fact that random forcing is ccc and ω^ω-bounding.

Theorem 3.18 (Pawlikowski [47])**.** *There is a proper-ω^ω-bounding generic extension W of V in which $\mathbb{E}^V$ and $(\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega))^V$ add dominating reals over W.*

Even more, in the case of random forcing:

Theorem 3.19 (Judah and Shelah [32])**.** *There is a ccc forcing extension W of V such that $(\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega))^V$ adds dominating reals over W.*

In the next section, we modify the forcing construction of Theorem 2.31 for $\alpha \in K_4$ to guarantee that the set of Cohen reals $\{c_\alpha^3 \colon \alpha < \lambda\}$ added by $\mathbb{C}_\lambda$ stays θ_3-$\langle \omega^\omega, \leq^* \rangle$-unbounded in the final extension.

4. FS iterations with measures and ultrafilters on the natural numbers

We show how to modify the iteration in Theorem 2.31 to force, in addition, $\mathfrak{b} \leq \theta_3$. We start by introducing the following good property for $\langle \omega^\omega, \leq^* \rangle$.

Definition 4.1 ([42, 12])**.** Let $F \subseteq \mathcal{P}(\omega)$ be a filter. We assume that all filters are *free*, i.e. they contain the *Frechet filter* $\mathrm{Fr} := \{\omega \smallsetminus a \colon a \in [\omega]^{<\aleph_0}\}$. A set $a \subseteq \omega$ is *F-positive* if it intersects every member of F. Denote by F^+ the collection of F-positive sets.

We define the linkedness property Λ_F, which we call *F-linked*: given a poset $\mathbb{P}$ and $Q \subseteq \mathbb{P}$, Q is F-linked if, for any $\langle p_n \colon n < \omega \rangle \in Q^\omega$, there is some $q \in \mathbb{P}$ such that

$$q \Vdash \{n < \omega \colon p_n \in \dot{G}\} \in F^+, \text{ i.e. it intersects every member of } F.$$

Note that, in the case $F = \mathrm{Fr}$, the previous equation is "$q \Vdash \{n < \omega \colon p_n \in \dot{G}\}$ is infinite".

We also define Λ_{uf}, which we call *uf-linked (ultrafilter-linked)*: $Q \in \Lambda_{\mathrm{uf}}(\mathbb{P})$ if $Q \in \Lambda_F(\mathbb{P})$ for every (ultra)filter F on ω.

If $F \subseteq F'$ are filters on ω, it is clear that $\Lambda_{\mathrm{uf}}(\mathbb{P}) \subseteq \Lambda_{F'}(\mathbb{P}) \subseteq \Lambda_F(\mathbb{P}) \subseteq \Lambda_{\mathrm{Fr}}(\mathbb{P})$. But, for ccc posets:

Lemma 4.2 ([42])**.** *If $\mathbb{P}$ is ccc then $\Lambda_{\mathrm{uf}}(\mathbb{P}) = \Lambda_{\mathrm{Fr}}(\mathbb{P})$.*

Proof. Assume that F is a filter on ω and that $Q \subseteq \mathbb{P}$ is not F-linked in $\mathbb{P}$. We show that Q is not Fr-linked. Since Q is not F-linked, there is some sequence $\langle p_n \colon n < \omega \rangle \in Q^\omega$ such that $\mathbb{P}$ forces that $\dot{w} := \{n < \omega \colon p_n \in \dot{G}\}$ does not intersect some member of F. Since F is in the ground model, we can find a maximal antichain A in $\mathbb{P}$ deciding a member of F disjoint with $\dot{w}$, namely, there is some $h \colon A \to F$ such that each $p \in A$ forces $\dot{w} \cap h(p) = \emptyset$.

Since A is countable and $\mathfrak{p}$ is uncountable, we can find some $c \in [\omega]^{\aleph_0}$ such that $c \subseteq^* h(p)$ for all $p \in A$. In detail: if A is finite, let $c := \bigcap_{p \in A} h(p)$; otherwise, enumerate $A = \{p_n \colon n < \omega\}$ and, by recursion, construct an increasing

sequence $\langle k_n \colon n < \omega \rangle$ of natural numbers such that $k_n \in \bigcap_{\ell \leq n} h(p)$ for each $n < \omega$, and set $c := \{k_n \colon n < \omega\}$.

Thus, each $p \in A$ forces $\dot{w} \cap c \subseteq^* \dot{w} \cap h(p) = \emptyset$, i.e. $\dot{w} \cap c$ is finite. Since A is a maximal antichain, $\mathbb{P}$ forces that $\dot{w} \cap c = \{n \in c \colon p_n \in \dot{G}\}$ is finite. Therefore, $\langle p_n \colon n \in c \rangle$ witnesses that Q is not Fr-linked in $\mathbb{P}$. $\qquad\square$

Example 4.3.

(1) Any singleton is uf-linked. Hence, any poset $\mathbb{P}$ is $|\mathbb{P}|$-uf-linked. In particular, Cohen forcing is σ-uf-linked.

(2) Random forcing is σ-uf-linked, in fact, any measure algebra is σ-uf-linked. Indeed, if $\mathbb{B}$ is a complete Boolean algebra and $\mu \colon \mathbb{B} \to [0,1]$ is a σ-additive measure such that $\mu(p) \neq 0$ for all $p \neq 0_{\mathbb{B}}$, then, for any $\delta > 0$, $B_\delta := \{p \in \mathbb{B} \colon \mu(p) \geq \delta\}$ is Fr-linked.

We use the characterization in Lemma 4.4 to show that B_δ is Fr-linked. Let $\dot{m}$ be a $\mathbb{B}$-name of a natural number. Then $\sum_{n < \omega} \mu(\|\dot{m} = n\|_{\mathbb{B}}) = 1$, so there is some $m' < \omega$ such that $\sum_{n \geq m'} \mu(\|\dot{m} = n\|_{\mathbb{B}}) < \delta$. As a consequence, $\mu(\|\dot{m} < m'\|_{\mathbb{B}}) > 1 - \delta$. Since any $q \in B_\delta$ is compatible with $\|\dot{m} < m'\|_{\mathbb{B}}$, it follows that q does not force $m' \leq \dot{m}$.

(3) The forcing $\mathbb{E}$ (see Definition 2.15) is σ-uf-linked. We show later that this poset satisfies a stronger property.

The following series of results indicate that Λ_{Fr} is good for $\langle \omega^\omega, \leq^* \rangle$.

Lemma 4.4. *Let $\mathbb{P}$ be a poset and $Q \subseteq \mathbb{P}$. Then Q is Fr-linked iff, for any $\mathbb{P}$-name $\dot{m}$ for a natural number, there is some $m' \in \omega$ (in the ground model) such that no $p \in Q$ forces $m' \leq \dot{m}$.*

Proof. $(\Rightarrow)$ [42] Assume that, for any $n < \omega$, there is some $p_n \in Q$ forcing $n \leq \dot{m}$. Then, if G is $\mathbb{P}$-generic over V, then $\{n < \omega \colon p_n \in G\}$ must be finite because $p_n \in G \Rightarrow n \leq \dot{m}[G] < \omega$. Therefore, in V, Q cannot be Fr-linked.

$(\Leftarrow)$ (with Cardona) Assume that Q is not Fr-linked, so there is some $\langle p_n \colon n < \omega \rangle \in Q^\omega$ such that $\Vdash$ "$\{n < \omega \colon p_n \in \dot{G}\}$ is finite". So pick some $\mathbb{P}$-name $\dot{m}$ of a natural number such that $\Vdash$ "$\{n < \omega \colon p_n \in \dot{G}\} \subseteq \dot{m}$. Note that $p_n \Vdash n < \dot{m}$. $\qquad\square$

Lemma 4.5. *Let $\mathbb{P}$ be a poset and Q be an Fr-linked subset of $\mathbb{P}$. If $\dot{y}$ is a $\mathbb{P}$-name of a member of ω^ω, then there is some $y' \in \omega^\omega$ (in the ground model) such that, for any $x \in \omega^\omega$*

$$x \not\leq^* y' \Rightarrow \forall n < \omega \ \forall p \in Q \colon p \not\Vdash \forall k \geq n \colon x(k) \leq \dot{y}(k).$$

Proof. Using Lemma 4.4, for each $k < \omega$ find $y'(k) < \omega$ such that no $p \in Q$ forces $y'(k) \leq \dot{y}(k)$. This defines $y' \in \omega^\omega$.

Now assume that $x \in \omega^\omega$ and $x \not\leq^* y'$. Let $n < \omega$ and $p \in Q$, so there is some $k \geq n$ such that $x(k) > y'(k)$. On the other hand, $p \not\Vdash y'(k) \leq \dot{y}(k)$, so there is some $q \leq p$ forcing $\dot{y}(k) < y'(k) < x(k)$, so $p \not\Vdash \forall k \geq n \colon x(k) \leq \dot{y}(k)$. $\qquad\square$

Theorem 4.6 ([42]). *Any μ-Fr-linked poset is μ^+-ω^ω-good.*

This theorem is an easy consequence of Lemma 4.5. However, we do not know how to modify the construction in Theorem 2.31 to obtain a θ_3-ω^ω-good iteration. But we have some other way to preserve unbounded families, as in the following result.

Theorem 4.7 ([12]). *Let θ be an uncountable regular cardinal. Then any θ-Fr-Knaster poset preserves θ-ω^ω-unbounded families.*

Proof. Assume that $\{x_i : i \in I\}$ is a θ-ω^ω-unbounded family, and that there is some $p \in \mathbb{P}$ forcing that it is not, i.e. for some $\mathbb{P}$-name $\dot{y}$ of a member of ω^ω, p forces that $|\{i \in I : x_i \leq^* \dot{y}\}| \geq \theta$. This implies that the set

$$I_0 := \{i \in I : \exists\, p' \leq p\colon p' \Vdash x_i \leq^* \dot{y}\}$$

has size $\geq\theta$. Pick $I_1 \subseteq I_0$ of size θ and, for each $i \in I_1$, choose $p_i \leq p$ and $n_i < \omega$ such that $p_i \Vdash x_i(k) \leq \dot{y}(k)$ for all $k \geq n_i$. Since $\mathrm{cf}(\theta) > \omega$, we can find $n < \omega$ and $I_2 \subseteq I_1$ of size θ such that $n_i = n$ for all $i \in I_2$.

Since θ is regular and $\mathbb{P}$ is θ-Fr-Knaster, there is some $I' \subseteq I_2$ such that the set $Q := \{p_i : i \in I'\}$ is Fr-linked. Now find $y' \in \omega^\omega$ as in Lemma 4.5 for $\dot{y}$ and Q. Then, $|\{i \in I : x_i \leq^* y'\}| < \theta$, so there is some $i \in I'$ such that $x_i \nleq^* y'$. Hence, by Lemma 4.5, no $p \in Q$ forces $\forall\, k \geq n\colon x_i(k) \leq \dot{y}(k)$. But p_i forces this, a contradiction. $\square$

Because of the previous theorem, the plan now is to modify the construction of Theorem 2.31 to obtain a θ_3-Fr-Knaster poset. To achieve this, we strengthen ultrafilter-linkedness as follows.

Definition 4.8 (cf. [24]). Given a (non-principal) ultrafilter D on ω, define the linkedness property $\Lambda_D^{\lim}$, called D-lim-*linked*: $Q \in \Lambda_D^{\lim}(\mathbb{P})$ if there are a $\mathbb{P}$-name $\dot{D}'$ of an ultrafilter on ω extending D and a map $\lim^D\colon Q^\omega \to \mathbb{P}$ such that, whenever $\bar{p} = \langle p_n : n < \omega \rangle \in Q^\omega$,

$$\lim{}^D \bar{p} \Vdash \{n < \omega : p_n \in \dot{G}\} \in \dot{D}'.$$

Define the linkedness property $\Lambda_{\mathrm{uf}}^{\lim}$, called *uf*-lim-*linked*, by $Q \in \Lambda_{\mathrm{uf}}^{\lim}(\mathbb{P})$ iff $Q \in \Lambda_D^{\lim}(\mathbb{P})$ for any ultrafilter D on ω.

In addition, for an infinite cardinal μ, we say that a poset $\mathbb{P}$ is *uniformly μ-D-lim-linked* if if is μ-$\Lambda_D^{\lim}$-linked witnessed by some $\langle Q_\alpha : \alpha < \mu \rangle$, but the $\dot{D}'$ above can be the same for any Q_α. And we say that $\mathbb{P}$ is *uniformly μ-uf-lim-linked* if there is some $\langle Q_\alpha : \alpha < \mu \rangle$ witnessing that $\mathbb{P}$ is uniformly μ-D-lim-linked for any ultrafilter D on ω.

Example 4.9. Any singleton is uf-lim-linked. As a consequence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$-uf-lim-linked, witnessed by its singletons: for $p \in P$, let $Q_p := \{p\}$, and $\lim^D$ on Q_p is just the constant map with value p, when D is an ultrafilter on ω. Since $\lim^D \bar{p} \Vdash \{n < \omega : p_n \in \dot{G}\} = \omega$ for all $\bar{p} \in Q_p^\omega$, $\dot{D}'$ can be any $\mathbb{P}$-name of an ultrafilter extending D.

Theorem 4.10 ([24, 12]). $\mathbb{E}$ *is uniformly σ-uf-lim-linked.*

Proof. We only indicate the components and the limit functions. For $s \in \omega^{<\omega}$ and $m \in \omega$, consider the set $E_{s,m}$ of conditions in $\mathbb{E}$ of the form (s, φ) with $\varphi \in \mathcal{S}(\omega, m)$. If D is an ultrafilter on ω and $\bar{p} = \langle p_n \colon n < \omega \rangle \in E_{s,m}^{\omega}$, $p_n = (s, \varphi_n)$, define $\lim^D \bar{p} := (s, \varphi)$ where

$$k \in \varphi(i) \text{ iff } \{n < \omega \colon k \in \varphi_n(i)\} \in D.$$

It is clear that $(s, \varphi) \in E_{s,m}$.

The sequence $\langle E_{s,m} \colon s \in \omega^{<\omega}, \ m < \omega \rangle$ witnesses that $\mathbb{E}$ is uniformly σ-D-lim-linked for any ultrafilter D on ω. This is proved by showing that, whenever G is $\mathbb{P}$-generic over V, the set

$$D \cup \bigcup_{s,m} \left\{ \{n < \omega \colon p_n \in G\} \colon \bar{p} \in E_{s,m}^{\omega} \cap V, \ \lim^D \bar{p} \in G \right\}$$

has the finite intersection property. $\qquad\square$

Based on [24, 50], we present a framework to construct FS iterations that allow ultrafilter limits. The candidates for such iterations can be presented in a more general fashion. For an infinite cardinal θ, denote

$$\theta^- = \begin{cases} \theta_0 & \text{if } \theta = \theta_0^+ \text{ for some cardinal } \theta_0, \\ \theta & \text{if } \theta \text{ is not a successor cardinal.} \end{cases}$$

Definition 4.11. Let θ be an uncountable cardinal. A θ-Γ-*iteration* is a FS iteration $\langle \mathbb{P}_\alpha, \mathbb{P}_\alpha^-, \dot{\mathbb{Q}}_\xi \colon \alpha \leq \pi, \ \xi < \pi \rangle$ satisfying:

(i) $\mathbb{P}_\xi^- \lessdot \mathbb{P}_\xi$ for all $\xi < \pi$, and

(ii) $\mathbb{P}_\xi^-$ forces that $\dot{\mathbb{Q}}_\xi$ is μ_ξ-Γ-linked witnessed by a sequence of $\mathbb{P}_\xi^-$-names $\langle \dot{Q}_{\xi,\zeta} \colon \zeta < \mu_\xi \rangle$, where $\mu_\xi < \theta$ (known from the ground model).

Associated with this iteration, we define the following notions.

(1) A function $h \colon d_h \to \theta^-$ with $\pi \subseteq d_h$ is usually called a *guardrail for the iteration*.[12]

(2) For $\alpha \leq \pi$ and h as above, let $\mathbb{P}_\alpha^h$ be the set of conditions $p \in \mathbb{P}_\alpha$ *following h*, i.e. for $\xi \in \mathrm{dom}\, p$, $h(\xi) < \mu_\xi$, $p(\xi)$ is a $\mathbb{P}_\xi^-$-name and $\Vdash_{\mathbb{P}_\xi^-} p(\xi) \in \dot{Q}_{\xi, h(\xi)}$.

A guardrail h can have coordinates ξ satisfying $h(\xi) \geq \mu_\xi$. Although they are not interesting, allowing them considerably simplifies writing the details of the construction of a θ-Γ-iteration.

(3) $\mathbb{P}_\alpha^* := \bigcup_{h \in \theta^{-\pi}} \mathbb{P}_\alpha^h$.

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[12]The assumption $\pi \subseteq d_h$ is for practicality, e.g. to have that it is still a guardrail of the iteration up to some $\xi < \pi$.

(4) Let L be a linear order and $\langle p_\ell : \ell \in L \rangle$ a sequence of conditions in $\mathbb{P}_\pi$. We say that $\langle p_\ell : \ell \in L \rangle$ is a *uniform Δ-system* if it satisfies the following:

   (i) All $\operatorname{dom} p_\ell$ ($\ell \in L$) have the same size n: $\operatorname{dom} p_\ell = \{\alpha_{\ell,k} : k < n\}$ (increasing enumeration).

   (ii) There is some $v \subseteq n$ such that, for each $k \in v$, the sequence $\langle \alpha_{\ell,k} : \ell \in L \rangle$ is constant with value $\alpha_{*,k}$.

   (iii) $\langle \operatorname{dom} p_\ell : \ell \in L \rangle$ forms a Δ-system with root $\{\alpha_{*,k} : k \in v\}$.

   (iv) For $k \in n \smallsetminus v$, the sequence $\langle \alpha_{\ell,k} : \ell \in L \rangle$ is increasing.

   (v) There is some guardrail h such that $\{p_\ell : \ell \in L\} \subseteq \mathbb{P}_\pi^h$.

By recursion on $\alpha \leq \pi$, we can show:

Fact 4.12. *For any θ-Γ-iteration as in Definition 4.11, $\mathbb{P}_\alpha^*$ is dense in $\mathbb{P}_\alpha$.*

We focus on the case $\Gamma = \Lambda_{\mathrm{uf}}^{\lim}$. We plan to construct a θ-$\Lambda_{\mathrm{uf}}^{\lim}$-iteration which is θ-Fr-Knaster (in our case, $\theta = \theta_3$).

Lemma 4.13. *For a θ-Λ_{Fr}-iteration as in Definition 4.11: Let H be a set of guardrails, $\theta' \geq \theta$ regular, and assume:*

   (i) *Any countable partial function from π into θ^- can be extended by some $h \in H$.*

   (ii) *If $h \in H$ and $\bar{p} = \langle p_n : n < \omega \rangle \subseteq \mathbb{P}_\pi^h$ forms a uniform Δ-system, then there is some $q \in \mathbb{P}_\pi$ forcing that $\{n < \omega : p_n \in \dot{G}\}$ is infinite.*

Then $\mathbb{P}_\pi$ is θ'-Fr-Knaster.

Proof. Let $A \subseteq \mathbb{P}_\pi$ have size θ'. Since θ' is regular uncountable, we can find an uniform Δ-system $B \subseteq A$ of size θ'. Condition (ii) implies that B is Fr-linked. $\qquad\square$

The q in (ii) is found as an *ultrafilter limit* similar to Definition 4.8, so this requires to construct ultrafilters along the iteration. For the successor step, the following lemma is useful.

$$
\begin{array}{ccccc}
D & \in & N & \longrightarrow & N^{\mathbb{Q}} \\
 & & \big\uparrow & & \big\uparrow \\
D^- & \in & M & \longrightarrow & M^{\mathbb{Q}} \ \ni D^+
\end{array}
$$

Figure 8: The situation in Lemma 4.14

Lemma 4.14 ([12, Lem. 3.20]). *Let $M \subseteq N$ be transitive models of* ZFC *and* $\mathbb{Q} \in M$ *be a poset. Assume that $M \models$ "D^- is an ultrafilter on ω", $M \models$ "$\dot{D}^+$ is a $\mathbb{Q}$-name of an ultrafilter on ω extending D^-", and $N \models$ "D is an ultrafilter on ω extending D^-". Then, in N, $\mathbb{Q}$ forces that $D \cup \dot{D}^+$ has the finite intersection property, i.e. it can be extended to an ultrafilter (see Figure 8).*

Definition 4.15. A θ-$\Lambda_{\mathrm{uf}}^{\lim}$-iteration as in Definition 4.11 *has ultrafilter limits for H* when:

(i) H is a set of guardrails,

(ii) for $h \in H$, $\langle \dot{D}_\xi^h : \xi \leq \pi \rangle$ is a sequence such that $\dot{D}_\xi^h$ is a $\mathbb{P}_\xi$ name of a non-principal ultrafilter on ω,

(iii) if $\xi < \eta \leq \pi$ then $\Vdash_{\mathbb{P}_\eta} \dot{D}_\xi^h \subseteq \dot{D}_\eta^h$,

(iv) $\mathbb{P}_\xi$ forces that $\dot{D}_\xi^h \cap V^{\mathbb{P}_\xi^-} \in V^{\mathbb{P}_\xi^-}$,

and whenever $h \in H$, $\langle \xi_n : n < \omega \rangle \subseteq \pi$ and $\Vdash_{\mathbb{P}_{\xi_n}^-} \dot{q}_n \in \dot{Q}_{\xi_n, h(\xi_n)}$:

(v) if $\langle \xi_n : n < \omega \rangle$ is constant with value ξ then

$$\Vdash_{\mathbb{P}_\xi} \lim_n^{\dot{D}_\xi^h} \dot{q}_n \Vdash_{\dot{\mathbb{Q}}_\xi} \{ n < \omega : \dot{q}_n \in \dot{G}(\xi) \} \in \dot{D}_{\xi+1}^h,$$

(vi) and if $\langle \xi_n : n < \omega \rangle$ is increasing, then

$$\Vdash_{\mathbb{P}_\pi} \{ n < \omega : \dot{q}_n \in \dot{G}(\xi_n) \} \in \dot{D}_\pi^h$$

Lemma 4.16. *Any iteration as in Definition 4.15 satisfies* (ii) *of Lemma 4.13 for H.*

Proof. Let $\langle p_n : n < \omega \rangle$ be an uniform Δ-system in $\mathbb{P}_\pi^h$. Let Δ be the root of the Δ-system and define $q \in \mathbb{P}_\pi$ with $\operatorname{dom} q := \Delta$ such that $q(\xi)$ is a $\mathbb{P}_\xi^-$-name of $\lim_n^{\dot{D}_\xi^h} p_n(\xi)$ for $\xi \in \Delta$. Then q forces that $\{ n < \omega : p_n \in \dot{G}_\pi \} \in \dot{D}_\pi^h$. $\square$

To obtain (i) of Definition 4.15 we could basically use $H = \theta^{-\pi}$. However, there are steps $\xi < \pi$ of the iteration where we want $\mathbb{P}_\xi^-$ to be quite small, so to guarantee (iv) of Definition 4.15 we need that H is also small. This is guaranteed by the following result.

Theorem 4.17 ([21, 48]). *Let ν, κ be infinite cardinals and L be a set such that $\nu \leq \kappa \leq |L| \leq 2^\kappa$. Then there exists an $H \subseteq {}^L\kappa$ such that $|H| \leq \kappa^{<\nu}$, and any partial function from L into κ with domain of size $<\nu$ can be extended by a function in H.*

The following two theorems indicate how to construct iterations as in Definition 4.15.

Theorem 4.18. *Let $\mathbb{P}_{\pi+1}$ be a θ-$\Lambda_{\mathrm{uf}}^{\lim}$-iteration of length $\pi+1$ and H be a set of guardrails such that, up to π, it has ultrafilter limits for H.*

Assume that $\mathbb{P}_\pi^- \Subset \mathbb{P}_\pi$ and $\mathbb{P}_\pi$ forces $\dot{D}_\pi^h \cap V^{\mathbb{P}_\pi^-} \in V^{\mathbb{P}_\pi^-}$. Then, for each $h \in H$, we can find a $\mathbb{P}_{\pi+1}$-name $\dot{D}_{\pi+1}^h$ of an ultrafilter extending $\dot{D}_\pi^h$, which together make $\mathbb{P}_{\pi+1}$ have ultrafilter limits for H.

Proof. Direct application of Lemma 4.14 to $M = V[G \cap \mathbb{P}_\pi^-]$ and $N = V[G]$, where G is $\mathbb{P}_\pi$-generic over V. $\qquad\square$

Theorem 4.19. *Assume that π is a limit ordinal and $\mathbb{P}_\pi$ is a θ-$\Lambda_{\mathrm{uf}}^{\lim}$-iteration of length π. Further assume that h is a guardrail and $\langle \dot{D}_\xi^h : \xi < \pi \rangle$ is a sequence witnessing that, for any $\xi < \pi$, $\mathbb{P}_\xi$ is an iteration with uf-limits for h.*

If, for any $\xi < \pi$, $\mathbb{P}_\xi^-$ forces that $\dot{Q}_{\xi, h(\xi)}$ is centered, then we can find a $\dot{D}_\pi^h$ that makes $\mathbb{P}_\pi$ have uf-limits for h.

Proof. Let E be the collection of sequences $\tau = \langle (\xi_n, \dot{q}_n) : n < \omega \rangle$ such that $\langle \xi_n : n < \omega \rangle$ is an increasing sequence of ordinals with limit π and $\Vdash_{\xi_n} \dot{q}_n \in \dot{Q}_{\xi_n, h(\xi_n)}$ for all $n < \omega$. For such a τ, let $\dot{d}_\tau$ be a $\mathbb{P}_\pi$-name of the set $\{ n < \omega : \dot{q}_n \in \dot{G}(\xi_n) \}$. It is enough to show that $\mathbb{P}_\pi$ forces that $\bigcup_{\xi < \pi} \dot{D}_\xi^h \cup \{ \dot{d}_\tau : \tau \in E \}$ has the finite intersection property.

Fix $p \in \mathbb{P}_\pi$, a $\mathbb{P}_\pi$-name $\dot{d}$ of a member of $\bigcup_{\xi < \pi} \dot{D}_\xi^h$, $i^* < \omega$ and, for each $i < i^*$, $\tau^i = \langle (\xi_n^i, \dot{q}_n^i) : n < \omega \rangle \in E$. Denote $\dot{d}_i := \dot{d}_{\tau^i}$. Without loss of generality, by strengthening p if necessary, we may assume that, for some $\alpha < \pi$, $\dot{d}$ is a $\mathbb{P}_\alpha$-name of a member of $\dot{D}_\alpha^h$. We can also increase α and say that $p \in \mathbb{P}_\alpha$. Since $\dot{D}_\alpha^h$ is forced non–principal, p forces in $\mathbb{P}_\alpha$ that $\dot{d} \cap \bigcap_{i < i^*} \{ n < \omega : \xi_n^i \geq \alpha \} \in \dot{D}_\alpha^h$, so there are $p' \leq p$ in $\mathbb{P}_\alpha$ and $k < \omega$ such that $\xi_k^i \geq \alpha$ for all $i < i^*$ and $p' \Vdash_\alpha k \in \dot{d}$. To conclude the proof, it is enough to find some $q \leq p'$ in $\mathbb{P}_\pi$ such that $q \Vdash \dot{q}_k^i \in \dot{G}(\xi_k^i)$ for all $i < i^*$.

Let $L := \{ \xi_j : j < m \}$ be the increasing enumeration of $\{ \xi_k^i : i < i^* \}$. For $j < m$, let $I_j := \{ i < i^* : \xi_k^i = \xi_j \}$. Since $\mathbb{P}_{\xi_j}$ forces that $\dot{Q}_{\xi_j, h(\xi_j)}$ is centered, there is some condition in $\dot{\mathbb{Q}}_{\xi_j}$ stronger than $\dot{q}_k^i$ for all $i \in I_j$. This allows to define $q \in \mathbb{P}_\pi$ with $\mathrm{dom}\, q = \mathrm{dom}\, p' \cup L$ such that $q(\xi) := p'(\xi)$ for $\xi \in \mathrm{dom}\, p'$ and, for $j < m$ and $i \in I_j$, $\Vdash_{\xi_j} q(\xi_j) \leq \dot{q}_k^i$. It is clear that q is as required. $\quad\square$

We are now ready to present the main forcing construction of this section.

Theorem 4.20 (cf. [24, 25])**.** *Let $\aleph_1 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ be regular cardinals, and assume λ is a cardinal such that $\lambda = \lambda^{\aleph_0}$ and $\mathrm{cf}([\lambda]^{<\theta_i}) = \lambda$ for $i = 1, \ldots, 4$. Further assume that one of the following holds:*

(i) $\theta_3 = \theta_4$.

(ii) $\theta_3^- < \theta_4$, $\theta^{\aleph_0} < \theta_4$ for every cardinal $\theta < \theta_4$, and $\lambda \leq 2^\kappa$ for some cardinal $\kappa < \theta_4$.

Then, we can construct a FS iteration of length (and size) λ of ccc posets forcing $\mathrm{add}(\mathcal{N}) = \theta_1$, $\mathrm{cov}(\mathcal{N}) = \theta_2$, $\mathfrak{b} = \theta_3$, $\mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{M}) = \theta_4$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$ (see Figure 6).

Proof. In case (i) the result follows directly from Theorem 2.31, so we focus on the assumptions of case (ii), in which we can further assume that $\kappa \geq \theta_3^-$. We proceed exactly as in the proof of Theorem 2.31 to construct a FS iteration of length $\pi := \lambda + \lambda$, using Cohen forcing at the first λ stages, but we modify the construction for $\alpha \in K_4$, the steps where we increase $\mathrm{non}(\mathcal{M})$ (and even $\mathrm{non}(\mathcal{E})$) using $\mathbb{E}$, to obtain a θ_3-$\Lambda_{\mathrm{uf}}^{\lim}$-iteration with ultrafilter limits on some H of size $<\theta_4$. We aim to apply Lemma 4.13 and 4.16 to conclude that the iteration is θ_3-Fr-Knaster, hence ensuring, by Theorem 4.7, that the first λ-many Cohen reals added in the iteration form a θ_3-ω^ω-unbounded family in the final extension, so the remaining $\mathfrak{b} \leq \theta_3$ will be forced.

Using that $\theta_3^- \leq \kappa < \theta_4$ and $\lambda \leq 2^\kappa$, by Theorem 4.17 we can find $H_0 \subseteq \kappa^\pi$ of size $\leq \kappa^{\aleph_0} < \theta_4$ (by (ii)) such that any countable partial function from π into κ can be extended by a function in H_0. For any $g \in \kappa^\pi$ define $g' \in \theta_3^{-\pi}$ by $g'(\xi) := g(\xi)$ if $g(\xi) < \theta_3^-$, and $g'(\xi) := 0$ otherwise. Then $H := \{g' : g \in H_0\} \subseteq \theta_3^{-\pi}$ has size $<\theta_4$ and any countable partial function from π into θ_3^- can be extended by a function in H. This guarantees requirement (i) of Lemma 4.13.

To construct the iteration, proceed by recursion, starting with an ultrafilter D_0^h on ω for $h \in H$. In the successor step $\xi \to \xi+1$, we do some work in the case $\xi = \lambda + \alpha$ with $\alpha \in K_4$ because in other cases we proceed as in Theorem 2.31 and just pick $\mu_\xi < \theta_3$ such that $\Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi = \{\dot{q}_\zeta^\xi : \zeta < \mu_\xi\}$, so we let $\mathbb{P}_\xi^- := \mathbb{P}_\xi$ and $\dot{Q}_{\xi,\zeta}$ be a $\mathbb{P}_\xi$-name of $\{\dot{q}_\zeta^\xi\}$, so any $\mathbb{P}_{\xi+1}$-name $\dot{D}_{\xi+1}^h$ of an ultrafilter extending $\dot{D}_\xi^h$ is suitable.

Using the book-keeping for K_4, in stage $\xi = \lambda + \alpha$ we have picked some $\mathbb{P}_\xi$-name $\dot{F}_\alpha$ of a subset of ω^ω of size $<\theta_4$, and aim to add an eventually different real over $\dot{F}_\alpha$ in the following step by using a restriction of $\mathbb{E}$. Since $\mathbb{P}_\xi$ has the ccc, we can find some $\nu_\alpha < \theta_4$ such that $\dot{F}_\alpha$ is represented by $\{\dot{x}_{\alpha,i} : i < \nu_\alpha\}$. Using the assumption (ii), for large enough χ we can find $M \prec H_\chi$ of size $<\theta_4$, closed under countable sequences, such that $\mathbb{P}_\xi$ and each $\dot{x}_{\alpha,i}$ ($i < \nu_\alpha$) and $\dot{D}_\xi^h$ ($h \in H$) are in M. Consider $\mathbb{P}_\xi^- := \mathbb{P}_\xi \cap M$, which is a complete suborder of $\mathbb{P}_\xi$ because the latter has the ccc and M is closed under countable sequences. Then, we force with $\dot{\mathbb{Q}}_\xi := \mathbb{E}^{V^{\mathbb{P}_\xi^-}}$ to advance to the next stage. Note that this is a $\mathbb{P}_\xi^-$-name (for $\mathbb{E}$). Enumerate $\omega^{<\omega} \times \omega = \{(s_k, m_k) : k < \omega\}$ and let $\mu_\xi := \omega$ and $\dot{Q}_{\xi,k}$ be a $\mathbb{P}_\xi^-$-name of E_{s_k, m_k} for $k < \omega$.

By the construction of $\mathbb{P}_\xi^-$, for any $h \in H$ we can find a $\mathbb{P}_\xi^-$-name $\dot{D}_\xi^{h,-}$ of $\dot{D}_\xi^h \cap V^{\mathbb{P}_\xi^-}$ (which exists because M is countably closed). Then, Theorem 4.18 applies.

Limit steps are guaranteed by Theorem 4.19, since all the components $\dot{Q}_{\xi,\zeta}$ are centered. $\square$

Notice that the condition "$2^\kappa \geq \lambda$ for some $\kappa < \theta_4$" in (ii) of Theorem 4.20 is incompatible with GCH in general, which would be a problem for applying the methods to force Cichoń's maximum in the following sections. This requirement can be weakened to a condition compatible with GCH, as below. Recall the

poset $\mathrm{Fn}_{<\kappa}(I, B)$ of partial functions from I into B of size $<\kappa$, ordered by $\supseteq$.

Theorem 4.21 ([25])**.** *The conclusion of Theorem 4.20 is valid for the case:*

(ii') $\theta_3^- < \theta_4$, $\theta^{\aleph_0} < \theta_4$ *for every cardinal* $\theta < \theta_4$, *and* $\kappa^{<\kappa} = \kappa$ *for some uncountable cardinal* $\kappa < \theta_4$.

Proof. Let V' be a generic extension of $\mathrm{Fn}_{<\kappa}(\lambda, \kappa)$. Notice that $2^\kappa \geq \lambda$ holds in V', and that the assumptions of this theorem are preserved. Then, the construction of Theorem 4.20 can be executed in V'. However, it is possible to construct the iteration $\mathbb{P} = \mathbb{P}_{\lambda+\lambda}$ in V even though the set of guardrails and the sequences of (names of) ultrafilters live in V'. The book-keeping can be executed in V, and the strong unbounded families live in $V^{\mathbb{P}}$, so $V^{\mathbb{P}}$ forces the desired conclusion. $\square$

Remark 4.22. Theorem 4.20 has been improved in [12] without need of the requirements (i) and (ii), even obtaining a model where $\theta_4 \leq \mathrm{cov}(\mathcal{M}) = \theta_5 \leq \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c}$ for a given regular θ_5. The construction comes from a two-dimensional iteration with ultrafilters.

A method preceding ultrafilter-limits, which is more powerful, is the method of iterations with finitely additive measures (fams). Shelah [49] introduced this method for random forcing to prove the consistency of ZFC with $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \omega$, and it was formalized in [37] with applications in Cichoń's diagram. Recently, Andrés Uribe-Zapata [50] formalized the general framework of iterations with fam-limits, which was further refined with Cardona and the author in [18].

Definition 4.23. Let $\mathbb{P}$ be a poset.

(1) Let $\Xi\colon \mathcal{P}(\omega) \to [0, 1]$ be a fam (with $\Xi(\omega) = 1$ and $\Xi(\{n\}) = 0$ for all $n < \omega$), $I = \langle I_n\colon n < \omega\rangle$ be a partition of ω into finite sets, and $\varepsilon > 0$.

A set $Q \subseteq \mathbb{P}$ is (Ξ, I, ε)-*linked* if there is a function $\lim\colon Q^\omega \to \mathbb{P}$ and a $\mathbb{P}$-name $\dot{\Xi}'$ of a fam on $\mathcal{P}(\omega)$ extending Ξ such that, for any $\bar{p} = \langle p_\ell\colon \ell < \omega\rangle \in Q^\omega$,
$$\lim \bar{p} \Vdash \int_\omega \frac{|\{\ell \in I_k\colon p_\ell \in \dot{G}\}|}{|I_k|} d\dot{\Xi}' \geq 1 - \varepsilon.$$

(2) The poset $\mathbb{P}$ is μ-*FAM-linked*, witnessed by $\langle Q_{\alpha,\varepsilon}\colon \alpha < \mu,\ \varepsilon \in (0, 1) \cap \mathbb{Q}\rangle$, if:

    (i) Each $Q_{\alpha,\varepsilon}$ is (Ξ, I, ε)-linked for any Ξ and I.

    (ii) For $\varepsilon \in (0, 1) \cap \mathbb{Q}$, $\bigcup_{\alpha < \omega} Q_{\alpha,\varepsilon}$ is dense in $\mathbb{P}$.

(3) The poset $\mathbb{P}$ is *uniformly* μ-*FAM-linked* if there is some $\langle Q_{\alpha,\varepsilon}\colon \alpha < \mu,\ \varepsilon \in (0, 1) \cap \mathbb{Q}\rangle$ as above, such that in (1) the name $\dot{\Xi}'$ only depends on Ξ (and not on any $Q_{\alpha,\varepsilon}$).

Example 4.24. The following are examples of the previous notions.

(1) Any singleton is (Ξ, I, ε)-linked. Hence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$-FAM-linked. In particular, Cohen forcing is uniformly σ-FAM-linked.

(2) Shelah [49] proved, implicitly, that random forcing is uniformly σ-FAM-linked. More generally, any measure algebra with Maharam type μ is uniformly μ-FAM-linked [44].

(3) The creature ccc forcing from [30] adding eventually different reals is (uniformly) σ-FAM-linked. This is proved in [37] and also in [43] in a more general setting.

Iterations with fam limits can be constructed as in Definition 4.15. For all the details, see [50, 18]. These can be used to prove:

Theorem 4.25 ([37]). *Let $\aleph_1 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ be regular cardinals, and assume λ is a cardinal such that $\lambda = \lambda^{\aleph_0}$ and $\mathrm{cf}([\lambda]^{<\theta_i}) = \lambda$ for $i = 1, \ldots, 4$. Further assume that one of the following holds:*

(i) *$\theta_2 = \theta_3$.*

(ii) *$\theta_2^- < \theta_3$, $\theta^{\aleph_0} < \theta_i$ for every cardinal $\theta < \theta_i$ and $i \in \{3, 4\}$, and $\kappa = \kappa^{<\kappa}$ for some uncountable cardinal $\kappa < \theta_3$.*

Then, we can construct a FS iteration of length (and size) λ of ccc posets forcing $\mathrm{add}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \theta_2$, $\mathrm{cov}(\mathcal{N}) = \theta_3$, $\mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{M}) = \theta_4$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$ (see Figure 9).
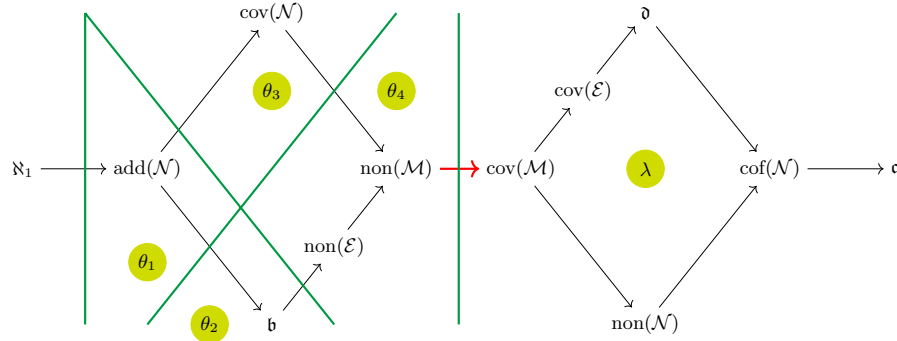

Figure 9: The constellation of Cichoń's diagram forced in Theorem 4.25.

Remark 4.26. Several extensions of Theorem 4.21 and Theorem 4.25 (and Cichoń's Maximum) have been obtained by separating more cardinals simultaneously, like $\mathfrak{m}$, $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$ and $\mathfrak{r}$ [26, 27, 29]. Very recently, Yamazoe [52] developed iterations with *closed ultrafilter limits* to separate $\mathfrak{c}$ (and its dual) in additon. He defines the linkedness notion $\Lambda_{cD}^{\lim}$ and $\Lambda_{\mathrm{cuf}}^{\lim}$ as a variation of Definition 4.8

where $\lim^D \colon Q^\omega \to Q$ (i.e., Q is closed under the ultrafilter limit), modifies the iteration theory with ultrafilters for *closed-ultrafilter limits* (i.e. Definition 4.15 for $\Lambda_{\mathrm{cuf}}^{\lim}$), and proves that closed-ultrafilter limits help control $\mathfrak{c}$.

Preceding Yamazoe's work, the author with Goldstern, Kellner and Shelah [38] claimed the same result, using that fam-limits help to control $\mathfrak{c}$ (and its dual). For their forcing iteration costruction, they claimed that ultrafilter-limits at limit steps (of countable cofinality) of the iteration can be obtained by weaking the "centered" requirement for $\dot{Q}_{\xi, h(\xi)}$ in Theorem 4.19. However, this way to proceed is still unclear, which represents a hole in their argument.

As a follow up, the author with Cardona [16, 18] proved that fam-limits help to control $\mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{E})$.

5. Boolean ultrapowers

Goldstern, Kellner and Shelah [25] proved that applying Boolean ultrapowers to the model from Theorem 4.21 yields a ccc poset that forces Cichoń's maximum.

The effect of Boolean ultrapowers to Tukey connections between relational systems is very relevant to this work. Indeed, we see how the main theorems proved in the previous sections can be reformulated in terms of Tukey connections. First notice:

Lemma 5.1. *Let K be a relational system, κ be an uncountable regular cardinal and let $\mathbb{P}$ be a κ-cc poset.*

(a) *If $\kappa \leq \mathfrak{b}(K)^V$ then $\mathbb{P}$ forces $\mathfrak{b}(K) = \mathfrak{b}(K)^V$.*

(b) *If $\kappa \leq \mathfrak{d}(K)^V$ then $\mathbb{P}$ forces $\mathfrak{d}(K) = \mathfrak{d}(K)^V$.*

(c) *If $\kappa \leq \theta \leq \lambda$ are cardinals, then $\mathbb{P}$ forces $[\lambda]^{<\theta} \cong_{\mathrm{T}} [\lambda]^{<\theta} \cap V$ and $\mathbf{C}_{[\lambda]^{<\theta}} \cong_{\mathrm{T}} \mathbf{C}_{[\lambda]^{<\theta} \cap V}$.*

Due to (c), since we mainly work with ccc forcing extensions, we can use $[\lambda]^{<\theta}$ with θ uncountable without specifying the model we are looking at.

Lemma 5.2 ([19, Fact 3.8]). *Let $\theta \leq \lambda$ be infinite cardinals. Then $\mathbf{C}_{[\lambda]^{<\theta}} \cong_{\mathrm{T}} [\lambda]^{<\theta}$ iff θ is regular and $\mathrm{cof}([\lambda]^{<\theta}) = \lambda$.*

Lemma 5.3. *For any relational system $R = \langle X, Y, \sqsubset \rangle$, if $\mathfrak{b}(R)$ exists, then $R \preceq_{\mathrm{T}} \mathbf{C}_{[X]^{<\mathfrak{b}(R)}}$.*

Proof. Use the maps $x \mapsto x$ and $B \in [X]^{<\mathfrak{b}(R)} \mapsto y_B$ where y_B is an upper bound of B. $\square$

Also recall from Lemma 3.2 that $\mathbf{C}_{[I]^{<\theta}} \preceq_{\mathrm{T}} R$ iff there is some θ-R-unbounded family indexed by I. As a consequence, we can reformulate Theorem 4.21:

Theorem 5.4. *Let $\aleph_1 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ be regular cardinals, and assume λ is a cardinal such that $\lambda = \lambda^{\aleph_0}$ and $\mathrm{cf}([\lambda]^{<\theta_i}) = \lambda$ for $i = 1, \ldots, 4$. Further assume that one of the following holds:*

(i) $\theta_3 = \theta_4$.

(ii) $\theta_3^- < \theta_4$, $\theta^{\aleph_0} < \theta_4$ for every cardinal $\theta < \theta_4$, and $\kappa = \kappa^{<\kappa}$ for some uncountable cardinal $\kappa < \theta_4$.

Then, we can construct a FS iteration of length (and size) λ of ccc posets forcing $\mathfrak{c} = \lambda$, $\mathcal{N} \cong_T \mathbf{C}_{[\lambda]^{<\theta_1}}$, $\mathbf{C}_\mathcal{N}^\perp \cong_T \mathbf{C}_{[\lambda]^{<\theta_2}}$, $\omega^\omega \cong_T \mathbf{C}_{[\lambda]^{<\theta_3}}$ and $\mathbf{C}_\mathcal{M} \cong_T \mathbf{C}_\mathcal{E} \cong_T \mathbf{C}_{[\lambda]^{<\theta_4}}$ and, for $i = 1, \ldots, 4$, $\mathbf{C}_{[\lambda]^{<\theta_i}} \cong_T \mathbf{C}_{[\lambda]^{<\theta_i} \cap V} \cong_T [\lambda]^{<\theta_i} \cap V \cong_T [\lambda]^{<\theta_i}$.

Tukey connections have certain types of witnesses as follows.

Lemma 5.5. *Let $R = \langle X, Y, \sqsubset \rangle$ and $K = \langle A, B, \lhd \rangle$ be relational systems. Then:*

(a) *$R \preceq_T K$ iff there is some $\langle y_b : b \in B \rangle \subseteq Y$ satisfying:*

$$\forall x \in X \, \exists a \in A \, \forall b \in B : a \lhd b \Rightarrow x \sqsubset y_b.$$

(b) *$K \preceq_T R$ iff there is some $\langle x_a : a \in A \rangle \subseteq X$ satisfying:*

$$\forall y \in Y \, \exists b \in B \, \forall a \in A : a \not\lhd b \Rightarrow x_a \not\sqsubset y.$$

In the context of the previous lemma, the role of K is a constant relational system which is not interpreted (in principle) in forcing generic extensions, i.e. we keep its meaning with respect to V. On the other hand, R is interpreted according to its definition.

Elementary embeddings modify Tukey connections as follows. Recall that a model M is *$<\kappa$-closed* if it is closed under sequences of length $<\kappa$.

Lemma 5.6. *Let θ be an uncountable regular cardinal and $\mathbb{P}$ a θ-cc poset. Let $M \subseteq V$ be a $<\theta^+$-closed transitive model, $j \colon V \to M$ an elementary embedding with critical point $>\theta$, $R = \langle X, Y, \sqsubset \rangle$ a relational system of the reals (Definition 2.1) and let $K = \langle A, B, \lhd \rangle$ be a relational system. Then:*

(a) *$j(\mathbb{P})$ has the θ-cc and $\mathbb{P} \sqsubset j(\mathbb{P})$.*

(b) *If $\Vdash_\mathbb{P} R \preceq_T K$ is witnessed by $\bar{y} = \langle \dot{y}_b : b \in B \rangle$, then $\Vdash_{j(\mathbb{P}), V} R \preceq_T j(K)$ is witnessed by $j(\bar{y}) = \langle \dot{y}_{b'}^* : b' \in j(B) \rangle$.*

(c) *If $\Vdash_\mathbb{P} K \preceq_T R$ is witnessed by $\bar{x} = \langle \dot{x}_a : a \in A \rangle$, then $\Vdash_{j(\mathbb{P}), V} j(K) \preceq_T R$ is witnessed by $j(\bar{x}) = \langle \dot{x}_{a'}^* : a' \in j(A) \rangle$.*

Proof. (a): If $C \subseteq j(\mathbb{P})$ has size θ, then $C \in M$ because M is $<\theta^+$-closed. Then, $M \models "C \subseteq j(\mathbb{P})"$. On the other hand, since $j \colon V \to M$ is an elementary embedding and θ is smaller than its critical point, $M \models "j(\mathbb{P})$ has the θ-cc", so C is not an antichain in $j(\mathbb{P})$ (both in M and in V). A similar argument shows that $\mathbb{P} \sqsubset j(\mathbb{P})$.

(b): Let $\dot{x}$ be a (nice) $j(\mathbb{P})$-name of a real in X. By (a), $\dot{x} \in M$. On the other hand, by elementarity, in M we have that $R \preceq_T J(K)$ is witnessed by $j(\bar{y})$,

so $\exists\, a' \in j(A)\ \forall\, b' \in j(B)\colon a'\ j(\lhd)\ b' \Rightarrow \dot{x} \sqsubset \dot{y}^*_{b'}$. The same holds in V by absoluteness, which concludes the proof.

(c) follows by (b) applied to $R^\perp$ and $K^\perp$. $\qquad\square$

Given the previous result, we plan to apply elementary embeddings to the poset of Theorem 5.4 to obtain a forcing that modifies Tukey equivalences in such a way that Cichoń's maximum is forced.

The elementary embeddings come from Boolean ultrapowers. Let κ be a strongly compact cardinal and $\mathbb{B}$ a $<\kappa$-distributive κ^+-cc complete Boolean algebra containing an antichain of size κ. Let $V_-^{\mathbb{B}}$ be the class of nice $\mathbb{B}$-names of members of the ground model V, i.e. they are coded by a function $h\colon A \to V$ with domain a maximal antichain A in $\mathbb{B}$. Given a κ-complete ultrafilter D on $\mathbb{B}$, define the following relations on $V_-^{\mathbb{B}}$:

$$\sigma =_D \tau \text{ iff } \|\sigma = \tau\| \in D,$$
$$\sigma \in_D \tau \text{ iff } \|\sigma \in \tau\| \in D.$$

It is easy to show that $=_D$ is an equivalence relation on $V_-^{\mathbb{B}}$, which defines the *Boolean ultrapower* $\mathrm{BUP}(\mathbb{B}, D) := V_-^{\mathbb{B}}/=_D$. The relation obtained from $\in_D$ makes $\mathrm{BUP}(\mathbb{B}, D)$ a well-founded model of ZFC, and $\iota\colon V \to \mathrm{BUP}(\mathbb{B}, D)$ is an elementary embedding, where $\iota(x)$ is the equivalence class of $\check{x} = \{(\check{y}, 1_{\mathbb{B}})\colon y \in x\}$. Let $c\colon \mathrm{BUP}(\mathbb{B}, D) \to M$ be the Mostowski collapse of $\mathrm{BUP}(\mathbb{B}, D)$. Then, $j := c \circ \iota\colon V \to M$ is an elementary embedding. The class M is a $<\kappa$-closed (even $<\kappa^+$-closed) transitive model of ZFC. Details can be found in [36].

Lemma 5.7 (cf. [36])**.** *Let κ be a strongly compact cardinal and $\lambda \geq \kappa$ a cardinal such that $\lambda^\kappa = \lambda$. Then, there is a κ-complete ultrafilter on the completion of $\mathrm{Fn}_{<\kappa}(\lambda, \kappa)$ such that its corresponding elementary embedding $j\colon V \to M$ satisfies:*

(a) *M is $<\kappa$-closed (and even $<\kappa^+$-closed).*

(b) *The critical point of j is κ.*

(c) *If $|a| < \kappa$ then $j(a) = j[a]$.*

(d) *If $\lambda' \geq \kappa$ then $\max\{\lambda, \lambda'\} \leq |j(\lambda')| \leq \max\{\lambda, \lambda'\}^\kappa$.*

(e) *If S is a directed preorder and $\mathfrak{b}(S) > \kappa$, then $j[S]$ is cofinal in $j(S)$. In particular, $S \cong_{\mathrm{T}} j(S)$.*

Corollary 5.8. *Under the framework of Lemma 5.7: if $\theta \leq \lambda$ are infinite cardinals, then:*

(a) *If $\theta < \kappa$ then $j([\lambda]^{<\theta}) = [j(\lambda)]^{<\theta}$.*

(b) *If $\mathrm{cf}(\theta) > \kappa$ then $j([\lambda]^{<\theta}) \cong_{\mathrm{T}} [\lambda]^{<\theta}$.*

We have developed enough material to prove Cichoń's maximum, first by using large cardinals.

Theorem 5.9. *Assume that* $\aleph_1 < \kappa_0 < \theta_1 < \kappa_1 < \theta_2 < \kappa_2 < \theta_3 < \kappa_3 < \theta_4 \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq \lambda_0$ *such that each* θ_i *is regular,* $\mathrm{cf}([\lambda_4]^{<\theta_i}) = \lambda_4 = \lambda_4^{\aleph_0}$, *condition* (ii) *of Theorem 5.4 holds,* κ_ℓ *is strongly compact and* $\lambda_\ell^{\kappa_\ell} = \lambda_\ell$ *for* $\ell \leq 3$. *Then there is a ccc poset of size* λ_0 *forcing* $\mathfrak{c} = \lambda_0$, $\mathcal{N} \cong_T \mathbf{C}_{[\lambda_1]^{<\theta_1}}$, $\mathbf{C}_{\mathcal{N}}^{\perp} \cong_T \mathbf{C}_{[\lambda_2]^{<\theta_2}}$, $\omega^\omega \cong_T \mathbf{C}_{[\lambda_3]^{<\theta_3}}$ *and* $\mathbf{C}_{\mathcal{M}} \cong_T \mathbf{C}_{\mathcal{E}} \cong_T \mathbf{C}_{[\lambda_4]^{<\theta_4}}$. *In particular (see Figure 10),*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) = \theta_1 < \mathrm{cov}(\mathcal{N}) = \theta_2 < \mathfrak{b} = \theta_3 < \mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{M}) = \theta_4$$
$$\leq \mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{E}) = \lambda_4 \leq \mathfrak{d} = \lambda_3 \leq \mathrm{non}(\mathcal{N}) = \lambda_2 \leq \mathrm{cof}(\mathcal{N}) = \lambda_1$$
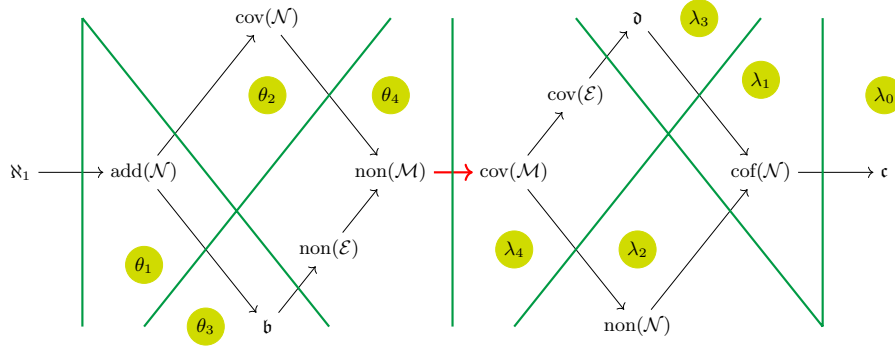$$\leq \mathfrak{c} = \lambda_0.$$

Figure 10: The constellation of Cichoń's maximum forced in Theorem 5.9.

The previous theorem is originally from [25] under the restriction that all λ_i are regular. In joint work of the author with Goldstern, Kellner, and Shelah, we discovered Lemma 5.6 and realized that all λ_i are allowed to be singular.

Proof. Let $\mathbb{P}_4$ be the ccc poset obtained in Theorem 5.4 for $\lambda = \lambda_4$. For each $\ell \leq 3$ let $j_\ell \colon V \to M_\ell$ be an elementary embedding as in Lemma 5.7 for $\kappa = \kappa_\ell$ and $\lambda = \lambda_\ell$, and define $\mathbb{P}_\ell := j_\ell(\mathbb{P}_{\ell+1})$ (by recursion). By using Lemma 5.6 and Corollary 5.8, we show that $\mathbb{P}_\ell$ forces the Tukey-equivalences and the value of $\mathfrak{c}$ as illustrated in Table 1. In particular, $\mathbb{P}_0$ is the desired ccc poset.

By Theorem 5.4, $\mathbb{P}_4$ forces the Tukey-equivalences and $\mathfrak{c} = \lambda_4$ as illustrated in Table 1. Then, by Lemma 5.6, $\mathbb{P}_3 = j_3(\mathbb{P}_4)$ has the ccc and it forces Tukey-equivalences with $j_3([\lambda_4]^{<\theta_i})$ for $1 \leq i \leq 4$. Since $\theta_4 > \kappa_3$, by Corollary 5.8 (b), $j_3([\lambda_4]^{<\theta_4}) \simeq_T [\lambda_4]^{<\theta_4}$; and for $1 \leq i \leq 3$, since $\theta_i < \kappa_3$, by Corollary 5.8 (a), $j_3([\lambda_4]^{<\theta_i}) = [j_3(\lambda_4)]^{<\theta_i}$. On the other hand, by Lemma 5.7, $|j_3(\lambda_4)| = \lambda_3^{\kappa_3} = \lambda_3$, so $j_3([\lambda_4]^{<\theta_i}) \cong_T [\lambda_3]^{<\theta_i}$. Finally, by elementarity, in M_3, $\mathbb{P}_3$ forces $\mathfrak{c} = |j_3(\lambda_4)| = \lambda_3$, which is also forced in V because $j_3(\mathbb{P}_3)$ has the ccc and M_3 is countably closed (so any $\mathbb{P}_3$-name of a real is in M_3).

Similarly, by using the elementary embedding j_2, $\mathbb{P}_2 = j_2(\mathbb{P}_3)$ has the ccc, $j_2([\lambda_i]^{<\theta_i}) \cong_T [\lambda_i]^{<\theta_i}$ for $i = 3, 4$, $j_2([\lambda_3]^{<\theta_i}) = [j_2(\lambda_3)]^{<\theta_i} \cong_T [\lambda_2]^{<\theta_i}$ for $i = 1, 2$, and $\mathbb{P}_2$ forces $\mathfrak{c} = j_2(\lambda_3) = \lambda_2$. Proceeding in the same way using j_1 and j_0, we can check the remaining information collected in Table 1. $\qquad\square$

	$\mathbb{P}_4$	$\mathbb{P}_3$	$\mathbb{P}_2$	$\mathbb{P}_1$	$\mathbb{P}_0$
$\mathbf{C}_{\mathcal{M}},\ \mathbf{C}_{\mathcal{E}}$	$[\lambda_4]^{<\theta_4}$	$[\lambda_4]^{<\theta_4}$	$[\lambda_4]^{<\theta_4}$	$[\lambda_4]^{<\theta_4}$	$[\lambda_4]^{<\theta_4}$
ω^ω	$[\lambda_4]^{<\theta_3}$	$[\lambda_3]^{<\theta_3}$	$[\lambda_3]^{<\theta_3}$	$[\lambda_3]^{<\theta_3}$	$[\lambda_3]^{<\theta_3}$
$\mathbf{C}_{\mathcal{N}}^{\perp}$	$[\lambda_4]^{<\theta_2}$	$[\lambda_3]^{<\theta_2}$	$[\lambda_2]^{<\theta_2}$	$[\lambda_2]^{<\theta_2}$	$[\lambda_2]^{<\theta_2}$
$\mathcal{N}$	$[\lambda_4]^{<\theta_1}$	$[\lambda_3]^{<\theta_1}$	$[\lambda_2]^{<\theta_1}$	$[\lambda_1]^{<\theta_1}$	$[\lambda_1]^{<\theta_1}$
$\mathfrak{c}$	λ_4	λ_3	λ_2	λ_1	λ_0

Table 1: This table illustrates that the forcing on the top forces a Tukey-equivalence with the relational systems in the first column for the first four rows, and the value it forces to the continuum in the last row.

In the same way, we can apply Boolean ultrapowers to the construction of Theorem 4.25 to force an alternative order of Cichoń's maximum. The result is originally from [37] for λ_i regular, but our methods allow them to be singular.
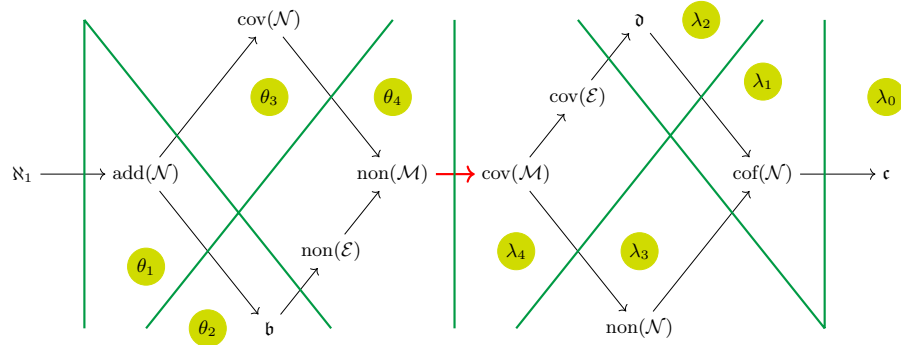

Figure 11: The constellation of Cichoń's maximum forced in Theorem 5.10.

Theorem 5.10. *Assume that* $\aleph_1 < \kappa_0 < \theta_1 < \kappa_1 < \theta_2 < \kappa_2 < \theta_3 < \kappa_3 < \theta_4 \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq \lambda_0$ *such that each* θ_i *is regular,* $\mathrm{cf}([\lambda_4]^{<\theta_i}) = \lambda_4$, *condition* (ii) *of Theorem 4.25 holds,* κ_ℓ *is strongly compact and* $\lambda_\ell^{\kappa_\ell} = \lambda_\ell$ *for* $\ell \leq 3$. *Then there is a ccc poset of size* λ_0 *forcing* $\mathfrak{c} = \lambda_0$, $\mathcal{N} \cong_T \mathbf{C}_{[\lambda_1]^{<\theta_1}}$, $\omega^\omega \cong_T \mathbf{C}_{[\lambda_2]^{<\theta_2}}$, $\mathbf{C}_{\mathcal{N}}^{\perp} \cong_T \mathbf{C}_{[\lambda_3]^{<\theta_3}}$ *and* $\mathbf{C}_{\mathcal{M}} \cong_T \mathbf{C}_{\mathcal{E}} \cong_T \mathbf{C}_{[\lambda_4]^{<\theta_4}}$. *In particular (see Figure 11),*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) = \theta_1 < \mathfrak{b} = \theta_2 < \mathrm{cov}(\mathcal{N}) = \theta_3 < \mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{M}) = \theta_4$$
$$\leq \mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{E}) = \lambda_4 \leq \mathrm{non}(\mathcal{N}) = \lambda_3 \leq \mathfrak{d} = \lambda_2 \leq \mathrm{cof}(\mathcal{N}) = \lambda_1$$
$$\leq \mathfrak{c} = \lambda_0.$$

There are four possible constellations of Cichoń's maximum with $\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$. We have proved two of them, but the consistency of each of the other two is not known.

Question 5.11. *Is each of the constellations in Figure 12 and Figure 13 consistent with* ZFC *(even under large cardinals)?*
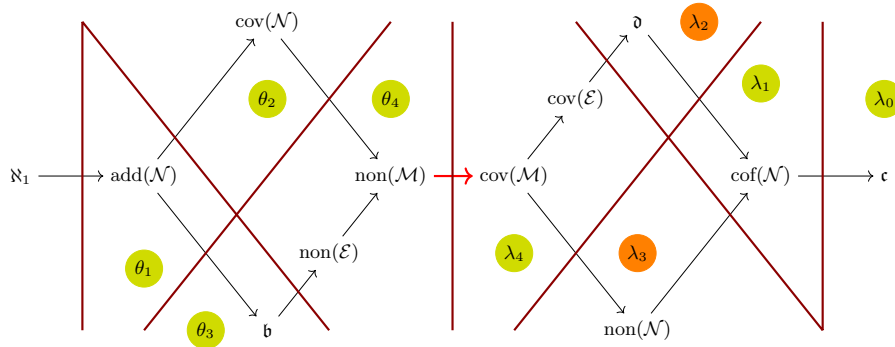

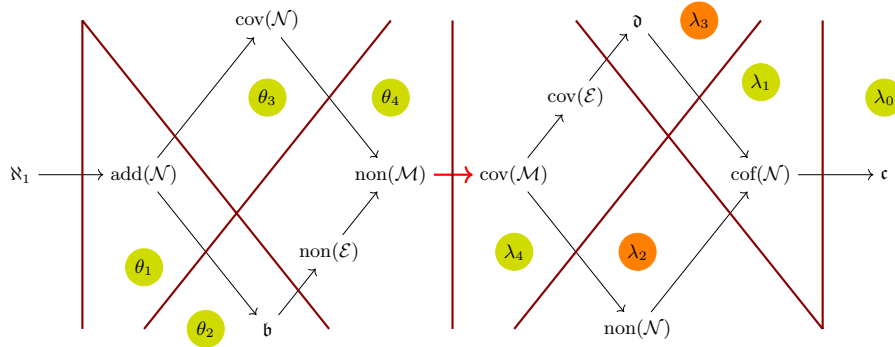Figure 12: Cichoń's maximum's constellation like in Theorem 5.9, but with the values of $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$ interchanged.

Figure 13: Cichoń's maximum's constellation like in Theorem 5.10, but with the values of $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$ interchanged.

On the other hand, no instance of Cichoń's maximum with $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M})$ is known to be consistent with ZFC.

Question 5.12. *Is Cichoń's maximum with* $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M})$ *consistent with* ZFC *(even under large cardinals)?*

To finish the paper, we briefly mention some facts around the method of intersection with submodels, originally from [28], which is used to prove the

consistency of Cichoń's maximum without using large cardinals. We only do a very brief presentation for comparison with the method of Boolean ultrapowers, but we recommend the reader to check and learn the details from [14, Sec. 4 & 5].

From now on, we fix:

(F1) an uncountable regular cardinal κ, a κ-cc poset $\mathbb{P}$;

(F2) a definable relational system $R = \langle X, Y, \sqsubset \rangle$ of the reals; and

(F3) a large enough regular cardinal χ such that $\mathbb{P} \in H_\chi$, and H_χ contains all the parameters defining R.

When intersecting a κ-cc poset with a $<\kappa$-closed model, we obtain a completely embedded subforcing.

Lemma 5.13. *If $N \preceq H_\chi$ is $<\kappa$-closed and $\mathbb{P} \in N$, then $\mathbb{P} \cap N \lessdot \mathbb{P}$.*

Note that this lemma was used implicitly in the proof of Theorem 4.20 (to define $\mathbb{P}^-_{\bar{\lambda}+\alpha}$ for $\alpha \in K_4$).

Semantically, there is a correspondence between some $\mathbb{P} \cap N$-names and $\mathbb{P}$-names belonging to N, and we can also have a correspondence for the forcing relation for some formulas.

Fact 5.14. *If N is $<\kappa$ closed then there is a one-to-one correspondence between:*

(i) $\mathbb{P}$-names $\tau \in N$ and

(ii) $\mathbb{P} \cap N$-names σ

of members of H_κ (in particular, reals). Thus, if G is $\mathbb{P}$-generic over V then $H_\kappa^{V[G]} \cap N[G] = H_\kappa^{V[G \cap N]}$.

Corollary 5.15. *For absolute $\varphi(\bar{x})$ (e.g. Borel on the reals) if $p \in \mathbb{P} \cap N$, $\bar{\tau} \in N$ is a finite sequence of $\mathbb{P}$-names of members of H_κ, and $\bar{\sigma}$ is the sequence of $\mathbb{P} \cap N$-names corresponding to $\bar{\tau}$, then*

$$p \Vdash_\mathbb{P} \varphi(\bar{\tau}) \Leftrightarrow p \Vdash_{\mathbb{P} \cap N} \varphi(\bar{\sigma}).$$

The following result illustrates the main motivation to intersect κ-cc posets with $<\kappa$-closed models, since it affects the Tukey relations forced by the posets. As before, K denotes a constant relational system from the ground model, while R is interpreted in the model of discourse.

Lemma 5.16. *Let $N \preceq H_\chi$ be $<\kappa$-closed and let $K = \langle A, B, \lhd \rangle$ be a relational system. Assume that $\mathbb{P}$, K and the parameters of R are in N.*

(a) If $\Vdash_\mathbb{P} R \preceq_T K$ then $\Vdash_{\mathbb{P} \cap N} R \preceq_T K \cap N$ where $K \cap N := \langle A \cap N, B \cap N, \lhd \rangle$. Moreover, if $\langle \dot{y}_b : b \in B \rangle$ is a sequence of $\mathbb{P}$-names of a witness of $R \preceq_T K$, then $\langle \dot{y}_b : b \in B \cap N \rangle$ is forced by $\mathbb{P} \cap N$ to witness $R \preceq_T K \cap N$.

(b) *If $\Vdash_{\mathbb{P}} K \preceq_{\mathrm{T}} R$ then $\Vdash_{\mathbb{P} \cap N} K \cap N \preceq_{\mathrm{T}} R$. Moreover, if $\langle \dot{x}_a \colon a \in A \rangle$ is a sequence of $\mathbb{P}$-names of a witness of $K \preceq_{\mathrm{T}} R$, then $\langle \dot{x}_a \colon a \in A \cap N \rangle$ is forced by $\mathbb{P} \cap N$ to witness $K \cap N \preceq_{\mathrm{T}} R$.*

Therefore, if $\mathbb{P}$ forces $R \cong_{\mathrm{T}} K$, then $\mathbb{P} \cap N$ forces $R \cong_{\mathrm{T}} K \cap N$, i.e. $\mathfrak{b}(R) = \mathfrak{b}(K \cap N)$ and $\mathfrak{d}(R) = \mathfrak{d}(K \cap N)$. This indicates that, to force different values of cardinal characteristics using intersection with submodels, one should construct an N such that $\mathfrak{b}(K \cap N)$ and $\mathfrak{d}(K \cap N)$ has the values one desires. In the case of Cichoń's diagram, one constructs a σ-closed $N \preceq H_\chi$ such that $\mathbb{P} \cap N$ forces Cichoń's maximum, where $\mathbb{P}$ is the poset obtained from Theorem 5.4 (and Theorem 4.25).

Theorem 5.17 ([28]). *Let $\aleph_1 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4 \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1$ be regular cardinals and $\lambda_0 = \lambda_0^{\aleph_0} \geq \lambda_1$. Then, for each one of the constellations in Figure 10 and Figure 11, there is a ccc poset of size λ_0 forcing it.*

We do not prove this theorem in detail but mention how the construction of N goes. For details, see [14, Sec. 5]. We pick a large enough regular cardinal $\kappa_0 > \lambda_0$ and assume, for the moment, that GCH holds above κ_0, i.e. $2^\kappa = \kappa^+$ for every cardinal $\kappa \geq \kappa_0$. We explain in Remark 5.18 how to remove this assumption.

For $j = 1, \ldots, 4$, pick a regular cardinal $\nu_j > \kappa_0$, which is the successor of another regular cardinal ν_j^-, such that $\nu_j < \nu_{j+1}^-$ for $j = 1, 2, 3$. Finally, pick a regular cardinal $\nu_\infty > \nu_4$.

To force Figure 10, we start with a forcing $\mathbb{P}$ as in Theorem 5.4 but applied to $\nu_1, \nu_2, \nu_3, \nu_4$, and ν_∞, i.e. it is a ccc poset of size ν_∞ forcing $\mathcal{N} \cong_{\mathrm{T}} [\nu_\infty]^{<\nu_1}$, $\mathbf{C}_\mathcal{N}^\perp \cong_{\mathrm{T}} [\nu_\infty]^{<\nu_2}$, $\omega^\omega \cong_{\mathrm{T}} [\nu_\infty]^{<\nu_3}$, $\mathbf{C}_\mathcal{M} \cong_{\mathrm{T}} \mathbf{C}_\mathcal{E} \cong_{\mathrm{T}} [\nu_\infty]^{<\nu_4}$, and $\mathfrak{c} = \nu_\infty$. Recall that, for $1 \leq j \leq 4$, $\mathfrak{b}([\nu_\infty]^{<\nu_j}) = \nu_1$ and $\mathfrak{d}([\nu_\infty]^{<\nu_j}) = \nu_\infty$.

Note that $\mathbb{P}$ forces the values in Cichoń's diagram as in the top part of Figure 14. Afterward, one constructs a σ-closed model N such that $\mathbb{P} \cap N$ forces the constellation at the bottom of Figure 14.

One constructs a sequence of elementary submodels of H_χ, containing all the relevant information, as below:

$$N^{\mathfrak{c}} \in N_1^{\mathfrak{b}} \in N_1^{\mathfrak{d}} \in N_2^{\mathfrak{b}} \in N_2^{\mathfrak{d}} \in N_3^{\mathfrak{b}} \in N_3^{\mathfrak{d}} \in N_4^{\mathfrak{b}} \in N_4^{\mathfrak{d}}.$$

For $1 \leq j \leq 4$, $N_j^{\mathfrak{d}}$ is constructed from a $\in$-increasing sequence of length λ_j of $<\nu_j$-closed elementary submodels of H_χ of size ν_j, and $N_j^{\mathfrak{b}}$ is constructed from a $\in$-increasing sequence of length θ_j of $<\nu_j^-$-closed elementary submodels of H_χ of size ν_j^-. The models are constructed by decreasing recursion on j, first $N_j^{\mathfrak{d}}$, and $N_j^{\mathfrak{b}}$ afterwards. At the end, $N^{\mathfrak{c}}$ is chosen as a σ-closed elementary submodel of H_χ of size λ_0.

Finally, $N := N^{\mathfrak{c}} \cap \bigcap_{j=1}^4 N_j^{\mathfrak{b}} \cap N_j^{\mathfrak{d}}$. For $1 \leq i \leq 4$, each model $\bigcap_{j=i}^4 N_j^{\mathfrak{b}} \cap N_j^{\mathfrak{d}}$, after intersection with $\mathbb{P}$, collapses the values forced at the top of Figure 14 to the values at the bottom, taking care of θ_j, λ_j for $i \leq j \leq 4$ in particular. The final intersection with $N^{\mathfrak{c}}$ is to force that the continuum is λ_0 and to conclude
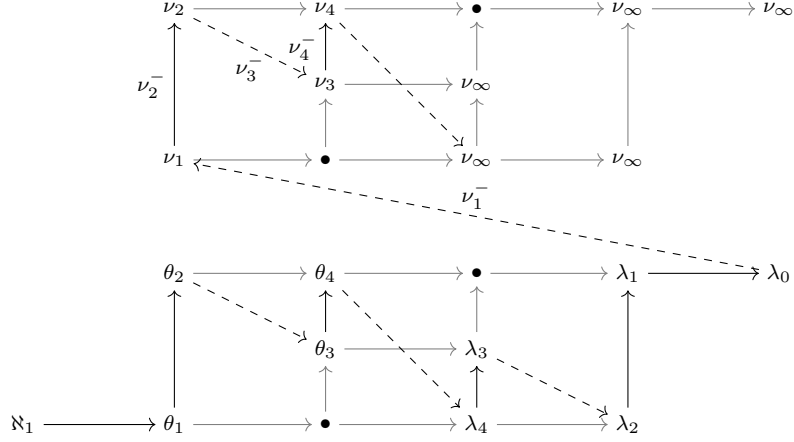
Figure 14: Strategy to force Cichoń's maximum: we construct a ccc poset $\mathbb{P}$ forcing the constellation at the top, and find a σ-closed model N such that $\mathbb{P} \cap N$ forces the constellation at the bottom.

$\mathfrak{b}([\nu_\infty]^{<\nu_i} \cap N) = \theta_i$ and $\mathfrak{d}([\nu_\infty]^{<\nu_i} \cap N) = \lambda_i$ for $1 \leq i \leq 4$. Moreover, it can be proved that $[\nu_\infty]^{<\nu_i} \cap N \cong_{\mathrm{T}} \prod_{j=i}^{4} \theta_j \times \lambda_j$ (with the coordinate-wise order).

Remark 5.18. The method from [28] (and [14]) uses *eventual GCH* (i.e. for some cardinal κ_0, $2^\kappa = \kappa^+$ for every cardinal $\kappa \geq \kappa_0$) as hypothesis to force Cichoń's maximum. However, thanks to an observation from Elliot Glazer (private communication), this assumption can be removed: Let κ^* be a large enough regular cardinal, and let W be a set of ordinals coding R_{κ^*}, the κ^*-th level of the Von Neumann hierarchy of the universe of sets. Since $L[W]$ models ZFC with eventual GCH, working inside $L[W]$ we can find a ccc poset $\mathbb{Q}$ forcing Cichoń's maximum, which has size λ_0, and hence, inside R_α for some α relatively small with respect to κ^*. As κ^* is *large enough*, we actually have that the collection of nice $\mathbb{Q}$-names of reals are in R_{κ^*}, so R_{κ^*}, and hence V, satisfies that $\mathbb{Q}$ is as required.

Remark 5.19. In contrast with the method of Boolean ultrapowers, we do not know how this method can be improved to allow λ_i singular for $1 \leq i \leq 4$.

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