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Geometric and topological characteristics of the n^{th}-order lemniscate

Características geométricas y topológicas de la lemniscata de n-ésimo orden

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ABSTRACT. The lemniscate of Bernoulli was, in a sense, what paved the way for modern Elliptic Function Theory. This curve can be generalized in the following way: $\mathscr{L}_n = |(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)| = r^n, r \in \mathbb{R}, \zeta_i \in \mathbb{C}$. In this paper, this generalized curve is meticulously studied when $r = 1$, and ζ_i is a n^{th}-root of the unity, which we call the n^{th}-order lemniscate. In the first section, the historical background of this curve is presented. In the second section, an analytic description of tangent lines and the singularity (in real plane $\mathbb{R}^2$) is presented together with a study of curvature, Schwarz function, and Joukowski maps applied to our curve. Finally, in the third section, calculations of some topological and geometric invariants (in the complex-projective plane $\mathbb{CP}^2$) are shown.

***Key words*:** n^{th}-order lemniscate, Schwarz function, Joukowski maps, ramification points, genus.

RESUMEN. La lemniscata de Bernoulli abrió las puertas al desarrollo de la teoría de funciones elípticas por propiedades geométricas elementales intrínsecas a la curva. Esta curva se puede generalizar como: $\mathscr{L}_n = |(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)| = r^n$, $r \in \mathbb{R}, \zeta_i \in \mathbb{C}$. En este artículo, esta curva es estudiada con detalle para el caso cuando $r = 1$ y ζ_i es una raíz n-ésima de la unidad, a la cual llamamos la lemniscata de orden n. En la primera sección, se presenta el contexto histórico de esta curva. En la segunda sección, se presenta una descripción analítica de las rectas tangentes y la singularidad de la curva (en el plano real $\mathbb{R}^2$) junto con un estudio de la curvatura, la función de Schwarz y los mapas de Joukowski aplicados a nuestra curva. Finalmente, en la tercera sección, se calculan algunos invariantes topológicos y geométricos (en el plano proyectivo complejo $\mathbb{CP}^2$).

***Palabras clave*:** lemniscata de orden n, función de Schwarz, mapas de Joukowski, punto de ramificación, género.

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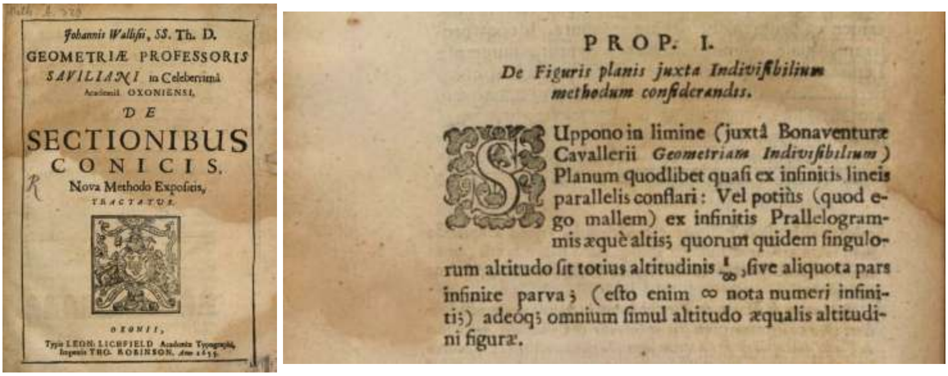
1 Introduction

Since antiquity, the notion of "infinity" has played a central role in the history of human thought. Aristotle, in his famous book *Physics*, exhibits the necessity of investigating whether infinity exists or not, and if it exists, what it is. Nevertheless, without a proper understanding of it, according to Aristotle, the Pythagoreans and geometers started to use this concept. It was not until the middle of the seventeenth century that the contemporary symbol was introduced. The symbol ∞ was first used by John Wallis in *De sectionibus conics* (Fig. 1). The choice of the symbol was unjustified, but very appropriate.



Johannis Wallisi, SS. Th. D.
GEOMETRIÆ PROFESSORIS
SAVILIANI in Celeberrimâ
Academiâ OXONIENSI,
DE
SECTIONIBUS
CONICIS,
Nova Methodo Expositis,
TRACTATUS.
OXONII,

PROP. I.

De Figuris planis juxta Indivisibilium methodum considerandis.

Suppono in limine (juxtà Bonaventuræ Cavallerii *Geometriam Indivisibilium*) Planum quodlibet quasi ex infinitis lineis parallelis conflari: Vel potiùs (quod ego mallem) ex infinitis Prallelogrammis æquè altis; quorum quidem singulorum altitudo sit totius altitudinis $\frac{1}{\infty}$, sive aliquota pars infinitè parva; (esto enim ∞ nota numeri infiniti;) adeóq; omnium simul altitudo æqualis altitudini figuræ.

Figure 1. Cover of *De sectionibus conics* (1655) and first documented appearance of ∞. Taken from archive.org.

The pertinence of the symbol lies in the fact that it resembles a curve known as the lemniscate (*Lemniscus*) of Bernoulli. While Wallis was studying conic sections, the curve was initially described by Perseus, a Greek geometer, as a toric section. Toric sections are the intersections of a torus with a plane parallel to the rotation axis of the torus.

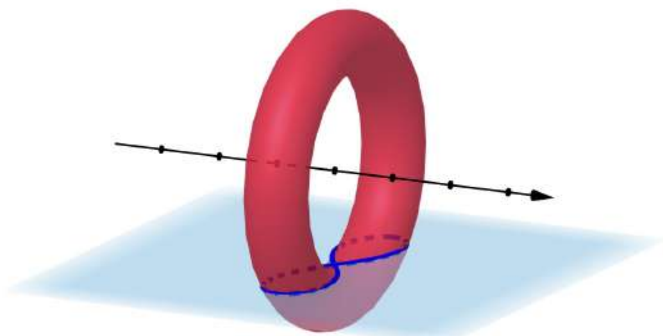


Figure 2. Lemniscate as toric section

One example of toric sections was proposed by the Italian astronomer Giovanni Domenico Cassini (1625-1712) in 1680. In an effort to describe the Sun's trajectory, Cassini fixed two points in the plane: f_1 and f_2 –called focal points– and considered the locus of all points p such that the product of its distance to these two points was constant, *i.e.,* points for which $|pf_1| \cdot |pf_2| = k$, where k is a positive constant. Each value of k determines a trajectory. Cassini believed that the Sun traveled around the Earth (located at one of the focal points) through one of those trajectories. Considering a torus with minor radius r and major radius R, the toric section corresponds to the lemniscate (Fig. 2) –as one of the Cassini trajectories– exactly when $r = \dfrac{R}{2}$. In this case, the distance between the focal points is $2R$ and the distance between the intersection plane and the rotation axis is $R - r$ (Fig. 3).

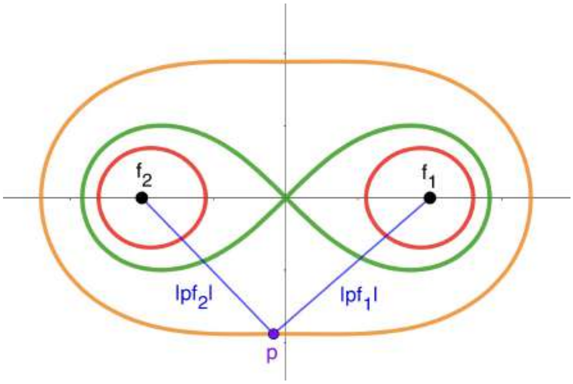


Figure 3. Lemniscate as Cassini's trajectory

Later, Jacob Bernoulli (1654-1705) in 1694 took up this work, but with a more analytical perspective. Bernoulli proved that the arc length of the lemniscate was given by the elliptic integral:

$$\int_0^x \frac{1}{\sqrt{1-t^4}}dt.$$

This result inspired a series of arguments that allowed the rigorous advance of Elliptic Function Theory. A remarkable case of this progress was the addition theorem by Leonhard Euler (1707-1783). Those results were motivated by Giovanni Fagnano's works in 1718 to double the arc length of the lemniscate. In fact, Carl G. J. Jacobi (1804-1851) named December 23, 1751 –the day Euler received Fagnano's work– as "the birthday of Elliptic Function Theory" [22, p. 232].

As we have seen, the lemniscate can be ascribed to a broad family of curves (toric sections, Cassini's trajectories). However, the lemniscate can also be thought of as a curve of two petals. In what follows, we denote it by $\mathscr{L}_2$. A naive generalization of $\mathscr{L}_2$ gives us a curve with n petals denoted by $\mathscr{L}_n$ (we will formalize this shortly).

In this paper, we study some of the main results of [12] and [13] regarding $\mathscr{L}_2$ and $\mathscr{L}_3$ in order to generalize some techniques and results to $\mathscr{L}_n$. Our presentation is divided into two sections. Each section is dedicated to the study of geometric and topological characteristics of $\mathscr{L}_n$ in the ambient space $\mathbb{R}^2$ and $\mathbb{CP}^2$, respectively. Both sections are part of the author's undergraduate thesis at Los Andes University. All results presented here are, to our best knowledge, new contributions to the subject, unless explicitly stated otherwise. The author would like to thank his advisor Alexander Getmanenko for his many helpful insights and his friendly encouragement.

2 Real geometry of $\mathscr{L}_n$

2.1 Analytic description of $\mathscr{L}_n$

Definition 1. Let $c \in (0, \infty)$ be a positive number, and let $f_1 = (c, 0)$ and $f_2 = (-c, 0)$ two points –which we call *focal points*[1]. If $p = (x, y)$ and $k > 0$, then, by the Pythagorean theorem, the locus such that $|pf_1| \cdot |pf_2| = k$ is given by

$$(y^2 + (x + c)^2)(y^2 + (x - c)^2) = k^2,$$

or equivalently by

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = k^2 - c^4. \tag{1}$$

This family of curves (parametrized by k) is the *Cassini's curves*.

Definition 2. The *lemniscate of Bernoulli* $\mathscr{L}_2$ is the Cassini's curve when $k^2 - c^4 = 0$.

Hence, $\mathscr{L}_2$ has Cartesian equation:

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = 0. \tag{2}$$

It can be shown that in the polar coordinates (r, θ) of $\mathbb{R}^2$ the equation (2) is equivalent to

$$r^2 = 2c^2 \cos(2\theta). \tag{3}$$

Without loss of generality, we take $c = 1$. The equation (3) is convenient because it is easily generalizable.

Definition 3. The *lemniscate of n^{th}-order* $\mathscr{L}_n$ is the curve with polar and Cartesian equations

$$r^n = 2\cos(n\theta), \tag{4}$$

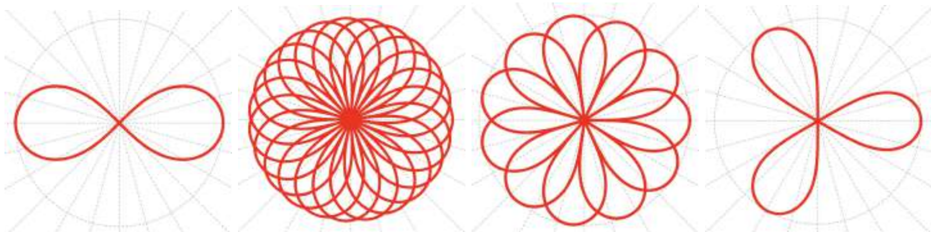
$$(x^2 + y^2)^n = 2 \sum_{\substack{k=0 \\ k \text{ even}}}^{n} (-1)^{k/2} \binom{n}{k} x^{n-k} y^k, \tag{5}$$

respectively.

Note that $\mathscr{L}_n$ is a curve of degree $2n$. This analytic description of $\mathscr{L}_n$ delineates the object under investigation here because there are many n-petals shaped curves. For instance, $\mathscr{L}_3$ is different from the well-known trifolium curve.

[1] In the next section we prove that the name corresponds to the classic definition. See Proposition 15

Remark 1. The curve $\mathscr{L}_n$ is an example of the classic curves *sinusoidal spirals* (Fig. 4). That name was given by Colin MacLaurin (1698-1746) in his book *Geometria Organica: Sive descriptio linearum curvarum universalis*, to curves with equation $r^\nu = a^\nu \cos(\nu\theta)$ with $\nu \in \mathbb{Q}$ (*cf.* [14, p. 184]).



(a) $\nu = 2$ **(b)** $\nu = 2.3$ **(c)** $\nu = 2.75$ **(d)** $\nu = 3$

Figure 4. Sinusoidal spirals for $a = 1$

Letting $z = x + iy$, where $x, y \in \mathbb{R}$, it can be shown that an equivalent form of the equation (5) is

$$|p_n(z)|^2 = |(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n)|^2 = r, \tag{6}$$

where ζ_i is a n^{th}-root of the unity and $r = 1$. This equation shows that our algebraic generalization (see equation 5) is indeed a generalization of Cassini's curves with n focal points at $\zeta_i \in \mathbb{C}$. We call this family of curves $|p_n(z)|^2 = r$, parametrized by r, *generalized lemniscates* (Fig. 5).

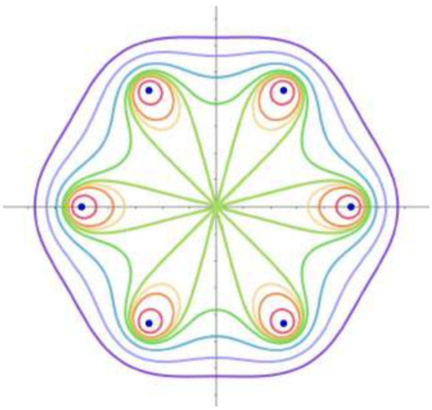


Figure 5. Family $|p_6(z)|^2 = r$

2.2 Singularity of $\mathscr{L}_n$

Definition 4. Let C be a curve defined by the polynomial[2] $f(x, y)$ and L a line such that $C \cap L \neq \emptyset$. Suppose the coordinate system on $\mathbb{R}^2$ is chosen such that $o = (0, 0) \in C \cap L$

[2]Unless otherwise indicated, we will work only with polynomials.

and $L = \{t(a_1, a_2) : t \in \mathbb{R}; (a_1, a_2) \neq (0, 0)\}$. Define the *intersection multiplicity of* L *and* C *at* o as the multiplicity m of the root $t = 0$ of $f(a_1 t, a_2 t)$ (*cf.* [21, pp. 15-16]).

Definition 5. Let C be a curve defined by $f(x, y) = 0$ and let $p = (a, b)$ be a point of C.

i. The *multiplicity* $\mathit{mult}_p(C) > 0$ *of* C *at* p is the order of the lowest non-vanishing term in the Taylor expansion of f at p.

ii. The *tangents to* C *at* p are the lines through p that cut C with multiplicity $m > \text{mult}_p(C)$. It is known that, counting multiplicities, C has at most $\text{mult}_p(C)$ (real) tangents to C at p.

iii. The point p is a *regular point* when $\text{mult}_p(C) = 1$. Otherwise, p is called a *singular point*.

Remark 2. In the previous definition, we have used affine coordinates. However, it is well known that it is independent of the choice of coordinates.

Remark 3. Our definition of singular points is different and, in fact, not equivalent to the definition of parametric singular points found in –for example– [20, p. 13]. Parametric singular points are those where the derivative of the parametrization of the curve is zero. We know that, for example, a straight line –which has no singular points according to Definition 5– has a parametrization with parametric singular points. The definition we adopted here coincides with the more general definition of singular algebraic varieties, but, because of the dimension we are working in, does not force us to introduce localization rings and tangent spaces yet (*cf.* [21, p. 234], [7, p. 62], [8, p. 227]).

Proposition 1. *The point* $o = (0, 0)$ *is a singular point of* $\mathscr{L}_n$*. Moreover,* $\mathit{mult}_o(\mathscr{L}_n) = n$.

Proof. This follows easily since equation (5) is the Taylor expansion of the curve. □

Remark 4. Actually, $\mathscr{L}_n$ has no other singular point besides o. This is easy to see because singular points are preserved under diffeomorphisms (in particular, under the change to polar coordinates) (see Eq. (4)).

Proposition 2. *The* n *tangent lines to* $\mathscr{L}_n$ *at* o *are*

$$y = \mu_j x,$$

where $\mu_j = \dfrac{\omega_j - 1}{i(1 + \omega_j)}$ *with* $j = 1, \cdots, n$ *for each* ω_j*, a* n^{th}*-root of* -1 *different from* -1*. For* n *odd, there exists* k *such that* $\omega_k = -1$*. This root corresponds to the tangent line* $x = 0$.

Proof. Following the method of [23, pp. 53-54], the tangents are given by

$$\mu x - \lambda y = 0,$$

where the ratios $\mu : \lambda$ satisfy

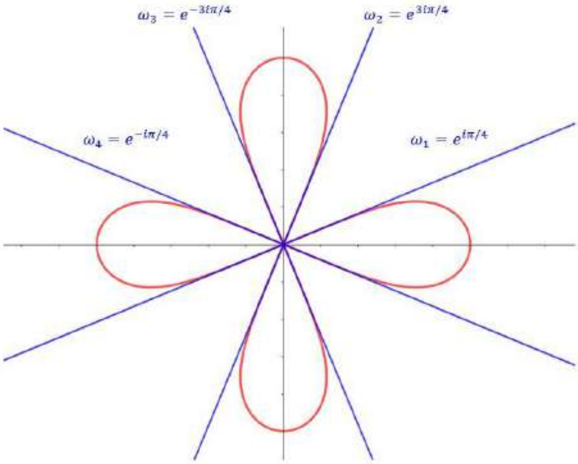


Figure 6. Tangent lines to $\mathscr{L}_4$ at o

$$\sum_{i=0}^{n} \binom{n}{i} \left[\frac{\partial^n f}{\partial x^{n-i} \partial y^i} \right] \Bigg|_{(0,0)} \lambda^{n-i} \mu^i = 0. \tag{7}$$

For $\mathscr{L}_n$ we have that, as in Proposition 1,

$$\begin{aligned} \frac{\partial^n f}{\partial x^{n-i} \partial y^i}\Bigg|_{(0,0)} = & \Bigg[\sum_{k=0}^{n} \binom{n}{k} (2n-2k)^{\underline{n-i}} (2k)^{\underline{i}} x^{n-2k+i} y^{2k-i} + \\ & - 2 \sum_{\substack{k=0 \\ k \text{ even}}}^{n} (-1)^{k/2} \binom{n}{k} (n-k)^{\underline{n-i}} (k)^{\underline{i}} x^{i-k} y^{k-i} \Bigg] \Bigg|_{(0,0)}. \end{aligned} \tag{8}$$

Here $a^{\underline{b}} = a(a-1)\cdots(a-b+1)$ denotes the falling factorial[3].

The first sum is a polynomial of degree $2n$ with no constant term. Therefore, when evaluating at o it vanishes. Similarly, for i odd, the second sum is a constant-term free polynomial. Because the powers of y in the equation (5) are even, for i even we have

$$\frac{\partial^n f}{\partial x^{n-i} \partial y^i}\Bigg|_{(0,0)} = (-1)^{i/2+1} 2 \binom{n}{i} (n-i)^{\underline{n-i}} = (-1)^{i/2+1} 2(n!).$$

Thus, the equation (7) for $\mathscr{L}_n$ is

[3] Indices must be understood as a positive expression or 0 otherwise. Analogously in the sequel.

$$\sum_{\substack{i=0\\ i \text{ even}}}^{n} (-1)^{i/2+1} 2\binom{n}{i} n!\lambda^{n-i}\mu^i = 0. \tag{9}$$

Case 1: $\lambda \neq 0$**.** Without loss of generality, we take $\lambda = 1$. It is sufficient to find the solution of

$$\sum_{\substack{i=0\\ i \text{ par}}}^{n} (-1)^{i/2+1} \binom{n}{i} \mu^i = 0. \tag{10}$$

It is straightforward to check that equation (10) is equivalent to

$$(1 + i\mu)^n + (1 - i\mu)^n = 0. \tag{11}$$

We find that

$$\mu_j = \frac{\omega_j - 1}{i(1 + \omega_j)}, j = 1, \cdots, n \tag{12}$$

solves the equation (11), where ω_j is a n^{th}-root of -1 different from -1. Observe that $\mu_j \in \mathbb{R}$ since

$$\begin{aligned}
\operatorname{Im}(\mu_j) &= \frac{1}{2i}(\mu_j - \overline{\mu_j})\\
&= -\frac{\omega_j - 1}{2(1+\omega_j)} - \frac{\overline{\omega_j} - 1}{2(1+\overline{\omega_j})}\\
&= \frac{-\omega_j - \omega_j\overline{\omega_j} + 1 + \overline{\omega_j} - \overline{\omega_j} - \overline{\omega_j}\omega_j + 1 + \omega_j}{2(1+\omega_j)(1+\overline{\omega_j})}\\
&= -\frac{2(\omega_j\overline{\omega_j} - 1)}{2(1+\omega_j)(1+\overline{\omega_j})}\\
&= 0.
\end{aligned} \tag{13}$$

Case 2: $\lambda = 0$. If n is even, the equation (9) implies that $\mu = 0$, but $0 : 0$ is undefined. For n odd, $x = 0$ is a tangent line (see below). Note that only for n odd $\omega_1 = -1$ is a n^{th}-root of -1. Consequently, $\mu_1 = \infty$ is undefined, but it is associated to $x = 0$.

Since the solutions to the equation (11) are simple, $\mathscr{L}_n$ has n different tangent lines at o given by the equations:

$$y = \mu_j x.$$

Now, we verify that the intersection multiplicity of $L = \{(1, \mu_j)t : t \in \mathbb{R}\}$ and $\mathscr{L}_n$ at o is greater than n. Because μ_j satisfies the equation (10), when we substitute $x = t$ and $y = \mu_j t$ in the equation of $\mathscr{L}_n$ we see that:

$$f(t, \mu_j t) = \left(t^2 + \mu_j^2 t^2\right)^n - \left(2 \sum_{\substack{k=0\\ k \text{ even}}}^{n} (-1)^{k/2} \binom{n}{k} \mu_j^k \right) t^n = \left(t^2 + \mu_j^2 t^2\right)^n.$$

Therefore, the multiplicity of the root $t = 0$ of $f(t, \mu_j t)$ is $m = 2n > n$. Analogously, for $L = \{(0,1)t : t \in \mathbb{R}\}$ when n is odd we have $f(0,t) = (t^2)^n$. Hence, $m = 2n$ in this case too. □

2.3 Curvature

One of the main intrinsic characteristics of a curve is its curvature function. In classical theory, curvature is defined using a parametrization of the curve. Since we are working over $\mathbb{R}$, $\mathscr{L}_n$ is the image of a regular parametrization (it is an immersion), so the curvature is defined. However, curvature –in this classical sense– is not defined at parametric singular points. In this subsection we derive a formula for the curvature of $\mathscr{L}_n$ using only its algebraic description. Our approach is based on the presentation in [6]. We will show that the curvature is well-defined at the singular point of $\mathscr{L}_n$.

Definition 6. Let $\gamma(s)$ be a unit-speed parametrization (and therefore, regular) of a curve in $\mathbb{R}^2$, and let $\vec{t} = \dfrac{d\gamma}{ds}$ be the unit tangent vector. The *signed normal vector* $\vec{n}_s$ is the anticlockwise rotation by $\dfrac{\pi}{2}$ of $\vec{t}$. Since $\dfrac{d\vec{t}}{ds}$ is parallel to $\vec{n}_s$ (because of the unit-speed parametrization), there exists $\kappa(s) \in \mathbb{R}$ such that $\dfrac{d\vec{t}}{ds} = \kappa(s)\vec{n}_s$. We define $\kappa = |\kappa(s)|$ as the *curvature of the curve at* s.

It is clear that $\vec{t}$ and $\vec{n}$ are perpendicular, so $\{\vec{t}, \vec{n}\}$ is an (orthonormal) basis of $\mathbb{R}^2$. This basis is called the *Frenet-Serret frame*. The natural implicit analogue of the preceding definition for a curve given by $f(x,y) = 0$ is as follows: take $N^f = \dfrac{\nabla f}{|\nabla f|} = (N_x^f, N_y^f)$ as the *normal vector* and $T^f = (-N_y^f, N_x^f)$ as the *unitary tangent vector* (which is a $\dfrac{\pi}{2}$-rotation of N^f).

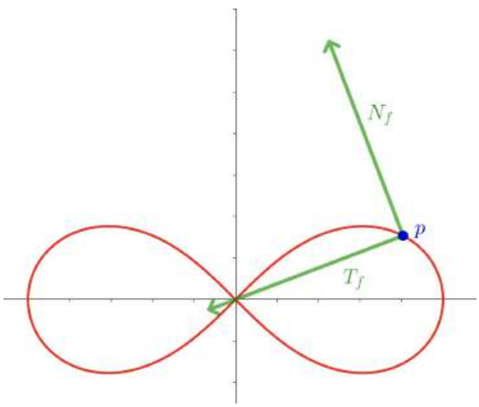


Figure 7. Frenet-Serret frame for $\mathscr{L}_2$ at p

In [6, p. 632, Proposition 3.1] it is proved that:

Proposition 3. *The curvature of a plane curve (at regular points) implicitly defined by $f(x,y)=0$ is*

$$\kappa(x,y) = \frac{|T^f \cdot Hess(f) \cdot (T^f)^t|}{|\nabla F|} = \left| \frac{\left(\frac{\partial f}{\partial y}\right)^2 \frac{\partial^2 f}{\partial x^2} - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial yx} + \left(\frac{\partial f}{\partial x}\right)^2 \frac{\partial^2 f}{\partial y^2}}{\left(\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2\right)^{3/2}} \right| \tag{14}$$

Proof. From the definition,

$$\kappa(s) = -\frac{d\vec{n}}{ds}\vec{t}.$$

The implicit analogous of that is

$$\kappa(s) = -\frac{dN^f}{ds}T^f.$$

By the chain rule,

$$\kappa(s) = -\frac{dN^f}{ds}T^f = -\left(\frac{\partial N^f}{\partial x}\frac{dx}{ds} + \frac{\partial N^f}{\partial y}\frac{dy}{ds}\right)T^f = -T^f \nabla N^f (T^f)^t.$$

Applying the quotient rule (and the fact that $\nabla f \cdot T^f = 0$) we obtain the desired formula. □

For points where $\frac{d\vec{t}}{ds} = 0$, the Frenet-Serret basis degenerates and κ is undefined. The same thing happens for singular points in the implicit Frenet-Serret basis. Nevertheless, there exists a notion of curvature at singular points along the tangent lines of the curve at those points. In general, this curvature depends on the tangent line chosen.

Definition 7. Let $p=(a,b)$ be a singular point of the curve C with $\mathrm{mult}_p(C) = r$. Consider the parametric curve (conic) Γ defined by

$$\vec{r} : \mathbb{R} \to \mathbb{R}^2; t \mapsto \vec{r}(t) = \left(a + a_1 t + \frac{1}{2}a_2 t^2, b + b_1 t + \frac{1}{2}b_2 t^2\right),$$

where $\frac{d\vec{r}}{dt}(0) = (a_1, b_1)$ is the director vector of one of the tangent lines, say L, of C at p. Since $(a_1,b_1) \neq (0,0)$, p is a parametric regular point of Γ. Hence, the curvature κ at $p \in \Gamma$ is (*cf.* [20, p. 31, Proposition 2.1.2]):

$$\kappa = \frac{\left|\det\left(\frac{d\vec{r}}{dt} \Big| \frac{d^2\vec{r}}{dt^2}\right)\right|}{\left|\frac{d\vec{r}}{dt}\right|^3} = \frac{|a_1 b_2 - a_2 b_1|}{(a_1^2 + b_1^2)^{3/2}}.$$

We define κ as the *curvature of C at p along the tangent L.*

In [15, p. 5, Theorem 1], the authors proved that this definition coincides with the curvature of the Proposition 3 when applied to points where $\frac{d\vec{t}}{ds} \neq 0$. Definition 7 allows us to calculate the curvature at points where $N^f = 0$, by measuring it through an auxiliary conic –this is the geometric meaning of curvature: the inverse of the radius of the best circular approximation. The following proposition not only gives an explicit formula for the curvature at regular points of $\mathscr{L}_n$, but also shows that the curvature is unambiguously determined at the singular point o. In [15, pp. 6-8] there are examples where, even though the values of the curvature is the same along all the tangent lines, the value does not continuously extend the curvature function around that point. For $\mathscr{L}_n$, however, it *is* a continuous extension.

Proposition 4. *The curvature function of $\mathscr{L}_n$ at regular points is*

$$\left| \frac{4n^3\left(\left(x^2+y^2\right)^n-(x-iy)^n\right)\left(\left(x^2+y^2\right)^n+(x+iy)^n\right)}{\left(x^2+y^2\right)^{(3n-1)/2}} \right.$$
$$\left. \times\left(2\left(x^2+y^2\right)^n+(n-1)\left((x-iy)^n-(x+iy)^n\right)\right)\right|,$$

and it is 0 at o. Thus, the curvature function of $\mathscr{L}_n$ is continuously extended to o.

Proof. The general formula is obtained after some calculations and simplifications. To calculate the curvature at o, substitute $x = a_1t + \frac{1}{2}a_2t^2$ and $y = b_1t + \frac{1}{2}b_2t^2$ into the equation (5):

$$\begin{aligned}
g(t) &= \left(\left(a_1t+\frac{1}{2}a_2t^2\right)^2+\left(b_1t+\frac{1}{2}b_2t^2\right)^2\right)^2 + \\
&- 2\sum_{\substack{k=0\\ k \text{ even}}}^{n}(-1)^{k/2}\binom{n}{k}\left(\sum_{i=0}^{n-k}\binom{n-k}{i}a_1^{n-k-i}\left(\frac{a_2}{2}\right)^i t^{n-k+i}\right)\left(\sum_{i=0}^{k}\binom{k}{i}b_1^{k-i}\left(\frac{b_2}{2}\right)^i t^{k-i}\right)\\
&= \left(\left(a_1t+\frac{1}{2}a_2t^2\right)^2+\left(b_1t+\frac{1}{2}b_2t^2\right)^2\right)^2 + \\
&- 2\sum_{\substack{k=0\\ k \text{ even}}}^{n}(-1)^{k/2}\binom{n}{k}\left(\sum_{\alpha=n-k}^{2n-2k}\binom{n-k}{\alpha-n+k}a_1^{\alpha}\left(\frac{a_2}{2}\right)^{\alpha-n+k}t^{\alpha}\right)\times\\
&\qquad\times\left(\sum_{\beta=0}^{k}\binom{k}{k-\beta}b_1^{\beta}\left(\frac{b_2}{2}\right)^{k-\beta}t^{\beta}\right).
\end{aligned} \tag{15}$$

We are interested in the coefficients of the powers t^n and t^{n+1} of this polynomial of degree $4n$. When we expand, we see that

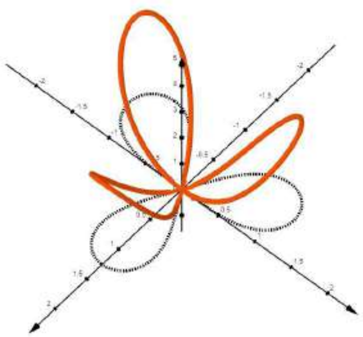
$$C_n = -2\sum_{\substack{k=0\\ k \text{ par}}}^{n}(-1)^{k/2}\binom{n}{k}a_1^{n-k}b_1^k, \tag{16}$$

and

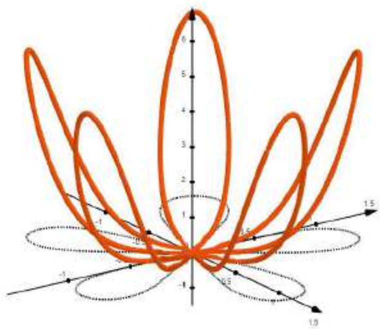
$$C_{n+1} = -2\sum_{\substack{k=0\\ k \text{ par}}}^{n}(-1)^{k/2}\binom{n}{k}\left[\binom{n-k}{0}\binom{k}{1}a_1^{n-k}b_1^{k-1}\left(\frac{b_2}{2}\right)+\right.$$
$$\left.+\binom{n-k}{0}\binom{k}{1}a_1^{n-k-1}b_1^{k}\left(\frac{a_2}{2}\right)\right] \tag{17}$$

are such coefficients, respectively. When $C_n = 0$, $a_1 = \mu_j b_1$, where μ_j is as in equation (12) (as expected!). Fix $j \in \{1, \cdots, n\}$ and take $b_1 = 1$. Then $C_{n+1} = 0$ implies (after straightforward calculations) that $b_2 = \dfrac{a_2}{\mu_j}$. Thus $\kappa = 0$ for all j. □

Figure 8 displays the curvature function of $\mathscr{L}_n$ for some values of n. We observe that the curvature attains maximum values proportional to n at points that minimize the distance from the focal points in each petal of the curve (*e.g.,* for $n = 2$, the maximum curvature is 2.1213203, and for $n = 5$ it is 5.223303). It is not hard to see that the coordinates of those points are $p_i = \left(Re(\sqrt[n]{2}), Im(\sqrt[n]{2})\right)$.



(a) Curvature of $\mathscr{L}_3$



(b) Curvature of $\mathscr{L}_5$

Figure 8. Curvature of $\mathscr{L}_n$

2.4 Schwarz function

Despite its name, the Schwarz function does not, to the best of our knowledge, appear explicitly in Schwarz's work. The concept was introduced systematically by Philip J. Davis (1923–2018) in [3]. In this subsection, we define the Schwarz function for $\mathscr{L}_n$ and describe the sense in which $\mathscr{L}_n$ is the multiplicative inverse of the n-hyperbola. This was suggested in [12] and [13], but we complete and generalize it.

Definition 8. Let $z = x + iy$ and $\overline{z} = x - iy$ with $x, y \in \mathbb{R}$. Since $x = \dfrac{z+\overline{z}}{2}$ and $y = \dfrac{z-\overline{z}}{2i}$, the coordinate system $(z, \overline{z})$ is referred to as the *conjugated coordinate system*.

Definition 9. Let $C \subset \mathbb{C}$ be a plane curve defined by $f(x,y) = 0$. In conjugated coordinates, $g(z,\overline{z}) = f\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$ is analytic[4]. If, for $z_0 \in C$, we have that $\left.\frac{\partial g}{\partial \overline{z}}\right|_{z_0} \neq 0$, by the implicit function theorem, we can (locally) solve $\overline{z}$ from $g(z,\overline{z})$:

$$\overline{z} = S(z).$$

The function

$$S_C : C \to \mathbb{C}; z \mapsto S(z). \tag{18}$$

is called the *Schwarz function associated to* C.

A priori, the Schwarz function can be analytically extended to many other points of $\mathbb{C} \setminus C$. The condition $\left.\frac{\partial g}{\partial \overline{z}}\right|_{z_0} \neq 0$ is only sufficient.

Remark 5. The uniqueness of the Schwarz function is evident, while its existence is established *a posteriori*. Moreover, the Schwarz function associated to C fixes the curve if it is symmetric with respect to the x-axis –just like $\mathscr{L}_n$.

Part of the following proposition can be found in [3, p. 27]

Proposition 5. *The Schwarz function associated to* $\mathscr{L}_n$ *is*

$$S_{\mathscr{L}_n} : \mathscr{L}_n \to \mathbb{C}; z \mapsto \sqrt[n]{\frac{z^n}{z^n-1}}. \tag{19}$$

Since it is a multivalued function, we choose the root such that $S_{\mathscr{L}_n}(p_i) = \overline{p_i}$ *for every point of maximum curvature* p_i, *with* $i = 1, \cdots, n$. *In fact,* $S_{\mathscr{L}_n}$ *analytically extends to* $D = \mathbb{C} \setminus \{\zeta_i : \zeta_i^n = 1\}$

Proof. In conjugated coordinates, $\mathscr{L}_n$ is given by

$$g(z,\overline{z}) = \left(\left(\frac{z+\overline{z}}{2}\right)^2 + \left(\frac{z-\overline{z}}{2i}\right)^2\right)^n - 2\sum_{\substack{k=0 \\ k \text{ even}}}^{n} (-1)^{k/2}\binom{n}{k}\left(\frac{z+\overline{z}}{2}\right)^{n-k}\left(\frac{z-\overline{z}}{2i}\right)^k = 0.$$

When we expand we get

$$g(z,\overline{z}) = (z\overline{z})^n - z^n - \overline{z}^n = 0. \tag{20}$$

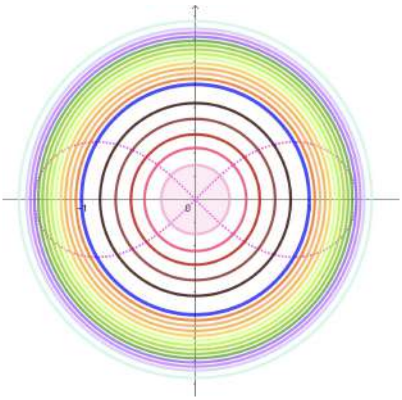
Note that this elegant identity exposes the inherent symmetry of $\mathscr{L}_n$, reducing the problem of solving for $\overline{z}$ to a straightforward algebraic manipulation:

$$\overline{z} = \sqrt[n]{\frac{z^n}{z^n-1}}.$$

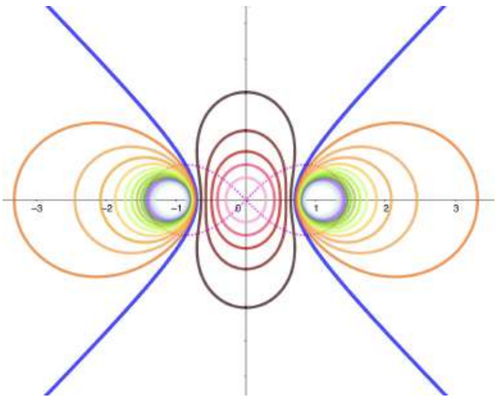
[4] In general, that is the condition we impose when f is not a polynomial

Nonetheless, $\sqrt[n]{\dfrac{z^n}{z^n-1}}$ represents n different complex numbers. Because of that, we must choose, for each petal (branch) of $\mathscr{L}_n$, the number that satisfies $S_{\mathscr{L}_n}(z) = \overline{z}$ (that was our geometric definition of S_C). It suffices to impose one condition over $p_i = \sqrt[n]{2} \in \mathbb{C}$, with $i = 1, \cdots, n$, the points of maximum curvature, and analytically extend the function. This uniquely associates each branch of $\mathscr{L}_n$ with an analytic branch of the complex function $\sqrt[n]{z}$. Note that $\dfrac{\partial g}{\partial \overline{z}} = nz^n\overline{z}^{n-1} - n\overline{z}^{n-1} = 0$ if and only if $\overline{z} = 0$ o $z^n - 1 = 0$. This finishes the proof. □

Figure 9 shows the action of $S_{\mathscr{L}_2}$ in a region $\mathcal{D} \subset D$ from its domain, formed by circles of different radii and $\mathscr{L}_2$. In panel (b), the image of each circle under $S_{\mathscr{L}_2}$ retains the color of its preimage shown in panel (a), providing visual coherence.



(a) Region $\mathcal{D}$ from the domain of $S_{\mathscr{L}_2}$



(b) $S_{\mathscr{L}_2}(\mathcal{D})$, with $\mathcal{D}$ as in **(a)**

Figure 9. Action of $S_{\mathscr{L}_n}$

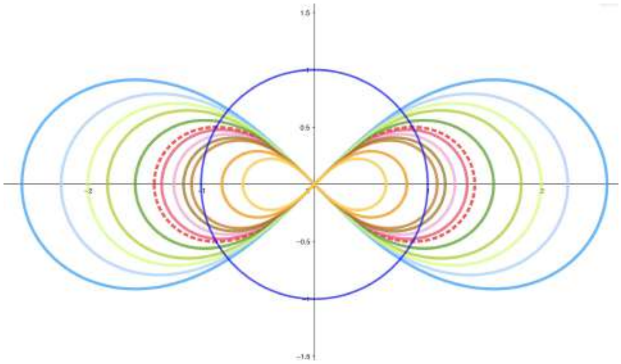
The previous figure shows, on the one hand, that $S_{\mathscr{L}_2}(\mathscr{L}_2) = \mathscr{L}_2$ as we mentioned early (because of the symmetry). On the other hand, it hints that $S_{\mathscr{L}_n} : D \to D$ is, actually, an involution. Finally, we can see that $S_{\mathscr{L}_2}(S^1) = \mathscr{H}_2$, where $S^1 = \{z : |z| = 1\}$ y $\mathscr{H}_2$ is the hyperbola defined by the equation $2(x^2 - y^2) - 1 = 0$. The following proposition formalize these observations.

Proposition 6. *The function $S_{\mathscr{L}_n}$, defined as in Proposition 5, is an involution.*

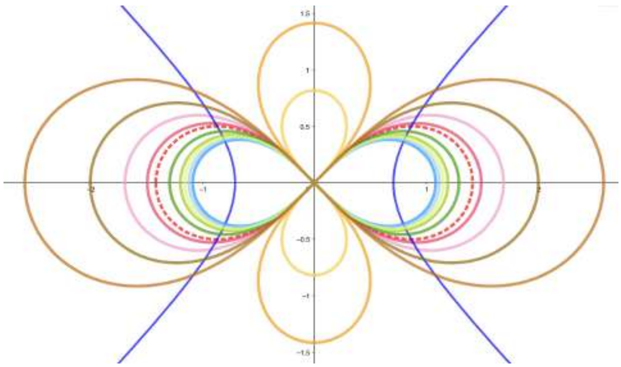
Proof. When we calculate the composition, we find out that

$$\left(S_{\mathscr{L}_n} \circ S_{\mathscr{L}_n}\right)(z) = \sqrt[n]{\dfrac{\dfrac{z^n}{z^n-1}}{\dfrac{z^n}{z^n-1} - 1}} = \sqrt[n]{z^n} = \zeta_i z,$$

with ζ_i a n^{th}-root of the unity. Since $S_{\mathscr{L}_n} \circ S_{\mathscr{L}_n}(p_i) = p_i$, it follows that $\zeta_i = 1$ for all i, and thus $S_{\mathscr{L}_n}$ is indeed an involution: $S_{\mathscr{L}_n} \circ S_{\mathscr{L}_n} = \mathrm{id}$. $\square$



(a) Region $\mathcal{E}$ from the domain of $S_{\mathscr{L}_2}$



(b) $S_{\mathscr{L}_2}(\mathcal{E})$, with $\mathcal{E}$ as in (a)

Figure 10. Action of $S_{\mathscr{L}_n}$ on $\mathscr{L}_n^\lambda$

In fact[5], $S_{\mathscr{L}_n}$ preserves the whole family of re-escalated lemniscates $\mathscr{L}_n^\lambda$ given by

$$\lambda(x^2+y^2)^n - 2\sum_{\substack{k=0\\ k \text{ even}}}^{n} (-1)^{k/2}\binom{n}{k} x^{n-k}y^k = 0 \tag{21}$$

according to the relation $S_{\mathscr{L}_n}(\mathscr{L}_n^\lambda) = \mathscr{L}_n^{2-\lambda}$. When $2-\lambda < 0$, the lemniscate is just a $\dfrac{\pi}{2}$-rotation of the corresponding curve $\mathscr{L}_n^{|2-\lambda|}$ (Fig. 10).

Definition 10. When we generalize the equation of a hyperbola with two foci, we get that the equation

$$2\sum_{\substack{k=0\\ k \text{ even}}}^{n} (-1)^{k/2}\binom{n}{k} x^{n-k}y^k - 1 = 0 \tag{22}$$

[5]This was a captivating remark by professor Getmanenko.

describe a "hyperbola" with n foci. We call the curve determined by this equation the *n-hyperbola with foci at ζ_i* –the n^{th}-root of the unity– and we denote it by $\mathscr{H}_n$.

We verify that the image of S^1 under $S_{\mathscr{L}_n}$ satisfies the defining equation of $\mathscr{H}_n$ in conjugated coordinates.

Proposition 7. *The image of S^1 under $S_{\mathscr{L}_n}$ is the n-hyperbola $\mathscr{H}_n$.*

Proof. In conjugated coordinates the equation (22) corresponds to

$$z^n + \overline{z}^n - 1 = 0. \tag{23}$$

Let $z_0 \in S^1$. Then $S_{\mathscr{L}_n}(z_0) = \sqrt[n]{\dfrac{z_0^n}{z_0^n - 1}}$ satisfies the equation (23), since

$$S_{\mathscr{L}_n}(z_0)^n + \overline{S_{\mathscr{L}_n}(z_0)}^n = \frac{z_0^n}{z_0^n - 1} + \frac{\overline{z_0}^n}{\overline{z_0}^n - 1} = \frac{\overbrace{z_0^n \overline{z_0}^n}^{1} - z_0^n + \overbrace{\overline{z_0}^n z_0^n}^{1} - \overline{z_0}^n}{\underbrace{z_0^n \overline{z_0}^n}_{1} - z_0^n - \overline{z_0}^n + 1} = 1.$$

This, together with the Proposition 6, completes the proof. □

As a matter of fact, the relation between $\mathscr{L}_n$ and $\mathscr{H}_n$ is stronger.

Proposition 8. *The image of S^1 under $S_{\mathscr{H}_n}$ is $\mathscr{L}_n$.*

Proof. From the equation (23) is easy to see that

$$S_{\mathscr{H}_n} : \mathbb{C} \to \mathbb{C}; z \mapsto \sqrt[n]{1 - z^n} \tag{24}$$

is the Schwarz function associated to $\mathscr{H}_n$. As we did with $S_{\mathscr{L}_n}$, we must establish some condition on $S_{\mathscr{H}_n}$ to define it unambiguously. In this case, it is enough that $S_{\mathscr{H}_n}(a_i) = \overline{a_i}$, where a_i are the vertices of $\mathscr{H}_n$.

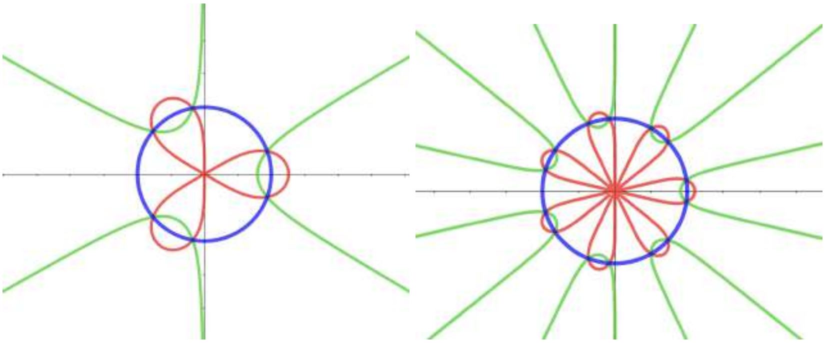
Let $z_0 \in S^1$. Then $S_{\mathscr{H}_n}(z_0) = \sqrt[n]{1 - z_0^n}$ satisfies the equation (20) because

$$\begin{aligned}\left(\sqrt[n]{1 - z_0^n}\right)^n \left(\overline{\sqrt[n]{1 - z_0^n}}\right)^n &= (1 - z_0^n)(1 - \overline{z_0}^n) = 1 - z_0^n - \overline{z_0}^n + z_0\overline{z_0}^n \\ &= (1 - \overline{z_0}^n) + (1 - z_0^n) = \left(\sqrt[n]{1 - z_0^n}\right)^n + \left(\overline{\sqrt[n]{1 - z_0^n}}\right)^n.\end{aligned}$$

It is straightforward to check that $S_{\mathscr{H}_n}$ is also an involution, which completes the proof. □

Definition 11. (informal)[6] Let $\mathfrak{X}$ be the space of algebraic plane curves. Given two curves $C_1, C_2 \in \mathfrak{X}$ with associated Schwarz function S_{C_1} and S_{C_2}, respectively, we define the operation (whenever defined) $\mathbf{Inv}_{C_1} C_2 = C$ where C is the curve with associated Schwarz function $S_{C_1} \circ S_{C_2}^{-1} \circ S_{C_1}$.

[6] The rigorous presentation of this definition requires the concept of Jordan arc, but the general idea is the same.



(a) $n = 3$ **(b)** $n = 7$

Figure 11. Triad S^1, $\mathscr{L}_n$ and $\mathscr{H}_n$

With this informal definition, $\mathfrak{X}$ can be understood with a non-associative operation in which the identity (base) element varies in the space $\mathfrak{X}$. This product satisfies the axioms of a symmetric space (*cf.* [16, p. 63, Definition 1]). Some of them are the following:

1. $\mathbf{Inv}_C C = C$,
2. $\mathbf{Inv}_{C_1} (\mathbf{Inv}_{C_1} C_2) = C_2$, and
3. $\mathbf{Inv}_{C_1} (\mathbf{Inv}_{C_2} C_3) = \mathbf{Inv}_{\mathbf{Inv}_{C_1} C_2} (\mathbf{Inv}_{C_1} C_3)$.

Proposition 9. *In the space $\mathfrak{X}$, the inverse curve of $\mathscr{L}_n$, with base S^1, is $\mathscr{H}_n$ and vice versa. That is, $\boldsymbol{Inv}_{\mathscr{L}_n} \mathscr{H}_n = \boldsymbol{Inv}_{\mathscr{H}_n} \mathscr{L}_n = S^1$.*

Proof. The result follows from the two precedent propositions, and the following calculation:

$$\sqrt[n]{\frac{z^n}{z^n - 1}} \circ \sqrt[n]{1 - z^n} \circ \sqrt[n]{\frac{z^n}{z^n - 1}} = \frac{1}{z}$$

$$\sqrt[n]{1 - z^n} \circ \sqrt[n]{\frac{z^n}{z^n - 1}} \circ \sqrt[n]{1 - z^n} = \frac{1}{z}.$$

Note that $\frac{1}{z}$ is the Schwarz function associated to S^1 because $\overline{z} = \frac{1}{z}$ if and only if $z\overline{z} = x^2 + y^2 = 1$. □

Remark 6. In section 1 we mentioned that $\mathscr{L}_n$ is an example of sinusoidal spiral with $\nu = n$ and $a^\nu = 2$. It is amusing to observe that $\mathscr{H}_n$ is so too, but with $\nu = -n$ and $a^\nu = \frac{1}{2}$.

To conclude this subsection, we present a compelling result from [3, pp. 41-45] that connects the Schwarz function with the curvature function of a curve.

Proposition 10. *The curvature of the curve C with associated Schwarz function $S_C(z)$ is*

$$\kappa = \frac{1}{2}|S''(z)| \tag{25}$$

Proof. Since $\overline{z} = S(z)$ along C,

$$S'(z) = \frac{d\overline{z}}{dz} = \frac{dx + idy}{dx - idy} = \frac{1 - i\dfrac{dy}{dx}}{1 + i\dfrac{dy}{dx}}.$$

Solving $\dfrac{dy}{dx}$ from the previous equation, we get that

$$\frac{dy}{dx} = -i\frac{1 - S'(z)}{1 + S'(z)}. \tag{26}$$

Now, in [20, p. 38, Proposition 2.2.3] it is proved that the signed curvature is given by

$$\kappa(s) = \frac{d\varphi}{ds},$$

where φ is the turning angle, *i.e.*, the angle the tangent line to C at a given point makes with the x-axis. Let $z_0 \in C$. It is easily seen that the tangent line of C at z_0 has equation

$$\overline{z} = S_C'(z_0)(z - z_0) + \overline{z_0}.$$

Additionally, we have that

$$\tan\varphi = \frac{dy}{dx} = -i\frac{1 - S_C'(z_0)}{1 + S_C'(z_0)}.$$

Therefore,

$$\frac{d\varphi}{ds} = \frac{d\tan^{-1}\left(\dfrac{dy}{dx}\right)}{dx}\frac{dx}{dz}\frac{dz}{ds}. \tag{27}$$

On the other hand, $ds^2 = dx^2 + dy^2 = dzd\overline{z} = S_C'(z)(dz)^2$. Thus,

$$\frac{dz}{ds} = \frac{1}{\sqrt{S_C'(z)}}. \tag{28}$$

Moreover, $\dfrac{dz}{dx} = \dfrac{dx + idy}{dx} = 1 + i\dfrac{dy}{dx} = \dfrac{2}{1 + S_C'(z)}$ and consequently,

$$\frac{d^2y}{dx^2} = \frac{d\left(\dfrac{dy}{dx}\right)}{dz}\frac{dz}{dx} = \frac{4iS_C''(z)}{(1 + S_C'(z))^3}. \tag{29}$$

Substituting the equations (28) and (29) into the equation (27) we obtain

$$\kappa(s) = \frac{d\varphi}{ds} = \frac{iS''_C(z)}{2\left(S'_C(z)\right)^{3/2}}. \tag{30}$$

Taking the absolute value, we get the curvature of C and the desired formula. $\square$

As a result, we have gotten a more compact expression for the curvature function of $\mathscr{L}_n$, namely,

$$\kappa = \frac{n+1}{2}\left|\frac{z^{n-2}\sqrt[n]{\dfrac{z^n}{z^n-1}}}{(z^n-1)^2}\right|,$$

where the choice of the root corresponds to the one made in the definition of $S_{\mathscr{L}_n}(z)$.

2.5 Joukowski maps

In this subsection, we introduce the Joukowski maps. Joukowski maps play a crucial role in the differential structure of $\mathscr{L}_n$. Our goal, then, is to consolidate a solid understanding of them in relation to $\mathscr{L}_n$ and other generalized lemniscates.

Definition 12. We define *positive and negative Joukowski maps*, respectively, as follows:

$$\begin{aligned} j_+ &: \mathbb{C}^* \to \mathbb{C}; z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right) \\ j_- &: \mathbb{C}^* \to \mathbb{C}; z \mapsto \frac{1}{2}\left(z - \frac{1}{z}\right) \end{aligned} \tag{31}$$

It is clear that these maps are not injective, but rather $2:1$ since $j_+(z) = j_+(1/z)$ and $j_-(z) = j_-(-1/z)$.

One of the visualization tools we have for complex-valued functions is the color domain technique. This method was popularized at the end of the twentieth century (*cf.* [19]), but Frank Farris was the one who named it. If we assign a color to each complex number, we could link a number $z \in \mathbb{C}$ (in the domain) with the color assigned to $w = f(z) \in \mathbb{C}$ (in the image). In the standard formulation of the color domain method, the assignment of the color to the complex number z follows the HSL model. The color of $z \in \mathbb{C}$ is formed by the Hue = $Arg(z)$, the Saturation= 100% and the Lightness = $\dfrac{|z|^a}{|z|^a+1} \cdot 100\%$ (where $a > 0$; here a =0.4). Thus, for example, the identity function $f(z) = z$ on the domain $\{z : -3 \leq \mathrm{Re}(z), \mathrm{Im}(z) \leq 3\}$ would have the representation of Figure 12.

Thus, the graphic representation of Joukowski maps in $\{z : -3 \leq \mathrm{Re}(z), \mathrm{Im}(z) \leq 3\}$ are shown in Figure 12.

In fact, as observed in [12], Joukowski maps give us back the structure of $\mathscr{L}_2, \mathscr{H}_2$ and S^1 from a symmetric configuration displayed in Figure 14.

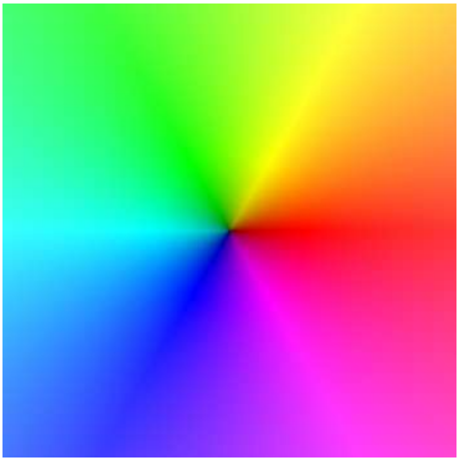
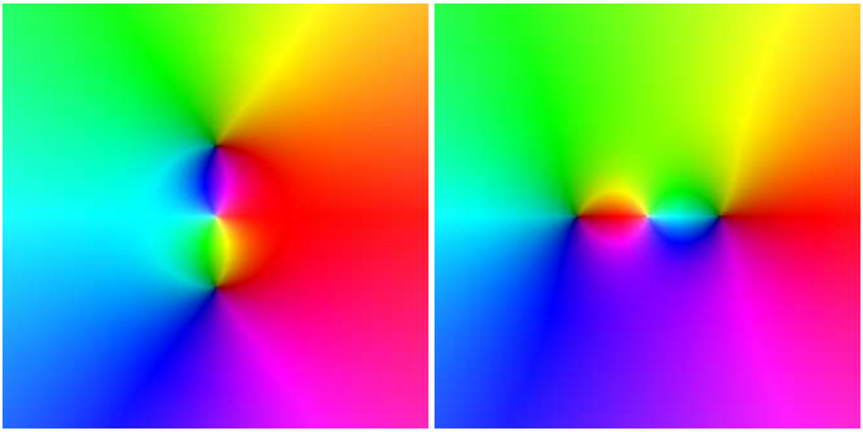


Figure 12. Color domain of $f(z) = z$

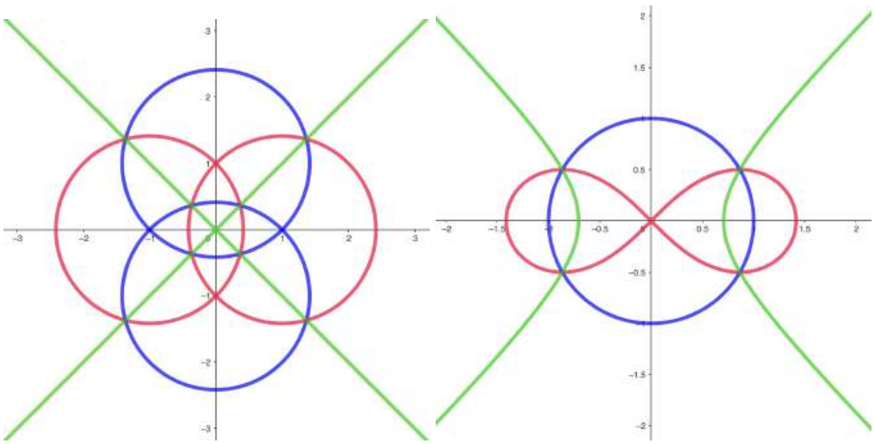


(a) Color domain of $j_+(z)$ **(b)** Color domain of $j_-(z)$

Figure 13. Color domain of Joukowski maps

The set $\mathcal{L}_6^1$ is formed by two circles centered at $\pm i$ with radius $\sqrt{2}$, two circles centered at ± 1 with radius $\sqrt{2}$ and the two lines $y = \pm x$ (which are the tangent lines to $\mathscr{L}_2$ at o). This explains the number 6. Symmetric results are easily obtained using j_-. It turns out that there exists a sense in which $\mathscr{L}_2$, and more generally, the triad $j_+\left(\mathcal{L}_6^1\right)$ is already contained in $\mathscr{L}_{2n}$. Let's consider the *generalized Joukowski maps*:

$$
\begin{aligned}
j_+^n &: \mathbb{C}^* \to \mathbb{C}; z \mapsto \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \\
j_-^n &: \mathbb{C}^* \to \mathbb{C}; z \mapsto \frac{1}{2}\left(z^n - \frac{1}{z^n}\right).
\end{aligned}
\tag{32}
$$



(a) Symmetric configuration $\mathcal{L}_6^1$ **(b)** Triad $j_+\left(\mathcal{L}_6^1\right)$

Figure 14. Configuration $\mathcal{L}_6^1$

In subsection 2.1, we indicated that $\mathscr{L}_{2n}$ is described by the equation $|p_{2n}(z)|^2 = 1$. On the other hand, the $2n^{th}$ roots of -1 can be split into two groups: $z_j^1 = \{\sqrt[n]{i}\}$ and $z_j^2 = \{\sqrt[n]{-i}\}$. With these two groups of roots, we can define two different curves in the family of generalized lemniscates $|p_n(z)|^2 = r$:

$$\mathcal{C}_1 : \left|\prod_{j=1}^{n}(z - z_j^1)\right|^2 = 2$$

$$\mathcal{C}_2 : \left|\prod_{j=1}^{n}(z - z_j^2)\right|^2 = 2$$

Strictly speaking, these curves are rotations of the originally defined generalized lemniscates $|p_n(z)|^2 = r$, to which $\mathscr{L}_n$ belongs. The $2n : 1$ image under j_+^n of $\mathcal{C}_1$ and $\mathcal{C}_2$ is precisely S^1. This new construction is, truly, a generalization of the case when $n = 1$. This completely elucidates the notation $\mathcal{L}_6^1$.

Similarly, the $2n^{th}$ roots of unity can be split into two groups: $w_j^1 = \{\sqrt[n]{1}\}$ and $w_j^2 = \{\sqrt[n]{-1}\}$. Once again, we define two curves, members of the family $|p_n(z)|^2 = r$ of generalized lemniscates:

$$\mathcal{C}_3 : \left|\prod_{j=1}^{n}(z - w_j^1)\right|^2 = 2$$

$$\mathcal{C}_4 : \left|\prod_{j=1}^{n}(z - w_j^2)\right|^2 = 2$$

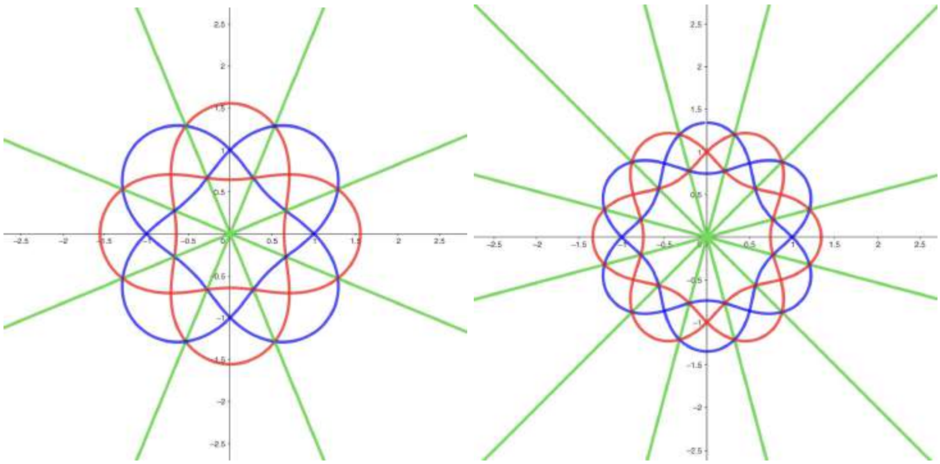
As before, $\mathcal{C}_4$ is a rotation of the actual curve of $|p_n(z)|^2 = r$. The $2n : 1$ image under j_+^n of $\mathcal{C}_3$ and $\mathcal{C}_4$ is $\mathscr{L}_2$. To complete the triad $j_+(\mathcal{L}_6^1)$, observe that $j_+^n(T_{2n}) = \mathscr{H}_2$, where T_{2n} denotes the tangent lines of $\mathscr{L}_{2n}$ at the origin o.

Accordingly, we have generalized the configuration $\mathcal{L}_6^1$. Defining

$$\mathcal{L}_6^n = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, T_{2n}\},$$

we conclude that:

$$j_+^n(\mathcal{L}_6^n) = \{\mathscr{L}_2, \mathscr{H}_2, S^1\}.$$



(a) Symmetric configuration $\mathcal{L}_6^2$ **(b)** Symmetric configuration $\mathcal{L}_6^3$

Figure 15. Configuration $\mathcal{L}_6^n$

3 Complex-projective geometry of $\mathscr{L}_n$

Definition 13. Define the *complex projective plane* as the quotient $\mathbb{CP}^2 = \mathbb{C}^3 \backslash \{(0,0,0)\}/_\sim$, where $(z_1, z_2, z_3) \sim (w_1, w_2, w_3)$ if and only if $z_j = \lambda w_j$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and all $j = 1, 2, 3$. Thus, $\mathbb{CP}^2 = \{[X : Y : Z]_\sim : X, Y, Z \in \mathbb{C},$ not all $0\}$. The coordinates X, Y, Z of the complex projective plane are the *homogeneous coordinates*.

We henceforth denote by $[X : Y : Z]$ the equivalence class $[X : Y : Z]_\sim$.

Definition 14. If $C \subseteq \mathbb{C}^2$ is an affine curve defined by $f(x, y) = 0$, the *homogenization of* f represents the projectivization of $C \subseteq \mathbb{CP}^2$. In other words, the projectivization of C is given by

$$F(X, Y, Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right) = 0,$$

where d is the degree of f. We say that C is a curve of *degree* d.

Therefore, by the equation (5), the complex projectivization of $\mathscr{L}_n$ is the locus of $[X : Y : Z]$ such that:

$$(X^2+Y^2)^n - 2Z^n \sum_{\substack{k=0 \\ k \text{ even}}}^{n} (-1)^{k/2} \binom{n}{k} X^{n-k} Y^k = 0. \tag{33}$$

Equation (33) is equivalent to

$$(X^2+Y^2)^n - Z^n(X+iY)^n - Z^n(X-iY)^n = 0. \tag{34}$$

Remark 7. When $Z=1$, the curve $C \subseteq \mathbb{CP}^2$ determined by F can be sent onto $\mathbb{C}^2$. When $Z=0$, the points $[X:Y:0]$ that satisfy $F(X,Y,Z)=0$ are called *points at infinity*. It is simple to see that $\mathscr{L}_n$ has only two points at infinity, namely, $I=[i:1:0]$ and $J=[-i:1:0]$, called *circular points* because every projective circle passes through these points.

Figure 16 is a visualization of $\mathscr{L}_2$ in a restricted domain (see Fig. 12).

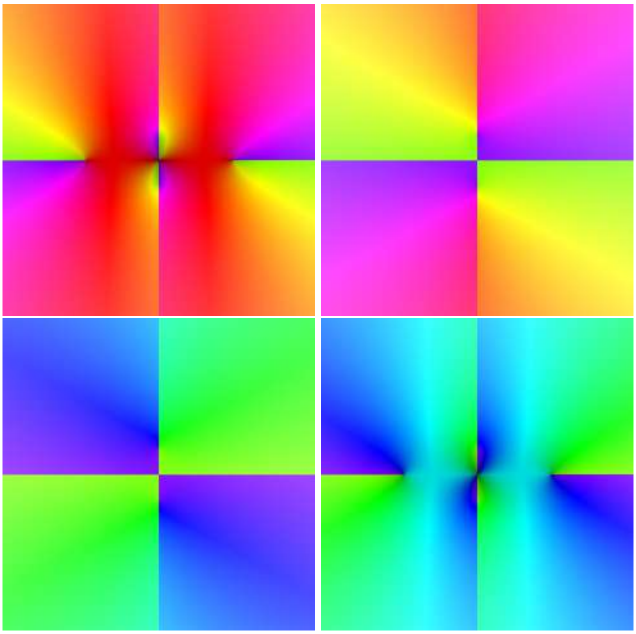


Figure 16. Projection of $\mathscr{L}_2 \subset \mathbb{CP}^2$ onto $\mathbb{C}^2$

Observe that the real part of $\mathscr{L}_2$ is split in two panels:

Definition 15. The coordinates $\boldsymbol{x}=X+iY, \boldsymbol{y}=X-iY$ and $\boldsymbol{z}=Z$ are called *isotropic coordinates*. In this new coordinate system, the points I, J, O become the *coordinate points* or *reference points* $A=[1:0:0]$ $B=[0:1:0]$ and $C=[0:0:1](=O)$, respectively.

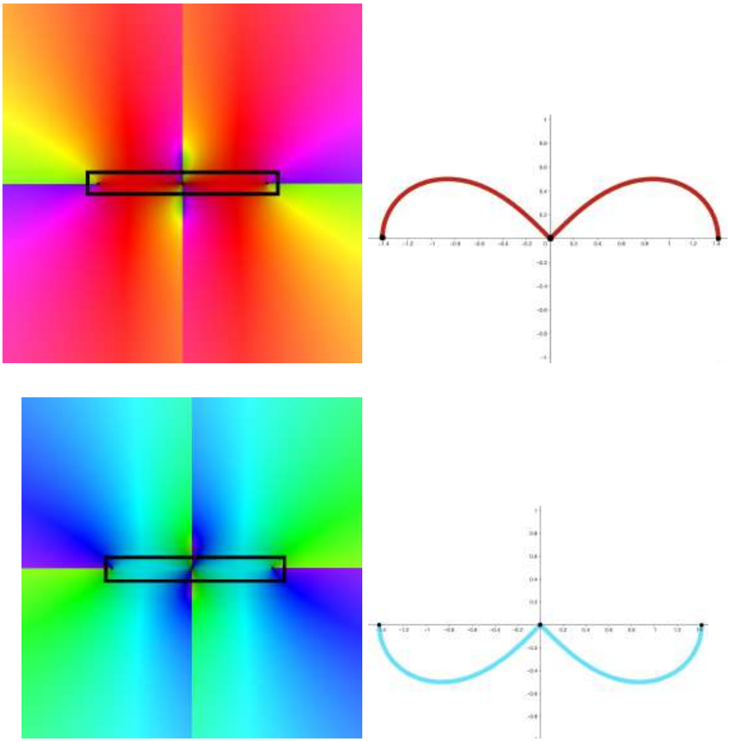


Figure 18. Real part of $\mathscr{L}_2 \subset \mathbb{C}^2$

In isotropic coordinates, the equation of $\mathscr{L}_n$ simplifies to

$$\boldsymbol{F} = (\boldsymbol{xy})^n - (\boldsymbol{zx})^n - (\boldsymbol{zy})^n = 0. \tag{35}$$

Consistently with our definition of singular points from the previous section, we have the following definition.

Definition 16. Let C be a curve defined by $F(X, Y, Z) = 0$ and let P be a point of C.

i. The *multiplicity* $\mathit{mult}_p(C) > 0$ *of* C *at* P is the order of the lowest non-vanishing term in the Taylor expansion of F at P.

ii. P is called a *regular point* whenever $\mathrm{mult}_p(C) = 1$. Otherwise, P is a *singular point*. The finite set of singular points (*cf.*, [10, p. 55, Corollary 3.10]) of C is denoted by $\mathrm{Sing}(C)$.

Proposition 11. *The points* $I = [i : 1 : 0]$, $J = [-i : 1 : 0]$ *and* $O = [0 : 0 : 1]$ *are singular points of* $\mathscr{L}_n$, *each one with multiplicity* n.

Proof. Straightforward using isotropic coordinates $\square$

Remark 8. Due to the fact that the multiplicity of $\mathscr{L}_n$ at the circular points I and J is n, $\mathscr{L}_n$ is known as a n-circular curve. This property is significant because the singular points I and J are the most interesting points at infinity.

Definition 17. Let $C \subset \mathbb{CP}^2$ be a curve defined by $F(X, Y, Z)$. The *Hessian curve associated to* C, $\mathrm{Hess}(C)$, is the locus that satisfies

$$\det\left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right)_{1\leq i,j\leq 3} = 0,$$

where $X_1 = X, X_2 = Y, X_3 = Z$. The regular points in $C \cap \mathrm{Hess}(C)$ are called *inflection points*.

It is known that $\mathrm{Hess}(C)$ is a curve of degree $3(d-2)$, if d is the degree of C. Moreover, if P is a singular point of C, $P \in \mathrm{Hess}(C)$. Indeed, by the Euler equation (*cf.* [4, p. 45]) applied to $\dfrac{\partial F}{\partial X_i} = F_{X_i}$ we get:

$$\det\left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right)_{0\leq i,j\leq 2} = \frac{n-1}{Z^2}\det\begin{pmatrix} nF & F_X & F_Y \\ (n-1)\,F_X & F_{XX} & F_{YX} \\ (n-1)\,F_Y & F_{XY} & F_{YY}\end{pmatrix}$$

And, given that P is a singular point of C, the first row of this matrix vanishes. This shows that $P \in \mathrm{Hess}(C)$.

Definition 18. Let C be a curve in $\mathbb{CP}^2$ determined by F. If F can be factorized into homogeneous polynomials F_i of positive degree $F = F_1F_2\cdots F_k$, the curves determined by F_i are called *components* of C. When F does not admit such a factorization, C is said to be *irreducible*.

We now turn to the structure of the Hessian curve $\mathrm{Hess}(\mathscr{L}_n)$ and examine its components.

For $\mathscr{L}_n$ the polynomial H that defines $\mathrm{Hess}(\mathscr{L}_n)$ is

$$\begin{aligned} H =& Z^{3n-2}(X^2+Y^2)^{n-2}\left((X+iY)^n + (X-iY)^n\right) + \\ & Z^{2n-2}(X^2+Y^2)^{n-2}\left[a(X^2+Y^2)^n - \left((X+iY)^{2n} + (X-iY)^{2n}\right)\right] + \\ & Z^{n-2}(X^2+Y^2)^{2n-2}\left((X+iY)^n + (X-iY)^n\right), \end{aligned} \tag{36}$$

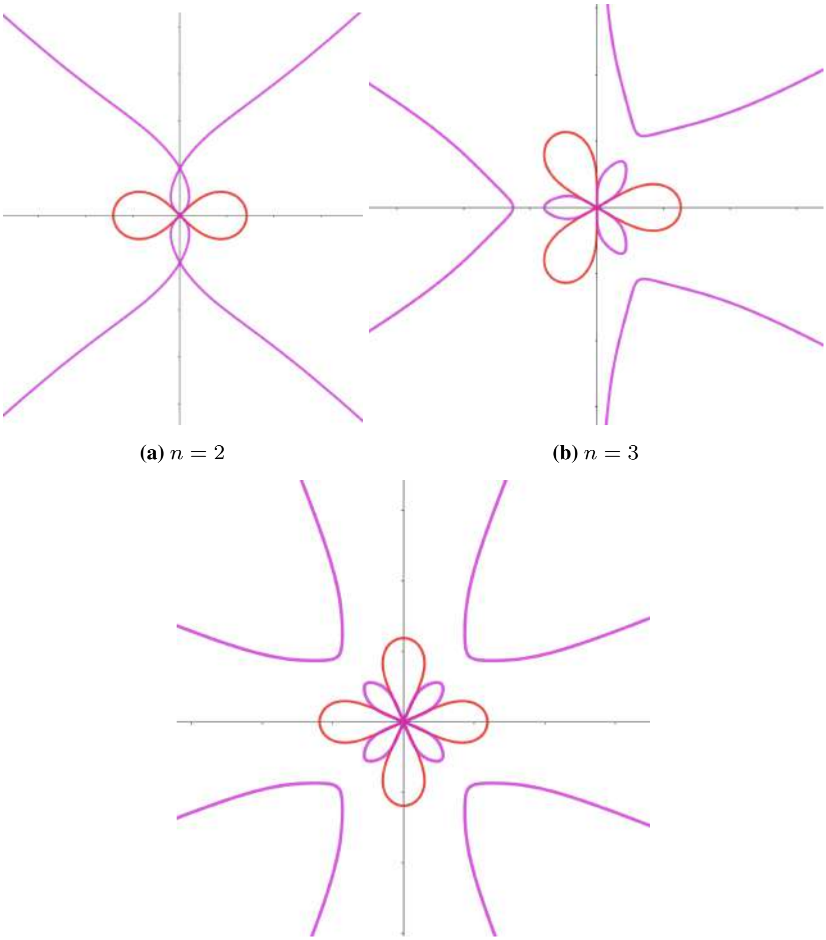
where $a = \dfrac{2n^2-2n+2}{n-1}$ (remember that $n > 1$).

In particular, $Z = 0$ is an irreducible component of $\mathrm{Hess}(\mathscr{L}_n)$ for $n > 2$. For $n = 2$ the only points at infinity of $\mathrm{Hess}(\mathscr{L}_2)$ are $[1:1:0], [1:-1:0], I$, and J. Figure 19 shows the real part of the curve $\mathrm{Hess}(\mathscr{L}_n)$ for some values of n.

In isotropic coordinates, $\mathrm{Hess}(\mathscr{L}_n)$ is given by

$$\boldsymbol{H} = n^3\boldsymbol{x}^{n-2}\boldsymbol{y}^{n-2}\boldsymbol{z}^{n-2}\Big[a(n)\boldsymbol{x}^n\boldsymbol{y}^n\boldsymbol{z}^n + b(n)\boldsymbol{x}^{2n}\boldsymbol{y}^n - b(n)\boldsymbol{x}^{2n}\boldsymbol{z}^n + b(n)\boldsymbol{y}^{2n}\boldsymbol{x}^n + \\ -b(n)\boldsymbol{y}^{2n}\boldsymbol{z}^n + b(n)\boldsymbol{z}^{2n}\boldsymbol{x}^n - b(n)\boldsymbol{z}^{2n}\boldsymbol{y}^n + \Big] \quad (37)$$

where $a(n) = 4n^3 - 6n^2 + 6n - 2$ and $b(n) = 2n^2 - 3n + 1$.



(a) $n = 2$ **(b)** $n = 3$

(c) $n = 4$

Figure 19. Real part of Hess($\mathscr{L}_n$)

Proposition 12. *$\mathscr{L}_n$ has no inflection points.*

Proof. We want to find the intersection points $C \cap \text{Hess}(C)$. Given that I and J are singular points, we can suppose $Z = 1$, then $(X^2 + Y^2) \neq 0$, and we can divide H by

$(X^2+Y^2)^{n-2}$. Substituting the relation $(X^2+Y^2)^n = (X+iY)^n + (X-iY)^n$ from equation (34) into the equation (36) we get:

$$(X+iY)^n + (X-iY)^n + a\left((X+iY)^n + (X-iY)^n\right) - (X+iY)^{2n} - (X-iY)^{2n} +$$
$$+\left((X+iY)^n + (X-iY)^n\right)\left((X+iY)^n + (X-iY)^n\right) = 0.$$

This equation is equivalent to:

$$\begin{aligned}
&(a+1)(X+iY)^n + (a+1)(X-iY)^n + 2(X+iY)^n(X-iY)^n \\
&= (a+1)(X+iY)^n + (a+1)(X-iY)^n + 2(X^2+Y^2)^n \\
&= (a+1)(X+iY)^n + (a+1)(X-iY)^n + 2\left((X+iY)^n + (X-iY)^n\right) \\
&= (a+3)(X+iY)^n + (a+3)(X-iY)^n \\
&= 0
\end{aligned} \tag{38}$$

Therefore, solving the resulting equation, we find that the solutions satisfy

$$Y = \mu_j X.$$

with μ_j as in equation (12). Substituting in the equation of $\mathscr{L}_n$ we note that

$$X = \sqrt[n]{\frac{(\mu_j i+1)^n + (1-\mu_j i)^n}{(\mu_j^2+1)^n}} = 0,$$

because μ_j are the roots of the polynomial $(1+iX)^n + (1-iX)^n$ (see Eq. (11)). Thus, the only affine point of intersection between $\mathscr{L}_n$ and Hess($\mathscr{L}_n$) is O, which completes the proof. $\square$

Corollary 1. Except for I, J and O, $\mathscr{L}_n$ has no other singular points.

Definition 19. Let C and D be curves determined by $P(X,Y,Z)$ and $Q(X,Y,Z)$, respectively, and $P = [P_1 : P_2 : P_3]$ be a point. Define $I_P(C,D)$, the *intersection multiplicity of C and D at P*, as follows:

i. $I_P(C,D) = \infty$, if P lies on a component that is common to both of C and D.
ii. $I_P(C,D) = 0$, if $p \notin C \cap D$.
iii. Assuming that C and D have no common components, and the coordinates are such that $[0:1:0] \notin C \cup D$ and does not lie in any line joining two points of $C \cap D$, we have that $I_P(C,D) = k$, where k is the multiplicity of the root (P_3, P_1) of $R_{P,Q}[Y]$, the resultant of P and Q with respect to Y (see [10, p. 59, Theorem 3.18]).

Part (iii) of the previous definition is satisfied by making a change of coordinates, if necessary. Nonetheless, computing these resultants can be algebraically intensive.

Definition 20. Let C be a curve in $\mathbb{CP}^2$. A line L is *tangent of C at P* if it passes through P and $I_P(C,L) > \text{mult}_p(C)$. This definition is analogous to the Definition 5 for the affine case.

The following useful result is proved in [1].

Proposition 13. *Let C and D two curves without common components and let $P \in C \cap D$. Then*

i. $I_p(C, D) \geq mult_p(C) mult_p(D)$

ii. $I_p(C, D) = mult_p(C) mult_p(D)$ *if and only if the tangent lines of C at P are pairwise different to the tangent lines of D at P.*

Proof. See [1, p. 235, Proposition 3]. □

Proposition 14. *Let D the curve determined by the polynomial $\dfrac{\partial F}{\partial Y}$. Then we have that $I_I(\mathscr{L}_n, D) = I_J(\mathscr{L}_n, D) = n(n-1)$. Moreover, for n even $I_O(\mathscr{L}_n, D) = n(n-1)$*

Proof. It is easily verified that $\{O, I, J\} \subset \mathscr{L}_n \cap D$. It is known that the tangent lines of $\mathscr{L}_n$ at P are given by

$$\sum_{l+j+k=m} \frac{1}{l!j!k!} \left[\frac{\partial^m F}{\partial X^l \partial Y^j \partial Z^k} \right] \Bigg|_P X^l Y^j Z^k = 0, \tag{39}$$

where $m = \text{mult}_p(\mathscr{L}_n)$. For $P = O$, Proposition 2 shows that the tangents of $\mathscr{L}_n$ at O are

$$\mu_j X - Y = 0, \tag{40}$$

where μ_j is as in equation (12).

Now let's consider the case $P = I$. If $0 < k < n$, from equation (34) follows that $\left[\dfrac{\partial^m F}{\partial X^l \partial Y^j \partial Z^k} \right] \Bigg|_P = 0$, and if $k = n$ we get $\left[\dfrac{\partial^m F}{\partial X^l \partial Y^j \partial Z^k} \right] \Bigg|_P = -n! 2^n i^n$. Thus, let $k = 0$. Then, by equation (8),

$$\frac{\partial^n F}{\partial x^{n-j} \partial y^j} \Bigg|_P = \sum_{k=0}^{n} \binom{n}{k} (2n-2k)^{\underline{n-j}} (2k)^{\underline{j}} (i)^{n-2k+j} = n! 2^n (i)^{n-j}. \tag{41}$$

With these observations, it can be seen that the equation (39) is equivalent to

$$(X - iY)^n - Z^n = 0. \tag{42}$$

Consequently, the tangent lines of $\mathscr{L}_n$ at I are

$$X - iY - \zeta_j Z = 0, \tag{43}$$

with ζ_j a n^{th}-root of the unity. Similarly, the tangents of $\mathscr{L}_n$ at J are

$$X + iY - \zeta_j Z = 0. \tag{44}$$

On the other hand, by Proposition 11, $\text{mult}_o(D) = \mu_I(D) = \mu_J(D) = n-1$. Now we calculate the tangents of D at O, I and J. We find out that they are given by:

$$(iX+Y)^{n-1} - (iX-Y)^{n-1} = 0, \tag{45}$$

$$(X-iY)^{n-1} = 0, \text{ and} \tag{46}$$

$$(Y-iX)^{n-1} = 0, \tag{47}$$

respectively. Therefore, the tangent lines are:

$$(i-i\zeta_j)X + (1+\zeta_j)Y = 0, \tag{48}$$

where ζ_j is a $(n-1)^{th}$-root of the unity[7],

$$X - iY = 0, \tag{49}$$

$$Y - iX = 0, \tag{50}$$

respectively. The Proposition 13 implies that $I_I(\mathscr{L}_n, D) = I_J(\mathscr{L}_n, D) = n(n-1)$, since it is clear that the tangents of $\mathscr{L}_n$ and D at I and J are pairwise different. Similarly, when n is even, we have that $I_O(\mathscr{L}_n, D) = n(n-1)$. When n is odd, -1 is a $(n-1)^{th}$-root of the unity, and then $X = 0$ is a *common* tangent line. Proposition 13 implies that $I_O(\mathscr{L}_n, D) > n(n-1)$. □

Definition 21. A point $P \in \mathbb{CP}^2$ is called *focal* of a curve C if the lines $\overline{IP}$ and $\overline{JP}$ are tangents of the curve. Equivalently, P is focal if $P = L_1 \cap L_2$, where L_1 is any tangent of C at I, and L_2 one at J

Proposition 15. *The points $[\zeta_j : 0 : 1]$ are n of the n^2 focal points of $\mathscr{L}_n$, where ζ_j is a n^{th}-root of the unity.*

Proof. From equations (43)-(44) it is easy to see that $[\zeta_j : 0 : 1]$ lies in the intersection of the j^{th} tangent of $\mathscr{L}_n$ at I and J. □

Remark 9. Note that these points can be projected onto the affine plane $\mathbb{C}^2$ to take the form $(\zeta_j, 0) \in \mathbb{C}^2$ which, at the same time, can be projected onto $\mathbb{C}$ as ζ_j. This justifies the use of the term in the previous section.

Definition 22. Let C be a curve defined by $F(X,Y,Z) = 0$ of degree $d > 1$. Making a change of coordinates, if necessary, the map

$$\varphi : C \to \mathbb{CP}^1; [X : Y : Z] \mapsto [X : Z]$$

is well-defined. Let $P = [P_1 : P_2 : P_3]$ be a point of C. We define the *ramification index* $\nu_\varphi P$ *of* φ *at* P as the multiplicity of the root $Y = P_2$ of the polynomial $F(P_1, Y, P_3)$. The point P is said to be a *ramification point* if $\nu_\varphi P > 1$.

[7]According to our definition, giving a line L, the tangent to L at a point on L is the line L itself.

Remark 10. The definition of ramification points is more general and depends on the map between, generally, the two Riemann surfaces (*cf.* [18, p. 45, Definition 4.5], [7, p. 217]). We are interested in the maps from the curve C to $\mathbb{CP}^1$ and the ramification points which, up to change of coordinates, are the points determined by φ.

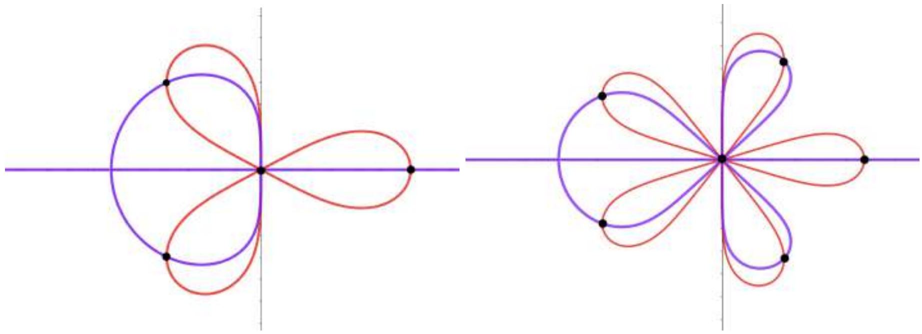
It is easy to see that $\nu_\varphi P > 1$ if and only if $F(P_1, P_2, P_3) = \dfrac{\partial F}{\partial Y}(P_1, P_2, P_3) = 0$. Therefore, if P is a singular point, then $\nu_\varphi P > 1$, but the converse is not true in general. In fact, it is easy to show that $\nu_\varphi P \geq \mathrm{mult}_p(C)$.

The following proposition describes the ramification points of $\mathscr{L}_n$. Its proof is simple but tedious. For this reason, we omit it.

Proposition 16. *$\mathscr{L}_n$ has exactly $2n^2 + 5n$ ramification points if n is even, and exactly $2n^2 + 4n$ if n is odd (counting multiplicities).*

Knowing all the ramification points, we are in a position to use the definition to complete Proposition 14. We omit the proof.

Proposition 17. *For n odd, $I_O(\mathscr{L}_n, D) = n^2$, where D is as in Proposition 14.*



(a) $n = 3$ **(b)** $n = 5$

Figure 20. real ramification points of $\mathscr{L}_n$

In what follows we make use of concepts of algebraic geometry following the terminology of [7] and [21]. Temporarily, we work in a different ambient space, but we will return to $\mathbb{CP}^2$.

Definition 23. A complex-valued function $\varrho : X \to \mathbb{C}$ of an affine variety $X \subset \mathbb{C}^2$ is *regular* if there exists a polynomial $f \in \mathbb{C}[x, y]$ such that $\varrho(p) = f(p)$ for all $p \in X$.

Definition 24. Let G be a finite group of automorphisms of $\mathbb{C}^2$. It is weel-known that $A = \mathbb{C}[x, y]$ is an algebra. Let $A^G = \{f \in A : g^*(f) = f \text{ for all } g \in G\}$ the subalgebra of invariants of G in A. It can be proved that there is a closed set (in the Zariski topology) Y such that $\mathbb{C}[Y] \cong A^G$ (here $\mathbb{C}[Y]$ denotes the ring of regular functions of Y) and a regular map $\eta : \mathbb{C}^2 \to Y$ such that $\eta^*(\mathbb{C}[Y]) = A^G$. The set Y is called *the quotient variety of* $\mathbb{C}^2$ *by the action of* G and it is denoted by $\mathbb{C}^2/G$.

Definition 25. Assume that G is a finite group of linear transformations of $\mathbb{C}^2$ such that $\det(g) = 1$ for every $g \in G$, that is, $G \leq SL(2, \mathbb{C})$. Define a *du Val singularity* as the pair $(\mathbb{C}^2/G, q)$, where q is the image under η of the origin $0 \in \mathbb{C}^2$.

du Val singularities is a very important class of singularities that refine, to some extent, the algebraic classification of singularities. In [5, p. 33], the singularities (or singular points) are *ordinary* if all the tangent lines are different, and they are *non-ordinary* otherwise. However, many singularities have a distinct geometric "origin" that is captured by du Val singularities.

Fortunately, the finite subgroups $G \leq SL(2, \mathbb{C})$ are well-studied. It turns out that G is one of the following form:

1. Cyclic group of order n: $\mathbb{Z}_n = \langle g \rangle$, where $g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ with $\zeta^n = 1$.
2. Binary dihedral group of order $4n$: $\mathbb{D}_n = \langle \sigma, \tau \rangle$, where $\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ with $\zeta^{2n} = 1$, and $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
3. Binary tetrahedral group of order 24: $2\mathbb{T} = \langle \sigma, \tau, \rho \rangle$, where $\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\rho = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$.
4. Binary octahedral group of order 48: $2\mathbb{O} = \langle \sigma, \tau, \rho \rangle$, where $\sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\rho = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$.
5. Binary icosahedral group of order 120: $2\mathbb{I} = \langle \sigma, \tau, \rho \rangle$, where $\sigma = \begin{pmatrix} \zeta^3 & 0 \\ 0 & \zeta^2 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\rho = \frac{1}{\sqrt{5}}\begin{pmatrix} -\zeta + \zeta^4 & -\zeta^2 + \zeta^3 \\ -\zeta^2 + \zeta^3 & \zeta - \zeta^4 \end{pmatrix}$ with $\zeta^5 = 1$.

Calculating the subalgebras of the invariants of each of these groups (we omit the calculations for brevity) we find that the du Val singularities (Fig. 21) are given, up to formal analytic automorphisms of $\mathbb{C}^2$, by the following equations, respectively:

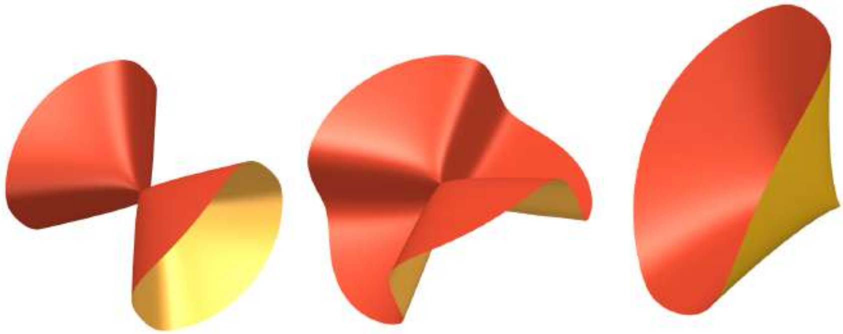
1. $A_{n-1} : \mathbb{C}[x, y]^{\mathbb{Z}_n} = \mathbb{C}[\alpha, \beta, \gamma]/\langle \alpha\beta - \gamma \rangle \cong \mathbb{C}[x, y, z]/\langle x^2 + y^2 + z^n \rangle$, where $\alpha = x^n, \beta = y^n$ and $\gamma = xy$ are the generators of the whole subalgebra of invariants.

Similarly, we get the singularities:

2. $D_{n+2} : x^2 + yz^2 + z^{n+1} = 0, n \geq 2$,
3. $E_6 : x^2 + y^3 + z^4 = 0$,
4. $E_7 : x^2 + y^3 + yz^3 = 0$,

5. $E_8 : x^2 + y^3 + z^5 = 0.$

It can be proved that du Val singularities are precisely simple hypersurface singularities $A - D - E$.



(a) A_3 **(b)** A_4 **(c)** A_6

(d) D_4 **(e)** D_5 **(f)** E_6

Figure 21. du Val singularities

Some singularities originate from others in a geometric (limit) way. They are known as *compound* singularities; the others are usually called *simple* singularities. Particular care must be taken here because all du Val singularities are called simple in other contexts, as they are the singularities of the lowest multiplicity (2 and 3). When n increases, A_n remains a singularity of multiplicity 2, but its geometric nature is much more complex. For example, A_5 is the composition of a tacnode (the degeneration of two nodes to only one point) and a node. Similarly, it is possible to describe D_n, E_6, E_7, and E_8, which are of multiplicity 3. Suffice it to mention that D_4 corresponds to the ordinary (simple) triple point and E_6 to the non-ordinary (simple) point with only one tangent.

Since this local description is invariant under analytic automorphisms, we can use them in the projective plane within a local chart of a given point. Therefore, we close the parentheses with a different ambient space and return to the projective plane.

Definition 26. A morphism $\pi : X \to Y$ between two algebraic varieties is birational if there exists a proper algebraic subvariety $Y' \subset Y$ such that π induces an isomorphism

$\pi|_{X \setminus \pi^{-1}(Y')} : X \setminus \pi^{-1}(Y') \to Y \setminus Y'$. The map π is called *proper* if the preimage of compact subsets is compact.

Definition 27. Let C be a singular curve. A *resolution of singularities of* C is a proper birational morphism

$$\pi : C' \to C$$

such that C' is a non-singular curve. C' is, in general, an abstract variety (not necessarily embedded in $\mathbb{CP}^2$).

The resolution of singularities for curves has been known since Newton (1676) and Riemann (1857), but a rigorous understanding was not achieved until 1944 with the works of Oscar Zariski (1899-1986). In fact, in 1970 the Japanese mathematician Heisuke Hironaka won the *Fields* medal for the proof of the following theorem:

Teorema 1. *Every complex variety (more generally, a variety over a field of characteristic zero) admits a resolution of singularities. Furthermore, the map can be taken to be a projection from a higher dimensional space [9].*

A resolution of the real singularity of $\mathscr{L}_n \subset \mathbb{R}^2$ as a projection from a higher dimensional space is displayed in Figure 22. There are many techniques to define the map π, but to resolve the singularities in a way that we guarantee that C' is again a plane curve requires a quadratic transformation (*cf.* [5, pp. 87-88]). Hereunder, we solve the singularities of $\mathscr{L}_n$. It is interesting to note that only one transformation is necessary to find a smooth model of $\mathscr{L}_n$, *i.e.,* a resolution of its singularities. In general, the first transformation only "improves" the singularities (reduces the multiplicity or makes them geometrically simpler). Returning to the isotropic coordinates equation for $\mathscr{L}_n$ (see Eq. (35)), a simple calculation shows that $\operatorname{mult}_p(\mathscr{L}_{\boldsymbol{n}}) = n$ for any $P \in \{A, B, C\}$. Now, let

$$F_2 = \boldsymbol{F}(\boldsymbol{yz}, \boldsymbol{xz}, \boldsymbol{xy}) = -\boldsymbol{x}^n \boldsymbol{y}^n \boldsymbol{z}^n (\boldsymbol{x}^n + \boldsymbol{y}^n - \boldsymbol{z}^n).$$

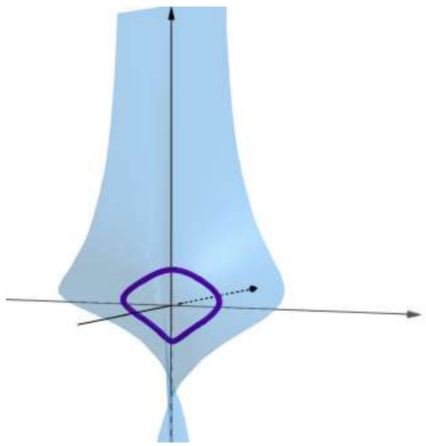
Thus, $\mathscr{L}_n'$ is determined by $\boldsymbol{F'} = \boldsymbol{x}^n + \boldsymbol{y}^n - \boldsymbol{z}^n$ (*cf.* [5, p. 88]). This renowned curve is the *Fermat curve* and we denote it by $\mathcal{F}_n$. Therefore, the resolution of the singularities of $\mathscr{L}_n$ is:

$$\pi : \mathscr{L}_n' = \mathcal{F}_n \to \mathscr{L}_n ; [\boldsymbol{x} : \boldsymbol{y} : \boldsymbol{z}] \mapsto [\boldsymbol{yz} : \boldsymbol{xz} : \boldsymbol{xy}]. \tag{51}$$

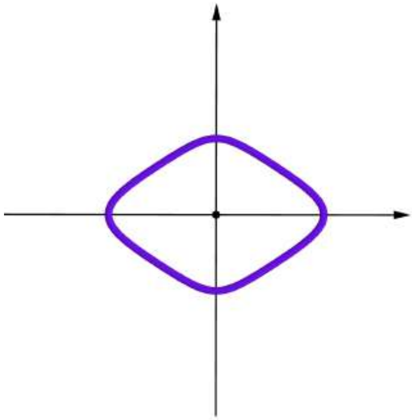
It is crystal clear that π is birational ($\pi^{-1} = \pi$). Since $\mathscr{L}_n$ is compact and Hausdorff, and $|\pi^{-1}(P)| < \infty$ we obtain that π is proper.

Remark 11. Surprisingly, when we return to homogeneous coordinates, the affine Fermat curve, *i.e.*, $\boldsymbol{x}^n + \boldsymbol{y}^n - 1 = 0$ is $\mathscr{H}_n$.

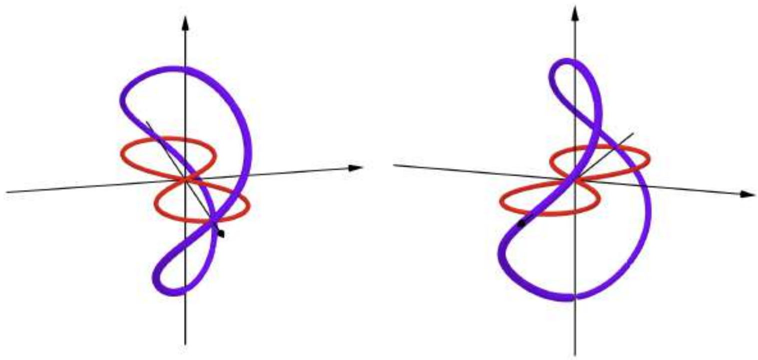
Proposition 18. *For* $P \in \{A : B : C\}$, $|\pi^{-1}(P)| = n$.



(a) $B = \{(x, y, z) \in \mathbb{R}^3 : y = xz\}$



(b) Projection of closure of $\mathscr{L}_2 \setminus 0$ in B



(c) Resolution $\mathscr{L}_2'$ of $\mathscr{L}_2$ as a projection map

Figure 22. Resolution of real singularity of $\mathscr{L}_2$

Proof. Let $P = C = [0 : 0 : 1] \in \mathscr{L}_n$. Then $\boldsymbol{z} = 0$ and $\boldsymbol{y} = \dfrac{1}{\boldsymbol{x}}$. It follows that

$$\pi^{-1}(P) = \left\{ [\boldsymbol{x} : \boldsymbol{y} : \boldsymbol{z}] : \boldsymbol{z} = 0, \boldsymbol{y} = \frac{1}{\boldsymbol{x}} \right\}.$$

But $\boldsymbol{x}^n + \dfrac{1}{\boldsymbol{x}^n} = 0$ if and only if $\boldsymbol{x} = \sqrt[2n]{-1}$. Thus,

$$\pi^{-1}(P) = \left\{ \left[\sqrt[2n]{-1} : \frac{1}{\sqrt[2n]{-1}} : 0 \right] \right\}.$$

Nevertheless, these are only n points because

$$[\omega_j : \omega_j^{-1} : 0] = [-\omega_j : -\omega_j^{-1} : 0] = [\omega_k : \omega_k^{-1} : 0]$$

with ω_i $2n^{th}$ roots of -1. A similar argument shows that $|\pi^{-1}(P)| = n$ for $P = A, B$. $\square$

Definition 28. Let C be a curve determined by F and $P \in C$ be a point in a coordinate system such that $[0:1:0]$ lies neither in C nor in the tangents of the inflection points. We define

$$\delta(P) = \frac{1}{2}\left(I_P\left(F, \frac{\partial F}{\partial Y}\right) - \nu_\varphi P + |\pi^{-1}(P)|\right).$$

In [1, pp. 601-627], it is proved that $\delta(P) \in \mathbb{N}$ and in [10, p. 219], that, under the mentioned hypothesis, $\delta(P)$ is independent of the choice of the coordinate system.

Remark 12. This number $\delta(P)$ has a variety of equivalent expressions in terms of the Milnor number at P or the multiplicity (in different senses) at P (*cf.* [1, pp. 601-627], [2, p. 111], [17, pp. 85-100])

It is well-known that every compact surface admits a triangulation (*cf.* [20, p. 350, Theorem 13.4.3]). The notion of triangulation allows us to associate with each surface an integer, defined below:

Definition 29. The *Euler characteristic* $\chi(S)$ of a compact surface S is

$$\chi(S) = |V| - |E| + |F|.$$

It can be shown that this definition is independent of the triangulation. For a proof, see [20, p. 351, Corollary 13.4.6]

Definition 30. The (topological) *genus* of a compact surface S is the positive integer

$$g = 1 - \frac{1}{2}\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

Proposition 19. *Let $C \subset \mathbb{CP}^2$ be a non-singular curve determined by F (of degree d). Then C has a holomorphic atlas, that is, C is a Riemann surface.*

Proof. See [10, p. 127, Proposition 5.28]. □

Definition 31. Since a non-singular curve C' has the Riemann surface structure, we define the *genus of a singular curve C* as the genus of its resolution of singularities.

Remark 13. As we mentioned, the non-singular model of C is not unique, but there is minimal resolution, that is, a normalization. Any other resolution is equivalent, as a Riemann surface, to this minimal resolution (*cf.* [11, p. 76, Theorem 2.16]). Therefore, the genus of a singular curve is well-defined. Nevertheless, we have been preparing the way to find the genus of $\mathscr{L}_n$ in a way that partially avoids the resolution.

Teorema 2. *(Noether, M.) The genus g of a projective curve C of degree d in $\mathbb{CP}^2$ is*

$$g = \frac{1}{2}(d-1)(d-2) - \sum_{P \in Sin(C)} \delta(P)$$

Proof. We prove that $\chi(C') = d(3-d) + \sum_{P \in Sing(C)} 2\delta(P)$, where C' is the resolution of singularities of C. Let $\mathcal{R} = \{P \in C : \nu_\varphi P > 1\}$ be the set of ramification points of C (with respect to φ) and $\{\{V\}, \{E\}, \{F\}\}$ be a triangulation of $\mathbb{CP}^1$ such that $\varphi(R) \subset V$. This triangulation induces a triangulation of C' such that $V' = (\varphi \circ \pi)^{-1}(V)$, $|E'| = d|E|$ and $|F'| = d|F|$.

By the Riemann-Hurwitz formula applied to φ (see [18, p.52, Theorem 4.16]) the preimage of any point $Q \in \mathbb{CP}^1$ under φ contains exactly $d - \sum_{P \in \varphi^{-1}(Q)} (\nu_\varphi P - 1)$ points; Now, if $P \notin \mathcal{R}$, $\nu_\varphi P = 1$, and since $\mathcal{R} \subset \varphi^{-1}(V)$,

$$|\varphi^{-1}(V)| = d|V| - \sum_{P \in \mathcal{R}} (\nu_\varphi(P) - 1).$$

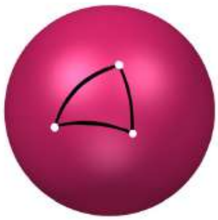
Using the fact that $\varphi^{-1}(V)$ contain all the singularities of C, we get

$$|V'| = |\pi^{-1}(\varphi^{-1}(V))| = d|V| - \sum_{P \in \mathcal{R}} (\nu_\varphi(P) - 1) + \sum_{P \in Sin(C)} (|\pi^{-1}(P)| - 1).$$

Therefore,

$$\begin{aligned} \chi(C') &= |V'| - |E'| + |F'| \\ &= d(|V| - |E| + |F|) - \sum_{P \in \pi(\mathcal{R})} (\nu_\varphi(P) - 1) + \sum_{P \in Sin(C)} (|\pi^{-1}(P)| - 1). \end{aligned}$$

$$\begin{array}{ccc} \mathcal{R} \subset \varphi^{-1}(V) \subset & C & \xrightarrow{\ \varphi\ } & \mathbb{CP}^1 \\ & \pi \Big\uparrow & & \\ & C' & & \end{array}$$



On the other hand, we know that $|V| - |E| + |F| = \chi(\mathbb{CP}^1) = 2$ and it is easy to see that if $P \in \mathcal{R} \setminus \mathrm{Sing}(C)$, $\nu_\varphi P = 2$ and $I_P\left(F, \dfrac{\partial F}{\partial Y}\right) = 1$. In particular,

$$\sum_{P \in \mathcal{R} \setminus \mathrm{Sing}(C)} (\nu_\varphi(P) - 1) = \sum_{P \in \mathcal{R} \setminus Sin(C)} I_P\left(F, \frac{\partial F}{\partial Y}\right).$$

By definition, $\mathrm{Sing}(C) \subset \mathcal{R}$, and since the degree of $\dfrac{\partial F}{\partial Y}$ is $d-1$ –by Bezout's theorem– we obtain that:

$$\sum_{P\in\mathcal{R}\setminus Sin(C)} I_P\left(F, \frac{\partial F}{\partial Y}\right) = d(d-1) - \sum_{P\in Sin(C)} I_P\left(F, \frac{\partial F}{\partial Y}\right).$$

Therefore,

$$\begin{aligned}\chi(C') =&2d - \sum_{P\in\pi(\mathcal{R})\setminus Sin(C)} I_P\left(F, \frac{\partial F}{\partial Y}\right) + \sum_{P\in Sin(C)} (|\pi^{-1}(P)| - 1 - \nu_\varphi P + 1)\\ =&2d - d(d-1) + \sum_{P\in Sin(C)} \left(I_P\left(F, \frac{\partial F}{\partial Y}\right) + |\pi^{-1}(P)| - \nu_\varphi P\right)\\ =& \, d(3-d) + \sum_{P\in Sing(C)} 2\delta(P).\end{aligned}$$

which completes the proof, because

$$\begin{aligned}g(C) = g(C') = 1 - \frac{1}{2}\chi(C') = 1 - \frac{1}{2}&\left(d(3-d) + \sum_{P\in Sing(C)} 2\delta(P)\right)\\ &= \frac{1}{2}(d-1)(d-2) - \sum_{P\in Sing(C)} \delta(P).\end{aligned}$$

□

Proposition 20. *The genus of* $\mathscr{L}_n$ *is* $\frac{1}{2}(n-1)(n-2)$.

Proof. It follows from Noether's theorem and the Propositions 14, 16, 17 and 18. □

Remark 14. Note that the genus of the Fermat curve, and therefore of $\mathscr{H}_n$ (by Clebsch's formula, see [4, p. 179]) is precisely $\frac{1}{2}(n-1)(n-2)$. This is, therefore, the genus of any smooth curve.

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