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An Introduction to Calculus in the $q-$ Real Spinor Variables

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Abstract. In this paper we introduce the calculus in $q-$ real spinor variables. We establish the $q-$ difference operator for $q-$ real spinor variables and the $q-$ spinor real integral formulas. We also define the differential equation on $q-$ real spinor variable, and the suggestions for further work at the end of the paper.

Keywords: $q-$ Spinor real variables, $q-$ differential operators, $q-$ spinor real integral formulas, differential equation in $q-$ spinor variables.

MSC2020: 81Q99, 46E99, 35A24, 15A66, 16T99, 17B37.

Una Introducción al $q-$ Cálculo en la $q-$ Variable Espinorial Real

Resumen. En este artículo introducimos el cálculo en la $q-$ variable espinorial real. Establecemos el $q-$ operador diferencial espinorial y las $q-$ formulas integrales reales spinoriales. También definimos la $q-$ ecuación diferencial en la variable espinorial real y las sugerencias para trabajos futuros al final del artículo.

Palabras clave: $q-$ Variable real spinorial, $q-$ operadores diferenciales, $q-$ formulas integrales reales espinoriales, ecuación diferencial en $q-$ variables espinoriales.

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1. Introduction

Based on the work of Beretetskii et al., Lachieze-Rey, Gori et al., and Cartan, the spinor ψ^α is defined as a magnitude components $\alpha = 1, 2$ expressed as $\psi^\alpha = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$ and its complex conjugate $\dot{\varphi}$ in terms of the rotation matrices (see [2],[1],[8],[5] for more details). Based on the work of the previously mentioned authors, there are two types of operations on spinors, which are reflections and rotations. In group theory, the set of rotations described by the matrices with complex entries is group $SU(2)$, whose generators are the Pauli matrices, described in the work of Zettili [16]. With respect to rotation matrices, Gori et al. mention, in their work, the rotation matrices that originated the Pauli matrices in the form:

$$R_x(\theta) = \begin{bmatrix}\cos(\theta/2) & i\sin(\theta/2)\\ i\sin(\theta/2) & \cos(\theta/2)\end{bmatrix}, R_y(\theta) = \begin{bmatrix}\cos(\theta/2) & \sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{bmatrix}, R_z(\theta) = \begin{bmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{bmatrix}, \tag{1}$$

being θ the angle of rotation [5]. Beretetskii et al. define the covariance and contravariance over the spinors by the relation $\psi'^1 = \psi_2, \psi'^2 = -\psi_1$ from the matrix $g_{\alpha\beta} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ and, similarly, for pointed spinors, $\psi_{\dot{1}} = \psi^{\dot{2}}, \psi_{\dot{2}} = -\psi^{\dot{1}}$ [1]. The same author defines bispinors as the pair $(\psi^\alpha, \varphi_{\dot{\alpha}})$, which form a broader group of Lorentz, and, with them, the scalar product is formulated as $(\psi^\alpha, \varphi_{\dot{\alpha}}) \cdot (f^\alpha, h_{\dot{\alpha}})$. The author of the reference [8] mentions algebra $Cl(3)$ as a space-time formulation generated by vectors e_μ, which form a basis for $R^{1,3}$ that satisfies the relation $e_\mu \cdot e_\nu = g_{\mu\nu}$, inducing a 16 -membered basis, as described below:

1. 1 scalar,
2. (e_0, e_1, e_2, e_3) 4-vector,
3. $(e_0e_1, e_0e_2, e_0e_3, e_1e_2, e_2e_3, e_3e_1)$ 6- bivectors,
4. $(e_1e_2e_3, e_0e_2e_3, e_0e_1e_3, e_0e_1e_2)$ 4-trivectors,
5. $e_5 \equiv e_0e_1e_2e_3$ pseudoscalar.

In accordance with the above, the same author describes the Weyl spinors as $\begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix}$ and the Dirac spinor $\begin{bmatrix}\psi^\alpha\\ \varphi^{\dot{\alpha}}\end{bmatrix}$, as defined in the work of Beretetskii et al [1].

The q - Lorentzian algebra was defined in the reference [11]. The quantum complex spinors have components ψ_1 and ψ^2 and conjugates $\varphi^{\dot{1}}$ and $\varphi_{\dot{2}}$. For all $q \in R - \{0\}$, they satisfy the following q - relations

$$\psi_1\psi^2 = q\psi^2\psi_1, \qquad \psi^2\varphi^{\dot{1}} = q\varphi^{\dot{1}}\psi^2, \tag{2}$$

$$\psi_1\varphi^{\dot{1}} = \varphi^{\dot{1}}\psi_1 - q(q+q^{-1})^{1/2}\varphi_{\dot{2}}\varphi^{\dot{1}}, \qquad \psi^2\varphi^{\dot{1}} = \varphi^{\dot{1}}\psi^2, \tag{3}$$

$$\psi_1\varphi_{\dot{2}} = q\varphi_{\dot{2}}\psi_1, \quad \varphi^{\dot{1}}\varphi_{\dot{2}} = q^{-1}\varphi_{\dot{2}}\varphi^{\dot{1}}. \tag{4}$$

Definition 1.1. Considering spinors $\psi^{\alpha}, \varphi^{\dot{\alpha}}$ and $\left\{\tau^1, T^2, S^1, \sigma^2\right\}$ as the generators of the q - Lorentzian algebra for the group $U_q(su(2))$ [12], we have:

1. For $\psi^{\alpha}, \alpha = 1, 2$

$$\begin{aligned}
\tau^1\psi_1 &= \psi_1\tau^1, \\
\tau^1\psi^2 &= \psi^2\tau^1 - q(q+q^{-1})^2\psi_1 T^2, \\
T^2\psi_1 &= q^{-1}\psi_1 T^2, \\
S^1\psi_1 &= q\psi_1 S^1, \\
T^2\psi^2 &= q\psi^2 T^2, \\
S^1\psi^2 &= q^{-1}\psi^2 S^1 - \psi_1\sigma^2, \\
\sigma^2\psi_1 &= \psi_1\sigma^2, \\
\sigma^2\psi^2 &= \psi^2\sigma^2.
\end{aligned}$$

2. Their complex conjugates $\varphi^{\dot{\alpha}}, \alpha = \dot{1}, \dot{2}$

$$\tau^1\varphi^{\dot{1}} = q^{-1}\varphi^{\dot{1}}\tau^1, \tag{5}$$
$$\tau^1\varphi_{\dot{2}} = q\varphi_{\dot{2}}\tau^1, \tag{6}$$
$$T^2\varphi^{\dot{1}} = \varphi^{\dot{1}}T^2 + q^{-1}\varphi_{\dot{2}}\tau^1, \tag{7}$$
$$T^2\varphi_{\dot{2}} = \varphi_{\dot{2}}T^2, \tag{8}$$
$$S^1\varphi^{\dot{1}} = \varphi^{\dot{1}}S^1, \tag{9}$$
$$\sigma^2\varphi^{\dot{1}} = q\varphi^{\dot{1}}\sigma^2 + (q+q^{-1})^2\varphi_{\dot{2}}S^1, \tag{10}$$
$$\sigma^2\varphi_{\dot{2}} = q\varphi_{\dot{2}}\sigma^2. \tag{11}$$

Deformed commutation relations for q - Lorentzian algebra are defined in the next proposition, on the quantum-symmetric plane and the quantum anti-symmetric plane.

Proposition 1.2. *Consider generator T of the set $\left\{\tau^1, T^2, S^1, \sigma^2\right\}$ for the algebra $U_q(su(2))$ and the relations 2, 3, and 4, defined in [11] and [12]. The q - Lorentzian algebra for spinors in the deformed space is defined through the following relations:*

$$T(\psi_1\psi^2 - q\psi^2\psi_1) = (\psi_1\psi^2 - q\psi^2\psi_1)T, \tag{12}$$
$$T(\psi_1\psi^2 - q\psi^2\psi_1) = (\psi_1\psi^2 - q\psi 2\psi^1)T, \tag{13}$$
$$T(\varphi^{\dot{1}}\varphi_{\dot{2}} + q^{-1}\varphi_{\dot{2}}\varphi^{\dot{1}}) = (\varphi^{\dot{1}}\varphi_{\dot{2}} + q^{-1}\varphi_{\dot{2}}\varphi^{\dot{1}})T, \tag{14}$$
$$T(\varphi^{\dot{1}}\varphi_{\dot{2}} + q^{-1}\varphi_{\dot{2}}\varphi^{\dot{1}}) = (\varphi^{\dot{1}}\varphi_{\dot{2}} + q^{-1}\varphi_{\dot{2}}\varphi^{\dot{1}})T. \tag{15}$$

Definition 1.3. The following are the bosonic q - deformed Minkowskian Pauli spin matrices defined in the Schmidt work [12]:

$$(\sigma^+)_{\alpha\dot{\beta}} = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}, \quad (\sigma^-)_{\alpha\dot{\beta}} = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, \tag{16}$$

$$(\sigma^3)_{\alpha\dot{\beta}} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\sigma^0)_{\alpha\dot{\beta}} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{bmatrix}. \tag{17}$$

Likewise, the conjugated Pauli matrices are:

$$(\overline{\sigma}^+)_{\dot{\alpha}\beta} = \begin{bmatrix} 0 & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad (\overline{\sigma}^-)_{\dot{\alpha}\beta} = \begin{bmatrix} q^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{18}$$

$$(\overline{\sigma}^3)_{\dot{\alpha}\beta} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\overline{\sigma}^0)_{\dot{\alpha}\beta} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{bmatrix}. \tag{19}$$

The inverse Pauli matrices

$$(\sigma_+^{-1})_{\alpha\dot{\beta}} = \begin{bmatrix} 0 & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad (\sigma_-^{-1})_{\alpha\dot{\beta}} = \begin{bmatrix} q^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{20}$$

$$(\sigma_3^{-1})_{\alpha\dot{\beta}} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\sigma_0^{-1})_{\alpha\dot{\beta}} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{bmatrix}. \tag{21}$$

Finally

$$(\overline{\sigma}_+^{-1})_{\dot{\alpha}\beta} = \begin{bmatrix} 0 & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad (\overline{\sigma}_-^{-1})_{\dot{\alpha}\beta} = \begin{bmatrix} q^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{22}$$

$$(\overline{\sigma}_3^{-1})_{\dot{\alpha}\beta} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\overline{\sigma}_0^{-1})_{\dot{\alpha}\beta} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{bmatrix}. \tag{23}$$

Definition 1.4. The q - Lorentzian spinor variables or q - spinor variables are defined according to the expressions (2), (3), and (4) as follows:

$$u_1^2 \equiv \psi_1\psi^2 - q\psi^2\psi_1, \tag{24}$$
$$v^{\mathrm{i}2} \equiv \psi^2\varphi^{\mathrm{i}} - q\varphi^{\mathrm{i}}\psi^2, \tag{25}$$
$$x_{1\dot{2}}^{\mathrm{i}} \equiv \psi_1\varphi^{\mathrm{i}} - \varphi^{\mathrm{i}}\psi_1 + q(q+1)^{-1/2}\varphi_{\dot{2}}\varphi^{\mathrm{i}}, \tag{26}$$
$$y^{2\mathrm{i}} \equiv \psi^2\varphi^{\mathrm{i}} - \varphi^{\mathrm{i}}\psi^2, \tag{27}$$
$$z_{\dot{2}}^{\mathrm{i}} \equiv \psi_1\varphi_{\dot{2}} - q^{-1}\varphi_{\dot{2}}\varphi^{\mathrm{i}}, \tag{28}$$
$$t_{1\dot{2}} \equiv \psi_1\varphi_{\dot{2}} - q\varphi_{\dot{2}}\psi_1. \tag{29}$$

Definition 1.5. We consider the set $U = \left\{u_1^2, v^{\mathrm{i}2}, x_{1\dot{2}}^{\mathrm{i}}, z_{\dot{2}}^{\mathrm{i}}, y^{2\mathrm{i}}, t_{1\dot{2}}\right\} \subset \mathbb{C}$. A function on the q - spinor variables is defined as $\Psi(U) = \Psi(u_1^2, v^{\mathrm{i}2}, x_{1\dot{2}}^{\mathrm{i}}, z_{\dot{2}}^{\mathrm{i}}, y^{2\mathrm{i}}, t_{1\dot{2}})$.

Definition 1.6. Let $f, g : U \longrightarrow \mathbb{C}$ be functions and $u^\beta \in U$. The following properties are satisfy on the q - spinor variables, we state some clear properties of the functions on the q - spinor variables.

1. $(f+g)(u^\beta) = f(u^\beta) + g(u^\beta)$.
2. $(f \cdot g)(u^\beta) = f(u^\beta) \cdot g(u^\beta)$.
3. $(f-g)(u^\beta) = f(u^\beta) - g(u^\beta)$.
4. $\left(\dfrac{f}{g}\right)(u^\beta) = \dfrac{f(u^\beta)}{g(u^\beta)}, \quad g(u^\beta) \neq 0.$

Definition 1.7. For a function $f : U \longrightarrow \mathbb{C}$ and $u^\beta \in \mathbb{C}$, the q - spinor derivative is defined as [7] :

$$\frac{\mathrm{d}_q f}{\mathrm{d}_q u^\beta} = \frac{f((qu)^\beta) - qf(u^\beta)}{(qu)^\beta - qu^\beta}, \tag{30}$$

and its conjugate complex

$$\frac{\mathrm{d}_q f}{\mathrm{d}_q v^{\dot{\alpha}}} = \frac{f((qv)^{\dot{\alpha}}) - qf(v^{\dot{\alpha}})}{(qv)^{\dot{\alpha}} - qv^{\dot{\alpha}}}. \tag{31}$$

1.1. Clifford algebra and Dirac operator

Let $\{\gamma_1, \gamma_2, \cdots, \gamma_n,\}$ be an orthonormal basis of R^n. The *Clifford algebra* is generated over $\mathbb{R}^n$ under the relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\delta_{\mu\nu}\gamma_0, \quad \gamma_\mu^2 = -|\gamma_\mu|^2\gamma_0, \quad \mu, \nu = 1, 2, ..., n, \tag{32}$$

where $\delta_{\mu\nu}$ is the Kronecker symbol (see [10], [3], [9] for more details). We will denote the Clifford algebra by Cl_n, and each element in Cl_n can be expressed by its components as

$\sum_a \gamma_a x_a$, where $a = (\mu_1, ..., \mu_n)$ with each $\mu_l \in \{1, 2, ..., n\}$. Any element $\boldsymbol{x} \in R^n$ can be identified with a 1-vector in the Clifford algebra [9]

$$(x_1, x_2, ..., x_n) \longrightarrow \boldsymbol{x} = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n. \tag{33}$$

On other hand, the Dirac operator used here is

$$D := \gamma_\mu \frac{\partial}{\partial x_\mu}, \tag{34}$$

we refer to reader to [3], [4], [**?**] for more details.

1.2. $q-$ deformed Dirac matrices

Definition 1.8. The $q-$ deformed Dirac matrices are defined in [13], and are given by

$$\gamma_\mu := \begin{bmatrix} 0 & (\sigma_\mu)^\alpha_{\dot\beta} \\ (\overline{\sigma}_\mu)^{\dot\alpha}_\beta & 0 \end{bmatrix}, \tag{35}$$

where $(\sigma_\mu)^\alpha_{\dot\beta}$ and $(\overline{\sigma}_\mu)^{\dot\alpha}_\beta$ denote the Pauli matrices of $q-$ deformed Minkowski space (e.g. [12] for more details), and are defined as

$$\begin{gathered} (\sigma_+)^\alpha_{\dot\beta} = \begin{bmatrix} 0 & 0 \\ 0 & kq^{1/2}\lambda_+^{1/2} \end{bmatrix}, \quad (\sigma_3)^\alpha_{\dot\beta} = k\begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix}, \\ (\sigma_-)^\alpha_{\dot\beta} = k\begin{bmatrix} q^{1/2}\lambda_+^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \quad (\sigma_0)^\alpha_{\dot\beta} = k\begin{bmatrix} 0 & -q^{-1} \\ 1 & 0 \end{bmatrix}, \end{gathered} \tag{36}$$

and their conjugated counterparts

$$\begin{gathered} (\overline{\sigma}_+)^\alpha_{\dot\beta} = \begin{bmatrix} 0 & 0 \\ 0 & \overline{k}q^{-1/2}\lambda_+^{1/2} \end{bmatrix}, \quad (\overline{\sigma}_3)^\alpha_{\dot\beta} = \overline{k}\begin{bmatrix} 0 & 1 \\ q^{-1} & 0 \end{bmatrix}, \\ (\overline{\sigma}_-)^\alpha_{\dot\beta} = \overline{k}\begin{bmatrix} q^{-1/2}\lambda_+^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \quad (\overline{\sigma}_0)^\alpha_{\dot\beta} = \overline{k}\begin{bmatrix} 0 & 1 \\ -q & 0 \end{bmatrix}, \end{gathered} \tag{37}$$

where $k, \overline{k}$ are characteristic parameters associated to bosons ($q = +1$) and fermions ($q = -1$).

1.3. $q-$ Spinor complex integral formulas

Definition 1.9. [7] Let Γ_q be the closed contour of the deformed quantum complex plane, and $u_0^\beta, v^{\dot\alpha} \subset \Gamma_q$ point spinors contained in the contour. The q - spinor complex integral formulas are defined by

$$\oint_{\Gamma_q} \frac{\Psi(u^\beta)\mathrm{d}_q u^\beta}{(qu)^\beta - qu_0^\beta} = \frac{1}{q}\sum_{n=0}^{\infty}\left[(\overline{\sigma}_\mu)_{\dot{\alpha}\beta}\Psi(u_0^\beta)\right]^n, \tag{38}$$

$$\oint_{\Gamma_q} \frac{\Psi((qu)^\beta)\mathrm{d}_q u^\beta}{(qu)^\beta - qu_0^\beta} = \sum_{n=0}^{\infty}\left[(\overline{\sigma}_\mu)_{\dot{\alpha}\beta}\Psi((qu_0)^\beta)\right]^n, \tag{39}$$

$$\oint_{\Gamma_q} \frac{\Psi(v^{\dot{\alpha}})\mathrm{d}_q v^{\dot{\alpha}}}{(qv)^{\dot{\alpha}} - qv_0^{\dot{\alpha}}} = \frac{1}{q}\sum_{m=0}^{\infty}\left[(\sigma_\mu)_{\alpha\dot{\beta}}\Psi(v_0^{\dot{\alpha}})\right]^m, \tag{40}$$

$$\oint_{\Gamma_q} \frac{\Psi((qv)^{\dot{\alpha}})\mathrm{d}_q v^{\dot{\alpha}}}{(qv)^{\dot{\alpha}} - qv_0^{\dot{\alpha}}} = \sum_{m=0}^{\infty}\left[(\sigma_\mu)_{\alpha\dot{\beta}}\Psi((qv_0)^{\dot{\alpha}})\right]^m. \tag{41}$$

1.4. Real spinors in the space

The real spinors in the space are defined based on the work of Zatloukal [15], which are 6 bivectors of the spacetime Clifford algebra

$$|\Psi\gamma_1\gamma_2\rangle = -i\hat{\gamma}_2|\Psi\rangle^*, \tag{42}$$
$$|\Psi\gamma_2\gamma_0\rangle = \hat{\gamma}_2|\Psi\rangle^*, \tag{43}$$
$$|\Psi\gamma_3\gamma_0\rangle = \hat{\gamma}_5|\Psi\rangle, \tag{44}$$
$$|\Psi\gamma_3\gamma_0\rangle = -\hat{\gamma}_2\hat{\gamma}_5|\Psi\rangle^*, \tag{45}$$
$$|\Psi\gamma_1\gamma_3\rangle = -i\hat{\gamma}_2\hat{\gamma}_5|\Psi\rangle^*, \tag{46}$$
$$|\Psi\gamma_2\gamma_1\rangle = i|\Psi\rangle, \tag{47}$$

where $|\Psi\rangle^* = (z_0^*, z_1^*, z_2^*, z_3^*)^T$, and we denoted $\hat{\gamma}_5 = i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ as common, and z_0^*, z_1^*, z_2^* and z_3^*

$$z_0^* = \langle\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{48}$$
$$z_1^* = \langle\gamma_1\gamma_3\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{49}$$
$$z_2^* = \langle\gamma_3\gamma_0\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{50}$$
$$z_3^* = \langle\gamma_1\gamma_2\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{51}$$

being $\hat{\gamma}$ the Dirac matrices in the standard representation respectively. Since $\gamma_0^2 = 1$ [15], [6], it follows readily that

$$\langle\gamma_\mu\gamma_\nu\Psi| = \langle\Psi|\hat{\gamma}_\mu\hat{\gamma}_\nu \quad \mu,\nu = 0,1,2,3. \tag{52}$$

Motivation

Though the topic of this paper is $q-$ real spinor calculus, the motivation comes from the study of $q-$ differential and integral calculus in spinor variables studied in [7] and the real spinors in the space based on the Zatloukal's work ([15]). According to the above, our interest here is to study and relate the $q-$ differential and integral calculus on $q-$ spinor variables with the real spinor in the space, and their implications with the differential equations. The main aim of this work is therefore to study the $q-$ differential and integral calculus and the differential equations on real spinor variables. Also it is found the solutions to the differential equation in $q-$ real spinor variables.

This paper is organized as follows. We briefly recall the preliminaries will be used in this paper in Sect.2. The $q-$ differential operators for $q-$ spinor variables, the $q-$ spinor chain rule, the new $q-$ differential operator, the $q-$ Dirac differential operator, and the integral formulas in $q-$ spinor variables are then proposed in Sect. 3. In Sect. 4 the differential equations in $q-$ real spinor variables are stablished. In the Sect. 5. Finally in the last Section some suggestions for further work are presented.

Notation

In the section 2, we will denote by $\boldsymbol{x}_{\dot{\alpha}}^{\beta}$ instead of $\gamma_\mu\gamma_\nu u_{\dot{\alpha}}^{\beta}$, and the $q-$ real spinor derivative by $\frac{\boldsymbol{\partial}_q\psi}{\boldsymbol{\partial}_q\boldsymbol{x}_{\dot{\alpha}}^{\beta}}$.

2. $q-$ Difference operators for $q-$ real spinor variables

In this section we will mention about the $q-$ difference operators for $q-$ real spinor variables considering the Section 1.4. To begin, first we define the function on $q-$ real spinor

2.5. Functions on $q-$ real spinor variables

Proposition 2.1. *We consider the set* $u_{\dot{\alpha}}^{\beta} = \left\{u_1^2, v^{\dot{1}2}, x_{1\dot{2}}^{\dot{1}}, z_{\dot{2}}^{\dot{1}}, y^{2\dot{1}}, t_{1\dot{2}}\right\} \subset C$. *A function on the q - real spinor variables is defined as*

$$\psi(\gamma_\mu\gamma_\nu u_{\dot{\alpha}}^{\beta}) = \psi(\gamma_\mu\gamma_\nu u_1^2, \gamma_\mu\gamma_\nu v^{\dot{1}2}, \gamma_\mu\gamma_\nu x_{1\dot{2}}^{\dot{1}}, \gamma_\mu\gamma_\nu z_{\dot{2}}^{\dot{1}}, \gamma_\mu\gamma_\nu y^{2\dot{1}}, \gamma_\mu\gamma_\nu t_{1\dot{2}}). \tag{53}$$

Proof. It is sufficient to use (52) together with the observation that $\langle\gamma_\mu\gamma_\nu u_{\dot{\alpha}}^{\beta}|\psi\rangle = \psi(\gamma_\mu\gamma_\nu u_{\dot{\alpha}}^{\beta})$. ☑

Remark 2.2. For convenience we will to denote the function on $q-$ real spinor variables as $\psi(\boldsymbol{x}_{\dot{\alpha}}^{\beta})$.

Taking into account the above remark, we can define the following properties for functions on the $q-$ real spinor variables similary to Definition 1.6

Definition 2.3. Let $f, g : u_{\dot{\alpha}}^{\beta} \longrightarrow R^m$ be functions and $\boldsymbol{x}_{\dot{\beta}}^{\alpha} \in u_{\dot{\alpha}}^{\beta}$. The following properties are satisfy on the functions of q - real spinor variables, and we state some clear properties of the functions on the q - spinor variables

1. $(f+g)(\boldsymbol{x}_{\dot{\beta}}^{\alpha}) = f(\boldsymbol{x}_{\dot{\beta}}^{\alpha}) + g(\boldsymbol{x}_{\dot{\beta}}^{\alpha})$.

2. $(f \cdot g)(\boldsymbol{x}^{\alpha}_{\dot{\beta}}) = f(\boldsymbol{x}^{\alpha}_{\dot{\beta}}) \cdot g(\boldsymbol{x}^{\alpha}_{\dot{\beta}})$.

3. $(f - g)(\boldsymbol{x}^{\alpha}_{\dot{\beta}}) = f(\boldsymbol{x}^{\alpha}_{\dot{\beta}}) - g(\boldsymbol{x}^{\alpha}_{\dot{\beta}})$.

4. $\left(\frac{f}{g}\right)(\boldsymbol{x}^{\alpha}_{\dot{\beta}}) = \frac{f(\boldsymbol{x}^{\alpha}_{\dot{\beta}})}{g(\boldsymbol{x}^{\alpha}_{\dot{\beta}})}, \quad g(\boldsymbol{x}^{\alpha}_{\dot{\beta}}) \neq 0.$

2.6. $q-$ *Real spinor derivative*

With the mathematical formalism of above section in hand, we are in a position to define the $q-$ real spinor derivative which is mentioned in the following proposition

Proposition 2.4. *Let $\psi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$. The $q-$ real spinor derivative can be expressed by*

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{54}$$

where $\boldsymbol{u}^{\beta}_{\dot{\alpha}} = (\gamma_\mu \gamma_\nu u)^{\beta}_{\dot{\alpha}}$.

Proof. From (30) we can see that

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \frac{\psi((q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot{\alpha}}) - q\psi(\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{(q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}}, \tag{55}$$

writing $(q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot{\alpha}} = \boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}$ yields $(q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}} = \boldsymbol{x}^{\beta}_{\dot{\alpha}}$, interchanging $q\psi(\boldsymbol{u}^{\beta}_{\dot{\alpha}})$ by $-\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})$, we can rewrite (55) as

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},$$

this is the desired conclusion ☑

Theorem 2.5. *Assume that $\psi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ and $\varphi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ are spinorial differentiable at $\boldsymbol{x}^{\beta}_{\dot{\alpha}} \in R^m$. Then,*

1. *the sum $\psi + \varphi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ is $q-$ differentiable at $\boldsymbol{u}^{\beta}_{\dot{\alpha}}$ and*

$$\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi + \varphi) = \frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\boldsymbol{\partial}_q \varphi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{56}$$

2. *the product $\psi\varphi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ is $q-$ differentiable at $\boldsymbol{u}^{\beta}_{\dot{\alpha}}$ and*

$$\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi\varphi) = \frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}})\frac{\boldsymbol{\partial}_q \varphi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{57}$$

3. $(\boldsymbol{x}^{\beta}_{\dot{\alpha}})^n : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$

$$\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}[(\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n = \sum_{k=1}^{n-1}(-1)^{k-1}(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{n-k}(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{k-1}. \tag{58}$$

Proof. 1.

$$\begin{aligned}
\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi+\varphi) = &\ \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},\\
= &\ \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},\\
= &\ \frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\boldsymbol{\partial}_q \varphi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}.
\end{aligned}$$

2.

$$\begin{aligned}
\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi\varphi) = &\ \frac{\psi(\boldsymbol{x}^{\beta} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})\varphi(\boldsymbol{x}^{\beta} - q\boldsymbol{u}^{\beta}_{\dot{\beta}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},\\
= &\ \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})\varphi(\boldsymbol{x}^{\beta} - q\boldsymbol{u}^{\beta}_{\dot{\beta}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},\\
= &\ \left[\frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right]\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}})\left[\frac{\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right],
\end{aligned}$$

for convenience we can interchange $\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})$ by $-\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})$ into the above expression, resulting

$$\left[\frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right]\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}})\left[\frac{\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right],$$

finally we obtain (57).

3.

$$\begin{aligned}
\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}[(\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n = &\ \frac{(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})^n + (q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^n}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},\\
= &\ (\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})^{n-1} - (\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})^{n-2}(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \dots + (q\boldsymbol{u}^{\beta}_{\dot{\alpha}}))^{n-1},\\
= &\ \sum_{k=1}^{n-1}(-1)^{k-1}(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{n-k}(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{k-1}.
\end{aligned}$$

☑

Example 2.6. *Let $\psi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ be a function on $q-$ real spinor variables of the form $\psi(\boldsymbol{v}^{\mathrm{i}2}) = (\boldsymbol{v}^{\mathrm{i}2})^2 + q\boldsymbol{v}^{\mathrm{i}2}$. Applying (56) and (58) we have*

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{v}^{\mathrm{i}2}} = (\boldsymbol{v}^{\mathrm{i}2} - q\boldsymbol{u}^{\mathrm{i}2})(\boldsymbol{v}^{\mathrm{i}2} - 2q\boldsymbol{u}^{\mathrm{i}2}) + q.$$

Remark 2.7. We assume that $\gamma_\mu \gamma_\nu u^{\beta}_{\dot{\alpha}} \neq (\gamma_\mu \gamma_\nu u)^{\beta}_{\dot{\alpha}}$.

This allows us to introduce the $q-$ chain rule for real spinor variables.

2.7. $q-$ chain rule for real spinor variables

Theorem 2.8. *If Ψ and $\boldsymbol{x}^{\beta}_{\dot{\alpha}}$ are both differentiable at $x_j, j = 1, .., n$ and $\Psi(x_j)$ is the composite function defined by $\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)]$, then Ψ is differentiable and $\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j}$ is given by the product*

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j} = \frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} \frac{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}{\boldsymbol{\partial}_q x_j}. \tag{59}$$

Proof. The following assumptions will be needed throughout the proof:

$$\frac{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}{\boldsymbol{\partial}_q x_j} = \frac{\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)}{x_j}, \tag{60}$$

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j} = \frac{\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j - qx_j)] + \Psi[q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j)]}{x_j}. \tag{61}$$

According to the above assumptions, we can claim that (54) can be written as

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \frac{\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j - qx_j)] + \Psi[q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j)]}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j)}, \tag{62}$$

replacing the denominator of (62) by $\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)$ yields

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \frac{\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j - qx_j)] + \Psi[q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j)]}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)}, \tag{63}$$

multipying both sides by $1/x_j$ we obtain

$$\frac{\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)}{x_j} \frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \frac{\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j - qx_j)] + \Psi[q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j)]}{x_j},$$

in virtue of (60) and (61), finally we get (59), and therefore the proof is complete. ☑

Remark 2.9. Similar considerations apply to $\boldsymbol{u}_{\dot\alpha}^{\beta}$, namely

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j} = \frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot\alpha}^{\beta}} \frac{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}{\boldsymbol{\partial}_q x_j}, \tag{64}$$

and $\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}$ is given by

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot\alpha}^{\beta}} = \frac{\Psi(\boldsymbol{u}_{\dot\alpha}^{\beta} - q\boldsymbol{x}_{\dot\alpha}^{\beta}) + \Psi(q\boldsymbol{x}_{\dot\alpha}^{\beta})}{\boldsymbol{u}_{\dot\alpha}^{\beta}}. \tag{65}$$

2.8. $q-$ Difference operators for $q-$ real spinor variables

Proposition 2.10. *Let us consider the $q-$ chain rule for $q-$ real spinor variables (59). According to the expressions (42), (43), (44), (45), (46), and (47), the $q-$ difference operator for $q-$ real spinor variables can be expressed as*

$$\boldsymbol{D}_2^q = \hat\gamma_2 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q x_2}, \tag{66}$$

$$\boldsymbol{D}_j^q = i\hat\gamma_5 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q x_j}, \tag{67}$$

$$\underline{\boldsymbol{D}}_j^q = i\hat\gamma_2\hat\gamma_5 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q x_j}, \quad j = 1, \ldots, 5. \tag{68}$$

Proof. If we prove that the square of (66), (67), and (68) are equivalent to $-\frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q x_j^2}$ and $-\frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q x_2^2}$, then the assertion follows. ☑

Remark 2.11. The expression (66) is called the *the q- Dirac real operator.*

Definition 2.12. From the (48), (49), (50), and (51), we define the $q-$ conjugated real spinor variables as

$$\boldsymbol{v}_{\dot 0} = \langle \boldsymbol{x}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot 0} = \langle \boldsymbol{u}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \tag{69}$$

$$\boldsymbol{v}_{\dot 1} = \langle \gamma_1\gamma_3 \boldsymbol{x}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot 1} = \langle \gamma_1\gamma_3 \boldsymbol{u}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \tag{70}$$

$$\boldsymbol{v}_{\dot 2} = \langle \gamma_3\gamma_0 \boldsymbol{x}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot 2} = \langle \gamma_3\gamma_0 \boldsymbol{u}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \tag{71}$$

$$\boldsymbol{v}_{\dot 3} = \langle \gamma_1\gamma_2 \boldsymbol{x}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot 3} = \langle \gamma_1\gamma_2 \boldsymbol{u}_{\dot\alpha}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle. \tag{72}$$

From the above definition we can construct a function on the $q-$ conjugated real spinor variables of the following form: $\psi : (\boldsymbol{u}_{\dot k}, \boldsymbol{p}_{\dot k}) \longrightarrow R^m$ for all $0 \leq \dot k \leq 3$, this is $\psi(\boldsymbol{v}_{\dot k}, \boldsymbol{p}_{\dot k})$.

Theorem 2.13. *For a function $\psi : (\boldsymbol{v}_{\dot k}, \boldsymbol{p}_{\dot k}) \longrightarrow R^m$, the $q-$ conjugated derivatives are defined as*

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot k}} = \frac{\psi(\boldsymbol{v}_{\dot k} - q\boldsymbol{x}_{\dot\alpha}^{\beta}) + \psi(q\boldsymbol{x}_{\dot\alpha}^{\beta})}{\boldsymbol{v}_{\dot k}}, \tag{73}$$

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot k}} = \frac{\psi(\boldsymbol{p}_{\dot k} - q\boldsymbol{u}_{\dot\alpha}^{\beta}) + \psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})}{\boldsymbol{p}_{\dot k}}. \tag{74}$$

Proof. It suffices to make the substitution $\boldsymbol{x}_{\dot{\alpha}}^{\beta}$ by $\boldsymbol{v}_{\dot{k}}$ and $\boldsymbol{p}_{\dot{k}}$ into (55), which the proof is complete ☑

Theorem 2.14. *The $q-$ difference operators associated to conjugated real spinor variables are given by*

$$\boldsymbol{D}_q = \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{0}}} + \gamma_1\gamma_3 \frac{\boldsymbol{\partial_q}}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{1}}} + i\gamma_3\gamma_{\dot{0}} \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{2}}} + \gamma_1\gamma_2 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{3}}}, \tag{75}$$

$$\boldsymbol{D}'_q = \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{0}}} + \gamma_1\gamma_3 \frac{\boldsymbol{\partial_q}}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{1}}} + i\gamma_3\gamma_0 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{2}}} + \gamma_1\gamma_2 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{3}}}. \tag{76}$$

Proof. Analysis similar to that in the proof of Proposition 5.8. shows that $\boldsymbol{D}_q^2 = \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{0}}^2} - \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{1}}^2} - \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{2}}^2} - \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{3}}^2}$ and $\boldsymbol{D}_q'^2 = \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{0}}^2} - \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{1}}^2} - \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{2}}^2} - \frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{3}}^2}$. ☑

Remark 2.15. The $q-$ difference operators (75) and (76) can be written in terms of (73) and (74) in explicit form as

$$\boldsymbol{D}_q^{\dot{1}}\psi = \gamma_1\gamma_3 \left[\frac{\psi(\boldsymbol{v}_{\dot{1}} - q\boldsymbol{x}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})}{\boldsymbol{v}_{\dot{1}}}\right], \quad \boldsymbol{D}_q^{\dot{2}}\psi = i\gamma_3\gamma_0 \left[\frac{\psi(\boldsymbol{v}_{\dot{2}} - q\boldsymbol{x}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})}{\boldsymbol{v}_{\dot{2}}}\right], \tag{77}$$

$$\boldsymbol{D}_q^{\dot{3}}\psi = \gamma_1\gamma_2 \left[\frac{\psi(\boldsymbol{v}_{\dot{3}} - q\boldsymbol{x}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})}{\boldsymbol{v}_{\dot{3}}}\right], \quad \boldsymbol{D}_q^{\dot{0}}\psi = \frac{\psi(\boldsymbol{v}_{\dot{0}} - q\boldsymbol{x}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})}{\boldsymbol{v}_{\dot{0}}}, \tag{78}$$

$$\boldsymbol{D}_q^{\dot{1}}\psi = \gamma_1\gamma_3 \left[\frac{\psi(\boldsymbol{p}_{\dot{1}} - q\boldsymbol{u}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{u}_{\dot{\alpha}}^{\beta})}{\boldsymbol{p}_{\dot{1}}}\right], \quad \boldsymbol{D}_q^{\dot{2}}\psi = i\gamma_3\gamma_0 \left[\frac{\psi(\boldsymbol{p}_{\dot{2}} - q\boldsymbol{u}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{u}_{\dot{\alpha}}^{\beta})}{\boldsymbol{p}_{\dot{2}}}\right], \tag{79}$$

$$\boldsymbol{D}_q^{\dot{3}}\psi = \gamma_1\gamma_2 \left[\frac{\psi(\boldsymbol{p}_{\dot{3}} - q\boldsymbol{u}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{u}_{\dot{\alpha}}^{\beta})}{\boldsymbol{p}_{\dot{3}}}\right], \quad \boldsymbol{D}_q^{\dot{0}}\psi = \frac{\psi(\boldsymbol{p}_{\dot{0}} - q\boldsymbol{u}_{\dot{\alpha}}^{\beta}) + \psi(q\boldsymbol{u}_{\dot{\alpha}}^{\beta})}{\boldsymbol{p}_{\dot{0}}}. \tag{80}$$

With these results, it is possible to define the $q-$ spinor real integral formulas by the following theorem.

3. $q-$ Spinor real integral formulas

Theorem 3.1. *Let $\psi : u_{\dot{\alpha}}^{\beta} \longrightarrow R^m$ and let Ω_q be a subset over a manifold $\mathcal{M}$ in R^m, the $q-$ spinor real integral formulas of the $q-$ spinor conjugated variables are given by*

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{\boldsymbol{v}_{\dot{k}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n, \tag{81}$$

$$\int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{\boldsymbol{v}_{\dot{k}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(1-q)\boldsymbol{x}^{\beta}_{\dot{\alpha}}]]^n, \tag{82}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{p}_{\dot{k}})\mathbf{d}_q\boldsymbol{p}_{\dot{k}}}{\boldsymbol{p}_{\dot{k}} + \boldsymbol{u}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})]^n, \tag{83}$$

$$\int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{p}_{\dot{k}}]\mathbf{d}_q\boldsymbol{p}_{\dot{k}}}{\boldsymbol{p}_{\dot{k}} + \boldsymbol{u}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(1-q)\boldsymbol{u}^{\beta}_{\dot{\alpha}}]]^n. \tag{84}$$

Proof. First, we present the following changes of variables in (73), $q\boldsymbol{v}_{\dot{k}} = \boldsymbol{v}_{\dot{k}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}$ and $(q^{-1}-1)\boldsymbol{v}_{\dot{k}} = \boldsymbol{x}^{\beta}_{\dot{\alpha}}$, obtaining

$$\frac{\boldsymbol{\partial}_q\psi}{\boldsymbol{\partial}_q\boldsymbol{v}_{\dot{k}}} = \frac{\psi(q\boldsymbol{v}_{\dot{k}}) + \psi[q(q^{-1}-1)\boldsymbol{v}_{\dot{k}}]}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{85}$$

multiplyng both sides by $\mathbf{d}_q\boldsymbol{v}_{\dot{k}}$ we get

$$\mathbf{d}_q\psi = \frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{86}$$

integrating both sides over Ω_q

$$\int_{\Omega_q}\mathbf{d}_q\psi = \int_{\Omega_q}\frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \int_{\Omega_q}\frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{87}$$

hence, to solve the integral $\int_{\Omega_q}\mathbf{d}_q\psi$, we will use similary the proof of the Theorem 2.9 of the reference [7], to obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n + \sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(\boldsymbol{x}^{\beta}_{\dot{\alpha}})(1-q)]]^n,\\
&\quad = \int_{\Omega_q}\frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \int_{\Omega_q}\frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}},
\end{aligned}$$

finally we get

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n, \tag{88}$$

$$\int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(1-q)(\boldsymbol{x}^{\beta}_{\dot{\alpha}})]]^n. \tag{89}$$

The same process is applied to get (83) and (84) from (74) using the changes of variables $q\boldsymbol{p}_k = \boldsymbol{p}_k - q\boldsymbol{u}_{\dot\alpha}^{\beta}$ and $\boldsymbol{u}_{\dot\alpha}^{\beta} = \boldsymbol{p}_k(q^{-1}-1)$. ☑

On other hand, we can stablish the $q-$ spinor real integral from (65), (66), (67), and (68) by our next theorem.

Theorem 3.2. *The $q-$ spinor real integral formulas over Ω_q associated to $q-$ difference operators (66), (67), and (68) are given by*

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} = \sum_{m=0}^{\infty}[\gamma^2\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))]^m, \tag{90}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} = \sum_{m=0}^{\infty}[\gamma^2\psi(\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))]^m, \tag{91}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^5\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_j))]^m, \tag{92}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_j))\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^5\psi(\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))]^m, \tag{93}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^2\gamma^5\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_j))]^m, \tag{94}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_j))\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^2\gamma^5\psi(\boldsymbol{x}_{\dot\alpha}^{\beta}(x_j))]^m. \tag{95}$$

Proof. For the operator (66), first we rewrite (65) replacing $\boldsymbol{u}_{\dot\alpha}^{\beta} - \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)$ by $q\boldsymbol{u}_{\dot\alpha}^{\beta}$ to obtain

$$\begin{aligned}\frac{\boldsymbol{\partial}_q\psi}{\boldsymbol{\partial}_q\boldsymbol{u}_{\dot\alpha}^{\beta}} &= \frac{\psi[\boldsymbol{u}_{\dot\alpha}^{\beta} - q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)] + \psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))}{\boldsymbol{u}_{\dot\alpha}^{\beta}}, \\ &= \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta}) + \psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)},\end{aligned}$$

substituting the above expression into (64)

$$\frac{\boldsymbol{\partial}_q\Psi}{\boldsymbol{\partial}_q x_2} = \left[\frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta}) + \psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)}\right]\frac{\boldsymbol{\partial}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{\boldsymbol{\partial}_q x_2}, \tag{96}$$

multiplying both sides by $\mathbf{d}_q x_2$ we get

$$\mathbf{d}_q\Psi = \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} + \frac{\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))\mathbf{d}_q\boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)}, \tag{97}$$

integrating both sides over Ω_q

$$\int_{\Omega_q} \mathbf{d}_q \Psi = \int_{\Omega_q} \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})\mathbf{d}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} + \int_{\Omega_q} \frac{\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))\mathbf{d}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)}, \tag{98}$$

we can now proceed analogously to the proof of above theorem, obtaining (90) and (91)

$$\begin{aligned} \int_{\Omega_q} \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})\mathbf{d}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} &+ \int_{\Omega_q} \frac{\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_j))\mathbf{d}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} \\ &= \sum_{m=0}^{\infty} \Big[\gamma^2 \psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))\Big]^m + \sum_{m=0}^{\infty} \Big[\gamma^2 \psi(\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))\Big]^m, \end{aligned} \tag{99}$$

therefore,

$$\begin{aligned} \int_{\Omega_q} \frac{\psi(q\boldsymbol{u}_{\dot\alpha}^{\beta})\mathbf{d}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} &= \sum_{m=0}^{\infty} \Big[\gamma^2 \psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))\Big]^m, \\ \int_{\Omega_q} \frac{\psi(q\boldsymbol{x}_{\dot\alpha}^{\beta}(x_j))\mathbf{d}_q \boldsymbol{u}_{\dot\alpha}^{\beta}}{q\boldsymbol{u}_{\dot\alpha}^{\beta} + \boldsymbol{x}_{\dot\alpha}^{\beta}(x_2)} &= \sum_{m=0}^{\infty} \Big[\gamma^2 \psi(\boldsymbol{x}_{\dot\alpha}^{\beta}(x_2))\Big]^m. \end{aligned}$$

The same reasoning applies to the operators (67) and (68), to obtain (92), (93), (94), and (95). ☑

Let us mention an important consequence of the above theorem.

4. Differential equations in $q-$ real spinor variables

Let us consider the following $q-$ real spinor differential equation

$$(\boldsymbol{D}_q^{\mathrm{i}} - m)\psi(\boldsymbol{v}_{\mathrm{i}}) = 0, \quad m \in R. \tag{100}$$

In order to get solution of (100), it is necessary to put the following condition on ψ

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \psi(\boldsymbol{x}_{\dot\alpha}^{\beta}), \quad \alpha, \dot\beta = 1, 2. \tag{101}$$

and the following lemma.

Lemma 4.1. *The integral (101) can be expressed in virtue of (81) as*

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\sum_{n=0}^{\infty} [\gamma^1\gamma^3 \psi(q\boldsymbol{x}_{\dot\alpha}^{\beta})]^n + \sum_{n=0}^{\infty} [\gamma^1\gamma^3 \psi((1-q)\boldsymbol{x}_{\dot\alpha}^{\beta})]^n\right]. \tag{102}$$

Proof. According to (77), the operator $\boldsymbol{D}_q^{\mathrm{i}}$ is defined as $\gamma_1\gamma_3 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\mathrm{i}}}$. Therefore (86) can be expressed as

$$\boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\frac{\psi(q\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} + \frac{\psi[(1-q)\boldsymbol{v}_{\mathrm{i}}]\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} \right], \tag{103}$$

integrating both sides over Ω_q,

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\int_{\Omega_q} \frac{\psi(q\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} + \int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{v}_{\mathrm{i}}]\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} \right], \tag{104}$$

the right side of (104) may be equalated to (88) and (89) finally we obtain

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\sum_{n=0}^{\infty} [\gamma^1\gamma^3\psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n + \sum_{n=0}^{\infty} [\gamma^1\gamma^3\psi((1-q)\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n \right],$$

which is our claim. ☑

Remark 4.2. The expressions (101) and (102) are equivalent.

The solution of (100) is established by our next theorem.

Theorem 4.3. *The solution of (100) over the subset Ω_q can be written as*

$$\int_{\Omega_q} \psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \frac{1}{m}\gamma_1\gamma_3 \left[\sum_{n=0}^{\infty} [\gamma^1\gamma^3\psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n + \sum_{n=0}^{\infty} [\gamma^1\gamma^3\psi((1-q)\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n \right]. \tag{105}$$

Proof. We begin by rewriting (100) as

$$\boldsymbol{D}_q^{\mathrm{i}} \psi(\boldsymbol{v}_{\mathrm{i}}) = m\psi(\boldsymbol{v}_{\mathrm{i}}), \tag{106}$$

multiplying by $\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}$,

$$\boldsymbol{D}_q^{\mathrm{i}} \psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = m\psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}, \tag{107}$$

integrating both sides over the subset Ω_q and using (102)

$$\begin{aligned} \int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} &= m\int_{\Omega_q} \psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}, \\ \gamma_1\gamma_3 \left[\sum_{n=0}^{\infty} [\gamma^1\gamma^3\psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n + \sum_{n=0}^{\infty} [\gamma^1\gamma^3\psi((1-q)\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n \right] &= m\int_{\Omega_q} \psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}, \end{aligned}$$

finally we get (105). ☑

However, the solution of (100) is not unique. According to (101) we can say that the solution of (100) is given by

$$\gamma_1\gamma_3\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}) = m\int_{\Omega_q}\psi(\boldsymbol{v}_{\dot{1}})\mathbf{d}_q\boldsymbol{v}_{\dot{1}}, \tag{108}$$

$$\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}) = \left[\sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n + \sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi((1-q)\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n\right]. \tag{109}$$

Let us to consider the following examples

Example 4.4. *Let $\boldsymbol{D}^{\dot{2}}_q\psi(\boldsymbol{p}_{\dot{2}}) = 0$ be a differential equation in $q-$ real spinor variable. This is a trivial case, where the solution is* $\mathbf{0}$. *For this case, is said that the solution is monogenic.*

Example 4.5. *For $a\boldsymbol{D}^{\dot{3}}_q\psi(\boldsymbol{v}_{\dot{3}}) + bf(\boldsymbol{v}_{\dot{3}}) = 0$ for all $a, b \in R$. Using the results previously mentioned, the solution is given by*

$$\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}) = -\frac{b}{a}\left[\sum_{n=0}^{\infty}[\gamma^1\gamma^2 f(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n + \sum_{n=0}^{\infty}[\gamma^1\gamma^2 f((1-q)\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n\right].$$

Example 4.6. *Let us consider the following differential equation*

$$(\boldsymbol{D}^q_j - \lambda)\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)) = 0, \quad \lambda \in R. \tag{110}$$

In order to solve (110), it is sufficient to consider the following condition taking into account (67)

$$\int_{\Omega_q}\boldsymbol{D}^q_j\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))\mathbf{d}_q\boldsymbol{x}^{\beta}_{\dot{\alpha}} = i\hat{\gamma}_5\psi(x_j), \tag{111}$$

integrating both sides over Ω_q

$$\begin{aligned}\int_{\Omega_q}\boldsymbol{D}^q_j\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))\mathbf{d}_q\boldsymbol{x}^{\beta}_{\dot{\alpha}} &= \lambda\int_{\Omega_q}\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))\mathbf{d}_q\boldsymbol{x}^{\beta}_{\dot{\alpha}}\\ i\hat{\gamma}_5\psi(x_j) &= i\hat{\gamma}_5\lambda\left\{\sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi(q(x_j)]^m + \sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi((1-q)x_j)]^m\right\}.\end{aligned}$$

Therefore,

$$\psi(x_j) = \lambda\left\{\sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi(q(x_j))]^m + \sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi((1-q)x_j)]^m\right\}.$$

5. Suggestions for further work

There is further topic arising from this paper which are worth investigation., there is the problem of describing the function that depends on the quadratic variables $x^2, y^2, z^2, u^2, ux, uy, uz, xy$ and xz, this is $f(ux, uy, uz, xy, xz, yz, x^2, y^2, z^2, u^2)$ and introducing the $q-$ quadratic difference operator over $f \in k\langle x, y, z, u\rangle$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(ux)} = \frac{f(q^2u^2 + ux) - f(ux)}{q^2u^2}, \quad \frac{\partial_{q^2} f}{\partial_{q^2}(uy)} = \frac{f(q^2u^2 + uy) - f(uy)}{q^2u^2}, \quad \frac{\partial_{q^2} f}{\partial_{q^2}(uz)} = \frac{f(q^2u^2 + zuz) - f(uz)}{q^2u^2},$$
$$\frac{\partial_{q^2} f}{\partial_{q^2}(yz)} = \frac{f(q^2y^2 + yz) - f(yz)}{q^2y^2}, \quad \frac{\partial_{q^2} f}{\partial_{q^2}(xy)} = \frac{f(q^2y^2 + xy) - f(xy)}{q^2y^2}, \quad \frac{\partial_{q^2} f}{\partial_{q^2}(xz)} = \frac{f(q^2z^2 + xz) - f(xz)}{q^2z^2},$$

and for a function $f : R^n \longrightarrow R$

$$\begin{aligned}
\frac{\partial_q f}{\partial_q x} &= \frac{f((x + q^2\mathrm{e}_0 u)u) - f(xu)}{qu}, & \frac{\partial_q f}{\partial_q y} &= \frac{f((y + q^2\mathrm{e}_0 u)u) - f(yu)}{qu},\\
\frac{\partial_q f}{\partial_q z} &= \frac{f(y(z + q^2\mathrm{e}_y y)) - f(yz)}{qy}, & \frac{\partial_q f}{\partial_q u} &= \frac{f((u - q^2\mathrm{e}_y y)y) - f(uy)}{qy},\\
\frac{\partial_q f}{\partial_q x} &= \frac{f((x + q^2\mathrm{e}_z z)z) - f(xz)}{qz}, & \frac{\partial_q f}{\partial_q z} &= \frac{f((z + q^2\mathrm{e}_0 u)u) - f(zu)}{qu},\\
\frac{\partial_q f}{\partial_q y} &= \frac{f((y + q^2\mathrm{e}_x x)x) - f(yx)}{qx}, & \frac{\partial_q f}{\partial_q z} &= \frac{f((z + q^2\mathrm{e}_x x)x) - f(zx)}{qx}.
\end{aligned}$$

Analogously, the above expressions can be written in terms of spinor variables.

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