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Extended b-metric preserving functions

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Abstract. In a previous investigation, we present the current state of the family of functions that preserve the weak ultrametric $\mathcal{UD}$ and the set of maps that preserve the extended b–metric $\mathcal{BE}$ and their relation to those existing in the literature. In this article, we continue with the investigation by providing a characterization for the space $\mathcal{BE}$, and this fact allows us to verify that the graph of the elements in $\mathcal{BE}$ are found in the region proposed by J. Doboš and Z. Piotrowski. Furthermore, we generalize some results from Tammatada Khemaratchatakumthorn, Prapanpong Pongsriiam and Suchat Samphavat.

Keywords: Metric spaces, ultrametric, inframetric, extended b-metric, metric preserving functions.

MSC2020: 54E40; 28D05, 28D15.

Funciones que preservan la b-métrica extendida

Resumen. En una investigación previa, presentamos el estado actual de la familia de funciones que preservan la ultramétrica débil $\mathcal{UD}$ y el conjunto de funciones que preservan la b-métrica extendida $\mathcal{BE}$ y su relación con las existentes en la literatura. En este artículo, continuamos con la investigación proporcionando una caracterización para el espacio $\mathcal{BE}$, y este hecho nos permite verificar que la gráfica de los elementos en $\mathcal{BE}$ se encuentran en la región propuesta por J. Doboš y Z. Piotrowski. Además, generalizamos algunos resultados de Tammatada Khemaratchatakumthorn, Prapanpong Pongsriiam and Suchat Samphavat.

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Palabras clave: Espacios métricos, ultramétricos, inframétricos, funciones que preservan la b-métrica extendida.

1. Introduction

Previously, we introduced two families of functions denoted by $\mathcal{UD}$ and $\mathcal{BE}$, we present the relationship between them and other known classes (see [9]). In this article, we continue the investigation by showing that many of the results of Tammatada Khemaratchatakumthorn and Prapanpong Pongsriiam (see [8]) can be generalized to the family $\mathcal{BE}$ (see Theorems 3.1, 3.3, 3.6) and obtain as Corollaries 3.2, 3.8 and 3.9. We providing a characterization for the space $\mathcal{BE}$, and this fact allows us to verify that the graph of the elements in $\mathcal{BE}$ are found in the region proposed by J. Doboš and Z. Piotrowski (see [5]). In Theorem 4.1 we establish that the graph of these elements is contained in said area, and replacing the interval (0,b] by (0,1], we obtain Corollary 4.2.

2. Preliminaries

To make this work self-contained, we will briefly expose the results and definitions necessary to read this work. The interested reader can consult the references [6], [7], and [8]. The definitions of metric, ultrametric, weak ultrametric, b–metric and extended b–metrics are as follows:

Definition 2.1. Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a *metric* if for all $x, y, z \in X$ it satisfies:

(MI) $d(x, y) = 0$ if and only if $x = y$,

(M2) $d(x, y) = d(y, x)$,

(M3) $d(x, y) \leq d(x, z) + d(z, y)$.

It is well known that the notion of metric currently has various generalizations; among others:

Definition 2.2. Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. We say that d is a *ultrametric* if for all $x, y, z \in X$ it satisfies:

(U1) $d(x, y) = 0$ if and only if $x = y$,

(U2) $d(x, y) = d(y, x)$,

(U3) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

The pair (X, d) is called an ultrametric space when d is an ultrametric in X.

Definition 2.3. Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. We say that d is a *weak ultrametric or inframetric* if for all $x, y, z \in X$ it satisfies:

(I1) $d(x,y) = 0$ if and only if $x = y$,

(I2) $d(x,y) = d(y,x)$,

(I3) there exists $C \geq 1$ such that $d(x,y) \leq C \max\{d(x,z), d(z,y)\}$.

The pair (X,d) is called a weak ultrametric space when d is a weak ultrametric in X.

Ultrametric spaces originate in the studio of p-adic numbers and non-archimedean analysis ([1] and [13]), topology and dynamical systems ([2]), topological algebra ([3]), and theoretical computer science ([12]).

Definition 2.4. Let X be a non-empty set. A function $d : X \times X \to [0,\infty)$ is called a *b–metric* if for all $x, y, z \in X$ it satisfies:

(B1) $d(x,y) = 0$ if and only if $x = y$,

(B2) $d(x,y) = d(y,x)$,

(B3) there exist $s \geq 1$ such that $d(x,y) \leq s[d(x,z) + d(z,y)]$ (s–triangle inequality).

If d is a $b-$metric on X, then (X,d) is called a $b-$metric space.

Previously, Tammatada Khemaratchatakumthorn, Prapanpong Pongsriiam and Suchat Samphavat show that definitions 2.3 and 2.4 are equivalent.

Theorem 2.5. *([8, Theorem 3.3]) Suppose X is a nonempty set and $d : X \times X \to \mathbb{R}$. Then d is a b–metric if and only if d is a weak ultrametric.*

As a consequence of the definitions 2.1, 2.2, 2.4 and the double inequality:

$$\begin{aligned} max\{d(x,z), d(z,y)\} &\leq d(x,z) + d(z,y) \\ &\leq s(d(x,z) + d(z,y)), \text{for all } x, y, z \in X \text{ and some } s \geq 1, \end{aligned} \tag{1}$$

we obtain the following observation.

Remark 2.6. Every ultrametric space (X,d) is a metric space and every metric space (X,d) is a b–metric space.

In [10, Examples 3.6 and 4.4], R. Martínez-Cruz, M. C. Cruz-Cruz, J. E. Pérez-Vázquez and R. López-Hernández verified that the reciprocals were not satisfied.

Definition 2.7. ([6]) Let X be a nonempty set and $\theta : X \times X \to [1,\infty)$. A function $d_\theta : X \times X \to [0,\infty)$ is called an *extended b–metric* if for all $x, y, z \in X$ it satisfies:

(d_θ1) $d_\theta(x,y) = 0$ if and only if $x = y$,

(d_θ2) $d_\theta(x,y) = d_\theta(y,x)$,

(d_θ3) $d_\theta(x,z) \leq \theta(x,z)(d_\theta(x,y) + d_\theta(y,z))$. The pair (X, d_θ) is called an extended b–metric space.

Remark 2.8. If for any $x, y \in X$, $\theta(x, y) = s$, for some $s \geq 1$, then we obtain the definition of a b-metric space.

In [10, Example 3.4], R. Martínez-Cruz, M. C. Cruz-Cruz, J. E. Pérez-Vázquez and R. López-Hernández verified that the reciprocal is not satisfied.

The existence of several generalizations to the notion of a metric and a function that preserves the metric naturally leads to generalizing the concept of a function that preserves the metric.

Definition 2.9. Let $f : [0, \infty) \to [0, \infty)$ a function and

$$\mathcal{P} \in \{\text{ultrametric}, \text{weak ultrametric}, \text{extended } b\text{–metric}\}.$$

We say that f is $\mathcal{P}$ preserving, if $f \circ d := \mathcal{P}$ with $d := \mathcal{P}$.

Definition 2.10. (1) A triangle triplet is a triple (a, b, c) with $a, b, c \geq 0$ such that $a \leq b+c$, $b \leq a + c$ and $c \leq a + b$,

(2) Let $s \geq 1$ and $a, b, c \geq 0$. A triplet (a, b, c) is said to be an s-triangle triplet if $a \leq s(b + c)$, $b \leq s(a + c)$, and $c \leq s(a + b)$.

(3) Let $\kappa : X \times X \to [1, \infty)$ and $a, b, c \geq 0$. A triple (a, b, c) is said to be a κ triplet if $a \leq \kappa(x, y)(b + c)$, $b \leq \kappa(x, z)(a + c)$, and $c \leq \kappa(z, y)(a + b)$, for all $x, y, z \in X$.

We denote for $\triangle$, $\triangle_s$ and $\triangle_\kappa$ the set of all triangle triplets, s-triangle triplets and κ-triangle triplets, respectively.

Remark 2.11.

(i) For all $(a, b, c) \in \triangle$, we obtain $(a, b, c) \in \triangle_s$.

(ii) If for any $x,\ y \in X$, $\kappa(x, y) = s$, for some $s \geq 1$, then we obtain the definition of an s-triangle triplet.

We will focus our attention on the ultrametric, weak ultrametric and extended b–metric.

Definition 2.12. Let $f : [0, \infty) \to [0, \infty)$. We said

(A) f *ultrametric-preserving*, if for all ultrametric (X, d) space, $f \circ d$ is an ultrametric, and we denote the set of all *ultrametric-preserving-functions* to the class $\mathcal{U}$;

(B) f *weak ultrametric-preserving*, if for all weak ultrametric (X, d) space, $f \circ d$ is a weak ultrametric, and we denote the set of all *weak ultrametric-preserving-functions* to the class $\mathcal{UD}$;

(C) f *extended b–metric-preserving*, if for all extended b–metric (X, d_θ) space, there exits an $\hat{\theta} : X \times X \to [1, \infty)$ such that $(f \circ d_\theta)_{\hat{\theta}}$ is an extended b–metric, and we denote the set of all *extended b–metric-preserving-functions* to the class $\mathcal{BE}$;

(D) f *metric-preserving*, if for all metric (X, d) space, $f \circ d$ is metric, and we denote the set of all *metric-preserving-functions* to the class $\mathcal{M}$;

(E) f *b–metric-preserving*, if for all b-metric space (X, d), $f \circ d$ is an b–metric on X, and we denote the set of all *b–metric-preserving-functions* to the class $\mathcal{B}$;

(F) f *metric-b–metric-preserving*, if for all metric spaces (X, d), $f \circ d$ is a b-metric on X, and we denote the set of all *metric-b–metric-preserving-functions* to the class $\mathcal{MB}$, and

(G) f *b–metric-metric-preserving*, if for all b-metric spaces (X, d), $f \circ d$ is a metric on X, and we denote the set of all *b–metric-metric-preserving-functions* to the class $\mathcal{BM}$.

Next, we recall classical results concerning the classes given in the previous definition.

Theorem 2.13. *([7, Theorem 15, Example 16] and [8, Theorem 3.1]) The following relationships are satisfied.*

1. $\mathcal{BM} \subseteq \mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{MB}$, $\mathcal{M} \nsubseteq \mathcal{BM}$ *and* $\mathcal{B} \nsubseteq \mathcal{M}$.
2. $\mathcal{MB} = \mathcal{B}$.

Definition 2.14. Let $f : [0, \infty) \to [0, \infty)$. We said

(A) f is amenable if and only if $f^{-1}(\{0\}) = \{0\}$,

(B) f is subadditive if for all $a, b \in [0, \infty)$, $f(a + b) \leq f(a) + f(b)$,

(C) f is *quasi*-subadditive if there exists $s \geq 1$ such that $f(a + b) \leq s(f(a) + f(b))$ for all $a, b \in [0, \infty)$,

(D) f is concave if $(1 - t)f(x) + tf(y) \leq f\big((1 - t)x + ty\big)$ for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

The following statements are easy to verify.

Remark 2.15. Let $f : [0, \infty) \to [0, \infty)$.

(i) If f is subadditive, then f is *quasi*-subadditive.

(ii) If f is amenable and concave, then f is increasing and subadditive.

Theorem 2.16. *([7, Theorem 20]) Let $f : [0, \infty) \to [0, \infty)$. If $f \in \mathcal{MB}$, then f is amenable and quasi-subadditive.*

Theorem 2.17. *([8, Corollary 3.2]) Let $f : [0, \infty) \to [0, \infty)$ amenable. Then the following statements are equivalent.*

(i) $f \in \mathcal{B}$.

(ii) $f \in \mathcal{MB}$.

(iii) *There exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \triangle_s$ for all $(a, b, c) \in \triangle$, where $\triangle$ and $\triangle_s$ are given in the Definition 2.10.*

Previously, R. Martínez-Cruz and E. Hernández-Piña ([9]) obtained the following results.

Proposition 2.18. *([9, Theorem 4.12]) We have $\mathcal{B} = \mathcal{UD}$.*

Proposition 2.19. *([9, Theorem 5.5]) We have $\mathcal{B} \subseteq \mathcal{BE}$.*

Proposition 2.20. *([9, Counterexample 4.16]) We have $\mathcal{U} \neq \mathcal{UD}$.*

Corollary 2.21. *([11, Erratum: Funciones que preservan la b–métrica y otras métricas relacionas]) $\mathcal{U} \neq \mathcal{B}$ and $\mathcal{U} \neq \mathcal{BE}$.*

Moreover, R. Martínez-Cruz, M. C. Cruz-Cruz, J. E. Pérez-Vazquez, and R. López-Hernández proved the following corollaries.

Corollary 2.22. *([10, Observación 4.18]) If $f \in \mathcal{UD}$, then $f \in \mathcal{BE}$.*

The following result was useful to demonstrate that the graph of the elements in the class $\mathcal{UD}$ is contained in the region proposed by J. Doboš and Z. Piotrowski (see Figure 1).

Corollary 2.23. *([10, Proposición 4.13]) Let $f : [0, \infty) \to [0, \infty)$ be amenable. Then the following statements are equivalent.*

(i) $f \in \mathcal{UD}$.

(ii) $f \in \mathcal{B}$.

(iii) $f \in \mathcal{MB}$.

(iv) *There exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \triangle_s$ for all $(a, b, c) \in \triangle$, where $\triangle$ and $\triangle_s$ are given in the Definition 2.10.*

3. Main Results

From now on, we will focus on the class $\mathcal{BE}$. We will see that the results given for class $\mathcal{B}$ are particular cases for the $\mathcal{BE}$ family (see [8]). In the following result, we establish that the elements of the $\mathcal{BE}$ class satisfy the amenable and quasi-subadditive properties.

Theorem 3.1. *If $f \in \mathcal{BE}$, then f is amenable and quasi-subadditive.*

Proof. Assume that $f \in \mathcal{BE}$. Then for any extended b-metric space (X, d_θ), there exists $\hat{\theta} : X \times X \to [1, \infty)$ such that $(f \circ d_\theta)_{\hat{\theta}}$ is a extended b–metric. In particular, if $d_\theta(x, y) = |y - x|$ for all $x, y \in \mathbb{R}$. Then $(f \circ d_\theta)_{\hat{\theta}}$ is an extended b–metric in $\mathbb{R}$. Then

$$f(0) = f(d_\theta(0, 0)) = (f \circ d_\theta)_{\hat{\theta}}(0, 0) = 0.$$

In addition, suppose $x \in [0, \infty)$ and $f(x) = 0$. Then

$$0 = f(x) = f(d_\theta(0, x)) = (f \circ d_\theta)_{\hat{\theta}}(0, x).$$

Since $(f \circ d_\theta)_{\hat{\theta}}(0,x) = 0$ and $(f \circ d_\theta)_{\hat{\theta}}$ is an extended b–metric, we have $x = 0$. This shows that f is amenable. Next, since f extended b–metric-preserving, there exists $\hat{\theta} : \mathbb{R} \times \mathbb{R} \to [1,\infty)$ such that, for all $x, y, z \in \mathbb{R}$,

$$(f \circ d_\theta)_{\hat{\theta}}(x,y) \leq \theta(x,y)((f \circ d_\theta)_{\hat{\theta}}(x,z) + (f \circ d_\theta)_{\hat{\theta}}(z,y)).$$

To show that f is quasi-subadditive, let $a, b \in [0,\infty)$ and $s = \theta(0, a+b)$. We have $s \geq 1$ and

$$\begin{aligned} f(a+b) &= f(d_\theta(0,a+b)) \\ &= (f \circ d_\theta)_{\hat{\theta}}(0,a+b) \\ &\leq \theta(0,a+b)((f \circ d_\theta)_{\hat{\theta}}(0,a) + (f \circ d_\theta)_{\hat{\theta}}(a,a+b)) \\ &= s(f(a) + f(b)). \end{aligned}$$

That's to say $f(a+b) \leq s(f(a) + f(b))$. This shows that f is quasi-subadditive. ☑

Corollary 3.2. *If $f \in \mathcal{B}$, then f is quasi-subadditive.*

Proof. Assume that $f \in \mathcal{B}$. Then by Proposition 2.19, $f \in \mathcal{BE}$, and by Theorem 3.1, f is amenable and quasi-subadditive. ☑

Of course, a natural question is:

If $f : [0,\infty) \to [0,\infty)$ is amenable and subadditive, then f is an element of $\mathcal{BE}$?

So far, the authors do not know an answer to the above question; however, if we add as a hypothesis that f is concave, the answer to the previous question is affirmative:

Theorem 3.3. *Let $f : [0,\infty) \to [0,\infty)$. If f is amenable and concave, then $f \in \mathcal{BE}$.*

Proof. Assume that f is amenable and concave. Let (X, d_θ) be a extended b–metric space, then there exists $\theta : X \times X \to [1,\infty)$ such that

$$d_\theta(x,y) \leq \theta(x,y)\big(d_\theta(x,z) + d_\theta(z,y)\big), \text{ for all } x, y, z \in X. \tag{2}$$

Next, we will verify that $(f \circ d_\theta)_{\hat{\theta}}$ is an extended b–metric on X. As f is amenable and d_θ is an extended b–metric,

$$0 = (f \circ d_\theta)_{\hat{\theta}}(x,y) \text{ if and only if } x = y.$$

The property $d_\theta 2$ is direct to verify. Leaving $x, y, z \in X$, we will see that the $d_\theta 3$ is satisfied. Since f is amenable and concave, then

$$\begin{aligned} \frac{1}{\theta(x,y)} f(d_\theta(x,y)) &= 0f(0) + \frac{1}{\theta(x,y)} f(d_\theta(x,y)) \\ &\leq f\Big(0 + \frac{1}{\theta(x,y)} d_\theta(x,y)\Big) \\ &= f\Big(\frac{1}{\theta(x,y)} d_\theta(x,y)\Big). \end{aligned} \tag{3}$$

By Observation 2.15, f is increasing and subadditive. By inequalities (2) and (3), we obtain

$$\begin{aligned}\frac{1}{\theta(x,y)}(f\circ d_\theta)_{\hat{\theta}}(x,y) &= \frac{1}{\theta(x,y)} f(d_\theta(x,y))\\ &\le f\Big(\frac{1}{\theta(x,y)} d_\theta(x,y)\Big)\\ &\le f(d_\theta(x,z)+d_\theta(z,y))\\ &\le f(d_\theta(x,z))+f(d_\theta(z,y)).\end{aligned} \tag{4}$$

The latter implies that

$$(f\circ d_\theta)_{\hat{\theta}}(x,y) \le \theta(x,y)((f\circ d_\theta)_{\hat{\theta}}(x,z)+(f\circ d_\theta)_{\hat{\theta}}(z,y)).$$

This shows that $(f\circ d_\theta)_{\hat{\theta}}$ is an extended b–metric and the proof is complete. ☑

Applying the triangular inequality and the previous definition, we obtain the following.

Remark 3.4. If (X, d_θ) is a extended b–metric spaces, then

$$(d_\theta(x,y), d_\theta(x,z), d_\theta(y,z)) \in \triangle_\theta \text{ for all } x,y,z\in X.$$

The following proposition is useful to prove Theorem 3.6.

Proposition 3.5. *([4, Proposition 1, pag. 15]) Let a, b and c be positive real numbers. Then $(a,b,c)\in\triangle$ iff there are $u,v,w\in\mathbb{R}^2$, $u\neq v\neq w\neq u$, such that $a=d(u,v)$, $b=d(u,w)$ and $c=d(v,w)$, where d denotes the Euclidean metric on $\mathbb{R}^2$.*

The following characterization is useful for showing that the graph of the elements of class $\mathcal{BE}$ is in the region proposed by Doboš and Piotrowski in section 4.

Theorem 3.6. *Suppose $f:[0,\infty)\to[0,\infty)$ is amenable. Then $f\in\mathcal{BE}$ if and only if there exists $\kappa: X\times X\to[1,\infty)$ such that $(f(a),f(b),f(c))\in\triangle_\kappa$ for all $(a,b,c)\in\triangle_s$.*

Proof. Assume that $f\in\mathcal{BE}$. Then for any extended b-metric space (X,d_θ), there exists $\hat{\theta}: X\times X\to[1,\infty)$ such that $(f\circ d_\theta)_{\hat{\theta}}$ is an extended b–metric. In particular, if d_θ is the Euclidean metric on $\mathbb{R}^2$. Then $f\circ d_\theta$ is an extended b–metric. So there exist $\theta: X\times X\to[1,\infty)$ such that

$$(f\circ d_\theta)_{\hat{\theta}}(x,y) \le \theta(x,y)((f\circ d_\theta)_{\hat{\theta}}(x,z)+(f\circ d_\theta)_{\hat{\theta}}(z,y)) \text{ for all } x,y,z\in\mathbb{R}^2.$$

Let $(a,b,c)\in\triangle_s$ and $\kappa(x,y)=\theta(x,y)$ for any $x,y\in X$. By the Proposition 3.5, there are $u,v,w\in\mathbb{R}^2$, with $u\neq v\neq w\neq u$, such that $d_\theta(u,w)=a$, $d_\theta(u,v)=b$ and $d_\theta(v,w)=c$. Then

$$\begin{aligned}f(a) &= f(d_\theta(u,w))\\ &= (f\circ d_\theta)_{\hat{\theta}}(u,w)\\ &\le \theta(u,w)((f\circ d_\theta)_{\hat{\theta}}(u,v)+(f\circ d_\theta)_{\hat{\theta}}(v,w))\end{aligned}$$

$$= \kappa(u,w)(f(b)+f(c)).$$

Similarly,

$$f(b) \le \kappa(u,v)(f(a)+f(c)) \text{ and } f(c) \le \kappa(v,w)(f(a)+f(b)).$$

Therefore, $(f(a), f(b), f(c)) \in \triangle_\kappa$.

For the converse, assume that there exists $\kappa : X \times X \to [1,\infty)$ such that

$$(f(a), f(b), f(c)) \in \triangle_\kappa \text{ for all } (a,b,c) \in \triangle_s.$$

Let (X, d_θ) be a extended b-metric space and $x, y, z \in X$. Since f is amenable, $(f \circ d_\theta)(x,y) = 0$ if and only if $x = y$. The condition $d_\theta 2$ is obvious. So it remains to show that $(f \circ d_\theta)_{\hat{\theta}}$ satisfies $d_\theta 3$. Since $(d_\theta(x,y), d_\theta(x,z), d_\theta(z,y)) \in \triangle_s$, it follows that $(f(d_\theta(x,y)), f(d_\theta(x,z)), f(d_\theta(z,y))) \in \triangle_\kappa$. Set $\hat{\theta}(x,y) = \kappa(x,y)$ for any $x, y \in X$. It follows that

$$(f \circ d_\theta)_{\hat{\theta}}(x,y) \le \theta(x,y)((f \circ d_\theta)_{\hat{\theta}}(x,z) + (f \circ d_\theta)_{\hat{\theta}}(z,y)).$$

Hence $f \circ d_\theta$ is an extended b–metric. This completes the proof. ☑

If we replace $\mathcal{BE}$ with $\mathcal{MB}$ in Theorem 3.6, we obtain as a corollary the results given by Khemaratchatakumthorn and Pongsriiam in [7, Theorem 17].

Corollary 3.7. *Let $f : [0,\infty) \to [0,\infty)$. If $f \in \mathcal{MB}$, then f is amenable and quasi-subadditive.*

Proof. Assume that $f \in \mathcal{MB}$. By the Theorem 2.13 and Propositions 2.18 and 2.19, we obtain that $f \in \mathcal{BE}$. By Theorem 3.1, f is amenable and quasi-subadditive. ☑

Corollary 3.8. *Suppose $f \in \mathcal{UD}$ and is amenable, then there exists $\kappa : X \times X \to [1,\infty)$ such that $(f(a), f(b), f(c)) \in \triangle_\kappa$ for all $(a,b,c) \in \triangle_s$.*

Proof. Assume that $f \in \mathcal{UD}$. By the Propositions 2.18 and 2.19, we obtain that $f \in \mathcal{BE}$. Now by Theorem 3.6, there exists $\kappa : X \times X \to [1,\infty)$ such that $(f(a), f(b), f(c)) \in \triangle_\kappa$ for all $(a,b,c) \in \triangle_s$. ☑

Another natural question is

If there exists $\kappa : X \times X \to [1,\infty)$ such that $(f(a), f(b), f(c)) \in \triangle_\kappa$ for all $(a,b,c) \in \triangle_s$, then $f \in \mathcal{UD}$?

Again, the authors do not know the answer to the above question.

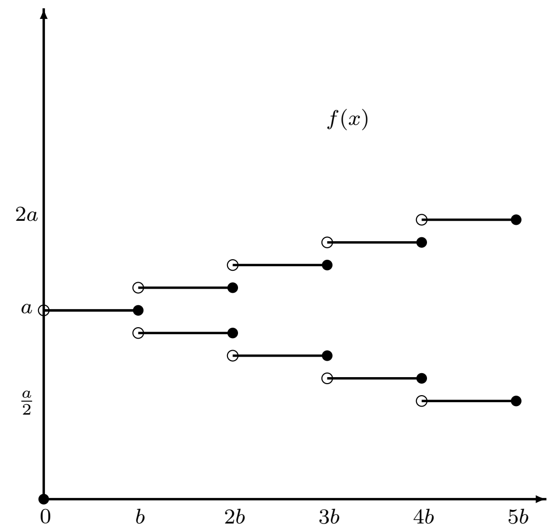

Figure 1. J. Doboš and Z. Piotrowski affirm that the graph of any function distance-preserving is contained in this linear region (Modified of [5, Fig. 6]).

4. Graph of the elements of class $\mathcal{BE}$

Through Proposition 2.19 and Corollary 2.21, we verify that the largest family that contains $\mathcal{B}$ and $\mathcal{UD}$ is $\mathcal{BE}$. Next, we verify that if f belongs to $\mathcal{BE}$, $\lim_{x\to 0^+} f(x) = a$ and $f(x) = a$ for all $x \in (0, b]$, then the graph of f is contained in the Doboš and Piotrowski region (Figure 1).

Theorem 4.1. *Suppose that $f \in \mathcal{BE}$, $\lim_{x\to 0^+} f(x) = a$ and $f(x) = a$ for all $x \in (0, b]$, then for each $n \in \mathbb{N}$ and each $x \in (nb, (n+1)b]$,*

$$\frac{a}{2} \leq f(x) \leq 2^n a.$$

Proof. We will apply the principle of mathematical induction. Let us see that, for $n = 1$, the conclusion is satisfied

$$\frac{a}{2} \leq f(x) \leq 2a \text{ for all } x \in (b, 2b].$$

To see this first inequality

$$\frac{a}{2} \leq f(x) \text{ for all } x \in (b, 2b],$$

suppose that there exists an $x \in (b, 2b]$ such that $f(x) < \frac{a}{2}$. Let $z \in (0, b]$.

We have (x, x, z) is a s–triangle triplet, while $(f(x), f(x), f(z))$ does not, since

$$f(x) + f(x) < \frac{a}{2} + \frac{a}{2} = a = f(z).$$

That is, for each, $\kappa : X \times X \to [1, \infty)$, there exist a triplet $(x, x, z) \in \triangle_s$ such that

$$f(z) \nleq \kappa(x, z)(f(x) + f(x)).$$

So by Theorem 3.6, f does not extended $b-$metric preserving, which contradicts the hypothesis.

To see this other inequality: $f(x) \leq 2a$ for all $x \in (b, 2b]$, suppose that there exists $x \in (b, 2b]$ such that $f(x) > 2a$.

We have $(\frac{x}{2}, \frac{x}{2}, x)$ is a s–triangle triplet, while

$$\left(f\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right), f(x)\right)$$

does not, since

$$f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right) = a + a = 2a < f(x).$$

That is, for each, $\kappa : X \times X \to [1, \infty)$ there exists a triplet $(\frac{x}{2}, \frac{x}{2}, x) \in \triangle_s$ such that

$$f(x) \nleq \kappa\left(\frac{x}{2}, x\right)\left(f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right)\right).$$

Again, Theorem 3.6 implies f does not extended $b-$metric preserving, which is a contradiction.

Assume that, for $n = k$ the inequality is satisfied,

$$f(x) \leq 2^k a \text{ for all } x \in (kb, (k+1)b].$$

We will show that for $n = k + 1$ it is also satisfied. Namely

$$f(x) \leq 2^{k+1} a \text{ for all } x \in ((k+1)b, (k+2)b].$$

Suppose that there exists an element $x \in ((k+1)b, (k+2)b]$ such that $f(x) > 2^{k+1}a$.

Note that $(\frac{x}{2}, \frac{x}{2}, x)$ is a s–triangle triplet and $(f(\frac{x}{2}), f(\frac{x}{2}), f(x))$ does not, since by the inductive hypothesis

$$f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right) \leq 2^k a + 2^k a = 2(2^k a) = 2^{k+1} a < f(x).$$

That is, for each, $\kappa : X \times X \to [1, \infty)$ there exists a triplet $(\frac{x}{2}, \frac{x}{2}, x) \in \triangle_s$ such that

$$f(x) \nleq \kappa\left(\frac{x}{2}, x\right)\left(f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right)\right).$$

So by Theorem 3.6, f does not extended $b-$metric preserving, which contradicts the hypothesis. ☑

Hence, this last result confirms the conjecture of Doboš and Piotrowski for the functions in $\mathcal{BE}$.

If in Theorem 4.1 we substitute the interval $(0, b]$ by $(0, 1]$ we obtain the following:

Corollary 4.2. *If $f \in \mathcal{BE}$, $\lim_{x\to 0^+} f(x) = a$ and $f(x) = a$ for each $x \in (0, 1]$, then for each $n \in \mathbb{N}$ and $x \in (n, (n+1)]$, $\frac{a}{2} \leq f(x) \leq 2^n a$.*

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