Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 43, N° 1, 2025, pág. 1–15

Conductors of fractional ideals in Burnside rings for cyclic p-groups and their zeta function

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Abstract. This study is part of the zeta function of the Burnside ring study. The main objective of this paper is to determine the conductors of all isomorphism classes of fractional ideals of finite index in $B_p(C_{p^n})$ the Burnside ring for cyclic groups of order p^n, which leads to a new explicit formula for $\zeta_{B_p(C_{p^n})}(s)$ the zeta function of $B_p(C_{p^n})$, and we present a conjecture in which we establish when a fractional ideal M of $B_p(C_{p^n})$ has a $\mathbb{Z}_p$-order structure, according to its $Z_{B_p(C_{p^n})}(M;s)$ function.

Keywords: Burnside ring, zeta function, fiber product.

MSC2020: 16H20, 19A22, 11S40.

Conductores de los ideales fraccionales en anillos de Burnside para p-grupos cíclicos y su función zeta

Resumen. Este trabajo es parte del estudio de la función zeta para anillos de Burnside. El objetivo principal de este artículo es determinar los conductores de todas las clases de isomorfismo de los ideales fraccionales de índice finito en $B_p(C_{p^n})$ el anillo de Burnside para grupos cíclicos de orden p^n, lo cual permite obtener una nueva fórmula para $\zeta_{B_p(C_{p^n})}(s)$ la función zeta de $B_p(C_{p^n})$, y se presenta una conjetura en la cual se establece cuando un ideal

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Received: 29 August 2024, Accepted: 13 March 2025.
To cite this article: D. Villa-Hernández, J. M. Ramírez-Contreras and C. Vázquez-Rosas, Conductors of fractional ideals in Burnside rings for cyclic $p-$groups and their zeta function, *Rev. Integr. Temas Mat.,* 43 (2025), No. 1, 1-14. doi: 10.18273/revint.v43n1-2025001

fraccional M de $B_p(C_{p^n})$ tiene estructura de $\mathbb{Z}_p$-orden, de acuerdo con su función $Z_{B_p(C_{p^n})}(M;s)$.

Palabras clave: Anillo de Burnside, función zeta, producto fibrado.

1. Introduction

The Burnside ring is an invariant of a group that detects solubility, and provides a framework for the induction theorems, with various applications in topology, see [1, 2]. On the other hand, the zeta function is an invariant of the ring that describes the distribution of prime ideals, see [5].

In Section 2, we recall the Burnside ring $B(G)$ of a finite group G, along with the zeta function $\zeta_{B(G)}(s)$ of $B(G)$.

In Section 3, our main goal is to determine all conductors of the fractional ideals of finite index in $B_p(C_{p^n})$.

According to the definition given by Solomon for the zeta function of an order, it is necessary to know all its ideals of finite index, which can be computationally challenging. From [11, Section 3, 4], through a method used by Bushnell C. J. and Reiner I. [4], which only depends on the finite set of the isomorphism classes of the ideals of finite index, we have that:

For $B_p(C_p)$ there are 2 isomorphism classes of fractional ideals of finite index;

For $B_p(C_{p^2})$ there are 9 isomorphism classes of fractional ideals of finite index.

In both cases, we can see that this is a better alternative than the method used in [10], where the same results were obtained by computing all the ideals. However, as we can see in [6, Section 3], for $B_p(C_{p^3})$ there are

$$82 + 7p + 5(p-1) + 3(p-2)$$

isomorphism classes of fractional ideals of finite index. So for $B_p(C_{p^n})$, this method used by I. Reiner quickly becomes unmanageable.

In this paper, we present a method that only depends on $n+1$ conductors for the general case $B_p(C_{p^n})$.

According to Theorem 3.8, we have that the only conductors of the isomorphisms classes of the fractional ideals of finite index in $B_p(C_{p^n})$ are:

$$\mathcal{N}_0 = B_p(C_{p^n}),$$

$$\mathcal{N}_i = (p, p^2, ..., p^{i-1}, p^i, \underbrace{p^i, ..., p^i}_{(n-i+1)-\text{times}}) \left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}})\right) \quad \text{for} \quad i = 1, ..., n-1, \quad \text{and}$$

$$\mathcal{N}_n = \left(p, p^2, ..., p^n, p^n\right) \mathbb{Z}_p^{n+1}.$$

This allows us to present in Theorem 3.9, a new expression for $\zeta_{B_p(C_{p^n})}(s)$ which depends on:

$$\mathcal{J}_i = \int\limits_{(\mathbb{Q}_p^*)^{n+1}\bigcap \mathcal{N}_i} \|x\|^s_{\mathbb{Q}_p^{n+1}}\, d^*x,$$

for $i = 0, ..., n$. In Theorem 3.10, we develop a recursion formula, to compute $\mathcal{J}_i$ and in Corollary 3.11 we deduce that:

$$\mathcal{J}_i = \frac{g_i(p^{-s})}{p^{\frac{(n-i)(n-i-1)}{2}}\,(p-1)^{n-i}\,(1-p^{-s})^{n+1}},$$

where $g_i(p^{-s}) \in \mathbb{Z}\,[p^{-s}]$ is a polynomial of degree:

$$\partial\left(g_i(p^{-s})\right) = \frac{n(n+3)}{2} \qquad \text{for} \quad i = 0, ..., n.$$

Finally, in the Conjecture, we establish when a fractional ideal M of $B_p(C_{p^n})$ has a $\mathbb{Z}_p$-order structure, according to its $Z_{B_p(C_{p^n})}(M; s)$ function.

2. Zeta Functions of Burnside Rings

Throughout this paper, let G be a finite group. Its Burnside ring $B(G)$ is the Grothendieck ring of the category of finite left G-sets. This is the free abelian group on the isomorphism classes of transitive left G-sets of the form G/H for subgroups H of G, two of which are identified if their stabilizers H are conjugate in G; addition and multiplication are given by the disjoint union and Cartesian product, respectively. That is:

$$B\,(G) = \bigoplus_{H\in\mathcal{C}(G)} \mathbb{Z}\,(G/H)\,,$$

where $\mathcal{C}(G)$ is a complete set of representatives of conjugacy classes of subgroups of G.

We study often the Burnside ring of a finite group G using the mark homomorphism $\varphi : B\,(G) \to \mathbb{Z}^{|\mathcal{C}(G)|}$, which is an injective map. The ring $\widetilde{B}\,(G) := \mathbb{Z}^{|\mathcal{C}(G)|}$ is the ring of super class functions $f : \mathcal{C}\,(G) \to \mathbb{Z}$ with multiplication given by coordinate-wise multiplication and it is called the ghost ring of G. For further information about the Burnside ring, see [3].

Let $p \in \mathbb{Z}$ be a rational prime and let $\mathbb{Z}_p$ be the ring of $p-$adic integers. We denote the following tensor products by:

$$B_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} B(G) = \bigoplus_{H\in\mathcal{C}(G)} \mathbb{Z}_p\,(G/H)$$

and

$$\widetilde{B}_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} \widetilde{B}(G) = \mathbb{Z}_p^{|\mathcal{C}(G)|},$$

where we have that $B_p(G)$ is a $\mathbb{Z}_p$-order, being $\widetilde{B}_p(G)$ its maximal order. For further information about orders, see [7, chapters 2, 3].

Let A be a finite-dimensional semisimple algebra over the rational field $\mathbb{Q}$ or over a $p-$adic field $\mathbb{Q}_p$, and let Λ be an order in A. When A is a $\mathbb{Q}$-algebra, Λ is a $\mathbb{Z}$-order in A; when A is a $\mathbb{Q}_p$-algebra, Λ is a $\mathbb{Z}_p-$ order in A. Let I be a left ideal of Λ, such that the index $(\Lambda : I)$ is finite. We use this index symbol in a general sense: if, for example, Y_1 and Y_2 are $\mathbb{Z}_p$-lattices spanning the same $\mathbb{Q}_p$-vector space, we put

$$(Y_1 : Y_2) = \frac{(Y_1 : Y_1 \cap Y_2)}{(Y_2 : Y_1 \cap Y_2)}.$$

The symbol $(Y_1 : Y_2)$ is therefore unambiguously defined whether or not Y_1 contains Y_2.

Definition 2.1. We define the Solomon's zeta function $\zeta_\Lambda(s)$ of an order Λ, as follows:

$$\zeta_\Lambda(s) := \sum_{\substack{I \leq \Lambda,\text{ left ideal} \\ (\Lambda : I) < \infty}} (\Lambda : I)^{-s},$$

which is a generalization of the classical Dedekind zeta function $\zeta_{\mathcal{K}}(s)$ of an algebraic number field $\mathcal{K}$. When Λ is a $\mathbb{Z}$-order in A, $\zeta_\Lambda(s)$ is the global zeta function; when Λ is a $\mathbb{Z}_p$-order in A, $\zeta_\Lambda(s)$ is the local zeta function.

For the commutative rings $B_p(G)$ and $\widetilde{B}_p(G)$, the sum extends over all the ideals of finite index and converges uniformly on compact subsets of

$$\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}.$$

For further information about Solomon's zeta function, see [9].

Theorem 2.2. *Let G be a finite group and $B(G)$ its Burnside ring, if $q \in \mathbb{Z}$ is a prime, we have:*

$$\zeta_{B_q(G)}(s) = f_G\left(q^{-s}\right)\zeta_{\widetilde{B}_q(G)}(s),$$

where $f_G(q^{-s})$ is a polynomial in $\mathbb{Z}[q^{-s}]$. See [9, Theorem 1].

Remark 2.3. If q does not divide $|G|$, then we have that $B_q(G) = \widetilde{B}_q(G)$, and we conclude that $f_G(q^{-s}) = 1$ when q does not divide $|G|$.

Definition 2.4. Let M be a full Λ-lattice in A. We define the zeta function $Z_\Lambda(M; s)$, as follows:

$$Z_\Lambda(M; s) = \sum (\Lambda : N)^{-s},$$

the sum extending over all full Λ-sublattices N in A, such that N, M are in the same isomorphism class.

So we can express

$$\zeta_\Lambda(s) = \sum_M Z_\Lambda(M; s),$$

the finite sum extending over all the representatives of the isomorphism classes of the full Λ-lattices in A.

We define the conductor of M in Λ, as follows:

$$\{M : \Lambda\} = \{x \in A : \quad Mx \subseteq \Lambda\}.$$

Let $\Phi_{\{M:\Lambda\}}$ be the characteristic function in A of $\{M : \Lambda\}$. Now we choose a Haar measure d^*x on the unit group A^*. For measurable sets $E \subset A$, $E' \subset A^*$, it will be convenient to write:

$$\mu(E) = \int_E dx, \qquad \mu^*(E') = \int_{E'} d^*x.$$

We have that:

$$Z_\Lambda(M; s) = \mu^*(\mathrm{Aut}_\Lambda M)^{-1} (\Lambda : M)^{-s} \int_{A^*} \Phi_{\{M:\Lambda\}}(x) \|x\|_A^s \, d^*x,$$

where $\|x\|_A = (Lx : L)$ for $x \in A^*$, which is independent of the choice of full $\mathbb{Z}_p$-lattice L in A, and we observe that it is multiplicative. Furthermore, we can see that $\|x\|_A = 1$ whenever x is a unit in some $\mathbb{Z}_p$-order in A. For further details on this result, see [4, 2.1 The Local Case, pp 138-139].

3. *The conductors of the fractional ideals in* $B_p(C_{p^n})$

Remark 3.1. Let p be a prime, and let C_{p^n} be a cyclic group of order p^n for $n \in \mathbb{N}$.

Due to the mark homomorphism, we can see $B_p(C_{p^n})$ in $\widetilde{B}_p(C_{p^n})$ as follows:

$$B_p(C_{p^n}) = \left\{(x_1, x_2, \ \dots\ , x_{n+1}) \in \mathbb{Z}_p^{n+1} \colon \ (x_i - x_{i+1}) \in p^i\mathbb{Z}_p \ \text{ for } \ i = 1, ..., n\right\},$$

and then we can give the following fiber product structure:

$$\begin{tikzcd}
(x_1, ..., x_n, x_{n+1}) \arrow[rrr] \arrow[ddd] & & & x_{n+1} \arrow[ddd] \\
& B_p(C_{p^n}) \arrow[r, "f_2"] \arrow[d, "f_1"'] & \mathbb{Z}_p \arrow[d, "g_2"] & \\
& B_p(C_{p^{n-1}}) \arrow[r, "g_1"'] & \mathbb{Z}_p/p^n\mathbb{Z}_p & \\
(x_1, ..., x_n) \arrow[rrr] & & & \overline{x_n} = \overline{x_{n+1}}
\end{tikzcd}$$

We observe that $\mathbb{Z}_p$ is a PID. Therefore, it has ideals of the form $p^r\mathbb{Z}_p$, for every integer $r \geq 0$, and according to the structure of the fiber product, we have that the ideals of finite index in $B_p(C_{p^n})$ are ideals of the form:

$$I = (\alpha, p^r) B_p(C_{p^n}) + (J, 0),$$

where α is an element of $B_p\left(C_{p^{n-1}}\right)$ and $J \leq B_p\left(C_{p^{n-1}}\right)$ is an ideal such that:

1. $g_1(J) = 0$,
2. $g_1(\alpha) = g_2(p^r)$, where α is uniquely determined mod J, and
3. if $D = p\mathbb{Z}_p \times p^2\mathbb{Z}_p \times \cdots \times p^n\mathbb{Z}_p \times \{0\}$ we have that:

$$f_1(D)\,\alpha \subseteq J.$$

For further details on this result, see [6, 8].

In this section we will denote

$$\mathcal{N}_i = (p, p^2, ..., p^{i-1}, p^i, \underbrace{p^i, \;...\;, p^i}_{(n-i+1)-\text{times}}\;)\left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}})\right) \quad \text{for} \quad i = 1, ..., n-1 \text{ and}$$

$$\mathcal{N}_n = \left(p, p^2, ..., p^n, p^n\right)\mathbb{Z}_p^{n+1}.$$

Lemma 3.2. *Let $\mathcal{N}_0 = B_p(C_{p^n})$. Then $\mathcal{N}_0, ..., \mathcal{N}_n$ are the conductors of some fractional ideals of finite index in $B_p(C_{p^n})$.*

Proof. First, let's note that $\boldsymbol{\{B_p(C_{p^n}) : B_p(C_{p^n})\} = \mathcal{N}_0}$ and then $\mathcal{N}_0$ is its own conductor.

Next, let $i \in \{1, ..., n-1\}$, and let $\mathbb{Z}_p^i \times B_p(C_{p^{n-i}})$ be the fractional ideal of finite index in $B_p(C_{p^n})$, which belongs to the same isomorphism class as $\mathcal{N}_i$. It is easy to see that $\mathbb{Z}_p^i \times B_p(C_{p^{n-i}}) = \langle e_1, e_2 ..., e_i, \beta_{i+1}, \beta_{i+2}, ..., \beta_{n+1}\rangle \subset \mathbb{Z}_p^{n+1}$ as $\mathbb{Z}_p$-module, where:

$$\begin{array}{ccc}
e_1 = (1, 0, ..., 0) & & \beta_{i+1} = (0, ..., 0, \underset{\underset{(i+1)-\text{th coordinate}}{\downarrow}}{1}, 1, ..., 1) \\
e_2 = (0, 1, 0, ..., 0) & \text{and} & \beta_{i+2} = (0, ..., 0, \underset{\underset{(i+2)-\text{th coordinate}}{\downarrow}}{p}, p, ..., p) \\
\vdots & & \vdots \\
e_i = (0, ..., 0, \underset{\underset{i-\text{th coordinate}}{\downarrow}}{1}, 0, ..., 0) & & \beta_{n+1} = (0, ..., 0, p^{n-i}).
\end{array}$$

We know that $\left\{\left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}})\right) :\; B_p(C_{p^n})\right\} =$

$$\left\{(y_1, ..., y_{n+1}) \in \mathbb{Q}_p^{n+1} :\; (y_1, ..., y_{n+1})\left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}})\right) \subseteq B_p(C_{p^n})\right\}.$$

If $(y_1, ..., y_{n+1}) \in \left\{\left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}})\right) :\; B_p(C_{p^n})\right\}$, then we have that:

$$\begin{array}{ccc}
(y_1, 0, ..., 0) \in B_p(C_{p^n}) & & \alpha_{i+1} = (0, ..., 0, y_{i+1}, ..., y_{n+1}) \in B_p(C_{p^n}) \\
(0, y_2, 0, ..., 0) \in B_p(C_{p^n}) & \text{and} & \alpha_{i+2} = (0, ..., 0, py_{i+2}, ..., py_{n+1}) \in B_p(C_{p^n}) \\
\vdots & & \vdots \\
(0, ..., 0, y_i, 0, ..., 0) \in B_p(C_{p^n}) & & \alpha_{n+1} = (0, ..., 0, p^{n-i}y_{n+1}) \in B_p(C_{p^n}). \\
\text{(i)} & & \text{(ii)}
\end{array}$$

From (i), it follows that $y_\kappa \in p^\kappa \mathbb{Z}_p$ for $\kappa \in \{1, ..., i\}$. From (ii) we have that $\alpha_{i+1} \in B_p(C_{p^n})$, then $y_\lambda \in p^i \mathbb{Z}_p$ for $\lambda \in \{i+1, ..., n+1\}$. It follows that $y_\lambda = p^i b_\lambda$ for some $b_\lambda \in \mathbb{Z}_p$. Furthermore, we have that $y_{l+1} - y_l \in p^l \mathbb{Z}_p$ for $l \in \{i+1, ..., n\}$, then $b_{l+1} - b_l \in p^{l-i}\mathbb{Z}_p$. Hence, we have:

$$\left\{ \left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}}) \right) : \ B_p(C_{p^n}) \right\} \subseteq \mathcal{N}_i.$$

It is easy to see the reverse inclusion $\left\{ \left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}}) \right) : \ B_p(C_{p^n}) \right\} \supseteq \mathcal{N}_i$. So that:

$$\boldsymbol{\left\{ \left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}}) \right) : \ B_p(C_{p^n}) \right\} = \mathcal{N}_i} \quad \text{for } i = 1, ..., n-1.$$

Finally, it is easy to see that $\boldsymbol{\left\{ \mathbb{Z}_p^{n+1} : B_p(C_{p^n}) \right\} = \mathcal{N}_n}$. Hence $\mathcal{N}_n$ is the conductor of the fractional ideal $\mathbb{Z}_p^{n+1}$ of finite index in $B_p(C_{p^n})$.

☑

Proposition 3.3. *Let M, N be full $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^{n+1}$. Then $\{M : B_p(C_{p^n})\} = N$ if and only if $\{\alpha M : B_p(C_{p^n})\} = \alpha^{-1}N$ for every $\alpha \in (\mathbb{Q}_p^*)^{n+1}$.*

Proof.

($\Rightarrow$) If $\{M : B_p(C_{p^n})\} = N$, then we have that

$$\{\alpha M : B_p(C_{p^n})\} =$$

$$\left\{ \beta \in \mathbb{Q}_p^{n+1} : \beta(\alpha M) \subseteq B_p(C_{p^n}) \right\} = \left\{ \beta \in \mathbb{Q}_p^{n+1} : (\alpha\beta) M \subseteq B_p(C_{p^n}) \right\} =$$

$$\left\{ \beta \in \mathbb{Q}_p^{n+1} : (\alpha\beta) \in \{M : B_p(C_{p^n})\} \right\} = \alpha^{-1} \{M : B_p(C_{p^n})\}$$

so:

$$\{\alpha M : B_p(C_{p^n})\} = \alpha^{-1} N.$$

($\Leftarrow$) If $\{\alpha M : B_p(C_{p^n})\} = \alpha^{-1} N$ for every $\alpha \in \mathbb{Q}_p^{n+1}$, then in particular, when $\alpha = (1, ..., 1) \in \mathbb{Q}_p^{n+1}$, this gives:

$$\{M : B_p(C_{p^n})\} = N.$$

☑

Remark 3.4. We will denote in this section $\mathcal{M}_i = \left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}}) \right)$ for $i = 1, ..., n-1$, and $\mathcal{M}_n = \mathbb{Z}_p^{n+1}$. Let M be a fractional ideal of finite index in $B_p(C_{p^n})$ such that:

$$\{M : B_p(C_{p^n})\} = \alpha \{\mathcal{M}_j : B_p(C_{p^n})\},$$

for some $j \in \{1, ..., n\}$ and $\alpha \in (\mathbb{Q}_p^*)^{n+1}$. Then, Proposition 3.3 gives:

$$\{\alpha M : B_p(C_{p^n})\} = \{\mathcal{M}_j : B_p(C_{p^n})\}.$$

This allows us to avoid the units $\alpha \in (\mathbb{Q}_p^*)^{n+1}$, when we work with isomorphism classes, since $[M] = [\alpha M]$.

Lemma 3.5. *Let M be a fractional ideal of finite index in $B_p(C_{p^n})$. If*

$$\{M : B_p(C_{p^n})\} = \{\mathcal{M}_i : B_p(C_{p^n})\},$$

for some $i \in \{1, ..., n\}$, then $M \subseteq \mathcal{M}_i$.

Proof. First, we observe that if $\{M : B_p(C_{p^n})\} = \{\mathcal{M}_n : B_p(C_{p^n})\}$ then it is clear that $M \subseteq \mathcal{M}_n$.

From the proof of Lemma 3.2, we have that $\{\mathcal{M}_i : B_p(C_{p^n})\} = \mathcal{N}_i$ for $i = 1, ..., n-1$, then we know that $(0, ..., 0, \underset{\underset{(i+1)-\text{th coordinate}}{\downarrow}}{p^i}, ..., p^i) \in \{\mathcal{M}_i : B_p(C_{p^n})\}$.

Now, if $\{M : B_p(C_{p^n})\} = \{\mathcal{M}_i : B_p(C_{p^n})\}$ for some $i \in \{1, ..., n-1\}$, then $(0, ..., 0, p^i, ..., p^i) \in \{M : B_p(C_{p^n})\}$. It follows that

$$(0, ..., 0, p^i, ..., p^i)(m_1, ..., m_{n+1}) \in B_p(C_{p^n})$$

for all $(m_1, ..., m_{n+1}) \in M$, then

$$(0, ..., 0, p^i m_{i+1}, ..., p^i m_{n+1}) \in B_p(C_{p^n}),$$

so

$$p^i(m_{k+1} - m_k) \in p^k\mathbb{Z}_p \text{ for } \ k = i+1, ..., n.$$

It follows that $(m_{k+1} - m_k) \in p^{k-i}\mathbb{Z}_p$ for $k = i+1, ..., n$, then we have

$$(m_1, ..., m_{n+1}) \in \mathcal{M}_i = \mathbb{Z}_p^i \times B_p(C_{p^{n-i}}),$$

and so

$$M \subseteq \mathcal{M}_i.$$

☑

Remark 3.6. With the above notation, we have the following set contentions

$$\mathcal{M}_0 = B_p(C_{p^n}) \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots \subseteq \mathcal{M}_n = \mathbb{Z}_p^{n+1}.$$

Let M be a fractional ideal of finite index in $B_p(C_{p^n})$ such that $M \nsubseteq B_p(C_{p^n})$ and $M \neq \mathcal{M}_i$ for all $i \in \{1, ..., n\}$. We have that $M \subseteq \mathcal{M}_n$, then we can choose $i \in \{1, ..., n\}$ the least index, such that $M \subseteq \mathcal{M}_i$ and so $M \nsubseteq \mathcal{M}_{i-1}$.

Lemma 3.7. *Let M be a fractional ideal of finite index in $B_p(C_{p^n})$ and let $i \in \{1, ..., n\}$ the least index, such that $M \subseteq \mathcal{M}_i$. Let $M = \mathcal{X} \cup \mathcal{Y}$, where $\mathcal{X} = M \cap \mathcal{M}_{i-1}^c$ and $\mathcal{Y} = M \cap \mathcal{M}_{i-1}$ (Here $\mathcal{M}_{i-1}^c$ is the complement of $\mathcal{M}_{i-1}$ in $\mathbb{Z}_p^{n+1}$). If there are $\gamma_1, ..., \gamma_{n+1} \in M$, such that $\mathcal{X} \subset (N = \langle \gamma_1, ..., \gamma_{n+1} \rangle$ as $\mathbb{Z}_p-$module) and $\{N : B_p(C_{p^n})\} \subseteq \{\mathcal{M}_i : B_p(C_{p^n})\}$, then $\{M : B_p(C_{p^n})\} = \{\mathcal{M}_i : B_p(C_{p^n})\}$.*

Proof. We have that $i \in \{1, ..., n\}$ is the least index, such that $M \subseteq \mathcal{M}_i$, then $\mathcal{X} \neq \emptyset$. Now, $N \subseteq M$, then $\{M : B_p(C_{p^n})\} \subseteq \{N : B_p(C_{p^n})\}$, and by hypothesis $\{N : B_p(C_{p^n})\} \subseteq \{\mathcal{M}_i : B_p(C_{p^n})\}$, so:

$$\{M : B_p(C_{p^n})\} \subseteq \{\mathcal{M}_i : B_p(C_{p^n})\}.$$

Finally, we have that $M \subseteq \mathcal{M}_i$, then:

$$\{\mathcal{M}_i : B_p(C_{p^n})\} \subseteq \{M : B_p(C_{p^n})\},$$

and so:

$$\{M : B_p(C_{p^n})\} = \{\mathcal{M}_i : B_p(C_{p^n})\}.$$

☑

In the following two theorems, we consider the hypotheses of the previous lemma.

Theorem 3.8. *The only conductors of the isomorphisms classes of the fractional ideals of finite index in $B_p(C_{p^n})$ are:*

$$\mathcal{N}_0 = B_p(C_{p^n}),$$

$$\mathcal{N}_i = (p, p^2, ..., p^{i-1}, p^i, \underbrace{p^i, ..., p^i}_{(n-i+1)-times})\left(\mathbb{Z}_p^i \times B_p(C_{p^{n-i}})\right) \quad \textit{for} \quad i = 1, ..., n-1, \quad \textit{and}$$

$$\mathcal{N}_n = \left(p, p^2, ..., p^n, p^n\right) \mathbb{Z}_p^{n+1}.$$

Proof. This follows from Lemmas 3.2, 3.5 and 3.7. ☑

Theorem 3.9. *There are polynomials $f_i(p^s) \in \mathbb{Z}[p^s]$ of degree:*

$$\partial\left(f_i(p^s)\right) \leq \frac{n(n+1)}{2},$$

for $i = 0, ..., n$ such that:

$$\zeta_{B_p(C_{p^n})}(s) = \sum_{i=0}^{n} f_i\left(p^s\right) \int\limits_{\left(\mathbb{Q}_p^*\right)^{n+1} \bigcap \mathcal{N}_i} \|x\|_{\mathbb{Q}_p^{n+1}}^s \, d^*x.$$

Proof. Let M be a fractional ideal of finite index in $B_p(C_{p^n})$. We know that:

$$Z_{B_p(C_{p^n})}(M; s) =$$
$$\mu^*\left(\mathrm{Aut}_{B_p(C_{p^n})} M\right)^{-1} \left(B_p(C_{p^n}) : M\right)^{-s} \int\limits_{\left(\mathbb{Q}_p^*\right)^{n+1} \bigcap \left\{M : B_p(C_{p^n})\right\}} \|x\|_{\mathbb{Q}_p^{n+1}}^s \, d^*x.$$

First, it is easy to see (as in [6, Remark 3.2]) that:

$$\boldsymbol{\mu}^{*}\left(\mathrm{Aut}_{\boldsymbol{B_p(C_{p^n})}} \boldsymbol{M}\right)^{\boldsymbol{-1}} = \left((\mathbb{Z}_p^*)^{n+1} : \mathrm{Aut}_{B_p(C_{p^n})} M\right) \in \mathbb{Z}. \tag{1}$$

Now, from Remark 3.1, we can proof by induction on n that the ideals of finite index in $B_p(C_{p^n})$ are of the form:

$$\left(p^{k_1}\beta_1, ..., p^{k_{n+1}}\beta_{n+1}\right)\left\{(x_1, ..., x_{n+1}) \in \mathbb{Z}_p^{n+1} : \quad \mathcal{R}_1 \in p^{m_1}\mathbb{Z}_p, ..., \mathcal{R}_\lambda \in p^{m_\lambda}\mathbb{Z}_p\right\},$$

where:

1. $0 \leq k_i,\ \ \beta_i \in \mathbb{Z}_p^*$ for $i = 1, ..., n+1$;
2. $1 \leq \lambda \leq n,\ \ 0 \leq m_j \leq n,\ \ \sum_{j=1}^{\lambda} m_j \leq \frac{n(n+1)}{2}$;
3. $\mathcal{R}_j = \sum_{i=1}^{n+1} \left(\alpha_{i,1} + p\alpha_{i,2} + ... + p^{m_j-1}\alpha_{i,m_j}\right) x_i$ for $\alpha_{i,\nu} \in \{0, ..., p-1\}$ and at least two $\alpha_{i,1}$ are nonzero.

Then M is of the form:

$$M = \left\{(x_1, ..., x_{n+1}) \in \mathbb{Z}_p^{n+1} : \quad \mathcal{R}_1 \in p^{m_1}\mathbb{Z}_p, ..., \mathcal{R}_\lambda \in p^{m_\lambda}\mathbb{Z}_p\right\},$$

for some $\mathcal{R}_j \in p^{m_j}\mathbb{Z}_p$ and $\sum_{j=1}^{\lambda} m_j \leq \frac{n(n+1)}{2}$. It is easy to see (as in [6, Remark 3.2]) that:

$$\frac{\mathbb{Z}_p^{n+1}}{M} \cong \frac{\mathbb{Z}}{p^{m_1}\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{p^{m_\lambda}\mathbb{Z}}$$

and so:

$$\boldsymbol{(B_p(C_{p^n}) : M)^{-s}} = \left(B_p(C_{p^n}) :\ \mathbb{Z}_p^{n+1}\right)^{-s} \left(\mathbb{Z}_p^{n+1} :\ M\right)^{-s} =$$

$$\left(\mathbb{Z}_p^{n+1} :\ B_p(C_{p^n})\right)^s p^{-\left(\sum_{j=1}^{\lambda} m_j\right)s} = p^{\left(\frac{n(n+1)}{2}\right)s} p^{-\left(\sum_{j=1}^{\lambda} m_j\right)s} = (p^s)^d \tag{2}$$

for some $d \in \mathbb{N}$, such that $0 \leq d \leq \frac{n(n+1)}{2}$.

Furthermore, from Remark 3.8, there is $i \in \{0, ..., n\}$ such that:

$$\boldsymbol{\{M : B_p(C_{p^n})\} = \mathcal{N}_i}, \tag{3}$$

and so, from Eq. (1), Eq. (2) and Eq. (3) it follows that:

$$Z_{B_p(C_{p^n})}(M; s) = f_M(p^s) \int\limits_{\left(\mathbb{Q}_p^*\right)^{n+1} \bigcap \mathcal{N}_i} \|x\|_{\mathbb{Q}_p^{n+1}}^s \, d^*x$$

for some polynomial $f_M(p^s) \in \mathbb{Z}[p^s]$ of degree $0 \leq \partial\left(f_M(p^s)\right) \leq \frac{n(n+1)}{2}$.

Finally, we know that

$$\zeta_{B_p(C_{p^n})}(s) = \sum_M Z_{B_p(C_{p^n})}(M; s),$$

where the sum extends over all the representatives of the isomorphism classes of fractional ideals of finite index in $B_p(C_{p^n})$. So, if we associate these representatives according to their conductors, we get the desired result. ☑

Theorem 3.10. *There is a recursion formula to compute:*

$$\int\limits_{\left(\mathbb{Q}_p^*\right)^{n+1} \bigcap \mathcal{N}_i} \|x\|_{\mathbb{Q}_p^{n+1}}^s \, d^*x,$$

for $i = 0, ..., n$.

Proof. First, we choose a Haar measure d^*x on $\left(\mathbb{Q}_p^*\right)^{n+1}$, such that $d^*x = (d^*\alpha)^{n+1}$, where $d^*\alpha$ is a Haar measure of $\mathbb{Q}_p^*$ such that $\int_{\mathbb{Z}_p^*} d^*\alpha = 1$. Thus:

$$\int\limits_{\mathbb{Q}_p^* \bigcap \mathbb{Z}_p} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha = \int\limits_{\bigcup\limits_{t=0}^{\infty} p^t \mathbb{Z}_p^*} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha = \sum_{t=0}^{\infty} \left[\left(p^{-s}\right)^t \int\limits_{\mathbb{Z}_p^*} d^*\alpha \right] = \frac{1}{(1-p^{-s})}.$$

Then, we have:

$$\int\limits_{\left(\mathbb{Q}_p^*\right)^{n+1} \bigcap \mathcal{N}_n} \|\boldsymbol{x}\|_{\mathbb{Q}_p^{n+1}}^s \, \boldsymbol{d^*x} =$$

$$\left\|\left(p, p^2, ..., p^n, p^n\right)\right\|_{\mathbb{Q}_p^{n+1}}^s \left[\int\limits_{\mathbb{Q}_p^* \bigcap \mathbb{Z}_p} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha \right]^{n+1} = \frac{\left(p^{-s}\right)^{\frac{n(n+3)}{2}}}{\left(1-p^{-s}\right)^{n+1}}. \tag{4}$$

Now, let $i \in \{1, ..., n-1\}$. We choose a Haar measure d^*y on $\left(\mathbb{Q}_p^*\right)^{n-i+1}$, such that $d^*y = (d^*\alpha)^{n-i+1}$, then, we have:

$$\int\limits_{\left(\mathbb{Q}_p^*\right)^{n+1} \bigcap \mathcal{N}_i} \|\boldsymbol{x}\|_{\mathbb{Q}_p^{n+1}}^s \, \boldsymbol{d^*x} =$$

$$\left\| \left(p, p^2, ..., p^i, \underbrace{p^i, ..., p^i}_{(n-i+1)-\text{times}} \right) \right\|_{\mathbb{Q}_p^{n+1}}^s \left[\int\limits_{\mathbb{Q}_p^* \bigcap \mathbb{Z}_p} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha \right]^i \int\limits_{\left(\mathbb{Q}_p^*\right)^{n-i+1} \bigcap B_p\left(C_{p^{n-i}}\right)} \|y\|_{\mathbb{Q}_p^{n-i+1}}^s \, d^*y =$$

$$\frac{\left(p^{-s}\right)^{\frac{i(2n+3-i)}{2}}}{\left(1-p^{-s}\right)^i} \left[\int\limits_{\left(\mathbb{Q}_p^*\right)^{n-i+1} \bigcap B_p\left(C_{p^{n-i}}\right)} \|y\|_{\mathbb{Q}_p^{n-i+1}}^s \, d^*y \right]. \tag{5}$$

Finally, we can compute recursively

$$\int\limits_{\left(\mathbb{Q}_p^*\right)^{n-i+1} \bigcap B_p\left(C_{p^{n-i}}\right)} \|y\|_{\mathbb{Q}_p^{n-i+1}}^s \, d^*y$$

as follows:

Let $2 \leq l$. We choose a Haar measure d^*y_{l+1} on $\left(\mathbb{Q}_p^*\right)^{l+1}$, such that $d^*y_{l+1} = (d^*\alpha)^{l+1}$. We know that, $B_p\left(C_{p^l}\right)$ is local, where:

$$\operatorname{rad}\left(B_p\left(C_{p^l}\right)\right) = (p, ..., p)\left[\mathbb{Z}_p \times B_p\left(C_{p^{l-1}}\right)\right].$$

Thus

$$B_p\left(C_{p^l}\right) = B_p^*\left(C_{p^l}\right) \bigcup (p, ..., p)\left[\mathbb{Z}_p \times B_p\left(C_{p^{l-1}}\right)\right],$$

and then:

$$\int_{(\mathbb{Q}_p^*)^{l+1}\cap B_p\left(C_{p^l}\right)} \|y_{l+1}\|^s_{\mathbb{Q}_p^{l+1}}\, d^*y_{l+1} =$$

$$\int_{(\mathbb{Q}_p^*)^{l+1}\cap B_p^*\left(C_{p^l}\right)} d^*y_{l+1} + \int_{(\mathbb{Q}_p^*)^{l+1}\cap(p,...,p)\left[\mathbb{Z}_p\times B_p\left(C_{p^{l-1}}\right)\right]} \|y_{l+1}\|^s_{\mathbb{Q}_p^{l+1}}\, d^*y_{l+1} =$$

$$\left((\mathbb{Z}_p^*)^{l+1}:\ B_p^*\left(C_{p^l}\right)\right)^{-1} +$$

$$\|(p,...,p)\|^s_{\mathbb{Q}_p^{l+1}} \left[\int_{\mathbb{Q}_p^*\cap\mathbb{Z}_p} \|\alpha\|^s_{\mathbb{Q}_p}\, d^*\alpha\right] \left[\int_{(\mathbb{Q}_p^*)^l\cap B_p\left(C_{p^{l-1}}\right)} \|y_l\|^s_{\mathbb{Q}_p^l}\, d^*y_l\right] =$$

$$\frac{1}{p^{\frac{l(l-1)}{2}}(p-1)^l} + \frac{(p^{-s})^{l+1}}{(1-p^{-s})}\left[\int_{(\mathbb{Q}_p^*)^l\cap B_p\left(C_{p^{l-1}}\right)} \|y_l\|^s_{\mathbb{Q}_p^l}\, d^*y_l\right]. \tag{6}$$

☑

Corollary 3.11.

$$\int_{(\mathbb{Q}_p^*)^{n+1}\cap\mathcal{N}_i} \|x\|^s_{\mathbb{Q}_p^{n+1}}\, d^*x = \frac{g_i(p^{-s})}{p^{\frac{(n-i)(n-i-1)}{2}}(p-1)^{n-i}(1-p^{-s})^{n+1}},$$

where $g_i(p^{-s})\in\mathbb{Z}[p^{-s}]$ *is a polynomial of degree:*

$$\partial\left(g_i(p^{-s})\right) = \frac{n(n+3)}{2},$$

for $i=0,...,n$.

Proof. From [6, Remark 3.6], we have that:

$$\int_{(\mathbb{Q}_p^*)^2\cap B_p(C_p)} \|z\|^s_{\mathbb{Q}_p^2}\, d^*z = \frac{1-2\left(p^{-s}\right)+p\left(p^{-s}\right)^2}{(p-1)\left(1-p^{-s}\right)^2},$$

then, from Eq. (6) it is easy to see by induction on $2\le l$ that:

$$\int_{(\mathbb{Q}_p^*)^{l+1}\cap B_p\left(C_{p^l}\right)} \|y_{l+1}\|^s_{\mathbb{Q}_p^{l+1}}\, d^*y_{l+1} = \frac{h_l(p^{-s})}{p^{\frac{l(l-1)}{2}}(p-1)^l(1-p^{-s})^{l+1}}, \tag{7}$$

where $h_l(p^{-s})\in\mathbb{Z}[p^{-s}]$ is a polynomial of degree:

$$\partial\left(h_l(p^{-s})\right) = \frac{l(l+3)}{2},$$

and so, from Eq. (4), Eq. (5) and Eq. (7) it follows that:

$$\int_{(\mathbb{Q}_p^*)^{n+1} \cap \mathcal{N}_i} \|x\|_{\mathbb{Q}_p^{n+1}}^s \, d^*x = \frac{g_i(p^{-s})}{p^{\frac{(n-i)(n-i-1)}{2}} (p-1)^{n-i} (1-p^{-s})^{n+1}},$$

where $g_i(p^{-s}) \in \mathbb{Z}[p^{-s}]$ is a polynomial of degree:

$$\partial \left(g_i(p^{-s}) \right) = \frac{n(n+3)}{2},$$

for $i = 0, ..., n$. ☑

Lemma 3.12. *Let M be a fractional ideal of finite index in $B_p(C_{p^n})$. If M is a $\mathbb{Z}_p$-order, then:*

$$\mu^* \left(Aut_{B_p(C_{p^n})} M \right)^{-1} (B_p(C_{p^n}) : M)^{-s} = p^b (p-1)^c (p^s)^d,$$

for $b + c + d = \frac{n(n+1)}{2}$.

Proof. From the proof of Theorem 3.9, we have that:

$$M = \left\{ (x_1, ..., x_{n+1}) \in \mathbb{Z}_p^{n+1} : \quad \mathcal{R}_1 \in p^{m_1} \mathbb{Z}_p, ..., \mathcal{R}_\lambda \in p^{m_\lambda} \mathbb{Z}_p \right\},$$

for some $\mathcal{R}_j \in p^{m_j} \mathbb{Z}_p$ and $\sum_{j=1}^{\lambda} m_j \leq \frac{n(n+1)}{2}$. From Eq. (2) we have:

$$(B_p(C_{p^n}) : M)^{-s} = (p^s)^{\left(\frac{n(n+1)}{2} \right) - \left(\sum_{j=1}^{\lambda} m_j \right)}, \tag{8}$$

and from Eq. (1) we have:

$$\mu^* \left(\mathrm{Aut}_{B_p(C_{p^n})} M \right)^{-1} = \left((\mathbb{Z}_p^*)^{n+1} : \mathrm{Aut}_{B_p(C_{p^n})} M \right).$$

Now, by hypothesis, we have that:

$$\mathrm{End}_{B_p(C_{p^n})} M = M,$$

and then its group of units is $\mathrm{Aut}_{B_p(C_{p^n})} M = M^*$. It is easy to see (as in [6, Remark 3.2]) that:

$$\frac{(\mathbb{Z}_p^*)^{n+1}}{M^*} \cong \left(\frac{\mathbb{Z}}{p^{m_1} \mathbb{Z}} \right)^* \times \cdots \times \left(\frac{\mathbb{Z}}{p^{m_\lambda} \mathbb{Z}} \right)^*$$

then:

$$\left((\mathbb{Z}_p^*)^{n+1} : \mathrm{Aut}_{B_p(C_{p^n})} M \right) = p^{\left(\sum_{j=1}^{\lambda} m_j \right) - \lambda} (p-1)^\lambda. \tag{9}$$

Finally, from Eq. (1), Eq. (8) and Eq. (9) we obtain:

$$\mu^* \left(\mathrm{Aut}_{B_p(C_{p^n})} M \right)^{-1} (B_p(C_{p^n}) : M)^{-s} = p^{\left(\sum_{j=1}^{\lambda} m_j \right) - \lambda} (p-1)^\lambda (p^s)^{\left(\frac{n(n+1)}{2} \right) - \left(\sum_{j=1}^{\lambda} m_j \right)}.$$

☑

Now, we present our conjecture:

Conjecture. *Let M be a fractional ideal of finite index in $B_p(C_{p^n})$. Then, M is a $\mathbb{Z}_p$-order if and only if:*

$$Z_{B_p(C_{p^n})}(M;s) = p^b(p-1)^c(p^s)^d \int_{(\mathbb{Q}_p^*)^{n+1}\cap\{M:B_p(C_{p^n})\}} \|x\|^s_{\mathbb{Q}_p^{n+1}}\, d^*x,$$

for $b+c+d = \frac{n(n+1)}{2}$.

Remark 3.13. Let M be a fractional ideal of finite index in $B_p(C_{p^n})$. We have that:

$$Z_{B_p(C_{p^n})}(M;s) =$$

$$\mu^*\left(\mathrm{Aut}_{B_p(C_{p^n})}M\right)^{-1}(B_p(C_{p^n}):M)^{-s} \int_{(\mathbb{Q}_p^*)^{n+1}\cap\{M:B_p(C_{p^n})\}} \|x\|^s_{\mathbb{Q}_p^{n+1}}\, d^*x.$$

If M is a $\mathbb{Z}_p$-order, then the result in the Conjecture, follows from Lemma 3.12. Conversely, if:

$$Z_{B_p(C_{p^n})}(M;s) = p^b(p-1)^c(p^s)^d \int_{(\mathbb{Q}_p^*)^{n+1}\cap\{M:B_p(C_{p^n})\}} \|x\|^s_{\mathbb{Q}_p^{n+1}}\, d^*x,$$

for $b+c+d = \frac{n(n+1)}{2}$, then M is a $\mathbb{Z}_p-$order for $n = 1, 2, 3$.

Acknowledgements: The authors would like to thank the referees for the valuable suggestions and comments.

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