ESSAYS ON GENERAL EQUILIBRIUM WITH ASYMMETRIC INFORMATION

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- AGOSTINHO DA SILVA -

Sumário

O ponto de partida para este trabalho é o modelo introduzido por Radner (1968), que estende a teoria de equilíbrio geral a situações nas quais os agentes têm informação diferentes sobre o estado da natureza. A ideia por detrás desta extensão consiste em restringir os agentes a produzir e consumir os mesmos cabazes, em estados da natureza que não distingam. Isto significa, essencialmente, que os contratos entre dois agentes só podem ser contingentes à ocorrência de eventos que ambos observam.

Uma propriedade importante de qualquer conceito de solução é a continuidade do resultado relativamente a variações nos parâmetros do modelo. Pequenas variações nos parâmetros devem conduzir a pequenas variações do resultado de equilíbrio. Mas, medindo as variações na informação dos agentes de acordo com as topologias introduzidas por Boylan (1971) e Cotter (1986), o conceito não cooperativo de *Equilíbrio de Expectativas Walrasianas* (Radner, 1968) e o conceito cooperativo de *núcleo privado* (Yannelis, 1991) não se comportam de forma contínua.

O problema crucial é que pequenas variações nos campos de informação privada podem provocar grandes variações no campo da informação comum. Como os contratos contingentes se baseiam na informação comum, pequenas variações na informação privada podem abrir ou fechar mercados contingentes, levando a variações significativas do resultado de equilíbrio.

Neste trabalho é introduzida uma topologia sobre a informação (σ-algebras finitas definidas no espaço de estados da natureza) que ultrapassa este problema. Nesta topologia, dois campos de informação estão próximos se ambos estiverem próximos da informação comum. Com esta topologia, passa a verificar-se a semicontinuidade superior do *núcleo privado* da economia.

Em seguida, procura-se generalizar o modelo de Radner (1968). A restrição que força os agentes a consumir o mesmo em estados que não distinguem é relaxada. Permite-se que os agentes façam *contratos de entrega incerta*. Isto significa que, além de poderem comprar bens contingentes, como uma "*bicicleta*" se estiver "*sol*", os agentes podem também comprar o direito a receber um dos cabazes que esteja numa lista. Por exemplo, uma "*bicicleta azul ou bicicleta vermelha*" se estiver "*sol*". Deste modo, o espaço de trocas é alargado, possibilitando melhorias de bem-estar no sentido de Pareto.

No contexto das economias com entrega incerta, estudam-se as expectativas prudentes/pessimistas. Estas levam os agentes a escolher estratégias *minimax*. São apresentadas diversas justificações. Com expectativas prudentes, o modelo da economia com entrega incerta é formalmente equivalente ao modelo de Arrow-Debreu (1954). Consequentemente, muitos resultados da teoria de equilíbrio geral se aplicam imediatamente a este modelo: existência de núcleo e equilíbrio competitivo, convergência núcleo-equilíbrio, propriedades de continuidade, etc.

Summary

The starting point for this work is the model introduced by Radner (1968), which extends general equilibrium theory to a setting in which agents have different private information. The idea underlying this extension is to restrict agents to produce and consume the same bundles, in states of nature that they do not distinguish. Essentially, this means that the contracts for contingent trade between two agents can only be contingent upon their common information.

An important property of any solution concept is the continuity with respect to the parameters of the model. That is, small changes in the parameters should lead to small changes in the equilibrium outcome. But measuring changes in the information of the agents according to the topologies introduced by Boylan (1971) and Cotter (1986), the non-cooperative *Walrasian Expectations Equilibrium* (Radner, 1968) and the cooperative *private core* (Yannelis, 1991) do not behave continuously.

The crucial problem is that small changes in the private information fields can lead to big changes in the field of common information. Since contingent contracts are based on common information, these small changes may open or close some contingent markets, leading to significant changes in the equilibrium outcome.

In this work is introduced a topology on information (finite σ-algebras defined over the space of states of nature) that overcomes this problem. In this topology, two information

fields are close if both are close to their common information. As a result, we find that the private core is upper semicontinuous with respect to variations in the information of the agents.

Afterwards, a generalization of the model of Radner (1968) is sought. The restriction that forces agents to consume the same in states of nature that they do not distinguish is alleviated. Agents are allowed to sign contracts for uncertain delivery. This means that, besides being able to buy state-contingent goods, for example, a "bicycle" if "weather is sunny", agents are also able to buy the right to receive one of the bundles that are included in a list. For example, a "blue bicycle or red bicycle" if "weather is sunny". In this way, the space of possible trades is enlarged, and welfare improvements in the sense of Pareto become possible.

In the context of uncertain delivery, the case is made for prudent/pessimistic expectations. These expectations lead agents to select *minimax* strategies. Several justifications are presented. With prudent expectations, the model of an economy with uncertain delivery is formally equivalent to the model of Arrow-Debreu (1954). As a result, many results in general equilibrium theory also apply in this model: existence of core and competitive equilibrium, core-convergence, continuity properties, etc.

Ah Love! could thou and I with Fate Conspire
To grasp this sorry Scheme of Things entire,
Would not we shatter it to bits - and then
Re-mould it nearer to the Heart’s Desire!

- OMAR KHAYYÁM -

Contents

Chapter 1

Words of caution

According to Adam Smith's (1776) idea of the *invisible hand*, in a market system, individuals contribute to the welfare of the society by seeking to maximize their well-being.

The economic outcome of a market system is the result of individuals independently trying to maximize their well-being, a notion known as *competitive equilibrium*.

A **competitive equilibrium** is a situation in which:

$i)$ each individual, taking prices as fixed, chooses the quantities of the different goods to produce and exchange, in order to obtain the most preferred bundle in the budget set;

$ii)$ equality between supply and demand holds, that is, for each good, the sum of the quantities supplied is equal to the sum of the quantities demanded.

These conditions must hold for all the commodities in an economy. This is what general equilibrium theory is concerned with: the determination of production, exchange and prices of all the commodities in an economy. The main contributor to this line of thought was Leon

Walras (1874), in spite of the independent contributions of Stanley Jevons (1871) and Carl Menger (1871).[1]

It is not obvious that such equilibrium situation is possible. Under general conditions, the **existence of competitive equilibria** was established by Arrow and Debreu (1954) and by McKenzie (1954). Their results mean that there exists a price system which induces individuals to choose quantities to produce and exchange which are consistent with equality between supply and demand.

Furthermore, Adam Smith's claim about the effectiveness of the *invisible hand* in promoting the welfare of the society was supported by two famous results. According to the **First Welfare Theorem**, a competitive equilibrium allocation is Pareto-optimal. This means that there isn't any situation that all individuals prefer to a competitive equilibrium. The **Second Welfare Theorem** holds that any Pareto-optimal allocation can be attained as a competitive equilibrium, if a certain redistribution of initial endowments is made.

The impossibility of measuring and comparing the well-being of different individuals could seem to prevent the measurement of society's welfare. But there are certain accepted criteria for comparing different economic outcomes, as the criterion of Pareto (1906). An outcome is designated as **Pareto-optimal** if there isn't any alternative that everyone prefers.[2]

If the model of Arrow-Debreu-McKenzie described the economy perfectly, we wouldn't observe any unemployment or price volatility. In fact, some strong hypothesis are imposed.

[1]As early as 1781, A.N. Isnard presented the first general equilibrium model, considering a pure exchange economy where in which each individual owned a single asset, with all demand functions having unit elasticity in income and own price.

[2]The criterion of Pareto is criticized for neglecting the question of distribution. Assume that there are 10 units to divide among 2 individuals. If one of them receives 10, and the other receives nothing, the outcome is optimal in the sense of Pareto.

It is assumed that the agents have **perfect and complete information**, and that there exists a **complete set of markets**. Let's analyze these and other limitations of classical general equilibrium theory.

The welfare theorems support the idea that market economies are efficient, but there are elements that lead to "non-fair" equilibrium prices, such as *adverse selection* and *moral hazard*. In fact, real market economies are almost never efficient, and Adam Smith's conjecture is not true in general.

For the two welfare results to be obtained, it is fundamental that each agent takes prices as fixed. This assumption is usually designated as *perfect competition*. Since there exists a large number of sellers and buyers in the markets, no one can influence prices.[3]

Another assumption is that there are markets for all the products. Even for commodities that will only be delivered in the distant future. These markets serve as a guide to the investment decisions of the firms. For example, the absence of a market for delivery of buildings in 2100 could prevent agents from optimizing their investment decisions.[4] Many decisions are actually based on bets, but the point is that even if we assume that agents behave rationally, the efficiency of the market system is not guaranteed in general in the absence of *complete markets*.

The conjecture of Coase (1960) focuses the importance of *property rights*, and the problems associated with the exploration of common resources. The example of fishing is quite illustrative, since it is an activity with private benefits which has some costs that are

[3]For a discussion and critique of the assumption of *perfect competition*, see Makowsky & Ostroy (2001).

[4]Suppose that a firm decides to construct a building based on an estimate of the value that it will have in 2015. The problem is that this value depends on the number of buildings that will be built in 2006, 2007, etc. And these depend on the estimate of the value of the buildings in 2016, 2017, etc. This extends to infinity!

supported by the whole society (decrease of the quantity of fish in the ecosystem). Consider a fishing zone shared by 100 firms, and a study that advises the use of 100 boats in order to maximize the total (present and future) volume of fishing. If each firm decided to use 1 boat, this advice would be followed. But imagine that when the firms weight benefits against costs, they conclude that it is better to use two boats and catch almost twice as much fish as with one single boat. Since all firms send two boats, too much fish is caught, and the ecosystem ends up being depleted.[5]

There are other strong assumptions in the model of Arrow-Debreu-McKenzie. One is the assumption of *linear prices*, that is, independence of prices from the quantities exchanged. And it is assumed that firms have no profits, otherwise a new competitor would enter the market (*free entry*).

Now let's turn to the crucial limitation which constitutes the motivation to our work: the problems related to information. In general, the agents do not possess all the relevant information for making economic decisions. There is usually some uncertainty about the environment, and there are events which only some of the agents can observe.

Our work is in the context of the literature on *differential information economies*, which developed from the seminal article of Radner (1968). This literature seeks to extend the model of general equilibrium to situations in which agents have asymmetric information. What follows are some questions that the reader should keep in mind.

The complexity associated with the issues related to information is such that it cannot be captured by any simple model. The market economy is too complicated to be fully described in simple terms. A realistic goal is to find simple models that give enlightening, although partial, descriptions.

[5]Hardin (1968) has a classical article on this problem.

Taking into account uncertainty about product quality introduces a lot of complexity. Even if quality is assumed to be purely objective, buyers should question the truthfulness of the claims made by sellers. This issue is usually referred to as the problem of *incentive compatibility*.

For many commodities, the cost of providing them depends on the behavior of the purchaser. **Moral hazard** arises when behavior of the demander that is not easily observed affects the cost of the supplier. An example is the case of insurance. After buying insurance, agents may become careless.[6]

Sometimes several goods are different in the eyes of the consumer, but are sold as if they were equal. When one side of the market treats certain commodities as different, while the other side treats them as equal, problems of **adverse selection** arise. An example occurs when buyers cannot observe the quality of the product, but sellers can. In this case, the sellers of high quality products withdraw from the market.

Akerlof (1970) analyzed a market in which the sellers could distinguish the quality level of a product, while the buyers did not. Initially the buyers may expect an average level of quality, and think of making a correspondent bid. But the sellers of good quality products would not be willing to sell them at this average price. So, the potential buyers reason that only the sellers of "bad" products will be willing to trade in the market. Expecting to receive a "bad" product, they offer a low price. As a consequence, only "bad" products are bought or sold, and all products are priced as if they were "bad".[7]

[6] Assuming that carefulness is, even if only slightly, costly.

[7] C. Wilson (1980) studies a variant of the model of Akerlof (1970) in which agents differ on the value that they attach to cars of the same quality. He finds that the results depend on whether it is an auctioneer, the buyers or the sellers that set the price.

Rothschild & Stiglitz (1976) analyzed an insurance market in which the sellers could not distinguish the risk level of the customers, which could be high or low. So, they cannot offer a better contract to the low-risk customers, because all the customers would pretend to be low-risk to get this contract. As a consequence, there was no competitive equilibrium.

In the model of Arrow-Debreu, supply equals demand when the economy is in equilibrium. But we observe frequently disparities between supply and demand of certain goods. The classical example is unemployment, or excess of supply of labor. In fact, problems of information may render the balance of supply and demand untenable.

Imagine a situation of full employment in which employers cannot observe perfectly the effort of the worker. Firing the worker does not work as a punishment, because she can find another job instantly. The incentives are for workers to shirk![8]

Another common situation is of excess demand for credit. The interest rate charged by a bank influences the risk of the loans that are proposed. The interest rate that an agent is willing to accept signals its risk-level. A bank which sets a high interest rate will only attract loans with high risk. Therefore, the optimal interest rate may not be equal to the one that balances supply and demand.[9]

The behavior of the agents varies with the interest rate. Higher interest rates diminish the value of investments, and induce individuals to take more risks (since the worst possible outcome is a return of zero, which corresponds to bankruptcy). With *perfect information*, these issues would be meaningless. But, since the bank cannot control the decisions of the firms, it is important to analyze the incentives that the loan contracts give to firms.

[8]This is described in the seminal article of Shapiro & Stiglitz (1984).

[9]This is analyzed by Stiglitz & Weiss (1981).

Spence (1973) studies actions outside the market that generate information which is then used by the market. In his model of *signalling*, agents engage in education because their potential employers see this a sign of good capabilities. If they had poor capabilities, engaging in education would be irrational. So, signalling is a kind of implicit guarantee. A situation in which beliefs are stable is a *signalling equilibrium*.[10]

Besides assuming that prices are linear, it is also assumed that they are *homogeneous*, that is, that all trades are made according to the same price system. Stigler (1961) questions this hypothesis and analyzes the the problem of *search*, that is, of buyers seeking costly information about the prices quoted by the different sellers.

The point of Grossman & Stiglitz (1980) is that if arbitrage is costly and gives no return in equilibrium, then no agents will engage in this activity. As a consequence, in equilibrium, the condition of null arbitrage profits should be substituted by one giving an "*equilibrium amount of disequilibrium*".

Besides analyzing whether an equilibrium situation exists or not, it is important to study the way agents reach this situation - the problem of *implementation*. In the seminal article of Schmeidler (1980), Walrasian equilibrium is implemented as a Nash equilibrium of a market game.

According to Hayek (1945), the perfect information model does not capture the fundamental role of prices and markets in processing and disseminating information. Sixty years have passed since he warned us that general equilibrium theory does not by itself solve the economic problem. The theory "only" gives a logical solution to a problem in which the relevant data is given.

[10]For a survey on signalling and general issues related to information see Riley (2001).

"*On certain familiar assumptions the answer is simple enough. If we possess all the relevant information, if we can start out from a given system of preferences, and if we command complete knowledge of available means, the problem which remains is purely one of logic. That is, the answer to the question of what is the best use of the available means is implicit in out assumptions.*"

But this relevant data is never given to a single mind.

"*[...] the economic calculus which we have developed to solve this logical problem, though an important step toward the solution of the economic problem of society, does not yet provide an answer to it. The reason for this is that the 'data' from which the economic calculus starts are never for the whole society 'given' to a single mind which could work out the implications, and can never be so given.*"

After these words of caution, we can start the study of general economic equilibrium with asymmetric information.

Chapter 2

Introduction

The treatment of uncertainty in the theory of general equilibrium is based upon two foundations: the *Expected Utility Theorem* of von Neumann and Morgenstern (1944); and the formulation of the ultimate goods or objects of choice in an uncertain universe as contingent consumption claims (Arrow, 1953).

The *Expected Utility Theorem* provides a convenient way to compare risky bundles, by establishing the existence of an utility function that represents preferences over lotteries.

Under the formulation of objects of choice as contingent consumption bundles, besides being defined by their physical properties and their location in space and time, commodities can also be defined by the *state* in which they are made available. For example, an "umbrella" that is delivered if the "weather is rainy" and an "umbrella" delivered if the "weather is sunny" are seen as two different commodities. This formulation allowed Debreu (1959, chapter 7) to extend the general equilibrium model to a situation of uncertainty.

There were essentially two lines of contribution to equilibrium theory with complete information: one of Cournot (1838) and Nash (1950), and that of K. J. Arrow & Debreu

(1954), and McKenzie (1954). But to take into account the problems of incomplete information introduces a great deal of complexity. The main advances were made by Harsanyi (1967), who extended the Cournot-Nash framework, and by Radner (1968), who did the same to the model of Arrow-Debreu-McKenzie.

The transformation of games with *incomplete information* into games with *imperfect information*, accomplished by Harsanyi (1967) was a giant step. In a game with *incomplete information*, agents are uncertain about the payoff functions. The meaning of *imperfect* information is that when information is that agents cannot perfectly observe the strategies chosen by the other players. Considering that there are many possible *types* of players, it may be assumed that an unobservable choice of nature at the beginning of the game selects the actual players from the set of possible types. So, from a problem of knowledge about payoffs, we move to a problem of knowledge about the type of player selected by nature. With agents having *prior* probabilities on the choice of nature, the game with incomplete information ends up being defined as a game of imperfect information. This type of game can be analyzed with standard techniques.

To model an economy in which agents have asymmetric information, it is considered that it extends over two time periods. In the first period, agents know their endowments and preferences, as a function of the state of nature, and have a partition of information, that tells them which events they can observe. In this period (*ex ante*), agents make contracts for delivery of goods in the second period, which can be contingent upon the state of nature. In the second period, agents get to know which set of their partition of information includes the actual state of nature. As a consequence, they are informed on their preferences and receive the corresponding endowments. Finally, contracts are enforced and consumption takes place.[1]

[1]For a survey on *differential information economies*, see Allen & Yannelis (2001).

This setup allowed Radner (1968) to propose an extension of the model of Arrow-Debreu-McKenzie to the case of private information. By *private information* it is meant a situation in which agents have asymmetric information and do not communicate.

The basic idea is that agents are not willing to pay for delivery that is contingent upon events that they do not observe. As a result, it is assumed that they will consume the same in states of nature that they do not distinguish. With this condition, the economy with private information is formally equivalent to the Arrow-Debreu-McKenzie economy. The equilibrium of prices and consumption vectors of the economy with private information is designated as a *Walrasian expectations equilibrium* (WEE).

The essential modification, with respect to the equilibrium notion without uncertainty, is this restriction of measurability, that is, of forcing agents to consume the same in states of nature that they do not distinguish. Formally, the consumption of an agent, as a function of the state of nature, has to be measurable with respect to the σ-field of its private information.

A corresponding cooperative notion of equilibrium is the it private core. This concept was introduced by Yannelis (1991), who also proved existence in general conditions. Relatively to the classical core notion, it has the same measurability restriction: allocations have to be informationally feasible.

Allowing for communication introduces a lot of complexity. But, actually, the first notions of a core in an economy with asymmetric information (Wilson, 1978) were based on the ideas of *common information* and *pooled information*. These are the *coarse core* and the *fine core*. If coalitions are only allowed to block allocations by using allocations which are measurable with respect to the common information among their members, the result is the *coarse core*. At the other extreme is the *fine core*, which is constituted by the *informationally feasible* allocations which cannot be blocked by any consumption vector that is measurable with respect to the pooled information of the members of a coalition.[2]

The notion of *incentive compatibility* (Hurwicz, 1972) is the focus of a recent survey by Forges, Minelli, & Vohra (2002) on the core of an exchange economy with asymmetric information. They consider the restriction of *informationally feasibility* to be too strong, and prefer to analyze incentives. The emphasis of their work is on incentive compatibility, and on convergence of the core to price equilibrium allocations.

In the case in which it is possible to communicate, it is crucial to know if information can be verified or not. If it can be verified, then we may be able to treat it as a commodity. This is the crux of the work of Allen (1990), who studied information as if it were an economic commodity, susceptible of production and exchange. There are non-convexities, because each partition is only interesting in integer quantities (half partition would be meaningless). With an infinite number of traders this problem disappears. A problem that arises if production of information is considered in the economy is that the costs associated to the production of information are essentially fixed costs.

The private core was shown to have nice properties. Koutsougeras & Yannelis (1993) proved that it is *coalitionally Bayesian incentive compatible* (*CBIC*). Einy, Moreno, & Shitovitz (2001) prove an equivalence theorem for the private core. Serrano, Vohra, & Volij (2001) present counter-examples to the core convergence theorems whenever expected utilities are interim.

Forges, Heifetz, & Minelli (2001) obtain a Debreu-Scarf analogue for a type-model where the space of allocations is defined as the set of incentive compatible state-contingent lotteries over consumption goods. They show that competitive equilibrium allocations

[2] Note that such consumption vector may not be an allocation, since it may not be *informationally feasible*.

exist and are elements of the (ex-ante incentive) core. This core is constituted by the allocations such that no coalition can propose a feasible incentive compatible allocation which improves the expected utility of all its members. Any competitive equilibrium is an element of the core of the n-fold replicated economy. The converse holds with the assumption of private values - equal preferences in states of nature that the agent does not distinguish. This is in the lines of Prescott and Townsend (1984a, 1984b), who also impose a finite "base" for lotteries and private values.

The main idea in Prescott and Townsend (1984a, 1984b) is that individuals trade state-contingent lotteries over the initial consumption goods. This ensures that the consumption set is convex. With objects of trade as incentive compatible state-contingent lotteries over the original goods, competitive equilibria can be defined in the usual way, using expected feasibility and constructing prices of lotteries as expectations of the prices of original goods.

To provide scope and context to this work, the first chapter was a discussion of the limits of these models to explain real economies.

In this second chapter, the state of the art of general equilibrium with asymmetric was presented. Special attention was given to the works on differential information economies. This is a line of research that follows the seminal work of Radner (1968), where general equilibrium theory was extended to a setting in which agents have different private information.

The third and fourth chapter review the theory of general equilibrium. In chapter 3, the analysis is restricted to perfect information. Chapter 4 extends the theory to the cases of symmetric uncertainty (Arrow and Debreu, 1954), and to a setting of asymmetric information (Radner, 1968). The idea underlying Radner's extension is to restrict agents to produce and consume the same bundles, in states of nature that they do not distinguish.

Essentially, this means that the net trades between two agents can only be contingent upon their common information.

In the fifth chapter, the problem of the continuity of equilibrium with respect to variations in the private information of the agents is studied. A new topology on finite information fields is introduced. This topology evaluates the similarity between information fields taking into account their compatibility, that is, the events that are commonly observed. With this "*topology of common information*", the Walrasian expectations equilibrium (Radner, 1968) and the private core (Yannelis, 1991) are upper semicontinuous.

In chapter 6, the model of Radner (1968) is generalized. Recall that Radner extended the model of Arrow-Debreu to the case of private information by constraining agents to consume the same in states of nature that they do not distinguish. But agents may be willing to buy different goods for delivery in states that they do not distinguish *ex ante*, if, in any case, they become better off. This suggests the introduction of *contracts for uncertain delivery*.

Finally, in chapter 7, economies with private information and uncertain delivery are studied. Agents are assumed to be *prudent*, that is, to follow *minimax* strategies. Many classical results still hold: existence of core and equilibrium, core convergence, continuity properties, etc. In a *prudent expectations equilibrium*, agents consume bundles with the same utility in states of nature that they do not distinguish *ex ante*. Since this restriction is weaker than equal consumption, efficiency of trade and welfare are improved.

In the appendix, both Von Neumann's and Savage's axiomatizations of expected utility are presented, and the value of information is defined accordingly.

Chapter 3

Equilibrium with Perfect Information

In the literature on *differential information economies*, two concepts of equilibrium predominate: one is the cooperative notion of the *core*; and the other is the non-cooperative notion of *competitive equilibrium*. The notion of competitive equilibrium has much in common with the famous concept of *Nash equilibrium*. But while Nash equilibrium applies to games in general, a competitive equilibrium makes sense in the context of a *market economy*.

3.1 Nash equilibrium of an n-player game

A **game in normal form** is defined by the strategies available to each player, and by the outcomes that correspond to every possible combination of strategies by the players. A **strategy** determines every action of a player throughout the game for all possible contingencies that the player may face. So, given the strategies of the players, it is possible to determine each player's outcome. We assume that each agent compares the outcomes according to an agent-specific **utility function** that assigns a real number to each point in the strategy space of the players.

Definition 1 (N-PLAYER GAME) *A game* $G \equiv (X_i, V_i)_{i=1}^n$ *in its normal form is defined by:*

- *the set of strategies available to each player* i, X_i*;*

- *the utility function of each player* i, V_i*.*

The space of the possible strategies of the game is $X = \prod_{i=1}^{n} X_i$. A possible strategy is $x = (x_1, x_2, ..., x_n) \in X$.

The utility function of each player, $V_i : X \times X_i \to \mathbb{R}$, is such that $V_i(x, x_i')$ is the utility of agent i playing x_i' while the others play x_j, $(j \neq i)$.

The idea of an equilibrium as a situation in which no agent has incentives to deviate, assuming the actions of the others as given, was first discussed by Augustin Cournot (1838) in a context of a duopoly. It was rediscovered by John Nash (1950), who proved the existence of such equilibrium solution for general n-player games. This is probably why it is referred as *Nash equilibrium*, although sometimes it is designated as *Cournot-Nash equilibrium*.

Definition 2 (NASH EQUILIBRIUM)

The strategy $x^* \in X \equiv \prod_{i=1}^{n} X_i$ *is a Nash Equilibrium of the game* $G \equiv (X_i, V_i)_{i=1}^n$ *if and only if, for every player* i, $V_i(x^*, x_i^*) \geq V_i(x^*, x_i')$, $\forall x_i' \in X_i$*. That is,* $x^* = (x_1^*, x_2^*, ..., x_n^*)$ *is composed by best responses of each agent to* x^**.*

In a Nash equilibrium, each agent's strategy is the best response to the strategies chosen by the other agents. Existence of Nash equilibrium means that the agents can reach a situation in which they consider their strategies to be simultaneously optimal.

Consider the illustrative *paper-rock-scissors game*. In this two-player game, the players choose among three possible actions: "paper", "rock" or "scissors". The paper beats the rock by enveloping it, the rock beats the scissors by breaking them, and the scissors beat the paper by cutting it. It is straightforward that, given the strategy of the opponent, there is a best response that makes one win every time (play paper against rock, rock against scissors, and scissors against paper). But then, given this new strategy, the opponent's best response will be to play differently, in order to be able to win every time. The change of strategy leads the opponent to choose a new optimizing strategy. From this circularity follows that there isn't any pair of mutually optimal pure strategies, that is, there isn't a Nash equilibrium of the game in the space of the pure strategies.

Yet, there exists a Nash equilibrium in the space of mixed strategies. If the players randomly play paper, rock or scissors with equal probabilities (1/3 each) in each repetition of the game, their strategies are optimal responses to the strategies of the opponent. As this example suggests, the famous result of existence of Nash Equilibrium demands a convex space of possible strategies (X_i convex for all i), otherwise, the theorem of Kakutani cannot be applied. Convexity can be obtained by allowing the agents to play mixed strategies.

3.2 Nash equilibrium of an n-player pseudo-game

A *pseudo-game* is a more general concept than that of a game. In a **pseudo-game**, the strategies that are available to a player may depend upon the strategies that are selected by the other players. A game does not allow this kind of interdependence.

Definition 3 (N-PLAYER PSEUDO-GAME)

A pseudo-game in its normal form, $PG \equiv (X_i, F_i, V_i)_{i=1}^n$, is defined by:

- the set of strategies potentially available to each player i, X_i;

- the set of strategies available to each player i, given the strategies chosen by the other players, $F_i : X \to X_i$, with $X = \prod_{i=1}^n X_i$;

- the utility or payoff function of each player i, $V_i : Gr(F_i) \to \mathbb{R}$.

In a Nash equilibrium of a pseudo-game, a player may have strategies that would be preferable, but which are inaccessible due to the choices of the other agents. This cannot occur in a game.[1]

When the correspondence F_i is continuous and convex-valued, existence of Nash equilibrium of a pseudo-game can be proved by direct application of the fixed point theorem of Kakutani and Berge's maximum theorem. Nash equilibrium of a game is a corollary of this for the case in which F_i is a constant correspondence.

Definition 4 (NASH EQUILIBRIUM OF A PSEUDO-GAME)

A strategy $x^* \in X = \prod_{i=1}^n X_i$ is a Nash Equilibrium of $PG \equiv (X_i, F_i, V_i)_{i=1}^n \Leftrightarrow$

a) $x_i^* \in F_i(x^*)$;

b) $V_i(x^*, x_i^*) \geq V_i(x^*, x_i')$, $\forall x_i' \in F_i(x^*)$.

Again, the vector of equilibrium strategies, $x^* = (x_1^*, x_2^*, ..., x_n^*)$, is composed by optimal responses of each agent to x^*, but only among those in the possibility set $F_i(x^*)$.

[1] Pseudo-games are useful in many contexts. For example, to model situations where imitation is excluded, as in the choice of location or in branding.

3.3 Competitive equilibrium of an exchange economy

An **exchange economy** is a system in which a finite number of agents exchanges an initial distribution of endowments, without incurring in any transaction cost. Each economic agent is characterized by: (1) a consumption possibility set; (2) preferences among alternative plans that are feasible; (3) initial endowments of physical resources. The objective of each agent is to maximize individual well-being.

The preferences of the agents are usually assumed to be representable by continuous and quasi-concave utility functions, in order to guarantee the convexity of the set of desirable bundles.

In general, each agent's choice of bundles, x_i, depends on what the other agents choose. The vector $x = (x_1, x_2, ..., x_n)$ is designated as an **allocation**. The interaction between the agents is mediated by a price-system. With a finite number of commodities, l, the price vector can be normalized to $p \in \Delta^l_+ = \{p \in \mathbb{R}^l_+ : \sum_j p_j = 1\}$ A common simplification that guarantees existence of competitive equilibrium consists in allowing only the consumption of non-negative quantities, $x \in \mathbb{R}^l_+$, and assuming non-negative prices, $p_i \geq 0$ (hypothesis of "*free disposal*").

Definition 5 (EXCHANGE ECONOMY)

An exchange economy is a triple, $\mathcal{E} \equiv (X_i, U_i, e_i)^n_{i=1}$, *where, for each agent* i*:*

- the space of possible consumption bundles is X_i*;*

- the utility function is $U_i : X_i \rightarrow \mathbb{R}$*;*

- the initial endowments are $e_i \in X_i$.

A Walrasian or **competitive equilibrium** is an equilibrium in a situation of perfect competition, as demand exactly matches supply with agents taking prices as fixed. In such a situation, every agent has an utility maximizing bundle, that is, each agent has the optimal quantities of each commodity at the prevailing price level. Therefore, each agent faces no utility-increasing trades.[2]

Definition 6 (COMPETITIVE EQUILIBRIUM)

(x^*, p^*) *is a Competitive Equilibrium of the economy* $\mathcal{E} \equiv (X_i, U_i, e_i)_{i=1}^n \;\Leftrightarrow$

1) $x^* = (x_1^*, ..., x_n^*)$ *is feasible, i.e.,* $x_i^* \in X_i, \forall i$*; and* $\sum\limits_{i=1}^{n} x_i^* \leq \sum\limits_{i=1}^{n} e_i^*$*.*

2) $p^* \in \Delta_+^l$ *is a price system such that for each agent* i*:*

2.1) $x_i^* \in B_i(p^*) = \{x \in X_i; p^* \cdot x \leq p^* \cdot e_i\}$*;*

2.2) $U_i(z) > U_i(x_i^*) \Rightarrow p^* \cdot z \leq p^* \cdot e_i$ *(*x *is* U_i*-maximal in* $B_i(p^*)$*).*

A competitive equilibrium is, by definition, a state in which every bundle, x_i', that an agent would prefer to x_i^* lies outside its budget set: $U_i(x_i') > U_i(x_i^*) \Leftrightarrow p \cdot x_i' > p \cdot e_i$. Such state must of course be feasible, that is, the sum of the quantities allocated cannot be higher than the sum of the initial endowments: $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$. It is also necessary that the cost of an agent's consumption bundle does not exceed the value of its initial endowments: $p \cdot x_i \leq p \cdot e_i$. This condition can be interpreted as excluding gains from speculation.

An assumption usually imposed to guarantee existence of competitive equilibrium is the "*hypothesis of survival*', which determines that for every price-system there is at least one bundle in the interior of the budget set: $\forall p, \exists x_p \in X_i : p \cdot x_p < p \cdot e_i$.

[2]Existence of equilibrium may be ruled out by indivisibities or rationings, which can prevent some agents from obtaining the bundle that they prefer at the prevailing market prices.

The proof of existence of competitive equilibrium is based on two results: Berge's theorem of the maximum and Kakutani's fixed point theorem.

The exchange economy is first characterized by two correspondences: one that assigns to given prices the utility-maximizing bundles of each agent; and another that assigns to given bundles the prices that maximize the difference between the value of the bundles and that of the initial endowments, that is, the value of the excess demand. These prices ensure that the allocations allowed by the agents' budgets are also feasible. By the theorem of the maximum, both correspondences are upper hemicontinuous, with non-empty compact values.

The product of these two correspondences, whose image is the product of the images of the two described correspondences, is a correspondence from the product space of prices and consumption possibilities into itself. The product correspondence retains the properties of upper hemicontinuity, and, given the quasi-concavity of the utility functions, has also convex values. In these conditions, the theorem of Kakutani ensures the existence of a fixed-point. The fixed point consists of an allocation and a price-system with the following properties: in this price-system, each agent's bundles is an utility-maximizer; and the prices are such that ensure that this allocation is feasible. Thus, the fixed point is a competitive equilibrium of the exchange economy.

3.4 The core of an exchange economy

Another known concept of equilibrium of an exchange economy is that of the *core*. An allocation is in the **core** if no *coalition* of agents can force a better outcome for themselves. If some group of agents can reach a better outcome, y, by trading only among them, we say that the coalition S **blocks** the allocation x via the feasible allocation y. By better we mean an outcome that is not worse for any member of the coalition and is better for at least one of them.

This concept of equilibrium is less restrictive than the competitive equilibrium. A competitive equilibrium is always in the core, while the converse is not true.

$$W(\mathcal{E}) \subseteq N(\mathcal{E})$$

Since an allocation in the core cannot be blocked by any individual coalition, the core satisfies the criteria of **individual rationality**. And, since the coalition of all agents does not block a core allocation, all the allocations in the core are **Pareto-optimal**.

$$W(\mathcal{E}) \ \subseteq \ N(\mathcal{E}) \ \subseteq \ IR(\mathcal{E}) \cap OP(\mathcal{E})$$

According to the old conjecture of Isidro Edgeworth (1881), in conditions of perfect competition, the core and the set of competitive equilibria coincide. In this context, perfect competition is modeled by considering an exchange economy with an infinite number of traders, so that the influence of each agent can be neglected.

This conjecture was proved by Debreu & Scarf (1963) for a market with an infinite number of traders, but with a finite number of types of traders. By replicating a finite economy they found that the core converges to the set of competitive equilibrium allocations.

Robert Aumann (1964) obtained a similar result for any infinite number of traders with different preferences and initial endowments. Instead of a finite number of types of traders, Aumann demanded neighboring preferences and endowments. With the further assumption of quasi-ordered preferences, Aumann (1966) also proved that the set of allocations satisfying the coinciding concepts of core and competitive equilibrium, in a market with an infinite number of traders, was not empty. A comprehensive study of economies with an infinite number of traders was provided by Hildenbrand (1970).

3.5 Exchange economies as pseudo-games

An exchange economy, $\mathcal{E} \equiv (X_i, U_i, e_i)_{i=1}^{n}$, can be modeled as a pseudo-game with $n+1$ players, $PG \equiv (X_i, F_i, V_i)_{i=1}^{n+1}$. The additional player is the *auctioneer*, that can also be designated as *market* or *price-setter*.

The space of possible strategies for the auctioneer is compact and convex:

$$X_{n+1} = \Delta_+^l = \{p \in \mathbb{R}_+^l : \sum p_j = 1\}.$$

This player is not restricted in its choice - F_{n+1} is a constant correspondence:

$$F_{n+1} : X \times \Delta_+^l \to \Delta_+^l$$

But the possible strategies of each agent are a function of the choice of the market. They must choose a bundle that belongs to their budget set. In this pseudo-game, the possible strategies of the agents are:

$$F_i(x,p) = B_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}.$$

The utility functions of the agents only change in terms of domain:

$V_i : Gr(F_i) \to \mathbb{R}$, with $V_i[(x,p); x_i'] = U_i(x_i')$.

The objective of the auctioneer is to maximize the cost of the excess demand:

$$V_{n+1} : Gr(F_{n+1}) = X \times \Delta \times \Delta \to \mathbb{R} \text{ , with } V_{n+1}[(x,p);q] = \sum_{i=1}^{n} q \cdot (x_i - e_i).$$

A Nash equilibrium of this pseudo-game is a competitive equilibrium of the original exchange economy.

Theorem 1 (EQUIVALENCE NASH-WALRAS)

The strategy (x^*, p^*) is a Nash equilibrium of the pseudo-game $PG \equiv (X_i, F_i, V_i)_{i=1}^{n+1}$ if and only if (x^*, p^*) is a competitive equilibrium of the exchange economy $\mathcal{E} \equiv (X_i, U_i, e_i)_{i=1}^{n}$

A corollary of this theorem is the existence of competitive equilibrium of the exchange economy $\mathcal{E} \equiv (X_i, U_i, e_i)_{i=1}^{n}$, since this pseudo-game is in the conditions of existence of Nash equilibrium.[3]

3.6 Existence of Nash equilibrium

Now we want to prove theorems of existence of Nash equilibrium and competitive equilibrium. To guarantee existence of a Nash Equilibrium, it is enough to assume a compact and convex space of strategies, and quasi-concave utility functions.

Theorem 2 (EXISTENCE OF NASH EQUILIBRIUM)

Consider a game defined in its normal form: $G \equiv (X_i, V_i)_{i=1}^{n}$. For every i, let X_i be compact and convex, and V_i be continuous and quasi-concave in the second variable.

$\Rightarrow$ *There exists a Nash Equilibrium.*

[3] Because $B_i(p)$ is a continuous and convex-valued correspondence.

Proof.

Consider the correspondence of the "best responses":

$$\psi_i(x) = \text{argmax } V_i(x) = \{z_i \in X_i; V_i(x, z_i) \geq V_i(x, x_i'), \forall x_i' \in X_i\}.$$

Since $V_i : X \times X_i \to \mathbb{R}$ is continuous, and the constant correspondence $F_i : X \to X_i$ is, of course, continuous and compact-valued, we can apply the Theorem of the Maximum. The correspondence of the "best responses" is non-empty and upper hemicontinuous. Furthermore, $\psi_i(x)$ is also closed because it is a u.h.c. correspondence with compact Hausdorff range.

We also need to show that $\psi_i(x)$ is convex. Since V_i is quasi-concave, for a given pair $z_1, z_2 \in \psi_i(x)$ and any $\lambda \in (0, 1)$:

$$V_i[x, \lambda z_1 + (1 - \lambda) z_2] \geq \min V_i(x, z_1), V_i(x, z_2) = V_M.$$

That is, $\psi_i(x)$ is convex. The product correspondence retains this property.

$$\psi : X \to X; \;\; \psi = \prod_{i=1}^{n} \psi_i; \;\; \psi(x) = \prod_{i=1}^{n} \psi_i(x).$$

The product correspondence retains also the properties of upper hemicontinuity (Aliprantis and Border, 1999, p. 537), and, by Tychonoff's Product Theorem (Aliprantis and Border, 1999, p. 52), of closedness.

The correspondence $\psi : X \to X$ is upper hemicontinuous and closed, with nonempty and convex values. Assuming that X is a convex and compact subset of a locally convex Hausdorff space (in particular, it may be a finite dimensional Euclidean space), we can apply the fixed point theorem of Kakutani.

This theorem establishes the existence of a fixed point of ψ. There exists a Nash equilibrium, x^*, composed by the best responses of each agent to the strategies of the others. QED

3.7 Existence of competitive equilibrium

First we prove existence of competitive equilibrium assuming a compact and convex space of possible bundles. Then we extend this result.

Theorem 3 *Let $\mathcal{E} \equiv (X_i, U_i, e_i)_{i=1}^n$ be such that, for every i:*

1) X_i is a compact and convex subset of $\mathbb{R}^l_+$;

2) U_i is continuous and quasi-concave;

3) for each $p \in \Delta^l_+$, there exists $x_p \in X_i$ such that $p \cdot x_p < p \cdot e_i$ ("hypothesis of survival").

$\Rightarrow$ *There exists a competitive equilibrium, (x^*, p^*).*

Proof.

For each i, define the utility functions, $V_i(p, x, x_i) = U_i(x_i)$. These functions are obviously continuous and quasi-concave as the U_i, but the domain is conveniently modified.

$V_i : \Delta^l_+ \times X \times X_i \rightarrow \mathbb{R}$.

The budget correspondence is defined by:

$B_i : \Delta^l_+ \times X \to X_i;$

$B_i(p, x) = \{x'_i \in X_i : p \cdot x'_i \leq p \cdot e_i\}.$

By 3), B_i is non-empty. To apply the Maximum Theorem, we also need the correspondence B_i to be continuous and compact-valued.

To see that B_i is upper hemicontinuous and compact-valued, consider its graph:

$Gr(B_i) = \{(p, x, x'_i) \in \Delta^l_+ \times X \times X_i : x'_i \in B_i(p)\}.$

Now consider an arbitrary sequence in the graph of B_i:

$\{(p, x, x'_i)\}^{\infty}_{n=1} : \forall n \in \mathbb{N}, (p_n, x_n, x'_{in}) \in Gr(B_i).$

We have: $p_n \cdot x'_{in} \leq p_n \cdot e_i$. With $\lim_{n\to\infty} p_n = p_\infty$:

$\lim_{n\to\infty}(p_n \cdot x'_{in}) \leq \lim_{n\to\infty}(p_n \cdot e_i) \Leftrightarrow p_\infty \cdot \lim_{n\to\infty} x'_{in} \leq p_\infty \cdot e_i.$

A point of the adherence is also a point of the graph. The graph of B_i is closed, and, therefore, closed-valued. It is also compact-valued, because $B_i(p)$ is a closed subset of the compact $X_i \in \mathbb{R}^l_+$. In these conditions, by the closed graph theorem (Aliprantis and Border, 1999, p. 529), B_i is upper hemicontinuous.

To apply the Maximum Theorem, all that is left to prove is that B_i is lower hemicontinuous. We will show that if some $x \in B_i(p)$ belongs to an open set V, then there is an open ball around p with radius δ such that for every $p' \in B(p, \delta) \bigcap \Delta$, there exists some $x' \in B_i(p')$ that also belongs to V.

By the *hypothesis of survival*, there exists $x_p \in X_i : p \cdot e_i - p \cdot x_p > 0$. Since X_i is convex, for some $\lambda_p \in (0, 1)$:

$\lambda_p \cdot x_p + (1-\lambda_p) \cdot x = x_{\lambda_p} \in X_i$. Of course that $p \cdot e_i - p \cdot x_{\lambda_p} = r > 0$.

Since $p' \in B(p, \delta)$, the minimum value of the initial endowments is:

$$p' \cdot e_i = p \cdot e_i - (p - p') \cdot e_i > p \cdot e_i - \delta \|e_i\|.$$

On the other hand, the maximum cost of x_{λ_p} is:

$$p' \cdot x_{\lambda_p} = p \cdot x_{\lambda_p} + (p' - p) \cdot x_{\lambda_p} < p \cdot x_{\lambda_p} + \delta \|x_{\lambda_p}\|.$$

As a result, we have:

$$p' \cdot e_i - p' \cdot x_{\lambda_p} > p \cdot e_i - \delta\|e_i\| - p \cdot x_{\lambda_p} - \delta \|x_{\lambda_p}\| > r - \delta(\|e_i\| + \|x_{\lambda_p}\|).$$

So it is enough to choose $\delta = \frac{r}{\|e_i\| + \|x_{\lambda_p}\|}$.

The particular case with $X_i = \mathbb{R}^l_+$ and $e_i >> 0$ satisfies 3). Since at least one of the commodities has a positive price $p_j > 0$, a bundle with $x_{pj} = e_i/2$ and $x_{pk} = e_i$ (with $k \neq j$) satisfies $p \cdot x_p < p \cdot e_i$.

The cost of excess demand is:

$$V_{n+1} : \Delta^l_+ \times X \times \Delta^l_+ \to \mathbb{R}; \quad \text{with } V_{n+1}(p, x, q) = \sum_{j=1}^{l} q_j \cdot (x - e_i).$$

The objective function of the auctioneer, V_{n+1}, is linear and, therefore, continuous.

Define also the constant correspondence:

$$B_{n+1} : \Delta^l_+ \times X \to \Delta^l_+ \; ; \text{ with } B_{n+1}(p, x) = \Delta^l_+.$$

We are in conditions of applying Berge's theorem to V_i and B_i for each i.

$$\psi_i : \Delta^l_+ \times X \to X_i \text{ , for } i = 1, ..., n;$$

$$\psi_{n+1} : \Delta^l_+ \times X \to \Delta^l_+.$$

All ψ_i, are u.h.c. with compact (closed subsets of the compact X) and convex (from quasi-concaveness) values.

The product correspondence: $\psi = \prod_{i=1}^{n+1} \psi_i$ retains these properties, satisfying the conditions of Kakutani's theorem.

$$\psi : \Delta^l_+ \times X \to \Delta^l_+ \times X.$$

Therefore, there exists $(p^*, x^*) \in \psi(p^*, x^*)$, which is a competitive equilibrium. In effect, with $p^* \in \Delta^l_+ \subseteq \mathbb{R}^l_+$, we have:

$$x_i^* \in \psi_i(p^*, x^*) \Rightarrow U_i(x_i^*) \geq U_i(z), \forall z \in B_i(p^*);$$

that is, $U_i(z) > U_i(x_i^*) \Rightarrow z \notin B_i(p^*)$.

The only thing that remains is to be confirmed is that x^* is feasible, i.e., that $\sum_{i=1}^{n} x_i^* \leq \sum_{i=1}^{n} e_i$. We know that the equilibrium prices maximize the cost of the excess demand:

$$p^* \in \psi_{n+1}(p^*, x^*) \Leftrightarrow, p^* \cdot \sum_{i=1}^{n}(x_i^* - e_i) \geq q \cdot \sum_{i=1}^{n}(x_i^* - e_i), \ \ \forall q \in \Delta^l_+.$$

Furthermore, we know that:

$$x_i^* \in B_i(p^*) \Leftrightarrow p^* \cdot (x_i^* - e_i) \geq 0 \Rightarrow$$

$$\Rightarrow 0 \geq p^* \cdot \sum_{i=1}^{n}(x_i^* - e_i) \geq e_j \cdot \sum_{i=1}^{n}(x_i^* - e_i) \equiv j\text{-coordinate of the sum.}^{4}$$

Each of the coordinates of $\sum_{i=1}^{n}(e_i - x_i^*)$ is not negative, that is, x^* is feasible. There exists a competitive equilibrium of the exchange economy, (x^*, p^*). QED

[4] $e_j = (0, ..., 1, ..., 0) \in \Delta^l_+$.

Now we extend this result to consumption sets which are not necessarily compact.

Theorem 4 (EXISTENCE OF COMPETITIVE EQUILIBRIUM)

Let $\mathcal{E} \equiv (X_i, U_i, e_i)_{i=1}^n$ be such that, for all i:

1) $X_i \subseteq \mathbb{R}_+^l$ is closed, convex and bounded from below.

2) U_i is continuous and quasi-concave.

3) for each $p \in \Delta$, there exists $x_p \in X_i$ s.t. $p \cdot x_p < p \cdot e_i$ (hypothesis of survival)

In particular, we have 3) if $e_i >> 0$ and $X_i = \mathbb{R}_+^l$.

$\Rightarrow$ *There exists a competitive equilibrium.*

Proof.

Since X_i is bounded from below, there exists $m \leq X_i$, $\forall i$.

With x^* being feasible, we have: $\displaystyle\sum_{i=1}^n x_i^* \leq \sum_{i=1}^n e_i$.

Then: $\overline{m} = n \cdot m \leq \displaystyle\sum_{i=1}^n x_i^* \leq \sum_{i=1}^n e_i \leq \overline{e}$, $\forall i$.

Let $R > 0$ be such that $\{x : \overline{m} \leq x \leq \overline{e}\} \subset B(0, R)$. By the previous theorem, there exists (x^*, p^*), a competitive equilibrium of $\overline{\mathcal{E}} \equiv (X_i \bigcap \overline{B}(0, R), U_i, e_i)_{i=1}^n$. We want to show that (x^*, p^*) is also a competitive equilibrium of $\mathcal{E}$.

The equilibrium allocation, x^*, belongs to the budget set:

$$x_i^* \in B_i(p^*) = \{x_i \in X_i \bigcap \overline{B}(0, 2R) : p^* \cdot x_i \leq p^* \cdot e_i\} \subset \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\}.$$

And maximizes utility in the restricted space:

$\forall z \in X_i \bigcap \overline{B}(0, 2R) : U_i(z) > U_i(x_i^*) \Rightarrow p^* \cdot z > p^* \cdot e_i$.

Let $z \in X_i$ be such that $U_i(z) > U_i(x_i^*)$ and $p^* \cdot z \leq p^* \cdot e_i$. The existence of such z would deny that (x^*, p^*) is an equilibrium in $\mathcal{E}$.

Consider a convex combination of z with the bundle that verifies the *hypothesis of survival*:

$z_\delta = \delta x_p + (1 - \delta) z$.

For all $\delta \in (0, 1)$, we know that $z_\delta \in X_i$, and that it is such that $p^* \cdot z_\delta < p^* \cdot e_i$.

The utility functions are continuous, so we can choose a small $\delta \in (0, 1)$ such that we also have $U_i(z_\delta) > U_i(x_i^*)$.

Now consider a convex combination of z_δ and x_i^*:

$z_\delta = \lambda x_i^* + (1 - \delta) z_\lambda$, with $\lambda \in (0, 1)$.

Since z_δ is in the interior of the budget set, z_λ also is, $\forall \lambda \in (0, 1)$:

$p^* \cdot z_\lambda < p^* \cdot e_i$.

The utility functions are quasi-concave, so $U_i(z_\lambda) \geq U_i(x_i^*)$.

Now consider a $\lambda \in (0, 1)$ such that $z_\lambda \in B(0, R)$. There exists a small $\epsilon >> 0$ such that $z' = z_\lambda + \epsilon$ is still in $B(0, R)$, and also in the budget set. Since preferences have the property of *no satiation*: $U_i(z') > Ui(z_\lambda)$.

This is a contradiction denying that (x^*, p^*) is an equilibrium in $\overline{\mathcal{E}}$. Therefore, such z does not exist, and (x^*, p^*) is also an equilibrium in $\mathcal{E}$. QED

Chapter 4

Equilibrium with Asymmetric Information

In an economic system, we may distinguish between *endogenous* and *exogenous* uncertainty. We deal exclusively with *exogenous uncertainty*. Only environmental variables are acceptable as contingencies to be included in the contracts. In this context, uncertainty can be seen as generated by an unobserved choice of *nature* between a set of possible states of nature. Our problem is of "*costless exchange at market clearing prices*".

When the relevant type of uncertainty is generated inside the economic system, that is, if it concerns the decisions of the agents, then the problem becomes one of "*market disequilibrium and price dynamics*".[1]

The economy extends over two time periods. In the first, there is uncertainty about the environment. Agents make contracts before (*ex ante stage*) and after they receive their

[1] For a review on the economics of uncertainty see Hirschleifer & Riley (1979).

information (*interim stage*). In the second period, contracts are enforced and consumption takes place.

4.1 Modeling information

By a **state of nature** is designated a complete specification (history) of the environmental variables from the beginning to the end of the economic system. An **event** is a set of states.

Agents have subjective beliefs about the probabilities of occurrence of the different states of nature. Each individual can assign to each state of nature a number between 0 and 1, with $\sum_{j=1}^{\Omega} q^j = 1$. **Subjective certainty** occurs when a probability of 100% is attributed to a single state. When the beliefs of the agent give strictly positive probabilities to at least two different states, we have **subjective uncertainty**.

The model is simpler when a finite number of possible states is assumed:

$\Omega = \{\omega^1, ..., \omega^\Omega\}$.[2]

Agents have a prior belief regarding the probability of occurrence of each state:

$q \subset \Delta_+^\Omega$, where $\Delta^\Omega = \{q \in \mathbb{R}_+^\Omega : \sum_{j=1}^{\Omega} q^j = 1\}$.

When an infinite set of states of nature is needed, we consider a compact and measurable space of states of nature: $(\Omega, \mathcal{F})$. In this case, the prior belief of an agent is represented by a probability measure on $(\Omega, \mathcal{F})$, with the density function denoted by $\mu(\cdot)$.

[2]Notice that here Ω denotes both the set of states and the number of states.

To illustrate the theory, we borrow an example from Laffont (1986). Assume that Ω has three elements: ω^1, ω^2 and ω^3. These states represent, respectively, *good*, *average*, and *bad* product quality.

The seller knows the actual quality of the product. With the *prior* beliefs written as $q = (q^1, q^2, q^3)$ the beliefs of the seller are:

$$\begin{cases} q = (1, 0, 0) & \text{, if the product is } \textit{good}; \\ q = (0, 1, 0) & \text{, if the product is } \textit{average}; \\ q = (0, 0, 1) & \text{, if the product is } \textit{bad}. \end{cases}$$

The buyer is uncertain about the product quality, having the following prior distribution: $q = (q^1, q^2, q^3)$, with $q^1 + q^2 + q^3 = 1$.

An **information structure without noise** consists of a σ-algebra on Ω, such that the agent knows whether the true state of nature belongs or not to each set of the σ-algebra. Dealing with finite Ω, we can also define information as a partition such that the agent cannot distinguish states of nature that belong to the same element of the partition.

After receiving its information, what the agent knows is which set of the partition includes the true state of nature. In the example above, the information structure of the seller is perfect. The seller knows the true state of nature: $P_S = \{\{\omega^1\}, \{\omega^2\}, \{\omega^3\}\}$. An expert that never makes a mistake, but who is unable to distinguish *good* from *average* quality, has the partition: $P_A = \{\{\omega^1, \omega^2\}, \{\omega^3\}\}$. Another expert that never makes a mistake, but who cannot distinguish *average* from *bad* quality has the partition: $P_B = \{\{\omega^1\}, \{\omega^2, \omega^3\}\}$.

4.2 Terminal acts and informational acts

While nature chooses among *states*, individuals choose among *acts*. Two classes of acts can be distinguished: *terminal* and *informational*. When making **terminal actions**, individuals make the best of their existing combination of information and ignorance to maximize their *utility*. With **informational actions**, individuals defer a final decision while waiting or actively seeking for new evidence which may reduce uncertainty.[3]

Upon receival of new information, agents adjust their *prior* beliefs. Higher *prior confidence* implies that *posterior* beliefs are more similar to the *prior*.[4] New information has less impact, so agents assign less value to their acquisition. A simple way to value new information is to equate it to the expected gain that results from revising the best action.

The idea of information emerging with time is a possible justification for the fact that real economic agents give value to flexibility and liquidity. The trade-off is between waiting and making an irreversible decision.

When thinking about informational acts, some keywords come to mind: dissemination, evaluation, espionage, monitoring, security, speculation, etc. These phenomena are very complex. In our study, we deal only with terminal acts, avoiding these more complex phenomena.

[3] The notion of *degree of confidence* is fundamental when dealing with informational acts. A higher *degree of confidence* implies that a lower value is assigned to the acquisition of new information.

[4] In many models, there are two trading periods: "prior" and "posterior" to receiving message. In complete market regimes, the price ratios are the same in both periods.

4.3 Arrow-Debreu equilibrium under uncertainty

Suppose that the state of nature becomes public information in the *interim stage*. As a result, agents cannot deceive each other about the state of nature. In this case, assuming the existence of complete markets for present and future contingent delivery, the model of Arrow-Debreu-McKenzie can be extended to a context of uncertainty.

The basic idea underlying this extension is to distinguish commodities not only by their physical characteristics, location, and dates of their availability, but also by the *state of nature* in which they are made available.

Existence of separate markets for each of these contingent commodities is assumed. An elementary contract in these markets consists of the purchase (or sale) of some specified number of units of a specified commodity to be delivered if and only if a specified *state of nature* occurs. Payment is made at the beginning.

Agents make a single choice, the choice of a consumption plan, which specifies consumption of each commodity in each *state of nature*.

Let X_i denote the set of feasible consumption plans for agent i, and let $x_i(\omega)$ denote the l-dimensional bundle consumed by agent i in state of nature ω. The function x_i maps the set of states of nature into $\mathbb{R}^l$, thus, consumption (and also initial endowments) can be written as a vector in $\mathbb{R}^{\Omega l}_{+}$.

The state-dependent utility function of agent i is a real-valued function on $\mathbb{R}^l$, and the expected utility of x_i is the expected value (with beliefs q_i) of $u_i(x_i, \omega)$:

$$U_i(x) = \sum_{j=1}^{\omega} q_i(\omega_j) u_i(x_i, \omega_j).$$

Besides their consumption possibility sets, preferences, and initial endowments, agents are also characterized by their *prior* beliefs, $q_i \in \Delta_+^\Omega$, about the probabilities of realization of the different states of nature.

Definition 7 (EXCHANGE ECONOMY WITH UNCERTAINTY)

An exchange economy with uncertainty, $\mathcal{E} \equiv (X_i, U_i, e_i, q_i)_{i=1}^n$*, is such that, for each agent* i*:*

- *the space of possible consumption bundles is* X_i*;*

- *the utility function is* $U_i : X_i \longrightarrow \mathbb{R}$*;*

- *the initial endowments are* $e_i \in X_i$*;*

- *the prior beliefs are* $q_i \in \Delta_+^\Omega$.

An equilibrium of the economy is a set of prices, and a set of consumption plans, such that: each consumer maximizes preferences inside the budget set; and, for each commodity in each state of nature, total demand equals total supply.

Agents are *price-takers*, so, there is no uncertainty about the value of the resource endowments, nor about the present cost of a consumption plan. This means that there is no uncertainty about a given agent's present net wealth.

Note that since a consumption plan may specify that, for a given commodity, quantity consumed is to vary according to the event that actually occurs, preferences reflect not only tastes, but also subjective beliefs about probabilities of different events and attitude towards risk (Savage, 1954).

All the assumptions that were necessary to prove existence of equilibrium are preserved. So, the existence theorem for exchange economies with perfect information still holds in economies with symmetric information.

This economy is formally equivalent to the exchange economy without uncertainty, so it is straightforward to establish:

(1) existence of equilibrium;

(2) Pareto-optimality of equilibrium;

(3) that every Pareto-optimum is an equilibrium relative to some price system and some distribution of resource endowments.

Theorem 5 (EXISTENCE OF COMPETITIVE EQUILIBRIUM)

Let $\mathcal{E} \equiv (X_i, q_i, u_i, e_i)_{i=1}^n$ *be such that, for all* i*:*

1) $X_i \subset \mathbb{R}_+^{\Omega l}$ *is closed, convex and bounded from below;*

2) the vector $q_i \in \Delta^\Omega$ *represents the subjective prior beliefs;*

3) the expected utility, $U_i = \sum_{\omega \in \Omega} q_i^\omega \; u_i^\omega(x_i)$*, is continuous and quasi-concave;*

4) for each $p \in \Delta\Omega l_+$*, there exists* $x_p \in X_i$ *s.t.* $p \cdot x_p < p \cdot e_i$ *(hypothesis of survival);*

In particular, we have 3) if $e_i >> 0$ *and* $X_i = \mathbb{R}_+^{\Omega l}$*.*

$\Rightarrow$ *There exists a competitive equilibrium.*

The model of Arrow-Debreu-McKenzie was easily extended to a context of uncertainty (with symmetric information). It was only necessary to expand the consumption space from a subset of $\mathbb{R}_+^l$ to one of $\mathbb{R}_+^{\Omega l}$, and to represent preferences by an *expected utility function*.[5]

[5] An analysis of the assumptions needed on the preferences of the agents for this representation to be possible is made in the Appendix.

4.4 Radner equilibrium under asymmetric information

Real economic agents have limited foresight. Some of them have more information, or better abilities to discern, than others. To take this into account, in general equilibrium theory, we talk about an *economy with asymmetric information*. If the information of the agents is *fixed* and purely *exogenous,* the extension of the model of Arrow-Debreu to this setting requires only a reinterpretation.

To say that the information of the agents is **fixed** means that it is independent of their actions. Introducing the possibility of acquisition of information is problematic, because this may be like a set-up cost which implies loss of convexity.

We still consider a finite number of possible states of the nature:

$\Omega = \{\omega^1, ..., \omega^\Omega\}$.

Each agent is endowed with a partition of information, $P = \{P^1, ..., P^T\}$, with $T \leq \Omega$, being unable to distinguish states of nature that are in the same set of the partition. What the agent knows is in which of the sets of the partition is included the actual state of nature. It is natural to represent the information of agent i by the σ-field, F_i, generated by the partition P_i.

The union of the sets P^j of a partition of information is equal to Ω, and any intersection of them is empty:

(1) $\bigcup_j P^j = \Omega$;

(2) $\forall j \neq k \; : \; \omega \in P^j \Rightarrow \omega \notin P^k$.

Let $e_i(\omega)$ denote agent i's endowment of commodities if state ω occurs. It is natural to assume the functions e_i and u_i to be *measurable* with respect to F_i.

Any action that the agent takes at that date must necessarily be the same for all elementary events in that set. This suggest that an agent should choose the same consumption in states of nature that she does not distinguish.

An agent would not want to go to the market to buy a commodity whose delivery is contingent upon the occurrence of an event that the agent cannot observe. To see this, suppose that the agent faces a seller that promises to deliver some bundle if a result of a toss of a coin is *heads*. If only the seller observes the coin toss, what should the agent expect? Well, the seller will say that the result was *tails*, and won't deliver anything.

This led Radner (1968) to restrict the consumption space of the agents. They are forced to make the same trades and consume the same bundles in states of nature that they do not distinguish. So it is also required that x_i be measurable with respect to F_i. This restriction is usually referred to as **informational feasibility**.

The concept of a *measurable function* provides a compact way of representing *allowable consumption* bundles. A commitment to deliver y units of commodity j if and only if event $E \subseteq \Omega$ occurs, can be regarded as a function defined on the space of states of nature, Ω, with value y in the set E, and zero elsewhere. Any sum of simple commitments that are allowable with respect to F_i would be a function defined on Ω, being constant on elements of the partition that generates F_i, the information σ-algebra of agent i. Such function has the property that, for any y and j, the set of elementary events in which the amount of commodity j that is delivered is y is a set in F_i. This is why we say that the function is measurable with respect to F_i.

Restricting consumption to be measurable with respect to information, we obtain a theory of existence and optimality of competitive equilibrium relative to a fixed structure of information.[6]

An allocation is feasible if each trader's consumption plan belongs to her consumption set and if total consumption does not exceed total endowments:

$$\forall \omega^j \in \Omega, \ \sum_{i=1}^{n} x_i^*(\omega^j) \leq \sum_{i=1}^{n} e_i(\omega^j).$$

This condition implies a kind of "*free disposal*". Observe that the amount to be disposed may not be measurable with respect to the information of any agent.

Each trader faces a single budget constraint:

$$\forall i, \quad x_i \in B_i(p) \ \Leftrightarrow \ p \cdot x_i \leq p \cdot e_i.$$

The model of Radner can now be seen as formally equivalent to the Arrow-Debreu model. A **Radner equilibrium** allocation maximizes the *expected utility* of the agents, is informationally feasible, and is physically feasible in all the states of nature.

Definition 8 (RADNER EQUILIBRIUM)

The pair (x^*, p^*) *is a Radner Equilibrium in the economy* $\mathcal{E} \equiv (X_i, P_i, q_i, u_i, e_i)_{i=1}^n$*, if and only if:*

1) $x^* = [x_1^*(\omega^1), ..., x_1^*(\omega^\Omega), x_2(\omega^1), ..., x_n(\omega^\Omega)]$ *is such that, for every agent i:*

1.1) x^* *is informationally feasible, that is,* $\omega^j, \omega^k \in P^m \Leftrightarrow x_i^*(\omega^j) = x_i^*(\omega^k)$*;*

1.2) x^* *is physically feasible, i.e.,* $\forall \omega^j \in \Omega, \sum_{i=1}^{n} x_i^*(\omega^j) \leq \sum_{i=1}^{n} e_i(\omega^j)$.

[6] For a complete presentation of this model, see Radner (1982).

2) $p^* = (p^*(\omega^1), ..., p^*(\omega^\Omega))$ is a non-zero, non-negative price system, such that, for every agent i:

$$x_i^* = arg \max_{B_i(p)} \{U_i(x_i^*)\} = arg \max_{B_i(p)} \{\sum_{j=1}^{\Omega} q_i^j u_i^j (x_i^*(\omega^j))\}.$$

With the *expected utility functions* being continuous and concave, we can apply Theorem 4 to establish **existence of Radner equilibrium** for quasi-concave, weakly monotone, and continuous expected utility functions. Such conditions are satisfied if the *state-dependent* utility functions are concave, weakly monotone, and continuous.

Theorem 6 (EXISTENCE OF RADNER EQUILIBRIUM)

Let $\mathcal{E} \equiv (X_i, q_i, u_i, e_i)_{i=1}^n$ be such that, for all i:

1) $X_i \subset \mathbb{R}_+^{\Omega l}$ is closed, convex and bounded from below.

2) the vector $q_i \in \Delta^\Omega$ represents the subjective prior beliefs.

3) the expected utility function, $U_i = \sum_{\omega \in \Omega} q^\omega \; u^\omega(x)$, is continuous, quasi-concave, and, for every feasible consumption plan, there is another, also feasible, that is strictly preferred;

4) for each $p \in \Delta$, there exists $x_p \in X_i$ s.t. $p \cdot x_p < p \cdot e_i$ (hypothesis of survival)

In particular, we have 4) if $e_i >> 0$ and $X_i = \mathbb{R}_+^{\Omega l}$.

$\Rightarrow$ There exists a Radner equilibrium.

4.5 Incomplete markets

To arrive at the notion of Arrow-Debreu equilibrium under uncertainty (Debreu, 1959, chapter 7), existence of *complete contingent markets* (CCM) was implicitly assumed. In this setting, agents can buy any contingent commodity. As a result, the welfare theorems hold.

When markets are not complete, the situation is more complicated. In general, there are some future dates and events for which it is not possible to contract for future contingent delivery. In this context, several concepts of equilibrium can be analyzed. To begin with, there are many possible patterns of market incompleteness.

One example is the *absence of prior-round markets*. Information arrives before any exchange takes place, preventing some risk sharing. An alternative is the consideration of *numeraire contingent markets* (NCM). Only one contingent commodity is available for each state. K. J. Arrow (1953) showed that equilibrium under CCM is achievable in this regime. Another possibility is to consider that it is only possible to trade in spot and future markets (FM).

Each of these possibilities has specific restrictions on the number of active markets. The CCM regime implies the existence of $C \times S$ markets, while NCM only demands $C + S$ markets, and FM demands $C + C$.

In a context of *emergent information* being inconclusive, *repeated rounds of trade* increase the effectiveness of FM relatively to CCM and NCM. A more sophisticated notion would be of a *reactive equilibrium*. If deviations are followed by reactions, then deviations may not occur in the first place. A set of offers is a **reactive equilibrium** if, for any additional offer that yields an expected gain to the agent making the offer, there is another that yields a gain to a second agent and losses to the first. Moreover, no further addition to or withdrawal from the set of offers generates losses to the second agent.

Suppose that the agents can use equilibrium prices to make inferences about the environment. An economic agent with a good understanding of the market is able to use market prices to make inferences about the (non-price) information of the other agents. These inferences are derived from the agent's model of the relationship between market prices and the non-price information received by the agents. Individuals successively revise their models and expectations. An equilibrium of this system, in which the individual models are identical with the true model, is called a “rational expectations equilibrium”.

The relation between equilibrium and informational acts is a complex one. We should keep in mind the thoughts of Schumpeter (1911): information generation is a disequilibrium creating process, while information dissemination is a disequilibrium repairing process.

Chapter 5

Topology of Common Information

5.1 Introduction

A classic problem in economic theory is that of the continuity of economic behavior with respect to variations in the characteristics of the agents. Economies with similar agents are expected to generate similar outcomes. In the Arrow-Debreu setting, where agents are characterized by preferences and initial endowments, Kannai (1970) and Hildenbrand & Mertens (1972) have, respectively, shown upper semicontinuity of the core and Walrasian equilibrium correspondences. In differential information economies, agents are also characterized by their private information, so similarity between agents also requires proximity of private information, evaluated by some topology on the information fields.

Information is modeled as a partition on the states of nature such that an agent distinguishes states of nature that belong to different sets of the partition. The question to answer is: How does an economy respond to small changes in the characteristics of the agents, including information? In differential information economies, this problem is not vacuous, since the Walrasian expectations equilibrium (also known as Radner equilibrium) set and the private core are not empty. Existence of W.E.E. in differential information economies was established by Radner (1968), while Yannelis (1991) proved the existence of the private core, and Glycopantis, Muir, & Yannelis (2001) gave it an extensive form interpretation.

To pursue this inquiry, a precise notion of proximity between information fields is needed. Boylan (1971) proposed a topology that is analogous to the Hausdorff metric on closed sets. Allen (1983) studied its properties and proved convergence of consumer demand and indirect utility with respect to this topology on information. Cotter (1986, 1987) introduced a weaker topology, based on the pointwise convergence metric, and showed that it retains the same continuity properties.

The metric of Boylan was used by Einy, Haimanko, Moreno, & Shitovitz (2005)[1] to establish upper semicontinuity of the W.E.E. correspondence. On the other hand, they present an example showing that the upper semicontinuity of the private core fails.[2] This unpleasant result suggests that small changes in information may have a big impact on the economy. But it may also be read as a sign of inadequacy of the topologies of Boylan and Cotter in the context of differential information economies.

A small perturbation in the information of an agent (in the sense of Boylan or Cotter) may render it incompatible with the information of the others. Thus, it can provoke a shift from a situation of no trade to one of full trade! These small perturbations that have significant impacts should actually be seen as big changes. Here is introduced a topology on information that accomplishes this. In the *topology of common information*, neighboring

[1]Referred to as Einy *et al.* in the rest of the paper.

[2]The example is also valid with the weaker topology of Cotter.

information fields are compatible, in the sense of allowing the agents to observe essentially the same events and, therefore, to make essentially the same contingent contracts.

This topology can be used to investigate the continuity properties of the private core and W.E.E. correspondences in differential information economies with a finite number of agents, where the private information of each agent is a finite partition of a compact and metrizable space of states of the world.[3] There are interesting positive results in the literature. Balder & Yannelis (2005) showed upper semicontinuity of the private core when the agents learn monotonically, and Einy et al. (2005) did this for the cases of convergence to the complete information economy and of convergence with decreasing information.

The *topology of common information* allows us to establish upper semicontinuity of the Walrasian expectations equilibrium and of the private core. Here this is done by recasting results of Einy et al. (2005). This is enough evidence of the intimate relation between convergence of equilibrium and convergence of information in the *topology of common information*.

The chapter is organized as follows: in section 2 the differential information economy is defined; in section 3 the *topology of common information* is introduced and characterized; in section 4, upper semicontinuity of the W.E.E. and private core correspondences is established; finally, in section 5 an example is presented as an illustration.

[3] As in the negative example of Einy et al. (2005) that excludes u.s.c. of the private core.

5.2 The Differential Information Economy

Our framework is the model of differential information economy with a finite number of agents. The economy extends over two time periods. In the first, agents make contracts that may be contingent on the state of nature that occurs in the second period (*ex-ante* contract arrangement). Consumption takes place in the second period. In every state of nature, the commodity space is the positive orthant of $\mathbb{R}^\ell$.

The exogenous uncertainty is described by the probability measure space $(\Omega, \mathcal{B}, \mu)$, where:

- Ω, compact and metrizable, denotes the possible states of nature;
- $\mathcal{B}$, a σ-algebra of subsets of Ω, denotes the set of all events;
- μ, a countably additive probability measure on $(\Omega, \mathcal{B})$, gives the (common) *prior* of every agent.

In the differential information economy, $\mathcal{E} \equiv (e^i, u^i, \mathcal{F}^i)_{i=1}^n$, for each agent i:

- A finite partition of Ω, P_i generates the σ-algebra $\mathcal{F}^i \subset \mathcal{B}$, the private information of agent i.
- $u^i : \Omega \times \mathbb{R}^\ell_+ \to \mathbb{R}_+$ is the random utility function of agent i. For all ω, the function $u^i(\omega, \cdot) : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ is continuous, strictly monotone and concave. For every x, $u^i(\cdot, x) : \Omega \to \mathbb{R}_+$ is continuous.
- $e^i : \Omega \to \mathbb{R}^\ell_+$, a function in L^l_p representing the random initial endowments of agent i, is $\mathcal{F}^i$-measurable and strictly positive: $e^i(\omega) \gg 0$ for all $\omega \in \Omega$.

Let L_{X^i} denote set of all $\mathcal{F}^i$-measurable functions in the random consumption set of agent i, that is: $L_{X^i} = \left\{ x^i \in L_p^l : \Omega \to \mathbb{R}_+^l, \text{ such that } x^i \text{ is } \mathcal{F}^i\text{-measurable} \right\}$.

The product of these sets, $L_X = \prod_{i=1}^{n} L_{X^i}$, is the space of allocations. With "free disposal", an allocation $x \in L_X$ is said to be feasible if:
$\sum_{i=1}^{n} x^i \leq \sum_{i=1}^{n} e^i$ for (μ-)almost every $\omega \in \Omega$.

The economic agents seek to maximize their *ex-ante* expected utility, given by:
$U^i(x) = \int_\Omega u^i \left(\omega, x^i(\omega)\right) d\mu.$

A coalition $S \subset N$ privately blocks an allocation $x \in L_X$ if there exists $(y^i)_{i \in S} \in \prod_{i \in S} L_{X^i}$ such that: $\sum_{i \in S} y^i \leq \sum_{i \in S} e^i$ and $U^i(y^i) > U^i(x^i)$ for every $i \in S$.

The private core of a differential information economy $\mathcal{E}$ is the set of all feasible allocations which are not privately blocked by any coalition. Although coalitions of agents are formed, information is not shared between them. The redistribution of the initial endowments is based only on each agent's private information.

A price system is a $\mathcal{B}$-measurable, non-zero function $\pi : \Omega \to \mathbb{R}_+^\ell$. Consider bundles in L_p^l, with $p \geq 1$, and, accordingly, restrict the price functions to the unit-sphere of L_q^l, with $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

For a price system π, the budget set of agent i is given by:
$B^i(\pi, e^i) = \left\{ x^i \in L_{X^i}, \text{ such that } \int_\Omega \pi(\omega) x^i(\omega) d\mu \leq \int_\Omega \pi(\omega) e^i(\omega) d\mu \right\}.$

A pair (π, x) is a Walrasian expectations equilibrium if π is a price system and $x = (x^1, \ldots, x^n) \in L_X$ is a feasible allocation such that, for every i, x^i maximizes U^i on $B^i(\pi, e^i)$.

5.3 The Topology of Common Information

The previous studies on the continuity of economic behavior with respect to information (Allen (1983) and Einy et al. (2005)) used the topology introduced by Boylan (1971). This topology is generated by a pseudometric d that assigns a finite distance to any pair of σ-algebras, x and y, contained in $\mathcal{B}$.

$$d(x, y) = sup_{A\in x}\, inf_{B\in y}\, \mu(A\Delta B) + sup_{B\in y}\, inf_{A\in x}\, \mu(A\Delta B).$$ [4]

In this model, the information of each agent is a σ-algebra generated by a finite partition of Ω such that the agent can tell in which of the sets of the partition lies the actual state of nature.

Let $\mathcal{X} = \{\ x \subset \mathcal{B}\ ;\ x$ is the σ-algebra generated by a finite partition of $\Omega\ \}$.

Although the possible information of the agents is restricted to this set $\mathcal{X}$, it is useful to include the σ-algebra of the total information, $\mathcal{B}$, in the topological space. Let $\overline{\mathcal{X}} = \mathcal{X} \cup \{\mathcal{B}\}$.

A stronger topology than Boylan's is constructed, having the particularity of taking into account the common information to establish similarity. Given two σ-algebras, $x, y \in \overline{\mathcal{X}}$, the σ-algebra that represents the "common information between x and y" is defined as:

$$x \wedge y = \{A \in x\ :\ \exists B \in y\ s.t.\ \mu(A\Delta B) = 0\}.$$ [5]

[4] $A\Delta B$ is the symmetric difference between sets A and B: $A\Delta B \equiv A\backslash B \cup B\backslash A$.

[5] To see that $x \wedge y$ is a σ-algebra, consider a countable family of sets $\{A_i\}$ that belong to $x \wedge y$. The difficulty lies in showing that the union of these sets also belongs to $x \wedge y$. For each A_i, there exists a $B_i \in y$

Observe that the σ-algebras $x \wedge y$ and $y \wedge x$ may be different. But also that they are equivalent in the sense of Boylan: $d(x \wedge y, y \wedge x) = 0$.

The topology is generated by a function, $d^* : \overline{\mathcal{X}} \times \overline{\mathcal{X}} \rightarrow \mathbb{R}^+$, defined as the sum of the Boylan distances from each information field to the common information.

Definition 9 $\forall x, y \in \overline{\mathcal{X}}$, $d^*(x, y) = d(x, x \wedge y) + d(x \wedge y, y)$, *where* $d(x, y)$ *is the Boylan distance between the information fields.*

This function is not a distance, but a related concept that can be designated as a *detachment*, since it satisfies the three following properties for all $x, y \in \overline{\mathcal{X}}$:

1. Positivity: $d^*(x, y) \geq 0$ and $d^*(x, x) = 0$;
2. Symmetry: $d^*(x, y) = d^*(y, x)$.[6]
3. Discrimination: $d^*(x, y) = 0$ implies that for every set in x there is a set in y that differs from it by at most a subset of Ω with μ-measure zero;

The *detachment* falls short of being a pseudometric because it violates the triangle inequality. It is not true that: $d^*(x, y) \leq d^*(x, z) + d^*(z, y) \;\; \forall x, y, z \in \overline{\mathcal{X}}$.

Observe that d^* defines an equivalence relation on $\overline{\mathcal{X}}$. Two σ-algebras $x, x' \in \overline{\mathcal{X}}$ are equivalent if and only if they have a null *detachment*:

such that $\mu(A_i \Delta B_i) = 0$. With some manipulation: $\mu(\bigcup A_i \Delta \bigcup B_i) = \mu(\bigcup A_i \backslash \bigcup B_i) + \mu(\bigcup B_i \backslash \bigcup A_i) \leq \sum_i \mu(A_i \backslash \bigcup B_j) + \sum_i \mu(B_i \backslash \bigcup A_j) \leq \sum_i \mu(A_i \backslash B_i) + \sum_i \mu(B_i \backslash A_i) = 0$. Since $\bigcup A_i \in x$ and $\bigcup B_i \in y$, we have $\bigcup A_i \in x \wedge y$.

[6] To see this, use $d(x \wedge y, y \wedge x) = 0$.

$x \sim x' \Leftrightarrow d^*(x, x') = 0.$

Let $\overline{\mathcal{Y}} = \overline{\mathcal{X}}/\sim$ denote the set of equivalence classes of $\overline{\mathcal{X}}$, that is, $\overline{\mathcal{Y}} = \{[x] : \; x \in \overline{\mathcal{X}}\}$, where $[x] = \{y \in \overline{\mathcal{X}} : \; d^*(x, y) = 0\}$. According to Proposition 1, the *detachment* and Boylan's pseudometric define the same equivalence classes.

Proposition 1 $\quad \forall x, y \in \overline{\mathcal{X}}\ ,\ d^*(x, y) = 0 \Leftrightarrow d(x, y) = 0.$

Proof. Since d is nonnegative and satisfies the triangle inequality, we have: $0 \leq d(x, y) \leq d(x, x \wedge y) + d(x \wedge y, y) = d^*(x, y)$. On the other hand, $d(x, y) = 0$ implies that for every set $A \in x$ there exists a set $B \in y$ such that $\mu(A \Delta B) = 0$. This also means that $x \wedge y = x$. So, $d^*(x, y) = d(x, x) + d(x, y) = 0 + 0.$ QED

Use "*open balls*", $B^*(x, \epsilon) = \{y \in \overline{\mathcal{X}} : d^*(x, y) < \epsilon\}$, to generate the topology. In the case of a metric, the triangle inequality ensures that the open balls generate a topology, that is, that the open sets are arbitrary unions of open balls. In this case, it has to be proved that the "*open balls*" produced by d^* also generate a topology.[7] This is done in three steps, each of them illustrative of the characteristics of the topology. Proposition 2 shows that in a small "*open ball*", all information partitions have more information than the center. And according to Proposition 4, in a small "*open ball*" all partitions have the same common information with a third partition. These two results allow to prove that a kind of local triangle inequality holds, implying that the "*open balls*" generated by d^* are open sets.

Now it is shown that all the information fields which are very close to a given finite σ-algebra x have more information than x.

[7] The concept of "*ball*" was generalized from distances to detachments, but, alternatively, a different designation can be used, like "*open zone*" with some "*reach*" around a center.

Proposition 2 $\forall x \in \mathcal{X}$, $\exists \delta(x) > 0$ *such that*
$d^*(x, y) < \delta(x) \Rightarrow x \wedge y = x$.

Proof. For the particular case of x having no information, $[x] = [\{\emptyset, \Omega\}]$, the proposition is trivial.
Now consider an arbitrary $x \in \mathcal{X}$ with some information. Since x is a finite σ-algebra, there exists a finite number of σ-algebras that are contained in x. Thus:
$$\min_{\substack{z \subset x \\ [z] \neq [x]}} d(x, z) = \delta(x) > 0.$$
By definition, $d^*(x, y) < \delta(x) \Rightarrow d(x, x \wedge y) < \delta(x)$.
Since $x \wedge y \subseteq x \Rightarrow [x \wedge y] = [x]$.
The way in which $x \wedge y$ is defined implies that $x \wedge y = x$. QED

The distance of Boylan and the *detachment* are locally equivalent in the following sense.

Proposition 3 $\forall x \in \overline{\mathcal{X}}$, $\exists \delta(x) > 0$ *such that*
$d^*(x, y) < \delta(x) \Rightarrow d^*(x, y) = d(x, y)$.

Proof. By definition, $d^*(x, y) = d(x, x \wedge y) + d(x \wedge y, y)$.
If x (or y) represents the total information, d and d^* are clearly equivalent: $d^*(x, y) = d(x, y) + d(y, y) = d(x, y)$.
With finite σ-algebras, in the small neighborhood as defined by Proposition 2, $x \wedge y = x$.
So, we have: $d^*(x, y) = 0 + d(x, y)$. QED

Observe that δ defines a d^*-ball where this equality holds, not a d-ball. It is not true that $\exists \delta(x) > 0$ such that $d(x, y) < \delta(x) \Rightarrow d^*(x, y) = d(x, y)$.

An important corollary is that convergence in the topology generated by d^* implies convergence in the topology of Boylan (defined by d). Note also that Proposition 1 is a particular case of Proposition 3.

According to the Proposition 4, given two finite information fields, if one of them varies slightly, the common information remains the same.

Proposition 4 $\forall x, y \in \mathcal{X}$, $\exists \delta(x,y) > 0$ *such that*
$d^*(y,z) < \delta(x,y) \Rightarrow x \wedge y = x \wedge z$.

Proof. Consider two arbitrary finite σ-algebras $x, y \in \mathcal{X}$. In the particular case of $[x] = [y]$, of course that $x \wedge y = x$. By Proposition 2, there exists a $\delta(y)$ such that $d^*(y,z) < \delta(y)$ implies that $y \wedge z = y$. It follows from the definition of common information that this operation is associative. So, we have $x \wedge y = x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z$.
In the general case of $[x] \neq [y]$, and because we deal with finite σ-algebras, there is only a finite number of subsets of Ω, $A_x \in x$ and $A_y \in y$. Thus:
$$\min_{\mu(A_x \Delta A_y) > 0} \mu(A_x \Delta A_y) = \epsilon.$$
Given x and y, consider $\delta(y)$ as in Proposition 2 and let $\delta(x,y) = \min\{\epsilon, \delta(y)\}$.
From Proposition 2, $d^*(y,z) < \delta(x,y) \Rightarrow y = y \wedge z$. Thus, $x \wedge y = x \wedge y \wedge z$.
Assuming that $x \wedge z \neq x \wedge y \wedge z$, then, by the way "common information" was defined, we are sure that $[x \wedge z] \neq [x \wedge y \wedge z]$. So there exist $A_z \in z$ and $A_x \in x$ with $\mu(A_z \Delta A_x) = 0$ such that there isn't any $A_y \in y$ with $\mu(A_z \Delta A_y) = 0$ or $\mu(A_x \Delta A_y) = 0$.
So, $\min_{A_y \in y} \mu(A_z \Delta A_y) = \min_{A_y \in y} \mu(A_x \Delta A_y) \geq \epsilon \geq \delta(x,y)$.
This implies that $d^*(y,z) \geq \delta(x,y)$.
This is a contradiction, so $x \wedge z = x \wedge y \wedge z = x \wedge y$. QED

Propositions 2 and 4 can be read together. By the first, in a small neighborhood of an information set, information does not decrease. According to the second, the possible increase of information has a bound, as the common information with another information set remains constant.

Theorem 1 is based on a kind of local triangle inequality which implies that all the points of an "*open ball*" are "*interior points*". A point x is "*interior*" to $A \Leftrightarrow \exists \epsilon > 0$ s.t. $B^*(x, \epsilon) \subset A$.

Use two kinds of "*open balls*" to define the topology:

(1) the open d^*-balls centered on finite σ-algebras with radius that are small enough for not including the σ-algebra of total information, $\mathcal{B}$;

(2) the open d^*-balls centered in $\mathcal{B}$.[8] These "*open balls*" constitute a base for the topology $\tau^* = \{A$: A is a union of open balls $\}$.

Theorem 7 *$\forall x \in \mathcal{X}$ and $0 < \epsilon < d^*(x, \mathcal{B})$: $B^*(x, \epsilon) = \{y \in \overline{\mathcal{X}} : d^*(x, y) < \epsilon\}$ are open d^*-balls (all points are interior). The d^*-balls defined by $B^*(\mathcal{B}, \epsilon) = \{y \in \overline{\mathcal{X}} : d^*(\mathcal{B}, y) < \epsilon\}$ are also open (all points are interior). This collection of open balls (consider only rational radius) is a base for τ^* and $(\overline{\mathcal{X}}, \tau^*)$ is a topological space.*

Proof. Given an arbitrary ball $B^*(x, \epsilon)$, we want to show that all points of this ball are interior points. Equivalently, that given $y \in B^*(x, \epsilon)$, there exists $\delta'(x, y) > 0$ such that $B^*(y, \delta'(x, y)) \subset B^*(x, \epsilon)$.
Consider first a finite center, $x \in \mathcal{X}$. With a $\delta(x, y)$ that is small enough for Proposition 4 to hold, and an arbitrary $z \in B^*(y, \delta(x, y))$:

[8]Considering only finite σ-algebras and using all the open d^*-balls we obtain a simpler topological space $(\mathcal{X}, \tau^*)$.

$$\begin{aligned} d^*(x,z) &= d(x, x\wedge z) + d(x\wedge z, z) = \text{ (by Proposition 4)}\\ &= d(x, x\wedge y) + d(x\wedge y, z) \leq \\ &\leq d(x, x\wedge y) + d(x\wedge y, y) + d(y,z) \leq \\ &\leq d^*(x,y) + d(y,z) \leq \\ &\leq d^*(x,y) + d^*(y,z). \end{aligned}$$

Let $\delta'(x,y) = \min\{\delta(x,y), \epsilon - d^*(x,y)\}$. For any $z \in B^*[y, \delta'(x,y)]$, we have:

$$d^*(x,z) \leq d^*(x,y) + d^*(y,z) \leq d^*(x,y) + \epsilon - d^*(x,y) = \epsilon.$$

Thus, the arbitrary y is an interior point. All points in the balls centered in finite σ-algebras (with small radius to prevent them from containing $\mathcal{B}$) are interior points.

With $x = \mathcal{B}$, and an arbitrary y in $B^*(x,\epsilon)$, let $d^*(x,y) = \alpha < \epsilon$, and let $\delta(B,y) = \epsilon - \alpha$. Given an arbitrary $z \in B^*(y, \delta(x,y))$:

$$\begin{aligned} d^*(\mathcal{B},z) &= d(\mathcal{B}, \mathcal{B}\wedge z) + d(\mathcal{B}\wedge z, z) = \\ &= d(\mathcal{B}, z) + d(z,z) = d(\mathcal{B}, z) \leq \\ &\leq d(\mathcal{B}, y) + d(y,z) = \alpha + d(y,z) < \\ &< \alpha + \epsilon - \alpha = \epsilon. \end{aligned}$$

Thus, all points in the balls that are considered are interior points.

The sets whose points are all interior are open sets (members of the topology), since they can be obtained by arbitrary unions of the members of the base. To see this, consider an arbitrary set, A, whose points are all interior.

$$A = int(A) \Rightarrow \forall x \in A\ , \exists B^*(x, \epsilon_x) \subset A \Rightarrow$$

$\Rightarrow \cup_{x\in A} B^*(x,\epsilon_x) \subset A \subset \cup_{x\in A} B^*(x,\epsilon_x) \Rightarrow A$ is a union of open balls.

Of course that all the points inside an open set are interior points. A point in a set that is an union of open sets is interior to at least one of the open sets, therefore it is also interior to the union.

For the topology to be well defined, a finite intersection of open sets A must be open. It is enough to prove that the intersection of two open sets is open. Consider an arbitrary point $a \in A = A_1 \cap A_2$. The point is interior to both open sets, so each of them contains a ball centered in a. Designate these balls by $B^*(a,r_1) \subset A_1$ and $B^*(a,r_2) \subset A_2$. Pick the smallest radius, w.l.o.g., r_1. Of course that $B^*(a,r_1) \subset A_1$ and $B^*(a,r_1) \subset A_2$. This open ball, $B^*(a,r_1)$, is contained in the intersection.

QED

The Boylan topology, defined by d, is a Hausdorff topology on the space of equivalence classes of information σ-algebras. The *topology of common information* is stronger, so it inherits this property.[9] Observe that the topology is first countable, as every point has a countable neighborhood base.[10] Thus, to prove upper semicontinuity of the equilibrium (or private core) correspondence, it suffices to show that given a convergent sequence of economies, the limit of a sequence of equilibrium (private core) allocations of the sequence of economies is an equilibrium (private core) allocation of the limit economy (see Theorem 16.20 of Aliprantis & Border (1999)).

[9] Given any two distinct points, there are Boylan neighborhoods of each point with null intersections (the Boylan and Cotter topologies are separated because every topology generated by a metric is separated). And for every Boylan neighborhood, there is a neighborhood in this topology that is contained in it, because: $\forall x \in \overline{\mathcal{X}}, \epsilon \in R^+ :\ B^*(x,\epsilon) \subset B(x,\epsilon)$. This implies that this topology is also separated.

[10] For every neighborhood of any x, there is an open ball with rational radius centered in x that is contained in the neighborhood.

The following example shows that this *detachment* does not generate a topology on the space of the infinite σ-algebras. Let the space of possible states of nature be $\Omega = [0,1]$ and all the states be "equally probable". Consider the simple information σ-algebra $x = \{\emptyset, [0, \frac{1}{2}],]\frac{1}{2}, 1], \Omega\}$. An infinite information σ-algebra y that is inside $B^*(x,\epsilon)$ is generated by the partition:

$y^p_\epsilon = \{[0, \frac{1}{2}],]\frac{1}{2} + \frac{\epsilon}{2^{n+1}}, \frac{1}{2} + \frac{\epsilon}{2^n}]_{n\in\mathbf{N}},]\frac{1}{2} + \frac{\epsilon}{2}, 1]\}$.[11]

Now construct a sequence of information fields that approaches y, but remains outside $B^*(x,\epsilon)$.

Let $z^p_n = \{[0, \frac{1}{2} + \frac{\epsilon}{2^{n+1}}],]\frac{1}{2} + \frac{\epsilon}{2^{m+1}}, \frac{1}{2} + \frac{\epsilon}{2^m}]_{m=n,\dots,1},]\frac{1}{2} + \frac{\epsilon}{2}, 1]\}$ be the partitions that generate the elements z_n of the sequence of information fields.

Observe that the difference between y and z_n is that $[0, \frac{1}{2} + \frac{\epsilon}{2^{n+1}}]$ is an elementary set in z_n, but appears subdivided in y. Since the set of y that is farther from $[0, \frac{1}{2} + \frac{\epsilon}{2^{n+1}}]$ is $[0, \frac{1}{2}]$, the Boylan distance between y and z_n is $\frac{\epsilon}{2^{n+1}}$.

So, we have: $d^*(y, z_n) = d(y, y \wedge z_n) + d(y \wedge z_n, z_n) = d(y, z_n) + d(z_n, z_n) = \epsilon/2^{n+1}$.

But, since x and z_n have no common information:[12]

$$\begin{aligned} d^*(x, z_n) &= d(x, x \wedge z_n) + d(x \wedge z_n, z_n) = d(x, 0) + d(0, z_n) = \\ &= \frac{1}{2} + \frac{1}{2} - \frac{\epsilon}{2^{n+1}} = 1 - \frac{\epsilon}{2^{n+1}} \; ! \end{aligned}$$

[11] Note that with some small ϵ, $d^*(x,y) = d(x, x \wedge y) + d(x \wedge y, y) = d(x,x) + d(x,y) = \sup_{A\in y} \inf_{B\in x} \mu(A \Delta B) = \epsilon/2$.

[12] With the σ-algebra of "no information" defined as $0 = \{\emptyset, \Omega\}$, to say that x and z_n have no common information means that $d(x \wedge z_n, 0) = 0$.

All the z_n are outside $B^*(x,\epsilon)$, while there isn't any open d^*-ball with center y that does not include any z_n. Therefore, there isn't any open d^*-ball centered in y contained in $B^*(x,\epsilon)$.

5.4 Upper Semicontinuity Results

In this section, a recasting of three upper semicontinuity results of Einy et al. (2005) is done. To illustrate the usefulness of the topology, a notion of convergence of economies is used, which differs from theirs only in what concerns convergence of private information fields. The use of the *topology of common information*, instead of the topology of Boylan, allows to establish the upper semicontinuity of the private core correspondence (Theorem 3).[13]

Definition 10 *Let $\{\mathcal{E}_k\}_{k=1}^{\infty} \equiv \{(e_k^i, u_k^i, F_k^i)_{i=1}^{n}\}_{k=1}^{\infty}$ be a sequence of economies with differential information that converges to $\mathcal{E}_0 \equiv (e_0^i, u_0^i, F_0^i)_{i=1}^{n}$. Precisely, convergence means that, for every agent $i \in N$:*

i) e_k^i converges to e_0^i in the L_1^{ℓ}-norm;

ii) u_k^i converges uniformly to u_0^i on every compact subset of $\Omega \times \mathbb{R}_+^{\ell}$;

iii) F_k^i converges to F_0^i in $(\overline{\mathcal{X}}, \tau^*)$.

[13]Allen (1983) showed continuity of consumer demand with respect to information using the topology of Boylan. Convergence in the *topology of common information* implies convergence in the Boylan topology: $x_n \in B(x_0,\epsilon) \Rightarrow x_n \in B^*(x_0,\epsilon)$. Therefore, the results obtained by Allen remain valid with this topology.

Note that what is needed is convergence of information fields for each agent in separate. The common information among the agents is not calculated. What is relevant is the common information between the information that an agent has in an economy of the sequence and the information that the same agent has in the limit economy.

Since convergence of information fields in the *topology of common information* implies convergence in the topology of Boylan, recasting Theorem 1 of Einy et al. (2005) establishes upper semicontinuity of the W.E.E. correspondence.

Theorem 8 *Let* $\{\mathcal{E}_k\}_{k=1}^{\infty} \equiv \{(e_k^i, u_k^i, F_k^i)_{i=1}^n\}_{k=1}^{\infty}$ *be a sequence of economies with differential information that converges to* $\mathcal{E}_0 \equiv (e_0^i, u_0^i, F_0^i)_{i=1}^n$.

Let $\{(x_k, \pi_k)\}_{k=1}^{\infty}$ *be a sequence such that* $(x_k, \pi_k) \in WEE(\mathcal{E}_k)$ *for every* k*, and for every* $i \in N$*:*

i) x_k^i *converges to* x_0^i *in the* L_p^ℓ*-norm;*

ii) π_k^i *converges to* π_0^i *in the* L_q^ℓ*-norm.*

Then $(x_0, \pi_0) \in WEE(\mathcal{E}_0)$.

From Proposition 2 we know that in a small neighborhood of a finite information field F_0^i, information fields are more rich,[14] in the sense that $F_0^i \wedge F_n^i = F_0^i$, that is, all sets in F_0^i have an equivalent set in F_n^i (that is, $\mu(A\Delta B) = 0$). This does not imply that $F_0^i \subset F_n^{i'}$. But, replacing all the sets in F_n^i by their equivalent in F_0^i, is obtained an information field,

[14]The limit economy differs of those in the sequence because markets for very unlikely contingencies may disappear.

$F_n^{i'}$, such that $F_n^{i'} \sim F_n^i$ and $F_0^i \subset F_n^{i'}$. Furthermore, changing the information fields from F_n^i to $F_n^{i'}$ has no impact on the solutions of the model. So, the sequence $F_n^{i'}$ can be used to apply Theorem 2 of Einy et al. (2005), establishing upper semicontinuity of the private core correspondence for all finite σ-algebras F_0^i.

Theorem 9 *Let* $\{\mathcal{E}_k\}_{k=1}^{\infty} \equiv \{(e_k^i, u_k^i, F_k^i)_{i=1}^n\}_{k=1}^{\infty}$ *be a sequence of economies with differential information that converges to* $\mathcal{E}_0 \equiv (e_0^i, u_0^i, F_0^i)_{i=1}^n$, *with* F_0^i *finite.*

If a convergent sequence of allocations in L_1^ℓ, $\{x_k\}_{k=1}^{\infty}$, *with* $\lim_{k\to\infty} x_k = x_0$, *is such that, for every* k, $x_k = (x_k^1, x_k^2, ..., x_k^n)$ *is a private core allocation in* $\mathcal{E}_k$, *then the limit of the sequence,* $x_0 = (x_0^1, x_0^2, ..., x_0^n)$, *is a private core allocation in* $\mathcal{E}_0$.

The private core correspondence also converges when the information fields of all the agents converge to the total information consisting of the σ-algebra of all Borel sets in Ω. From Proposition 3, convergence of information fields in the *topology of common information* implies convergence in the topology of Boylan. Thus, with $\mathcal{B}$ equal to the complete information field, Theorem 3 of Einy et al. (2005) establishes convergence of the private core.

Theorem 10 *Let* $\{\mathcal{E}_k\}_{k=1}^{\infty} \equiv \{(e_k^i, u_k^i, F_k^i)_{i=1}^n\}_{k=1}^{\infty}$ *be a sequence of economies with differential information that converges to* $\mathcal{E}_B \equiv (e_B^i, u_B^i, \mathcal{B})_{i=1}^n$, *with* $\mathcal{B}$ *defined as the* σ-*algebra of all Borel sets in* Ω.

If a convergent sequence of allocations in L_1^ℓ, $\{x_k\}_{k=1}^{\infty}$, *with* $\lim_{k\to\infty} x_k = x_0$, *is such that, for every* k, $x_k = (x_k^1, x_k^2, ..., x_k^n)$ *is a private core allocation in* $\mathcal{E}_k$, *then the limit of the sequence,* $x_B = (x_B^1, x_B^2, ..., x_B^n)$, *is a private core allocation in* $\mathcal{E}_B$.

A negative counterpart of this result is given by Krasa & Shafer (2001): if the complete information is approached by changing priors instead of expanding fields, upper semicontinuity fails.

Theorem 4 is also related to a result of Balder & Yannelis (2005) that establishes upper semicontinuity of the private core for sequences of economies with increasing information (learning), that is, when $F_k \subseteq F_{k+1}$ for every k. It may seem at first that $F_k \subseteq F_{k+1}$ together with $\bigvee_k F_k = F_\infty$ implies that $\lim\limits_{k\to\infty} d(F_k, F_\infty) = 0$. This would allow us to recast their result, because in the case of monotonic learning, convergence of information fields in the topology of Boylan is equivalent to convergence in our topology. But, in fact, monotone convergence in the sense of Balder and Yannelis does not imply convergence in the sense of Boylan, so the results are complementary.[15]

5.5 An Illustrative Example

In the introduction, was mentioned an example presented by Einy et al. (2005) which excluded the upper semicontinuity of the private core. This example is reproduced in this section to illustrate the problem of the continuity of the private core with respect to variations in information. The sequence of information fields considered in this example does not converge in the topology of common information.

[15]To see this, consider $\Omega = P_0 = [0,1]$ and $P_k = \{[0, \frac{1}{2^k}], ..., [\frac{j}{2^k}, \frac{j+1}{2^k}], ..., [\frac{2^k-1}{2^k}], 1\}$. Observe that the partition P_{k+1} is obtained by dividing each element of P_k in half. It may be shown that $d(F_{k+1}, F_k) = 1/2$ by selecting from F_{k+1} the set $A = \bigcup_{j=1,3,...,2^k-1}[\frac{j}{2^k}, \frac{j+1}{2^k}]$, since $\min\limits_{B\in F_k} \mu(A\Delta B) = 1/2$.

Consider a sequence of economies, $\mathcal{E}_\epsilon$, with two agents and one commodity, where only one of the private information fields varies. The space of possible states of nature is $\Omega = [0,1] \cup [2,3]$. The agents have equal initial endowments, independent of the state of nature: $e = \frac{1}{2}$. The private information of the agents are generated by the finite partitions:

$$\begin{cases} F^1_\epsilon = [0,1] \cup [2, 2+\epsilon],]2+\epsilon, 3]; \\ F^2_\epsilon = [0,1], [2,3]. \end{cases}$$

Agent 1 only values consumption in $[0,1]$, while agent 2 only values consumption in $[2,3]$. Their preferences are given by:

$$u^1(\omega, x) = \begin{cases} x & \text{, if } \omega \in [0,1]; \\ 0 & \text{, if } \omega \in [2,3]; \end{cases} \quad \text{and } u^2(\omega, x) = \begin{cases} 0 & \text{, if } \omega \in [0,1]; \\ x & \text{, if } \omega \in [2,3]. \end{cases}$$

The economies differ only in the parameter ϵ, which converges to zero. Private allocations are of the form:

$$x_\epsilon = \left(a^1_\epsilon \cdot \chi_{[0,1] \cup [2,2+\epsilon]} + a^2_\epsilon \cdot \chi_{]2+\epsilon,3]}, b^1_\epsilon \cdot \chi_{[0,1]} + b^2_\epsilon \cdot \chi_{[2,3]} \right).$$

Feasibility in $[0,1]$, $[2, 2+\epsilon]$, and in $]2+\epsilon, 3]$ implies that:

$$\begin{cases} a^1_\epsilon + b^1_\epsilon \leq 1; \\ a^1_\epsilon + b^2_\epsilon \leq 1; \\ a^2_\epsilon + b^2_\epsilon \leq 1. \end{cases}$$

Since x_ϵ is a core allocation, $a^1_\epsilon \geq \frac{1}{2}$, or else $U^1(e^1) > U^1(x^1_\epsilon)$. For the same reason, $b^2_\epsilon \geq \frac{1}{2}$. So, $a^1_\epsilon + b^2_\epsilon \leq 1$ implies that $a^1_\epsilon = b^2_\epsilon = \frac{1}{2}$, that $b^1_\epsilon \leq \frac{1}{2}$, and $a^2_\epsilon \leq \frac{1}{2}$.

Therefore, the initial endowments form a (constant) sequence of private core allocations, converging, of course, to $x = e = \frac{1}{2}$. In a limit economy, where $F^1 = F^2 =$

$\{[0,1],[2,3]\}$[16], the only private core allocation is $x_\epsilon = (\chi_{[0,1]}, \chi_{[2,3]})$, corresponding to a situation in which agent 1 consumes everything in $[0,1]$ and agent 2 consumes everything in $[2,3]$. Upper semicontinuity of the private core correspondence fails.

Observe that in the sequence of economies $\mathcal{E}_\epsilon$, even for a very small ϵ, the common information of the agents is null. So, agent 1 cannot trade worthless consumption in $[2,3]$ for consumption in $[0,1]$ (which agent 2 doesn't value). Their information fields are incompatible, in the sense that they do not allow contingent trade. In the limit economy, the agents have the same information, so they are able to make contingent trades. This is the source of the discontinuity.

According to Boylan's topology on information, the fields generated by the partitions $\{[0,1] \cup [2, 2+\epsilon],]2+\epsilon, 3]\}$ and $\{[0,1],[2,3]\}$ are neighbor. Nevertheless, these fields imply substantially different economic outcomes. The first has no information in common with agent 2's information field, so it does not allow contingent trade. It is as useless for agent 1 as would be the null information field $\{\emptyset, \Omega\}$. The second is compatible with the information of agent 2, that is, agents have common information, $\{\emptyset, [0,1], [2,3], \Omega\}$, based on which they are able to make contingent trades.

This means that a very small perturbation can lead to incompatibilities in the information of the agents, and have a big impact on the economic outcome. This motivates the introduction of a new topology that can grasp the compatibility of the information of the agents. According to the *topology of common information*, if the information fields become incompatible, the perturbation could not have been a small one. Compatibility of information fields is preserved under small perturbations.

[16]Note that the information partitions $F^1 = \{[0,1],[2,3]\}$ and $F^{1'} = \{[0,1] \cup \{2\},]2,3]\}$ are equivalent in the sense of Boylan.

With this new topology, the example does not show any failure of continuity, because the sequence of information fields $F^1_\epsilon = \{[0,1] \cup [2, 2+\epsilon],]2+\epsilon, 3]\}$ does not converge.[17] A sequence that would actually converge to $F^1_0 = \{[0,1], [2,3]\}$ in our topology is, for example, $F^{1'}_\epsilon = \{[0,1], [2, 2+\epsilon],]2+\epsilon, 3]\}$.[18] But, in this case, contingent trades would also be allowed in the sequence of economies, not just in the limit economy.

In the *topology of common information*, two information fields that are neighbor may differ only in events that are very unlikely. Notice that F^1_0 and $F^{1'}_\epsilon$ differ because while F^1_0 observes $[2,3]$, $F^{1'}_\epsilon$ can distinguish the unlikely event $[2, 2+\epsilon]$ from $]2+\epsilon, 3]$. Trades contingent on realization of $[2,3]$ are allowed. Only trades that are contingent on a very "unlikely" event, $[2, 2+\epsilon]$, are excluded.

Information fields that are very close in the topology of Boylan may also differ in an additional way, by distinguishing different but very correlated events. In fact, F^1_0 and F^1_ϵ differ because they allow the observation of very correlated events: $[0,1] \cup [2, 2+\epsilon]$ is similar to $[0,1]$; and $]2+\epsilon, 3]$ is similar to $[2,3]$. Nevertheless, the common information is null and so contingent agreements are not allowed.

The differences of the first kind only imply that agreements cannot be contingent on the very unlikely events that are not commonly observed, and therefore, have a small impact on economic outcomes. Differences of the second kind may prevent valuable agreements, contingent on events that are not commonly observed but nevertheless probable, and thus may imply very different economic outcomes. This second type of differences between

[17]Consider $\Omega = [0,2]$, and a strictly decreasing sequence $\{\epsilon_n\}$ that converges to zero. The limit of a sequence of information fields $F_n = \{\emptyset, [0, 1-\epsilon_n],]1-\epsilon_n, 2], \Omega\}$ must include all the sets $[0, 1-\epsilon_n]$ after some n (or sets that are Boylan-equivalent). Otherwise, there is no common information between F_n and F_0, and therefore $d^*(F_n, F_0) = 1 - \epsilon$. This means that the limit cannot be a finite information field.

[18]Note that $F^1_0 \wedge F^{1'}_\epsilon = F^1_0$, so $d^*(F^1_0, F^{1'}_\epsilon) = d(F^1_0, F^{1'}_\epsilon) = \epsilon$.

information fields that are neighbor is allowed by the topology of Boylan but not by the topology of common information.

Chapter 6

Economies with Uncertain Delivery

6.1 Introduction

Uncertainty and private information are crucial in modern economies. Agents know that their decisions can lead to different outcomes, depending on the decisions of others, and on the state of the environment. The complexity associated with these issues is such that it cannot be completely captured by any simple model. A realistic goal is to find simple models that give enlightening, although partial, descriptions.

In general equilibrium theory, several proposals have been made regarding the introduction of private information. A first one was made by Radner (1968), who restricted agents to consume the same in states of nature that they did not distinguish. With this condition, the model of K. J. Arrow & Debreu (1954) could be reinterpreted in a way that took into account each agent's private information.

After this first solution, another concept came to dominate the literature: the *rational expectations equilibrium* (Muth, 1961). But to assume that agents have rational expectations and take prices as fixed can be problematic. If, by observing prices, an agent can infer all the information of the others, then it is useless to have more information *ex ante*. Agents do not care about producing and gathering information, therefore, insights on these economic processes do not arise. Furthermore, this kind of inference also seems to require agents to have incredible knowledge and cognitive abilities.

Another alternative approach was taken by Prescott and Townsend (1984a, 1984b), who restricted trade contracts to be incentive compatible. But how can the incentive compatibility of the contracts be guaranteed? Again, agents would have to know everything about the whole economy in order to evaluate whether the contracts are incentive compatible or not.

The objective of this work is not to provide an equilibrium concept that is "better" than these in all instances. The goal is to present an equilibrium concept that fits a situation in which agents know only their characteristics (endowments and preferences in each state of nature) and the prevailing prices. The economy is not assumed to be common information. Agents do not know the endowments, preferences and private information of the others, and aren't able to figure them out.

The notion that is propose is a *prudent expectations equilibrium*. In this model, agents are allowed to make *contracts for uncertain delivery*, that is, contracts that may give them different bundles in states of nature that they do not distinguish *ex ante*. Agents buy the right to receive one of these different bundles, and expect to receive the worst of the possibilities contracted. This leads them to select bundles with the same utility for consumption in states that they do not distinguish. So, agents actually end up receiving the worst possibility, which is as good as any of the others.

In sum, economies with incomplete private information are modeled, in which agents are assumed to follow a simple rule-of-thumb: to be *prudent*. The model of Arrow-Debreu can be reinterpreted to cover this situation, therefore, many classical results still hold: existence of core and equilibrium, core convergence, continuity properties, etc.

In a *prudent expectations equilibrium*, agents obtain the same utility in states of nature that they do not distinguish, instead of equal consumption. This is a weaker restriction, therefore, efficiency of trade and welfare are improved.

The chapter is organized as follows: in section 2, contracts for uncertain delivery are defined, section 3 includes examples that motivate the model; and, in section 4, an interpretation of the economy with uncertain delivery is given.

6.2 Contracts for uncertain delivery

The theory of general equilibrium under uncertainty has developed upon the formulation of objects of choice as contingent consumption claims (Arrow, 1953). Under this formulation, besides being defined by their physical properties and their location in space and time, commodities can also be defined by the *state of nature* in which they are made available. For example, a “*bicycle*” in “*rainy weather*” and a “*bicycle*” in “*sunny weather*” are seen as two different commodities. This incorporation of uncertainty in the commodity space allows an interpretation of the Walrasian model that covers the case of uncertainty.

The Arrow-Debreu (1954) economy extends over two time periods. In the first period, agents know their preferences and endowments, which depend on the state of nature. In this *ex ante* stage, agents trade state-dependent endowments for state-dependent consumption.

In the second period, the state of nature becomes public information, trade is realized, and consumption takes place.

Now suppose that the state of nature does not become public information. In this case, agents have to be careful when trading contingent goods. Consider a seller that offers the following game:

"*I will toss a coin. If the result is heads, you receive a bicycle; if it is tails, you don't receive anything.*"

How much would an agent pay for this contingent good, which can be described as a "*bicycle*" if the state of nature is "heads"? If it is common information that the agent does not observe the coin toss, this contingent good has no value. The seller is able to avoid delivery. This suggests that agents are only willing to pay for goods which are contingent upon events that they can observe.

This restriction allowed Radner (1968) to extend the model of Arrow and Debreu to the case of private information. Agents are constrained to consume the same in states of nature that they do not distinguish. That is, consumption is measurable with respect to the private information of each agent. This restriction trivially implies incentive compatibility. Whatever the state of nature that occurs, agents are always sure about the bundle that will be delivered to them, so they can never be deceived. On the other hand, incentive compatibility does not imply measurability, so this restriction may be seen as too strong.[1]

Relaxing this restriction could allow agents to achieve better outcomes, in the sense of Pareto. But does it make sense to buy the right to receive different bundles in states of

[1]For a thorough analysis of the problem of incentive compatibility in exchange economies with private information, see Forges, Minelli, & Vohra (2002).

nature that the agent does not distinguish? Suppose now that the seller offers a different game:

"*I will toss a coin. If the result is heads, you receive a blue bicycle; if it is tails, you receive a red bicycle.*"

Even if it is common information that the agent does not observe the coin toss, this is a valuable *uncertain* contingent good, because the delivery of a "*bicycle*" is guaranteed. An agent is probably willing to pay for the right to receive a "*blue bicycle or red bicycle*".

Here the notion of objects of choice as uncertain consumption bundles is proposed, and these uncertain bundles are designated as "*lists*". If the specified contingency occurs, a contingent list gives an agent the right to receive one of the bundles in the list. Agents are now allowed to sign "*contracts for uncertain delivery*", which specify a list of bundles out of which a single one will be selected for delivery. These contracts can be contingent, so, in general, agents buy the right to receive one of the bundles in the list if the specified contingency occurs. The selection of the bundle that is delivered is made by the seller, but the buyer is certain about receiving one of the bundles in the list.[2]

Agents are able to sign more general contracts, so allowing contracts for uncertain delivery may be seen as opening additional markets.[3] A supplier may not be able to guarantee the delivery of neither a "blue bicycle" nor a "red bicycle", while being able to ensure the delivery of one of the two. In the Radner model, there would be *no trade*, but *contracts for uncertain delivery* allow trade to take place.

[2]Contracts commonly known as "options" are covered by this definition.

[3]Obviously, contracts for contingent delivery (Arrow, 1953) can be seen as "*contracts for uncertain delivery*" with lists of only one element.

6.3 Examples

Two situations will be presented in which there aren't any commonly observed events. As a consequence, if agents are constrained to consume the same in states that they do not distinguish, there will be *no trade* in equilibrium. Allowing agents to sign *contracts for uncertain delivery* leads to welfare improvements in the sense of Pareto. In these two examples, agents actually reach the full information (first-best) outcome.

Example 1: Perfect substitutes

This economy has two agents and four commodities: "ham sandwiches", "cheese sandwiches", "orange juices" and "apple juices".

Both agents need to eat and drink. Sandwiches are perfect substitutes, as well as the juices. Agents want to maximize expected utility, having the same preferences in every state, described by a Cobb-Douglas utility function:

$$u = (s_h + s_c)^{0.5} \cdot (j_o + j_a)^{0.5}.$$

There are four possible states of nature, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

- In ω_1, agent A is endowed with two "ham sandwiches" and agent B with two "orange juices": $e_A(\omega_1) = (2, 0, 0, 0)$ and $e_B(\omega_1) = (0, 0, 2, 0)$;

- In ω_2, agent A is endowed with two "ham sandwiches" and agent B with two "apple juices": $e_A(\omega_2) = (2, 0, 0, 0)$ and $e_B(\omega_2) = (0, 0, 0, 2)$;

- In ω_3, agent A is endowed with two "cheese sandwiches" and agent B with two "orange juices": $e_A(\omega_3) = (0, 2, 0, 0)$ and $e_B(\omega_3) = (0, 0, 2, 0)$;

- In ω_4, agent A is endowed with two "cheese sandwiches" and agent B with two "apple juices": $e_A(\omega_4) = (0, 2, 0, 0)$ and $e_B(\omega_4) = (0, 0, 0, 2)$.

Each agent observes only its endowments. Their information partitions are:

$P_A = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $P_B = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$.

Agents want to guarantee that they will eat and drink in the future. The problem is that they are unable to buy any specific good for future delivery contingent upon events (sets of states) that they observe.

For example, agent A wants to buy orange juice. For consumption to be the same across undistinguished states, the delivery of orange juice must be contingent upon events that A can observe. The possibilities are: (i) delivery in all states, (ii) delivery in $\{\omega_1, \omega_2\}$, (iii) delivery in $\{\omega_3, \omega_4\}$.

None of these possibilities is feasible, because agent B only has orange juice in the states ω_1 and ω_3. The same reasoning applies to each of the other commodities, so there is no trade in this economy. From another angle, suppose that agent A consumed some quantity of "orange juice" in ω_1. The same consumption would have to take place in ω_2, but in ω_2 there isn't any "orange juice" in the economy.[4]

There is *no trade* if equal consumption in undistinguished states is imposed. Nevertheless, contracts for uncertain delivery allow agents to guarantee future consumption of a sandwich and a juice.

[4] We can assume strictly positive endowments, substituting every zero for a small ϵ, and reach the same conclusions.

An agent can buy a sandwich (or a juice), as an uncertain bundle with two possibilities. The agents trade a "ham sandwich or cheese sandwich" for an "orange juice or apple juice". Since agent A is able to ensure the delivery of a sandwich and agent B is able to ensure the delivery of a juice, contracts for uncertain delivery allow them to attain the optimal outcome, which is:

$$x_A = x_B = \begin{cases} (1,0,1,0) \text{ in } \omega_1, \\ (1,0,0,1) \text{ in } \omega_2, \\ (0,1,1,0) \text{ in } \omega_3, \\ (0,1,0,1) \text{ in } \omega_4. \end{cases}$$

Both agents obtain an utility that is equal to 1 in all states of nature. This constitutes an improvement in the sense of Pareto relatively to the Walrasian expectations equilibrium solution, which resulted in an utility of zero to both agents.[5]

In states of nature that an agent does not distinguish, the consumption vectors are different, but note that the correspondent utility is always the same.

Example 2 - Risk sharing

Consider now an economy with two agents and two commodities. There are three possible states of nature: ω_1, ω_2 and ω_3. The state ω_2 has a probability of 0,2%, while ω_1 and ω_3 have a probability of 49,9%.

The initial endowments depend on the state of nature:

$$e_A = \begin{cases} (199,100) \text{ in } \omega_1, \\ (1,100) \text{ in } \{\omega_2,\omega_3\}. \end{cases} \qquad e_B = \begin{cases} (1,100) \text{ in } \{\omega_1,\omega_2\}, \\ (199,100) \text{ in } \omega_3. \end{cases}$$

[5]Together with the price vector $p = \frac{1}{24}[(1,2,1,2);(1,2,2,1);(2,1,1,2);(2,1,2,1)]$, this allocation is an equilibrium of the economy with uncertain delivery.

Again, agents observe only their endowments, and there isn't any event that is observed by both agents:

$P_A = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ and $P_B = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$.

Agents want to maximize expected utility, having the same preferences in all states of nature. The marginal utility of good 1 is diminishing, while that of good 2 is constant:

$$u_A(x_1, x_2) = u_B(x_1, x_2) = 10\sqrt{x_1} + x_2.$$

Observe that the game is symmetric. Agent A wants to sell good 1 in ω_1 and to buy in $\{\omega_2, \omega_3\}$. Agent B wants good 1 in $\{\omega_1, \omega_2\}$ and to sell it in ω_3.

The total resources in the economy are:

$$e_{total} = \begin{cases} (200, 200) \text{ in } \omega_1, \\ (2, 200) \text{ in } \omega_2, \\ (200, 200) \text{ in } \omega_3. \end{cases}$$

In the least probable state, ω_2, physical feasibility implies that $x_1^A + x_1^B = 2$. This restriction is crucial.

In a symmetric solution, $x_1^A(\omega_2) = x_1^B(\omega_2) = 1$. Measurability implies that $x_1^A(\omega_3) = 1$ and $x_1^B(\omega_1) = 1$. Agents retain their endowments, and there is no trade. The resulting expected utilities are:

$$U_1 = U_2 = 0.499 \cdot (10\sqrt{199} + 100) + 0.501 \cdot 110 = 175.$$

Without symmetry, we would have (w.l.o.g.):

$$\begin{cases} x_1^A(\omega_2) = x_1^A(\omega_3) = 1 + e, \\ x_1^B(\omega_2) = x_1^B(\omega_1) = 1 - e. \end{cases}$$

Physical feasibility implies that:

$$\begin{cases} x_1^A(\omega_1) \leq 200 - x_1^B(\omega_1) \leq 199 + e, \\ x_1^B(\omega_3) \leq 200 - x_1^A(\omega_3) \leq 199 - e. \end{cases}$$

The only measurable and efficient allocations are of the form:

$$\begin{cases} x^A(\omega_1) = (199 + e, 100 - p), \\ x^A(\omega_2) = (1 + e, 100 - p), \\ x^A(\omega_3) = (1 + e, 100 - p). \end{cases} ; \quad \begin{cases} x^B(\omega_1) = (1 - e, 100 + p), \\ x^B(\omega_2) = (1 - e, 100 + p), \\ x^B(\omega_3) = (199 - e, 100 + p). \end{cases}$$

Trade is constant across states of nature. To receive an additional quantity, e, of good 1, agent A pays p units of good 2. Then:

$$U_A = 0.499 \cdot (10 \cdot \sqrt{199 + e} + 100 - p) + 0.501 \cdot (10 \cdot \sqrt{1 + e} + 100 - p) =$$

$$= 4.99 \cdot \sqrt{199 + e} + 5.01 \cdot \sqrt{1 + e} + 100 - p.$$

$$U_B = 0.499 \cdot (10 \cdot \sqrt{199 - e}) + 0.501 \cdot (10 \cdot \sqrt{1 - e}) + 100 + p =$$

$$= 4.99 \cdot \sqrt{199 - e} + 5.01 \cdot \sqrt{1 - e} + 100 + p.$$

$$U_A + U_B = 4.99 \cdot (\sqrt{199 + e} + \sqrt{199 - e}) + 5.01 \cdot (\sqrt{1 + e} + \sqrt{1 - e}) + 200.$$

$$\frac{d(U_A + U_B)}{de} = 4.99 \cdot [\frac{1/2}{\sqrt{199 + e}} - \frac{1/2}{\sqrt{199 - e}}] + 5.01 \cdot [\frac{1/2}{\sqrt{1 + e}} - \frac{1/2}{\sqrt{1 - e}}] < 0$$

It is not possible to increase the sum of the utilities, therefore Pareto improvements relatively to the initial endowments are not possible. The only "core" allocation corresponds to the initial endowments, so there is "*no trade*" in this economy.

Can the agents improve this situation? Let's restrict the contracts that agents can make to those that give them the same utility in states that they do not distinguish.

In a symmetric allocation, each agent gets $(1, 100)$ in ω_2. The correspondent utilities are $u_A = u_B = 110$. An allocation with measurable utility for agent A must have the same utility in ω_3:

$$10\sqrt{x_1^A(\omega_3)} + x_2^A(\omega_3) = 110 \;\Rightarrow\; x_2^A(\omega_3) = 110 - 10\sqrt{x_1^A(\omega_3)}.$$

Thus, $x^A(\omega_3)$ must be of the form:

$$x^A(\omega_3) = (X, 110 - 10\sqrt{X}).$$

Without waste of resources, we have $x^B(\omega_3) = (200 - X, 90 + 10\sqrt{X})$. By symmetry: $x^B(\omega_1) = (X, 110 - 10\sqrt{X})$ and $x^A(\omega_1) = (200 - X, 90 + 10\sqrt{X})$.

The utility of agent A in $\{\omega_2, \omega_3\}$ has to be equal to 110. To arrive at an optimal solution, it is enough to maximize utility in ω_1:

$$U = 10\sqrt{200 - X} + 90 + 10\sqrt{X} \;\Rightarrow\; U' = -5 \cdot (200 - X)^{-1/2} + 5 \cdot X^{-1/2}.$$

$$U' = 0 \;\Rightarrow\; (200 - X)^{-1/2} = X^{-1/2} \;\Rightarrow\; X = 100.$$

This gives the following state-contingent and expected utilities:

$$u_A(\omega_1) = 10 \cdot \sqrt{100} + 90 + 10\sqrt{100} = 290.$$

$$U_A = U_B = 0.499 \cdot 290 + 0.501 \cdot 110 = 200.$$

The symmetric optimal solution is:

$$x^A = \begin{cases} (100, 190) \text{ in } \omega_1, \\ (1, 100) \text{ in } \omega_2, \\ (100, 10), \omega_3. \end{cases} \qquad u^A = \begin{cases} 290 \text{ in } \omega_1, \\ 110 \text{ in } \{\omega_2, \omega_3\}. \end{cases}$$

$$x^B = \begin{cases} (100, 10) \text{ in } \omega_1, \\ (1, 100) \text{ in } \omega_2, \\ (100, 190), \omega_3. \end{cases} \qquad u^B = \begin{cases} 110 \text{ in } \{\omega_1, \omega_2\}, \\ 290 \text{ in } \omega_3. \end{cases}$$

They can obtain this allocation by signing a contract under which, in every state of nature, each agent would deliver to the other one of two bundles: $(99, -90)$ or $(0, 0)$. It is straightforward to see that agents would deliver $(99, -90)$ if their endowments are $(199, 100)$, ending up with (100,190) in that state of nature.

This solution can also be achieved as a competitive equilibrium with the prevailing price vector $p = [(1, 2); \frac{2}{499}(10, 2); (1, 2)]$, leading agents to select the non-measurable bundles x^A and x^B.

The resulting expected utility is close to 200, higher than the 175 which correspond to the classical solution. Again, the introduction of contracts for uncertain delivery allowed a Pareto improvement in the exchange economy.

6.4 Economies with Uncertain Delivery

To clarify the modification that is proposed to the model of the economy, first an economy that leads to the model of Radner (1968) is described. This economy has separate markets for each commodity, and extends over two periods. In the first period, agents make contingent trade agreements in each market. In the second period, agents receive their information, trade is realized, and consumption takes place.[6]

In this interpretation of the model, a very demanding notion of information is used. To have information about an event is actually to be able to prove in a court of law that the event occurred.[7] For the goods to be delivered in the second period, the agent has to be able to prove that the specified contingency occurred. This has the obvious implication that, in this economy, contracts for contingent delivery are enforceable.

Agents are not able to buy a “blue bicycle or red bicycle”, because the markets are separated. To see this, suppose that an agent buys a “blue bicycle” if the “coin toss result is heads” and buys a “red bicycle” if the “coin toss result is tails”. The agent ought to receive a “bicycle”, in any case. But notice that the agent cannot neither prove that the state is “heads”, nor that it is “tails”. So, the sellers in both markets evade the law, and the agent gets nothing. In this economy, contracts for uncertain delivery are not enforceable.

As a result, agents only will demand goods contingent on events included in their private information. This resembles what occurs in the model of Radner (1968).

The modification that is introduced is to lump markets together, so that there is only one representative of the market dealing with the agent. Thus, an agent can take the “market” to a court of law, in order to receive a “blue bicycle or red bicycle”.

[6]Information is received in the second period, but in the first period the agents already know which events they are able to observe.

[7]Another way to see private information is as allowing agents to “distinguish” between states of nature. So this would be a very demanding notion of “distinguishing” between two states: to be able to prove in a court of law that the state cannot be one of the two.

Instead of expanding the market structure to have different markets for each *list*, the structure of *complete contingent markets* is kept. But now it is as if the obligation of delivering the contingent goods rested in the market as a whole.

Another natural way to conceive the Radner economy, and the modified one, is to consider a much weaker notion of "distinguishing", based on awareness. An agent that does not distinguish between ω_1 and ω_2 is in fact not aware that these are two different states. In this case, the agent does not participating in the complete markets for contingent delivery. Instead of observing prices in ω_1 and ω_2, the agent only observes prices for delivery in the event $\{\omega_1, \omega_2\}$, which are equal to $p(\omega_1) + p(\omega_2)$. Consequently, the agent makes the same net trades (and consumes the same) in undistinguished states, simply because the agent is not aware that these undistinguished states are actually more than a single state.

But when the seller appears and says that this event contains two different states (for example: "heads" and "tails"), then the agent becomes aware of the existence of two states. But the agent is still not able to know which of the states occurred. This corresponds to the economy with uncertain delivery. Agents become aware of the existence of all the states, and observe all the state-contingent prices. Being allowed to participate in the complete contingent markets, agents can choose non-measurable bundles, that is, uncertain consumption.

In both interpretations, a *list* can be seen as a bundle that is not measurable with respect to the information of the agent. Consider three possible states of nature: $\Omega = \{\omega_1, \omega_2, \omega_3\}$. An agent who does not distinguish ω_1 from ω_2 may select a random consumption bundle that delivers x_1 in ω_1, x_2 in ω_2 and x_3 in ω_3. With $x_1 \neq x_2$, this consumption bundle is not measurable with respect to private information. In ω_1 and ω_2, the agent will have to accept delivery of x_1 or x_2. She may prefer x_1 and the real state of nature may be ω_1, but since she cannot prove that the state of nature is ω_1, she has to accept

x_2 if this is the bundle that is delivered. In ω_1 and ω_2, she receives "x_1 or x_2", an uncertain bundle that is denoted as $(x_1 \vee x_2)$. Instead of writing the consumption bundle as $x = (x_1, x_2, x_3)$, from the perspective of the agent it would be more adequate to use the notation $\bar{x} = [(x_1 \vee x_2), (x_1 \vee x_2), x_3]$. Observe that this construction implies measurability of the vector of contingent lists with respect to the information of the agent. Instead of contingent bundles, agents are constrained to select contingent lists.

Observe that if consumption is measurable with respect to the information of the agents, then the contingent bundles may be seen as contingent lists with only one element. Let the information of an agent be $P = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, and consider the measurable consumption bundle $x = (x_1, x_1, x_3)$. The correspondent list is $\bar{x} = [(x_1 \vee x_1), (x_1 \vee x_1), x_3] = (x_1, x_1, x_3)$.

In the economy with uncertain delivery, the prices of the lists are restricted to be based on the prices of the contingent commodities. The intermediaries are prevented from doing speculation. Their role is simply to offer to the agent a list composed by contingent goods.

Consider the right to receive a "blue bicycle or red bicycle". It is weaker than the right to receive a "blue bicycle", in the sense that delivery of a "blue bicycle" implies delivery of a "blue bicycle or red bicycle", while the converse is not true. Thus, uncertain delivery of a "blue bicycle or red bicycle" should not be more expensive than the delivery of a "blue bicycle". If it were more expensive, there would be an opportunity for arbitrage. An intermediary could buy a "blue bicycle" and sell it as a "blue bicycle or red bicycle" with profit.

To see how prices are assigned to the lists consider an agent with information $P = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$. The agent can obtain the list $\bar{x} = [(x_1 \vee x_2), (x_1 \vee x_2), x_3]$ by buying any of the following bundles: $x_a = (x_1, x_1, x_3)$, $x_b = (x_1, x_2, x_3)$, $x_c = (x_2, x_1, x_3)$, or

$x_d = (x_2, x_2, x_3)$. It only makes sense to buy the cheapest of these bundles, so the price of a list is actually the price of the cheapest alternative.

Chapter 7

Prudent Expectations Equilibrium

This chapter is organized as follows: in section 1, the idea of prudent preferences is justified; the model of general equilibrium with uncertain delivery is formalized in section 2, and characterized in section 3; in section 4, concepts of core in economies with uncertain delivery are introduced and commented; and, finally, in section 5 we conclude the paper with some remarks.

7.1 Prudent preferences

It is necessary to extend the domain in which preferences are defined, to include the non-measurable bundles. What is the utility of receiving "x_1 or x_2"?

7.1.1 Prudence as a rule-of-thumb

We assume that agents are not able to figure out the true probabilities of getting x_1 and x_2. So, they cannot use expected utility. They use a simple rule of thumb, instead. Consider, for example, a seller which has a “bicycle” to sell. Instead of selling a “bicycle”, the seller could gain by selling a “bicycle or car”, and always deliver the “bicycle” and never the car. In order to defend themselves against being deceived by the sellers, the agents expect the worst outcome:[1]

$$\forall x_1, ..., x_k : u(x_1 \vee ... \vee x_k) = \min_{j=1,...,k} u(x_j).$$

This is our proposal: the utility of an uncertain bundle (a list) is equal to the utility of the worst possible outcome.

Contingent bundles which are constant in states that the agent does not distinguish can be seen as contingent lists with only one element. In this case, prudent utility is equal to the primitive utility. Now we extend preferences to a domain that include also the non-measurable bundles (contingent lists), preserving the values in the space of measurable bundles.

7.1.2 Prudence as a result

The following analysis resembles the work of K. Arrow & Hurwicz (1972) on optimality criteria for decision under ignorance. By “ignorance” it is meant that the agent has no prior probabilities on the occurrence of different states, perceiving them as a single state.

[1]These preferences have some relation with Choquet expected utilities (Schmeidler, 1989), but they are a degenerate case since an infinite weight is placed on the lowest utilities.

The "*prudent behavior*" is necessary to generate preferences that have the two following properties: (1) if $x' \succeq x$, then substituting x by x' in a list does not decrease the utility of the list; and (2) indifference between a bundle x_1 and a list with the alternatives x_1 and x_2, where $x_2 \geq x_1$.

The first property is a kind of monotonicity:

(P1) $x_j \succeq y_j \ , \ \forall j = 1, ..., k \ \Rightarrow \ (x_1 \vee ... \vee x_k) \succeq (y_1 \vee ... \vee y_k)$.

Suppose that an agent is indifferent between receiving a "blue bicycle" and a "red bicycle". What utility should be assigned to the delivery of a "blue bicycle or red bicycle"? An "uncertainty averse" agent would prefer to know what will be delivered in the future. Although she is indifferent between the two bicycles, she wants to know which bicycle will be delivered. On the other hand, an agent with "taste for uncertainty" may prefer to be surprised. A corollary of P1 is that agents are neutral with respect to uncertainty:

$\forall x_1, ..., x_k : x_1 \sim ... \sim x_k \Rightarrow (x_1 \vee ... \vee x_k) \sim x_1 \sim ... \sim x_k$.

It is easy to see that P1 implies that the utility of a list cannot be lower than the utility of the worst possibility. Assuming that the least preferred bundle is x_1:

$x_j \succeq x_1 \ , \ \forall j = 1, ..., k \ \Rightarrow \ (x_1 \vee ... \vee x_k) \succeq (x_1 \vee ... \vee x_1) \sim x_1$.

The second property means that an agent is indifferent between a bundle x_1 and a list with x_1 and $x_1 + a$, where $a \geq 0$. The seller complies with the contract by delivering x_1, so why would she make the effort to deliver the additional a? The agent realistically expects to receive always x_1, and never $x_1 + a$.

(P2) $\forall x_1, ..., x_k : \ x_{k+1} \geq x_k \Rightarrow (x_1 \vee ... \vee x_k \vee x_{k+1}) \sim (x_1 \vee ... \vee x_k)$.

Assume that preferences satisfy P1 and P2. Introducing an alternative such as x_{k+1} does not increase the utility of the list, and, by P1, introducing an alternative that is less attractive than x_{k+1} also does not. In fact, there isn't any additional alternative that increases the utility of the list.

$$\forall x_1, ..., x_k, x_{k+1} : (x_1 \vee ... \vee x_k \vee x_{k+1}) \preceq (x_1 \vee ... \vee x_k).$$

This implies that agents behave with prudence, being indifferent between the uncertain bundle and the worst possibility:

$$x_1 \preceq x_j \ , \ \forall j = 1, ..., k \ \Rightarrow \ (x_1 \vee ... \vee x_k) \sim x_1 \ \Leftrightarrow$$

$$\Leftrightarrow \ \forall x_1, ..., x_k : u(x_1 \vee ... \vee x_k) = \min_{j=1,...,k} u(x_j).$$

Suppose that uncertainty is between two possible bundles: a "bicycle" and "$1 million". Since it is the (hypothetic) seller that selects the bundle after the observation of the state of nature, the buyer should prudently ignore the possibility of receiving "$1 million". By delivering a "bicycle", the seller complies with the contract. The "$1 million" may have been included in the contract just to make it more attractive, while, in any circumstance, the seller plans to deliver a "bicycle".

7.1.3 Prudence by construction

A further justification for pessimism may be given. Take the utility functions defined over lists and make the following transformation:[2]

$$u'(x_1) = \max\{u(x_1 \vee ...)\}$$

The transformed utility of a list x_1 is equal to the maximum of the utilities of lists containing x_1. The reasoning for this transformation is that, knowing the preferences of the agent, the

[2] Here x_1 denotes a list, and $x_1 \vee ...$ denotes any list containing x_1.

seller may sell x_1 as the most preferred list which contains x_1. In any case, x_1 will be delivered. So, u' is a kind of virtual utility of x_1. Observe that under the assumption of prudent preferences, we have $u' = u$.

Under this framework, notice that if the seller has a product to deliver which is $(x_1 \vee x_2)$, then this product cannot have more virtual utility to the seller than either x_1 or x_2. The seller has to sell this product as list containing $(x_1 \vee x_2)$, which cannot have more virtual utility than x_1 or x_2:

(A) $u'(x_1 \vee x_2) = \max\{u(x_1 \vee x_2 \vee ...)\} \leq \min\{u'(x_1), u'(x_2)\}$.

Assume that, when faced with the possibility of receiving the list x_1 or the list x_2, the agents do not prefer to receive the worst possibility with certainty.

$$u(x_1 \vee x_2) \geq \min\{u(x_1), u(x_2)\}.$$

Denote by y a list containing both x_1 and x_2 that maximizes utility, that is, such that $u(y) = u'(x_1 \vee x_2)$. Similarly, find the list y_1 such that $u(y_1) = u'(x_1)$ and the list y_2 such that $u(y_2) = u'(x_2)$. Since y is a maximizer:

$$u(y) \geq u(y_1 \vee y_2) \geq \min\{u(y_1), u(y_2)\} \Leftrightarrow u'(x_1 \vee x_2) \geq \min\{u'(x_1), u'(x_2)\}.$$

Together with (A), this implies that the transformed preferences are *prudent*:

$$u'(x_1 \vee x_2) = \min\{u'(x_1), u'(x_2)\}.$$

So, over the (virtual) preferences u', prudence seems to be a weak restriction. But for the transformation to be applied in the context of our results, it is necessary that the transformation from u to u' preserves continuity, weak monotonicity, and concaveness. Continuity and weak monotonicity should normally be preserved. The assumption of concaveness is harder to interpret.

$$u'(\lambda x_1 + (1-\lambda)x_2) \geq \lambda u'(x_1) + (1-\lambda)u'(x_2) \Leftrightarrow$$

$$\max\{u(\lambda x_1 + (1-\lambda)x_2 \vee ...)\} \geq \lambda \max\{u(x_1 \vee ...)\} + (1-\lambda)\max\{u(x_2 \vee ...)\}.$$

Having a the product (lottery) $\lambda x_1 + (1-\lambda)x_2$, the seller can sell it as one of the elements of a list with utility $u'(\lambda x_1 + (1-\lambda)x_2)$. Alternatively, the product can be sold as a lottery between two lists. One that gives $u'(x_1)$, and other that gives $u'(x_2)$. Concaveness means that the buyer weakly prefers the first "package".

If these hypothesis are accepted, prudence is ensured by construction.

7.1.4 Prudence as realism

It seems pessimistic to consider that the utility of a "blue bicycle or red bicycle" is equal to the worst possibility. The seller should deliver the bundle that has the lowest (*ex post*) value. But it is a big step from the lowest value to the lowest utility for the buyer. So, in some situations, prudence may be seen as overly conservative. But when the interests of the seller and the buyer are perfectly aligned, then the buyer should reasonably expect the worst possibility.

If the bundles are actually portfolios that give a money return, and the bicycle is what the agent buys with this money, then it is perfectly realistic to consider that the minimum is going to be delivered. Suppose that there is a second round of trade. In this case, agents should evaluate the utility of bundles (portfolios) by their indirect utility. Sellers do exactly the same valuation. In this case, the seller always delivers the bundle with the lowest (*ex-post*) value, so this worst bundle is what the buyer always receives.

7.2 General Equilibrium with Uncertain Delivery

The model of the *economy with uncertain delivery* is actually the model known as a *differential information economy*, but now agents are allowed to select non-measurable consumption bundles. The economy extends over two time periods. In the first, agents trade their state-contingent endowments for state-contingent bundles. In the second period agents receive (and consume) one of the bundles that corresponds to a state that the agent does not distinguish from the actual state of nature. This is equivalent to assume that agents select measurable *lists*.

Trade takes place in markets for contingent goods, where agents select bundles which do not need to be measurable with respect to their information. These bundles are, in turn, equivalent to lists. For example, suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and that the agent's partition of information is $P_i = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$. The agent may select a consumption bundle that is not P_i-measurable, $x = (x_1, x_2, x_3)$. Observing $\{\omega_1, \omega_2\}$, the agent has the right to receive x_1 or x_2, while the observation of ω_3 ensures consumption of x_3. So, from the perspective of the agent, this bundle is seen as the following P_i-measurable list: $x = [(x_1 \vee x_2), (x_1 \vee x_2), x_3]$.

Nothing essential is lost by considering that the choice is between non-measurable bundles instead of measurable lists. For convenience, restrict lists to a maximum of K alternatives, and divide each state of nature into K identical sub-states. This transformed economy is equivalent to the original economy. But now agents can select any consumption list with a maximum of K alternatives for each original state of nature, simply by selecting different bundles in the K sub-states.

Consider a finite number of agents, commodities and states of nature. In the economy with uncertain delivery, $\mathcal{E} \equiv (e_i, u_i, P_i, q_i)_{i=1}^{n}$, for each agent i:

- A partition of Ω, P_i, represents the private information. Sets that belong to P_i are denoted A_i^j. The set of states of nature that agent i does not distinguish from ω_k is denoted $P_i(\omega_k)$.

- Agents assign subjective probabilities to the different elementary events that they observe. To each set $A_i^j \in P_i$ corresponds a prior probability q_i^j, with $\sum_j q_i^j = 1$.

- Preferences are the same in undistinguished states, represented by the Von Neumann-Morgenstern (1944) utility functions $u_i^j : \mathbb{R}_+^l \rightarrow \mathbb{R}_+$, which are assumed to be continuous, weakly monotone and concave.

- The initial endowments are constant across undistinguished states, and strictly positive: $e_i^j \gg 0$ for all j.

After receiving information, agent i knows that the state of nature that occurred belongs to A_i^j, one of the sets of P_i. In this *interim* stage, the agent is sure of receiving one of the bundles $x_i(\omega)$ with $\omega \in A_i^j$. Under prudent expectations, the utility that the agent expects is the lowest:

$$v_i^j(x_i) = \min_{\omega \in A_i^j} u_i^j\,(x_i(\omega)).$$

The objective function, that may be designated as *prudent expected utility,* is simply the expected *interim* utility:

$$U_i(x_i) = \sum_{A_i^j \in P_i} q_i^j\; v_i^j(x_i).$$

From the properties of the state-dependent utility functions, u_i^j, it is shown below that the *prudent expected utility* function is also concave.

$$
\begin{aligned}
&U_i(\lambda x_i + (1-\lambda)y_i) = \sum_{A_i^j \in P_i} q_i^j \; v_i^j(\lambda x_i + (1-\lambda)y_i) = \\
&= \sum_{A_i^j \in P_i} q_i^j \; \min_{\omega \in A_i^j} \{u_i^j(\lambda x_i(\omega) + (1-\lambda)y_i(\omega))\} \geq \\
&\geq \sum_{A_i^j \in P_i} q_i^j \; \min_{\omega \in A_i^j} \{\lambda u_i^j(x_i(\omega)) + (1-\lambda)u_i^j(y_i(\omega))\} \geq \\
&\geq \sum_{A_i^j \in P_i} q_i^j \; \min_{\omega \in A_i^j} \{\lambda u_i^j(x_i(\omega))\} + \sum_{A_i^j \in P_i} q_i^j \; \min_{\omega \in A_i^j} \{(1-\lambda)u_i^j(y_i(\omega))\} = \\
&= \lambda \sum_{A_i^j \in P_i} q_i^j \; v_i^j(x_i) + (1-\lambda) \sum_{A_i^j \in P_i} q_i^j \; v_i^j(y_i) = \\
&= \lambda U_i(x_i) + (1-\lambda)U_i(y_i).
\end{aligned}
$$

With this property, we can interpret the model of Arrow-Debreu to cover the case of an economy with uncertain delivery in which agents are *prudent*. The economy with uncertain delivery is transformed in the Arrow-Debreu economy, $\mathcal{E}_{AD} \equiv (e_i, U_i)_{i=1}^n$, where, for each agent i:[3]

- The utility function, $U_i : \mathbb{R}_+^{\Omega l} \to \mathbb{R}_+$, is continuous, weakly monotone and concave.

- The vector of initial endowments, $e_i \in \mathbb{R}_+^{\Omega l}$, is strictly positive.

We impose exact feasibility with free disposal:

$$\sum x \leq \sum e \;\Leftrightarrow\; \forall \omega : \sum x(\omega) \leq \sum e(\omega).$$

We normalize the price functions to the simplex of $\mathbb{R}^{\Omega l}$, that is:

$$\sum_{\omega \in \Omega} \sum_{k=1,\ldots,l} p_k(\omega) = 1.$$

[3]Note that if x is P_i-measurable, then prudent expected utility is equal to classical expected utility. If agents are perfectly informed this obviously occurs. So, with symmetric information, the transformed model is equivalent to the classical model of Arrow and Debreu.

The "budget set" of agent i is given by:

$$B_i(p, e_i) = \left\{ x_i \in \mathbb{R}^{\Omega l}, \text{ such that } \sum_{\Omega} p(\omega) x_i(\omega) \leq \sum_{\Omega} p(\omega) e_i(\omega) \right\}.$$

A pair (p^*, x^*) is a *competitive equilibrium with prudent expectations* if p^* is a price system and $x^* = (x_1^*, ..., x_n^*)$ is a feasible allocation such that, for every i, $x_i^* \in \mathbb{R}_+^{\Omega l}$ maximizes U_i on $B_i(p^*, e_i)$.

Private information is introduced in the model of Arrow and Debreu by a transformation of the preferences. This transformation preserves the properties of continuity, weak monotonicity and concaveness. Everything else in the model remains unchanged. Therefore, several classical results still hold: existence of core and competitive equilibrium, core convergence, continuity properties, etc.[4]

7.3 Characteristics of Equilibrium

On the measurability of utility

In this extended Arrow-Debreu model, competitive equilibrium allocations are characterized by the fact that in states of nature that an agent does not distinguish, the utility of the contingent bundles tends to be the same. Instead of imposing a measurability

[4]This framework also allows the analysis of continuity properties of equilibrium with respect to information as a problem of continuity with respect to preferences, which was settled by Hildenbrand & Mertens (1972).

restriction on the consumption space, as in Radner (1968) and Yannelis (1991), we see the weaker restriction of measurable utility arising naturally in our model.

Actually, if equilibrium prices aren't strictly positive, then some equilibrium allocations may have non-measurable utility. But, removing a *free* component of *excess supply*, we can obtain consumption vectors with measurable utility, which are also equilibrium allocations. The following results make this precise.

Theorem 11 *Let (x^*, p^*) be a competitive equilibrium with prudent expectations. Then, for each agent i, $x_i^* = y_i^* + z_i$, with y_i^* having measurable utility, and z_i being "free". Precisely:*

$y_i^* \in \mathbb{R}_+^{\Omega l}$ *and such that:* $\omega' \in P_i(\omega) \Rightarrow u_i^\omega(y_i^*(\omega)) = u_i^{\omega'}(y_i^*(\omega'))$.

$z_i \in \mathbb{R}_+^{\Omega l}$ *and such that:* $p^* \cdot z_i = 0$.

Proof. Recall that for any $\omega' \in P_i(\omega)$, preferences are equal: $u_i^\omega = u_i^{\omega'}$. Now suppose that for some $\omega' \in P_i(\omega)$, we have different utilities, that is: $u_i^\omega(x_i^*(\omega)) > u_i^\omega(x_i^*(\omega'))$. Then, there exists some $\delta < 1$ such that $u_i^\omega(\delta \cdot x_i^*(\omega)) = u_i^\omega(x_i^*(\omega'))$. Whenever this occurs, modify the allocation accordingly to obtain $y_i^* \leq x_i^*$. This allocation has measurable utility. If y_i^* belongs to the interior of the budget set, there exists a positive ϵ such that the allocation $(1+\epsilon) \cdot y_i^*$ belongs to the budget set and has higher utility than x_i^*. In this case, x^* would not be an equilibrium allocation, and we would have a contradiction. Therefore, y_i^* is not in the interior of the budget set, that is: $z_i = x_i^* - y_i^*$ is such that $p^* \cdot z_i = 0$. QED

With utility being measurable with respect to the information of the agents, *prudent expected utility* is equal to expected utility, for any prior probabilities over states of nature consistent with the given prior probabilities over observed events.

$$\sum_{A_i^j \in P_i} q_i^j \min_{\omega \in A_i^j} u_i^j (x_i(\omega)) = \sum_{\omega \in \Omega} q_i^\omega \ u_i^\omega (x_i(\omega)).$$

The pair (y^*, p^*) is also a *competitive equilibrium with prudent expectations*. But, since y^* has measurable utility, the prudent behavior is not shown to have been unjustified. *Prudent expected utility* is equivalent to the classical expected utility, so the prudent expectations were, in a certain sense, self-fulfilled. A natural refinement of the concept of equilibrium is to demand expectations to be fulfilled, that is, to restrict equilibrium to allocations with measurable utility.

A pair (y^*, p^*) is a "*prudent expectations equilibrium*" if p^* is a price system and $y^* = (y_1^*, ..., y_n^*)$ is a feasible allocation such that, for every i, $y_i^* \in \mathbb{R}_+^{\Omega l}$ maximizes U_i on $B_i(p^*, e_i)$ with $u_i(y_i^*)$ being P_i-measurable.

Corollary 1 *Given any competitive equilibrium with prudent expectations,* (x^*, p^*)*, there exists a prudent expectations equilibrium,* (y^*, p^*) *under the same price system (*y^* *as defined in Theorem 1).*

Proof. The allocation y^* has the same prudent expected utilities as the equilibrium allocation x^*: $U_i(y^*) = U_i(x^*)$, for all i. Under the price system p^*, both allocations cost the same: $p^* \cdot y_i^* = p^* \cdot x_i^*$, for all i. Thus, y^* is also allowed by each agent's budget restriction, and maximizes utility. Furthermore, since $y^* \leq x^*$, y^* is feasible. QED

An important consequence is the existence of equilibrium allocations with measurable utility. If instead of forcing agents to consume the same in states that they do not distinguish, as in Radner (1968) and Yannelis (1991), we force them to consume bundles with the same utility, equilibrium existence is preserved.

Corollary 2 *There exists a prudent expectations equilibrium.*

There exists a competitive equilibrium of the Arrow-Debreu economy, so this is an obvious consequence of Corollary 1.

From Theorem 1, it is straightforward that with strictly positive prices, then $z = 0$ and $x^* = y^*$. That is, all *competitive equilibrium with prudent expectations* have measurable utility. Any conditions that guarantee strict positivity of prices are sufficient to guarantee that equilibrium allocations are *prudent expectations equilibria*.

On incentive compatibility

Remember the seller that offered the game:

"*I will toss a coin. If the result is heads, you receive a blue bicycle; if it is tails, you receive a red bicycle.*"

If the agent is indifferent between the two colors, the impossibility of observing the state of nature is not a problem. The agent does not fear being "tricked", because the delivery of a bicycle is guaranteed.

After receiving private information, the agent can prove that the state of nature belongs to, for example, A_i^j. Whatever the state in A_i^j, the bundles that are supposed to be delivered have the same utility. So, agents cannot be deceived to receive consumption bundles with lower utility. Contracts can be enforced and issues of incentive compatibility do not arise.

In sum, the consideration of contracts for uncertain delivery allows us to relax in a natural way the measurability assumption, while preserving (trivial) incentive compatibility. This enlarges the space of allocations, improving the efficiency of exchange, relatively to economies in which consumption has to be measurable with respect to private information.

On welfare

Compared with measurable consumption, measurable utility is less restrictive, as it allows agents to select different consumption bundles in order to take advantage of variations in prices across states that they do not distinguish.

Theorem 12 *Let* (x^*, p^*) *be a competitive equilibrium with prudent expectations.*

$\omega' \in P_i(\omega) \Rightarrow p^*(\omega) \cdot x_i^*(\omega) \leq p^*(\omega) \cdot x_i^*(\omega')$.

Proof. Suppose that for some $\omega' \in P_i(\omega)$, we had $p^*(\omega) \cdot x_i^*(\omega) > p^*(\omega) \cdot x_i^*(\omega')$. Designate by y_i a modified bundle with $y_i^*(\omega) = x_i^*(\omega')$ being the only difference relatively to x_i^*. This bundle has the same utility and allows the agent to retain some income. There exists a positive ϵ such that $(1 + \epsilon) \cdot y_i$ belongs to the budget set and has higher utility than x_i^*. Contradiction! QED

In spite of the penalization implied by prudence, in equilibrium, *prudent expected utility* is higher in the sense of Pareto than that which is attainable under the classical restriction of equal consumption in states of nature that are not distinguished.

Theorem 13 *Let* (x^*, p^*) *be an equilibrium in the sense of Radner (1968).*
There are Pareto optima of the economy with uncertain delivery, z*, such that* $U_i(z_i) \geq U_i(x_i^*)$*, for every agent. The improvement may be strict (see section 3).*

Proof. The proof is straightforward. If (x^*, p^*) is an equilibrium in the sense of Radner (1968), the allocation x^* is still feasible in the economy with prudent expectations. QED

If preferences are strictly concave and relative prices vary across states that at least one agent does not distinguish, then, in a *competitive equilibrium with prudent expectations,* consumption is not measurable. In these cases, welfare improvements are strict in the sense of Pareto.

7.4 Cooperative Solutions: the Prudent Cores

Cooperative solutions can be defined in a similar way. Instead of constraining allocations to be measurable with respect to information, we introduce again the *prudent expectations* regarding consumption in undistinguished states.

Remember that the economy with uncertain delivery and prudent expectations was transformed into an Arrow-Debreu economy. The core of an Arrow-Debreu economy exists, and we designate it as the "*prudent private core*".

A coalition $S \subseteq N$ "*privately blocks*" an allocation x if there exists $(y_i)_{i \in S}$ such that: $\sum_{i \in S} y_i \leq \sum_{i \in S} e_i$ and $U_i(y_i) > U_i(x_i)$ for every $i \in S$, where U_i is the prudent expected utility of agent i.

The "*prudent private core*" is the set of all feasible allocations which are not *privately blocked* by any coalition. Although coalitions of agents are formed, information is not shared between them. The *prudent expected utility* is based only on each agent's private information.

The *prudent private core* is very similar to a modified private core where measurable utility is required instead of measurable consumption. Given an allocation in the *prudent private core*, there exists another with the same utility for every agent, which has measurable utility and requires less resources.

Theorem 14 *Let $x \in PPC(\mathcal{E})$.*

There exists some $x' \in PPC(\mathcal{E})$ such that, $\forall i = 1, ..., n$:

a) $x'_i \leq x_i$;

b) $U_i(x'_i) = U_i(x_i)$;

c) $u_i(x'_i)$ is P_i-measurable.

Proof. If $u_i(x_i)$ isn't P_i-measurable, we can multiply the $x_i(\omega)$ that have higher utilities in each element of P_i by a factor smaller than 1 to obtain a modified allocation with measurable utility. These higher utilities are not taken into account in the calculation of prudent expected utility, because only the worst outcome is considered. Therefore, expected utility remains unchanged and this allocation satisfies $x'_i \leq x_i$. QED

Even being penalized by the prudence, allocations in the *prudent private core* dominate, in the sense of Pareto, those in the *private core* (Yannelis, 1991). The latter are always feasible in the economy with uncertain delivery, while the converse is not true.

The coarse core and the fine core introduced by R. Wilson (1978), also have correspondent concepts with prudent expectations: the "*prudent coarse core*" and the "*prudent fine core*".

To find the *prudent coarse core*, consider a *strong block*, in which prudence is based on common information.

A coalition $S \subseteq N$ *strongly blocks* an allocation x if there exists $(y_i)_{i \in S}$ such that: $\sum_{i \in S} y_i \leq \sum_{i \in S} e_i$ and $U^s_{i,S}(y_i) > U^s_{iS}(x_i)$ for every $i \in S$, where U^s_{iS} is the "*strongly S-prudent expected utility*" of agent i in the coalition S. The *interim* utility for agent i in coalition S is calculated using the minimum utility across states that the coalition cannot distinguish using only the common information among the members.

To find the *prudent fine core*, use a weak notion of block, based on pooled information.

A coalition $S \subseteq N$ *weakly blocks* an allocation x if there exists $(y_i)_{i \in S}$ such that: $\sum_{i \in S} y_i \leq \sum_{i \in S} e_i$ and $U^w_{i,S}(y_i) > U^w_{iS}(x_i)$ for every $i \in S$, where U^w_{iS} is the "*weakly S-prudent expected utility*" of agent i in the coalition S. The interim utility for agent i in coalition S is calculated using the minimum utility across states that the coalition cannot distinguish using the pooled information of its members.

In any case, welfare is improved in the prudent cores.

7.5 Concluding Remarks

We model economies with private information and *uncertain delivery*, in which agents are assumed to follow a simple rule-of-thumb: to be *prudent*. The model of Arrow-Debreu can be reinterpreted to cover this situation, therefore, many classical results still hold: existence of core and equilibrium, core convergence, continuity properties, etc.

The inclusion of *contracts for uncertain delivery* allows agents to improve their welfare. In any case, they are better off relatively to allocations with measurable consumption.

Expecting to receive the worst of the possibilities contracted, agents behave prudently by selecting bundles with the same utility for consumption in states that they do not distinguish. Instead of consuming the same, as in Radner (1968) and Yannelis (1991), agents consume bundles with the same utility.

In certain situations, such prudence may not be appropriate, but there are others in which it is absolutely justified. For example, if there is a second round of trade, agents should evaluate portfolios by their indirect utility. In this case, the seller always delivers the bundle with the lowest value.

While an intuition for the concept of rational expectations equilibrium was the idea that "*agents cannot be fooled*", in the prudent expectations equilibrium it is the market that cannot be fooled. Agents use a rule of thumb which is related to Murphy's law: "*if anything can go wrong, it will*".

An advantage of this concept with respect to the *rational expectations equilibrium* is that it is useful to have more information. *Markets for information* can be studied with two-stage games: in the first stage, agents trade information; in the second, they maximize prudent expected utility. This way, economic insights on information production and dissemination could be obtained.

Real economic agents follow simple rules of decision, instead of making huge amounts of calculations (Tversky and Kahneman, 1974). This further justifies the study of equilibrium with agents constructing expectations in a simple way.

Appendix A

The Expected Utility Hypothesis

An **act** is a mapping of a probability space $(\Omega, \mathcal{F}, \mu)$ into a space of consequences, C. This may be simply $\mathbb{R}$, representing utility. Each act induces a probability measure q on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borelian σ-algebra.

For simplicity, assume a finite number of possible states of nature: $\Omega = \{\omega_1, ..., \omega_\Omega\}$. The σ-algebra $\mathcal{F}$ consists of the subsets of Ω.

Under the **expected utility hypothesis**, there exist functions $u(\cdot)$ that are nondecreasing and bounded.[1] such that it is possible to represent the rational behavior of an agent by the maximization of:

$$\sum_{j=1}^{\Omega} q^j u^j(x(\omega^j)).$$

[1]Non-boundedness of $u(\cdot)$ leads to the St. Petersburg paradox, according to which agents are willing to pay an infinite value to play a game that pays 2^n units of utility if a head appears for the first time on the n^{th} toss.

A.1 Von Neumann's Axiomatization

Maximization of expected utility can be viewed as a consequence of rationality. Consider the space of lotteries $\mathcal{M}$ over the finite set of bundles [$x(\omega^1), ..., x(\omega^\Omega)$].[2]

Assume that the choices of a rational agent between lotteries are represented by the complete and continuous preordering $\succeq$. That is, that the binary preference relation $\succeq$ satisfies:

(a) **reflexivity** - $a^1 \succeq a^1, \forall a^1 \in \mathcal{M}$;

(b) **transitivity** - $a^1 \succeq a^2$ and $a^2 \succeq a^3 \Rightarrow a^1 \succeq a^3, \forall a^1, a^2, a^3 \in \mathcal{M}$;

(c) **completeness** - $\forall a^1, a^2 \in \mathcal{M}$, either $a^1 \succeq a^2 or a^2 \succeq a^1$;

(d) **continuity** - $\forall a^1 \in \mathcal{M}, \{a : a \succeq a^1\}$ are closed sets.

In a seminal paper, Samuel Eilenberg (1941) showed that every continuous total preorder given on a connected and separable topological space admits a continuous utility representation. Debreu (1964) showed that the assumption of connectedness could be replaced by second countability. A negative result was presented by Estévez & Hervés-Beloso (1995): in every non-separable metric space, there exists a continuous total preorder which doesn't have a continuous utility representation.

So, with $\mathbb{R}^l$ as the commodity space, there exists a continuous and nondecreasing function $U(\cdot)$ (defined up to a monotone increasing transformation) that represents $\succeq$.

$a^1 \succeq a^2 \Leftrightarrow U(a^1) \geq U(a^2)$;

$a^1 \succ a^2 \Leftrightarrow U(a^1) > U(a^2)$.

[2]Note that $\mathcal{M}$ is equivalent to the simplex of $\mathbb{R}^\Omega$.

Von Neumann & Morgenstern (1944) proposed axioms that imply that there exists some $U(\cdot)$ that is linear in the probabilities:

$$U(a) = \sum_{j=1}^{\Omega} q^j u^j (x(\omega^j)),$$

where $u(\cdot)$, defined up to an increasing affine transformation, is the *Von Neumann-Morgenstern utility function*. These axioms of rational choice are three:

Axiom 1 (Completeness) - The agent has a complete preordering on the space of lotteries $\mathcal{M}$, defined over the consequences.

This first axiom may be interpreted as the indifference of the agent regarding the means that lead to the consequences.

Axiom 2 (Continuity) - $\forall a^1, a^2, a^3 \in \mathcal{M}$ such that $a^1 \succeq a^2$ and $a^2 \succeq a^3$ there exists $\alpha \in [0,1]$ such that $\alpha a^1 + (1-\alpha)a^3 \sim a^2$.

Axiom 3 (Independence) - $\forall \alpha \in]0,1[, \forall a \in \mathcal{M}$:
$a^1 \succ a^2 \Rightarrow \alpha a^1 + (1-\alpha)a \succ \alpha a^2 + (1-\alpha)a$;
$a^1 \sim a^2 \Rightarrow \alpha a^1 + (1-\alpha)a \sim \alpha a^2 + (1-\alpha)a$.

A.2 Savage's Axiomatization

Let's drop the assumption of existence of a measure of objective probability. Savage (1954) considers only as given the space of acts, $\mathcal{A}$, associating consequences to the events in a measurable space $(\Omega, \mathcal{F})$, and a complete preordering, $\succeq$, on the space of acts. Rational behavior under uncertainty is specified by seven axioms on this preordering. From these axioms, Savage derives a subjective probability distribution and an utility function such that the preordering is represented by the expected value of this function.

Define first the conditional preferences. Let $E \in \mathcal{F}$ be an event. Comparison between acts depends only of the consequences when E occurs:

$a \succeq_{|E} a' \Rightarrow \bar{a} \succeq \bar{a}'$,

with $\omega \in E \Rightarrow \bar{a}(\omega) = a(\omega) \wedge \bar{a}'(\omega) = a'(\omega)$, and $\omega \notin E \Rightarrow \bar{a}(\omega) = \bar{a}'(\omega)$.

Axiom 1 (Existence) - Conditional preferences exist.

For any $x \in C$, the constant act a_x is defined by: $a_x = x, \ \forall \omega \in \Omega$.

Axiom 2 (Constant acts) - $\forall x \in C, a_x \in \mathcal{A}$.

There may not be such acts, it is sufficient to imagine their existence.

Axiom 3 (Independence) - If $E \neq \emptyset, a_x \succeq_{|E} a_{x'} \Leftrightarrow x \succeq x'$.

Let $E, E' \in \mathcal{F}$ be two events, and $x, x' \in C$ two consequences with $x \succ x'$. Construct acts $a, a' \in \mathcal{A}$ as follows:

$\omega \in E \Rightarrow a(\omega) = x$ and $\omega \notin E \Rightarrow a(\omega) = x'$

$\omega \in E' \Rightarrow a'(\omega) = x$ and $\omega \notin E' \Rightarrow a'(\omega) = x'$

If $a \succeq a'$, we say that the qualitative probability of E is at least as great as that of E': $E \hat{\succeq} E'$.

So it is possible to infer subjective probabilities from the preordering over lotteries. For the relation $\hat{\succeq}$ to be well defined, we need:

Axiom 4 (Comparability) - All the events are comparable in qualitative probability.

And we also need some technical axioms.

Axiom 5 (No indifference) - $\exists a, a' \in \mathcal{A}$ such that $a \succ a' \vee a' \succ a$.

Axiom 6 (Continuity - implies infinite Ω) - If $a \succ a'$, for any $x \in C$ there exists a finite partition of Ω such that if a or a' is modified on an event of the partition so that x becomes the consequence in this event, the strict preference of a over a' is preserved.

Axiom 7 (Independence II) - Let $a \in \mathcal{A}$. Then:

$a' \succeq_{|E} a_a(\omega) \;\; \forall \omega \in E$ implies $a' \succeq_{|E} a$;

$a_a(\omega) \succeq_{|E} a' \;\; \forall \omega \in E$ implies $a \succeq_{|E} a'$.

These seven axioms allowed Savage (1954) to obtain the following result.

Theorem Given axioms 1-7, there exists a unique probability measure μ defined on $(\Omega, \mathcal{F})$, and a continuous, nondecreasing and bounded function $u(\cdot)$ defined up to an affine transformation such that:

$a \succeq a' \Leftrightarrow \int_\Omega u(a(\omega))d\mu(\omega) \geq \int_\Omega u(a'(\omega))d\mu(\omega)$.

μ is such that $E' \hat{\succeq} E \Leftrightarrow \mu(E') \geq \mu(E)$, and it is called the *agent's subjective probability*.

A.3 The value of information

Consider a rational decision maker with imperfect information, seeking to maximize expected utility:

$$\max_{a \in A} \int_{\omega \in \Omega} u^{\omega}(a)\, q^{\omega}\, d\omega.$$

Let a^* be the solution of this problem, and let P be the partition of information of the agent. With the agent revising expectations according to the Theorem of Bayes, after receiving its information, $P^j \in P$, the (*posterior*) beliefs are:

$$Pr(\omega|P^j) = 0 \text{ , if } \omega \notin P^j;$$

$$Pr(\omega|P^j) = \frac{q^{\omega}}{\int_{P^j} q^{\omega}\, d\omega} = \frac{q^{\omega}}{q(P^j)} \text{ , if } \omega \in P^j.$$

For each $P^j \in P$, the agent solves the problem:

$$\max_{a \in A} \int_{\omega \in \Omega} u^{\omega}(a)\, Pr(\omega|P^j)\, d\omega;$$

Which is equivalent to:

$$\max_{a \in A} \int_{\omega \in P^j} u^{\omega}(a)\, q^{\omega}\, d\omega = V(P^j).$$

So, the value of having the information structure P can be estimated, for finite and infinite partitions, as:

$$U(P, q, u(\Delta)) = \sum_j V(P^j) q(P^j),$$

$$\text{or } U(P, q, u(\Delta)) = \int_j V(P^j) q(P^j) dj.$$

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