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TESIS DOCTORAL

INFORMACIÓN DESCONOCIDA EN ANÁLISIS DE CONCEPTOS FORMALES

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Undoubtedly, this thesis would not have been possible without the long road that has led me to it. Fortunately, it has not been a lonely road and I have been fortunate to be surrounded by people who have helped me to consolidate this thesis.

To start in some order. Without a doubt, I wouldn't be here if it wasn't for my family. I have been fortunate to be able to count on them whenever I needed them and to have their support and affection at all times, from childhood to the last moments of this thesis. I am also thinking at this time of some who are not with us, but I know that if they were with us today they would also share the joy of having completed this step in my life.

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Last but not least, I have to talk about my colleagues and I have to say that I have been very lucky to be in this department. I've always felt supported, I can't forget the morning chats over a cup of coffee. I am thinking not only of colleagues from the department, but also of colleagues from outside the department, and even outside the university, who have made the journey more pleasant with their scientific meetings, from which I have learnt a great deal.

To made a long story short: thanks to all of you who have made this possible. As you can see, I have avoided naming names and simply divided everyone into three groups (not disjoint) because I would have so many names to give that, for sure, I might forget some of them. Undoubtedly, everyone will know that, at some point, I have referred to each of them.

I will make an exception, three exceptions in fact. They have contributed a lot to this thesis and without their understanding, help, patience and knowledge it would not be possible. I am referring to my tutor and director, Pablo José Cordero Ortega, my second director, Manolo Enciso, and Ángel Mora, with whom I did most of my research. Thank you for welcoming me into your research group and making me feel like one of you from the very first day, and I hope that this thesis is just the beginning and will be followed by many more researches together.

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Contents

Resumen

Hoy en día vivimos rodeados de datos. Constantemente en nuestro día a día estamos recibiendo datos de diferentes fuentes. Sin ir más lejos, esta misma tesis contiene diferentes datos. Los datos siempre han existido, aunque no siempre han estado a nuestro alcance, por ejemplo, no siempre hemos sido capaces de obtener la temperatura en nuestra ciudad, aunque, hoy en día, podemos obtenerla en tiempo real. Sin embargo, podemos afirmar que siempre tendremos datos que son desconocidos en un momento dado, aunque en un futuro estos puedan ser obtenidos.

Debido a la cantidad enorme de datos que tenemos a nuestra alcance, también tenemos mucha información disponible. Esta información por sí sola no es útil. Por ejemplo, si disponemos de 1000 ofertas diferentes sobre un viaje, es necesario hacer un análisis de todas ellas para poder escoger la que más se adapte a nuestras necesidades. Este análisis se realiza buscando patrones, estableciendo relación entre los datos, etc. Este proceso puede realizarse usando la lógica mediante tres pasos.

En el primer paso realizaremos la representación del conocimiento. Este paso es necesario para poder compartir el conocimiento con otras personas. Para ello, necesitaremos definir la sintaxis de la lógica. En el segundo paso debemos asegurarnos de que todos los usuarios que reciben los datos den una interpretación común, para ello, usaremos la semántica de la lógica. Por último, debemos razonar con la información para extraer el conocimiento

de estos datos. Para este último paso necesitaremos el sistema axiomático de la lógica.

Es importante tener en cuenta que la manipulación de los datos no es un proceso lineal sino cíclico. Cuando recibimos nuevos datos, debemos procesarlos y extraer nuevo conocimiento. Este nuevo conocimiento debe ser integrado en el anterior y, de esta manera, crear nuevo conocimiento. Este proceso es cíclico y nunca termina, es decir, siempre estaremos recibiendo nuevos datos y, por tanto, consiguiendo nuevo conocimiento.

Cuando trabajamos con datos y en estos tenemos información desconocida, normalmente, esta información desconocida no es considerada. Por otro lado, la información está en continuo cambio, es decir, lo que es desconocido hoy mañana podría ser conocido y viceversa. Esto nos lleva a la cuestión principal de esta tesis doctoral: ¿Cómo podemos tratar esa información desconocida y trabajar con ella de forma sólida y consistente?

Para ello, pensamos que debemos considerar tanto las situaciones de la información desconocida como también su impacto en los sistemas que recopilan la información. En nuestro caso estamos especialmente centrados en las implicaciones de atributos, las reglas sí-entonces, las reglas de asociación y conceptos similares que permiten representar información como conexiones entre dos conjuntos de propiedades. El siguiente ejemplo ilustra dos formas en las que la información desconocida puede repercutir en este tipo de representación de la información.

Pongamos el caso en el que estemos trabajando con un conjunto de datos sobre pacientes de un hospital. A los pacientes se le han realizado pruebas para comprobar si tienen fiebre y dolor de cabeza. Un primer paciente tiene ambos síntomas mientras que un segundo no presenta ninguno de los dos. Hay un tercer paciente que tiene fiebre pero no ha declarado nada sobre el dolor de cabeza. Con los datos de estos tres pacientes, podemos inferir dos implicaciones: (1) “Dolor de cabeza implica Fiebre” y (2) “No tener dolor de cabeza implica No tener Fiebre". Observemos que estas implicaciones tienen un comportamiento diferente con respecto a la información desconocida del tercer paciente. La implicación (1) se seguirá cumpliendo incluso cuando llegue la información del tercer paciente, independientemente de si éste declara tener dolor de cabeza o no. Sin embargo, la implicación (2) se cumple con la información disponible en este momento. Sin embargo, si la tercera persona nos informa de que no tiene dolor de cabeza, entonces, la implicación se deja de cumplir, ya que el tercer paciente será un contraejemplo de ella. Así, de alguna manera, podemos decir que la primera implicación es más fuerte que la segunda.

Planteamiento del problema

Esta tesis doctoral está enmarcada en el área conocida como Análisis de Conceptos Formales (FCA), una disciplina introducida originalmente por Rudolf Wille en 1984 [70]. FCA proporciona técnicas y herramientas para descubrir conocimiento a partir de datos, representar el conocimiento descubierto y, finalmente, razonar sobre él.

Los datos son almacenados en tablas de doble entrada que representan una relación binaria entre objetos o individuos (por filas) y atributos o propiedades (por columnas). Llamamos a esta representación contexto formal. En la primera forma de representar proporcionada por FCA encontramos las nociones (o conceptos) que surgen de los contextos formales, además de generar una estructura ordenada con ellos (un retículo). La segunda forma de representar el conocimiento con FCA es a través de las implicaciones de atributos, que son relaciones binarias entre dos conjuntos de atributos. La principal ventaja de las implicaciones de atributos es que es posible manipularlas simbólicamente y razonar con ellas.

Además, FCA nos proporciona una colección sólida de algoritmos y técnicas para construir el retículo de conceptos y para la obtención de implicaciones de un modo automático y eficiente. Es posible también construir de forma automática el retículo de conceptos de las implicaciones y viceversa.

En FCA clásico sólo trataba la información positiva, es decir, la información que nos dan. En algunas ocasiones, se interpreta la información no dada como información negativa, es decir, si no sabemos que una luz está encendida, se puede considerar que la misma está apagada.

Este conocimiento negativo puede ser extraído usando FCA clásico tal y como propuso Rokia Missaoui [52]: duplicamos las columnas del contexto formal y estas nuevas columnas se consideran como las negadas de las anteriores. De esta forma, podemos interpretar información positiva y negativa. Esta solución no es la más eficiente ya que podemos obtener información redundante, además, al duplicar el número de columnas, estamos incrementando el coste computacional necesario para extraer el conocimiento y para su representación. En [65] se presentó un modo alternativo que también trabaja con información negativa en la que se usa directamente el contexto formal, trabajando con la información positiva y negativa. De la misma forma, se modificaron las implicaciones y los conceptos para que éstos considerasen información positiva y negativa.

Desde nuestro punto de vista, hay ciertas circunstancias que no se pueden tratar con información positiva y negativa y es necesario considerar un tercer valor, es decir, la información desconocida. Algunos de los ejemplos en los que esta información desconocida es necesaria son los siguientes: cuando estamos trabajando en un hospital con pacientes y tenemos un paciente varón. Si uno de los atributos que tenemos que considerar es si el paciente tiene o no tiene el periodo regularmente, no podremos dar ningún valor (positivo o negativo) ya que no tendría sentido. Otro ejemplo es en una valoración a un hotel cuando el cliente no quiere rellenar alguna información, quizás porque no ha usado ese servicio. Está claro que no podemos asumir que la valoración es positiva, es decir, que al cliente que le gustó el servicio. Pero tampoco podemos considerar que la valoración es negativa, es decir, que al cliente no le gustó el servicio. Un último ejemplo es cuando la información no está disponible en el instante en el que estamos trabajando. Por ejemplo, Si tenemos un conjunto de datos sobre estudiantes, si el examen está programado para la semana que viene, no podemos disponer todavía de las calificaciones.

Para nuestro trabajo, como hemos mencionado, deberemos considerar tres valores y, como consecuencia, tanto los conceptos como las implicaciones deberán ser revisados y modificados para que contengan estos tres valores.

Trabajos previos

No somos los primeros en trabajar sobre análisis de datos con información desconocida, por tanto, es necesario un estudio de los trabajos que hay anteriormente sobre esta situación ya que nos puede servir como punto de partida. Este estudio será realizado enfocándonos en tres áreas bastante cercanas a nuestro trabajo: las bases de datos, la lógica y el análisis de conceptos formales.

La información desconocida en bases de datos fue explorada al final de los setenta por varios autores [14, 49, 69]. El primero fue Lipski [49], el cual expuso que la información desconocida puede presentarse entre dos casos: el caso en el que nada es conocido y el caso en el que todo es conocido, por tanto,para poder responder consultas realizadas en una base de datos con datos desconocidos debemos explorar todas los casos posibles. Vassiliou [69] distinguía entre dos tipos de información desconocida: información de la que no tiene sentido hablar y la información desconocida como tal y, por tanto, se debe trabajar con cuatro valores en vez de con tres: la información positiva, la negativa, la desconocida y la inconsistente. Codd [14] fue uno de los autores que más influencia tuvo sobre la información desconocida en bases de datos. Codd llamó *"valor nulo"* a todo valor desconocido y trabajó con una lógica trivaluada, es decir, con tres valores. C.J Date [23], al igual que Vassiliou, definió más de un significado para el valor nulo o desconocido, además, expuso que estos valores deben ser tratados de diferente manera y tener diferentes propiedades. Hoy en día, el tratamiento de información desconocida en bases de datos sigue siendo un tema de investigación activo como se afirma en [73]. Algunos ejemplos de trabajos presentes sobre la información desconocida en base de datos son los siguientes trabajos: Alattar et al. [1], Console et al. [15], Libkin [48], Greco et al. [38], Guagliardo et al. [39] y Geerts et al [37].

En cuanto a la lógica, el primer enfoque sobre información desconocida fue realizado por Łukasiewicz [51], donde el valor desconocido significa *posible*, para considerar este tercer valor, presentó un primer marco de trabajo lógico rompiendo algunas de las leyes clásicas de la lógica, particularmente, el principio del tercero excluido. Para Kleene [43], el valor desconocido es el resultado de un fallo al discernir si una proposición dada es verdadera o falsa y puede ser considerada como un ejemplo de las lógicas parciales. Estas lógicas consideran solo dos valores de verdad y las evaluaciones se definen sobre una función parcial de las expresiones del lenguaje de ese conjunto de verdad ([67] y [10]). De Finetti [25] distinguió la veracidad de una proposición y su conocimiento sobre ella. En el primer caso sólo podemos considerar dos valores posibles (verdadero o falso) mientras en el segundo la cantidad de valores dependen del agente. Este enfoque se considera en la lógica epistémica, la cual, considera que los valores de verdad deben ser dados por el conocimiento de un agente. Este agente podría devolvernos valores desconocidos. Uno de los artículos más citados relacionados con el trabajo de información desconocida en la lógica es el trabajo de Belnap [5], el cual consideró cuatro valores de verdad (*desconocido*, *verdadero*, *falso* y *contradicción*) siguiendo un punto de vista epistémico. Además, Belnap consideró dos órdenes diferentes entre los valores: el orden de veracidad (considerando falso como el valor más bajo y verdadero como el más alto) y el orden de información (considerando desconocido como el valor más bajo y la contradicción como el más alto). Por último, [16] presenta un estudio sobre algunas cuestiones de lógicas con diferentes valores y puede ser visto como un ejemplo para probar que el estudio de la información desconocida o de valores nulos sigue siendo importante en la lógica hoy en día.

Por último, presentamos los antecedentes sobre el análisis de conceptos formales con información desconocida. El primer trabajo que considera información desconocida en retículo de conceptos es [47]. En este artículo Lex introdujo un álgebra de cuatro valores y estudió cómo debe ser los conceptos formales cuando estamos trabajando con información desconocida. Holzer [41] presentó los "contextos formales incompletos" que son contextos formales con información desconocida. También presento los "intents posibles" y los "extents posibles" que son intents y extents que se tienen en, al menos, una compleción del contexto incompleto, es decir, contextos formales en los que hemos cambiado los valores desconocidos por valores positivos o negativos. Este enfoque es bastante similar al enfoque previamente mencionado en la sección anterior presentado por Lipski [49] para bases de datos. Además, Holzer también extendió las implicaciones para tratar la información desconocida, presentando las "implicaciones de atributos satisfacibles" que son implicaciones que son ciertas en, al menos, una compleción del contexto formal incompleto, y las llamada "implicaciones de atributos Kripke" que son implicaciones de atributos que se cumplen en todas las compleciones del contexto formal incompleto. Estas ideas, como se puede observar, están bastante relacionadas con las del ejemplo del dolor de cabeza y fiebre mencionado anteriormente. En [32], Ganter consideró la información desconocida en análisis de contextos formales cuando compactamos objetos en uno solo y estos tienen información distinta (positiva y negativa) sobre un atributo. En su artículo, Ganter presenta los llamados "contextos formales parciales", que son contextos con tres valores distintos (positivos, negativos y desconocidos). En [72], los autores introducen un contexto formal trivaluado, es decir, no solo consideran información positiva y desconocida, sino que además, consideran la información desconocida ordenando estos valores y añadiendo que la información negativa es menor que la desconocida y esta, a su vez, es menor que la positiva. También trataron el problema de construir el conjunto de compleciones de un contexto formal parcial. Ahora, mencionamos algunos trabajos recientes bastantes relacionados con nuestra línea de trabajo, estos trabajos también estudian el análisis de conceptos formales usando tres valores. En [63] los autores consideraron una estructura algebraica muy similar a nuestro enfoque, sin embargo, sólo se dedicaron a la construcción del retículo de conceptos formales. En [4] los autores trabajan con análisis de conceptos formales con valores difusos con conceptos formales conteniendo información positiva y negativa y pudiendo realizar un modelo de la incertidumbre. Podemos observar que, en general, los artículos que hemos comentado se centran en el retículo de conceptos. Sin embargo, nosotros estamos interesados en la manipulación simbólica de la información desconocida que se puede abordar con las implicaciones de atributos y los métodos de razonamiento. Como antecedente a esta línea de trabajo, consideramos el trabajo [44], donde el autor presentó implicaciones de atributos que captura información positiva y negativa, pero a un nivel de complejidad más alto, usando valores difusos. Por otro lado, en [28] los autores presentan implicaciones de atributos pero sin preservar la interpretación conjuntiva (que es la usual) de las implicaciones y considerando en su lugar una interpretación disyuntiva. Ésta induce un crecimiento significativo en la complejidad de los métodos de razonamiento.

Acabamos esta sección mencionando dos aspectos importantes para nuestro trabajo y relacionados con la representación del conocimiento.

El primero es dar el significado a ese valor que representa la información desconocida. El estudio descrito aquí está basado en el artículo [12] donde Dubois presentó una dicotomía entre dos significados: desconocidos y frontera.

En el caso del valor de frontera es cuando no podemos discernir claramente si el valor que se debe dar es el positivo (verdadero) o el negativo (falso). Este valor frontera se considera en otras áreas, por ejemplo, la lógica difusa o la teoría de "rough set" (también llamados conjuntos aproximados).

El uso de la interpretación de frontera en algunos casos no tiene ningún sentido, por ejemplo, para la frase "estar embarazada". En estos casos, se necesita la segunda representación y esta representa que no tenemos información.

La segunda cuestión de esta discusión es cómo podemos representar la información desconocida. Obsérvese que la representación es independiente del significado escogido.

Una de las representaciones más populares es usando conjuntos trivaluados [11] que son funciones $\rho : \Omega \to \{0, 1, \frac{1}{2}\}$ donde Ω es el conjunto de los símbolos proposicionales. Otra posibilidad es considerar los llamados ortopares (consistentes) [12] que son parejas (P, N) de subconjuntos de Ω donde se cumple que $P \cap N = \varnothing$. Los símbolos proposicionales en P son aquellos que son verdaderos (*Positivos*) y los símbolos en N son aquellos que son falsos (*Negativos*). Existe una tercera representación, introducida por [46], llamada evaluaciones de pares Booleanos consistentes (BVP). En esta representación se consideran pares $\vec{v} = (\underline{v}, \overline{v})$ donde $\underline{v}, \overline{v} : \Omega \to \{0, 1\}$ cumpliendo que $\underline{v} \leqslant \overline{v}$ se cumple punto a punto. En este enfoque los símbolos proposicionales que cumplan que $\underline{v} = 1$ son aquellos que son verdaderos (*Positivos*) mientras aquellos que son falsos (*Negativos*) cumplen que $\overline{v} = 0$.

Los demás se consideran como desconocidos. Es fácil de comprobar que los conjuntos trivaluados, los ortopares y las BVPs son equivalentes.

En algunos casos, tres valores no son suficientes para trabajar con la información desconocida, y es necesario introducir un cuarto valor, denotando la contradicción. Hay dos representaciones que aparecen normalmente en la bibliografía. Estas representaciones están basadas en los conjuntos tetravaluados y en los ortopares, donde la condición $P \cap N = \varnothing$ ha sido eliminada para permitir expresar la contradicción.

Objetivos de la tesis

El tratamiento de la información desconocida en el análisis de conceptos formales es un problema abierto y que merece ser explorado. En nuestro opinión, no hay duda sobre que la inclusión de la información desconocida otorga una gran ventaja desde el punto de vista de las aplicaciones. Sin embargo, el punto clave es el proporcionar un enfoque completo y uniforme que considere los principales aspectos de FCA.

El principal objetivo de nuestra tesis es **desarrollar un marco de trabajo general para FCA que permita trabajar con información desconocida** cubriendo algunas carencias detectadas en la bibliografía existente. En particular, una de estas carencias es la formalización de las implicaciones de atributos y sus métodos de razonamiento. Siguiendo los trabajos previos en FCA, pretendemos proporcionar nuevas lógicas basadas en el pardigma de Simplificación y sus correspondientes métodos de inferencias, basados en eliminar redundancia. Este problema podría ser considerado como la principal motivación de esta tesis doctoral cuyos objetivos podrían ser descritos de la siguiente manera:

- Desarrollar un nuevo álgebra con tres valores que nos permita extender la lógica y todos las nociones de FCA con información positiva, negativa y desconocida en vez de solamente información positiva y negativa.

- Debido a que el núcleo de FCA es la noción de conexión de Galois, nuestro segundo objetivo es introducir una nueva conexión de Galois que capture diferentes aspectos de la información desconocida. Más concretamente, estamos interesados en la definición de una conexión de Galois que refleje la información que tenemos en el momento actual y una segunda conexión de Galois que capture la información que pueda venir en un futuro.

- Introducir dos nuevas implicaciones que enriquezcan nuestro marco de trabajo. El primer tipo de implicaciones, llamadas implicaciones débiles, son aquellas que se cumplen con la información presente. Sin embargo, estas pueden dejar de cumplirse cuando recibamos nueva información. Incluso, podrían aparecer nuevas según la información se va completando. Por otro lado, el segundo tipo de implicaciones, llamadas implicaciones fuertes, son aquellas que se cumplen no solo con la información disponible, sino que se mantiene cuando recibimos nueva información.

- Una parte significativa de este trabajo es el desarrollo de diferentes sistemas axiomáticos que nos permitan desarrollar un método de razonamiento automático para los nuevos tipos de implicaciones mencionados anteriormente. Por supuesto, debemos probar que los sistemas axiomáticos son correctos y completos. Nuestro objetivo principal es la extensión del paradigma de simplificación, que se ha definido para diferentes marcos de trabajo, a estas nuevas implicaciones. En particular, construiremos métodos que están bastantes influenciados por la simplificación: conseguir una reducción de las implicaciones mientras preservamos su conocimiento completo.

Estructuración de la tesis

Una vez que hemos introducido el problema, las investigaciones previas y los principales objetivos de esta tesis doctoral, vamos a introducir la estructuración de ésta y los aportes principales de cada capítulo.

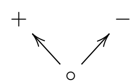
Preliminares

En esta sección describimos las definiciones preliminares y resultados previos que facilitarán la comprensión y la lectura de esta tesis doctoral. Entre estos resultados se encuentran nociones algebraicas (como, por ejemplo, orden, retículo completo,conexión de Galois, sistemas de clausura,etc.) y nociones básicas del análisis de conceptos formales (como, por ejemplo, contexto formal, concepto formal, implicaciones de atributos, etc.)

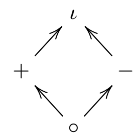
Primeros pasos para extender FCA para considerar información desconocida

En esta sección presentamos la estructura de los valores de verdad que vamos a considerar para trabajar con información positiva, negativa y desconocida. En particular, se presentarán tres valores y se considerará que la información desconocida es menor que la positiva y negativa, siendo estas incomparables (ver Figura 1a). A continuación, observamos que necesitamos un nuevo valor para marcar la contradicción, es decir, tener, a la misma vez, información positiva e información negativa sobre el mismo asunto. Este valor será denotado por ι y quedará como el supremo de la información positiva y negativa (ver Figura 1b).

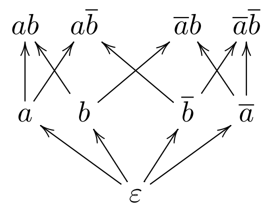
Por otro lado, usando conjuntos trivaluados y la extensión punto a punto del orden anterior, daremos una semirretículo que contendrá, dado un conjunto de atributos, todas las posibles combinaciones que podemos



(a) $\wedge$-semirretículo $\underline{\mathbf{3}}$



(b) El retículo $\underline{\mathbf{4}}$



(a) El $\wedge$-semirretículo $\underline{\mathbf{3}}^{\{a,b\}}$

encontrar con los tres valores. Por ejemplo, considerando el conjunto de atributos $M = \{a, b\}$ el semirretículo de la Figura 2a sería el que contiene todas las opciones. En él, a representa que tenemos el atributo a tiene el valor positivo y b desconocido, mientras que $\overline{a}b$ muestra que tenemos valor negativo para a y positivo para b.

Como de una información contradictoria podemos deducir cualquier información, vamos a unificar con un único símbolo cualquier situación que contenga, al menos, una contradicción a través de un operador de clausura. Este símbolo será $\dot{\iota}$. Denotaremos por $\dot{\mathbf{3}}^M$ a todos los elementos que nos devuelve este operador de clausura, es decir, todos los elementos que no tienen contradicción y $\dot{\iota}$.

Finalmente, introducimos los "contextos formales parciales" que son los contextos formales donde no toda la información es conocida. Para definir estos contextos es necesaria toda la estructura algebraica previamente definida. Podemos ver un ejemplo de contexto formal parcial en la figura 3.

Estos contextos formales parciales se relacionan con los contextos formales clásicos pudiendo construir uno del otro y viceversa. Estos resultados se publicaron en [56] y se presentaron en el 16º Congreso internacional de conceptos de análisis formales (ICFCA) , en julio de 2021.

$\mathbb{P}$	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$+$	$+$
3	$-$	$-$	$\circ$

Figure 3: Contexto formal parcial $\mathbb{P}$

Conceptos necesarios e implicaciones débiles

En esta sección introducimos la base para poder trabajar con el análisis de conceptos formales: la conexión de Galois. Como hemos comentado anteriormente, en esta tesis, vamos a definir dos conexiones de Galois diferentes. La primera de ella captura la información presente en el momento actual y viene dada por el Teorema 4.1.1 que nos asegura que:

Dado un contexto formal parcial $\mathbb{P} = (G, M, I)$ y los operadores $(\ )^{\uparrow} \colon \mathbf{2}^G \to \dot{\mathbf{3}}^M$ y $(\ )^{\downarrow} \colon \dot{\mathbf{3}}^M \to \mathbf{2}^G$ definidos por

$$X^{\uparrow} = \bigwedge_{g \in X} I(g,\ ), \quad y \quad Y^{\downarrow} = \mathrm{Pos}(Y)^{+} \cap \mathrm{Neg}(Y)^{-},$$

Donde, $\bigwedge$ denota el ínfimo de los valores y $^+$ y $^-$ son operadores de derivación de contextos que contienen, solamente, la información positiva y negativa (respectivamente). Se cumple que el par $(\uparrow, \downarrow)$ forma una conexión de Galois entre los retículos $\underline{\mathbf{2}}^G$ y $\underline{\dot{\mathbf{3}}}^M$.

Con esta conexión de Galois introduciremos los "conceptos necesarios" que son los conceptos que, de alguna manera, se mantienen cuando recibimos nueva información. Estos conceptos son aquellos pares $(A, B) \in \mathbf{2}^G \times \dot{\mathbf{3}}^M$ que cumplen que $A^{\uparrow} = B$ and $B^{\downarrow} = A$.

En este mismo capítulo presentamos las "implicaciones débiles" que son implicaciones que son ciertas con la información disponible en el contexto formal parcial pero que, cuando recibimos nueva información, podrían cambiar, es decir, algunas de las que son ciertas podrían dejar de ser ciertas y viceversa, algunas que no lo son podrían ser ciertas con la nueva información. Estas implicaciones se pueden capturar usando la conexión de Galois anterior. Dado un conjunto de atributos M y $A, B\dot{\mathbf{3}}^M$ se define una implicación débil como la siguiente expresión: $A \rightsquigarrow B$.

Finalmente, definimos un sistema axiomático basado en los axiomas de Armstrong para implicaciones débiles, el cual, considera un axioma y dos reglas de inferencia. Que son las siguientes: para todo $A, B, C \in \underline{\dot{\mathbf{3}}}^M$,

`[Inc]` Inclusión: Inferimos $AB \rightsquigarrow A$.

`[Augm]` Aumento: De $A \rightsquigarrow B$ inferimos $AC \rightsquigarrow BC$.

`[Trans]` Transitividad: De $A \rightsquigarrow B$ y $B \rightsquigarrow C$ inferimos $A \rightsquigarrow C$.

Probamos que este sistema axiomático es correcto y completo para las implicaciones débiles. Estos resultados se publicaron en [56] y se presentaron en el 16º Congreso internacional de conceptos de análisis formales (ICFCA) , en julio de 2021.

Álgebra de Heyting completa dual débil

En esta sección observamos la evolución de las estructuras necesarias para poder definir una lógica de Simplificación. Cuando se definió por primera vez, la lógica de Simplificación [53] se basaba en un álgebra de Boole de conjuntos. Sin embargo, en [6] se extendió la lógica de Simplificación al marco teórico difuso usando una álgebra de Heyting completa dual. En nuestro marco de trabajo, podemos observar (véase, de nuevo, la figura 3) que podemos encontrar estructuras que no permiten una álgebra de Heyting completa dual (si completamos esa figura con un elemento máximo, el retículo que obtenemos no es distributivo), por tanto, necesitamos una estructura que tenga algunas propiedades de las álgebras de Heyting completas dual, aunque no las tenga todas. A esta estructura la llamaremos álgebra de Heyting completa dual débil y la denotaremos por wcdHa por sus siglas en inglés.

Dado un retículo completo $(L, \leqslant)$, se dirá que ese retículo junto a una función $\setminus\colon L \times L \to L$ (que llamaremos diferencia) es un wcdHa si se cumplen las siguientes condiciones:

`[wH1]` $x \vee y \neq \top$ implica que $(x \vee y) \setminus x \leqslant y$, para todo $x, y \in L$.

`[wH2]` $x \setminus y \leqslant x$, para todo $x, y \in L$.

`[wH3]` $x \setminus y = \bot$ si y solo si $x \leqslant y$, para todo $x, y \in L$.

`[wH4]` $x \vee y = x \vee (y \setminus x)$, para todo $x, y \in L$.

Donde $\top$ y $\bot$ denotan el máximo y el mínimo del retículo y $\wedge$ y $\vee$ los operadores ínfimo y supremo, respectivamente.

Finalmente, en este capítulo, realizamos un estudio sobre esta estructura que hemos definido dando, entre otras cosas, la siguiente caracterización de esta:

Dados un retículo completo $(L, \leqslant)$ y una operación diferencia $\setminus\colon L \times L \to L$. Diremos que $(L, \leqslant, \setminus)$ es un wcdHa si y solo si las siguientes condiciones se cumplen:

$$x \smallsetminus y = \min\{z \mid z \vee y = x \vee y\} \text{ para todo } x, y \in L \text{ con } x \not\parallel y \text{ y } x \neq \top \quad (1)$$

$$\top \smallsetminus \top = \bot. \quad (2)$$

$$\top \smallsetminus y \in \{z \mid z \vee y = \top\} \text{ para todo } y \in L \text{ con } y \neq \top. \quad (3)$$

$$x \smallsetminus y \in \{z \mid z \leqslant x \text{ y } z \vee y = x \vee y\} \text{ para todo } x, y \in L \text{ con } x \parallel y. \quad (4)$$

Donde, $x \not\parallel y$ indica que x es comparable con y, es decir, que o bien $x \leq y$ o bien $y \leq x$ mientras que $x \parallel y$ indica lo contrario, es decir que, $x \nleq y$ y que $y \nleq x$.

Estos resultados se publicaron en [58] y un estudio más exhaustivo del wcdHa se ha sometido al " International Journal of Intelligent Systems".

Lógica de Simplificación para implicaciones débiles

En esta sección presentamos dos sistemas axiomáticos basados en el paradigma de simplificación.

El primer sistema axiomático se llama *sistema axiomático de Simplificación* y es $\{[\texttt{Inc}], [\texttt{Key}], [\texttt{Simp}]\}$ donde $[\texttt{Inc}]$, $[\texttt{Key}]$ y $[\texttt{Simp}]$ se definen, respectivamente, como sigue: para todo $A, B, C, D \in \dot{\mathbf{3}}^M$ y todo para todo $b \in \dot{\mathbf{3}}^M$ unitario,

Inclusión: Inferimos $AB \rightsquigarrow A$,

Clave: De $A \rightsquigarrow b$ inferimos $A\bar{b} \rightsquigarrow i$,

Simplificación: De $A \rightsquigarrow B$ y $C \rightsquigarrow D$ inferimos $A(C \smallsetminus B) \rightsquigarrow D$,

La segunda propuesta reemplaza la regla de inferencia "clave" por una versión clásica de la regla Unión.

Llamamos *sistema axiomático* $\texttt{U}$*-simplificación* a $\{[\texttt{Inc}], [\texttt{Simp}], [\texttt{Un}]\}$ donde $[\texttt{Un}]$ se define como sigue: para todo $A, B, C \in \dot{\mathbf{3}}^M$,

Unión: De $A \rightsquigarrow B$ y $A \rightsquigarrow C$ inferimos $A \rightsquigarrow BC$.

A continuación, usando una wcdHa probamos el teorema 6.1.4 que demuestra que estos dos sistemas axiomáticos son equivalentes al sistema axiomático inspirado en los axiomas de Armstrong para implicaciones débiles definido anteriormente.

Sea M un conjunto de atributos, $\Sigma \subseteq \mathcal{L}_M$ y $A \rightsquigarrow B \in \mathcal{L}_M$. Tenemos que

$$\Sigma \vdash_{\mathbb{A}} A \rightsquigarrow B \quad \textit{si y solo si} \quad \Sigma \vdash_{\mathbb{S}_{\mathbb{U}}} A \rightsquigarrow B \quad \textit{si y solo si} \quad \Sigma \vdash_{\mathbb{S}} A \rightsquigarrow B.$$

Como consecuencia, tenemos que los dos sistemas axiomáticos basados en el paradigma de simplificación son correctos y completos. A continuación, en el teorema 6.2.2, probamos que las reglas de inferencia de los sistemas de Simplificación son, de hecho, reglas de equivalencias:

Las siguientes reglas de equivalencias se tienen: para todo $A, B, C, D \in \dot{\mathbf{3}}^M$,

`[FragEq]`: $\{A \rightsquigarrow B\} \equiv \{A \rightsquigarrow B \setminus A\}$.

`[UnEq]`: $\{A \rightsquigarrow B,\ A \rightsquigarrow C\} \equiv \{A \rightsquigarrow BC\}$.

`[ε-Eq]`: $\{A \rightsquigarrow \varepsilon\} \equiv \varnothing$.

`[ί-Eq]`: $\{A \rightsquigarrow \acute{\imath}\} \equiv \{A \rightsquigarrow \acute{\imath}\}$ *cuando* $A \sqcup B = \acute{\imath}$.

`[SimpEq]`: $\{A \rightsquigarrow B,\ C \rightsquigarrow D\} \equiv \{A \rightsquigarrow B,\ C \setminus B \rightsquigarrow D \setminus B\}$ *cuando* $A \sqsubseteq C \setminus B$.

Para finalizar, usando estas reglas de equivalencias definimos el algoritmo 1 que nos comprueba si una implicación débil puede ser inferida, en tiempo polinomial, de un conjunto de implicaciones débiles dado.

Todos estos resultados fueron publicados en [58].

Algorithm 1: `Cierre sintáctico de` A `con respecto a` Σ

Input: Σ siendo un conjunto de implicaciones débiles, A siendo un conjunto de $\mathbf{3}^U$

Output: $[A]_\Sigma$

repeat

  $\Sigma_{\texttt{old}} := \Sigma;\ \Sigma := \varnothing$

  foreach $B \rightsquigarrow C \in \Sigma_{\texttt{old}}$ **do**

    $B_{\texttt{new}} := B \smallsetminus A;\ C_{\texttt{new}} := C \smallsetminus A$ `// Por [SimpEq]`

    if $B_{\texttt{new}} = \varepsilon$ **then**

      $A := A \sqcup C_{\texttt{new}}$ `// Por [UnEq]`

    else if $C_{\texttt{new}} \not\sqsubseteq B_{\texttt{new}}$ **then**

      Add $B_{\texttt{new}} \rightsquigarrow (C_{\texttt{new}} \smallsetminus B_{\texttt{new}})$ to Σ `// Por [FragEq] y [ε-Eq]`

  end

until $\Sigma = \Sigma_{\texttt{old}}$ *or* $A = \imath$

return A

Conceptos posibles e implicaciones fuertes

En este capítulo presentamos la segunda conexión de Galois para trabajar con información positiva, negativa y desconocida. Esta segunda conexión de Galois captura toda la información posible, es decir, la presente y la que podría llegar a venir. Para poder definir esta conexión de Galois, previamente necesitamos introducir un orden entre diferentes contextos parciales. Dicho orden se presenta a continuación:

Dados dos contextos formales parciales $\mathbb{P}_1 = (G_1, M_1, I_1)$ y $\mathbb{P}_2 = (G_2, M_2, I_2)$, decimos que $\mathbb{P}_1$ es un *refinamiento* de $\mathbb{P}_2$ (denotado por $\mathbb{P}_1 \preceq \mathbb{P}_2$) si

$$G_1 \subseteq G_2,\ M_1 = M_2,\ \text{ y } I_2(g,\ ) \sqsubseteq I_1(g,\ ) \text{ para todo } g \in G_1 \tag{5}$$

En la figura 4, presentamos una cadena de contextos formales parciales

$$\begin{array}{|c||c|c|c|} \hline & a & b & c \\ \hline 1 & + & \circ & - \\ \hline 2 & \circ & \circ & + \\ \hline 3 & - & - & \circ \\ \hline \end{array} \succeq \begin{array}{|c||c|c|c|} \hline & a & b & c \\ \hline 1 & + & \circ & - \\ \hline 2 & + & \circ & + \\ \hline \end{array} \succeq \begin{array}{|c||c|c|c|} \hline & a & b & c \\ \hline 1 & + & - & - \\ \hline 2 & + & - & + \\ \hline \end{array} \succeq \begin{array}{|c||c|c|c|} \hline & a & b & c \\ \hline 1 & + & - & - \\ \hline \end{array}$$

Figure 4: Una cadena de contextos formales parciales.

Dado un contexto formal parcial $\mathbb{P}_0$, demostramos que el conjunto de todos los contextos parciales que son un refinamiento de él (denotado por $\mathfrak{P}(\mathbb{P}_0)$) forman un retículo completo. Con esto, dado un contexto parcial $\mathbb{P}_0 = (G_0, M, I_0)$, definimos los operadores de derivación de conceptos como sigue:

- $^{\Uparrow} : \mathfrak{P}(\mathbb{P}_0) \to \dot{\mathbf{3}}^M$ que asigna a $\mathbb{X} = (G, M, I) \in \mathfrak{P}(\mathbb{P}_0)$ con $\mathbb{X}^{\Uparrow} = \prod_{g \in G} I(g,\ )$.
- $^{\Downarrow} : \dot{\mathbf{3}}^M \to \mathfrak{P}(\mathbb{P}_0)$ que asigna cualquier $\dot{\mathbf{3}}$-conjunto $A \in \dot{\mathbf{3}}^M$ a $A^{\Downarrow} = (G, M, I)$ donde $G = \{g \in G_0 \colon I_0(g,\ ) \sqcup A \neq i\}$ y $I(g,\ ) = I_0(g,\ ) \sqcup A$, para cada $g \in G$.

Este par de operadores generan una conexión de Galois que nos permiten definir los conceptos necesarios que son pares $(\mathbb{X}, Y) \in \mathfrak{P}(\mathbb{P}_0) \times \dot{\mathbf{3}}^M$ con $\mathbb{X}^{\Uparrow} = Y$ y $Y^{\Downarrow} = \mathbb{X}$. Estos conceptos son verdad sólo con la información disponible, es decir, cuando recibamos nueva información estos conceptos podrían dejar de mantenerse.

Por otro lado, usando también esta conexión de Galois, definimos las implicaciones fuertes que son implicaciones que son ciertas no sólo con la información disponible sino también cuando aparece nueva información. A continuación, usando los contextos parciales y su orden, se introduce la noción de modelo de una implicación fuerte (Definición 7.3.2):

Sea $\mathbb{P} = (G, M, I)$ un contexto parcial y $A \Rightarrow B$ una implicación fuerte, decimos que $\mathbb{P}$ es modelo de $A \Rightarrow B$, si $A \sqsubseteq \{\mathbb{X}_i\}^{\Uparrow}$ implica que $B \sqsubseteq \{\mathbb{X}_i\}^{\Uparrow}$ para todo $\mathbb{X}_i \in \mathfrak{P}(\mathbb{P})$.

También debemos modificar los sistemas axiomáticos para que sean completos y correctos para las implicaciones fuertes. En este caso, introduciremos un sistema axiomático basado en los axiomas de Armstrong y otro basado en el paradigma de simplificación. A continuación presentamos ambos:

El sistema axiomático basado en Armstrong para implicaciones débiles, considera $\{[[\mathtt{Inc}]], [[\mathtt{Augm}]], [[\mathtt{Trans}]], [[\mathtt{Rft}]], [[\mathtt{Tru}]]\}$ que son reglas conocidas como *inclusión*, *aumento*, *transitividad*, *reflexión* y *confianza* respectivamente,y se define para todo $A \in \dot{\mathbf{3}}^M$ y todo conjunto unitario $a, b \in \mathbf{3}^M$,

$[[\mathtt{Inc}]]$ Inferimos $AB \Rightarrow A$.

$[[\mathtt{Augm}]]$ De $A \Rightarrow B$ inferimos $AC \Rightarrow BC$.

$[[\mathtt{Trans}]]$ De $A \Rightarrow B$ y $B \Rightarrow C$ inferimos $A \Rightarrow C$.

$[[\mathtt{Rft}]]$ De $Aa \Rightarrow b$ inferimos $A\overline{b} \Rightarrow \overline{a}$

$[[\mathtt{Tru}]]$ De $a \Rightarrow \overline{a}$ inferimos $\varepsilon \Rightarrow \overline{a}$

El *sistema axiomático de simplificación para implicaciones fuertes* considera

$\{[[\texttt{Inc}]], [[\texttt{Key}]], [[\texttt{Simp}]], [[\texttt{InKy}]]\}$ que son reglas conocidas como *inclusión*, *clave*, *simplificación* y *clave inversa* respectivamente, y se definen como sigue: para todo $A \in \dot{\mathbf{3}}^M$ y todo $b \in \mathbf{3}^M$ unitario,

[[Inc]] Inferimos $AB \Rightarrow A$.

[[Key]] De $A \Rightarrow b$ inferimos $A\bar{b} \Rightarrow \imath$.

[[Simp]] De $A \Rightarrow B$ y $C \Rightarrow D$ inferimos $A(C \smallsetminus B) \Rightarrow D$.

[[Inky]] De $Ab \Rightarrow \imath$ inferimos $A \Rightarrow \bar{b}$.

Finalmente, el teorema 7.4.3 asegura que ambos sistemas axiomáticos son equivalentes:

Sea M un conjunto de atributos , $\Sigma \subseteq \mathcal{L}_M^s$ y $A \Rightarrow B \in \mathcal{L}_M^s$. Tenemos que

$$\Sigma \vdash_{\mathcal{A}_s} A \Rightarrow B \quad \textit{si y solo si} \quad \Sigma \vdash_{\mathcal{S}_s} A \Rightarrow B$$

En los teoremas 7.5.1 y 7.5.7 probamos la corrección y completitud (para el caso finito), respectivamente, de ambos sistemas axiomáticos.

Para toda implicación fuerte $A \Rightarrow B \in \mathcal{L}_M^s$ y todo conjunto $\Sigma \subseteq \mathcal{L}_M^s$, tenemos que

$$\Sigma \vdash A \Rightarrow B \quad \textit{si y solo si} \quad \Sigma \models A \Rightarrow B.$$

Por último, en el teorema 7.6.1 se demuestra que, como viene siendo habitual con el paradigma de la lógica de Simplificación, las reglas de inferencia pueden ser vistas como reglas de equivalencias.

Las siguientes reglas de equivalencia se cumplen: para todo $A, B, C, D \in \dot{\mathbf{3}}^M$ y todo $b \in \mathbf{3}^M$ unitario, tenemos que:

[[FragEq]] $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \smallsetminus A\}$.

[[$\texttt{UnEq}$]] $\{A \Rightarrow B,\ A \Rightarrow C\} \equiv \{A \Rightarrow BC\}$.

[[ε-$\texttt{Eq}$]] $\{A \Rightarrow \varepsilon\} \equiv \varnothing$.

[[$\imath$-$\texttt{Eq}$]] $\{A \Rightarrow B\} \equiv \{A \Rightarrow \imath\}$ *cuando* $A \sqcup B = \imath$.

[[$\texttt{SimpEq}$]] $\{A \Rightarrow B,\ C \Rightarrow D\} \equiv \{A \Rightarrow B,\ C \smallsetminus B \Rightarrow D \smallsetminus B\}$ *cuando* $A \sqsubseteq C \smallsetminus B$.

[[$\texttt{RdEq}$]] $\{Ab \Rightarrow C, A\overline{b} \Rightarrow C\} \equiv \{A \Rightarrow C\}$ *cuando* $\mathrm{Spp}(C)$ *es finito.*

[[$\texttt{KyEq}$]] $\{A \Rightarrow b\} \equiv \{A\overline{b} \Rightarrow \imath\}$.

El estudio de esta última conexión de Galois fue publicado en [57] y presentada en el 16º congreso de retículo de conceptos y sus aplicaciones (CLA) en Junio de 2022.

Trabajos futuros

En esta tesis doctoral, consideramos tres posibles valores (positivo, negativo y desconocido). Sin embargo, en nuestro mundo real hay muchos más valores posibles. Además, existen valores que, solamente con estos tres valores no se pueden capturar correctamente, por ejemplo, el atributo ser alto. Así, si decimos que una persona es alta si mide más de 1.80 metros, ¿qué podríamos decir de una persona que mide 1.79? Si decimos que no es alta estamos perdiendo información ya que, aunque no podemos catalogarla como alta, está mucho más cerca de serlo que una persona que mide 1.20 metros.

Para estos casos, trabajaremos en una extensión para considerar, en vez de solo tres valores, infinitos valores. Es decir, trabajar en un marco difuso. Más en concreto, trabajaremos con pares de valores difusos donde el primer valor dirá la información positiva que tenemos sobre el atributo, mientras que la segunda nos devolverá la información negativa sobre el atributo, es decir, el valor $(1,0)$ se corresponderá con nuestro valor $+$, el valor $(0,1)$ con nuestro valor $-$ y el $(0, 0)$ con nuestro desconocido. Finalmente, estamos interesados en diferenciar valores consistentes de valores inconsistentes como, por ejemplo, el valor $(1, 1)$ y, tal y como hemos hecho aquí, usaremos un sólo símbolo para denotar todos aquellos que no son consistentes. Para esto, usaremos conjuntos de Atanassov y combinaremos nuestro trabajo con la línea de investigación de Jan Koneckny [45]. El trabajo presentado en esta tesis doctoral puede ser el puente necesario para movernos desde el caso particular presentado en [58] a un marco de trabajo más general.

Por otro lado, también queremos considerar otras interpretaciones para conseguir un marco de trabajo único donde razonar con información desconocida. Para ello, combinaremos nuestro trabajo con la línea de investigación abierta en [28] para definir métodos de razonamiento automáticos para lógicas disyuntivas.

Por otro lado, queremos ampliar nuestra investigación para conseguir el marco formal para cada situación. En particular, nos centramos en las wcdHa. Sobre este asunto, es interesante estudiar qué wcdHa son algebraicas y cuales no. También es interesante buscar si, aparte de la importancia que tiene en este trabajo, esta estructura podría ser útil en otras áreas o en otros problemas.

Finalmente, sobre la aplicación práctica de estos resultados, hay aún un gran trabajo por hacer. De hecho, poder trabajar con tres valores nos aporta ventajas con respecto a trabajar con solo dos valores. Queremos extender este trabajo definiendo sistemas de recomendación, siguiendo la línea marcada por trabajos previos [17, 21]. El sistema de recomendación incorporará la posibilidad de trabajar con información desconocida, por ejemplo, cuando un usuario no aporta información acerca de su estancia en un hotel (sobre si se ha sentido cómodo o no, si le ha gustado la limpieza, etc.), o cuando no ha visto una película y, por tanto, no puede dar su opinión sobre ella. Para la implementación de este sistema de recomendación, como paso previo, queremos incorporar nuestros resultados al paquete fcaR [18], un paquete para el lenguaje R que ha sido implementado por nuestro grupo de investigación y que incluye todos los métodos y algoritmos de FCA para diferentes extensiones. En particular, planeamos definir un nuevo operador de clausura de implicaciones de atributos para esta nueva extensión, siguiendo nuestro orden usual: necesitamos probar que las reglas de inferencias se pueden ver como reglas de equivalencias y, después, definir un algoritmo basado en estas equivalencias que permita una construcción iterativa del conjunto cerrado y, al mismo tiempo, realizar una reducción del tamaño del conjunto de implicaciones. El primer paso ha sido completado en esta tesis doctoral, mientras que el segundo ha sido iniciado pero no completado, para ello, es necesario un estudio más profundo de la interpretación disyuntiva.

Publications

1. F. Pérez-Gámez, P. Cordero, M. Enciso, Á. Mora. A New Kind of Implication to Reason with Unknown Information. Proceedings of *The 16th International Conference on Formal Concept Analysis, ICFCA 2021*, Strasbourg, France, June 29–July 2, 2021, (pp. 74-90). Cham: Springer International Publishing.

2. F. Pérez-Gámez; D. López-Rodríguez; P. Cordero; Á. Mora ; M. Ojeda-Aciego. Simplifying Implications with Positive and Negative Attributes: A Logic-Based Approach. *Mathematics*, 2022.

   Journal impact in J.C.R in the year 2021: 2.592, 21° position of 333 **(Q1)** in the category Mathematics.

3. F. Pérez-Gámez, P. Cordero, M. Enciso, Á. Mora, M. Ojeda-Aciego. Partial formal contexts with degrees. Procedings of *The* 16^{th} *International Conference on Concept Lattices and Their Applications, CLA 2022, Tallinn, Estonia, June 20–22, 2022*, pp. 35–44.

4. F. Pérez-Gámez, P. Cordero, M. Enciso, Á. Mora. A Galois connection between partial formal contexts and attribute sets. Procedings of *The* 16^{th} *International Conference on Concept Lattices and Their Applications, CLA 2022, Tallinn, Estonia, June 20–22, 2022*, pp. 45–55.

5. F. Pérez-Gámez, P. Cordero, M. Enciso , D. López-Rodríguez , Á. Mora.

Computing the Mixed Concept Lattice. Procedings of *Information Processing and Management of Uncertainty in Knowledge-Based Systems: 19th International Conference, IPMU 2022, Milan, Italy, July 11–15, 2022, Proceedings, Part I* (pp. 87-99). Cham: Springer International Publishing

6. F. Pérez-Gámez, P. Cordero, M. Enciso, Á. Mora. Simplification logic for the management of unknown information. *Information Sciences*, 634, 505-519.

   Journal impact in J.C.R in the year 2021: 8.233, 16º position of 164 **(Q1)** in the category Computer sciences, Information systems.

7. F. Pérez-Gámez, P. Cordero, M. Enciso, Á. Mora, M. Ojeda-Aciego. Analisis de conceptos formales bajo una visión intuicionista. *Actas del XXI Congreso de Tecnologías y Lógica Fuzzy (ESTYLF'22)*, 5-7 September, 2022, Toledo.

8. F. Pérez-Gámez, P. Cordero, M. Enciso, Á. Mora, M. Ojeda-Aciego. Grading the unknown information via intuitionistic approach. *Book of abstracts of* 14^{th} *European Symposium on Computational Intelligence and Mathematics*, 2-5 October, Naples, Italy

Chapter 1

Introduction

Nowadays we have available more information than ever. In the data era, some issues continue to be addressed: the management of big data, the extraction of useful knowledge from the data, ensuring data privacy, the development of further intelligent methods to process the data, understanding of the unknown information also stored in the data, etc. This thesis focused on this last problem by using a solid mathematical background and, particularly, a new logical framework, covering the three standard pillars: syntax, semantics and axiomatic system.

When we work with data stored in data frames, the unknown information appears as blank cells, and usually, those blank cells are not considered. Furthermore, information is dynamic and is usually in continuous changes, i.e. what is unknown today may be known in the future and vice-versa. This situation leads us to the main question of this work: how to manage those blank cells and how to deal with them when the information changes. The following question comes to our mind: we have to consider not only the situation of the blank cells themselves but also the impact on their relationships. In particular, we focus on how data is related by means of the well-known notion of binary relationships defined in different areas with different names and definitions: if-then rule, functional dependency, association rule, attribute implication, etc. In the following example, we illustrate how unknown information strongly impacts on these definitions and notions.

We are developing some tests for several patients in a hospital. The patients are tested for fever and headache. The first patient has both symptoms whereas the second one doesn't present any of them. There is a third patient with fever and no information about the second symptom. Focusing on these three patients, we can infer two implications: (1) "Headache implies Fever" and (2) "No Headache implies No Fever". Remark that these implications behave in a different way regarding the unknown information of the third patient. Implication (1) remains true whatever future information about the third patient having or not headache. Implication (2) is now true since, with the current information, no counterexample exists. However, if the third person informs us that he has no headache, then the implication becomes false. In some way, we can say that the first implication is stronger than the second one, which conversely is weaker than the first one.

1.1 Problem statement

This work is mainly related to Formal Concept Analysis (FCA), a discipline first introduced by Rudolf Wille in 1984 [70]. It provides techniques and tools to discover knowledge from raw data, representing the discovered knowledge and, finally, reasoning with it.

In the first representation, notions (or concepts) arise from the data and an order structure (a lattice) is provided over them. The second way to represent the knowledge, attribute implication, is similar to some if-then rules appearing in other areas. The main advantage of attribute implications is that it allows symbolic manipulation and reasoning.

In addition, FCA provides a solid collection of algorithms and techniques to extract knowledge from the data, building the concept lattice and the attribute implications in an automatic and efficient way. In addition, they also allow building the concept lattice from the attribute implications and vice versa.

Classical FCA only deals with positive information, that is, the information that it is given. However, sometimes we can extract some knowledge from some not given information. For instance, when we have that a light is not switched on we can assume that it is switched off. To extract this knowledge while remaining in classical FCA, Rokia Missaoui [52] proposed to duplicate the dataset by considering two versions of each column, representing the positive and negative evidence of this property. This solution tends to be inefficient as it gets redundancy in the implications and we are duplicating the number of columns, inducing a worst computational behavior when extracting the knowledge and managing its further representation. In [65] Rodríguez Jiménez introduced an alternative way to also manage with the negative information. He presented the Mixed formal context which is a dataset containing positive and negative information. He consistently defines new implications and concepts extending the classic definitions (just with positive information) to a mixed framework (positive and negative information).

However, from our point of view, there are some circumstances that can not be fulfilled just with positive and negative information, and unknown information has to be considered. Let's see some examples where this information appears. The first example is when we are working with patients in a hospital and we have a male patient if one of the attributes that we have to take into consideration is whether or not the patient has a regular period, because it does not have sense. Another example is when we have reviews about a hotel and there is some information that the customer does not want to fill in (maybe because he has not used that service). It is obvious that we cannot assume that the review is positive (that is, the customer likes that service) or consider that the review is negative (that is, the customer does not like that service). One last example is when the information is not available at the present instant. For example, in a data frame about students, if one exam is scheduled next week, we can not have yet the information about the grades.

To capture the unknown information together with the positive and negative one we need to consider a new value which means unknown and we have to study how to deal with it. With this new value, in particular, new concepts and implications appear considering positive, negative and unknown values.

1.2 State of the art

Here, we survey previous works that have addressed the treatment of unknown information stored in data frames. As far as we know, the pioneers works to address this problem in the database area were presented in the 70's, strongly related with others that consider the logic point of view. Later, in the 80's, the same problem was also addressed in the area of formal concept analysis, influenced by the previously mentioned works. In the following, we briefly summarize these different approaches.

1.2.1 Database approaches

In databases, the study of missing or partial information was traditionally considered a key problem since the relational model was introduced.

In the late 1970s, several authors addressed, almost simultaneously, the treatment of unknown values [14, 49, 69] with different approaches. In the following, we present them in chronological order.

Lipski [49] presented a general theory for working with incomplete information. He describes two different cases that enclose unknown information: the case in which everything is known and the case in which nothing is known. Thus, when a table with stored unknown information is queried, the known information renders true or false whereas the unknown information is not true or false but it is possible. Thus, to fully complete the knowledge we need to study all the possible cases for the unknown information and, consequently, efficiency takes action because of the computational complexity of the study of all the possible completions in the table.

Vassiliou [69] distinguished between two cases of unknown information: the case where the value is missing and the case where it has no sense to have information about the value. For instance, in a table, if one person declared to be single it has no sense to write the name of the couple. Thus, it is necessary to have two different symbols one for the missing information and other for the so-called *inconsistent information*. In addition, Vassiliou also claimed that when a query in a database involves an inconsistent value for an object then the answer has to be *inconsistent*, thus, Vassiliou considered four truth values: *true*, *false*, *unknown* and *inconsistent*. Finally, he also introduced two operators "**AND**" and "**OR**" providing their corresponding truth tables (see Table 1.1 and Table 1.2 respectively).

In this same period, the work that most influenced data management was due to Codd [14], mainly because of its integration with the relational data model in the field of databases. Codd considered the so-called *"null value"*, to represent the unknown information. In order to perform database queries he extended the bivalued logic by incorporating this unknown value and defining the corresponding operators as Table 1.3 shows. Codd also defined the so-called relational algebra, inspired by his three-valued logic, as a formal underlying framework for the query language. In particular, he

AND	*true*	*false*	*unknown*	*inconsistent*
true	*true*	*false*	*unknown*	*inconsistent*
false	*false*	*false*	*false*	*inconsistent*
unknown	*unknown*	*false*	*unknown*	*inconsistent*
inconsistent	*inconsistent*	*inconsistent*	*inconsistent*	*inconsistent*

Table 1.1: “**AND**” operator in Vassiliou view

OR	*true*	*false*	*unknown*	*inconsistent*
true	*true*	*true*	*true*	*inconsistent*
false	*true*	*false*	*unknown*	*inconsistent*
unknown	*true*	*unknown*	*unknown*	*inconsistent*
inconsistent	*inconsistent*	*inconsistent*	*inconsistent*	*inconsistent*

Table 1.2: “**OR**” operator in Vassiliou view

discusses the true value of the comparison “$x = y$" when x or y, or both of them, are null values. His proposal is a three-valued logic to manage the NULL value, defining the truth tables for the “AND”, “OR”, and “NOT” operators (Table 1.3).

AND	*true*	*false*	*null*
true	*true*	*false*	*null*
false	*false*	*false*	*false*
null	*null*	*false*	*null*

OR	*true*	*false*	*null*
true	*true*	*true*	*true*
false	*true*	*false*	*null*
null	*true*	*null*	*null*

	NOT
true	*false*
false	*true*
null	*null*

Table 1.3: “**AND**”, “**OR**” and “**NOT**” operators in Codd view.

C.J. Date [23] supports the idea of Vassiliou about having more than one meaning for the NULL value. Actually, he claims that there could be different meanings for the NULL value with different properties treated in different ways. As a consequence, he introduced different operators like, for instance, the “MAYBE" operator or the “DUPLICATE" operator.

Nowadays, unknown information in databases still remains an active research topic as [73] stated. Thus, Alattar et al. [1], Console et al. [15], Libkin [48], Greco et al. [38], Guagliardo et al. [39] and Wolf et al. [71] deal with query processing in incomplete databases, Geerts et al [37] focussed on cleaning databases, a hot topic related with data quality for AI methods, Luna Dong et al. [26] studied data integration which received a lot of attention when databases have moved to data lakes and Benjelloun et al. [7] stills go deeper in the model theory.

1.2.2 Logic-based approaches

In this subsection, we are going to consider some of the authors that addressed the unknown information from a logical point of view. In [27] a wide study of this approach is shown. Here, we are going to extract those proposals that are closer to the framework used in this work, which will be presented in the following subsection.

The logicians usually tend to denote the truth value *true* by 1, the truth value *false* by 0, and some of them, use $\frac{1}{2}$ to indicate the truth value *unknown*. These values are ordered as follows: $0 < \frac{1}{2} < 1$.

The first approach to this problem was the Łukasiewicz's work [51], where the unknown value mean *possible*. The $\wedge$ and $\vee$ are, respectively, the minimum and maximum. Łukasiewicz also extended the implication as Table 1.1 shows and negation by considering $p = \neg(p) = \frac{1}{2}$ when p is unknown.

Thus, Łukasiewicz presented a first logical framework breaking some of the classical logic laws, particularly the excluded middle one. The implication also has a particular treatment: when p is certainly known (whether it

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

Figure 1.1: implication table of Łukasiewicz

is true or false) then we assign a proper truth value to the expression $p \rightarrow p$ (true) and $p \rightarrow \neg p$ (false). However, when p is unknown, we have that both $p \rightarrow p$ and $p \rightarrow \neg p$ are true.

Kleene proposed an alternative to this situation. His three-valued logic [43] consider the same three values algebraic framework but considering $\frac{1}{2} \rightarrow \frac{1}{2} = \frac{1}{2}$. In this semantics, the truth values arise from a computation, so that the value $\frac{1}{2}$ is used when the computation fails. Thus, the concept of possibility is replaced by indefiniteness. In a consistent way, in Kleene's approach, $p \vee \neg p$ remains indefinite when p is indefinite (as in Łukasiewicz logic).

Kleene logic can be considered to be an instance of the so-called partial logic. They only considered two truth values and evaluations are defined to be partial functions from language expressions to these truth values set (see [67] and [10] for further information).

Another point of view was given by De Finetti [25] who distinguishes between the veracity of a proposition and our knowledge about it. In the first case, only two truth values are considered, whereas, in the second one, the truth set and its structure are a matter of representation conventions.

Uncertainty or unknown information appears, naturally, in epistemic logic. It follows the De Finetti approach: truth values should be given by the knowledge of an agent, whose beliefs can report unknown information. Some authors approached this situation with a modal logic [40] whereas others [36] defined a belief set for each agent as a deductively closed set of formulas, and the modalities are omitted.

Also in the framework of partial logics, Van Fraassen [68] considered two truth values and introduced the notion of supervaluations. A supervaluation VS over a coherent situation s (a partial evaluation) is a function that assigns to each proposition the super-truth value 1 (0) if the proposition is *true* (*false*) in all the Boolean completions of s.

One of the most cited papers related to the logic with unknown information is Benalp's work [5]. He considered four possible values (*unknown*, *true*, *false* and *contradiction*), and, following the epistemic view, he called these four values "told values". It should be seen as the present state of information, that is, given a sentence P it takes one of these truth values:

True Having information that P is *true* and no information that P is *false*.

False Having information that P is *false* and no information that P is *true*.

Contradiction We have both information: P is *true* and at the same time we have information that P is *false*.

Unknown We have no information about whether P is *true* or P is *false*.

In addition, Belnap offers two different orders between these four values. The first one is induced by the "amount of information" whereas the second one is the so-called truthfulness order. These two orders can be represented as a structure called bilattice [30, 31] which is a lattice that contains two lattices. In Figure 1.2 we see the bilattice for both Benalp's orders: the information order is depicted bottom-up whereas the truthfulness order is represented right to left.

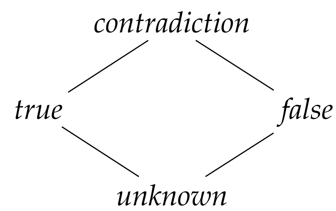


Figure 1.2: Belnap's bilattice

Furthermore, to remark that the study of unknown information or null values in the logic still being an important issue, we rely on [16] where the authors made a study about some questions of the logic with different values.

1.2.3 Formal concept analysis approaches

In this subsection, we summarize the antecedents about unknown information and FCA.

The work [47] could be considered one of the first articles that work with unknown information in the concept lattice. In that paper, Lex introduced a four-values algebra (see Figure 1.3 and its relationship with Figure 1.2) and he also studied how should be the formal concepts when dealing with unknown information.

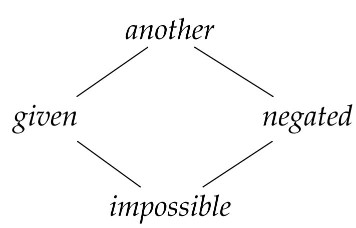


Figure 1.3: Lex's lattice

In [41], Holzer et al. presented the so-called *incomplete formal contexts*:

formal contexts with unknown information. They also presented the so-called *possible intents* and *possible extents* which are intents and extents in one of the completion of the incomplete formal context, i.e. formal context changing the unknown value by positive or negative values, which a similar approach as the one mentioned in the above section presented by [49] for databases.

Furthermore, Holzer et al. also can be considered one of the first approaches to extend the notion of attribute implication to deal with unknown information. They defined the so-called *satisfiable attribute implications*, which are attribute implications that hold in, at least, one completion of the incomplete context, and the *Kripke attribute implications* which are the attribute implications that hold within all the completions of the incomplete formal context. Remark that this approach is strongly related to the illustrative example mentioned above.

In [32], Ganter et al. considered the unknown information to represent an aggregate value of an attribute when some grouping operator is performed in the formal context and the corresponding grouping function for this column does not render a unique value. In that paper, the authors analyze the relationships held in the original formal context by analyzing the implications in the new so-called *partial formal context*, introducing an association rule-like notion.

In [72], the authors work with a formal context with unknown information named three-way formal context. In that paper, the authors not only used the positive information but they also used the negative. However, they did not take both of them at the same level, that is, the positive truth value is bigger than the negative one, as Łukasiewicz did [51]. With this idea, they consistently built the set of completions of a partial formal context, that classical context “containing" the original nonclassical formal context, and considered the pointwise order. Furthermore, in their work, they finally reduce the study by just considering the maximum context (fully completed with positive values) and the minimum one (filled with negative values).

Now, we mention some very recent works strongly related to our line, they also work on formal concept analysis and use three truth values as the ground layer. In [63], the authors considered an algebraic structure very similar to our approach but they just addressed the construction of the lattice of formal concepts. In [4], the authors work with formal concept analysis with fuzzy elements, whose concepts collect positive and negative information to model uncertainty.

The works cited in this subsection focused on the concept lattice. However, we are particularly interested in the symbolic manipulation of unknown information, which completes our view also with attribute implications and reasoning methods. In this line, we remark on the works of [44] where the author presented attribute implications that capture positive and negative information, but they used a more complex ground level, by using fuzzy values. In [28], the authors presented attribute implications but they do not preserve the usual conjunctive interpretation of implications, considering also a disjunction one, which induces a significant growth in the complexity of the reasoning methods.

1.2.4 Knowledge repesentation

Once we have summarized the most outstanding works related to dealing with unknown information, and from different points of view, we discuss here how to represent the unknown information and its meaning. In particular, we consider here the closest approach to our work, being those works considering three or four truth values.

We begin with the meaning of the value representing unknown information. The study described here is based on the paper [12] where Dubois introduced a dichotomy between two meanings: borderline or unknown.

Firstly, he considered the case where the third value is a borderline value. In this case, it is assumed that all the models are complete, that is, each variable is assigned a truth value (*true*, *false* or *borderline*). The last value corresponds to a vague zone between truthfulness and falsity, which can be an unclear border as other areas did, for instance, fuzzy logic or rough set theory. Lawry and González-Rodríguez explored this approach in [46], considering *vague* propositions in Kleene three-valued logics [43]. In their interpretation, $\frac{1}{2}$ means *borderline*, 0 means *clearly false* and 1 means *clearly true*.

The use of the borderline interpretation in some cases is meaningless, for instance, in the sentence "being pregnant". In these cases, the second interpretation is needed and it represents that we have no information.

The second issue of this section is how unknown information can be represented. Remark that the representation is independent of the chosen meaning.

One of the most popular representations is by using three-valued sets [11] which are mappings $\rho : \Omega \to \{0, 1, \frac{1}{2}\}$ where Ω is the set of propositional symbols. Another possibility is to consider the so-called (consistent) orthopair [12] which are couples (P, N) of subsets of Ω where we have that $P \cap N = \varnothing$. The propositional symbols in P are those being true (*Positives*) and symbols in N are those being false (*Negatives*). There is a third representation, that was introduced in [46], named consistent Boolean valuation pairs (BVP). They considered pairs $\vec{v} = (\underline{v}, \overline{v})$ where $\underline{v}, \overline{v} : \Omega \to \{0, 1\}$ such that $\underline{v} \leqslant \overline{v}$ holds pointwisely. In this approach, the propositional symbols having $\underline{v} = 1$ are those being true (*Positives*) whereas those being false (*Negatives*) fulfills $\overline{v} = 0$. The others are considered to be unknown. It is straightforwardly that three-valued sets, orthopairs, and BVPs are equivalent.

In some cases, three values are not enough to address unknown information, and a fourth value, denoting the contradiction, is introduced. Two representations usually appear in the literature. They are based on the four-valued sets and on the orthopairs, where the condition $P \cap N = \varnothing$ has been removed allowing to express the contradiction.

1.3 Objectives of the work

The management of unknown information in formal concept analysis is an open problem that deserves to be explored. In our opinion, there is no doubt that the inclusion of this kind of information is a great advantage from the point of view of the applications. However, the key point is to provide a uniform and complete approach that considers the main facets of FCA.

The main objective of our work is **to develop a general FCA framework that allows us to work with unknown information** covering some lacks detected in the literature. In particular, one of these lacks is the formalization of attribute implication and its reasoning methods. Following our previous work in FCA, we intend to provide new logics based on the Simplification paradigm and the corresponding inference methods, based on redundancy removal. This issue can be considered the main motivation of this PhD Thesis whose objectives can be described as follows:

- Developing a new algebra with three values that allow us to extend the logic and all the notions of FCA with positive, negative, and unknown information instead of just positive and negative.
- Being the notion of Galois connection the core of FCA, our second objective is to introduce new Galois connections which capture different aspects of the unknown information issue. More specifically, we are interested in the definition of a Galois connection to reflect the information we currently have and a second Galois connection to capture the upcoming information, when available.

- Introducing two new kinds of implications in the enriched information framework. The first kind of implications, called weak implications, are those that hold with the current information. The fulfilment of these implications may change as new information becomes available. On the other hand, the second kind of implications, called strong implications, are those that either hold with the current information and remain so when new information arrives.

- A significant part of this work is the development of several axiomatic systems that allow us to develop an automatic reasoning method for the new kinds of implications just mentioned above. Of course, these systems should be proven to be sound and complete. Our main motivation is to extend the simplification paradigm, that was successfully used in several different frameworks, to the new implications. In particular, we will build methods that are strongly influenced by the simplification issue: performing a reduction in the implications while preserving its knowledge.

1.4 Summary and methodology

Once we have introduced the problem, the "state of art" and the goals of this work, we briefly summarize its contributions.

Chapter 2 describes the preliminary definitions and results to eases its readability.

In Chapter 3 we present the truthfulness value structure to deal with positive, negative and unknown information. In addition, a new algebraic framework has been presented. Furthermore, we are going to add a new value that represents contradictory information, that is, having positive and negative information regarding the same thing. Since from contradiction anything can be deduced, we will unite in a single symbol all situations involving a contradiction. Finally, the so-called partial formal context, where not all the information is known, is formalized by using our framework. We relate the partial formal contexts with the classical ones providing a transformation between them. This results were published in [56] and presented in the 16^{th} International Conference on Formal Concept Analysis, ICFCA 2021, on July 2021.

In Chapter 4 we present the first Galois connection that captures information from a partial formal context. This Galois connection considers the current information, that is, those available at the present moment. It allows to introduce the so-called necessary concepts: the concepts, in some sense, that remain when new information arrives. In addition, with the same Galois connection, we present the so-called weak implications, that is, implications that hold with the information available in the partial formal context but it can stop holding when new information appears and vice-versa some that are not holding now but they can hold when new information appears. Finally, an Armstrong's style axiomatic system is introduced to reason with these weak implications. This results were published in [56] and presented in the 16^{th} International Conference on Formal Concept Analysis, ICFCA 2021, on July 2021.

In [6] a complete dual Heyting algebra was used to extend the Simplification logic to the fuzzy framework. In this work, we go deeper and make a formal study of the conditions required for the underlying structure, concluding that we need some properties from the complete dual Heyting algebra, but not all of them. Chapter 5 presents the new structure and characterizes it. These results were published in [58] and a further exhaustive study about the weak complete dual Heyting algebras has been submitted to the International Journal of Intelligent Systems.

In Chapter 6 we provide two axiomatic systems based on Simplification paradigm and, by using the weak complete dual Heyting algebra, we prove it soundness and completeness. A starting point about this issue was presented in [62]. In addition, we introduce an automatic reasoning method that check if a weak implication can be inferred, in polynomial time, from a given set of weak implications. This results were published in Information Sciences [58].

In Chapter 7 we present another Galois Connection which gives us the concepts that are true with the current information available, but could not hold any longer when new information arises. A starting point about this issue was [55]. This Galois Connection was published in [57] and presented in the 16^{th} International Conference on Concept Lattices and Their Applications, CLA2022. In addition, with that Galois Connection, we present the strong implications, that is, the implications that are true not only with the information available, but also when new information arises. We show that with this Galois Connection, we can get the same information as when we complete the Formal Concept with all the possible information but in a lazy way. In addition, again by using a weak Heyting dual algebra, axiomatic systems based on Armstrong's axioms and Simplification logic are proved to be sound and complete for strong implications.

Finally, Chapter 8 presents the conclusions of this PhD, some discussions and some future works that we would like to develop in a future. We have presented some of these future works in some conferences, for instance, in [61] in the 16^{th} International Conference on Concept Lattices and Their Applications, in [60] in the XXI Congreso de Tecnologías y Lógica Fuzzy and in the 14^{th} European Symposium on Computational Intelligence and Mathematics [59].

Chapter 2

Preliminaries

In this chapter, we present the preliminary notions necessary to understand this work. The main objective is to make the research as self-contained as possible. We present the notions of order, Galois Connection and closure operators. We recommend [9, 24, 35] as the main source of information for these topics. Furthermore, we add some definitions and ideas about FCA that will be the basis for building our research. As a reference for the definitions of FCA, we recommend [33, 34].

2.1 Orders and Lattices

A *binary relation* between two sets A and B is a subset

$$R \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

When $A = B$ we say that we have an inner binary relation in A and, in this case, we usually used index notation $a\ R\ b$ instead of $(a, b) \in R$.

Definition 2.1.1 *We say that a pair* $(A, \leqslant)$ *is a* partially ordered set *if A is a not empty set and* $\leqslant$ *is an inner binary relation in* A *holding the following properties:*

- *It is* reflexive*: $a \leqslant a$ for all $a \in A$.*
- *It is* antisymmetric*: $a \leqslant b$ and $b \leqslant a$ implies that $a = b$ for all $a, b \in A$.*
- *It is* transitive*: $a \leqslant b$ and $b \leqslant c$ implies that $a \leqslant c$ for all $a, b, c \in A$.*

We say that $\leqslant$ is a partial order *in A.*

Note that there may be elements $a, b \in A$ that satisfy $a \nleqslant b$ and $b \nleqslant a$, in which case we say that a and b are *incomparable* and denote this situation by $a \parallel b$. Otherwise, we say they are *comparable* and denote it by $a \not\parallel b$. When all elements are comparable to each other, we will say that $\leqslant$ is a *total order*.

As usual, we use the symbol $<$ for the strict order, i.e. $a < b$ means $a \leqslant b$ and $a \neq b$.

When possible, especially in the finite case, the standard way to represent partial orders is to use Hasse diagrams. Given a partially ordered set $(A, \leqslant)$, its *Hasse diagram* is the directed graph $(A, \prec)$ where the elements of A are the vertices and the edges are given by the covering relation $a \prec b$ if and only if it holds the following conditions:

1 $a < b$

2 $a \leqslant c < b$ implies that $a = c$ for all $c \in A$.

To ease the reading, the vertices, sorted from smallest to largest, are plotted from bottom to top; that is, the smallest is at the bottom and the biggest is at the top.

The interpretation of the Hasse diagram is based on the fact that the order relation $\leqslant$ is a reflexive and transitive closure of the covering relation $\prec$: given two elements $a, b \in A$ we have that $a \leqslant b$ if and only if $a = b$ or there is a sequence $x_1, x_2, x_3, \ldots, x_n$ such that $x_1 = a$, $x_n = b$ and $x_i \prec x_{i+1}$ for all $0 < i < n - 1$, that is, there is a upward chain from a to b in the graph.

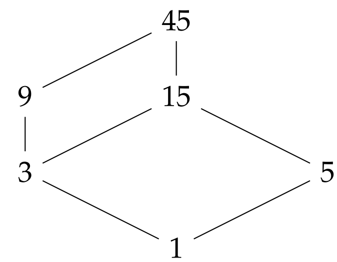


Figure 2.1: Hassen Diagram dividers of 45

Example 2.1.2 *The Hasse diagram of the partial ordered set* $(A, \leqslant)$ *being* A *the set* $\{1, 3, 5, 9, 15, 45\}$ *and* $\leqslant$ *is the divisibility relation (*$a \leqslant b$ *if and only if there is* c *such that* $ac = b$*) is the graph of Figure 2.1.*

Definition 2.1.3 *Let* $(A, \leqslant)$ *a partial ordered set,* $B \subseteq A$ *and* $a \in A$*. We say that*

- a *is a* lower bound *of* B *if* $a \leqslant x$ *for all* $x \in B$*.*
- a *is a* upper bound *of* B *if* $x \leqslant a$ *for all* $x \in B$*.*

If there is at least one lower and one upper bound of B we say that B is *bounded*.

Example 2.1.4 *In the partially ordered set of Figure 2.1, if we consider* $B = \{3, 5, 15\}$ *we have that* 1 *is a lower bound, whereas* 15 *and* 45 *are upper bound.*

Definition 2.1.5 *Let* $(A, \leqslant)$ *a partial ordered set,* $B \subseteq A$ *and* $a \in A$*. We say that:*

- a *is a* maximum *of* B *if* $a \in B$ *and* a *is a upper bound of* B*.*
- a *is a* minimum *of* B *if* $a \in B$ *and* a *is a lower bound of* B*.*

Due to the antisymmetric property, we have that the maximum and minimum of a set, if they exist, are unique. We are going to denote them like $\max B$ and $\min B$, respectively. In addition, if in a partially ordered set A, the set A is bounded, then the maximum and the minimum of A will be denoted as $\top$ and $\perp$ respectively.

Definition 2.1.6 *Let $(A, \leqslant)$ be a partial ordered set and $B \subseteq A$. We say that $a \in A$ is:*

- *the* supremum *of B if a is the minimum of the upper bounds of B.*
- *the* infimum *of B if a is the maximum of the lower bounds of B.*

The supremum and the infimum of a set B will be denoted by $\bigvee B$ and $\bigwedge B$, respectively.

Observe that $\bigvee \varnothing$ exists if and only if there exists the infimum of A, in that case, $\bigvee \varnothing = \perp$. In the same way, $\bigwedge \varnothing$ exists if and only if there exists the supremum of A, in that case, $\bigwedge \varnothing = \top$.

2.1.1 Lattices and complete lattices

We are ready to introduce the notion of lattice.

Definition 2.1.7 *An* ordered lattice *is a partially ordered set $(L, \leqslant)$ in which there exist $\bigvee B$ and $\bigwedge B$ for all $B \subseteq L$ with B finite and not empty.*

The above conditions are equivalent to asking the existence of the supremum and infimum of all the pairs of elements, that is, for all $a, b \in L$ there exist $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$. There is an equivalent algebraic definition for the structure of lattice.

Definition 2.1.8 *An* algebraic semilattice *is a pair $(A, *)$ where A is a not empty set and $*$ is an internal binary operation in A with the following properties:*

1. Idempotency*: $a * a = a$ for all $a \in A$.*

2. Commutativity*: $a * b = b * a$ for all $a, b \in A$.*

3. Associativity*: $a * (b * c) = (a * b) * c$ for all $a, b, c \in A$.*

An algebraic lattice *is a triple $(A, \wedge, \vee)$ such that $(A, \wedge)$ and $(A, \vee)$ are algebraic semilattices and the following property is fulfilled:*

4. Absorption*: $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in A$.*

The equivalence between ordered lattice and algebraic lattice is the following one: Each ordered lattice $\mathbb{L} = (L, \leqslant)$ forms an algebraic lattice $\mathbb{L}^a = (L, \wedge, \vee)$ where $a \wedge b = \bigwedge\{a, b\}$ and $a \vee b = \bigvee\{a, b\}$. Reciprocally, if we have an algebraic lattice $\mathbb{L} = (L, \wedge, \vee)$, we can form an ordered lattice $\mathbb{L}^o = (L, \leqslant)$ where $a \leqslant b$ if and only if $\bigwedge\{a, b\} = a$ or, equivalently, if $\bigvee\{a, b\} = b$. In addition, it holds that $(\mathbb{L}^o)^a = \mathbb{L}$ and $(\mathbb{L}^a)^o = \mathbb{L}$. As a consequence, from this moment on, we use the word lattice without making any difference between ordered or algebraic lattices.

Definition 2.1.9 *A partially ordered set is a* complete lattice *if all subset has a supremum.*

It is easy to prove that a partially ordered set $(L, \leqslant)$ is a complete lattice if and only if it holds any of the following properties (that are equivalent to each other):

1. There exists $\bigwedge H$ for all $H \subseteq L$.

2. There exists $\top = \max L$ and, for all $\varnothing \neq H \subseteq L$, there exists $\bigwedge H$.

3. There exists $\bigvee H$ for all $H \subseteq L$.

4. There exists $\perp = \min L$ and, for all $\varnothing \neq H \subseteq L$, there exists $\bigvee H$.

As a consequence, any complete lattice is bounded.

In a complete lattice $\mathbb{L} = (L, \leqslant)$ for each $x \in L$ we define:

$$x_* = \bigvee\{y \in L \mid y < x\} \quad \text{and} \quad x^* = \bigwedge\{y \in L \mid x < y\} \tag{2.1}$$

It is clear that we have that $x_* \leqslant x \leqslant x^*$ for all $x \in L$.

Definition 2.1.10 *Let $\mathbb{L} = (L, \leqslant)$ be a complete lattice. We say that $x \in L$ is* $\wedge$-irreducible *if $x \neq x_*$ and x is* $\vee$-irreducible *if $x \neq x^*$.*

The set of the $\wedge$-irreducible elements of $\mathbb{L}$ will be denoted by $M(\mathbb{L})$ and the set of $\vee$-irreducible elements of $\mathbb{L}$ will be denoted by $J(\mathbb{L})$.

As a consequence of the definition, if $j \in J(\mathbb{L})$ then: $l_1 \vee l_2 = j$ implies that $l_1 = j$ or $l_2 = j$ for all $l_1, l_2 \in L$.

In the case of finite lattices, the above property characterizes the $\vee$-irreducible elements. In addition, we have that j_* is the only one neighbour below it, that is, the only one that holds $j_* \prec j$.

In the same way, if $m \in M(\mathbb{L})$ we have that: $l_1 \wedge l_2 = m$ implies that $l_1 = m$ or $l_2 = m$ for all $l_1, l_2 \in L$ and, in the case of finite lattices, m^* is the only one neighbour above it, i.e. there is no other element $a \in L$ that fulfils $m \prec a$.

Definition 2.1.11 *Let $\mathbb{L} = (L, \leqslant)$ be a bounded lattice and $a \in L$. We say that a is an* atom *if $\bot \prec a$ and we say that a is a* superatom *if $a \prec \top$.*

The set of all the atoms of $\mathbb{L}$ and all superatoms of $\mathbb{L}$ will be denoted by $At(\mathbb{L})$ and $SupAt(\mathbb{L})$, respectively.

Example 2.1.12 *In the partial ordered set whose Hasse diagram is shown in Figure 2.1 we have that $J(\mathbb{L}) = \{3, 5, 9\}$, $M(\mathbb{L}) = \{5, 9, 15\}$, $At(\mathbb{L}) = \{3, 5\}$ and $SupAt(\mathbb{L}) = \{9, 15\}$.*

Definition 2.1.13 *Let $\mathbb{L} = (L, \leqslant)$ be a complete lattice. We say that a subset $H \subseteq L$ is* $\vee$-dense *(respectively* $\wedge$-dense*) if for all $l \in L$ there is a subset $H \subseteq L$ such that $l = \bigvee H$ (respectively $l = \bigwedge H$).*

It is clear that, if H is a $\vee$-dense set, for all $l \in L$ we have that $l = \bigvee\{x \in H \mid x \leqslant l\}$ and, if H is a $\wedge$-dense set, for all $l \in L$ we have that $l = \bigwedge\{x \in H \mid l \leqslant x\}$.

Proposition 2.1.14 *Let $\mathbb{L} = (L, \leqslant)$ a finite lattice and $H \subseteq L$. We have that:*

- *H is $\vee$-dense if and only if $J(\mathbb{L}) \subseteq H$.*
- *H is $\wedge$-dense if and only if $M(\mathbb{L}) \subseteq H$.*

Observe that we can not affirm that the above proposition holds when we work with infinite lattices. For instance, in the lattice $(\mathbb{R}, \leqslant)$ we have that $J(\mathbb{R}) = M(\mathbb{R}) = \varnothing$ and, if the proposition applies, all the subsets would be $\vee$- dense and $\wedge$-dense. However, it has some subsets that are $\vee$-dense and $\wedge$-dense like, for instance, $\mathbb{Q}$, and it has some subsets that are not $\vee$-dense and $\wedge$-dense like, for instance, $\mathbb{Z}$.

2.1.2 Sublattices and homomorphisms of lattices

Now we introduce some notations about sets that we use in the following chapters.

Definition 2.1.15 *Let $\mathbb{L}$ be a lattice and $\varnothing \neq M \subseteq L$. We say that M is a* $\vee$-subsemilattice *of $\mathbb{L}$ if it is closed under the supremum, that is, $a, b \in M$ implies that $a \vee b \in M$. In the same way, we say that M is a* $\wedge$-subsemilattice *of $\mathbb{L}$ if it is closed under the infimum, that is, $a, b \in M$ implies that $a \wedge b \in M$. The set M is a* sublattice *of $\mathbb{L}$ if it is a $\vee$-subsemilattice and $\wedge$-subsemilattice of $\mathbb{L}$.*

In the case of complete lattices, the definition of subsemilattice, $\vee$-subsemilattice and $\wedge$-subsemilattice are obtained by changing the condition of being closed, instead of pairs of elements, to arbitrary subsets, that is, if $\mathbb{L}$ is a complete lattice, a subset $H \subseteq L$ is a $\vee$-subsemilattice of $\mathbb{L}$ if $\bigvee T \in H$ for all subset $T \subseteq H$ and it is a $\wedge$-subsemilattice of $\mathbb{L}$ if $\bigwedge T \in H$ for all subset $T \subseteq H$.

Now we are focused on the functions between partially ordered sets or between lattices:

Definition 2.1.16 *Let $(A, \leqslant)$ and $(B, \leqslant)$ two partial ordered sets. A function $f : A \to B$ is said to be:*

- Isotone *if $a \leqslant b$ implies that $f(a) \leqslant f(b)$ for all $a, b \in A$.*
- Antitone *if $a \leqslant b$ implies that $f(b) \leqslant f(a)$ for all $a, b \in A$.*

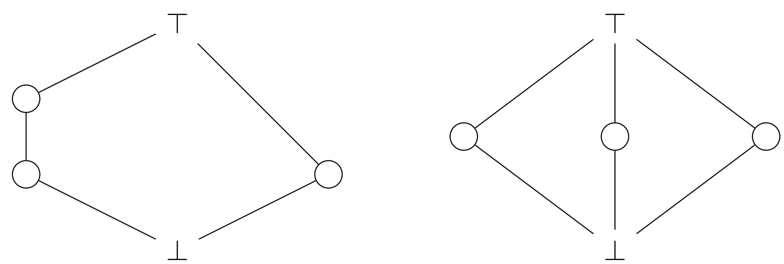
When $A = B$, that is, $f : A \to A$ we say that f is:

- Inflationary *if $a \leqslant f(a)$ for all $a \in A$.*
- Deflationary *if $f(a) \leqslant a$ for all $a \in A$.*
- Idempotent *if $f(f(a)) = f(a)$.*

Finally, we can introduce the notion of homomorphism between lattices.

Definition 2.1.17 *Let $\mathbb{L}_1 = (L_1, \leqslant)$ and $\mathbb{L}_2 = (L_2, \leqslant)$ two (complete) lattices. A function $f : L_1 \to L_2$ is said to be a* homomorphism *between the (complete) lattices if it fulfills that $f(A \vee B) = f(A) \vee f(B)$ and $f(A \wedge B) = f(A) \wedge f(B)$ for all finite (arbitrary) subsets $A, B \in L_1$. In addition, if f is bijective, then we say that f is a* isomorphism *between $\mathbb{L}_1$ and $\mathbb{L}_2$.*

We say that two (complete) lattices are *isomorphic*, and we denote it by $\mathbb{L}_1 \cong \mathbb{L}_2$, if there exists an isomorphism between them.



(a) Pentagon lattice (b) Diamond lattice

Figure 2.2: Complemented lattices

As a consequence of the definition, each homomorphism between lattices is isotone. In addition, $f : \mathbb{L}_1 \to \mathbb{L}_2$ is a isomorphism between complete lattices if and only if f is bijective and for all $a, b \in \mathbb{L}_1$ we have that

$$a \leqslant b \quad \text{if and only if} \quad f(a) \leqslant f(b)$$

2.1.3 Boolean Lattices

Definition 2.1.18 *Let $\mathbb{L} = (L, \vee, \wedge, \top, \bot)$ a bounded lattice and $a \in L$. We say that $b \in L$ is the* complement *of a if and only if $a \vee b = \top$ and $a \wedge b = \bot$.*

We say that a lattice is complemented *if all the elements have a complement.*

For instance, the lattices in Figure 2.2 are complemented:

Definition 2.1.19 *A lattice $\mathbb{L} = (L, \wedge, \vee)$ is* distributive *if for all $a, b, c \in L$ it fulfils the following properties:*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \textit{and} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

It is well known that a lattice is distributive if and only if it has no sublattice isomorphic to the pentagon or diamond lattice (see Figure 2.2).

Definition 2.1.20 *We say that a lattice is* Boolean *if it is bounded, complemented and distributive.*

In the Boolean lattices, the complement of each element exists and it is unique. So we can give an algebraic definition of Boolean lattice: $(L, \wedge, \vee, {}^c, \top, \bot)$ where $(L, \wedge, \vee, \top, \bot)$ is a bounded lattice and $(-)^c : L \to L$ is a unary operation such that $a \wedge a^c = \bot$ and $a \vee a^c = \top$. This algebraic structure is known as *Boolean algebra*.

2.1.4 Complete dual Heyting algebras

A *complete dual Heyting algebra* is a complete lattice $(L, \leqslant)$ endowed with an operation $\smallsetminus$ that satisfies the following adjoint property for all $a, b, c \in L$:

$$a \leqslant b \vee c \qquad \text{if and only if} \qquad a \smallsetminus b \leqslant c, \tag{2.2}$$

The operation $\smallsetminus$ is called *difference operation*

It is well known that, given a complete lattice $(L, \leqslant)$, a necessary and sufficient condition for the existence of a difference operation $\smallsetminus$ is the following:

$$\min\{x \mid a \leqslant b \vee x\} \text{ exists, for all } a, b \in L \tag{2.3}$$

In fact, Equation 2.2 is equivalent to:

$$a \smallsetminus b = \min\{x \mid a \leqslant b \vee x\} \text{ for all } a, b \in L \tag{2.4}$$

Remark 2.1.21 *It is well known that if $(L, \leqslant, \smallsetminus)$ is complete dual Heyting algebra, then the complete lattice $(L, \leqslant)$ is distributive. In fact, in the finite case, $(L, \leqslant)$ is distributive if and only if $(L, \leqslant, \smallsetminus)$ is a complete dual Heyting algebra where $a \smallsetminus b = \bigwedge\{x \mid a \leqslant b \vee x\}$. This construction of the complete dual Heyting algebra can be extended to complete (infinite) lattices by requiring the property of infinite distributivity.*

In [6], it was also proved that, in any complete dual Heyting algebra, the following property holds:

$$a \vee ((a \vee b) \smallsetminus c) = a \vee (b \smallsetminus c) \tag{2.5}$$

The next proposition gives us a known characterisation of the complete dual Heyting algebras [66]:

Proposition 2.1.22 *An algebra $(L, \wedge, \vee, \smallsetminus, \top, \bot)$ is a complete dual Heyting algebra if and only if $(L, \wedge, \vee, \top, \bot)$ is a complete lattice and the following properties hold:*

`[H1]` $x \vee (y \smallsetminus x) = x \vee y$ *for all* $x, y \in L$.

`[H2]` $x \vee (z \smallsetminus y) = x \vee [(x \vee z) \smallsetminus (x \vee y)]$ *for all* $x, y, z \in L$.

`[H3]` $x \smallsetminus (x \vee y) = \bot$ *for all* $x, y \in L$.

2.2 Closure operators and Galois connections

In this section, we introduce two concepts of considerable importance for formal concept analysis: the closure operators and the Galois connection. In addition, we remind some relevant results about closure operators, closure systems and Galois connections.

Definition 2.2.1 *Let $(A, \leqslant)$ be a partial ordered set. A* closure operator *on A is a map $c : A \to A$ such that c is isotone, inflationary and idempotent.*

Definition 2.2.2 *Let $(A, \leqslant)$ be a partial ordered set. A subset $S \subseteq A$ is said to be a* closure system *(also known as Moore family) if for all $a \in A$ the set $\{s \in S \mid a \leqslant s\}$ has a minimum. That is, if there exists $\min(\{s \in A \mid a \leqslant s\} \cap S)$ for all $a \in A$.*

The following proposition shows the biunivocal relation between closure operators and closure systems of a partially ordered set.

Proposition 2.2.3 *Let $(A, \leqslant)$ be a partial ordered set.*

1. *If $c : A \to A$ is a closure operator, then the image set of c, which coincides with $S_c = \{a \in A \mid c(a) = a\}$, is a closure system.*

2. *If S is a closure system then the map $c_S : A \to A$ defined as $c_S(a) = \min\{s \in S \mid a \leqslant s\}$, for each $a \in A$, is a closure operator.*

In addition, $c = c_{S_c}$ and $S = S_{c_S}$.

In the above proposition, c_s is called the closure operator associated with S and S_c is called the closure system associated with c.

In the case of the lattices, the following proposition is an alternative characterisation for the closure systems.

Proposition 2.2.4 *Let $(L, \leqslant)$ be a complete lattice. A subset $S \subseteq L$ is a closure system if and only if S is a $\wedge$-subsemilattice.*

Observe that, in the case of complete lattices, given a closure system S, we have that $c_S(a) = \bigwedge\{s \in S \mid a \leqslant s\}$ for all $a \in A$, that $\top \in S$ and that $(S, \leqslant)$ is also a complete lattice. However, it is not necessary a sublattice of $(L, \leqslant)$. On the other hand, all complete lattice can be seen as a closure system.

Definition 2.2.5 *Given two partial ordered sets $(L_1, \leqslant)$ and $(L_2, \leqslant)$, a* Galois connection *is pair of mappings $\phi : L_1 \to L_2$ and $\psi : L_2 \to L_1$ such that both of them are antitone and both compositions, $\phi \circ \psi$ and $\psi \circ \phi$, are extensive. It is well-known that the pair (ϕ, ψ) is a Galois connection if and only if, for all $\ell_1 \in L_1$ and $\ell_2 \in L_2$,*

$$\ell_1 \leqslant \psi(\ell_2) \quad \text{if and only if} \quad \ell_2 \leqslant \phi(\ell_1) \tag{2.6}$$

As a consequence of the definition, we have the following theorem:

Theorem 2.2.6 *Given a Galois connection* (ϕ, ψ) *between* $\mathbb{L}_1$ *and* $\mathbb{L}_2$*, the map* $\psi \circ \phi$ *is a closure operator on* $\mathbb{L}_1$ *and the map* $\phi \circ \psi$ *is a closure operator on* $\mathbb{L}_2$*. The maps* ϕ *and* ψ*, respectively, define dual isomorphisms between the corresponding closure systems. Specifically, the set* $\mathfrak{B} = \{(x, y) \mid \phi(x) = y, \psi(y) = x\}$ *with the order* $\leqslant$ *defined as*

$$(x_1, y_1) \leqslant (x_2, y_2) \quad \textit{iff} \quad x_1 \leqslant x_2 \quad \textit{or, equivalently, iff} \quad y_2 \leqslant y_1$$

form a complete lattice such that, for any family $\{(x_j, y_j) \in \mathfrak{B} \colon j \in J\}$*, the supremum and the infimum are given by:*

$$\sup_{j \in J}(x_j, y_j) = \Big(\psi\phi\Big(\bigvee_{j\in J} x_j\Big), \bigwedge_{j \in J} y_j\Big) \qquad \inf_{j\in J}(x_j, y_j) = \Big(\bigwedge_{j\in J} x_j, \phi\psi\Big(\bigvee_{j \in J} y_j\Big)\Big)$$

2.3 Formal Concept Analysis

Formal Concept Analysis (FCA) was first introduced by Rudolf Wille [70] in the 80s. It is a useful tool to collect knowledge from a piece of given information stored as a binary relation between two sets: objects and attributes. FCA has been used and researched in recent years for many different purposes like, for instance in social media analysis, marketing, medical diagnosis, etc.

First, as the starting point, FCA considers a formal context where the relationship between objects and attributes is captured.

Definition 2.3.1 *A* formal context*, denoted by* $\mathbb{K} = (G, M, I)$*, consists in two not empty sets* G *(whose elements are called objects) and* M *(whose elements are called attributes) and a relation* I *between* G *and* M*. The meaning of* $(g, m) \in I$ *is that the object* g *has the attribute* m*.*

The notation of G comes from *Gegenstand* in german and the notation of M comes from *Merkmal*.

$\mathbb{K}$	a	b	c
1	$\times$		$\times$
2		$\times$	
3	$\times$	$\times$	

Figure 2.3: Example of formal context

Example 2.3.2 *In the formal context given by Table 2.3, we have that the object* 1 *has the attributes* a *and* c*, the object* 2 *has the attribute* b*, and the object* 3 *has the attributes* a *and* b*.*

The knowledge we can extract from the formal context can be shown with a structure called concept lattice or a set of attribute implications. Actually, both ways of showing the knowledge are equivalent in the sense that they represent the same knowledge, and we can build one from the other without using the formal context. In the following subsections, we show both definitions.

2.3.1 The lattice of formal concepts

In this subsection, we present the lattice of formal concepts. Formal concepts are subsets of objects and attributes which satisfy some properties in the formal context. First, we need to introduce the concept forming operators to define the formal concepts.

Definition 2.3.3 *Given a formal context* (G, M, I)*, we call concept forming operators to the functions* $^{\uparrow}: 2^G \to 2^M$ *and* $^{\downarrow}: 2^M \to 2^G$ *such that, for a set* $A \subseteq G$*, we have that* $A^{\uparrow} = \{m \in M \mid (g, m) \in I \quad \forall g \in A\}$ *and, for a set* $B \in M$*, we have that* $B^{\downarrow} = \{g \in G \mid (g, m) \in I \quad \forall m \in B\}$*.*

Namely, $A^{\uparrow}$ is the set of properties that are shared by all the objects in A, and $B^{\downarrow}$ is the set of objects having all the attributes in B.

Example 2.3.4 *In the formal context given in Figure 2.3 we have that*

$$\{1\}^{\uparrow} = \{a, c\}, \qquad \{1,3\}^{\uparrow} = \{a\}, \qquad G^{\uparrow} = \varnothing, \qquad \textit{and} \qquad \varnothing^{\uparrow} = M$$

$$\{b\}^{\downarrow} = \{2,3\}, \qquad \{a,b\}^{\downarrow} = \{3\} \qquad M^{\downarrow} = \varnothing \qquad \textit{and} \qquad \varnothing^{\downarrow} = G$$

Definition 2.3.5 *Given a formal context (G, M, I) the* formal concepts *are the fixed points of the concept forming operators(closed sets), namely the pairs (A, B) with $A \subseteq G$ and $B \subseteq M$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$.*

These "are formal abstractions of concepts of human thought allowing meaningful and comprehensible interpretation". The prefix formal emphasises that "they are mathematical entities and must not be identified with concepts of the mind" [35].

In other words, a formal context is a pair of sets (A, B) being A, a set of objects that share all the attributes in B, and do not share any other attribute. So an alternative way to define formal contexts is as pair of maximal subsets A and B such that $A \times B \subseteq I$.

Example 2.3.6 *In the formal context given by Figure 2.3 we have that the pair $(\{2,3\}, \{b\})$ is a formal concept.*

Theorem 2.3.7 *Let $\mathbb{K} = (G, M, I)$ a formal context. The pairs of concept forming operators $(\uparrow, \downarrow)$ form a Galois connection between $(2^G, \subseteq)$ and $(2^M, \subseteq)$.*

As consequence of the above theorem and Theorem 2.2.6, the formal concepts with the order $\leqslant$ defined as

$$(A_1, B_1) \leqslant (A_2, B_2) \text{ iff } A_1 \subseteq A_2 \text{ or, equivalently, iff } B_2 \subseteq B_1,$$

form a complete lattice, which is called the *concept lattice* and denoted by $\underline{\mathbb{B}}(G, M, I)$, whose supremum and infimum are described in the following theorem which is known as *the Basic Theorem of Concept Lattices* [34, Theorem 3, Chapter 1].

Theorem 2.3.8 *The concept lattice $\underline{\mathbb{B}}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t\in T}(A_t, B_t) = \left(\bigcap_{t\in T} A_t, (\bigcup_{t\in T} B_t)^{\downarrow\uparrow}\right)$$

$$\bigvee_{t\in T}(A_t, B_t) = \left((\bigcup_{t\in T} A_t)^{\uparrow\downarrow}), \bigcap_{t\in T} B_t\right)$$

A complete lattice $\mathbb{L} = (L, \leqslant)$ is isomorphic to $\underline{\mathbb{B}}(G, M, I)$ if and only if there are mappings $\overline{\gamma} : G \to L$ and $\overline{\mu} : M \to L$ such that $\overline{\gamma}(G)$ is $\vee$-dense in $\mathbb{L}$, $\overline{\mu}(M)$ is $\wedge$-dense in $\mathbb{L}$ and $(g, m) \in I$ is equivalent to $\overline{\gamma}(g) \leqslant \overline{\mu}(m)$ for all $g \in G$ and all $m \in M$. In particular, $\mathbb{L} \cong \underline{\mathbb{B}}(L, L, \leqslant)$.

The order presented for formal concepts show us which concept is more specify, that is, $(A_1, B_1) \leqslant (A_2, B_2)$ is equivalent to that (A_1, B_1) is more specific than (A_2, B_2), and we say that it is a *subconcept*, or equivalently, that (A_2, B_2) is more general and we call it *superconcept*.

Example 2.3.9 *In Figure 2.4 we present the formal concept for the formal context given in Table 2.3. The concepts in Figure 2.4 are the following ones:*

$$C_0 = (\varnothing, M), \qquad C_1 = (\{1\}, \{a, c\}), \qquad C_2 = (\{3\}, \{a, b\}),$$

$$C_3 = \{1, 3\}, \{a\}, \qquad C_4 = (\{2, 3\}, \{b\}) \qquad \textit{and} \qquad C_5 = (G, \varnothing).$$

In conclusion, every concept lattice is a complete lattice, and every complete lattice is isomorphic to several concept lattices. In particular, given

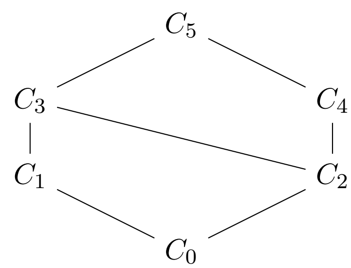


Figure 2.4: Formal concept of the formal context given in Figure 2.3

a complete lattice $(L, \leqslant)$, it is isomorphic to the concept lattice of the context $(L, L, \leqslant)$. Now, we are wondering if there are smaller contexts that are isomorphic to the given complete lattice. We remind that, in case of finite lattices, $\overline{\gamma}(G)$ is $\wedge$-dense if and only if $J(\mathbb{L}) \subseteq \overline{\gamma}(G)$ and $\overline{\mu}$ is $\vee$- dense if and only if $M(\mathbb{L}) \subseteq \overline{\mu}(G)$ (see Proposition 2.1.14)

It is easy to check that if we duplicate rows and/or columns in the context, the new concept lattice will be isomorphic to the initial one. This idea let us to introduce the following new definition:

Definition 2.3.10 *A formal context $\mathbb{K} = (G, M, I)$ is said to be* clarified *if it satisfies that:*

1. $\{g_1\}^{\uparrow} = \{g_2\}^{\uparrow}$ *implies that* $g_1 = g_2$ *for all* $g_1, g_2 \in G$.

2. $\{m_1\}^{\downarrow} = \{m_2\}^{\downarrow}$ *implies that* $m_1 = m_2$ *for all* $m_1, m_2 \in M$.

Using similar reasoning, we can conclude that, in a finite context, if a row or column is the intersection of others, it can be removed from the formal context without affecting the structure of the concept lattice. In particular, this can be done if we have an empty or complete row/ column.

Definition 2.3.11 *We say that a finite formal context $\mathbb{K} = (G, M, I)$ is* reduced *if, for all $A \subseteq G$, $g \in G$, $B \subseteq M$ and $m \in M$ we have that:*

1. *$\{g\}^{\uparrow} \neq \varnothing$ and, if $\{g\}^{\uparrow} = A^{\uparrow}$ then $g \in A$.*

2. *$\{m\}^{\downarrow} \neq \varnothing$ and, if $\{m\}^{\downarrow} = B^{\downarrow}$ then $m \in B$.*

Finally, the following theorem answer the question presented about the smaller context such that its concept lattice is isomorphic to a given complete lattice.

Theorem 2.3.12 *For all finite lattice $\mathbb{L} = (L, \leqslant)$ we have that the formal context given by $\mathbb{K}(\mathbb{L}) = (J(\mathbb{L}), M(\mathbb{L}), \leqslant)$ satisfies that $\mathbb{L} \cong \underline{\mathbb{B}}(\mathbb{K}(\mathbb{L}))$. In addition, this is the unique reduced context, except isomorphisms, that holds this condition.*

This theorem can be not extended for all complete lattices, for instance, for $(\mathbb{R}, \leqslant)$ we have that $J(\mathbb{R}) = M(\mathbb{R}) = \varnothing$.

Example 2.3.13 *The just one reduced context which concept lattice is isomorphic to the lattice in Figure 2.1 is $(\{3, 5, 9\}, \{5, 9, 12\}, I)$ where I is given in Figure 2.5.*

$\mathbb{K}$	5	9	15
3		$\times$	$\times$
5	$\times$		$\times$
9		$\times$	

Figure 2.5: Reduced context for positive divisors of 45

2.3.2 Attribute implications

In this section, we present an alternative way to obtain and show the knowledge from a formal context: the attribute implications.

We also show the equivalence between attribute implications and concept lattices in the sense that both have the same knowledge, and we can obtain one from the other without going through the formal context. The advantage of the concept lattice is that it allows browsing through it, passing from subconcepts to superconcepts and vice-versa. The advantage of the attribute implications is that it allows us to use logical processing and, even more, automate reasoning. On the other hand, it is quite common that the knowledge in terms of attribute implications can be characterised by using less information than when we use concept lattices.

We show the results related to attribute implications by considering the usual components of logic: language, semantics, axiomatic system or syntactic inference, and automatic demonstrator.

Definition 2.3.14 *Given a set of attributes M we define the* language *as*

$$\mathcal{L}_M = \{X \to Y \mid X, Y \subseteq M\}$$

The formulas in $\mathcal{L}_M$ are called *attribute implications* or implications. Given an attribute implication $X \to Y$ we say that X is the *premise* and Y is the *conclusion*. We call implicational system to any subset of $\Sigma \subseteq \mathcal{L}_M$.

Usually, the attribute implications are expressed without brackets. Also, we remove the coma in the premise and in the conclusion. In the same way, the union of sets in the attribute implications is denoted by juxtaposition with the two sets. We are going to take this notation. For instance, we write $m_1m_2 \to m_3$ instead of $\{m_1, m_2\} \to \{m_3\}$ and $AB \to CD$ instead of $A \cup B \to C \cup D$. We take this notation with the goal of reducing the size of the formulas and easing its reading.

Now we can speak about *semantic*:

Definition 2.3.15 *We say that a subset $T \subseteq M$ is a* model *of $A \to B$ if $A \not\subseteq T$ or $B \subseteq T$ (in that case, we denote it by $T \models A \to B$).*

T is model of a set Σ of implications ($T \models \Sigma$) if T is model of every single implication in Σ.

We say that $A \to B$ holds in a context (G, M, I) if $\{g\}^{\uparrow}$ is a model of $A \to B$ for all $g \in G$, that is, if each object that has all the attributes from A has all the attributes from B as well. In that case, we say that $A \to B$ is a (valid) implication *of (G, M, I).*

The following proposition characterizes the validity of implications.

Proposition 2.3.16 *An implication $A \to B$ holds in (G, M, I) if and only if $B \subseteq A^{\downarrow\uparrow}$, which is equivalent to $A^{\downarrow} \subseteq B^{\downarrow}$.*

Example 2.3.17 *We can consider the formal context $\mathbb{K} = (G, M, I)$ where $G = \{1, 2, 3\}$, $M = \{a, b, c\}$ and I is the binary relation given by Figure 2.6. It is easy to prove that $\mathbb{K} \models \{c \to a, \varnothing \to b\}$. However, we have that $\mathbb{K} \not\models b \to a$.*

$\mathbb{K}$	a	b	c
1		$\times$	
2	$\times$	$\times$	$\times$
3	$\times$	$\times$	

Figure 2.6: Formal context $\mathbb{K} = (G, M, I)$

Definition 2.3.18 *Let M be a set of attributes, $\Sigma \subseteq \mathcal{L}_M$ a set of attribute implications and $A \to B \in \mathcal{L}_M$. We say that $A \to B$* follows semantically *from Σ (and we denote it by $\Sigma \models A \to B$) if each subset of M that is model of Σ is also model of $A \to B$.*

On the other hand, we say that two implicational systems $\Sigma_1, \Sigma_2 \subseteq \mathcal{L}_M$ are *semantically equivalent* (and we denote it by $\Sigma_1 \equiv \Sigma_2$) if for each subset of M we have that M is model of Σ_1 if and only if M is model of Σ_2. Observe that the meaning of being semantically equivalent is that both implicational systems denote the same knowledge.

Example 2.3.19 *Let $M = \{m_1, m_2, m_3\}$ be a set of attributes. We have that $\{m_1 \to m_2, m_1 \to m_3\} \equiv \{m_1 \to m_2m_3\}$ observe that all set of implication that is model for $\{m_1 \to m_2, m_1 \to m_3\}$ or it has m_1 and m_2 and m_3 or it has not m_1 or it has m_2 and m_3 in any case, we have that it is model of $\{m_1 \to m_2m_3\}$ and, equivalently, we can see that the models of this last implication are the same that the set of implications $\{m_1 \to m_2, m_1 \to m_3\}$.*

There are different axiomatic systems, being the most popular the so-called *Armstrong's Axioms* [2] that consider a scheme of an axiom and two inference rules: Let $A, B, C \subseteq M$,

`[Inc]` Inclusion: $\vdash_{\mathcal{A}} AB \to A$

`[Aum]` Augmentation: $A \to B \vdash_{\mathcal{A}} AC \to BC$

`[Trans]` Transitivity $A \to B, B \to C \vdash_{\mathcal{A}} A \to C$.

We introduce the definition of syntactic derivation as usual:

Definition 2.3.20 *An implication $A \to B$ is said to be* derived *from a set $\Sigma \subseteq \mathcal{L}_M$, denoted by $\Sigma \vdash A \to B$, if there exists a sequence of implications $A_i \to B_i$ with $1 \leqslant i \leqslant n$ such that $A_n = A$, $B_n = B$, and each $A_i \to B_i$ is either an axiom or $A_i \to B_i \in \Sigma$ or it is obtained from the formulas in $\{A_j \to B_j \mid j < i\}$ by using one of the inference rules.*

When $A \to B$ is derived from $\Sigma \subseteq \mathcal{L}_M$ we say that the sequence is a proof for $\Sigma \vdash A \to B$. Now we show an example of derivation.

Example 2.3.21 *Let $M = \{m_1, m_2, m_3, m_4, m_5\}$. The following sequence prove* *that* $\{m_1m_2 \to m_3m_4, m_2m_4 \to m_5\} \vdash_{\mathcal{A}} m_1m_2 \to m_5$*:*

$\varphi_1 = m_1m_2 \to m_3m_4$	*By hypothesis.*
$\varphi_2 = m_2m_4 \to m_5$	*By hypothesis.*
$\varphi_3 = m_1m_2 \to m_2m_3m_4$	*By using* `[Augm] to` φ_1 *with* m_2.
$\varphi_4 = m_2m_3m_4 \to m_2m_4$	*By* `[Inc]`.
$\varphi_5 = m_2m_3m_4 \to m_5$	*By using* `[Trans] to` φ_4 *and* φ_2.
$\varphi_6 = m_1m_2 \to m_5$	*By using* `[Trans] to` φ_3 *and* φ_5.

Armstrong's Axioms were first introduced in [2], where it is used to study the properties of functional dependencies in Codd's relational model [13]. In several papers like ([3, 29, 42, 54]), different axiomatic systems equivalent to Armstrong are presented. However, we have chosen the original (although we have called the axiom inclusion when usually it is known as reflexivity. The reason is that we use that name for another axiom that fits better with the mathematical idea of reflexivity).

The following theorem shows that the syntactic derivation and the semantic derivation match; that is, everything you can derive from the axiomatic system of Armstrong can be semantically derived (is sound) and vice-versa (is complete).

Theorem 2.3.22 (Soundness and completeness) *Let M be a finite set of attributes, then:*

$$\Sigma \models A \to B \qquad \textit{if and only if} \qquad \Sigma \vdash A \to B.$$

Although the axiomatic system of Armstrong is referenced in many different papers, in the practice is used for the theoretical study of attribute implications and it is not used in the development of applications or algorithms. The problem is that the proofs are not easily automated. To solve this, the Simplification Logic was developed; it is more suitable for automated reasoning [53].

The *Simplification Logic* considers reflexivity as scheme of axiom and three inference rules: Let $A, B, C \subseteq M$,

[Ref] Reflexivity: $\vdash_{\mathcal{S}} A \to A$.

[Frag] Fragmentation: $A \to BC \vdash_{\mathcal{S}} A \to B$.

[Comp] Composition: $A \to B, C \to D \vdash_{\mathcal{S}} AC \to BD$.

[Simp] Simplification: $A \to B, C \to D \vdash_{\mathcal{S}} A(C \smallsetminus B) \to D$.

Actually, we could use just `[Ref]` and `[Simp]` because `[Comp]` and `[Frag]` are derived rules from that two.

In the following theorem we establish a strong connection between the Armstrong's Axioms, $\mathcal{A}$, and the Simplification Logic, $\mathcal{S}$.

Theorem 2.3.23 *Let M be a finite set of attributes. For all $\Sigma \subseteq \mathcal{L}_M$ and for all attribute implication $A \to B \in \mathcal{L}_M$ we have that*

$$\Sigma \vdash_{\mathcal{S}} A \to B \quad \textit{if and only if} \quad \Sigma \vdash_{\mathcal{A}} A \to B$$

As a result of the above theorem, from now on we will drop the subscript and just write $\vdash$, since the two axiomatic systems can be used interchangeably. In addition, as a consequence of the above theorem and Theorem 2.3.22 we have the following corollary:

Corollary 2.3.24 *The Simplification Logic is sound and complete.*

For each implicational system, the axiomatic system defines a closure operator in 2^M. We call this closure operator the *syntactic closure*.

Definition 2.3.25 *Let M be a set of attributes and $\Sigma \subseteq \mathcal{L}_M$. We say that a set $X \subseteq M$ is closed with respect to Σ if, for all $A \to B \in \Sigma$ we have that $A \subseteq X$ implies $B \subseteq X$.*

The set of closed with respect to Σ form a closure system (see Definition 2.2.2) and, by Proposition 2.2.3 it defines a closure operator on $(2^M, \subseteq)$.

Definition 2.3.26 *Let M be a set of attributes and $\Sigma \subseteq \mathcal{L}_M$. For each $X \subseteq M$ we define the syntactic closure of X with respect to Σ as:*

$$X_\Sigma^+ = \bigcap\{C \subseteq M \mid X \subseteq C \ \textit{ and } \ C \ \textit{ is closed with respect to } \Sigma\}.$$

2.3.3 Attribute implications versus concept lattices

Apart from the fact that we can reason automatically from the implications, the main advantage of working with implications over working with the concept lattices is that we do not have to compute and work with all the implications that are held in the context. The size of the set of all the implications held in a context can be exponential with respect to the size of the context. However, we can obtain some smaller subsets from which we can derive all the other implications at a low cost. These subsets are called complete implicational systems.

Definition 2.3.27 *Let $\mathbb{K} = (G, M, I)$ a formal context. We say that a set of implications $\Sigma \subseteq \mathcal{L}_M$ is complete for $\mathbb{K}$, if for all $A \to B \in \mathcal{L}_M$ we have that*

$$\mathbb{K} \models A \to B \quad \textit{if and only if} \quad \Sigma \vdash A \to B$$

As a consequence of the above definition, we have that, for all formal context $\mathbb{K}$ and for all complete implicational set Σ both of the closure operators coincide, that is,

$$A^{\downarrow\uparrow} = A_\Sigma^+ \quad \text{for all} \quad A \subseteq M$$

We also have that the concept lattices and the implicational systems show the same knowledge about the context. There are different algorithms for the construction of the concept lattice from a complete implicational system. For example, in [19,20] there are some methods to compute all the closed sets and their minimal generators, which are strongly based on the closure algorithm described in the following subsection.

For the reverse problem, note that there are different complete implicational systems for the same context, and therefore for the same concept lattice. In [8] there is a comprehensive study of different properties related to the size of the complete implicational system. An implicational system is considered to be the basis of a context if it is complete and, in addition, satisfies some criterion of minimalism (in terms of cardinality, in terms of the number of attributes that are in premises and/or conclusions, etc.), i.e. if redundancy is reduced to a minimum in some sense.

2.3.4 Simplification paradigm and automated reasoning

The main advantage of Simplification Logic is that the inference rules can be considered as equivalence rules, which allows us to remove redundancy and to capture all the derivations (see [53] for more details and proofs).

Theorem 2.3.28 *The following equivalence rules hold: for all* $A, B \subseteq M$*,*

`[FragEq]` *Equivalence of Fragmentation:* $\{A \to B\} \equiv \{A \to B \smallsetminus A\}$.

`[CompEq]` *Equivalence of Composition:* $\{A \to B,\ A \to C\} \equiv \{A \to BC\}$.

`[SimpEq]` *Equivalence of Simplification: if* $A \cap B = \varnothing$ *and* $A \subseteq C$ *then*

$$\{A \to B,\ C \to D\} \equiv \{A \to B,\ C \smallsetminus B \to D \smallsetminus B\}.$$

As we can see, if we read these equivalences from left to right we are removing redundancy which justifies the name of the logic.

We are now ready to present the automatic reasoning method, which is strongly based on the following theorem. Note that this theorem relates syntactic derivation to the syntactic closure operator, and also gives a characterisation reminiscent of the deduction theorem of classical propositional logic.

Theorem 2.3.29 *Let M be a finite set of attributes, $\Sigma \subseteq \mathcal{L}_M$ it is fulfilled that:*

$$\Sigma \vdash A \to B \text{ if and only if } B \subseteq A^+_\Sigma \text{ if and only if } \Sigma \cup \{\varnothing \to A\} \vdash \varnothing \to B.$$

As a consequence of the above theorem, we have that

$$X^+_\Sigma = \max\{Y \subseteq M \mid \Sigma \vdash X \to Y\}.$$

On the other hand, the linear method shown in [53] to compute the closure of $A \subseteq M$ with respect to $\Sigma \subseteq \mathcal{L}_M$, or equivalently to determine whether $A \to B$ can be inferred from Σ, consists in:

1. The implication $\varnothing \to A$ will be called *the guide* and will be the seed for the process.

2. As far as possible, we systematically apply the equivalence rules seen in Theorem 2.3.28 to our guide and each of the remaining implications. Specifically, the equivalence rules that we have to apply are given below:

   2.1. If $B \subseteq A$ then $\{\varnothing \to A, B \to C\} \equiv \{\varnothing \to AC\}$.

   2.2. If $C \subseteq A$ then $\{\varnothing \to A, B \to C\} \equiv \{\varnothing \to A\}$.

   2.3. In other case $\{\varnothing \to A, B \to C\} \equiv \{\varnothing \to A, B \smallsetminus A \to C \smallsetminus A\}$.

3. When we get to a fixed point, i.e. when Σ does not change, we look at the guide $\varnothing \to A$, which can change during the process, and conclude that $X^+_\Sigma = A$.

If we want to determine if $\Sigma \vdash A \to B$, by Theorem 2.3.29, the answer will be affirmative if and only if $B \subseteq X^+_\Sigma$. Observe that, in this case, we can stop the process before obtaining the fixed point as soon as we get $B \subseteq X^+_\Sigma$.

Example 2.3.30 *Let be* $\Sigma = \{bd \to eg, a \to f, cd \to ae, af \to e, de \to f\}$*. In order to calculate* $[ac]^+_\Sigma$*, we consider the guide* $\varnothing \to ac$ *and proceed step by step as follows:*

1. *For the implication* $bd \to eg$*, rule 2.3 applies and nothing changes in either the guide or* Σ*.*
2. *For the implication* $a \to f$*, rule 2.1 applies, the guide becomes* $\varnothing \to acf$ *and* Σ *becomes* $\{bd \to eg, cd \to ae, af \to e, de \to f\}$*.*
3. *For the implication* $cd \to ae$*, rule 2.3 applies, the guide remains* $\varnothing \to acf$ *and* Σ *becomes* $\{bd \to eg, d \to e, af \to e, de \to f\}$*.*
4. *For the implication* $af \to e$*, rule 2.1 applies, the guide becomes* $\varnothing \to acef$ *and* Σ *becomes* $\{bd \to eg, d \to e, de \to f\}$*.*
5. *For the implication* $de \to f$*, rule 2.2 applies, the guide remains* $\varnothing \to acef$ *and* Σ *becomes* $\{bd \to eg, d \to e\}$*.*
6. *For the implication* $bd \to eg$*, rule 2.3 applies, the guide remains* $\varnothing \to acef$ *and* Σ *becomes* $\{bd \to g, d \to e\}$*.*
7. *For the implication* $d \to e$*, rule 2.2 applies, the guide remains* $\varnothing \to acef$ *and* Σ *becomes* $\{bd \to g\}$*.*

Since we get a fixed point ($\Sigma = \{bd \to g\}$*), the process is finished and we conclude that* $[ac]^+_\Sigma = acef$*.*

If we wanted to determine whether $\Sigma \vdash ac \to e$, we would terminate after the fourth iteration and answer in the affirmative.

This process not only computes the closure of a set of attributes in the guide, but also the set of implications returned when the process terminates gives us some relevant information, as shown in [19, 20], for example.

Chapter 3

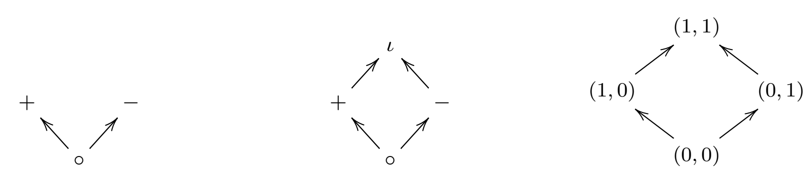
First step to extend FCA to consider unknown information

This chapter will be the basis for working with unknown information, as here we will build, step by step, the algebraic framework that will allow us to extend FCA to consider the unknown values.

3.1 Extending the algebraic structure of truthfulness values

As mentioned in the preliminaries, the starting point of classical FCA is a binary relationship between objects and their properties. If G is the set of objects and M is the set of attributes, this relation can be viewed as a mapping (the characteristic function of the relation) of the type $I\colon G \times M \to \{0, 1\}$ where $I(g, m) = 1$ indicates that object g has attribute m. The meaning of $I(g, m) = 0$ is more debatable. Some authors assume that this means that object g does not have attribute m, while others consider that no information is available on whether the object has the attribute or not. To distinguish both possibilities, we need to enrich the truth-value structure by emphasizing the meaning of these values. Thus, we use the set of three elements $\{+, -, \circ\}$, denoted by $\mathbf{3}$. The element $+$ represents the information that we know to be true (we call it positive information), the element $-$ represents the information that we know to be false (we call it negative information) and the element $\circ$ represents the information that we do not know whether it is true or false (we call it unknown information).

The next step should be to establish an order among these elements. The most usual way to order them is to consider the unknown value as an intermediate value between the negative and the positive ($- < \circ < +$). This is considered, for example, in Łukasiewicz logic or the most usual fuzzy logic [27]. However, the approach we intend to give to this work leads us to an ordering in the style of the order of knowledge considered by Belnap when defining the bilattices [5]. Thus, we endow this set with a $\wedge$-semilattice structure by considering the reflexive clousure of $\{(\circ, +), (\circ, -)\}$ (see Fig. 3.1a). This $\wedge$-semilattice will be denoted by $\underline{\mathbf{3}} = (\mathbf{3}, \leq)$.



(a) $\wedge$-semilattice $\underline{\mathbf{3}}$ (b) lattice $\underline{\mathbf{4}}$ (c) Boolean Algebra $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$

Figure 3.1: Truthfulness's values

In the following section, we study sets valued in $\mathbf{3}$ with a conjunctive interpretation. This interpretation leads us to introduce a fourth element, denoted ι, which is called oxymoron and represents inconsistent or contradictory information. This element will be the maximum of the completion of $\mathbf{3}$ to be a lattice. This lattice, denoted by $\underline{\mathbf{4}} = (\mathbf{4}, \leq)$, is shown in Fig. 3.1b and is isomorphic to the Boolean algebra $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$ (see Fig. 3.1c).

As we have already anticipated, the lattice $\underline{\mathbf{4}}$ can be considered as the order of the amount of information in the sense of Belnap. Observe that this order is the dual of the Lex's one considered in [47] mentioned in Section 1.2.

3.1.1 The $\wedge$-semilattice of 3-sets

In this section we extend the notion of set, as usual, from the equivalent notion of characteristic mapping. Thus, a $\mathbf{3}$-set in a universal set U is a mapping $X\colon U \to \mathbf{3}$. We denote by $\underline{\mathbf{3}}^U$ the $\wedge$-semilattice of the $\mathbf{3}$-sets with the structure of $\wedge$-semilattice considering the point-wise extension of the order $\leq$:

$$X \sqsubseteq Y \text{ if and only if } X(u) \leq Y(u) \text{ for all } u \in U.$$

A $\mathbf{3}$-set X provides information about the knowledge that we have from each element in U. These elements usually correspond to attributes or properties that can hold.

We call *support* of a $\mathbf{3}$-set X to the set $\mathrm{Spp}(X) = \{u \in U \mid X(u) \neq \circ\}$. The support collects those elements that we have absolute knowledge about them. We use the term *support* by analogy with the usual terminology in fuzzy set theory. It should not be confused with the notion of support in association rules or concept metrics.

Following the habit of fuzzy set theory, when the support of a $\mathbf{3}$-set X is finite, $\mathrm{Spp}(X) = \{u_1, \ldots, u_n\}$, we could denote it as follows:

$$\{u_1/X(u_1), \ldots, u_n/X(u_n)\}.$$

However, to simplify the reading and to keep the notation used in previous works [64], we will usually represent the $\mathbf{3}$-set X as a sequence in which only appears elements belonging to the support and, for each $u \in \mathrm{Spp}(X)$, in the sequence must appear u or $\overline{u}$, meaning $X(u) = +$ or $X(u) = -$, respectively. In particular, the unique $\mathbf{3}$-set having the empty support is denoted by ε.

Example 3.1.1 *Given the universe $U = \{a, b\}$, the $\mathbf{3}$-sets $X = \{a/_+, b/_-\}$ and $Y = \{a/_+\}$ are denoted by $X = a\overline{b}$ and $Y = a$ respectively.*

We also consider the functions $\mathrm{Pos}, \mathrm{Neg}, \mathrm{Unk}\colon \mathbf{3}^U \to \mathbf{2}^U$ defined as follows:

$$\begin{aligned}
\mathrm{Pos}(X) &= X^{-1}(+) = \{u \in U \mid X(u) = +\},\\
\mathrm{Neg}(X) &= X^{-1}(-) = \{u \in U \mid X(u) = -\},\\
\mathrm{Unk}(X) &= X^{-1}(\circ) = \{u \in U \mid X(u) = \circ\},
\end{aligned}$$

for each $X \in \mathbf{3}^U$. We can see that Pos and $\mathrm{Neg}(X)$ are isotone functions between $\underline{\mathbf{3}}^U$ and $\underline{\mathbf{2}}^U = (\mathbf{2}^U, \subseteq)$, while that Unk is antitone. In addition, we have that, for all $X \in \mathbf{3}^U$,

$$\mathrm{Spp}(X) = \mathrm{Pos}(X) \cup \mathrm{Neg}(X) = U \smallsetminus \mathrm{Unk}(X) \quad \text{and} \quad \mathrm{Pos}(X) \cap \mathrm{Neg}(x) = \varnothing.$$

We can see that this view matches with the idea of the consistent orthopairs [12] introduced in Section 1.2, that is, given a $\mathbf{3}^U$ set X, we have that it can be seen as an orthopair (P, N) where $P = \mathrm{Pos}(X)$ and $N = \mathrm{Neg}(X)$, and it is consistent since $P \cap N = \varnothing$.

Finally, we define $\overline{(\,)}\colon \mathbf{3}^U \to \mathbf{3}^U$ where, for all $X \in \mathbf{3}^U$ and $u \in U$,

$$\overline{X}(u) = \begin{cases} - & \text{if } X(u) = + \\ \circ & \text{if } X(u) = \circ \\ + & \text{if } X(u) = - \end{cases}$$

Given $X \in \mathbf{3}^U$, $\overline{X}$ is named the *opposite* of X. Obviously, $\mathrm{Pos}(\overline{X}) = \mathrm{Neg}(X)$, $\mathrm{Neg}(\overline{X}) = \mathrm{Pos}(X)$, and $\mathrm{Unk}(\overline{X}) = \mathrm{Unk}(X)$.

3.1.2 The lattices of 4-sets and $\dot{3}$-sets

In this section, first, we extend $\mathbf{3}^U$ to $\mathbf{4}^U$; that is, we consider the possibility of having contradictory information about the belongingness of some elements to the set. Finally, since anything can be derived from a contradiction, it allows us to reduce the size of the structure by considering $\dot{\mathbf{3}}$-sets.

As mentioned, we conceive the $\mathbf{3}$-sets as the knowledge we have about the universal set's elements (properties in the case of FCA). Since we consider a conjunctive interpretation, we can find inconsistencies when we join two different $\mathbf{3}$-sets: a property can be positive in one of the sets and negative in the other set. Thus, in the final set, we have an inconsistent element. Due to these inconsistencies, we must introduce a new kind of set that may contain contradictions, that is, sets valued in $\mathbf{4}$. Thus, we denote by $\mathbf{4}^U$ the set of the $\mathbf{4}$-sets, that is, the set of functions $X\colon U \to \mathbf{4}$. We can assume that $\mathbf{3}^U \subseteq \mathbf{4}^U$. In addition, we point-wise extend the order of $\underline{\mathbf{4}}$ to the set of the $\mathbf{4}$-sets as usual:

$$X \leq Y \qquad \text{if and only if} \qquad X(u) \leq Y(u) \text{ for all } u \in U,$$

obtaining a complete lattice where

$$\Big(\bigvee_{i\in I} X_i\Big)(u) = \bigvee_{i\in I} X_i(u) \qquad \Big(\bigwedge_{i\in I} X_i\Big)(u) = \bigwedge_{i\in I} X_i(u) \qquad \text{for all } u \in U.$$

The minimum of this complete lattice is ε, i.e. the $\mathbf{4}$-set with empty support, whereas the maximum, denoted by $\dot{\imath}$, is the $\mathbf{4}$-set such that $\dot{\imath}(u) = \iota$ for all $u \in U$.

Example 3.1.2 *In $\underline{\mathbf{4}}^{\{a,b\}}$, we have that:*

$$\{a/_+\} \leq \{a/_+, b/_-\} \leq \{a/_\iota, b/_-\} \leq \dot{\imath},$$

In addition, $\{a/_+\} \vee \{b/_-\} = \{a/_+, b/_-\}$, $\{a/_+\} \wedge \{a/_-, b/_-\} = \varepsilon$ *and* $\{a/_+\} \vee \{a/_-, b/_-\} = \{a/_\iota, b/_-\}$.

Observe that this view matches with the idea of orthopairs that may be non consistent, thus, given a $\mathbf{4}^U$ set X, we can see it as the paraconsistent orthopair $(\mathrm{Pos}(X), \mathrm{Neg}(X))$ [12].

As we shall see, our logic is not a classical logic, since it does not fulfill the *law of excluded middle*, but it is neither a paraconsistent logic since the principle of explosion or *ex contradictione sequitur quodlibet* (Latin, "from a contradiction, anything follows") holds. Since anything can be derived from a contradiction, to avoid redundancy in the sets of implications, we consider equivalent all the sets that have some contradiction, i.e., those $X \in \mathbf{4}^U$ with $X(u) = \iota$ for one or more elements $u \in U$. These sets are called *inconsistent sets* whereas the rest of the sets are called *consistent sets*. To formalize it, we define the function:

$$\mathcal{O}\colon \mathbf{4}^U \to \mathbf{4}^U \text{ where } \mathcal{O}(X) = \begin{cases} X & \text{if } X \in \mathbf{3}^U, \\ \dot{\iota} & \text{otherwise.} \end{cases}$$

This function is a closure operator in $\underline{\mathbf{4}}^U$, so it is an $\wedge$-homomorphism. The set of their images, $\mathbf{3}^U \cup \{\dot{\iota}\}$, denoted by $\dot{\mathbf{3}}^U$, is a closure system and, as a consequence, it is also a complete lattice.

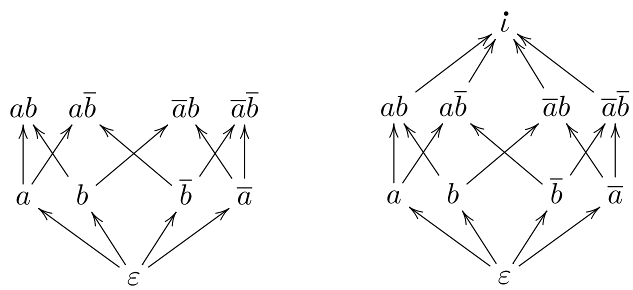
Specifically, if we consider the relation $\leq$ of $\underline{\mathbf{4}}^U$ restricted to $\dot{\mathbf{3}}^U$ we have that $(\dot{\mathbf{3}}^U, \sqsubseteq)$ is a $\wedge$-subsemilattice of $(\mathbf{4}^U, \leq)$, but not a sublattice. Since both infima coincide, they will be denoted by the same symbol: $\wedge$. However, the supremum in $(\mathbf{4}^U, \leq)$ is denoted by the $\vee$, whereas in $(\dot{\mathbf{3}}^U, \sqsubseteq)$ is denoted by $\sqcup$. Thus, we have that, for all $\{A_j : j \in J\} \subseteq \dot{\mathbf{3}}^U$,

$$\bigsqcup_{j \in J} A_j = \mathcal{O}(\bigvee_{j \in J} A_j)$$

and, in particular,

$$\bigsqcup_{j \in J} A_j \neq \dot{\iota} \qquad \text{implies} \qquad \bigsqcup_{j \in J} A_j = \bigvee_{j \in J} A_j. \tag{3.1}$$

We can check that $(\dot{\mathbf{3}}^U, \sqsubseteq)$ is the completion to lattice from $\wedge$-semilattice $\underline{\mathbf{3}}^U$, that is, $(\dot{\mathbf{3}}^U, \sqsubseteq)$ is obtained by adding the element $\dot{\iota}$ as supremum to



(a) The $\wedge$-semilattice $\underline{\mathbf{3}}^{\{a,b\}}$ (b) The lattice $\underline{\dot{\mathbf{3}}}^{\{a,b\}}$

Figure 3.2

the $\wedge$-semilattice $\underline{\mathbf{3}}^U$ (see Fig.3.2). The complete lattice $(\dot{\mathbf{3}}^U, \sqcup, \wedge, \varepsilon, \dot{\imath})$ will be denoted by $\underline{\dot{\mathbf{3}}}^U$, being our target algebraic structure.

Example 3.1.3 *In $(\dot{\mathbf{3}}^{\{a,b\}}, \sqsubseteq)$, we have that $a \sqsubseteq a\overline{b} \sqsubseteq \dot{\imath}$, $a \sqcup \overline{b} = a\overline{b}$, $a \wedge \overline{a}\overline{b} = \varepsilon$ and $a \sqcup \overline{a}\overline{b} = \dot{\imath}$.*

The maximal sets of $\underline{\mathbf{3}}^U$, i.e. those having support U, are the super-atoms of $\underline{\dot{\mathbf{3}}}^U$. These sets are named *full sets* and the set of all of them will be denoted by $\mathcal{F}\mathrm{ull}(U)$. On the other hand, the atoms of $\mathbf{3}^U$ are those $\mathbf{3}$-sets whose support has cardinality one, and are named *singletons*. These are usually denoted by lowercase letters.

Now, we extend the functions Pos, Neg and Unk considering

$$\mathrm{Pos}(\dot{\imath}) = \mathrm{Neg}(\dot{\imath}) = U, \qquad \mathrm{Unk}(\dot{\imath}) = \varnothing.$$

To finish this section, we present the following proposition that gives some properties about the maps Pos and Neg in $\dot{\mathbf{3}}^U$.

Proposition 3.1.4 *Let U be a non-empty set and $X, Y \in \dot{\mathbf{3}}^U$. Then,*

1 $X \sqsubseteq Y$ if and only if $\mathrm{Pos}(X) \subseteq \mathrm{Pos}(Y)$ and $\mathrm{Neg}(X) \subseteq \mathrm{Neg}(Y)$.

2 $X \in \mathcal{F}\mathrm{ull}(U)$ *iff* $\mathrm{Unk}(X) = \varnothing$ *iff* $\mathrm{Pos}(X) \cup \mathrm{Neg}(X) = U$.

3 *The restriction of the functions* Pos *and* Neg *to* $\mathcal{F}\mathrm{ull}(U)$ *are bijections in* $\mathbf{2}^U$.

The proof of this proposition is straightforward from the definitions.

3.2 Partial formal contexts

In this section, we extend the starting point of FCA, that is, the formal contexts, to consider unknown information using the algebraic framework introduced in the previous section. In addition, we compare the extended formal contexts with some classical formal contexts.

We begin by defining a *partial formal context* as a triple $\mathbb{P} = (G, M, I)$ where G and M are non-empty sets, whose elements are called *objects* and *attributes* respectively, and $I\colon G \times M \to \mathbf{3}$ is called the *incidence relation*. $I(g, m) = +$ means that we know that the attribute m is present in the object g whereas $I(g, m) = -$ means that we know that the attribute m is not present in the object g, and, finally, $I(g, m) = \circ$ means that we do not know whether the attribute m is present in the object g or not. We represent these contexts as tables (see Figure 3.3 for instance).

$\mathbb{P}$	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$+$	$+$
3	$-$	$-$	$\circ$

Figure 3.3: Partial formal context $\mathbb{P}$

Given a partial formal context $\mathbb{P} = (G, M, I)$, for each $g \in G$, we consider the **3**-sets $I(g,\ )$ defined as

$$I(g,\ )(x) = I(g, x) \text{ for all } x \in M$$

which gathers the knowledge we have about the attributes of g; and, for each $m \in M$, the **3**-set $I(\ , m)$ defined as

$$I(\ , m)(x) = I(x, m) \text{ for all } x \in G.$$

with our knowledge about the objects having the attribute m.

If a partial formal context $\mathbb{P} = (G, M, I)$ satisfies that $I(g,\ )$ is a full set for all $g \in G$ we say that it is a *total formal context*. Moreover, any (classic) formal context $\mathbb{K} = (G, M, I)$ can be seen as a partial formal context $\mathbb{P} = (G, M, I')$ where $I'(g, m) = +$ if and only if $g\ I\ m$, and $I'(g, m) = \circ$ otherwise. In addition, a partial formal context $\mathbb{P} = (G, M, I)$ can induce the following formal contexts:

- $\mathbb{K}^+_{\mathbb{P}} = (G, M, I^+)$ where $I^+ = I^{-1}(+)$, that is gI^+m iff $I(g, m) = +$.

  Its concept forming operators are denoted by the symbol $+$, that is, for all $X \subseteq G$ and $Y \subseteq M$

  $$X^+ = \bigcap_{g \in X} gI^+(\ ) = \{m \in M \mid gI^+m, \forall g \in X\}.$$
  $$Y^+ = \bigcap_{m \in Y} (\ )I^+m = \{g \in G \mid gI^+m, \forall m \in Y\}.$$

- $\mathbb{K}^-_{\mathbb{P}} = (G, M, I^-)$ where $I^- = I^{-1}(-)$ and its concept forming operators are denoted by the symbol $-$ and defined in a similar way.

- $\mathbb{K}^\circ_{\mathbb{P}} = (G, M, I^\circ)$ where $I^\circ = I^{-1}(\circ)$ and its concept forming operators are denoted by the symbol $\circ$.

- $\mathbb{K}^\oplus_{\mathbb{P}} = (G, M, I^\oplus)$ where $I^\oplus = I^{-1}(+) \cup I^{-1}(\circ)$ and its concept forming operators are denoted by the symbol $\oplus$.

- $\mathbb{K}_{\mathbb{P}}^{\ominus} = (G, M, I^{\ominus})$ where $I^{\ominus} = I^{-1}(-) \cup I^{-1}(\circ)$ and its concept forming operators are denoted by the symbol $\ominus$.
- $\mathbb{K}_{\mathbb{P}}^{\pm} = (G, M, I^{\pm})$ where $I^{\pm} = I^{-1}(-) \cup I^{-1}(+)$ and its concept forming operators are denoted by the symbol $\pm$.

Chapter 4

Necessary concepts and weak implications

In this chapter, we introduce a first Galois connection that generalizes the classical one to give a specific treatment to the unknown information. As in the classical case, this Galois connection induces two forms of knowledge representation: on the one hand, a generalized notion of formal concept with its corresponding hierarchy, and, on the other hand, a notion of attribute implication with its corresponding inference mechanism.

As we will discuss throughout the chapter, there is a duality between these two views if we take into account what can happen when new knowledge is incorporated, that is, when unknown data becomes true or false. On the one hand, concepts defined in this way remain in some sense (although new concepts may appear); therefore we will call them *necessary concepts*. On the other hand, the opposite happens to implications: they may cease to be true; therefore we call them *weak implications*.

4.1 Necessary concepts

In this section, we start by proposing new concept forming operators for partial formal contexts that allow us to capture part of the information stored in the partial formal contexts. As in the classical case, these concept forming operators form a Galois connection and, by using Theorem 2.2.6, the notion of (necessary) formal concepts and their properties can be induced.

Theorem 4.1.1 *Given a partial formal context $\mathbb{P} = (G, M, I)$ and the concept forming operators $(\ )^{\uparrow}\colon \mathbf{2}^G \to \dot{\mathbf{3}}^M$ and $(\ )^{\downarrow}\colon \dot{\mathbf{3}}^M \to \mathbf{2}^G$ defined as*

$$X^{\uparrow} = \bigwedge_{g \in X} I(g,\ ), \quad \textit{and} \quad Y^{\downarrow} = \mathrm{Pos}(Y)^{+} \cap \mathrm{Neg}(Y)^{-},$$

the pair $(\uparrow, \downarrow)$ is a Galois connection between the lattices $\underline{\mathbf{2}}^G$ and $\underline{\dot{\mathbf{3}}}^M$.

PROOF: We prove Condition (2.6), i.e. for all the subsets $X \subseteq G$ and $Y \in \dot{3}^M$ we have that $X \subseteq Y^{\downarrow}$ if and only if $Y \sqsubseteq X^{\uparrow}$.

Assume that $X \subseteq Y^{\downarrow} = \mathrm{Pos}(Y)^{+} \cap \mathrm{Neg}(Y)^{-}$, i.e. $X \subseteq \mathrm{Pos}(Y)^{+}$ and $X \subseteq \mathrm{Neg}(Y)^{-}$. Since $\mathbb{K}^{+}_{\mathbb{P}}$ and $\mathbb{K}^{-}_{\mathbb{P}}$ are (classical) formal contexts, $X \subseteq Y^{\downarrow}$ holds if and only if $\mathrm{Pos}(Y) \subseteq X^{+}$ and $\mathrm{Neg}(Y) \subseteq X^{-}$. By Proposition 3.1.4, we can ensure that it is equivalent to $Y \sqsubseteq X^{\uparrow}$ because it is straightforwardly proved that $X^{+} \subseteq \mathrm{Pos}(X^{\uparrow})$ and $X^{-} \subseteq \mathrm{Neg}(X^{\uparrow})$. $\square$

Definition 4.1.2 *Given a partial formal context $\mathbb{P} = (G, M, I)$, we call* necessary concept *to a pair $(A, B) \in \mathbf{2}^G \times \dot{\mathbf{3}}^M$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. The set of necessary concepts will be denoted by $\mathfrak{B}_{\star}(\mathbb{P})$.*

The fixed pairs of this Galois connection are called necessary concepts because, as the following proposition establishes, these concepts, in some sense, must remain so even when new information appears, i.e., when unknown information (the $\circ$ values) is replaced by positive ($+$ value) or negative ($-$ value) information.

Proposition 4.1.3 *Let $\mathbb{P}_1 = (G, M, I_1)$ and $\mathbb{P}_2 = (G, M, I_2)$ be two partial formal contexts such that $I_1(g,m) \leq I_2(g,m)$ for all $(g,m) \in G \times M$. Then, for all $(A_1, B_1) \in \mathfrak{B}_\star(\mathbb{P}_1)$ there exists $(A_2, B_2) \in \mathfrak{B}_\star(\mathbb{P}_2)$ such that $A_1 \subseteq A_2$ and $B_1 \sqsubseteq B_2$.*

PROOF: It is a direct consequence of the fact that, for all $X \in \mathbf{2}^G$,

$$\bigwedge_{g \in X} I_1(g,\ ) \leq \bigwedge_{g \in X} I_2(g,\ )$$

and, for all $Y \in \dot{\mathbf{3}}^M$, we have that

$$\mathrm{Pos}(Y)^{+_1} \subseteq \mathrm{Pos}(Y)^{+_2} \quad \text{and} \quad \mathrm{Neg}(Y)^{-_1} \subseteq \mathrm{Neg}(Y)^{-_2}$$

where $+_1$ and $+_2$ are the concept forming operators of $\mathbb{K}^+_{\mathbb{P}_1}$ and $\mathbb{K}^+_{\mathbb{P}_2}$, respectively, and $-_1$ and $-_2$ are the concept forming operators of $\mathbb{K}^-_{\mathbb{P}_1}$ and $\mathbb{K}^-_{\mathbb{P}_2}$, respectively. □

The next corollary, which extends the classical FCA's fundamental theorem, is a consequence of Theorems 4.1.1 and 2.2.6.

Corollary 4.1.4 *Given a partial formal context $\mathbb{P} = (G, M, I)$, the set $\mathfrak{B}_\star(\mathbb{P})$ with the order defined as*

$$(A_1, B_1) \leq (A_2, B_2) \quad \textit{if and only if} \quad A_1 \subseteq A_2 \quad \textit{if and only if} \quad B_2 \sqsubseteq B_1$$

is a complete lattice, denoted by $\underline{\mathfrak{B}}_\star(\mathbb{P})$, such that, for all the families of concepts $\{(A_j, B_j) \in \mathfrak{B}_\star(\mathbb{P}) \colon j \in J\}$, the join and the meet are given by:

$$\sup_{j \in J}(A_j, B_j) = \left(\Big(\bigcup_{j \in J} A_j \Big)^{\uparrow\downarrow}, \bigwedge_{j \in J} B_j \right); \quad \inf_{j \in J}(A_j, B_j) = \left(\bigcap_{j \in J} A_j, \Big(\bigsqcup_{j \in J} B_j \Big)^{\downarrow\uparrow} \right)$$

In the following theorem, we can see a connection between the concepts of a formal context and the necessary concepts of a partial formal context:

Theorem 4.1.5 *Given a partial formal context $\mathbb{P} = (G, M, I)$, we have that $\underline{\mathfrak{B}}_\star(\mathbb{P})$ is isomorphic to $\underline{\mathfrak{B}}(\mathbb{K}^-_\mathbb{P} \mid \mathbb{K}^+_\mathbb{P})$.*

Conversely, for any formal context $\mathbb{K} = (G, M, I)$ and $X \subseteq M$, one has that $\underline{\mathfrak{B}}(\mathbb{K})$ is isomorphic to $\underline{\mathfrak{B}}_\star(\mathbb{P}_X)$ where $\mathbb{P}_X = (G, M, I_X)$ with

$$I_X(g,m) = \begin{cases} + & \text{if } g\ I\ m \text{ and } m \in X, \\ - & \text{if } g\ I\ m \text{ and } m \notin X, \\ \circ & \text{otherwise.} \end{cases}$$

PROOF: We just have to see that $h\colon \underline{\mathfrak{B}}_\star(\mathbb{P}) \to \underline{\mathfrak{B}}(\mathbb{K}^-_\mathbb{P} \mid \mathbb{K}^+_\mathbb{P})$ given by $h(A,B) = (A, B_0 \cup B_1)$ (being $B_0 = \{x0\colon x \in \mathrm{Neg}(B)\}$ and being $B_1 = \{x1\colon x \in \mathrm{Pos}(B)\}$) is an isomorphism of lattices. For the converse result, we can define the isomorphism $h\colon \underline{\mathfrak{B}}(\mathbb{K}) \to \underline{\mathfrak{B}}_\star(\mathbb{P}_X)$ such that $h(A,B) = (A, B')$ where, for each $m \in M$, we have that $B'(m) = \circ$ if $m \notin B$, $B'(m) = +$ if $m \in B \cap X$, and $B'(m) = -$ otherwise.

□

$\mathbb{P}$	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$+$	$+$
3	$-$	$-$	$\circ$

$\mathbb{K}^-_\mathbb{P} \mid \mathbb{K}^+_\mathbb{P}$	$a0$	$b0$	$c0$	$a1$	$b1$	$c1$
1			$\times$	$\times$		
2					$\times$	$\times$
3	$\times$	$\times$				

Figure 4.1: Partial formal context $\mathbb{P}$ and formal context $(\mathbb{K}^-_\mathbb{P} \mid \mathbb{K}^+_\mathbb{P})$

Example 4.1.6 *Figure 4.1 shows a partial formal context $\mathbb{P}$, and the classical formal context $(\mathbb{K}^-_\mathbb{P} \mid \mathbb{K}^+_\mathbb{P})$ built from it. The lattice $\underline{\mathfrak{B}}_\star(\mathbb{P})$ is shown in Figure 4.2, which is isomorphic to the lattice $\underline{\mathfrak{B}}(\mathbb{K}^-_\mathbb{P} \mid \mathbb{K}^+_\mathbb{P})$, as Theorem 4.1.5 ensures.*

Corollary 4.1.7 *Let $\mathbb{L} = (L, \leq)$ be a complete lattice and G and M be not empty sets. If there exist mappings $\overline{\gamma}\colon G \to L$ and $\overline{\mu}\colon M \to L$ such that $\overline{\gamma}(G)$ is $\vee$-dense*

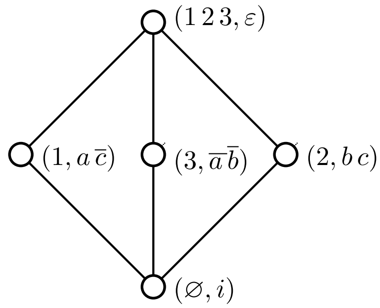


Figure 4.2: The lattice $\underline{\mathfrak{B}}_\star(\mathbb{P}) \cong \underline{\mathfrak{B}}(\mathbb{K}^-_{\mathbb{P}} \mid \mathbb{K}^+_{\mathbb{P}})$

and $\overline{\mu}(G)$ is $\wedge$-dense in $\mathbb{L}$, then $\mathbb{L} \cong \underline{\mathfrak{B}}_\star(\mathbb{P})$ where $\mathbb{P} = (G, M, I)$ with

$$I(g,m) = \begin{cases} + & \textit{if } \overline{\gamma}(g) \leq \overline{\mu}(m), \\ \circ & \textit{otherwise.} \end{cases}$$

That is, every concept lattice of a partial formal context is a complete lattice, and every complete lattice $(L, \leq)$ is isomorphic to some concept lattice of partial formal contexts.

These results were presented at the ICFCA 2021, *16th International Conference on Formal Concept Analysis*, held in June 2021 in Strasbourg (France) [56]. A few days later, Qi et al. published a paper [63] introducing a very similar construction to the one we mentioned above, but their approach only addressed the algebraic structure. Thus, they do not address the issue of attribute implications while we, in [56], defined them and presented a first Armstrong-style axiomatic system. The rest of the chapter is devoted to this issue.

4.2 Weak implications

In this section, we present the first kind of implications that we can extract from a partial formal context by using the previously defined Galois connection. We also present an axiomatic system inspired on Armstrong's axioms and prove its soundness and completeness.

First, we define the notion of weak implications:

Definition 4.2.1 *Given a not empty set of attributes M, we call* weak implication *(of attributes) to any expression $A \rightsquigarrow B$ where $A, B \in \dot{\mathbf{3}}^M$. The set of weak implications will be denoted by*

$$\mathcal{L}_M^w = \{A \rightsquigarrow B\colon A, B \in \dot{\mathbf{3}}^M\}.$$

In this set, which we consider to be the language of the logic, we introduce the semantics as follows:

Definition 4.2.2 *Let C be a $\dot{\mathbf{3}}$-set on M. We say that C is model of a weak implication $A \rightsquigarrow B \in \mathcal{L}_M^w$ if it satisfies that $A \sqsubseteq C$ implies $B \sqsubseteq C$. The set of the models of $A \rightsquigarrow B$ is denoted by $\mathcal{M}\mathrm{od}(A \rightsquigarrow B)$.*

We say that C is model of a theory $\Sigma \subseteq \mathcal{L}_M^w$ if it is model of all weak implication $A \rightsquigarrow B \in \Sigma$, that is,

$$\mathcal{M}\mathrm{od}(\Sigma) = \bigcap_{A \rightsquigarrow B \in \Sigma} \mathcal{M}\mathrm{od}(A \rightsquigarrow B)$$

As usual, we can consider that a partial formal context is a model of a weak implication when the attribute set (which is a $\dot{\mathbf{3}}$-set) of any object is a model of the weak implication.

Definition 4.2.3 *Let $\mathbb{P} = (G, M, I)$ be a partial formal context. We say that $\mathbb{P}$ is model of a weak implication $A \rightsquigarrow B \in \mathcal{L}_M^w$, or that $A \rightsquigarrow B$ holds in $\mathbb{P}$, if $\{g\}^\uparrow \in \mathcal{M}\mathrm{od}(A \rightsquigarrow B)$ for all $g \in G$. It will be denoted by $\mathbb{P} \models A \rightsquigarrow B$.*

We say that a partial formal context $\mathbb{P}$ is model of a set $\Sigma \subseteq \mathcal{L}_M^w$, denoted by $\mathbb{P} \models \Sigma$, if, for all $X \rightsquigarrow Y \in \Sigma$, we have that $\mathbb{P} \models X \rightsquigarrow Y$.

In words, $\mathbb{P}$ is a model of a weak implication $A \rightsquigarrow B$ if for all object that we know that it has all attributes in A, then we also know that it also has all attributes in B. That is, every object g satisfies at least one of the following conditions:

i. The assertion "we know that g has all attributes in A" is false.

ii. We know that g has all attributes in B.

Conversely, there is two possible situations in which a partial context $\mathbb{P}$ is not a model of a weak implication $A \rightsquigarrow B$:

iii. There is an object g in $\mathbb{P}$ that is a counterexample of $A \rightsquigarrow B$ with the information we currently have, that is, we know that g has all attributes in A and lacks any of those in B.

iv. There is an object g in $\mathbb{P}$ that has all attributes in A, but we have no information about if g satisfies some attribute in B.

The following example illustrates these situations.

Example 4.2.4 *Let $\mathbb{P}$ the partial formal context given by Figure 4.3.*

$\mathbb{P}$	a	b	c
1	$\circ$	$+$	$+$
2	$+$	$+$	$-$

Figure 4.3: Partial formal context $\mathbb{P}$

We have that $\mathbb{P} \models a \rightsquigarrow \overline{c}$ because

$$1^{\uparrow} = bc \in \mathcal{M}\mathrm{od}(a \rightsquigarrow \overline{c}) \textit{ since } a \not\sqsubseteq bc$$

$$2^{\uparrow} = ab\overline{c} \in \mathcal{M}\mathrm{od}(a \rightsquigarrow \overline{c}) \textit{ since } a \sqsubseteq ab\overline{c} \textit{ and } \overline{c} \sqsubseteq ab\overline{c}$$

However, $\mathbb{P} \not\models a \rightsquigarrow c$, because 2 is a counterexample since $a \sqsubseteq 2^{\uparrow}$ and $c \not\sqsubseteq 2^{\uparrow}$, and $\mathbb{P} \not\models b \rightsquigarrow a$, because for 1 we have that $b \sqsubseteq 1^{\uparrow}$ and $a \not\sqsubseteq 1^{\uparrow}$.

Notice that, $\mathbb{P}$ is a model for $a \rightsquigarrow \overline{c}$ because 1 satisfies the above condition i. and 2 satisfies the condition ii. That is, every object satisfies either i. or ii. On the other hand, $\mathbb{P}$ is not a model for $a \rightsquigarrow c$ because iii. holds (see the information about the object 2), and $\mathbb{P}$ is not a model for $b \rightsquigarrow a$ because iv. holds (see the object 1).

Note that, if we obtain new information about whether object 1 has attribute a or not, the implication $a \rightsquigarrow \overline{c}$ may or may not still be fulfilled. Similarly, the implication $b \rightsquigarrow a$ could switch from false to true. This is why we call this kind of implication as weak implication.

As in the classical case, we can easily characterize the implications that are satisfied by a context by using the concept forming operators.

Proposition 4.2.5 *Let $\mathbb{P} = (G, M, I)$ be a partial formal context, $\uparrow$ and $\downarrow$ be its concept forming operators, and $A \rightsquigarrow B \in \mathcal{L}_M^w$. Then,*

$$\mathbb{P} \models A \rightsquigarrow B \quad \text{if and only if} \quad A^{\downarrow} \subseteq B^{\downarrow}, \quad \text{if and only if} \quad B \sqsubseteq A^{\downarrow\uparrow}.$$

Now, we introduce the notion of semantic entailment.

Definition 4.2.6 *Let $A \rightsquigarrow B \in \mathcal{L}_M^w$ and $\Sigma \subseteq \mathcal{L}_M^w$. We say that $A \rightsquigarrow B$ is semantically entailed from Σ, denoted by $\Sigma \models A \rightsquigarrow B$, when $\mathbb{P} \models \Sigma$ implies $\mathbb{P} \models A \rightsquigarrow B$ for all partial formal context $\mathbb{P}$.*

In order to simplify the notation, when there is not any possible confusion, we denote the sets of implications without curly brackets. In the same way, we write $\models A \rightsquigarrow B$ instead of $\varepsilon \models A \rightsquigarrow B$.

Proposition 4.2.7 *Let $\mathbb{P} = (G, M, I)$ be a partial formal context and $A, B \subseteq M$.*[1]

[1]Note that we are abusing the notation here, but there is no place for confusion. A set $A \subseteq M$ can also be seen as a $\dot{\mathbf{3}}$-set, the one in which $A(m) = +$ for all $m \in A$ and $\circ$ in any other case.

1. *If* $\mathbb{K}_{\mathbb{P}}^{+} \models A \to B$ *then* $\mathbb{P} \models A \rightsquigarrow B$.
2. *If* $\mathbb{K}_{\mathbb{P}}^{-} \models A \to B$ *then* $\mathbb{P} \models \overline{A} \rightsquigarrow \overline{B}$.

PROOF: If $A \subseteq M$, we have that $\mathrm{Pos}(A) = A$ and $\mathrm{Neg}(A) = \varnothing$. Then, as a consequence, $A^{\downarrow} = A^{+} \cap \varnothing^{-} = A^{+} \cap G = A^{+}$. Analogously, we have that $B^{\downarrow} = B^{+}$. Finally, from $\mathbb{K}_{\mathbb{P}}^{+} \models A \to B$ we have that $A^{+} \subseteq B^{+}$ and, so, $A^{\downarrow} \subseteq B^{\downarrow}$, i.e. $\mathbb{P} \models A \rightsquigarrow B$. The second item is a consequence of

$$\overline{A}^{\downarrow} = \mathrm{Pos}(\overline{A})^{+} \cap \mathrm{Neg}(\overline{A})^{-} = \mathrm{Neg}(A)^{+} \cap \mathrm{Pos}(A)^{-} = \varnothing^{+} \cap A^{-} = G \cap A^{-} = A^{-},$$

and, analogously, $\overline{B}^{\downarrow} = B^{-}$. □

4.2.1 Armstrong-style axiomatic system

The third pillar of the logic is the axiomatic system. In this case, we consider an axiomatic system called 'Armstrong-style' and inspired on Armstrong's axioms, which we will prove to be correct and complete. The difference with this axiomatic system and the Armstrong's Axioms resides in the meaning of AB: whereas in the Armstrong's axioms AB means $A \cup B$, in our logic it means $A \sqcup B$, that is, it is not an union of sets but it is the supremum of two $\dot{\mathbf{3}}$-sets.

Definition 4.2.8 *The* Armstrong-style *axiomatic system for weak implications, denoted by $\mathcal{A}$, considers one axiom and two inference rules. They are the following: for all $A, B, C \in \underline{\dot{\mathbf{3}}}^{M}$,*

[`Inc`] *Inclusion: Infer $AB \rightsquigarrow A$.*

[`Augm`] *Augmentation: From $A \rightsquigarrow B$ infer $AC \rightsquigarrow BC$.*

[`Trans`] *Transitivity: From $A \rightsquigarrow B$ and $B \rightsquigarrow C$ infer $A \rightsquigarrow C$.*

The notion of syntactic derivation is introduced in the standard way.

Definition 4.2.9 *A weak implication ϕ is said to be syntactically derived, or it is inferred, from a set of weak implications Σ, denoted by $\Sigma \vdash_{\mathcal{A}} \phi$, if there is a sequence of weak implications $\phi_1, ..., \phi_n$ such that $\phi_n = \phi$ and, for all $1 \leq i \leq n$, one of the following conditions is satisfied: $\phi_i \in \Sigma$, ϕ_i is an axiom, or ϕ_i is obtained from implications belonging to $\{\phi_j \mid 1 \leq j < i\}$ by applying the inferences rules of $\mathcal{A}$. In this case, we say that the sequence $\{\phi_i \mid 1 \leq i \leq n\}$ is a proof for $\Sigma \vdash_{\mathcal{A}} \phi$.*

As usual, we consider some derived rules from the Armstrong's axioms which are easily proved.

Proposition 4.2.10 *Let M be a finite set of attributes. The following inference rules are derived from the Amstrong's axioms: for all $A, B, C \in \dot{\mathbf{3}}^M$,*

[`Frag`] *Fragmentation: $A \rightsquigarrow BC \vdash_{\mathcal{A}} A \rightsquigarrow B$*

[`Un`] *Union: $A \rightsquigarrow B, A \rightsquigarrow C \vdash_{\mathcal{A}} A \rightsquigarrow BC$*

[`gTr`] *Generalized Transitivity: $A \rightsquigarrow BC, B \rightsquigarrow D \vdash_{\mathcal{A}} A \rightsquigarrow D$.*

PROOF: A proof for [`Frag`] is the sequence

$$\phi_1 = (A \rightsquigarrow BC), \quad \phi_2 = (BC \rightsquigarrow B), \quad \phi_3 = (A \rightsquigarrow B),$$

where ϕ_1 is the hypothesis, ϕ_2 is the axiom [`Inc`] and ϕ_3 is obtained from ϕ_1 and ϕ_2 by using [`Trans`].

The sequence $\phi_1 = (A \rightsquigarrow B)$, $\phi_2 = (A \rightsquigarrow C)$, $\phi_3 = (AC \rightsquigarrow BC)$, $\phi_4 = (A \rightsquigarrow AC)$, $\phi_5 = (A \rightsquigarrow BC)$ is a proof for [`Un`], where ϕ_1 and ϕ_2 are the hypothesis, ϕ_3 is obtained from ϕ_1 by using [`Augm`], ϕ_4 is obtained from ϕ_2 by using [`Augm`] and, finally, ϕ_5 is inferred from ϕ_4 and ϕ_3 by using [`Trans`].

Finally, the sequence $\phi_1 = (A \rightsquigarrow BC)$, $\phi_2 = (BC \rightsquigarrow B)$, $\phi_3 = (A \rightsquigarrow B)$, $\phi_4 = (B \rightsquigarrow D)$ and $\phi_5 = (A \rightsquigarrow D)$ is a proof for [`Un`], where ϕ_1 and ϕ_4 are the hypothesis, ϕ_2 is the axiom [`Inc`], ϕ_3 is obtained by using [`Trans`] with ϕ_1 and ϕ_2, ϕ_5 is obtained by using [`Trans`] with ϕ_3 and ϕ_4. $\square$

Now, we focus on proving that the axiomatic system $\mathcal{A}$ is sound and complete.

Theorem 4.2.11 (Soundness) *For all weak implication $A \rightsquigarrow B \in \mathcal{L}_M^w$ and all set $\Sigma \subseteq \mathcal{L}_M^w$, we have that $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow B$ implies $\Sigma \models A \rightsquigarrow B$.*

PROOF: It is a direct consequence of Proposition 4.2.5 and the fact that $(\ )^{\downarrow\uparrow}$ is a closure operator in $\underline{\dot{\mathbf{3}}}^M$. □

In order to prove the completeness of the axiomatic system, we first introduce some necessary results.

Theorem 4.2.12 *Let M be a finite set and $\Sigma \in \mathcal{L}_M^w$. The mapping*

$$(\ )^{\sharp}_{\Sigma} \colon \dot{\mathbf{3}}^M \to \dot{\mathbf{3}}^M \quad \textit{defined as} \quad A^{\sharp}_{\Sigma} = \bigsqcup\{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash_{\mathcal{A}} A \rightsquigarrow X\}$$

is a closure operator in $\underline{\dot{\mathbf{3}}}^M$ that we name the syntactic closure *with respect to Σ. In addition, for all $A \in \dot{\mathbf{3}}^M$, we have that $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow A^{\sharp}_{\Sigma}$.*

PROOF: By [`Inc`], $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow A$ and, then, $A \sqsubseteq A^{\sharp}_{\Sigma}$.

Let's prove the isotonicity of $(\ )^{\sharp}_{\Sigma}$. Assume that $A \sqsubseteq B$. By [`Inc`], $\Sigma \vdash_{\mathcal{A}} B \rightsquigarrow A$, and, by [`Trans`], for all $X \in \dot{\mathbf{3}}^M$ such that $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow X$, we have that $\Sigma \vdash_{\mathcal{A}} B \rightsquigarrow X$. Thus,

$$\{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash_{\mathcal{A}} A \rightsquigarrow X\} \subseteq \{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash_{\mathcal{A}} B \rightsquigarrow X\}$$

and, therefore, $A^{\sharp}_{\Sigma} \sqsubseteq B^{\sharp}_{\Sigma}$.

In order to prove the idempotency of the mapping $(\ )^{\sharp}_{\Sigma}$, we previously demonstrate that $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow A^{\sharp}_{\Sigma}$ for all $A \in \dot{\mathbf{3}}^M$. As M is finite, the set $\mathcal{X} = \{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash_{\mathcal{A}} A \rightsquigarrow X\}$ is also finite. It is enough to prove that $X, Y \in \mathcal{X}$ implies $X \sqcup Y \in \mathcal{X}$ and it is straightforward from [`Un`]. Therefore, $A^{\sharp}_{\Sigma} = \bigsqcup \mathcal{X} \in \mathcal{X}$ and $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow A^{\sharp}_{\Sigma}$.

Finally, $(A^\sharp_\Sigma)^\sharp_\Sigma \sqsubseteq A^\sharp_\Sigma$ because $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow A^\sharp_\Sigma$, $\Sigma \vdash_{\mathcal{A}} A^\sharp_\Sigma \rightsquigarrow (A^\sharp_\Sigma)^\sharp_\Sigma$ and, by transitivity, $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow (A^\sharp_\Sigma)^\sharp_\Sigma$. Therefore, since $(\ )^\sharp_\Sigma$ is inflationary, we have that $(\ )^\sharp_\Sigma$ is idempotent. $\square$

Corollary 4.2.13 *Let M be a finite set of attributes. For all $\Sigma \subseteq \mathcal{L}^w_M$ and all $A, B \in \dot{\mathbf{3}}^M$, we have that*

$$\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow B \quad \textit{if and only if} \quad B \sqsubseteq A^\sharp_\Sigma.$$

PROOF: The direct implication is a consequence of Theorem 4.2.12, and the converse result is obtained by using [`Inc`] and [`Trans`]. $\square$

Lemma 4.2.14 *Let M be a finite set of attributes. For all $\Sigma \subseteq \mathcal{L}^w_M$ and $A \in \dot{\mathbf{3}}^M$, we have that*

$$A^\sharp_\Sigma = \min\{X \in \mathcal{M}\mathrm{od}(\Sigma) \mid A \sqsubseteq X\}.$$

PROOF: Let $X \in \mathcal{M}\mathrm{od}(\Sigma)$ such that $A \sqsubseteq X$. From Theorem 4.2.11, $X \in \mathcal{M}\mathrm{od}(\Sigma)$ implies that $X \in \mathcal{M}\mathrm{od}(B \rightsquigarrow C)$ for all $B \rightsquigarrow C \in \mathcal{L}^w_M$ with $\Sigma \vdash_{\mathcal{A}} B \rightsquigarrow C$ and, particularly, by Theorem 4.2.12, $X \in \mathcal{M}\mathrm{od}(A \rightsquigarrow A^\sharp_\Sigma)$. Thus, $A \sqsubseteq X$ implies $A^\sharp_\Sigma \sqsubseteq X$.

Moreover, we prove that $A^\sharp_\Sigma \in \mathcal{M}\mathrm{od}(\Sigma)$. For all $B \rightsquigarrow C \in \Sigma$, if $B \sqsubseteq A^\sharp_\Sigma$, by Corollary 4.2.13, we have that $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow B$. Then, by [`Trans`], $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow C$ and, again, by Corollary 4.2.13, we have that $C \sqsubseteq A^\sharp_\Sigma$. $\square$

With all this, we now have the necessary tools to prove the completeness of the axiomatic system.

Theorem 4.2.15 (Completeness) *Let M be a finite set of attributes. For all $A \rightsquigarrow B \in \mathcal{L}^w_M$ and $\Sigma \subseteq \mathcal{L}^w_M$, we have that $\Sigma \models A \rightsquigarrow B$ implies $\Sigma \vdash_{\mathcal{A}} A \rightsquigarrow B$.*

PROOF: Let's prove that $\Sigma \nvdash_{\mathcal{A}} A \rightsquigarrow B$ implies $\Sigma \not\models A \rightsquigarrow B$. Using the Corollary 4.2.13, we have that $\Sigma \nvdash_{\mathcal{A}} A \rightsquigarrow B$ implies that $B \not\sqsubseteq A^\sharp_\Sigma$, and, therefore, $A^\sharp_\Sigma \neq ¿$.

Let us consider the partial formal context $\mathbb{P} = (G, M, I)$ being $G = \mathcal{M}\mathrm{od}(\Sigma) \smallsetminus \{i\}$ and $I\colon G \times M \to \mathbf{3}$ where $I(g, \ ) = g$ for each $g \in G$.

It is straightforward that $\mathbb{P} \models \Sigma$ because $\{g\}^{\uparrow} = I(g, \ ) = g \in \mathcal{M}\mathrm{od}(\Sigma)$. However, $\mathbb{P} \not\models A \rightsquigarrow B$ because, by Lemma 4.2.14, $A^{\sharp}_{\Sigma} \in G$, $A \sqsubseteq A^{\sharp}_{\Sigma}$ and $B \not\sqsubseteq A^{\sharp}_{\Sigma}$. $\square$

Chapter 5

Weak complete dual Heyting algebras

Classical Simplification Logic [53] is based on the Boolean algebra of sets. However, in [6] it has been extended to the fuzzy framework using a weaker structure, the complete dual Heyting algebra. As mentioned in the preliminaries (see Remark 2.1.21), in order to build a complete dual Heyting algebra from a complete lattice by endowing it with the convenient difference operation, it is necessary that the lattice be completely distributive. However, the lattices introduced in Chapter 3, on which we base all the theory developed in this work, are not necessarily distributive. See, for instance, the lattice $\underline{\dot{\mathbf{3}}}^{\{a,b\}}$ shown in Figure 3.2. Therefore, we will need a weaker structure than the complete dual Heyting algebra in order to have a logic based on the simplification paradigm in this framework that incorporates unknown information.

For this purpose, in [58], we introduced the structure of "weak complete dual Heyting algebra", which we present in this chapter. In addition, we study what conditions a complete lattice must fulfil so that, by adding a suitable difference operation to it, it has this structure. Then, in Chapter 6, we will show that, relying on this structure, we can define sound and complete axiomatic systems based on the simplification paradigm.

5.1 Definition and first properties

Throughout this chapter, $\mathbb{L} = (L, \leqslant)$ will denote a complete lattice, and, as usual, the symbols $\wedge$ and $\vee$ will be used for the infimum and supremum, and the symbols $\top$ and $\perp$ for the maximum and minimum, respectively. We begin with the definition of a weak complete dual Heyting algebra:

Definition 5.1.1 *Given a complete lattice $(L, \leqslant)$ and a mapping $\smallsetminus\colon L \times L \to L$, we say that $(L, \leqslant, \smallsetminus)$ is a* weak complete dual Heyting algebra*, for short* weak-cdHa*, if the following conditions hold:*

`[wH1]` $x \vee y \neq \top$ *implies* $(x \vee y) \smallsetminus x \leqslant y$*, for all* $x, y \in L$.

`[wH2]` $x \smallsetminus y \leqslant x$*, for all* $x, y \in L$.

`[wH3]` $x \smallsetminus y = \perp$ *if and only if* $x \leqslant y$*, for all* $x, y \in L$.

`[wH4]` $x \vee y = x \vee (y \smallsetminus x)$*, for all* $x, y \in L$.

The following proposition gives an alternative characterization of the weak-cdHas.

Proposition 5.1.2 *Let $(L, \leqslant)$ be a complete lattice. The algebra $(L, \leqslant, \smallsetminus)$ is a weak-cdHa, if and only if it satisfies* `[wH1]`, `[wH2]` *and the following ones:*

`[wH3′]` $x \leqslant y$ *implies* $x \smallsetminus y = \perp$*, for all* $x, y \in L$.

`[wH4′]` $y \leqslant x \vee (y \smallsetminus x)$*, for all* $x, y \in L$.

PROOF: It is straightforward that [wH3′] and [wH4′]hold in any weak-cdHa. Now, assume that [wH1], [wH2], [wH3′], and [wH4′] hold and prove [wH3] and [wH4]. If $x \setminus y = \bot$, by [wH4′], we have that $x \leqslant y \vee (x \setminus y) = y \vee \bot = y$. The converse implication is due to [wH3′]. Finally, by [wH4′] and [wH2], we have that $x \vee y \leqslant x \vee (y \setminus x) \leqslant x \vee y$. □

The following proposition gives one derived property that we will use later.

Proposition 5.1.3 *Let $(L, \leqslant, \setminus)$ be a weak-cdHa. Then $x \setminus \bot = x$ for all $x \in L$.*

PROOF: Since $\bot$ is the infimum of L and, by using [wH4], we have that, for all $x \in L$, $x \setminus \bot = \bot \vee (x \setminus \bot) = \bot \vee x = x$. □

The following proposition establishes the relationship between the complete dual Heyting algebras and the weak-cdHa.

Proposition 5.1.4 *Any complete dual Heyting algebra is a weak-cdHa.*

PROOF: Assume that $(L, \leqslant, \setminus)$ is a complete dual Heyting algebra. See its characterization in Proposition 2.1.22. We have to prove that [wH1], [wH2], [wH3] and [wH4] hold.

1. Consider $x, y \in L$ satisfying $x \vee y \neq \top$, by [H2],

   $$y \vee \Big((x \vee y) \setminus x\Big) = y \vee \Big((x \vee y) \setminus (y \vee x \vee y)\Big) = y \vee \Big((x \vee y) \setminus (x \vee y)\Big).$$

   By [H3] , $(x \vee y) \setminus (x \vee y) = \bot$, so we have that

   $$y \vee \Big((x \vee y) \setminus x\Big) = y \vee \bot = y,$$

   or, equivalently, $(x \vee y) \setminus x \leqslant y$. That is, [wH1] holds.

2. For all $x, y \in L$, by [H2], we have that

   $$x \vee (x \setminus y) = x \vee \Big((x \vee x) \setminus (x \vee y)\Big) = x \vee \Big(x \setminus (x \vee y)\Big).$$

By [H2] again, $x \setminus (x \vee y) = \bot$, and then

$$x \vee (x \setminus y) = x \vee \bot = x,$$

or, equivalently, $x \setminus y \leqslant x$. That is, [wH2] holds.

3. If $x \leq y$, by [H2], $x \setminus y = \bot$. Conversely, let us suppose that $x \setminus y = \bot$. On the one hand,

$$y \vee (x \setminus y) = y \vee \bot = y.$$

On the other hand, by [H1], we have that

$$y \vee (x \setminus y) = y \vee x.$$

Therefore, $y = y \vee x$ and consequently, $x \leqslant y$.

4. [wH4] is equal to [H1].

□

The above Proposition can not be improved, that is, there are weak-cdHas that are not complete dual Heyting algebras. See, for instance, Example 5.2.1.

5.2 Characterization of the weak complete dual Heyting algebras

The aim of this section is to find, given a complete lattice $(L, \leqslant)$, necessary and sufficient conditions that must be satisfied to establish an operation $\setminus$ that allows $(L, \leqslant, \setminus)$ to be a weak-cdHa. We will also study the uniqueness, or not, of such an operation.

As previously proved, any complete dual Heyting algebra is a weak-cdHa. Thus, Condition (2.3). i.e.

$$\text{for all } a, b \in L,\ \min\{x \mid a \leqslant b \vee x\} \text{ exists},$$

it is enough, but not mandatory. The following example shows a lattice in which Condition (2.3) is not satisfied and in which, not only can we define an operation that converts it to weak-cdHa, but there is more than one.

Example 5.2.1 *Consider the complete lattices depicted in Figure 5.1a and Figure 5.1b, denoted by* $(L, \leqslant_1)$ *and* $(L, \leqslant_2)$*, which are known as Diamond and Pentagon, respectively.*

(a) $(L, \leqslant_1)$ (b) $(L, \leqslant_2)$

Figure 5.1: Diamond and Pentagon lattices

Both lattices do not satisfy Condition (2.3)*, because in both cases*

$$\min\{x \mid c \leqslant a \vee x\} = \min\{b, c, \top\}$$

do not exist. However, over $(L, \leqslant_1)$*, we can define not only one weak-cdHa, but we can define, at least, two different weak-cdHas:* $(L, \leqslant_1, \diagdown_1)$ *and* $(L, \leqslant_1, \diagdown_2)$*, where*

$\diagdown_1$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	a	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	c	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

$\diagdown_2$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	a	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	c	$\bot$	$\bot$
$\top$	$\top$	b	c	b	$\bot$

Analogously, $(L, \leqslant_2, \smallsetminus_3)$ *and* $(L, \leqslant_2, \smallsetminus_4)$ *are also two different weak-cdHas over the same lattice* $(L, \leqslant_2)$*, where*

$\smallsetminus_3$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	$\bot$	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	c	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

$\smallsetminus_4$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	$\bot$	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	a	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

Furthermore, Remark 2.1.21 establishes that a necessary condition for $(L, \leqslant, \smallsetminus)$ being a complete dual Heyting algebra is that $(L, \leqslant)$ need to be a distributive lattice. In addition, infinite distributivity is a sufficient condition to build a complete dual Heyting algebra, which is also a weak-cdHa. However, it is not a necessary condition for being a weak-cdHa, as previous Example 5.2.1 shows.

We have shown complete lattices in which different operations define different weak-cdHas, in contrast to the situation with complete dual Heyting algebras. The next question is whether, for any complete lattice $(L, \leqslant)$, there is a difference operation $\smallsetminus : L \times L \to L$ that converts it into a weak-cdHa $(L, \leqslant, \smallsetminus)$. The example below shows that this is not true.

Example 5.2.2 *The desired difference operation does not exist for the complete lattice shown in Figure 5.2, denoted by* $(L, \leqslant)$*.*

Assume that there exists $\smallsetminus : L \times L \to L$ *such that* $(L, \leqslant, \smallsetminus)$ *is a weak-cdHa.*

First, by `[wH4]`*, we have that* $d = d \vee a = a \vee (d \smallsetminus a)$ *and, therefore,*

$$d \smallsetminus a \in \{c, b, d\}. \tag{5.1}$$

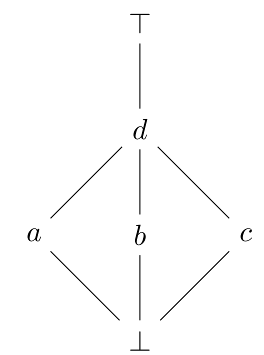


Figure 5.2: The Diamond lattice with one extra vertex

Second, by `[wH1]`*, we have that* $d \smallsetminus a = (a \vee b) \smallsetminus a \leqslant b$*,* $d \smallsetminus a = (a \vee c) \smallsetminus a \leqslant c$*, and* $d \smallsetminus a = (a \vee d) \smallsetminus a \leqslant d$*. As a consequence* $d \smallsetminus a \leqslant b \wedge c \wedge d = \bot$*, which contradicts* (5.1).

Notice that, as expected, the lattice of the previous example is not distributive.

In summary, we have seen that there are lattices in which we can define more than one weak-cdHas and there are other lattices in which we cannot define any. Now, the following characterisation theorem for weak-cdHas will later provide us sufficient and necessary conditions for the existence and uniqueness issues (see Corollary 5.2.4, Corollary 5.2.5 and Theorem 5.2.6, respectively.)

Theorem 5.2.3 (Characterisation) *Consider a complete lattice* $(L, \leqslant)$ *and a difference operation* $\smallsetminus\colon L \times L \to L$*. Then,* $(L, \leqslant, \smallsetminus)$ *is a weak-cdHa if and only if* *the following conditions are satisfied:*

$$x \setminus y = \min\{z \mid z \vee y = x \vee y\} \textit{ for all } x, y \in L \textit{ with } x \not\parallel y \textit{ and } x \neq \top \quad (5.2)$$

$$\top \setminus \top = \bot. \quad (5.3)$$

$$\top \setminus y \in \{z \mid z \vee y = \top\} \textit{ for all } y \in L \textit{ with } y \neq \top. \quad (5.4)$$

$$x \setminus y \in \{z \mid z \leqslant x \textit{ and } z \vee y = x \vee y\} \textit{ for all } x, y \in L \textit{ with } x \parallel y. \quad (5.5)$$

PROOF: First, assume that $(L, \leqslant, \setminus)$ is a weak-cdHa and prove that it holds the four assertions:

To prove (5.2), consider $x, y \in L$ such that x and y are comparable, being $x \neq \top$, and distinguish two cases:

- If $x \vee y = y$, by `[wH3]`, one has that $x \setminus y = \bot = \min\{z \mid z \vee y = x \vee y\}$.
- In a different situation, i.e. if $x \vee y = x \neq \top$, then $x \setminus y \in \{z \in L \mid z \vee y = x \vee y\}$ by `[wH4]`. In addition, for all $z \in \{z \in L \mid z \vee y = x \vee y\}$, one has that $z \vee y = x \neq \top$ and, by `[wH1]`, $x \setminus y \leqslant z$. Consequently, $x \setminus y = \min\{z \in L \mid z \vee y = x \vee y\}$.

(5.3) is straightforward from `[wH3]`.

(5.4) is equal to $(\top \setminus y) \vee y = \top$ for all $y \in L$ with $y \neq \top$, which is a particular case of `[wH4]`.

Finally, to prove (5.5) assume that $x, y \in L$ being x not comparable with y. By `[wH2]` and `[wH4]`, one has that $x \setminus y \in \{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\}$.

Conversely, consider a complete lattice $(L, \leqslant)$ and a difference operantion $\setminus : L \times L \to L$ fulfilling (5.2)–(5.5), and let us prove that $(L, \leqslant, \setminus)$ is a weak-cdHa, i.e. `[wH1]`, `[wH2]`, `[wH3]` and `[wH4]` hold.

Let $x, y \in L$ with $x \vee y \neq \top$. By (5.2), $(x \vee y) \setminus y = \min\{z \in L \mid z \vee y = x \vee y\}$. Therefore, $(x \vee y) \setminus y \leqslant z$ for all $z \in \{z \in L \mid z \vee y = x \vee y\}$ and, in particular, $(x \vee y) \setminus y \leqslant x$, i.e. `[wH1]` holds.

In all the cases, (5.2)–(5.5), it is straightforward that `[wH2]` holds. Notice that (5.2)–(5.5) exhaustively describe all the situations for $x \setminus y$, depicting a classification in four disjoints cases.

Let's prove `[wH3]`. On the one hand, assume that $x \leqslant y$. If $x = \top$, by (5.3), $\top \setminus y = \top \setminus \top = \bot$. In other case, $x \setminus y = \min\{z \in L \mid z \vee y = x \vee y\}$, by (5.2), and this minimum element is $\bot$ because $\bot \vee y = y = x \vee y$. Thus, $x \leqslant y$ implies $x \setminus y = \bot$.

On the other hand, assume $x \nleqslant y$ and prove $x \setminus y \neq \bot$. If $y < x \neq \top$, by (5.2), $x \setminus y = \min\{z \in L \mid z \vee y = x \vee y\}$, which is not $\bot$ because $\bot \vee y = y \neq x \vee y = x$. Analogously, by (5.4), it is proved that $y < x = \top$ implies $x \setminus y \neq \bot$. Finally, if x is not comparable with y, by (5.5), $x \setminus y \in \{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\}$ and, therefore, $x \setminus y \neq \bot$ because $\bot \vee y \neq x \vee y$.

Finally, it is straightforward that `[wH4]` holds in all the cases (5.2)–(5.5).

□

Given a complete lattice, since the sets $\{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\}$ and $\{z \in L \mid z \vee x = \top\}$ are always non empty, we can always define a $\setminus$ operation holding (5.3)–(5.5). Thus, we focus on (5.2); i.e., on the existence of $\min\{z \in L \mid z \vee y = x \vee y\}$ for all $x, y \in L$ with x comparable with y and being $x \neq \top$. In addition, if $x \leqslant y$, that minimum always exists and it is $\bot$. Otherwise, if x is $\vee$-irreducible, $\{z \in L \mid z \vee y = x \vee y\} = \{x\}$ and the minimum also exists. In summary, there exists just one situation where (5.2) are not guaranteed: $x > y$ and x is a $\vee$-reducible element. The following corollary presents this situation that we have justified above:

Corollary 5.2.4 *Consider a complete lattice $(L, \leqslant)$. There exists a difference operation $\setminus : L \times L \to L$ such that $(L, \leqslant, \setminus)$ is a weak-cdHa if and only if the following holds:*

$$\min\{z \in L \mid z \vee y = x\} \textit{ exists for all } \vee\textit{-reducible } x \neq \top \textit{ and all } y < x \tag{5.6}$$

Once we have characterised the existence, we focus on the uniqueness issue. Recall that in Example 5.2.1 we show two lattices where uniqueness does not fulfil. In the first one, $\{z \in L \mid z \vee a = \top\}$ is not a singleton (see (5.4)) whereas in the second one, $\{z \in L \mid z \leqslant c$ and $z \vee b = c \vee b\}$ is either not a singleton (see (5.5)).

Since (5.2)–(5.5) exhaustively describe all the situations for $x \smallsetminus y$, depicting a classification in four disjoints cases, we have the following corollary from Theorem 5.2.3.

Corollary 5.2.5 *Consider a complete lattice $(L, \leqslant)$ such that there exists a difference operation $\smallsetminus : L \times L \to L$ satisfying that $(L, \leqslant, \smallsetminus)$ is a weak-cdHa. Then, this operation $\smallsetminus$ is the unique one with $(L, \leqslant, \smallsetminus)$ being a weak-cdHa if and only if the following properties hold:*

$$\{z \in L \mid z \vee y = \top\} = \{\top\} \text{ for all } y \in L \text{ with } y \neq \top. \tag{5.7}$$

$$\{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\} = \{x\} \text{ for all } x, y \in L \text{ with } x \parallel y. \tag{5.8}$$

The subsequent theorem provides a criterion that establishes both the necessary and sufficient conditions for the uniqueness of the weak-cdHas.

Theorem 5.2.6 (Uniqueness) *Consider a complete lattice $(L, \leqslant)$ satisfying* (5.6). *There is just one difference operation $\smallsetminus : L \times L \to L$ being $(L, \leqslant, \smallsetminus)$ a weak-cdHa if and only if the following conditions hold:*

$$\top \text{ is } \vee\text{-irreducible.} \tag{5.9}$$

$$\text{For all } x, y, z \in L,\ x \parallel y \parallel z \text{ and } z \vee y = x \vee y \text{ implies } z = x. \tag{5.10}$$

PROOF: From the $\vee$-irreducibleness definition, (5.9) and (5.7) are equivalent. It is also straightforward that (5.10) implies (5.8). We conclude the proof showing that (5.6), (5.7) (or, equivalently, (5.9)) and (5.8) imply (5.10).

Let $x, y, z \in L$ with $x \parallel y \parallel z$ and $z \vee y = x \vee y$. Consider $w = x \vee y$, which is $\vee$-reducible (because, the opposite contradicts that $x \parallel y$) and, therefore, by (5.9), we have that $w \neq \top$ and $y < w$. In addition, by (5.6) we have that there exists $v = \min\{t \mid t \vee y = w\}$. Thus, $v \leqslant x$ and $v \leqslant z$ because $x, z \in \{t \mid t \vee y = w\}$. Furthermore, $v \vee y = w = x \vee y$. From (5.8), we have that $v = x$ and, as consequence, $x \leqslant z$.

Repeating each step of the previous paragraph, but swapping the roles of x and z, we have that $z \leqslant x$, concluding that $z = x$.

□

5.3 The weak complete dual Heyting algebra $\underline{\dot{\mathbf{3}}}^U$

In this section we will show an example of a weak complete dual Heyting algebra that will be used along of this document to work with FCA for unknown values. This is another example that show that the above Proposition 5.1.4 can not be improved, that is, this weak-cdHa is not a complete dual Heyting algebra.

First, we introduce two complete dual Heyting algebras, denoted by $\underline{\mathbf{4}}$ and $\underline{\mathbf{4}}^U$ respectively, from which we build the complete weak dual Heyting algebra $\underline{\dot{\mathbf{3}}}^U$ that will be the key point to define the Simplification Logic.

From the lattice $(\mathbf{4}, \vee, \wedge, \circ, \iota)$ introduced in Section 3.1, since it is distributive, we can define the complete dual Heyting algebra $\underline{\mathbf{4}} = (\mathbf{4}, \vee, \wedge, \smallsetminus, \circ, \iota)$ where $\smallsetminus\colon \mathbf{4} \times \mathbf{4} \to \mathbf{4}$ with $a \smallsetminus b = \min\{x \mid a \leqslant b \vee x\}$ (see Equation (2.4)). The following table describes this operation:

$\smallsetminus$	$\circ$	$+$	$-$	ι
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$+$	$+$	$\circ$	$+$	$\circ$
$-$	$-$	$-$	$\circ$	$\circ$
ι	ι	$-$	$+$	$\circ$

The complete dual Heyting algebra $\underline{\mathbf{4}}^U$ is defined by pointwise extending the previous one to $\mathbf{4}^U$, i.e. $\underline{\mathbf{4}}^U = (\mathbf{4}^U, \vee, \wedge, \setminus, \varepsilon, \imath)$ where $\Big(X \setminus Y\Big)(u) = X(u) \setminus Y(u),$ for all $u \in U$.

The following straightforward proposition shows that the restriction of the difference operation to $\dot{\mathbf{3}}$ can be considered as an operation in $\dot{\mathbf{3}}$.

Proposition 5.3.1 *If $X, Y \in \dot{\mathbf{3}}^U$, then $X \setminus Y \in \dot{\mathbf{3}}^U$. In particular,*

1 $X \setminus \imath = \varepsilon$ for all $X \in \dot{\mathbf{3}}^U$.

2 If $Y \in \mathcal{F}\mathrm{ull}(U)$ then $\imath \setminus Y = \overline{Y} \in \mathcal{F}\mathrm{ull}(U)$.

3 If $Y \in \mathbf{3}^U \setminus \mathcal{F}\mathrm{ull}(U)$ then $\imath \setminus Y = \imath$.

4 If $X, Y \in \mathbf{3}^U$ then $X \setminus Y \in \mathbf{3}^U$.

Finally, we prove that $\underline{\dot{\mathbf{3}}}^U = (\dot{\mathbf{3}}^U, \sqcup, \wedge, \setminus, \varepsilon, \imath)$ is a weak-cdHa and show with an example that it is not a complete dual Heyting algebra.

Proposition 5.3.2 *The structure $\underline{\dot{\mathbf{3}}}^U = (\dot{\mathbf{3}}^U, \sqcup, \wedge, \setminus, \varepsilon, \imath)$ is a weak-cdHa.*

PROOF: As has been shown before, $(\dot{\mathbf{3}}^U, \sqcup, \wedge, \varepsilon, \imath)$ is a complete lattice and, by Proposition 5.3.1, the set $\dot{\mathbf{3}}^U$ is closed for the operation $\setminus$ defined in $\mathbf{4}^U$. Thus, we prove that properties given in Definition 5.1.1 hold.

First, we have that $X \sqcup Y \neq \imath$ implies $X \sqcup Y = X \vee Y$ and, since $\underline{\mathbf{4}}^U$ is a complete dual Heyting algebra, one has $(X \vee Y) \setminus X \sqsubseteq Y$.

Second, from the fact that $\underline{\mathbf{4}}^U$ is a complete dual Heyting algebra and using Proposition 5.3.1, it is straightforward that $X \setminus Y \sqsubseteq X$, and $X \setminus Y = \varepsilon$ if and only if $X \sqsubseteq Y$.

Third prove $X \sqcup Y = X \sqcup (Y \setminus X)$ we have two situations, namely $X \sqcup Y \neq \imath$ or $X \sqcup Y = \imath$. In the first one, the proof is straightforward from the fact that $\underline{\mathbf{4}}^U$ is a complete dual Heyting algebra. In the second situation, we distinguish the following cases. If either $X = \dot{\imath}$ or $Y = \dot{\imath}$ then we have that $X \sqcup (Y \smallsetminus X) = \dot{\imath} = X \sqcup Y$. Otherwise, we have that there exists an element $a \in X$ such that $\overline{a} \in Y$. Then $\overline{a} \in Y \smallsetminus X$ and $X \sqcup (Y \smallsetminus X) = \dot{\imath} = X \sqcup Y$. $\square$

The following technical result will be helpful throughout this work.

Proposition 5.3.3 *Let $X, Y, Z \in \dot{\mathbf{3}}^M$, the following assertions are fulfilled:*

1 $X \sqcup Y = \dot{\imath}$ if and only if $\overline{X} \wedge Y \neq \varepsilon$.

2 If $X \sqcup (Z \smallsetminus Y) \neq \dot{\imath}$ and $X \sqcup Z = \dot{\imath}$, then $X \sqcup Y = \dot{\imath}$.

3 If $X \sqcup ((X \sqcup Z) \smallsetminus Y) \neq \dot{\imath}$ then $X \sqcup ((X \sqcup Z) \smallsetminus Y) = X \sqcup (Z \smallsetminus Y)$

4 If $X \neq \dot{\imath}$ then $Y \wedge (X \smallsetminus Y) = \varepsilon$.

PROOF: Item 1 is straightforward from the definition.

For item 2, assume that $X \sqcup (Z \smallsetminus Y) \neq \dot{\imath}$ and $X \sqcup Z = \dot{\imath}$. From item 1, we have that $\overline{X} \wedge (Z \smallsetminus Y) = \varepsilon$ and $\overline{X} \wedge Z \neq \varepsilon$. Therefore, $\overline{X} \wedge Y \neq \varepsilon$ or, equivalently, from item 1, $X \sqcup Y = \dot{\imath}$.

Let us prove item 3. Assume $X \sqcup ((X \sqcup Z) \smallsetminus Y) \neq \dot{\imath}$. Since $\underline{\mathbf{4}}^U$ is a complete dual Heyting algebra, from (3.1) and (2.5), we have that:

$$X \sqcup ((X \sqcup Z) \smallsetminus Y) = X \vee ((X \vee Z) \smallsetminus Y) = X \vee (Z \smallsetminus Y) = X \sqcup (Z \smallsetminus Y).$$

Finally, let $X \in \mathbf{3}^U$ and $Y \in \dot{\mathbf{3}}^U$. If $Y = \dot{\imath}$ we have that $X \smallsetminus Y = \varepsilon$ so $Y \wedge (X \smallsetminus Y) = \varepsilon$. Let suppose now that $Y \neq \dot{\imath}$ and denote $Z = Y \wedge (X \smallsetminus Y)$ we have to prove that for all $u \in U$ we have that $Z(u) = \circ$. Given $u \in U$, if $X(u) = \circ$ or $Y(u) = \circ$ or $X(u) = Y(u)$ we have straightforward that $Z(u) = \circ$. If $X(u) \neq Y(u)$, $X(u) \neq \circ$ and $Y(u) \neq \circ$ we have that $X(u) = +$ and $Y(u) = -$ or $X(u) = -$ and $Y(u) = +$, in any case, we have that $Z(u) = Y(u) \wedge X(u) = \circ$.

$\square$

We conclude this section with an example that shows that $\dot{\underline{\mathbf{3}}}^U$ is not necessarily a complete dual Heyting algebra.

Example 5.3.4 *Given $U = \{a, b\}$, the algebra $\dot{\underline{\mathbf{3}}}^U$ is not a complete dual Heyting algebra because it does not satisfy the adjoint property* (2.2). *For instance, for $X = ab$ and $Y = a\bar{b}$, we have that $\dot{\imath} \sqsubseteq X \sqcup Y$ but $\dot{\imath} \smallsetminus X = \bar{a}\bar{b} \not\sqsubseteq Y$.*

Chapter 6

Simplification logic for weak implications

As we mentioned in the introduction, one of the most important contributions in this work is the Simplification logic. Thus, we are going to introduce two axiomatic systems based on the Simplification logic that are sound and complete for the weak implications by using the weak complete dual Heyting algebra $\dot{\mathbf{3}}^M$ presented in Section 5.3.

6.1 Axiomatic system based in Simplification for weak implications

We started presenting our first proposal.

Definition 6.1.1 Simplification Axiomatic System *for weak implications is* $\{[\texttt{Inc}], [\texttt{Key}], [\texttt{Simp}]\}$ *where these rules are defined as follows: for all* $A, B, C \in \underline{\dot{\mathbf{3}}}^M$ *and all singleton* $b \in \underline{\dot{\mathbf{3}}}^M$*,*

$[\texttt{Inc}]$ *Infer* $AB \rightsquigarrow A$*,*

[Key] *From* $A \rightsquigarrow b$ *infer* $A\overline{b} \rightsquigarrow i$,

[Simp] *From* $A \rightsquigarrow B$ *and* $C \rightsquigarrow D$ *infer* $A(C \smallsetminus B) \rightsquigarrow D$,

These rules are named inclusion, key *and* simplification, *respectively.*

Our second proposal replaces the "Key" inference rule with a version of the classical union rule.

Definition 6.1.2 U-Simplification Axiomatic System *is* $\{[\texttt{Inc}], [\texttt{Simp}], [\texttt{Un}]\}$ *where the last rule, named* union, *is defined as follows: for all* $A, B, C \in \dot{\mathbf{3}}^M$,

[Un] *From* $A \rightsquigarrow B$ *and* $A \rightsquigarrow C$ *infer* $A \rightsquigarrow BC$.

We extend the notion of syntactic derivation that we have in Chapter 4 to consider, not only the axiomatic system $\mathcal{A}$, but the three axiomatic systems together. The notion is introduced in the standard way.

Definition 6.1.3 *Let* $\varphi \in \mathcal{L}_M$ *and* $\Sigma \subseteq \mathcal{L}_M$. *We say that* φ *is syntactically derived, or inferred, from* Σ *by using Armstrong-style (or Simplification or* U-*Simplification, resp.) Axiomatic System if there is a sequence* $(\varphi_i \mid 1 \leq i \leq n)$ *such that* $\varphi_n = \varphi$ *and, for all* $1 \leq i \leq n$, *either* $\varphi_i \in \Sigma$ *or* φ_i *is obtained by applying one of the rules of Armstrong-style (or Simplification or* U-*Simplification, resp.) Axiomatic System to implications belonging to* $\{\varphi_j \mid 1 \leq j < i\}$. *In this situation, the above sequence is said to be a proof for the derivation.*

In the same way that we did in the previous section, $\Sigma \vdash_{\mathbb{S}} \varphi$ and $\Sigma \vdash_{\mathbb{S}_{\mathtt{U}}} \varphi$ denote, respectively, that φ is syntactically derived from Σ by using Simplification and U-Simplification Axiomatic System.

In the following theorem, we prove that the three axiomatic systems are equivalent.

Theorem 6.1.4 *Let M be a set of attributes, $\Sigma \subseteq \mathcal{L}_M$ and $A \rightsquigarrow B \in \mathcal{L}_M$. We have that*

$$\Sigma \vdash_{\mathbb{A}} A \rightsquigarrow B \quad \textit{if and only if} \quad \Sigma \vdash_{\mathbb{S}_{\mathtt{U}}} A \rightsquigarrow B \quad \textit{if and only if} \quad \Sigma \vdash_{\mathbb{S}} A \rightsquigarrow B.$$

PROOF: First, we prove $\Sigma \vdash_{\mathbb{A}} A \rightsquigarrow B$ implies $\Sigma \vdash_{\mathbb{S}_{\mathtt{U}}} A \rightsquigarrow B$, and to do this, it is enough to prove that [Un] and [Simp] are derived rules from Armstrong-style Axiomatic System.

The following sequence prove that [Un] is obtained from Armstrong-style Axiomatic System:

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = A \rightsquigarrow C$	By hypothesis.
$\varphi_3 = AC \rightsquigarrow BC$	By [Augm] to φ_1 with C.
$\varphi_4 = A \rightsquigarrow AC$	By [Augm] to φ_2 with A.
$\varphi_5 = A \rightsquigarrow BC$	By [Trans] to φ_4 and φ_3.

For [Simp], we provide the following proof:

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = C \rightsquigarrow D$	By hypothesis.
$\varphi_3 = A(C \smallsetminus B) \rightsquigarrow BC$	Applying [Augm] to φ_1 with $C \smallsetminus B$.
$\varphi_4 = BC \rightsquigarrow BD$	Applying [Augm] to φ_2 with B.
$\varphi_5 = A(C \smallsetminus B) \rightsquigarrow BD$	Applying [Trans] to φ_3 and φ_4.
$\varphi_6 = BD \rightsquigarrow D$	By [Inc].
$\varphi_7 = A(C \smallsetminus B) \rightsquigarrow D$	Applying [Trans] to φ_5 and φ_6.

Notice that the derivation of φ_3 is based on the weak dual Heyting structure, and particularly the [wH4] property.

Second, we prove $\Sigma \vdash_{\mathbb{S}_{\mathtt{U}}} A \rightsquigarrow B$ implies $\Sigma \vdash_{\mathbb{S}} A \rightsquigarrow B$, and to do this, it is enough to prove that [Key] is derived from U-Simplification Axiomatic

System, and it is obtained with the following sequence:

$\varphi_1 = A \rightsquigarrow b$	By hypothesis.
$\varphi_2 = A\overline{b} \rightsquigarrow A$	By [Inc].
$\varphi_3 = A\overline{b} \rightsquigarrow b$	Applying [Simp] to φ_2 and φ_1.
$\varphi_4 = A\overline{b} \rightsquigarrow \overline{b}$	By [Inc].
$\varphi_5 = A\overline{b} \rightsquigarrow \imath$	Applying [Un] to φ_3 and φ_4.

Notice that, once again, the algebraic structure is needed for the derivation of φ_3. In this case, [wH3] is used.

Finally, to end the chain of equivalences, we have to prove that $\Sigma \vdash_{\mathbb{S}} A \rightsquigarrow B$ implies $\Sigma \vdash_{\mathbb{A}} A \rightsquigarrow B$. Specifically, to do this it is enough to prove that [Trans] and [Augm] are derivated rules from Simplification Axiomatic System.

[Trans] is straightforwardly obtained from Simplification Axiomatic System because it is a particular case of [Simp] applied to $A \rightsquigarrow B$ and $B \rightsquigarrow C$ by using the property [wH3].

To prove that [Augm] is obtained from Simplification Axiomatic System, we distinguish two cases depending on whether $B \sqcup C$ is $\imath$ or not. On the one hand, if $B \sqcup C = \imath$, we have that there is a singleton x with $x \sqsubseteq B$ such that $\overline{x} \sqsubseteq C$. Then, the following sequence proves $A \rightsquigarrow B \vdash_{\mathbb{S}} A \sqcup C \rightsquigarrow \imath$:

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = B \rightsquigarrow x$	By [Inc].
$\varphi_3 = A \rightsquigarrow x$	Applying [Simp] to φ_1 and φ_2.
$\varphi_4 = A\overline{x} \rightsquigarrow \imath$	Applying [Key] to φ_3.
$\varphi_5 = AC \rightsquigarrow A\overline{x}$	By [Inc].
$\varphi_6 = AC \rightsquigarrow \imath$	Applying [Simp] to φ_5 and φ_4 and using [wH3].

On the other hand, the following sequence proves $A \rightsquigarrow B \vdash_{\mathbb{S}} AC \rightsquigarrow BC$

when $B \sqcup C \neq \imath$:

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = BC \rightsquigarrow BC$	By [Inc].
$\varphi_3 = A(BC \smallsetminus B) \rightsquigarrow BC$	Applying [Simp] to φ_1 and φ_2.
$\varphi_4 = C \rightsquigarrow \varepsilon$	By [Inc].
$\varphi_5 = AC \rightsquigarrow BC$	Applying [Simp] to φ_4 and φ_3.

In the last step, Proposition 5.1.3 and [wH1] have been used. □

As a consequence of the above theorem and the soundness and completeness of Armstrong-style Axiomatic System, which was proved above in Theorem 4.2.11 and Theorem 4.2.15, we have the following theorem.

Theorem 6.1.5 *The Simplification and* U*-Simplification Axiomatic Systems are sound and complete.*

Another direct consequence of Theorem 6.1.4 is that we can write $\Sigma \vdash \varphi$ without the need to indicate as subscript the axiomatic system used.

6.2 The Simplification paradigm

The family of logics named Simplification have the common property that the inference rules can be seen as equivalence rules that allow simplifying a set of implications preserving the knowledge, i.e. the set of implications that can be derived is the same.

First, we introduce the *generalized augmentation* rule, denoted by [gAug], that will be used in the proof of the equivalence rules: for all $A, B, C, D \in \dot{\mathbf{3}}^M$,

[gAug] If $A \sqsubseteq C$ and $D \sqsubseteq C \sqcup B$, then $A \rightsquigarrow B \vdash C \rightsquigarrow D$.

Proposition 6.2.1 [gAug] *is a derived inference rule from Simplification Axiomatic System.*

PROOF: The following sequence is a proof for [gAug]:

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = CB \rightsquigarrow D$	By [Inc].
$\varphi_3 = A(CB \setminus B) \rightsquigarrow D$	By using [Simp] to φ_1 and φ_2.
$\varphi_4 = C \rightsquigarrow A(CB \setminus B)$	By [Inc] and [wH1].
$\varphi_5 = C \rightsquigarrow D$	By using [Simp] to φ_4 and φ_3.

□

The following theorem gives the set of equivalences that allow to simplify the set of implications, i.e. to reduce the size of the set of implications while preserving the equivalence. By size of a set of implications Σ we mean

$$\|\Sigma\| = \sum_{A \rightsquigarrow B \in \Sigma} \Big(|A| + |B| \Big)$$

where $|\imath| = 1$ and $|X|$ is the sum of the cardinality of $\mathrm{Spp}(X)$ for all $X \in \mathbf{3}^U$.

Theorem 6.2.2 *The following equivalence rules hold: for all $A, B, C, D \in \dot{\mathbf{3}}^M$,*

[FragEq] $\{A \rightsquigarrow B\} \equiv \{A \rightsquigarrow B \setminus A\}$.

[UnEq] $\{A \rightsquigarrow B,\ A \rightsquigarrow C\} \equiv \{A \rightsquigarrow BC\}$.

[ε-Eq] $\{A \rightsquigarrow \varepsilon\} \equiv \varnothing$.

[$\imath$-Eq] $\{A \rightsquigarrow B\} \equiv \{A \rightsquigarrow \imath\}$ *when* $A \sqcup B = \imath$.

[SimpEq] $\{A \rightsquigarrow B,\ C \rightsquigarrow D\} \equiv \{A \rightsquigarrow B,\ C \setminus B \rightsquigarrow D \setminus B\}$ *when* $A \sqsubseteq C \setminus B$.

PROOF: First, a proof for $A \rightsquigarrow B \vdash A \rightsquigarrow B \setminus A$ is the following sequence:

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = B \rightsquigarrow B \setminus A$	By `[wH2]` and [`Inc`].
$\varphi_3 = A \rightsquigarrow B \setminus A$	Applying [`Simp`] to φ_1 and φ_2 using `[wH3]`.

The opposite direction can be proved applying [`Augm`] to $A \rightsquigarrow B \setminus A$ (which is the hypothesis) and using `[wH4]`.

Second, to prove that from $\{A \rightsquigarrow B,\ A \rightsquigarrow C\}$ we can derive $\{A \rightsquigarrow BC\}$ we use [`Un`] to both hypothesis. The opposite direction is straightforward from [`gAug`].

[ε-`Eq`] is due to [`FragEq`] and [`Inc`]. And [$\imath$-`Eq`] is due to [`UnEq`], [`Inc`] and [`Frag`].

Finally, we prove that $\{A \rightsquigarrow B,\ C \rightsquigarrow D\} \equiv \{A \rightsquigarrow B,\ C \setminus B \rightsquigarrow D \setminus B\}$ when $A \sqsubseteq C \setminus B$. The following sequence proves that, if $A \sqsubseteq C \setminus B$, from $A \rightsquigarrow B$ and $C \rightsquigarrow D$ we derive $C \setminus B \rightsquigarrow D \setminus B$:

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = C \setminus B \rightsquigarrow C$	By applying [`gAug`] to φ_1.
$\varphi_3 = C \rightsquigarrow D$	By hypothesis.
$\varphi_4 = C \setminus B \rightsquigarrow D$	By applying [`Simp`] to φ_2 and φ_3.
$\varphi_5 = C \setminus B \rightsquigarrow D \setminus B$	By applying [`gAug`] to φ_4, and [`wH2`].

where, to get φ_2, we have used that $A \sqsubseteq C \setminus B$ and $C \sqsubseteq B \sqcup (C \setminus B)$, which is due to `[wH4']`. To prove the opposite direction, we use the following sequence

$\varphi_1 = A \rightsquigarrow B$	By hypothesis.
$\varphi_2 = C \setminus B \rightsquigarrow D \setminus B$	By hypothesis.
$\varphi_3 = C \setminus B \rightsquigarrow DB$	By applying [`Un`] to φ_1 and φ_2.
$\varphi_4 = C \rightsquigarrow D$	By applying [`gAug`] to φ_3, and [`wH2`].

Notice that in φ_3 we use $A \sqsubseteq C \smallsetminus B$ and $(D \smallsetminus B) \sqcup B = D \sqcup B$ by `[wH4]`. $\square$

In the following example, we apply these equivalences, left-to-right read, to reduce the size of the set of implications without losing any knowledge, that is, preserving the equivalences.

Example 6.2.3 *Consider the set of weak implications Σ over the universe $U = \{a,b,c,d\}$, $\Sigma = \{b \rightsquigarrow c, \imath \rightsquigarrow b, bc \rightsquigarrow d\overline{a}, bd \rightsquigarrow c\overline{a}, bcd \rightsquigarrow \imath, a\overline{bd} \rightsquigarrow c\overline{a}, c \rightsquigarrow d\}$. Let us see how the size of Σ can be reduced using the equivalences given in Theorem 6.2.2.*

- *By* [`FragEq`] *and* [ε-`Eq`], *we have that* $\{\imath \rightsquigarrow b\} \equiv \{\imath \rightsquigarrow \varepsilon\} \equiv \varnothing$ *and*

$$\Sigma \equiv \{b \rightsquigarrow c, bc \rightsquigarrow d\overline{a}, bd \rightsquigarrow c\overline{a}, bcd \rightsquigarrow \imath, a\overline{bd} \rightsquigarrow c\overline{a}, c \rightsquigarrow d\}$$

- *Applying* [`SimpEq`] *and* [`UnEq`], *we have that*

$$\{b \rightsquigarrow c,\ bc \rightsquigarrow d\overline{a}\} \equiv \{b \rightsquigarrow c,\ b \rightsquigarrow d\overline{a}\} \equiv \{b \rightsquigarrow cd\overline{a}\}$$

  Therefore, $\Sigma \equiv \{b \rightsquigarrow cd\overline{a}, bd \rightsquigarrow c\overline{a}, bcd \rightsquigarrow \imath, a\overline{bd} \rightsquigarrow c\overline{a}, c \rightsquigarrow d\}$.

- *Applying* [`SimpEq`] *and* [`FragEq`], *we have that*

$$\{b \rightsquigarrow cd\overline{a}, bd \rightsquigarrow c\overline{a}\} \equiv \{b \rightsquigarrow cd\overline{a}, b \rightsquigarrow \varepsilon\} \equiv \{b \rightsquigarrow cd\overline{a}\}$$

  Therefore, $\Sigma \equiv \{b \rightsquigarrow cd\overline{a}, bcd \rightsquigarrow \imath, a\overline{bd} \rightsquigarrow c\overline{a}, c \rightsquigarrow d\}$.

- *Applying* [`SimpEq`] *and* [`UnEq`], *we have that*

$$\{b \rightsquigarrow cd\overline{a}, bcd \rightsquigarrow \imath\} \equiv \{b \rightsquigarrow cd\overline{a}, b \rightsquigarrow \imath\} \equiv \{b \rightsquigarrow \imath\}$$

  Then, $\Sigma \equiv \{b \rightsquigarrow \imath, a\overline{bd} \rightsquigarrow c\overline{a}, c \rightsquigarrow d\}$.

- *Finally, by* [$\imath$-`Eq`], *we obtain* $\Sigma \equiv \{b \rightsquigarrow \imath, a\overline{bd} \rightsquigarrow \imath, c \rightsquigarrow d\}$.

6.3 Automatic reasoning method

In this section, we present how the introduced logic leads to the design of an automated reasoning method. To achieve this, we will first generalize the notion of syntactic closure of a set of attributes with respect to a set of implications. This notion is well known in classical FCA and allows us to check whether from a set of implications Σ an implication $A \to B$ is derived by simply checking whether B is contained in the syntactic closure of A with respect to Σ.

Definition 6.3.1 *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. The syntactic closure with respect to Σ is the map $[-]^w_\Sigma \colon \dot{\mathbf{3}}^M \to \dot{\mathbf{3}}^M$ defined as*

$$[A]^w_\Sigma = \bigsqcup\{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash A \rightsquigarrow X\}.$$

The following Theorem shows that the previous definition generalizes the so-called syntactic closure operator.

Theorem 6.3.2 *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For any $A \rightsquigarrow B \in \mathcal{L}_M$ we have that $\Sigma \vdash A \rightsquigarrow B$ if and only if $B \sqsubseteq [A]^w_\Sigma$. In addition, the mapping $[-]^w_\Sigma$ is a closure operator in $\underline{\dot{\mathbf{3}}}^M$.*

PROOF: It is straightforward that $\Sigma \vdash A \rightsquigarrow B$ implies $B \sqsubseteq [A]^w_\Sigma$, from the definition of $[A]^w_\Sigma$.

Assume that $B \sqsubseteq [A]^w_\Sigma$. On the one hand, by [`Inc`] we have that $\Sigma \vdash [A]^w_\Sigma \rightsquigarrow B$. On the other hand, as M is finite, the set $\chi = \{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash A \rightsquigarrow X\}$ is also finite and, from [`Un`], we have that $\Sigma \vdash A \rightsquigarrow \bigsqcup \chi$. Thus, from the Definition 6.3.1, we have that $\Sigma \vdash A \rightsquigarrow [A]^w_\Sigma$. Finally, applying [`Simp`] to $\Sigma \vdash A \rightsquigarrow [A]^w_\Sigma$ and $\Sigma \vdash [A]^w_\Sigma \rightsquigarrow B$, we have that $\Sigma \vdash A \rightsquigarrow B$.

Let us prove that $[-]^w_\Sigma$ is a closure operator in $\dot{\mathbf{3}}^M$. First, by [`Inc`], it is inflationary, i.e. $A \sqsubseteq [A]^w_\Sigma$ for all $A \in \dot{\mathbf{3}}^M$.

If $A \sqsubseteq B$ then, by [Inc], we have that $\Sigma \vdash B \rightsquigarrow A$ and, since $\Sigma \vdash A \rightsquigarrow [A]^w_\Sigma$ and using [Simp], we have that $\Sigma \vdash B \rightsquigarrow [A]^w_\Sigma$. Therefore, $[A]^w_\Sigma \sqsubseteq [B]^w_\Sigma$.

Finally, let us prove the idempotency of the $[-]^w_\Sigma$ mapping: $[[A]^w_\Sigma]^w_\Sigma \sqsubseteq [A]^w_\Sigma$ because $\Sigma \vdash A \rightsquigarrow [A]^w_\Sigma$, $\Sigma \vdash [A]^w_\Sigma \rightsquigarrow [[A]^w_\Sigma]^w_\Sigma$ and, by [Simp], $\Sigma \vdash A \rightsquigarrow [[A]^w_\Sigma]^w_\Sigma$. Therefore, since $[-]^w_\Sigma$ is inflationary, we have that $[-]^w_\Sigma$ is idempotent. $\square$

The second step necessary to achieve our goal of designing an automatic reasoning method based on Simplification logic is to obtain a result analogous to the deduction theorem of classical propositional logic. Recall that this classical theorem of propositional logic says that $\Sigma \vdash \varphi \Rightarrow \psi$ if and only if $\Sigma \cup \{\varphi\} \vdash \psi$. Using the fact that any propositional formula χ is equivalent to $\top \Rightarrow \chi$ where $\top$ denotes a tautology, the classical deduction theorem can be equivalently restated as $\Sigma \vdash \varphi \Rightarrow \psi$ if and only if $\Sigma \cup \{\top \Rightarrow \varphi\} \vdash \top \Rightarrow \psi$.

The automatic reasoning method we propose here is intended to answer the question of whether $a \rightsquigarrow b$ can be inferred from a theory Σ based on two pillars: one is a theorem of deduction reminiscent of propositional logic, and the other is a set of transformations that *simplify* the theory $\Sigma \cup \{\varepsilon \rightsquigarrow a\}$ by using Theorem 6.2.2, where the element $\varepsilon \in \dot{\mathbf{3}}^M$ will play the same role as the tautology $\top$ of propositional logic.

Before giving the above-mentioned version of the deduction theorem, we introduce a notation and a previous result necessary in order to ease the reading of its proof.

For all $A \in \dot{\mathbf{3}}^M$:

- If $\varphi = X \rightsquigarrow Y$, then φ_A denotes $AX \rightsquigarrow Y$.
- If $\Sigma \in \mathcal{L}_M$, then Σ_A denotes $\{\varphi_A \colon \varphi \in \Sigma\}$.

Lemma 6.3.3 *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For all $A \in \dot{\mathbf{3}}^M$ and all $\varphi \in \mathcal{L}_M$,*

$$\Sigma \vdash \varphi \quad \textit{implies} \quad \Sigma_A \vdash \varphi_A.$$

PROOF: By Theorem6.1.4, we can prove it by using Armstrong-style, Simplification or U-Simplification Axiomatic System. We consider here the last one. From Definition 6.1.3, $\Sigma \vdash \varphi$ if there is a sequence $\varphi_1, ..., \varphi_n \in \mathcal{L}_M$ such that $\varphi_n = \varphi$ and, for all $1 \leq i \leq n$, either $\varphi_i \in \Sigma$ or φ_i is obtained by applying one of the rules of U-Simplification Axiomatic System to implications belonging to $\{\varphi_j \mid 1 \leq j < i\}$.

We prove by induction that, for all $1 \leq i \leq n$, we have that $\Sigma_A \vdash \varphi_{iA}$. If $\varphi_i \in \Sigma$, it is straightforward that $\varphi_{iA} \in \Sigma_A$. Assume, as induction Hypothesis, that $\Sigma_A \vdash \varphi_{jA}$ for all $1 \leq j < i$, and let $\varphi_i = B \rightsquigarrow C$ with $B, C \in \dot{\mathbf{3}}^M$, which is obtained by using one of the rules of U-Simplification Axiomatic System. We distinguish three cases:

- If φ_i is obtained by [Inc], then $C \sqsubseteq B$. Therefore, $C \sqsubseteq A \sqcup B$ and $\Sigma_A \vdash AB \rightsquigarrow C$ also by [Inc].

- If φ_i is obtained by [Simp], then there exist $X \rightsquigarrow Y, Z \rightsquigarrow C \in \{\varphi_j \colon 1 \leq j < n\}$ such that $B = X \sqcup (Z \smallsetminus Y)$. By the induction hypothesis, we have that $\Sigma_A \vdash AX \rightsquigarrow Y$ and $\Sigma_A \vdash AZ \rightsquigarrow C$.

  - If $A \sqcup B = \imath$, we have that $\Sigma_A \vdash AB \rightsquigarrow C$ from [Inc].
  - If $A \sqcup B \neq \imath$ and $A \sqcup Z = \imath$, then $\Sigma_A \vdash \imath \rightsquigarrow C$, from item 1 in Proposition 5.3.3, since $A \sqcup B = A \sqcup X \sqcup (Z \smallsetminus Y)$, we have that $A \sqcup X \sqcup Y = \imath$. Now, applying [$\imath$-Eq] to the hypothesis $\Sigma_A \vdash A \sqcup X \rightsquigarrow Y$, we have that $\Sigma_A \vdash A \sqcup X \rightsquigarrow \imath$. Thus, by [Simp] from $\Sigma_A \vdash A \sqcup X \rightsquigarrow \imath$ and $\Sigma_A \vdash \imath \rightsquigarrow C$, we obtain that $\Sigma_A \vdash A \sqcup X \rightsquigarrow C$. Finally, by using [gAug] we have that $\Sigma_A \vdash A \sqcup X \sqcup (Z \smallsetminus Y) \rightsquigarrow C$, concluding $\Sigma_A \vdash A \sqcup B \rightsquigarrow C$.
  - If $A \sqcup B \neq \imath$ and $A \sqcup Z \neq \imath$, by item 2 in Proposition 5.3.3,

$$A \sqcup X \sqcup ((A \sqcup Z) \smallsetminus Y) = A \sqcup X \sqcup (Z \smallsetminus Y) = A \sqcup B$$

Now, by [Simp], from $\Sigma_A \vdash AX \rightsquigarrow Y$ and $\Sigma_A \vdash AZ \rightsquigarrow C$, we obtain that $\Sigma_A \vdash AX(AZ \smallsetminus Y) \rightsquigarrow C$, concluding also that $\Sigma_A \vdash A \sqcup B \rightsquigarrow C$.

- If φ_i is obtained by [Un], then there are $B \rightsquigarrow X, B \rightsquigarrow Y \in \{\varphi_j : 1 \leq j < i\}$ such that $C = X \sqcup Y$. By induction hypothesis, $\Sigma_A \vdash AB \rightsquigarrow X$ and $\Sigma_A \vdash AB \rightsquigarrow Y$ applying [Un] we obtain that $\Sigma_A \vdash AB \rightsquigarrow XY$, concluding $\Sigma_A \vdash AB \rightsquigarrow C$.

$\square$

We are now in a position to state and prove the deduction theorem.

Theorem 6.3.4 *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For any $A \rightsquigarrow B \in \mathcal{L}_M$:*

$$\Sigma \vdash A \rightsquigarrow B \qquad \textit{if and only if} \qquad \Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow B$$

PROOF: Assume that $\Sigma \vdash A \rightsquigarrow B$. Then $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \{\varepsilon \rightsquigarrow A, A \rightsquigarrow B\}$ and by using [Simp], we have $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow B$.

For the converse implication, Let us suppose that $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow B$ then by Lemma 6.3.3 we have that $\Sigma_A \cup \{A \rightsquigarrow A\} \vdash A \rightsquigarrow B$. By [Inc], $\Sigma_A \cup \{A \rightsquigarrow A\} \equiv \Sigma_A$, so we have that $\Sigma_A \vdash A \rightsquigarrow B$. Now, observe that for all $\psi \in \Sigma_A$ there exists $\varphi \in \Sigma$ such that $\psi = \varphi_A$ and, by [gAug], we have that $\Sigma \vdash \varphi_A$. Therefore, $\Sigma \vdash A \rightsquigarrow B$. $\square$

Finally, as we have already advanced, we propose an algorithm, Algorithm 2, which, based on the deduction theorem (Theorem 6.3.4) and the equivalences of Theorem 6.2.2, allows to compute the syntactic closure of a **3**-set A with respect to a set of implications Σ. In addition to that, thanks to Theorem 6.3.2, this algorithm also solves the deduction problem, i.e., it allows to discern whether $\Sigma \vdash A \rightsquigarrow B$ is satisfied.

Finally, let us prove that Algorithm 2 always ends, and it is sound and complete.

Algorithm 2: `Syntactic closure of` A `with respect to` Σ

Input: Σ being a set of weak implications, A being a set of $\mathbf{3}^U$
Output: $[A]^w_\Sigma$
repeat
  $\Sigma_{\texttt{old}} := \Sigma;\ \Sigma := \varnothing$
  foreach $B \rightsquigarrow C \in \Sigma_{\texttt{old}}$ **do**
    $B_{\texttt{new}} := B \smallsetminus A;\ C_{\texttt{new}} := C \smallsetminus A$ `// By [SimpEq]`
    if $B_{\texttt{new}} = \varepsilon$ **then**
      $A := A \sqcup C_{\texttt{new}}$ `// By [UnEq]`
    else if $C_{\texttt{new}} \not\sqsubseteq B_{\texttt{new}}$ **then**
      Add $B_{\texttt{new}} \rightsquigarrow (C_{\texttt{new}} \smallsetminus B_{\texttt{new}})$ to Σ `// By [FragEq] and [ε-Eq]`
  end
until $\Sigma = \Sigma_{\texttt{old}}$ *or* $A = \wr$
return A

Theorem 6.3.5 *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For all $A \in \dot{\mathbf{3}}^M$, the algorithm returns $[A]^w_\Sigma$ and its cost is, in the worst case, $|\Sigma| \cdot \|\Sigma\|$.*

PROOF: Let A_0 and Σ_0 be the input parameters and A_j and Σ_j their values after the j-th iteration of the repeat loop. First, the algorithm ends because $\|\Sigma_{j-1}\| \geq \|\Sigma_j\|$ for each iteration j, and $\|\Sigma_{j-1}\| = \|\Sigma_j\|$ implies $\Sigma_{j-1} = \Sigma_j$, which is one of the stop conditions. Let n be the number of iterations that is lower or equal to $\|\Sigma\|$. In the worst case, the cost of the algorithm is $|\Sigma| \cdot \|\Sigma\|$.

At this point, we want to guarantee that we do not lose information in each step of the algorithm. Since only the equivalences given in Theorem 6.2.2 are used, we have that

$$\Sigma_{j-1} \cup \{\varepsilon \rightsquigarrow A_{j-1}\} \equiv \Sigma_j \cup \{\varepsilon \rightsquigarrow A_j\}$$

for all iteration j, and, by Theorems 6.3.4 and 6.3.2, we have that $\Sigma \vdash \varepsilon \rightsquigarrow A_n$ and $A_n \sqsubseteq [A]^w_\Sigma$. In order to prove the reverse inclusion, i.e. $[A]^w_\Sigma \sqsubseteq A_n$, we demonstrate that $\Sigma \vdash A \rightsquigarrow X$ implies $X \sqsubseteq A_n$.

Assume that $\Sigma \vdash A \rightsquigarrow X$, and $A_n \neq \wr$ because it is straightforward otherwise. By Theorem 6.3.4, we have that $\Sigma \vdash A \rightsquigarrow X$ is equivalent to $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow X$ and, hence, to $\Sigma_n \cup \{\varepsilon \rightsquigarrow A_n\} \vdash \varepsilon \rightsquigarrow X$. Let $\varphi_1 \cdots \varphi_k$ be a proof for $\Sigma_n \cup \{\varepsilon \rightsquigarrow A_n\} \vdash \varepsilon \rightsquigarrow X$. We prove by induction that, for all $1 \leq r \leq k$,

$$\text{if } \varphi_r = Y \rightsquigarrow Z \text{ and } Y \sqsubseteq A_n, \text{ then } Z \sqsubseteq A_n. \tag{6.1}$$

Notice that, by using Item 4 of Proposition 5.3.3, we have that:

$$A_n \wedge B = \varepsilon, \text{ for all } B \rightsquigarrow C \in \Sigma_n \tag{6.2}$$

In addition, by Algorithm 1, we have that:

$$B \neq \varepsilon \text{ and } C \neq \varepsilon, \text{ for all } B \rightsquigarrow C \in \Sigma_n \tag{6.3}$$

If $\varphi_r = Y \rightsquigarrow Z \in \Sigma_n \cup \{\varepsilon \rightsquigarrow A_n\}$ and $Y \sqsubseteq A_n$, then, by (6.2) and (6.3), $\varphi_r = \varepsilon \rightsquigarrow A_n$ and $Z = A_n$. Assume, as induction hypothesis, that φ_s satisfies (6.1), for all $1 \leq s < r$.

- If $\varphi_r = Y \rightsquigarrow Z$ is obtained by [`Inc`] and $Y \sqsubseteq A_n$, we have straightforwardly that $Z \sqsubseteq Y \sqsubseteq A_n$.
- If φ_r is obtained by [`Simp`] and $Y \sqsubseteq A_n$, then there exist $U \rightsquigarrow V$, $W \rightsquigarrow Z \in \{\varphi_s \colon 1 \leq s < r\}$ such that $Y = U \sqcup (W \smallsetminus V) \sqsubseteq A_n$, which implies $U \sqsubseteq A_n$ and $W \smallsetminus V \sqsubseteq A_n$. By induction hypothesis, we have that $V \sqsubseteq A_n$ and, by `[wH4']`, $W \sqsubseteq V \sqcup (W \smallsetminus V) \sqsubseteq A_n$ concluding, by induction hypothesis, $Z \sqsubseteq A_n$.
- If $\varphi_r = Y \rightsquigarrow Z$ is obtained by [`Un`] and $Y \sqsubseteq A_n$, there exist $Y \rightsquigarrow V$, $Y \rightsquigarrow W \in \{\varphi_s \colon 1 \leq s < r\}$ such that $Z = V \sqcup W$. By induction hypothesis, $V \sqsubseteq A_n$ and $W \sqsubseteq A_n$ so we have that $Z = W \sqcup V \sqsubseteq A_n$.

Finally, since $\varphi_k = \varepsilon \rightsquigarrow X$ and $\varepsilon \sqsubseteq A_n$, we conclude that $X \sqsubseteq A_n$. $\square$

We conclude the section with an illustrative example of the Algorithm's execution.

Example 6.3.6 *Let $M = \{a, b, c, d, e, f\}$ and let Σ be the following set of weak implications*

$$\Sigma = \{a\overline{e} \rightsquigarrow bc, cd \rightsquigarrow \overline{a}\overline{b}, \imath \rightsquigarrow \overline{e}\overline{f}, de \rightsquigarrow f, \overline{a} \rightsquigarrow de, \overline{b}f \rightsquigarrow \overline{c}ab, f \rightsquigarrow \overline{c}\}.$$

We show how Algorithm 2 computes $[adf]^w_\Sigma$.

First we have that $\Sigma_{old} = \Sigma$, $\Sigma = \varnothing$ and $A = adf$.

1. *For $a\overline{e} \rightsquigarrow bc \in \Sigma_{old}$, Algorithm 2 adds $\overline{e} \rightsquigarrow bc$ to Σ, having $\Sigma = \{\overline{e} \rightsquigarrow bc\}$.*

   Notice that $\{\varepsilon \rightsquigarrow adf, a\overline{e} \rightsquigarrow bc\} \equiv \{\varepsilon \rightsquigarrow adf, \overline{e} \rightsquigarrow bc\}$ by [`SimpEq`].

2. *For $cd \rightsquigarrow \overline{a}\overline{b} \in \Sigma_{old}$, Algorithm 2 adds $c \rightsquigarrow \overline{a}\overline{b}$ to Σ.*

   Thus, $\Sigma = \{\overline{e} \rightsquigarrow bc,\ c \rightsquigarrow \overline{a}\overline{b}\}$. Observe that, by [`SimpEq`], *we have that $\{\varepsilon \rightsquigarrow adf, cd \rightsquigarrow \overline{a}\overline{b}\} \equiv \{\varepsilon \rightsquigarrow adf, c \rightsquigarrow \overline{a}\overline{b}\}$.*

3. *For $\imath \rightsquigarrow \overline{e}\overline{f} \in \Sigma_{old}$, Algorithm 2 does not change neither A nor Σ.*

   Notice that, in this case, if we apply [`FragEq`] *and* [ε-`Eq`], *we have $\{\varepsilon \rightsquigarrow adf, \imath \rightsquigarrow \overline{e}\overline{f}\} \equiv \{\varepsilon \rightsquigarrow adf\}$.*

4. *For $de \rightsquigarrow f$, Algorithm 2 changes neither A nor Σ.*

   In this case, applying also the equivalences [`SimpEq`], [`FragEq`] *and* [ε-`Eq`], *we have $\{\varepsilon \rightsquigarrow adf, de \rightsquigarrow f\} \equiv \{\varepsilon \rightsquigarrow adf\}$.*

5. *For $\overline{a} \rightsquigarrow de \in \Sigma_{old}$, Algorithm 2 adds $\overline{a} \rightsquigarrow e$ to Σ, in such a case, we have that $\Sigma = \{\overline{e} \rightsquigarrow bc, c \rightsquigarrow \overline{a}\overline{b}, \overline{a} \rightsquigarrow e\}$.*

   By [`SimpEq`] *we have $\{\varepsilon \rightsquigarrow adf, \overline{a} \rightsquigarrow de\} \equiv \{\varepsilon \rightsquigarrow adf, \overline{a} \rightsquigarrow e\}$.*

6. *For $\overline{b}f \rightsquigarrow \overline{c}ab \in \Sigma_{old}$, Algorithm 2 adds $\overline{b} \rightsquigarrow \imath$ to Σ, in such a case, we have $\Sigma = \{\overline{e} \rightsquigarrow bc, c \rightsquigarrow \overline{a}\overline{b}, \overline{a} \rightsquigarrow e, \overline{b} \rightsquigarrow \imath\}$. Observe that, if we apply* [`SimpEq`] *and* [$\imath$-`Eq`]*we have $\{\varepsilon \rightsquigarrow adf, \overline{b}f \rightsquigarrow \overline{c}ab\} \equiv \{\varepsilon \rightsquigarrow adf, \overline{b} \rightsquigarrow \imath\}$.*

7. *For $f \rightsquigarrow \overline{c} \in \Sigma_{old}$, Algorithm 2 adds $\overline{c}$ to A having $A = adf\overline{c}$. In this case, Σ does not change, having $\Sigma = \{\overline{e} \rightsquigarrow bc, c \rightsquigarrow \overline{a}\overline{b}, \overline{a} \rightsquigarrow e, \overline{b} \rightsquigarrow \imath\}$. Notice*

that, by applying [`SimpEq`] *and* [`UnEq`], *we have* $\{\varepsilon \rightsquigarrow adf, f \rightsquigarrow \overline{c}\} \equiv \{\varepsilon \rightsquigarrow adf\overline{c}\}$.

8. *Now, the first iteration of the "repeat"-loop has finished and, as stated in the proof of Theorem 6.3.5, we have that*

$$\begin{aligned}&\{\varepsilon \rightsquigarrow adf\} \cup \Sigma_{old} =\\ =&\{\varepsilon \rightsquigarrow adf, a\overline{e} \rightsquigarrow bc, cd \rightsquigarrow \overline{a}\overline{b}, i \rightsquigarrow \overline{e}\overline{f}, de \rightsquigarrow f, \overline{a} \rightsquigarrow de, \overline{b}f \rightsquigarrow \overline{c}ab, f \rightsquigarrow \overline{c}\} \equiv\\ \equiv&\{\varepsilon \rightsquigarrow adf\overline{c}, \overline{e} \rightsquigarrow bc, c \rightsquigarrow \overline{a}\overline{b}, \overline{a} \rightsquigarrow e, \overline{b} \rightsquigarrow i\} \;=\; \{\varepsilon \rightsquigarrow adf\overline{c}\} \cup \Sigma.\end{aligned}$$

9. *Since* $\Sigma_{old} \neq \Sigma$, *Algorithm 2 changes* Σ_{old} *by* Σ, *sets* Σ *to* $\varnothing$, *and repeats the process, but does not modify neither* A *nor* Σ *(i.e.* $\Sigma = \Sigma_{old}$*). Therefore, Algorithm 2 finishes and returns* $[A]^w_\Sigma = adf\overline{c}$.

Chapter 7

Possible concepts and strong implications

So far, we have focused on attribute implications that are satisfied in context with the knowledge we currently have, although they may no longer be satisfied when new information becomes available: weak implications. In this chapter, we focus on a new Galois connection that will allow us to define implications, which we will call strong implications, that are necessarily true in all possible configurations after acquiring new knowledge. For this, we will have to take into consideration the whole possible universe for the partial formal context. From a computational point of view, the worst way to do this would be to complete the partial context with all possible extensions (see, for example, Figure 7.1). Thus, given a partial formal context $\mathbb{P} = (G, M, I)$, we define its *completion* as the total formal context $\mathbb{K}_*(\mathbb{P}) = (G', M, I')$ where

$$G' = \{(g, X) \in G \times \mathbf{3}^M : \mathrm{Pos}(X) \cup \mathrm{Neg}(X) = \mathrm{Unk}(I(g,\ ))\}$$

and $I'((g, X),\ ) = I(g,\ ) \sqcup X$ for all $(g, X) \in G'$. Finally, this total formal context can be analyzed and managed with the tools introduced in [65]. The

$\mathbb{P}$	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$\circ$	$+$
3	$-$	$-$	$\circ$

(a) $\mathbb{K} = (G, M, I)$

$\mathbb{K}_*$	a	b	c
$1.b$	$+$	$+$	$-$
$1.\bar{b}$	$+$	$-$	$-$
$2.ab$	$+$	$+$	$+$
$2.\bar{a}b$	$-$	$+$	$+$
$2.a\bar{b}$	$+$	$-$	$+$
$2.\bar{a}\bar{b}$	$-$	$-$	$+$
$3.c$	$-$	$-$	$+$
$3.\bar{c}$	$-$	$-$	$-$

(b) $\mathbb{K}_*(\mathbb{P}) = (G', M, I')$

Figure 7.1: Completion of a partial formal context.

main problem of this approach is that the growth of the size of $\mathbb{K}_*(\mathbb{P})$ with respect to the initial $\mathbb{P}$ is exponential. Specifically, $|G'| = \sum_{g \in G} 2^{|\mathrm{Unk}(I(g,\ ))|}$.

In the following section we give a different approach that helps to extract the knowledge in an efficient way, without having to fully complete the partial formal context. The proposal we make is based on the idea of using a lazy methodology that computes only what is strictly necessary.

7.1 The lattice of partial formal contexts

An important feature of FCA is that, although the concept lattice can be exponentially large with respect to the context, concepts can be lazily computed with algorithms whose cost is "polynomial delay". In the following, we describe how to extend this idea to partial formal contexts by lazily computing the concepts of $\mathbb{K}_*(\mathbb{P})$ without having computed $\mathbb{K}_*(\mathbb{P})$. To do this, we introduce a lattice of partial contexts on which we will navigate when searching for concepts.

Given two partial contexts $\mathbb{P}_1 = (G_1, M_1, I_1)$ and $\mathbb{P}_2 = (G_2, M_2, I_2)$, we say that $\mathbb{P}_1$ is a *refinement* of $\mathbb{P}_2$ (denoted by $\mathbb{P}_1 \preceq \mathbb{P}_2$) if

$$G_1 \subseteq G_2,\ M_1 = M_2,\ \text{ and } I_2(g,\ ) \sqsubseteq I_1(g,\ ) \text{ for all } g \in G_1 \tag{7.1}$$

In the Figure 7.2, a chain of partial formal contexts is shown.

	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$\circ$	$+$
3	$-$	$-$	$\circ$

$\succeq$

	a	b	c
1	$+$	$\circ$	$-$
2	$+$	$\circ$	$+$

$\succeq$

	a	b	c
1	$+$	$-$	$-$
2	$+$	$-$	$+$

$\succeq$

	a	b	c
1	$+$	$-$	$-$

Figure 7.2: A chain of partial formal contexts.

Theorem 7.1.1 *Let $\mathbb{P}_0$ be a partial formal context and $\mathfrak{P}(\mathbb{P}_0) = \{\mathbb{P} \colon \mathbb{P} \preceq \mathbb{P}_0\}$. Then $\underline{\mathfrak{P}}(\mathbb{P}_0) = (\mathfrak{P}(\mathbb{P}_0), \preceq)$ is a complete lattice.*

The infimum and the supremum in the complete lattice $\underline{\mathfrak{P}}(\mathbb{P}_0)$ are defined as follow:

- The infimum of $\{\mathbb{P}_j = (G_j, M, I_j) \colon j \in J\} \subseteq \mathfrak{P}(\mathbb{P}_0)$ is $\underset{j \in J}{\curlywedge} \mathbb{P}_j = (G, M, I)$ with

  $$G = \Big\{ g \in \bigcap_{j \in J} G_j \colon \bigsqcup_{j \in J} I_j(g,\ ) \neq \text{¿} \Big\} \text{ and, for all } g \in G,\ I(g,\ ) = \bigsqcup_{j \in J} I_j(g,\ )$$

- The supremum of $\{\mathbb{P}_j = (G_j, M, I_j) \colon j \in J\} \subseteq \mathfrak{P}(\mathbb{P}_0)$ is $\underset{j \in J}{\curlyvee} \mathbb{P}_j = (G, M, I)$ with

  $$G = \bigcup_{j \in J} G_j \text{ and, for all } g \in G,\ I(g,\ ) = \bigwedge_{j \in J_g} I_j(g,\ )$$

  being $J_g = \{j \in J \colon g \in G_j\}$.

PROOF: The relation $\preceq$ is an order in $\mathfrak{P}(\mathbb{P})$ because it pointwise combines the set inclusion and the order relation $\sqsubseteq$ in $\underline{\mathbf{3}}^M$.

- Reflexivity holds trivially.
- Antisymmetry: Let $\mathbb{P}_1 = (G_1, M, I_1), \mathbb{P}_2 = (G_2, M, I_2) \in \mathfrak{P}(\mathbb{P})$ be two partial formal contexts. Assume that $\mathbb{P}_1 \preceq \mathbb{P}_2$ and $\mathbb{P}_2 \preceq \mathbb{P}_1$.

  First we prove that $G_1 = G_2$. As $\mathbb{P}_1 \preceq \mathbb{P}_2$ we have that $G_1 \subseteq G_2$ and, as $\mathbb{P}_2 \preceq \mathbb{P}_1$, we have that $G_2 \subseteq G_1$. Therefore, $G_1 = G_2$.

  Now we have to check that $I_1 = I_2$. For all $g \in G_1 = G_2$, since $\mathbb{P}_1 \preceq \mathbb{P}_2$, we have that $I_2(g, \ ) \sqsubseteq I_1(g, \ )$ and, since $\mathbb{P}_2 \preceq \mathbb{P}_1$, we have that $I_1(g, \ ) \sqsubseteq I_2(g, \ )$. Then, since $(\mathbf{3}^m, \sqsubseteq)$ is a poset, we have that $I_1(g, \ ) = I_2(g, \ )$, which, together with $G_1 = G_2$, is equivalent to $\mathbb{P}_1 = \mathbb{P}_2$.

- Transitivity: Let $\mathbb{P}_1 = (G_1, M, I_1), \mathbb{P}_2 = (G_2, M, I_2), \mathbb{P}_3 = (G_3, M, I_3) \in \mathfrak{P}(\mathbb{P})$ be three partial formal contexts such that $\mathbb{P}_1 \preceq \mathbb{P}_2$ and $\mathbb{P}_2 \preceq \mathbb{P}_3$. We have to prove that $\mathbb{P}_1 \preceq \mathbb{P}3$.

  First, $\mathbb{P}_1 \preceq \mathbb{P}_2 \preceq \mathbb{P}_3$ implies $G_1 \subseteq G_2 \subseteq G_3$, and therefore $G_1 \subseteq G_3$.

  Finally, for all $g \in G$, we have that $\mathbb{P}_1 \preceq \mathbb{P}_2$ implies $I_2(g, \ ) \sqsubseteq I_1(g, \ )$ and $\mathbb{P}_2 \preceq \mathbb{P}_3$ implies $I_3(g, \ ) \sqsubseteq I_2(g, \ )$. Then, since $(\mathbf{3}^m, \sqsubseteq)$ is a poset, we have that $I_3(g, \ ) \sqsubseteq I_1(g, \ )$. Thus, we conclude that $\mathbb{P}_1 \preceq \mathbb{P}_3$.

Let $\{\mathbb{P}_j = (G_j, M, I_j) \colon j \in J\} \subseteq \mathfrak{P}(\mathbb{P})$ be a family of formal partial contexts and let's see that the infimum is $\mathbb{P}_0 = (G_0, M, I_0)$ with

$$G_0 = \left\{ g \in \bigcap_{j \in J} G_j \colon \bigsqcup_{j \in J} I_j(g, \ ) \neq \boldsymbol{i} \right\} \text{ and } I_0(g, \ ) = \bigsqcup_{j \in J} I_j(g, \ ), \text{for all } g \in G_0.$$

By the definition of $\preceq$, it is straightforward that $\mathbb{P}_0 \preceq \mathbb{P}_j$ for all $j \in J$. Assume that $\mathbb{P}' = (G', M, I') \preceq \mathbb{P}_j$ for all $j \in J$. On the one hand, $G' \subseteq \bigcap_{j \in J} G_j$. On the other hand, for all $g \in G'$, we have that $\bigsqcup_{j \in J} I_j(g, \ ) \sqsubseteq I'(g, \ ) \in \mathbf{3}^M$

and, then, $\bigsqcup_{j\in J} I_j(g,\ ) \neq \imath$. Therefore, $G' \subseteq G_0$ and $I_0(g,\ ) \sqsubseteq I'(g,\ )$ for all $g \in G'$. That is, $\mathbb{P}' \preceq \mathbb{P}_0$.

Now we prove that the supremum of $\{\mathbb{P}_j = (G_j, M, I_j) \colon j \in J\}$ is $\mathbb{P}_0 = (G_0, M, I_0)$ with

$$G_0 = \bigcup_{j\in J} G_j \ \text{ and } \ I_0(g,\ ) = \bigwedge_{j\in J_g} I_j(g,\ ), \text{for all } g \in G_0,$$

being $J_g = \{j \in J \colon g \in G_j\}$. For all $j \in J$, it is clear that $G_j \subseteq \bigcup_{j\in J} G_j = G_0$ and $I_0(g,\ ) \sqsubseteq I_j(g,\ )$ for all $g \in G_0$ and $j \in J_g$. Therefore, $\mathbb{P}_j \preceq \mathbb{P}_0$ for all $j \in J$. Assume now that $\mathbb{P}' = (G', M, I')$ is an upper bound of $\{\mathbb{P}_j = (G_j, M, I_j) \colon j \in J\}$. Trivially, $G_0 = \bigcup_{j\in J} G_j \subseteq G'$. Moreover, for all $g \in G'$, if $g \in G_j$ we have that $I'(g,\ ) \sqsubseteq I_j(g,\ )$. Therefore, $I'(g,\ ) \sqsubseteq I_0(g,\ )$. □

In addition, straightforwardly we have that the upper bound and the lower bound of $\underline{\mathfrak{P}}(\mathbb{P}_0)$ are $\mathbb{P}_0$ and $(\varnothing, M, \varepsilon)$ respectively.

7.2 A Galois connection between partial formal contexts and $\dot{\mathbf{3}}$-sets of attributes

Now we present the Galois connection that will allow us to collect the formal concepts in a lazy way. Given a partial formal context $\mathbb{P}_0 = (G_0, M, I_0)$, we define two concept forming operators as follows:

- $^{\Uparrow} \colon \mathfrak{P}(\mathbb{P}_0) \to \dot{\mathbf{3}}^M$ that maps any $\mathbb{X} = (G, M, I) \in \mathfrak{P}(\mathbb{P}_0)$ to
$$\mathbb{X}^{\Uparrow} = \bigwedge_{g\in G} I(g,\ ).$$
- $^{\Downarrow} \colon \dot{\mathbf{3}}^M \to \mathfrak{P}(\mathbb{P}_0)$ that maps any $\dot{\mathbf{3}}$-set $A \in \dot{\mathbf{3}}^M$ to $A^{\Downarrow} = (G, M, I)$ where
$$G = \{g \in G_0 \colon I_0(g,\ ) \sqcup A \neq \imath\} \ \text{ and}$$
$$I(g,\ ) = I_0(g,\ ) \sqcup A, \ \text{for each } g \in G.$$

Example 7.2.1 *Given the following partial formal context* $\mathbb{P}_0$ *and* $\mathbb{X}_1, \mathbb{X}_2 \in \mathfrak{P}(\mathbb{P}_0)$

$\mathbb{P}_0$	a	b	c
1	$+$	$\circ$	$-$
2	$-$	$+$	$\circ$

$\mathbb{X}_1$	a	b	c
1	$+$	$+$	$-$
2	$-$	$+$	$\circ$

$\mathbb{X}_2$	a	b	c
1	$+$	$\circ$	$-$

we have ${\mathbb{X}_1}^{\Uparrow} = b$ *and* $a\overline{c}^{\Downarrow} = \mathbb{X}_2$*.*

Theorem 7.2.2 *The pair* $(\Uparrow, \Downarrow)$ *is a Galois connection between* $\underline{\mathfrak{P}}(\mathbb{P}_0)$ *and* $\underline{\dot{\mathbf{3}}}^M$*.*

PROOF: First, assume $\mathbb{X}_1 = (G_1, M, I_1) \preceq \mathbb{X}_2 = (G_2, M, I_2)$, i.e. $G_1 \subseteq G_2$ and $I_2(g,\ ) \sqsubseteq I_1(g,\ )$ for all $g \in G_1$. Then

$$ {\mathbb{X}_2}^{\Uparrow} = \bigwedge_{g \in G_2} I_2(g,\ ) \sqsubseteq \bigwedge_{g \in G_1} I_2(g,\ ) \sqsubseteq \bigwedge_{g \in G_1} I_1(g,\ ) = {\mathbb{X}_1}^{\Uparrow} $$

and, therefore, $\Uparrow$ is an antitone mapping.

Let's prove that $\Downarrow$ is also antitone. Assume that $A_1, A_2 \in \dot{\mathbf{3}}^M$ satisfy $A_1 \sqsubseteq A_2$, and let ${A_1}^{\Downarrow} = (G_1, M, I_1)$ and ${A_2}^{\Downarrow} = (G_2, M, I_2)$. On the one hand, since $A_1 \sqsubseteq A_2$, we straightforwardly have that

$$ G_2 = \{g \in G \colon I_0(g,\ ) \sqcup A_2 \neq \imath\} \subseteq \{g \in G \colon I_0(g,\ ) \sqcup A_1 \neq \imath\} = G_1. $$

On the other hand, for all $g \in G_2 \subseteq G_1$, we have that

$$ I_1(g,\ ) = I_0(g,\ ) \sqcup A_1 \sqsubseteq I_0(g,\ ) \sqcup A_2 = I_2(g,\ ). $$

Therefore, ${A_2}^{\Downarrow} \preceq {A_1}^{\Downarrow}$.

Now we prove that $\mathbb{X}_1 \preceq {\mathbb{X}_1}^{\Uparrow\Downarrow}$ for all $\mathbb{X}_1 \in \mathfrak{P}(\mathbb{P}_0)$. If $\mathbb{X}_1 = (G_1, M, I_1)$ and ${\mathbb{X}_1}^{\Uparrow\Downarrow} = \mathbb{X}_2 = (G_2, M, I_2)$, we have that

$$ G_2 = \left\{ g \in G_0 \colon I_0(g,\ ) \sqcup \bigwedge_{g_1 \in G_1} I_1(g_1,\ ) \neq \imath \right\}. $$

Then, for all $g \in G_1$, we have

$$I_0(g,\ ) \sqcup \bigwedge_{g_1 \in G_1} I_1(g_1,\ ) \sqsubseteq I_0(g,\ ) \sqcup I_1(g,\ ) = I_1(g,\ )$$

and, therefore $g \in G_2$ and $I_2(g,\ ) \sqsubseteq I_1(g,\ )$.

Finally, let's prove that $A \sqsubseteq A^{\Downarrow\Uparrow}$ for all $A \in \dot{\mathbf{3}}^M$. Since $A^{\Uparrow} = (G_1, M, I_1)$ where $G_1 = \{g \in G\colon I_0(g,\ ) \sqcup A \neq \dot{\imath}\}$ and $I_1(g,\ ) = I_0(g,\ ) \sqcup A$ for each $g \in G_1$, we have that $A \sqsubseteq \bigwedge_{g \in G_1} I_1(g,\ ) = A^{\Downarrow\Uparrow}$. □

As a consequence, we have that both compositions of these maps are closure operators and their fixed points provide dually isomorphic lattices.

Corollary 7.2.3 *Given a partial formal context $\mathbb{P}_0 = (G_0, M, I_0)$, the set*

$$\mathfrak{S}(\mathbb{P}_0) = \{(\mathbb{X}, Y) \in \mathfrak{P}(\mathbb{P}_0) \times \dot{\mathbf{3}}^M : \mathbb{X}^{\Uparrow} = Y \ \textit{ and } \ Y^{\Downarrow} = \mathbb{X}\}$$

with the order

$$(\mathbb{X}_1, Y_1) \preceq (\mathbb{X}_2, Y_2) \ \textit{ if and only if } \ \mathbb{X}_1 \preceq \mathbb{X}_2 \ (\textit{or, equivalently, iff } \ Y_2 \sqsubseteq Y_1)$$

form a complete lattice denoted by $\underline{\mathfrak{S}}(\mathbb{P}_0)$.

The couples $(\mathbb{X}, Y) \in \mathfrak{S}(\mathbb{P}_0)$ are named *possible formal concept* on $\mathbb{P}_0$, and its name is due to that these concepts may change when new information is available, that is, some of them may stop holding and some new possible formal concepts may appear. As classically, given a possible formal concept $(\mathbb{X}, Y) \in \mathfrak{S}(\mathbb{P}_0)$, its components $\mathbb{X}$ and Y are named *extent* and *intent* of the concept, respectively.

Example 7.2.4 *In Figure 7.3 we present the lattice $\underline{\mathfrak{S}}(\mathbb{P}_0)$ obtained from the following partial formal context $\mathbb{P}_0$*

$\mathbb{P}_0$	a	b	c
1	$+$	$\circ$	$-$
2	$-$	$+$	$\circ$

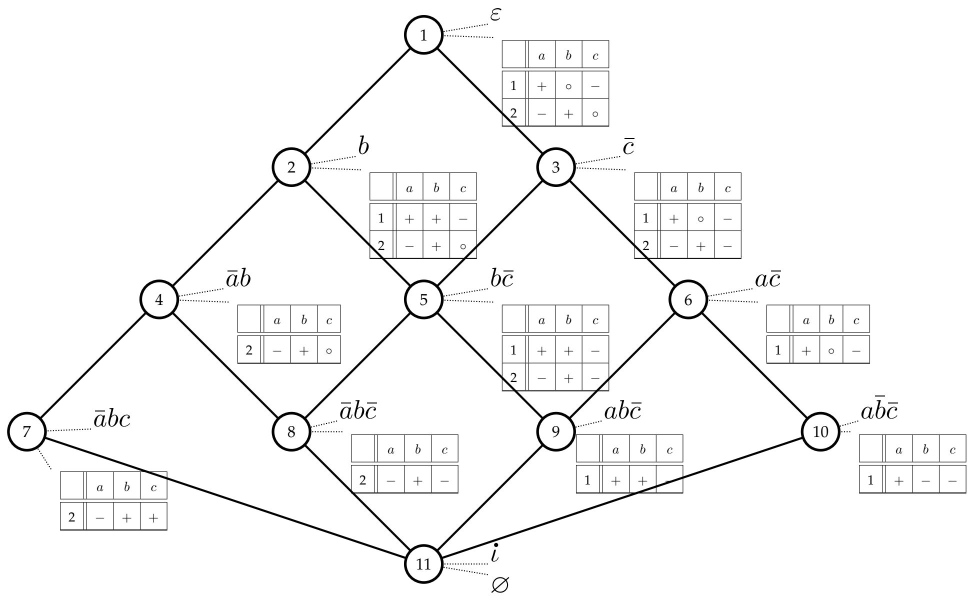


Figure 7.3: Lattice $\underline{\mathfrak{S}}(\mathbb{P})$

The following theorem gives us an equivalence between the lattice of possible formal concepts of a partial formal context and the mixed concept lattice obtained from $\mathbb{K}_*(\mathbb{P})$, which was defined in [64]. Before doing so, we rewrite some of the definitions given there to our notation in order to compare the results.

A mixed formal context can be seen as a partial context $\mathbb{M} = (G, M, I)$ in which there is no unknown information, i.e. such that $I(g, m) \neq \circ$ for all $g \in G$ and all $m \in M$, or, equivalently, $I(g,\ ) \in \mathfrak{F}\text{ull}(M)$ for all $g \in G$. Then, a Galois connection was defined by using the concept forming operators that we rewrite as follows: given $X \subseteq G$ and $Y \in \dot{\mathbf{3}}^M$,

$$X^{\Uparrow} = \bigwedge_{g \in G} I(g,\ ) \text{ and } Y^{\Downarrow} = \{g \in G \mid Y \sqsubseteq I(g,\ )\}.$$

The fixed pairs of this Galois connection were called mixed formal concepts and the set of all of them was denoted by $\mathfrak{W}(\mathbb{M})$. In addition, the lattice of mixed formal concepts was denoted by $\underline{\mathfrak{W}}(\mathbb{M})$.

Theorem 7.2.5 *Given a partial formal context $\mathbb{P} = (G, M, I)$, the set of atoms of $\underline{\mathfrak{S}}(\mathbb{P})$ is $\{(A^{\Downarrow}, A) \colon A \in \mathcal{M}(\mathbb{P})\}$ where*

$$\mathcal{M}(\mathbb{P}) = \{A \in \mathfrak{F}\mathrm{ull}(M) \colon I(g, \ ) \sqsubseteq A \text{ for some } g \in G\} \tag{7.2}$$

In addition, if the completion of $\mathbb{P}$ is $\mathbb{K}_*(\mathbb{P}) = (G', M, I')$, then

$$\mathcal{M}(\mathbb{P}) = \{I'((g, X), \ ) \colon (g, X) \in G'\}$$

and the lattices $\underline{\mathfrak{S}}(\mathbb{P})$ and $\underline{\mathfrak{W}}(\mathbb{K}_*(\mathbb{P}))$ are isomorphic.

PROOF: First, we prove that $A \in \mathcal{M}(\mathbb{P})$ implies that $(A^{\Downarrow}, A)$ is an atom of $\mathfrak{S}(\mathbb{P})$. We have that $A^{\Downarrow} = (G_1, M, I_1)$ where $G_1 = \{g \in G \colon I(g, \ ) \sqsubseteq A\}$ and $I_1(g, \ ) = A$ for all $g \in G_1$. Therefore, it is clear that $A^{\Downarrow\Uparrow} = A$. On the other hand, it cannot exist a $(B, C) \in \mathfrak{S}(\mathbb{P})$ such that $(\varnothing, \dot{\imath}) \prec (B, C) \prec (A^{\Downarrow}, A)$ because in that case we would have that $A \sqsubset C \sqsubset \dot{\imath}$ and this is not possible because A is a full set and, thus, a super-atom of $\underline{\dot{\mathbf{3}}}^M$.

Let's consider now $\mathbb{P}' = (\mathcal{M}(\mathbb{P}), M, I')$ where I' is defined as $I'(X, m) = X(m)$ for each $X \in \mathcal{M}(\mathbb{P})$ and $m \in M$; and we define $h \colon \mathfrak{S}(\mathbb{P}) \to \mathfrak{W}(\mathbb{P}')$ as $h(A, B) = (\mathcal{M}(A), B)$. Now we have to prove that h is a isomorphism between $\underline{\mathfrak{S}}(\mathbb{P})$ and $\underline{\mathfrak{W}}(\mathbb{P}')$.

First, we have to prove that h is well defined, that is, $h(A, B) \in \underline{\mathfrak{W}}(\mathbb{P}')$ for all $(A, B) \in \underline{\mathfrak{S}}(\mathbb{P})$. Let $(A, B) \in \underline{\mathfrak{S}}(\mathbb{P})$, that is, $A = (G_A, M, I_A) \in \mathfrak{P}(\mathbb{P})$ and $B \in \dot{\mathbf{3}}^M$ such that $G_A = \{g \in G : I(g, \ ) \sqcup B \neq \dot{\imath}\}$, $I_A(g, \ ) = I(g, \ ) \sqcup A$ for all $g \in G_A$, and $B = \displaystyle\bigwedge_{g \in G_A} I_A(g,)$.

Now consider $(\mathcal{M}(A), B)$, we need to prove that $\mathcal{M}(A)^{\uparrow\uparrow} = B$ and $B^{\downarrow\downarrow} = \mathcal{M}(A)$.

We have that $\mathcal{M}(A) = \{X \in \mathfrak{F}\mathrm{ull}(M) \mid I_A(g, \ ) \sqsubseteq X \text{ for some } g \in G\}$ and therefore

$$\mathcal{M}(A)^{\uparrow\uparrow} = \bigwedge_{X \in \mathcal{M}(A)} I'(X, \ ) = \bigwedge_{X \in \mathcal{M}(A)} X \overset{(i)}{=} \bigwedge_{g \in G_A} I_A(g, \ ) = B$$

where (i) is due to the definition of $\mathcal{M}(A)$ and the fact that, for all $X \in \mathcal{M}(A)$, there exists $g \in G$ such that $I_A(g, \ ) \sqsubseteq X$ and, therefore, we have that $Y = I_A(g, \ ) \sqcup \overline{X \setminus I_A(g, \ )} \in \mathcal{M}(A)$, and then $X \wedge Y = I_A(g, \ )$.

On the other hand, since $B = \bigwedge\limits_{g \in G_A} I_A(g,)$ we have that

$$\begin{aligned} B^{\downarrow\downarrow} &= \{X \in \mathcal{M}(\mathbb{P}) \mid B \sqsubseteq I'(X, \ )\} = \{X \in \mathcal{M}(\mathbb{P})\colon B \sqsubseteq X\} \\ &= \{X \in \mathfrak{Full}(M)\colon I_A(g, \ ) \sqsubseteq X \text{ for some } g \in G\} = \mathcal{M}(A) \end{aligned}$$

Once we have proved that h is well defined, we need to prove that it is an homomorphism. That is, if $(A, B), (C, D) \in \underline{\mathfrak{S}}(\mathbb{P})$ then $h((A, B) \vee (C, D)) = h(A, B) \vee h(C, D)$. We have that if $(E, F) = (A, B) \vee (C, D)$ then $(E, F) \in \underline{\mathfrak{S}}(\mathbb{P})$, being $E = A \curlyvee C$ and $F = B \wedge D$ so $h((A, B) \vee (C, D)) = h(E, F) = (\mathcal{M}(E), F)$. We need to prove that $\mathcal{M}(E) = \mathcal{M}(A) \cup \mathcal{M}(C)$.

It is clear that if $x \in \mathcal{M}(A)$ or $x \in \mathcal{M}(C)$ then $x \in M(E)$ because we have that $I_E(g,) \sqsubseteq I_x(g,)$ for all $g \in G_x$ being $x \in \{A, C\}$ so we can affirm that $\mathcal{M}(A) \cup \mathcal{M}(C) \subseteq \mathcal{M}(E)$.

On the other hand, if $x \in \mathcal{M}(E)$ we have that there is a $g \in G_E$ such that $I_E(G,) \sqsubseteq x$, but, $G_E = G_A \cup G_C$ and $I_E(g, \ ) = I_A(g, \ ) \wedge I_C(g, \ )$, thus, $g \in G_E$ such that $I_E(G,) \sqsubseteq x$, implies that there is a $g \in G_A$ ($g \in G_C$) such that $I_A(g,) \sqsubseteq x$ ($I_C(g,) \sqsubseteq x$), that is, $x \in \mathcal{M}(A)$ ($x \in \mathcal{M}(C)$ concluding that $\mathcal{M}(E) \subseteq \mathcal{M}(A) \cup \mathcal{M}(C)$.

Finally, we have to prove that h is bijective, that is, given $(A, B) \in \mathfrak{W}(\mathbb{P}')$ there is one, and just one, $(C, B) \in \mathfrak{S}(\mathbb{P})$ such that $h(C, B) = (A, B)$. We can define $C = B^{\Uparrow}$ then we have that $C^{\Downarrow} = B$, thus, we have that $(C, B) \in \mathfrak{S}(\mathbb{P})$ we have to check that $h(C, B) = (A, B)$ given $x \in \mathcal{M}(A)$ we have that there is $g \in A$ such that $I_A(g, \ ) \sqsubseteq x$ but $I_C(g, \ ) \sqsubseteq I_A(g, \ )$ and thus, $x \in \mathcal{M}(C)$ and as consequence we have that $h(C, B) = (A, B)$ in addition, by the uniqueness of the concept with B as set of attributes, we have that there is not other element and we can conclude that h is a isomorphism between $\underline{\mathfrak{S}}(\mathbb{P})$ and $\underline{\mathfrak{W}}(\mathbb{P}')$. $\square$

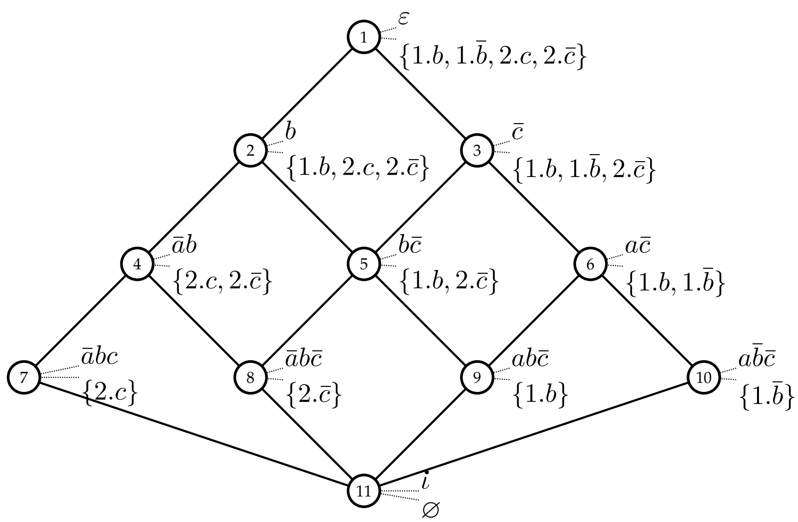


Figure 7.4: The mixed concept lattice defined by $\mathbb{K}_*(\mathbb{P}_0)$

Example 7.2.6 *For the partial formal context $\mathbb{P}_0$ defined in Example 7.2.4 provided in Section 4, the atoms of the lattice $\underline{\mathfrak{S}}(\mathbb{P}_0)$ are $\mathcal{M}(\mathbb{P}) = \{\bar{a}bc, \bar{a}b\bar{c}, ab\bar{c}, a\bar{b}\bar{c}\}$ (see Fig. 7.3) and, from the completion of $\mathbb{P}_0$,*

$\mathbb{K}_*(\mathbb{P}_0)$	a	b	c
$1.b$	$+$	$+$	$-$
$1.\bar{b}$	$+$	$-$	$-$
$2.c$	$-$	$+$	$+$
$2.\bar{c}$	$-$	$+$	$-$

we obtain the mixed concept lattice depicted in Figure 7.4.

As a consequence of Theorem 7.2.5 and [64, Theorem 6] we have the following result that characterizes the lattices $\underline{\mathfrak{S}}(\mathbb{P})$ obtained from partial formal contexts $\mathbb{P}$.

Corollary 7.2.7 *A finite lattice $\mathbb{L}$ is isomorphic to $\underline{\mathfrak{S}}(\mathbb{P})$ for some partial formal context $\mathbb{P}$ if and only if the following conditions hold:*

1. *$\mathbb{L}$ is $\wedge$-complemented, i.e. for all $\ell \in L$ we have that $\ell \vee \ell^{op} = \top$ and $\ell \wedge \ell^{op} = \bot$ being $\ell^{op} = \bigvee\{x \in L \mid \ell \wedge x = \bot\}$.*

2. *$\mathbb{L}$ is atomistic, i.e. any $\vee$-irreducible element is an atom.*

7.3 Strong implications: Semantics

In this section, we introduce the notion of strong implications, which hold not only in the current partial formal context but also when new information is available. Later, in the following section, our interest will be focus on the definition of axiomatic systems, which allows us to reason and infer new implications, collecting new knowledge. As stated, there are two main paradigms for designing axiomatic systems for implications: Armstrong's axioms [2], more oriented to describe the semantics of the implications, and Simplification paradigm [53], presenting a practical orientation that facilitates the design of automated methods. Here, we also present these two approaches by introducing two strong implication logics, each of them based on one of the paradigms. Furthermore, we will prove that the two axiomatic systems are equivalent.

Given a non-empty set of attributes M, we call *strong implication* (of attributes) to the expression $A \Rightarrow B$ where $A, B \in \dot{\mathbf{3}}^M$. The set of strong implications will be denoted by

$$\mathcal{L}_M^s = \{A \Rightarrow B\colon A, B \in \dot{\mathbf{3}}^M\}$$

Over the set $\mathcal{L}_M^s$, which we consider to be the language of the logic, we introduce the semantics as follows:

Definition 7.3.1 *Let $C \in \dot{\mathbf{3}}^M$. We say that C is model of a strong implication $A \Rightarrow B \in \mathcal{L}_M^s$ if it satisfies that $A \sqsubseteq C$ implies $B \sqsubseteq C$. The set of the models of $A \Rightarrow B$ is denoted by $\mathrm{Mod}(A \Rightarrow B)$.*

We say that C is model of a theory $\Sigma \subseteq \mathcal{L}^s_M$ if it is model of all strong implication $A \Rightarrow B \in \Sigma$, that is, $\mathcal{M}\mathrm{od}(\Sigma) = \bigcap_{A \Rightarrow B \in \Sigma} \mathcal{M}\mathrm{od}(A \Rightarrow B)$.

Note that the above definition is analogous to the one introduced for weak implications (see Definition 4.2.2). That definition was extended in Definition 4.2.3 so that a partial context was a model of an implication if all its object-intents $\{g\}^{\uparrow}$ were. This is where the difference comes in. When considering the Galois connection introduced in the previous section, now, instead of objects we now have partial contexts, and we will ask that all the context-intents $\{\mathbb{X}_i\}^{\Uparrow}$ are models.

Definition 7.3.2 *Let $\mathbb{P} = (G, M, I)$ be a partial formal context and $A \Rightarrow B \in \mathcal{L}^s_M$. $\mathbb{P}$ is said to be model of $A \Rightarrow B$, or that $A \Rightarrow B$ is satisfied in $\mathbb{P}$, and it will be denoted by $\mathbb{P} \models A \Rightarrow B$, if $\{\mathbb{X}_i\}^{\Uparrow} \in \mathcal{M}\mathrm{od}(A \Rightarrow B)$ for all $\mathbb{X}_i \in \mathfrak{P}(\mathbb{P})$.*

A partial formal context $\mathbb{P}$ is said to be model of a set $\Sigma \subseteq \mathcal{L}^s_M$, denoted by $\mathbb{P} \models \Sigma$, if $\mathbb{P} \models X \Rightarrow Y$ for all $X \Rightarrow Y \in \Sigma$.

The following proposition allows us to characterize the implications that are satisfied by a partial formal context by using the new concept forming operators. The proof is analogous to the classical case in that it is based on the properties of the Galois connection.

Proposition 7.3.3 *Let $\mathbb{P} = (G, M, I)$ be a partial context and $A \Rightarrow B \in \mathcal{L}^s_M$.*

$$\mathbb{P} \models A \Rightarrow B \quad \textit{if and only if} \quad A^{\Downarrow} \preceq B^{\Downarrow} \quad \textit{if and only if} \quad B \sqsubseteq A^{\Downarrow\Uparrow}.$$

The next proposition shows that, for a given strong implication $A \Rightarrow B$, the set $\{\mathbb{X}_i \in \mathfrak{P}(\mathbb{P}) \mid \mathbb{X}_i^{\Uparrow} \in \mathcal{M}\mathrm{od}(A \Rightarrow B)\}$ is a kernel system or dual closure system in $\underline{\mathfrak{P}}(\mathbb{P})$.

Proposition 7.3.4 *Let $\mathbb{P} = (G, M, I)$ be a partial formal context and $A \Rightarrow B \in \mathcal{L}^s_M$. The following assertions are satisfied:*

- $(\varnothing, M, \varepsilon)^{\Uparrow} \in \mathcal{M}\mathrm{od}(A \Rightarrow B)$.
- $\mathbb{X}_1^{\Uparrow}, \mathbb{X}_2^{\Uparrow} \in \mathcal{M}\mathrm{od}(A \Rightarrow B)$ *implies* $(\mathbb{X}_1 \curlyvee \mathbb{X}_2)^{\Uparrow} \in \mathcal{M}\mathrm{od}(A \Rightarrow B)$ *for all* $\mathbb{X}_1, \mathbb{X}_2 \in \mathfrak{P}(\mathbb{P})$.

PROOF: The first assertion is straightforward due that $(\varnothing, M, \varepsilon)^{\Uparrow} = \imath$ so $A \sqsubseteq \imath$ implies $B \sqsubseteq \imath$.

Assume that $\mathbb{X}_i^{\Uparrow} = (G_i, M, I_i)^{\Uparrow} \in \mathcal{M}\mathrm{od}(A \Rightarrow B)$ for $i = 1, 2$. That is, $A \sqsubseteq \mathbb{X}_i^{\Uparrow}$ implies $B \sqsubseteq \mathbb{X}_i^{\Uparrow}$ for $i = 1, 2$. Since $\mathbb{X}_1 \curlyvee \mathbb{X}_2 = (G_1 \cup G_2, M, I_0)$ where

$$I_0(g, \ ) = \begin{cases} I_1(g, \ ) & \text{if } g \in G_1 \setminus G_2, \\ I_2(g, \ ) & \text{if } g \in G_2 \setminus G_1, \\ I_1(g, \ ) \wedge I_2(g, \ ) & \text{otherwise,} \end{cases}$$

we have that

$$\begin{aligned} (\mathbb{X}_1 \curlyvee \mathbb{X}_2)^{\Uparrow} &= \bigwedge_{g \in G_1 \setminus G_2} I_1(g, \ ) \wedge \bigwedge_{g \in G_2 \setminus G_1} I_2(g, \ ) \wedge \bigwedge_{g \in G_1 \cap G_2} (I_1(g, \ ) \wedge I_2(g, \ )) \\ &= \bigwedge_{g \in G_1} I_1(g, \ ) \wedge \bigwedge_{g \in G_2} I_2(g, \ ) = \mathbb{X}_1^{\Uparrow} \wedge \mathbb{X}_2^{\Uparrow}. \end{aligned}$$

Thus, if $A \sqsubseteq \mathbb{X}_1^{\Uparrow} \wedge \mathbb{X}_2^{\Uparrow}$ then $B \sqsubseteq \mathbb{X}_1^{\Uparrow} \wedge \mathbb{X}_2^{\Uparrow}$, i.e. $(\mathbb{X}_1 \curlyvee \mathbb{X}_2)^{\Uparrow} \in \mathcal{M}\mathrm{od}(A \Rightarrow B)$. $\Box$

As consequence of the above proposition we have the following theorem, which ensures that strong implications have the desired semantics, i.e. that they are the implications that are not only satisfied by the currently available information, but which will continue to be satisfied when we obtain new information about the objects that is currently unknown.

Theorem 7.3.5 *Let* $\mathbb{P} = (G, M, I)$ *be a partial formal context and* $A \Rightarrow B \in \mathcal{L}_M^s$. *The following assertions are equivalent:*

1. $\mathbb{P} \models A \Rightarrow B$.

2. $\mathcal{M}(\mathbb{P}) \subseteq \mathrm{Mod}(A \Rightarrow B)$ *where* $\mathcal{M}(\mathbb{P})$ *is the set defined in* (7.2).

PROOF: Assume that $\mathbb{P} \models A \Rightarrow B$. By Theorem 7.2.5, $\{(X^{\Downarrow}, X) \mid X \in \mathcal{M}(\mathbb{P})\} \subseteq \mathfrak{S}(\mathbb{P})$ where $\mathcal{M}(\mathbb{P}) = \{X \in \mathcal{F}\mathrm{ull}(M) \mid I(g, \ ) \sqsubseteq X \text{ for some } g \in G\}$. Therefore, $X^{\Downarrow\Uparrow} = X$ for all $X \in \mathcal{M}(\mathbb{P})$. Since $\mathbb{P} \models A \Rightarrow B$, if $X \in \mathcal{M}(\mathbb{P})$ and $A \sqsubseteq X$, then $B \sqsubseteq A^{\Downarrow\Uparrow} \sqsubseteq X^{\Downarrow\Uparrow} = X$. That is, $X \in \mathrm{Mod}(A \Rightarrow B)$ for all $X \in \mathcal{M}(\mathbb{P})$.

Conversely, as a consequence of Theorem 7.3.4, to prove $\mathbb{P} \models A \Rightarrow B$ it is enough to study the $\vee$-irreducible elements of $\underline{\mathfrak{S}}(\mathbb{P})$ and, by Corollary 7.2.7, they are precisely the atoms of $\underline{\mathfrak{S}}(\mathbb{P})$. In addition, by Theorem 7.2.5, the atoms of the lattice $\underline{\mathfrak{S}}(\mathbb{P})$ are the pairs $\{(X^{\Downarrow}, X) \mid X \in \mathcal{M}(\mathbb{P})\}$. Therefore, $\mathcal{M}(\mathbb{P}) \subseteq \mathrm{Mod}(A \Rightarrow B)$ implies $\mathbb{P} \models A \Rightarrow B$. $\square$

The following proposition relates classical attribute implications to strong implications.

Proposition 7.3.6 *Given a partial formal context* $\mathbb{P} = (G, M, I)$ *and* $A, B \subseteq M$.

1. *If* $\mathbb{K}^{+}_{\mathbb{P}} \models A \to B$ *then* $\mathbb{P} \models A \Rightarrow B$.
2. *If* $\mathbb{K}^{-}_{\mathbb{P}} \models A \to B$ *then* $\mathbb{P} \models \overline{A} \Rightarrow \overline{B}$.

PROOF: If $A \subseteq M$, we have that $\mathrm{Pos}(A) = A$ and $\mathrm{Neg}(A) = \varnothing$. Then, as a consequence, $A^{\downarrow} = A^{+} \cap \varnothing^{-} = A^{+} \cap G = A^{+}$. Analogously, we have that $B^{\downarrow} = B^{+}$. Now if we build $A^{\Uparrow}$ we find the partial formal context with all the objects than have the attributes from A. If we consider $B^{\Uparrow}$ we have all the objects than have the attributes from B. Since $\mathbb{K}^{+}_{\mathbb{P}} \models A \to B$ we have that all the objects that have the attributes from A have all the attributes from B as well so we have that $B \sqsubseteq A^{\Downarrow\Uparrow}$.

The second item is a consequence of the following equality:

$$\overline{A}^{\downarrow} = \mathrm{Pos}(\overline{A})^{+} \cap \mathrm{Neg}(\overline{A})^{-} = \mathrm{Neg}(A)^{+} \cap \mathrm{Pos}(A)^{-} = \varnothing^{+} \cap A^{-} = G \cap A^{-} = A^{-}$$

and, analogously, $\overline{B}^{\downarrow} = B^{-}$. $\square$

As in the case of weak implications, the notion of semantic entailment is introduced in terms of the partial formal contexts that are models of the implications.

Definition 7.3.7 *Let $A \Rightarrow B \in \mathcal{L}_M^s$ and $\Sigma \subseteq \mathcal{L}_M^s$. We say that $A \Rightarrow B$ is semantically entailed from Σ, denoted by $\Sigma \models A \Rightarrow B$, when $\mathbb{P} \models \Sigma$ implies $\mathbb{P} \models A \Rightarrow B$ for all partial formal context $\mathbb{P}$.*

Regarding the notation, if there's no confusion, we denote the sets of implications without curly brackets. In the same way, we denote by $\models A \Rightarrow B$ when we have that $\varepsilon \models A \Rightarrow B$.

7.4 Axiomatic systems for strong implications

As we have done in Chapter 4 and Chapter 6, the next step is to define sound and complete axiomatic systems for the strong implications. In this section we focus only on the syntactic treatment of implications, and in the following section we will focus on the relationship between the semantic and the syntactic aspects.

We start with an axiomatic system based on Armstrong's axioms.

Definition 7.4.1 *The* Armstrong-style *axiomatic system for strong implications, denoted as $\mathcal{A}_s$, is* $\{[[\texttt{Inc}]], [[\texttt{Augm}]], [[\texttt{Trans}]], [[\texttt{Rft}]], [[\texttt{Tru}]]\}$ *where these rules are called* inclusion, augmentation, transitivity, reflection *and* trust *respectively, and are defined as follows: for all $A \in \dot{\mathbf{3}}^M$ and all singletons $a, b \in \mathbf{3}^M$,*

[[Inc]] *Infer $AB \Rightarrow A$.*

[[Augm]] *From $A \Rightarrow B$ infer $AC \Rightarrow BC$.*

[[`Trans`]] *From* $A \Rightarrow B$ *and* $B \Rightarrow C$ *infer* $A \Rightarrow C$.

[[`Rft`]] *From* $Aa \Rightarrow b$ *infer* $A\overline{b} \Rightarrow \overline{a}$

[[`Tru`]] *From* $a \Rightarrow \overline{a}$ *infer* $\varepsilon \Rightarrow \overline{a}$

Notice that the Armstrong-style axiomatic system for strong implications, $\mathcal{A}_s$, is an extension of the Armstrong-style axiomatic system for weak implications, $\mathcal{A}$, which was introduced in Definition 4.2.8. Specifically, we have that $\mathcal{A}_s = \mathcal{A} \cup \{[[\texttt{Rft}]], [[\texttt{Tru}]]\}$. The aim of adding new inference rules to the axiomatic system is to make it complete, as we will see in the next section.

Our second approach is based on the Simplification paradigm. The new axiomatic system is an extension of the Simplification axiomatic system for weak implications introduced in Definition 6.1.1. Specifically, the new axiomatic system is obtained by adding a new inference rule, called the inverse key rule.

Definition 7.4.2 *The* Simplification Axiomatic System for strong implications, *denoted by* $\mathcal{S}_s$, *is* $\{[[\texttt{Inc}]], [[\texttt{Key}]], [[\texttt{Simp}]], [[\texttt{InKy}]]\}$ *where these rules are called* inclusion, key, simplification *and* inverse key *respectively, and are defined as follows: for all* $A \in \dot{\mathbf{3}}^M$ *and all singleton* $b \in \mathbf{3}^M$,

[[`Inc`]] *Infer* $AB \Rightarrow A$.

[[`Key`]] *From* $A \Rightarrow b$ *infer* $A\overline{b} \Rightarrow i$.

[[`Simp`]] *From* $A \Rightarrow B$ *and* $C \Rightarrow D$ *infer* $A(C \smallsetminus B) \Rightarrow D$.

[[`Inky`]] *From* $Ab \Rightarrow \dot{\imath}$ *infer* $A \Rightarrow \overline{b}$.

Now, our aim is to proof that both axiomatic systems are sound and complete. First, the following theorem ensures that both axiomatic systems $\mathcal{A}_s$ and $\mathcal{S}_s$ are equivalent.

Theorem 7.4.3 *Let M be a set of attributes, $\Sigma \subseteq \mathcal{L}_M^s$ and $A \Rightarrow B \in \mathcal{L}_M^s$. We have that*

$$\Sigma \vdash_{\mathcal{A}_s} A \Rightarrow B \quad \textit{if and only if} \quad \Sigma \vdash_{\mathcal{S}_s} A \Rightarrow B$$

PROOF: First, we prove $\Sigma \vdash_{\mathcal{A}_s} A \Rightarrow B$ implies $\Sigma \vdash_{\mathcal{S}_s} A \Rightarrow B$ and, by Theorem 6.1.4, it is enough to prove that [[`Inky`]] is a derived rule from $\mathcal{A}_s$. We distinguish two cases:

If $A = \varepsilon$ the following sequence is a proof:

$\phi_1 = b \Rightarrow \imath$	By hypothesis.
$\phi_2 = \imath \Rightarrow \overline{b}$	By [[`Inc`]]
$\phi_3 = b \Rightarrow \overline{b}$	Applying [[`Trans`]] to ϕ_1 and ϕ_2.
$\phi_4 = \varepsilon \Rightarrow \overline{b}$	Applying [[`Tru`]] to ϕ_3.

If $A \neq \varepsilon$ the following sequence is a proof:

$\phi_1 = Ab \Rightarrow \imath$	By hypothesis.
$\phi_2 = \imath \Rightarrow \overline{a}$	By [[`Inc`]] and being a an element of A.
$\phi_3 = Ab \Rightarrow \overline{a}$	Applying [[`Trans`]] to ϕ_1 and ϕ_2.
$\phi_4 = A \Rightarrow \overline{b}$	Applying [[`Rft`]] to ϕ_3.

Conversely, to prove that $\Sigma \vdash_{\mathcal{S}_s} A \Rightarrow B$ implies $\Sigma \vdash_{\mathcal{A}_s} A \Rightarrow B$, by Theorem 6.1.4, is enough to prove that the rules [[`Rft`]] and [[`Tru`]] are derived from $\mathcal{S}_s$.

The following sequence proves that [[`Rft`]] can be derived from $\mathcal{S}_s$:

$\phi_1 = Aa \Rightarrow b$	By hypothesis.
$\phi_2 = Aa\overline{b} \Rightarrow \imath$	Applying [[`Key`]] to ϕ_1.
$\phi_3 = A\overline{b} \Rightarrow \overline{a}$	Applying [[`Inky`]] to ϕ_2.

Finally, the following sequence proves that [[`Tru`]] can be derived from $\mathcal{S}_s$:

$\phi_1 = a \Rightarrow \overline{a}$	By hypothesis.
$\phi_2 = a \Rightarrow \dot{\imath}$	Applying [[`Key`]] to ϕ_1.
$\phi_3 = \varepsilon \Rightarrow \overline{a}$	Applying [[`Inky`]] to ϕ_2.

$\square$

As a consequence of the above theorem, the results regarding inference rules can be equally applied to any of the two systems. Thus, from now on, we don't use the subindex referencing the specific axiomatic system.

Lemma 7.4.4 *Indistinctly, from $\mathcal{S}_s$ and from $\mathcal{A}_s$ we can derive the following inference rules: for all $A, B, C, D \in \dot{\mathbf{3}}^M$ and all singleton $b \in \mathbf{3}^M$*

[[`Un`]] $A \Rightarrow B$, $A \Rightarrow C \vdash A \Rightarrow BC$.

[[`gAug`]] *If* $A \sqsubseteq C$ *and* $D \sqsubseteq C \sqcup B$, *then* $A \Rightarrow B \vdash C \Rightarrow D$.

[[`Red`]] *If* $\mathrm{Spp}(C)$ *is finite, then* $Ab \Rightarrow C$, $A\overline{b} \Rightarrow C \vdash A \Rightarrow C$.

These rules are called union, generalized augmentation *and* reduction*, respectively.*

PROOF: As we have seen in Theorem 7.4.3 that both axiomatic systems are equivalent, in the proof we can use indistinctly all the inference rules for strong implications that we have already presented. Proposition 4.2.10 proved that [[`Un`]] is derived from $\mathcal{A}$, and therefore this rule is also derived from $\mathcal{A}_s$. Similarly, from Proposition 6.2.1, we have that [[`gAug`]] is a derived rule from $\mathcal{S}_s$. Finally, assume $c \in M \cup \overline{M}$ is such that $c \sqsubseteq C$. The following sequence proves $Ab \Rightarrow C$, $A\bar{b} \Rightarrow C \vdash A \Rightarrow c$.

$\phi_1 = Ab \Rightarrow C$	By hypothesis.
$\phi_2 = A\bar{b} \Rightarrow C$	By hypothesis.
$\phi_3 = C \Rightarrow c$	By [[Inc]] and $c \sqsubseteq C$.
$\phi_4 = Ab \Rightarrow c$	Applying [[Trans]] to ϕ_1 and ϕ_3.
$\phi_5 = A\bar{b} \Rightarrow c$	Applying [[Trans]] to ϕ_2 and ϕ_3.
$\phi_6 = A\bar{c}\bar{b} \Rightarrow \imath$	Applying [[Augm]] to ϕ_5.
$\phi_7 = A\bar{c} \Rightarrow b$	Applying [[Inky]] to ϕ_6.
$\phi_8 = A\bar{c} \Rightarrow \bar{b}$	Applying [[Rft]] to ϕ_4.
$\phi_9 = A\bar{c} \Rightarrow \imath$	Applying [[Un]] to ϕ_7 and ϕ_8.
$\phi_{10} = A \Rightarrow c$	Applying [[Inky]] to ϕ_8.

Since it holds for all $c \sqsubseteq C$ and $\mathrm{Spp}(C)$ is finite, by iteratively applying [[Un]], we have that [[Red]] is a derived inference rule. $\square$

7.5 Soundness and completeness

To conclude the formal introduction of our logic framework, we present here the two fundamental properties: soundness and completeness. These two properties established that the semantics and the inference systems are strongly tied, i.e. all the formulae that can be proved to be semantically derived, can be syntactically inferred, and *vice verse*. This result is crucial since it ensures that automated reasoning methods can be further designed for the new framework to manage missing information properly.

Theorem 7.5.1 (Soundness) *For all strong implication $A \Rightarrow B \in \mathcal{L}^s_M$ and all set $\Sigma \subseteq \mathcal{L}^s_M$, we have that $\Sigma \vdash A \Rightarrow B$ implies $\Sigma \models A \Rightarrow B$.*

PROOF: By Theorem 7.4.3, it is enough to prove the soundness of one of the axiomatic systems $\mathcal{A}_s$ or $\mathcal{S}_s$. In particular, we are going to prove the soundness of the first one. In the proof we will also use the two characterizations given in Proposition 7.3.3 for a context to be a model of a strong implication, and the fact that the composition ${}^{\Downarrow\Uparrow}$ is a closure operator in $(\dot{\mathbf{3}}^M, \sqsubseteq)$.

First, for all $A, B \in \dot{\mathbf{3}}^M$, since $A \sqsubseteq AB \sqsubseteq AB^{\Downarrow\Uparrow}$, we have that $\models AB \Rightarrow A$. Therefore, [[Inc]] is sound.

Second, assume that $\mathbb{P} \models A \Rightarrow B$, i.e. $B \sqsubseteq A^{\Downarrow\Uparrow}$. Then, we have that $BC \sqsubseteq A^{\Downarrow\Uparrow} \sqcup C^{\Downarrow\Uparrow} \sqsubseteq (A \sqcup C)^{\Downarrow\Uparrow}$. Therefore, $\mathbb{P} \models A \Rightarrow B \models AC \Rightarrow BC$ and [[Augm]] is proved.

To prove the soundness of [[Trans]], assume that $\mathbb{P} \models A \Rightarrow B$ and $\mathbb{P} \models B \Rightarrow C$ and, then, $B \sqsubseteq A^{\Downarrow\Uparrow}$ and $C \sqsubseteq B^{\Downarrow\Uparrow}$. Therefore, we have that $C \sqsubseteq B^{\Downarrow\Uparrow} \sqsubseteq (A^{\Downarrow\Uparrow})^{\Downarrow\Uparrow} = A^{\Downarrow\Uparrow}$, i.e. $\mathbb{P} \models A \Rightarrow C$.

Now we prove that [[Rft]] is sound. Obviously, we can assume that $A \neq \text{¿}$, because in this case [[Rft]] could be seen as a particular case of [[Inc]], which has already been proved.

Let $\mathbb{P} = (G, M, I)$ be a partial formal context. If $\mathbb{P} \models Aa \Rightarrow b$, then $Aa^{\Downarrow} \preceq b^{\Downarrow}$, i.e.

$$G_1 = \{g \in G \mid I(g,\ ) \sqcup Aa \neq \text{¿}\} \subseteq G_2 = \{g \in G \mid I(g,\ ) \sqcup b \neq \text{¿}\} \text{ and}$$
$$I(g,\ ) \sqcup b \sqsubseteq I(g,\ ) \sqcup Aa \ \text{ for all } \ g \in G_1.$$

We are going to prove that this implies $\mathbb{P} \models A\overline{b} \Rightarrow \overline{a}$, i.e.

$$G_3 = \{g \in G \mid I(g,\ ) \sqcup A\overline{b} \neq \text{¿}\} \subseteq G_4 = \{g \in G \mid I(g,\ ) \sqcup \overline{a} \neq \text{¿}\} \text{ and} \tag{7.3}$$
$$I(g,\ ) \sqcup \overline{a} \sqsubseteq I(g,\ ) \sqcup A\overline{b} \ \text{ for all } \ g \in G_3. \tag{7.4}$$

We previously prove that $g \in G_3$ implies $g \notin G_1$. Assume that $g \in G_3 \cap G_1$. From $g \in G_3$, we have that $I(g,\ ) \sqcup A\overline{b} \neq \text{¿}$ and, therefore, $b \not\sqsubseteq I(g,\ )$. On the other hand, from $g \in G_1$, we have that $b \sqsubseteq I(g,\ ) \sqcup b \sqsubseteq I(g,\ ) \sqcup Aa \neq \text{¿}$. Therefore, $b \sqsubseteq A$ and $A\overline{b} = \text{¿}$, which contradicts $g \in G_3$.

Now, we prove (7.3) and (7.4). Let $g \in G_3$ and, then $g \notin G_1$. Thus, we have that $I(g,\ ) \sqcup A\overline{b} \neq \text{¿}$ and $I(g,\ ) \sqcup Aa = \text{¿}$.

- If $Aa = \dot{\imath}$, from the assumption that $A \neq \dot{\imath}$, we have that $\overline{a} \sqsubseteq A$. Therefore, $I(g,\ ) \sqcup \overline{a} \sqsubseteq I(g,\ ) \sqcup A\overline{b} \neq \dot{\imath}$ and $g \in G_4$. That is, (7.3) and (7.4) hold.

- Otherwise, if $Aa \neq \dot{\imath}$, we have $I(g,\ ) \sqcup a = \dot{\imath}$, which implies $\overline{a} \sqsubseteq I(g,)$ and, then, $g \in G_4$. In addition, $I(g,\ ) \sqcup \overline{a} = I(g,\ ) \sqsubseteq I(g,\ ) \sqcup A\overline{b}$.

Finally, we prove the soundness of [[`Tru`]]. Let $\mathbb{P} = (G, M, I)$ be a partial formal context such that $\mathbb{P} \models a \Rightarrow \overline{a}$ that means $a^{\Downarrow} \preceq \overline{a}^{\Downarrow}$. That is,

$$G_1 = \{g \in G \mid I(g,\ ) \sqcup a \neq \dot{\imath}\} \subseteq G_2 = \{g \in G \mid I(g,\ ) \sqcup \overline{a} \neq \dot{\imath}\} \text{ and} \tag{7.5}$$

$$I(g,\ ) \sqcup \overline{a} \sqsubseteq I(g,\ ) \sqcup a \ \text{ for all } \ g \in G_1. \tag{7.6}$$

First, we prove that $G_2 = G$. Assume that $g \notin G_2$. Then $I(g,\ ) \sqcup \overline{a} = \dot{\imath}$. Then $a \sqsubseteq I(g,\ )$ and $I(g,\ ) \sqcup a \neq \dot{\imath}$. Therefore, $g \in G_1 \subseteq G_2$, which is contradictory.

Second, we prove that $G_1 = \varnothing$. If $g \in G_1$ then, by (7.6) and (7.5), we have that $I(g,\ ) \sqcup \overline{a} \sqsubseteq I(g,\ ) \sqcup a \neq \dot{\imath}$. The inclusion $I(g,\ ) \sqcup \overline{a} \sqsubseteq I(g,\ ) \sqcup a$ implies $\overline{a} \sqsubseteq I(g,\ )$, but this contradicts $I(g,\ ) \sqcup a \neq \dot{\imath}$.

In summary, $G \subseteq G_2$ and, since $G_1 = \varnothing$, we have that $I(g,\ ) \sqcup \overline{a} \sqsubseteq I(g,\ )$ for all $g \in G$. That is, $\mathbb{P} \models \varepsilon \Rightarrow \overline{a}$. $\square$

We need some necessary results in order to prove the completeness of the axiomatic systems.

Theorem 7.5.2 *Let M be a finite set and $\Sigma \in \mathcal{L}^s_M$. The map $[\ ]^s_\Sigma \colon \dot{\mathbf{3}}^M \to \dot{\mathbf{3}}^M$ defined as*

$$[A]^s_\Sigma = \bigsqcup \{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash A \Rightarrow X\}$$

is a closure operator in $\underline{\dot{\mathbf{3}}}^M$, which is called syntactic closure with respect to Σ. *In addition, $\Sigma \vdash A \Rightarrow [A]^s_\Sigma$ for all $A \in \dot{\mathbf{3}}^M$.*

PROOF: By [[`Inc`]], $\Sigma \vdash A \Rightarrow A$ and, then, $A \sqsubseteq [A]^s_\Sigma$.

Let's prove that $A \sqsubseteq B$ implies $[A]^s_\Sigma \sqsubseteq [B]^s_\Sigma$. By [[`Inc`]], $\Sigma \vdash B \Rightarrow A$. For all $X \in \dot{\mathbf{3}}^M$ such that $\Sigma \vdash A \Rightarrow X$, by [[`Trans`]] with $\Sigma \vdash B \Rightarrow A$, we have that $\Sigma \vdash B \Rightarrow X$.

In order to prove the idempotency of the mapping $[\ ]^s_\Sigma$, we previously demonstrate that $\Sigma \vdash A \Rightarrow [A]^s_\Sigma$ for all $A \in \dot{\mathbf{3}}^M$. As M is finite, the set $\chi = \{X \in \dot{\mathbf{3}}^M \mid \Sigma \vdash A \Rightarrow X\}$ is also finite. It is enough to prove that $X, Y \in \chi$ implies $X \sqcup Y \in \chi$ and it is straightforward from [[`Un`]]. Therefore, $[A]^s_\Sigma = \bigsqcup \chi \in \chi$ and $\Sigma \vdash A \Rightarrow [A]^s_\Sigma$.

Finally, $[[A]^s_\Sigma]^s_\Sigma \sqsubseteq [A]^s_\Sigma$ because $\Sigma \vdash A \Rightarrow [A]^s_\Sigma$, $\Sigma \vdash [A]^s_\Sigma \Rightarrow [[A]^s_\Sigma]^s_\Sigma$ and, by transitivity, $\Sigma \vdash A \Rightarrow [[A]^s_\Sigma)^s_\Sigma$. Therefore, since $[\ ]^s_\Sigma$ is inflationary, we have that $[\ ]^s_\Sigma$ is idempotent.

□

Corollary 7.5.3 *Let M be a finite set of attributes. For all $\Sigma \subseteq \mathcal{L}^s_M$ and all $A, B \in \dot{\mathbf{3}}^M$, we have that $\Sigma \vdash A \Rightarrow B$ if and only if $B \sqsubseteq [A]^s_\Sigma$.*

PROOF: From Theorem 7.5.2, one has $\Sigma \vdash A \Rightarrow B$ implies $B \sqsubseteq [A]^s_\Sigma$. We can obtain the converse result by using [[`Inc`]] and [[`Trans`]]. □

In the rest of this section, given a set of strong implications Σ, we denote the set of syntactically closed $\dot{\mathbf{3}}$-sets with respect to Σ by:

$$\mathcal{C}(\Sigma) = \{X \in \dot{\mathbf{3}}^M \colon [X]^s_\Sigma = X\}$$

Thus, as a consequence of Proposition 2.2.3, we have the following result.

Lemma 7.5.4 *Let M be a finite set of attributes. For all $\Sigma \subseteq \mathcal{L}^s_M$ and $A \in \dot{\mathbf{3}}^M$, one has $[A]^s_\Sigma = \min\{X \in \mathcal{C}(\Sigma) \mid A \sqsubseteq X\}$.*

The following lemma and theorem characterize the syntactic closure with respect to Σ in terms of fullsets which will be the key in the proof of the completeness issue.

Lemma 7.5.5 *Let M be a finite set and $\Sigma \subseteq \mathcal{L}^s_M$. For all $C \in \mathcal{C}(\Sigma) \smallsetminus \{\imath\}$ there exists $X \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M)$ such that $C \sqsubseteq X$.*

PROOF: Consider $C \in \mathcal{C}(\Sigma) \smallsetminus \{\imath\}$. As M is finite, there is a maximal set $C_1 \in \mathcal{C}(\Sigma) \smallsetminus \{\imath\}$ such that $C \sqsubseteq C_1$. Let's prove $C_1 \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M)$. Assume that $C_1 \notin \mathfrak{Full}(M)$, then there exists $m \in M$ such that $C_1(m) = \circ$ and, from the maximality of C_1 in $\mathcal{C}(\Sigma) \smallsetminus \{\imath\}$, we have that $[C_1 m]^s_\Sigma = \imath$. By using Corollary 7.5.3 we have that $\Sigma \vdash C_1 m \Rightarrow \imath$ and we distinguish two cases:

- If $C_1 = \varepsilon$ them $\Sigma \vdash m \Rightarrow \imath$, on the other hand, by [[`Inc`]] $\Sigma \vdash \imath \Rightarrow \overline{m}$, and applying [[`Trans`]] to both implications we have that $\Sigma \vdash m \Rightarrow \overline{m}$. Now applying [[`Tru`]] we have that $\Sigma \vdash \varepsilon \Rightarrow \overline{m}$ which contradicts that $C_1 \in \mathcal{C}(\Sigma) \smallsetminus \{\imath\}$.

- If $C_1 \neq \varepsilon$, let $c \sqsubseteq C_1$. From $\Sigma \vdash C_1 m \Rightarrow \imath$ and from $\Sigma \vdash \imath \Rightarrow \overline{c}$ (which is given by [[`Inc`]]), we conclude, by [[`Trans`]], that $\Sigma \vdash C_1 m \Rightarrow \overline{c}$. Now, by using [[`Rft`]] we have that $\Sigma \vdash C_1 \Rightarrow \overline{m}$. As consequence, $\overline{m} \sqsubseteq [C_1]^s_\Sigma$ which contradicts that $C_1 \in \mathcal{C}(\Sigma) \smallsetminus \{\imath\}$.

□

Now, the following theorem prove that the syntactic closure can be described in terms of full sets.

Proposition 7.5.6 *Let M be a finite set of attributes, $\Sigma \subseteq \mathcal{L}^s_M$ and $A \in \dot{\mathbf{3}}^M$. Then*

$$[A]^s_\Sigma = \bigwedge \{X \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M) \mid A \sqsubseteq X\}.$$

PROOF: From Lemma 7.5.4, it is sufficient to prove that $B \in \mathcal{C}(\Sigma)$ implies $B = \bigwedge \{X \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M) \mid B \sqsubseteq X\}$. Let $B \in \mathcal{C}(\Sigma)$.

If $B = \imath$, then it is straightforward because $\bigwedge \varnothing = \imath$.

If $B = \varepsilon$, we need to prove that $\bigwedge \Big(\mathcal{C}(\Sigma) \cap \mathfrak{Full}(M)\Big) = \varepsilon$. Assume that it is not true, i.e. there exists c such that $c \sqsubseteq X$ for all $X \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M)$.

In that case, there is not X in $\mathcal{C}(\Sigma) \cap \mathfrak{Full}(M)$ such that $\overline{c} \sqsubseteq X$. By using Lemma 7.5.5 we have that $[\overline{c}]^s_\Sigma = \imath$ and, by using, Corollary 7.5.3 we have that $\Sigma \vdash \overline{c} \Rightarrow \imath$ and, from [[Inc]], we have that $\Sigma \vdash \imath \Rightarrow c$. Now, by using [[Trans]] we have that $\Sigma \vdash \overline{c} \Rightarrow c$. Finally, by applying [[Tru]] we have that $\Sigma \vdash \varepsilon \Rightarrow c$, which contradicts that $B = \varepsilon \in \mathcal{C}(\Sigma)$.

Finally, consider the case that $B \neq \imath$ and $B \neq \varepsilon$. It is straightforward that $B \sqsubseteq \bigwedge\{X \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M) \mid B \sqsubseteq X\}$. Now, we prove the other inclusion. Let $c \sqsubseteq \bigwedge\{X \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M) \mid B \sqsubseteq X\}$. Then, there is not $X \in \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M)$ such that $B \sqcup \overline{c} \sqsubseteq X$. By using Lemma 7.5.5, we have that $[B \sqcup \overline{c}]^s_\Sigma = \imath$ and, by using, Corollary 7.5.3 we have that $\Sigma \vdash B\overline{c} \Rightarrow \imath$. If we consider an arbitrary $b \in B$, by [[Inc]], we have that $\Sigma \vdash \imath \Rightarrow b$ and, applying [[Trans]], we have that $\Sigma \vdash B\overline{c} \Rightarrow b$. By [[Rft]] we have that $\Sigma \vdash B \Rightarrow c$. As $B \in \mathcal{C}(\Sigma)$ we have that $c \in B$.

□

These previous results allow us to establish the completeness of the axiomatic systems as follows:

Theorem 7.5.7 (Completeness) *Let M be a finite set of attributes. For all $A \Rightarrow B \in \mathcal{L}^s_M$ and $\Sigma \subseteq \mathcal{L}^s_M$, we have that $\Sigma \models A \Rightarrow B$ implies $\Sigma \vdash A \Rightarrow B$.*

PROOF: Let's prove that $\Sigma \nvdash A \Rightarrow B$ implies $\Sigma \not\models A \Rightarrow B$. Using the Corollary 7.5.3, we have that $\Sigma \nvdash A \Rightarrow B$ implies that $B \not\sqsubseteq [A]^s_\Sigma$, and, therefore, $[A]^s_\Sigma \neq \imath$.

Let $\mathbb{P} = (G, M, I)$ be the partial formal context such that the set of objects is $G = \mathcal{C}(\Sigma) \cap \mathfrak{Full}(M)$ and $I\colon G \times M \to \mathbf{3}$ being $I(g, m) = g(m)$ for all $g \in G$ and $m \in M$. Notice that G is a subset of $\mathbf{3}^M$. From Theorem 7.5.1, it is straightforward that $\mathbb{P} \models \Sigma$. However, we are going to prove that $X^{\Downarrow\Uparrow} = [X]^s_\Sigma$ for all $X \in \dot{\mathbf{3}}^M$, which implies that $\mathbb{P} \not\models A \Rightarrow B$.

Given $X \in \dot{\mathbf{3}}^M$, we have that $X^{\Downarrow} = (G_1, M, I_1)$ where

$$G_1 = \{g \in G\colon I(g, \ ) \sqcup X \neq \imath\} = \{g \in G\colon X \sqsubseteq g\}$$

and, for each $g \in G_1$,

$$I_1(g, \;) = I(g, \;) \sqcup Y = I(g, \;) = g$$

By applying the other concept forming operator we have that

$$X^{\Downarrow\Uparrow} = \bigwedge_{g \in G_1} I(g, \;) = \bigwedge\{g \in G \colon X \sqsubseteq g\}$$

Finally, by Proposition 7.5.6, we have that $X^{\Downarrow\Uparrow} = [X]^s_\Sigma$ for all $X \in \dot{\mathbf{3}}^M$.

Therefore, we have proved that $\mathbb{P} \models \Sigma$ but $\mathbb{P} \not\models A \Rightarrow B$ and, as a consequence, we have that $\Sigma \not\models A \Rightarrow B$. □

7.6 Simplification paradigm

This chapter ends with the motivation of the Simplification paradigm as a solid basis to built automated methods. This further development relies on the characteristic that some inference rules can be seen as equivalence rules. In particular, $\mathcal{S}$ was designed to have all its primitive rules as equivalence rules. This situation allows us to manage set of implications by reducing the size of the set of implications whereas the equivalence is preserved.

Theorem 7.6.1 *The following equivalence rules holds: for all $A, B, C, D \in \dot{\mathbf{3}}^M$ and all singleton $b \in \mathbf{3}^M$,*

[[`FragEq`]] $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \smallsetminus A\}$.

[[`UnEq`]] $\{A \Rightarrow B,\; A \Rightarrow C\} \equiv \{A \Rightarrow BC\}$.

[[ε-`Eq`]] $\{A \Rightarrow \varepsilon\} \equiv \varnothing$.

[[$\dot{\imath}$-`Eq`]] $\{A \Rightarrow B\} \equiv \{A \Rightarrow \dot{\imath}\}$ *when* $A \sqcup B = \dot{\imath}$.

[[`SimpEq`]] $\{A \Rightarrow B,\; C \Rightarrow D\} \equiv \{A \Rightarrow B,\; C \smallsetminus B \Rightarrow D \smallsetminus B\}$ *when* $A \sqsubseteq C \smallsetminus B$.

[[`RdEq`]] $\{Ab \Rightarrow C, A\bar{b} \Rightarrow C\} \equiv \{A \Rightarrow C\}$ *when* $\mathrm{Spp}(C)$ *is finite.*

[[`KyEq`]] $\{A \Rightarrow b\} \equiv \{A\bar{b} \Rightarrow \imath\}$.

PROOF: The proofs for [[`FragEq`]], [[`UnEq`]], [[ε-`Eq`]], [[$\imath$-`Eq`]] and [[`SimpEq`]] are analogous to those given in Theorem 6.2.2.

Let's prove [[`RdEq`]]. Since $\mathrm{Spp}(C)$ is finite, by [[`Red`]], we have that $Ab \Rightarrow C, A\bar{b} \Rightarrow C \vdash A \Rightarrow C$. In the opposite direction, by using [[`gAug`]], we have that $A \Rightarrow C \vdash Ab \Rightarrow C$ and $A \Rightarrow C \vdash A\bar{b} \Rightarrow C$.

The equivalence [[`KyEq`]] is a direct consequence of [[`Key`]] and [[`Inky`]]. □

In the following example, we apply these equivalences, left-to-right read, to reduce the size of the set of implications without losing any knowledge, that is, preserving the equivalenceness.

Example 7.6.2 *On* $M = \{a, b, c, d\}$*, consider the set of strong implications* $\Gamma = \{ab \Rightarrow c, \imath \Rightarrow de, bd \Rightarrow c\bar{b}, d\bar{c} \Rightarrow b\bar{a}, dc \Rightarrow b\bar{a}, abc \Rightarrow \imath, b\bar{c}\bar{d} \Rightarrow a\bar{d}, a \Rightarrow b\}$*. Let's see how the size of* Γ *can be reduced using the equivalences given in Theorem 7.6.1.*

- *By* [[`FragEq`]] *and* [[ε-`Eq`]]*, we have that* $\{\imath \Rightarrow de\} \equiv \{\imath \Rightarrow \varepsilon\} \equiv \varepsilon$ *and*

$$\Gamma \equiv \{ab \Rightarrow c, bd \Rightarrow c\bar{b}, d\bar{c} \Rightarrow b\bar{a}, dc \Rightarrow b\bar{a}, abc \Rightarrow \imath, b\bar{c}\bar{d} \Rightarrow a\bar{d}, a \Rightarrow b\}.$$

- *Applying* [[$\imath$-`Eq`]]*, we have that* $\{bd \Rightarrow c\bar{b}\} \equiv \{bd \Rightarrow \imath\}$*. Therefore,*

$$\Gamma \equiv \{ab \Rightarrow c, bd \Rightarrow \imath, d\bar{c} \Rightarrow b\bar{a}, dc \Rightarrow b\bar{a}, abc \Rightarrow \imath, b\bar{c}\bar{d} \Rightarrow a\bar{d}, a \Rightarrow b\}.$$

- *Applying* [[`KyEq`]]*, we have* $\{d\bar{c} \Rightarrow b\bar{a}, dc \Rightarrow b\bar{a}\} \equiv \{d \Rightarrow b\bar{a}\}$*. Therefore,*

$$\Gamma \equiv \{ab \Rightarrow c, bd \Rightarrow \imath, d \Rightarrow b\bar{a}, abc \Rightarrow \imath, b\bar{c}\bar{d} \Rightarrow a\bar{d}, a \Rightarrow b\}.$$

- *Applying* [[`SimpEq`]] *and* [[`UnEq`]] *we have that*

$$\{a \Rightarrow b, ab \Rightarrow c\} \equiv \{a \Rightarrow b, a \Rightarrow c\} \equiv \{a \Rightarrow bc\},$$

and therefore

$$\Gamma \equiv \{a \Rightarrow bc, bd \Rightarrow \imath, d \Rightarrow b\overline{a}, abc \Rightarrow \imath, b\overline{c}\overline{d} \Rightarrow a\overline{d}\}.$$

- *Applying* [[`SimpEq`]] *and* [[`UnEq`]] *we have that*

$$\{d \Rightarrow b\overline{a}, bd \Rightarrow \imath\} \equiv \{d \Rightarrow b\overline{a}, d \Rightarrow \imath\} \equiv \{d \Rightarrow \imath\},$$

and therefore

$$\Gamma \equiv \{a \Rightarrow bc, d \Rightarrow \imath, abc \Rightarrow \imath, b\overline{c}\overline{d} \Rightarrow a\overline{d}\}.$$

- *Applying* [[`FragEq`]] *we have that* $\{b\overline{c}\overline{d} \Rightarrow a\overline{d}\} \equiv \{b\overline{c}\overline{d} \Rightarrow a\}$. *Then,*

$$\Gamma \equiv \{a \Rightarrow bc, d \Rightarrow \imath, abc \Rightarrow \imath, b\overline{c}\overline{d} \Rightarrow a\}.$$

- *Finally, applying* [[`SimpEq`]] *and* [[`UnEq`]] *we have that*

$$\{a \Rightarrow bc, abc \Rightarrow \imath\} \equiv \{a \Rightarrow bc, a \Rightarrow \imath\} \equiv \{a \Rightarrow \imath\},$$

and therefore

$$\Gamma \equiv \{a \Rightarrow \imath, d \Rightarrow \imath, b\overline{c}\overline{d} \Rightarrow a\}.$$

Chapter 8

Conclusions, some fruitful discussions and future works

8.1 Conclusions

The main motivation of this Ph.D. Thesis is to deal with unknown information by providing a formal framework for specifying and managing this kind of information. Our main idea is to build as complete an approach as possible and therefore, we work in the area of Formal Concept Analysis, which provides a strong formalization and, at the same time, an excellent orientation to the practical dimension.

In this work, we have extended FCA to consider not only positive but also negative and unknown information in a natural way. The classical FCA paradigm studies the presence of an attribute for an object and does not explicitly consider the attribute's absence as information. In some extensions, positive and negative information are considered.

The starting point is a three-valued relationship between objects and attributes, which we call partial formal context (see Chapter 3). Since the interpretation of attribute sets is conjunctive, when we join different sets together, contradictions may arise. Therefore, we need to enrich the structure of truthfulness values to take this into account. The essence of this approach is to replace the attribute powerset with a new structure, $(\dot{\mathbf{3}}^M, \sqsubseteq)$, which is very close to bilattices. Considering this structure, we present a Galois connection which we use to capture the information that is available in the partial formal context. With this Galois connection, we presented the necessary concepts, which are formal concepts that hold with the currently available information, and we built the corresponding concept lattice. Furthermore, we also establish the relationship between this concept lattice obtained and the classical one.

The key to this was the choice of the algebraic structure on which the semantics are to be defined, and the derivation operators by ensuring that they still form a Galois connection.

Furthermore, we have presented a new type of attribute implication, which we name weak implications because they can change when new information is added (Chapter 4). This change could cause some of the so-called weak implications to no longer hold but also cause new implications to emerge with the new information. In FCA, implications describe the information extracted from the formal concept, equivalent to that described by means of the corresponding concept lattice. The main advantage of implications is that they can be managed in a symbolic way.

For these implications, we first consider an Armstrong's axioms-like approach, which have the same appearance as the classical Armstrong's axioms but being defined on the new structure. We incorporate the semantics that allows dealing with positive/negative information, as well as with unknown information and even contradiction or inconsistency. We also prove the soundness and completeness of this new axiomatic system.

In addition, we also developed a new axiomatic system in the framework of Simplification Logic [20] , that is closer to applications in the sense that, unlike the previous one, it can be considered as an executable logic. The development of the Simplification paradigm to this new scenario can be approached using a guideline with several well-defined steps. In this guideline, after considering complete lattice as the base structure to build the semantics up, the key point is the definition of the difference operation over the complete lattice. For the Simplification Logic to be deemed sound and to establish a foundation for the corresponding inference engine, the operation in question must satisfy certain properties and define a set of equivalence rules. In this work, we are working with a non distributive lattice, thus, we can not use Boolean algebra as in the classical case [20] nor complete dual Heyting algebras as in the fuzzy case [6]. We define a new structure called weak complete dual Heyting algebra, which was made in Chapter 5, to ensure the necessary properties to have a sound and complete axiomatic system. Observe that there are other structures that are not complete dual Heyting algebra but we can not use them because they do not hold these properties that we demand. In particular, the difference operation for our new structure requires some of the conditions to be a complete dual Heyting algebra, but not all of them.

We have introduced, Chapter 6, two new axiomatic systems based on the simplification paradigm for reasoning about the weak implications introduced in [56]. In addition, by using the weak complete dual Heyting algebra defined in [58], we prove that both axiomatic systems are equivalent to the Armstrong-style one and, as consequence, both are sound and complete.

As a common feature of the family of so-called Simplification logics, we prove that in this case, too, inference rules can be described as equivalence rules. These equivalence rules allow reducing the size of implicational systems without loss of knowledge, i.e., simplifying implicational systems by removing redundant information (by means of the difference operator).

This result, together with Theorems 6.3.2 and 6.3.4 that resembles the so-called classical deduction theorem, allowed us to provide an algorithm for computing the syntactic closure (see Algorithm 2) and, consequently, for defining an automated reasoning method about weak implications. This Algorithm follows the same schema as those proposed for other extensions of the Simplification Logic [6,22,50]. Finally, we have proved the correctness of the Algorithm and studied its cost in the worst case (see Theorem 6.3.5).

Moreover, in this work we also consider another Galois connection that captures information from a partial formal context; in particular, this Galois connection allows us to combine our partial formal context with the idea of granularity that appears in [32]. We present an order with the Partial formal contexts, which can be seen as the order having less granularity or more unknown information. Unlike the weak issue, whose true knowledge can be modified when new information is provided, in Chapter 7 we studied the idea of being necessarily true in all possible configurations after acquiring new knowledge. This will require us to consider the whole possible universe for the partial formal context.

Furthermore, with the new Galois connection, we introduce another definition of attribute implication that we name strong implications. The idea is that they would remain unchanged when new information is provided. The ultimate goal in this point is to establish a logic that allows reasoning simultaneously with both types of implications defining an axiomatic system based on Armstrong for the strong implications and another one based on Simplification logic. We prove that both of them are sound and complete.

8.2 Discussions

In this point, we would like to present some additional questions that we've been discussing while working on it. We structure this section with some separate subsections for each issue.

8.2.1 Truth value set structure

In Chapter 3, we have presented the truth values that we have considered in this Ph.D. thesis to extend FCA to consider not only positive but also negative and, even more, unknown information in a natural way.

In the literature, we can find several approaches to this issue being, the closest one by Belnap [5]. The order that we use is the information order of the bilattice of Belnap (see Figure 1.2), and it is dual to the order considered by Lex [47] (see Figure 1.3). Observe that there are other frameworks where different orders have been considered, for instance, Holzer et al. [41], that considered the veracity order (excluding the value contradiction).

In [32], Ganter and Meschke presented the so-called partial formal context: formal contexts where the incidence maps go to a three-valued set instead of a Boolean one. In this Ph.D. thesis, we have used this partial formal context, and we have analysed its equivalences with some classical formal contexts. In Chapter 4, we presented a Galois connection which we use to capture the information that is available in the partial formal context. With this Galois connection, we presented the necessary concepts, which are formal concepts that hold with the information available, and they are contained in the formal concepts formed when new information is received, that is when unknown information is changed by positive or negative one. Observe that a few days later than when we published these results, Qi et al. published a paper [63] introducing a very similar construction to the one we developed.

In addition, using the same Galois connection, we have presented a new kind of attribute implication, the so-called weak implications. Observe that these implications can be compared with the *satisfiable attribute implications* introduced by Holzer [41]. It is therefore worth pausing at this point to look at the differences between the two types of implications.

The first difference between the implications appears in their semantics. The satisfiable attribute implications are those attribute implications that hold in, at least, one completion of the partial formal context, whereas the weak implications are attribute implications that hold with just the information that we have at that moment, that is, not completions are considered at this moment.

There is also a difference in the syntax. Holzer just considers positive values in the implications (although negative values can appear in the partial formal context), whereas in our implications, we use positive, negative, and unknown values.

The following example will facilitate the understanding of the differences expressed.

Example 8.2.1 *Let $\mathbb{P} = (G, M, I)$ be the partial formal context given in Figure 8.1:*

$\mathbb{P}$	a	b	c
1	$+$	$\circ$	$-$
2	$-$	$+$	$\circ$

Figure 8.1: Partial formal context $\mathbb{P}$

In this partial context, the satisfiable attribute implications as Holzer gets in his article will be:

$$\{a \to b, b \to c, c \to a, c \to b\}$$

Our weak implications are:

$$\{\overline{c} \rightsquigarrow a, \overline{b} \rightsquigarrow i, \overline{a} \rightsquigarrow b, c \rightsquigarrow i, b \rightsquigarrow \overline{a}, a \rightsquigarrow \overline{c}\}$$

Observe that, if Holzer would consider the negative values as-well in the implications, he would get some of our weak implications like, for instance, $\overline{c} \rightsquigarrow a$.

8.2.2 Dual Heyting algebras

We have developed sound and complete axiomatic systems based on the Simplification paradigm. We are working with a non-distributive lattice, thus, we can not use Boolean algebra nor complete dual Heyting algebras. To look for the necessary properties to have a sound and complete axiomatic system, we define a new structure called weak complete dual Heyting algebra which was made in Chapter 5. Observe that there are other structures that are not complete dual Heyting algebra, but we can not use them because they do not hold the properties we use.

In this way, we compare our weak complete dual Heyting algebra with the semi dual Heyting algebra [66], which is a previous algebra with some properties from the complete dual Heyting algebra. Let's see first the definition of the Semi dual Heyting algebra.

Definition 8.2.2 *An algebra* $\mathbb{L} = (L, \leqslant, \setminus)$ *is a* semi dual Heyting algebra *if the following conditions hold:*

[sH1] $(L, \vee, \wedge, 0, 1)$ *is a lattice.*

[sH2] $x \vee (y \setminus x) = x \vee y$, *for all* $x, y \in L$.

[sH3] $x \vee (z \setminus y) = x \vee [(x \vee z) \setminus (x \vee y)]$, *for all* $x, y, z \in L$.

[sH4] $x \setminus x = 0$, *for all* $x \in L$.

As expected, this algebra does not hold all the properties to prove the soundness and completeness of the Simplification paradigm as it does not hold the property [wH2]. With this information, we can say that the semi dual Heyting algebras are not weak complete dual Heyting algebras. We can build a semi-dual Heyting algebra that does not satisfy `[wH2]`, for example, the following:

We consider the complete lattice $(\{0,1\}, \leqslant)$ with $0 \leqslant 1$, we also consider the difference operation $\smallsetminus$ defined by the following table:

$\smallsetminus$	0	1
0	0	1
1	1	0

It is easy to prove that $(\{0,1\}, \leqslant, \smallsetminus)$ holds the properties `[sH1]`–`[sH4]` and it does not hold the property `[wH2]` because $0 \smallsetminus 1 = 1$ and $0 \leqslant 1$.

Now, we are wondering if the weak complete dual Heyting algebra is a particular case of the semi dual Heyting algebra. In Chapter 5, we prove that $\underline{\dot{\mathbf{3}}}^U$ is a complete dual Heyting algebra. The following example can be used as a counterexample to show that there are weak complete dual Heyting algebras that are not semi dual Heyting algebras.

Example 8.2.3 *Consider $U = \{a, b\}$ and the sets $X, Y, Z \in \dot{\mathbf{3}}^U$ defined by $X = \overline{a}b$, $Y = a$ and $Z = \dot{\imath}$ we have that:*

$$X \sqcup (Z \smallsetminus Y) = \overline{a}b \sqcup \dot{\imath} = \dot{\imath},$$

on the other hand we have that

$$X \sqcup [(X \sqcup Z) \smallsetminus (X \sqcup Y)] = \overline{a}b \sqcup [\dot{\imath} \smallsetminus \dot{\imath}] = \overline{a}b.$$

Thus, we can conclude that the property `[sH3]` does not necessarily hold in $\underline{\dot{\mathbf{3}}}$, and we have that, for this U, $\underline{\dot{\mathbf{3}}}^U$ is a weak complete dual Heyting algebra (it is shown in the section above), but it is not a semi dual Heyting algebra.

Observe that the complete dual Heyting algebras are both semi dual Heyting algebra and weak complete dual Heyting algebra. In addition, the following theorem establishes that this is a necessary and sufficient condition to have both algebras at the same time.

Theorem 8.2.4 *Let $(L, \leqslant)$ be a complete lattice, $\mathbb{L} = (L, \leqslant, \setminus)$ is a complete dual Heyting algebra if and only if it is a weak complete dual Heyting algebra and a semi dual Heyting algebra.*

The proof of this theorem comes from the fact that if we put the properties of semi dual Heyting algebras and weak complete dual Heyting algebras together, we have the properties of complete dual Heyting algebra. That is, from $\mathbb{L} = (L, \leqslant, \setminus)$ being a weak complete dual Heyting algebra, we cover `[H1]` and `[H3]` and from the fact of $\mathbb{L} = (L, \leqslant, \setminus)$ being a semi dual Heyting algebra we have that `[H2]` holds, thus, $\mathbb{L} = (L, \leqslant, \setminus)$ holds all the properties of complete dual Heyting algebra.

To finish Chapter 5 we have characterised the weak-cdHas to differentiate the lattices in which we can define this structure from the lattices in which we cannot. In addition, we also characterise the properties of the difference operation to build such structure in a consistent way.

8.2.3 Partial Formal Concepts

In Chapter 7, we present a second Galois connection that captures information from a partial formal context. In particular, this Galois connection allows us to combine our partial formal context with the idea of granularity that appears in [32]. We present an order with the Partial formal contexts, which can be seen as the order having less granularity or more unknown information.

About this issue, we can compare our proposal with the one made by Zhi et al. [72]. We start showing that the order between the contexts is not the same. Let's consider the partial formal context $\mathbb{K}$ and their two completions $\mathbb{K}_1$ and $\mathbb{K}_2$ in 8.2.

If we take the order defined in our work, we would have that both completions are incomparable and all of them are lower than the partial formal context with unknown information, that is, $\mathbb{K}_1 \preceq \mathbb{K}$ and $\mathbb{K}_2 \preceq \mathbb{K}$. However, Zhi et al. just order the completions, and its order is the pointwise order considering that $- \leqslant +$, as a consequence, it has that $\mathbb{K}_1 \preceq \mathbb{K}_2$.

$\mathbb{K}$	a	b	c
o_1	$+$	$+$	$-$
o_2	$-$	$\circ$	$+$
o_3	$+$	$-$	$\circ$

$\mathbb{K}_1$	a	b	c
o_1	$+$	$+$	$-$
o_2	$-$	$+$	$+$
o_3	$+$	$-$	$+$

$\mathbb{K}_1$	a	b	c
o_1	$+$	$+$	$-$
o_2	$-$	$-$	$+$
o_3	$+$	$-$	$-$

Figure 8.2: A partial formal context and two of its completions

With this Galois connection, we present the so-called possible concepts, which are concepts that hold in the partial formal context but could stop holding when new information is received, that is when unknown information changes to positive or negative. Again, in this point can compare our approach with the Zhi's one. Zhi et al. defined concepts in the partial formal context, like a combination of the concepts of the biggest and lowest completion in his order. Observe that these concepts are quite different from our approach for the possible formal concepts as we do not use just two completions but analyse all of the cases.

8.2.4 Strong implications

In addition, we present the strong implications that match with the idea of Kripke implication defined by Holzer [41]. However, here again we have some differences, the main one being the syntax as we use positive, negative and unknown values in the implications while Kripke only uses the positive.

Thus, we would have more implications that they would capture. In addition, in their paper, they do not show how to capture such implications. Concluding that our work, again, differs from their work.

8.3 Future works

Here, we just consider three possible values (positive, negative, and unknown). However, in the real world, there are many more possible values. In addition, some attributes cannot be adequately shown using just these values. For instance, the attribute “being tall": if we consider a person to be tall if it is taller than 1.80 meters, what can we say about a person who is 1.79,9 meters tall? If we just say that the person is not tall, we lose information because it is really close to being tall and is quite different from a person that is 1.2 tall.

To solve this problem, as future work, we will work in an extension of the three values considering (possible) infinitive values, that is, working in a fuzzy framework. Specifically, we are going to work with pairs of fuzzy values, following Atanassov’s approach to fuzzy sets; the first value is the information that we have about if the object has the attribute, the second one is the information that we have about if the object has not the attribute. Thus, the value $(0,0)$ will be “we have no information about if the object has the attribute or not”, the value $(1,0)$ “will be our $+$, while $(0,1)$ will be our $-$. Finally, we are going to have so many inconsistent pairs, and we are going to denote all the inconsistent pairs with the same value $(1,1)$, which is our ι. In this line, we would like to combine our work about this issue with the line opened by Jan Koneckny [45]. Thus, the work presented in this Ph.D. Thesis can be considered the needed bridge to move from a particular case presented in [58] to a more general framework.

In addition, we will generalize our results by considering other information interpretations and looking for a unified framework for reasoning with missing information. In particular, we would like to combine our work with the lines opened by D. Dubois, J. Medina, H. Prade, and E. Ramírez-Pousa in [28] to define automated methods for an FCA disjunctive logic properly. Remark that this is not a trivial task since a disjunctive interpretation significantly impacts the reasoning methods and their efficiency.

Another point where we can extend our work is in the formal framework needed adequately for each situation. In particular, as a particular problem, we plan to focus on the structure of the weak complete dual Heyting algebra. About this issue, it is relevant to study which weak-cdHas are algebraic and which are not. Another open line in the formalization issue that arises from this work is the extension of the weak dual Heyting algebras for a given lattice (not necessarily with four elements) and if they can be used for different issues.

Finally, regarding the practical application of these results, there is also a long work to do. In fact, it is a great advantage to have a three value approach compared with the previous two value approaches. We plan to extend this work to build a recommendation system, in line with previous works that we have developed in the past [17,21]. The recommender system will incorporate in this way the possibility to deal with unknown information, for instance, when a user does not declare anything about a facility in a hotel or when he has not seen some film, and he has no opinion about it. To tackle the implementation of the recommender system, as a previous step, we intend to incorporate the results of this work in the package fcaR [18], a package that has been developed in our group to include all FCA methods and algorithms in different extensions. In particular, we plan to define a new attribute closure operator for the new extension, following our usual guideline: we need to show the inference rules as equivalences and, later, to define an algorithm based on these equivalences allowing an iterative construction of the closure set and, at the same time, developing a reduction in the set of implications. The first stage has been fully developed in this Ph.D. Thesis, while the second one has been initiated, but a further study of the disjunction interpretation has to be fulfilled.

Index of terms

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