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GEOMETRIC INVARIANTS AND INNER STRUCTURE

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GEOMETRIC INVARIANTS AND INNER STRUCTURE

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A mis padres,

por hacer de mí la persona que soy.

A mi hermano y a mis abuelos,

por hacer de mí la persona que quiero ser.

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Suena *A Cruel Angel's Thesis* - Yōko Takahashi.

Contents

CHAPTER

1

Abstract

First thing, a summary of this dissertation in both English and Spanish will be presented.

1.1 Abstract

The geometry of the unit ball of Banach spaces is defined as the central object of study in this dissertation. In particular, we will deeply study the geometry of the inner structure and the extremal structure of this unit ball. First, we establish some conditions for a convex set to hold the equality between its inner points and non-support points, and we present a non-trivial weakly compact and convex subset of an infinite-dimensional Banach space lacking inner points. Then we focus on extremal structure by means of a deep study of facets and frames of the unit ball: we present novel definitions, such as the P-property, flat set, starlike envelope or the pre-maximal face definition. Also, we relate the different types of points and well-known subsets in a unit ball. A new reformulation of the frame of the unit ball is presented by using the relative interior of its facets. In the same chapter, several new terms concerning flatness are defined, for instance, the mentioned starlike envelope, almost flat or flat subsets, and we go over their relations and behaviour as convex components in the unit sphere. The last part of this thesis is devoted to determine how the structure of the unit ball is affected under different operators. In the first place, the mappings considered are surjective isometries defined between unit spheres and new (and shorter) proofs for known results are presented. In addition, we prove the invariance of flatness, faces, and segments under particular circumstances, where the original new properties will be employed. Also, we show the invariance of antipodal rotund points and antipodal maximal faces. Here, the inner structure will play a fundamental role. The second part, related to Operator Theory, concludes this dissertation with the analysis of the geometry of the unit ball under projections. A new type of projection is introduced, named as S-projection. This kind of projections will help us to extend the extremal subsets of the projected space into the large one. Finally, we define the term of strongly maximal face and we show its preservation under a particular type of 1-projections, namely, L^2-projections.

1.2 Resumen

El principal objeto de estudio dentro de esta tesis es la geometría de la bola unidad de un espacio de Banach. En particular, estudiaremos en profundidad la estructura interna y extremal de dicha bola. Para empezar, establecemos algunas condiciones para las cuales se tiene la igualdad entre los puntos internos y los puntos de no-soporte de subconjuntos convexo y presentamos un subconjunto no trivial débilmente compacto y convexo sin puntos interiores en un espacio de Banach de dimensión infinita. Después, nos centramos en la

estructura extremal con el estudio en profundidad de las “facetas” y los “marcos” de la bola unidad: presentamos nuevas definiciones, como la Propiedad P, los conjuntos planos, la “envolvente estrellada” o una cara pre-maximal. Asimismo, relacionamos los diferentes tipos de puntos y subconjuntos conocidos de la bola unidad. Presentamos una reformulación nueva del “marco” de la bola unidad usando el interior relativo de sus “facetas”. En el mismo capítulo, se definen varios términos nuevos relacionados con la planitud, como la envoltura estrellada, los subconjuntos planos y casi planos, y haremos un estudio sobre sus relaciones e interacciones como componentes convexas de la esfera unidad. La última parte de la tesis está destinada al estudio del comportamiento de la estructura de la bola unidad bajo distintos tipos de operadores. En primer lugar, las aplicaciones consideradas serán las isometrías sobreyectivas definidas entre las esferas unidad y presentaremos nuevas pruebas para resultados ya conocidos. Además, probaremos la invarianza de la planitud, caras y segmentos bajo ciertas hipótesis, donde serán de ayuda las nuevas propiedades definidas en la tesis. También mostraremos la invarianza de los puntos antípodas cuando sean redondos y de las caras maximales antípodas. Aquí, la estructura interna jugará un papel fundamental. La segunda parte relacionada con teoría de operadores concluye la tesis con el análisis de la geometría de la bola unidad bajo proyecciones. Presentaremos las S-proyecciones como una nueva aplicación de este tipo. Estas proyecciones nos serán de gran ayuda para extender subconjuntos extremales del espacio projectado al espacio grande. Finalmente, definiremos el nuevo término de caras fuertemente maximales y veremos cómo se mantienen bajo un tipo particular de 1-proyecciones, a saber, las L^2-proyecciones.

CHAPTER

2

Introduction

Contents

2.1 State of the art

This work is entirely framed in the General Theory of real Topological Vector Spaces. We consider important to make the reader beware that the category of real Topological Vector Spaces naturally includes the category of real vector spaces by simply endowing the vector space with the locally convex vector topology.

There are many different versions of geometry, depending on the category we are working on. For example, the relevant category in Differential Geometry is constituted by differential manifolds. The category of algebraic manifolds is crucial in Algebraic Geometry. In Metric Geometry, the main category is the one of metric spaces. A particular version of geometry, Analytical Geometry, makes the category of real normed spaces come into play as one of its typical and characteristic categories.

Then Analytical Geometry is the field of geometry taking care of the geometrical aspects of real topological vector spaces. One of the most important and impacting branches of Analytical Geometry is the Extremal Theory, where some known subjects of this study are faces, extremal subsets, facets, rotund points or exposed faces, among others.

Extremal Theory encompasses a large research and a huge amount of results. On of these results is the invariance under surjective isometries between unit spheres of Banach spaces for maximal faces, proved by Tanaka in (1, Lemma 3.5). Geometrically speaking, maximal faces are just convex components of the unit sphere. Tanaka used this invariance together with the invariance of proper exposed points to fully determine the surjective isometries for the operator norm unit sphere of $n \times n$ matrices (2, Theorem 6.1), where proper exposed points are those elements of the unit sphere formed by the intersection of all the maximal faces containing it. This idea motivates our original definition of premaximal faces and Property P (3). The latter condition allows us to show the invariance of faces (3, Theorem 13), which is not proved in the general case . Another subject of study in Extremal Theory is the behaviour between the objects mentioned before. For example, rotund points (which are singleton maximal faces of the unit ball) are proper exposed points, which are in fact exposed points, and they also satisfy the extremal condition. We want to highlight the notion of exposed face between all the mentioned objects: informally speaking, exposed faces are those subsets of the unit ball shared with a particular hyperplane, and if we consider the boundary respect to that hyperplane, we have what we called an edge (4). The union of all the edges is called the frame of the unit ball. In (1, Theorem 3.7) the invariance of the frame under surjective isometries was also proved. Nevertheless, we present a new reformulation of the frame by using the relative interior of facets respect to the unit sphere in (3, Theorem 7), what makes simpler to prove the invariance of the frame (3, Theorem 11). As well as all the terms cited above, facets (understood as faces with no empty interior relative to the unit sphere) are into the research of Extremal Theory and its invariance under surjective isometries is a direct consequence of the fact that this types of operators are in particular homeomorphisms.

Our background about Extremal Theory comes to an end with the study of the starlike structures. The term of starlike set was originally introduced, just for finite dimensional vector spaces, by Tingley in (5), where the concept was characterized in Lemma 4 and Corollary 5. This kind of subsets where defined in general normed spaces (1; 6; 7) as those elements of the unit sphere which are at distance 2 from the opposite of the center of the starlike set, and this set is characterized as the set of points in the unit sphere whose segment, with the center as the other extreme, is contained in the sphere, and also as the union of all the maximal faces containing the center of the starlike set. Even more, for separable Banach spaces, in (1, Lemma 3.3) it was demonstrated that every maximal convex subset of the unit sphere is a starlike set. Compared to this result, in (3, Theorem 9) we have shown that, for a general normed space, a convex starlike set is the only maximal face containing any of its elements. This Tingley's paper is the birth of what is known as Tingley's problem, which asks when it is possible to extend a surjective isometry defined between the unit spheres of two Banach spaces to a linear surjective one defined in the whole space. The amount of literature generated around this question in the last thirty years (1; 2; 4; 7; 8; 9; 10; 11; 12; 13; 14; 15) has made this topic considered as a long-lasting opened problem (the two-dimensional case has been already proved barely last year (16; 17; 18; 19) and the proof is made up by partial solutions due to Banakh and Cabello-Sánchez), and it has conceived even new generalized questions, such as the Mazur-Ulam property or the Strong Mankiewicz property, which are Tingley's problem for every surjective isometry and for arbitrary convex subsets instead of the unit spheres, respectively.

The main result in the quoted Tingley's paper (5) shows that surjective isometries between the unit spheres of finite dimensional Banach spaces preserves antipodal points. A version of this result is established in the case of two compact C^*-algebras by Tanaka and Peralta in (14, Theorem 3.7), and as far as we know, there is no generalization of this Tingley's result for arbitrary dimensional spaces. An extension of it was provided in (3, Theorems 14 and 15), where this antipodal preservation is proved for rotund points and for maximal faces with inner points respectively.

As it turns, inner and extremal structure are notably related, but the latter is greatly more developed than the first one. By this inner structure we mean the existence of inner points, internal points or even interior points. The inner structure of a (not necessarily convex) set was formally introduced in (20). The interest in those inner points is that they are independent of the ambient space, for example, it is possible to provide an intrinsic characterization of linear manifolds through inner points, which was not possible by using internal points (20). Regarding the existence, in (20; 21; 22), non-trivial convex sets with no inner points are presented. One of those papers shows that it is not an easy task to find non-trivial convex sets lacking inner points, example found in (20, Theorem 5.4). These facts brings the question if it is possible to find a non-singleton compact and convex subset free of inner points in an arbitrary Hausdorff locally convex topological vector space. An approach to its answer is exposed in (23, Theorem 4.8), where we provide a full solution to this question in the normed space setting endowed with the weak topology. In order to prove this theorem, all the background about inner points will be used, such as the fact that inner and internal points coincide for convex sets. In fact, a new characterization of the set of inner points for convex sets is presented in (23, Theorem 3.3) by using the set of non-supporting vectors. Supporting vectors, as well as all the applications they had not only in Functional Analysis but also in Optimization Theory applied to Engineering, simply because many optimization problems in Physics and Engineering can be reformulated as the problem of finding the unit vectors at which a certain matrix attains its Euclidean norm (24; 25), are extremely related with the Geometry of the unit ball: exposed faces are convex sets of 1-supporting vectors (a special subset of supporting vectors considered when the linear operator is a functional) or the fact that a norm-1 operator is an isometry if and only if the unit sphere coincides with the set of supporting vectors of that isometry. Respect to this research, in (26) there are plenty of characterizations for supporting vectors, but what is more appealing in this paper is that L^{∞}-projections are characterized in terms of supporting vectors (26, Proposition 3.1) or the characterization of 1-projections in strictly convex Banach spaces through supporting vectors (26, Corollary 3.4). This is why we decided to entail an study of the geometry of the unit ball under projections.

2.2 Objectives

The main objectives pursued by this work are the following:

1. To revisit the geometric properties of 2-dimensional real Banach spaces, mostly those properties directly involved with the Mazur-Ulam property and Tingley's problem.

2. To study the relation between the Strong Mankiewicz Property and the strict convexity

in arbitrary Banach spaces with special attention for the 2-dimensional ones.

3. To study whether the facial structure of the unit ball is a geometric invariant under surjective isometries defined between unit spheres of Banach spaces.

4. To determine the behaviour of facets under surjective isometries.

5. To analyse the behaviour of surjective isometries after equivalent renormings.

6. To provide a partial or total solution to Tingley's problem.

Keeping in mind all the research and articles generated around Tingley's problem, Extremal Theory and Inner Structure, the achievement of any of these specifics objectives is a huge advance in this line of research and produces enough material for a dissertation.

2.3 Methodology

The methodology employed in taking this research work into a successful project is mainly described in the fourth, fifth and sixth chapters, where the reader can realize that the objectives proposed previously have been successfully achieved, and all the necessary background and results are gathered in Chapter 3.

There is a general methodology employed that we can describe here though. This methodology is simply the Purely Mathematical Method, which consists in axiomatic-deductive method of the first-order language Zermelo-Fraenkel with the Axiom of Choice (ZFC).

We have also employed a hybrid submethodology consisting in following the Bourbaki School for results more oriented to the inner and extremal structure, and the Category Theory for results more oriented to the relations, via morphisms, of different mathematical objects. All our results as well as all the background employed in this dissertation is framed within the Category of real Topological Vector Spaces. The subcategory of real normed spaces will be mostly used.

2.4 Feasibility and broader impact

The feasibility of this research work is primarily supported by the contingency plan consisting of dividing the objectives into sub-objectives in case the main objectives could not be accomplished. However, as the reader may check in the Conclusions chapter, the main objectives have been accomplished and then the feasibility is proved.

Respect to the broader impact, every purely mathematical work always finds some trouble to have a broader impact than the simply direct one on its field. However, this purely mathematical research work differentiates from others in the fact that some of the obtained results can be directly applied to other areas of Mathematics, such as Operator Theory, see (27; 28), and to solve optimization problems in Physics and Engineering (24; 25; 29; 30).

CHAPTER

3

Materials and Methods

Contents

As we have mentioned in the Methodology of the previous chapter, we start by gathering the necessary background concerning Banach Space Theory and well-known terms for Banach Space Geometers. This chapters is split into two branches: Extremal Theory and Operator

Theory. The first one is aimed to make a survey about the geometrical terms that we will work with in the following chapters, such as extremal subsets, faces, exposed faces, facets, rotund points, among others, and well-known results about them, for example, the equivalence between convex components and maximal faces of the unit ball (Proposition 3.2.7), the invariance of maximal faces under surjective isometries (Theorem 3.2.9), or the characterizations of those geometrical terms (Theorem 3.2.13, Equation 3.2.6, Theorem 3.2.32, etc.), whereas the second one is focused in operators defined between normed spaces, such as projections. We want the reader to notice that all the Geometrical Inner Structure is summarized in the Extremal Theory section, as it is expected after the motivation given in the state of the art, but we introduce this kind of structure with more detail in its corresponding subsection. In addition, we will briefly talk about supporting vectors and for what they are useful in the last section of this chapter.

3.1 Topological vector spaces

All vector spaces considered throughout this manuscript will be over the reals. If X is a normed space, then B_X, U_X, S_X will stand for its (closed) unit ball, its open unit ball, and its unit sphere, respectively. If $x \in X$ and $r > 0$, then $B_X(x,r), U_X(x,r), S_X(x,r)$ will denote, as expected, the (closed) ball of center x and radius $r > 0$, the open ball of center x and radius $r > 0$, and the sphere of center x and radius $r > 0$. For metric spaces, we will keep using the same notation for the closed balls, the open balls and the spheres.

If X is a vector space, and $A \subseteq X$, $\mathrm{int}(A), \mathrm{cl}(A), \mathrm{bd}(A)$ will denote the interior, the closure and the boundary of A respectively, and if $B \subseteq A$, the relative interior of B respect to A, the relative closure of B respect to A and the relative boundary of B respect to A will be denoted by $\mathrm{int}_A(B), \mathrm{cl}_A(B), \mathrm{bd}_A(B)$.

A topological vector space (TVS) just means that the vector addition and the scalar multiplication are continuous with respect to the given topology. Important examples of topological vector spaces are the Hausdorff locally convex topological vector spaces (there exists a local basis of zero-neighborhoods which are balanced, convex, and absorbing). Besides, if we consider the topology induced by the family of all seminorms on X we have what we call the finest locally convex vector topology, which is locally convex and Hausdorff.

Notice that every vector space can be endowed with the finest locally convex vector topology by taking the collection of all absorbing and absolutely convex as local basis of zero-neighborhoods. In other words, every vector space can be turned into a Hausdorff locally convex topological vector space (independently of the linear dimension). Even more, any vector space can actually be endowed with a norm the following way: fix a Hamel basis and take the norm of a vector as the summation of the absolute values of the coordinates of the vector with respect to such basis. Note that the previous constructed norm is not complete in general.

3.2 **Extremal Theory**

The concepts from this section are well known among the Banach Space Geometers and belong to the folklore of classic literature of Banach Space Theory. For further reading on these topics, we refer the reader to the classical texts (31; 32; 33).

3.2.1 **Extremal structure**

Definition 3.2.1 (Extremal subset)**.** *Let X be a normed space, and $E \subseteq F \subseteq X$. We say that E satisfies the extremal condition with respect to F provided the following condition holds:*

$$\forall x, y \in F, \forall t \in (0,1) \text{ such that } tx + (1-t)y \in E \implies x, y \in E. \tag{3.2.1}$$

In this case, we call E extremal in F. In the particular case that $E = \{e\}$ is a singleton, e is called an extremal point in F. The set of all the extremal points of F is denoted by $\operatorname{ext}(F)$.

The non-empty intersection of extremal subsets is extremal as well. Besides, if E is extremal in F and D is extremal in E, then D is extremal in F, and $\operatorname{ext}(D) \subseteq \operatorname{ext}(E) \subseteq \operatorname{ext}(F)$.

An important example of extremal subsets is the supporting hyperplane.

Definition 3.2.2 (Supporting hyperplane)**.** *Let A be a non-empty subset of a vector space X and $f \in X^*$. The supporting hyperplane relative to f in A is defined as*

$$F(f, A) := \{x \in X : f(x) = \max f(A)\}.$$

A supporting hyperplane is an example of an extremal subset, provided that $F(f, A) \neq \varnothing$. If X is also a normed space and A is the closed unit ball B_X, we will simply write $F(f)$.

The extremal condition allows to define other geometrical concepts, as face, exposed faces, etc. The notions of face and facet defined in (5) are what we call in this work exposed face and maximal face, respectively.

Definition 3.2.3 (Face, extreme point)**.** *Let X be a normed space, and $A \subseteq B_X$. We say that A is a face of B_X if it is convex and extremal. The extremal points of a convex subset $A \subseteq B_X$ are* *called extreme points.*

It is easy to check that every extremal subset E of B_X satisfies that either $E = B_X$ or $E \subseteq S_X$ (it is sufficient to consider $e \in E$ and $x \in B_X \setminus S_X$, the segment $[tx + (1-t)e, x]$ with $t < 0$ and close enough to 0 and apply the extremal condition of E). As a consequence, proper faces of the unit ball are always contained in the unit sphere. Also, a point x is an extreme point of a convex set A if and only if $A \backslash \{x\}$ is also convex.

Definition 3.2.4 (Exposed face, exposed point, Edge)**.** *Let X be a normed space. An exposed face of B_X is a set of the form $F(f) = \{x \in B_X : f(x) = \max f(B_X)\}$, for some $f \in S_{X^*}$. An equivalent form of an exposed face that will be used along this work is $F(f) = f^{-1}(\{1\}) \cap B_X$, for some $f \in S_{X^*}$. In case that an exposed face $F(f) = \{x\}$ is a singleton, $\{x\}$ is called an exposed point of B_X, and we will denote the set of all exposed points of B_X as $\exp(B_X)$. Besides, we define an edge $E(f)$ of the unit ball with respect to $f \in S_{X^*}$ as $E(f) = \mathrm{bd}_{f^{-1}(\{1\})}(F(f))$.*

Note that exposed faces are examples of proper faces, and the inclusion $\exp(B_X) \subseteq \mathrm{ext}(B_X)$ is straightforward from their definitions (relaying on the fact that 1 is an extreme point to prove it). We also want to remark to the reader that the definition of edge give above is the same as the one seen in (4, Theorem 1.1) and (1, Section 2), and it differs from the one provided in (34, Definition 1.2.).

Remark 3.2.5. *If $(C_n)_{n\in\mathbb{N}}$ is a family of exposed faces in B_X with $\cap_{n\in\mathbb{N}} C_n \neq \varnothing$, then the intersection $\cap_{n\in\mathbb{N}} C_n$ is also an exposed face of B_X. Indeed, for each C_n, take $f_n \in S_{X^*}$ such that $C_n = F(f_n)$, for all $n \in \mathbb{N}$. Then $\cap_{n\in N} C_n = F(f)$, with $f := \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$.*

Definition 3.2.6 (Maximal face, rotund point)**.** *Let X be a normed space. A proper face C is called a maximal face if it is a maximal element respect to the set of proper faces of B_X endowed with the inclusion. In case that $C = \{x\}$ is a singleton, we called $\{x\}$ a rotund point, and the* *set of all rotund points of B_X is denoted by* $\mathrm{rot}(B_X)$.

Notice that every maximal face of the unit ball is an exposed face: let C be a maximal face and consider the open unit ball, U_X. Since C is a proper (maximal) face, $C \subseteq S_X$, therefore C and U_X are convex subsets, disjoint and U_X is open, then, relaying on Hahn-Banach Separation Theorem (see Theorem 4i), there exists a functional $f \in X^*$ such that $\inf f(C) \geq \sup f(U_X)$. On the other side, $c \subseteq B_X$, then $\sup f(B_X) \geq \inf f(C)$, and $\sup f(B_X) = \|f\| = \sup f(U_X)$, then $f(C) = \|f\|$, which is the definition of an exposed face. At that point, if C is a maximal face of B_X, then C is a convex component of S_X, and a maximal exposed face of B_X. Since being a maximal exposed face of B_X trivially implies that it is a maximal face of B_X, we have proved the next equivalences:

Proposition 3.2.7. *For a subset $C \subseteq S_X$, where X is a normed space, the following are equivalent:*

$i)$ *C is a convex component of B_X,*

$ii)$ *C is a maximal face of B_X,*

$iii)$ *C is a maximal exposed face of B_X.*

Therefore, we will indistinctly talk about maximal convex subsets of the unit sphere and maximal faces of the unit ball. In addition, notice that

$$\mathrm{rot}(B_X) \subseteq \exp(B_X) \subseteq \mathrm{ext}(B_X). \tag{3.2.2}$$

The following examples show that the contentions in Equation (3.2.2) are strict for some kind of Banach spaces.

Example 3.2.8. *In $\ell_\infty^2 = (\mathbb{R}^2, \|\cdot\|_\infty)$, each corner of the unit ball is an exposed point which is not rotund. If we renormed the space so that we smoothen the corners of $B_{\ell_\infty^2}$, then each extreme of any edge is an extreme point which is not an exposed point.*

One of the invariants studied along this memory are maximal convex sets of the unit sphere, which trivially implies the invariance of maximal faces and maximal exposed faces of the unit ball. This invariability is proved on detail in (1, Lemma 3.5) and (2, Lemma 6.3).

Theorem 3.2.9. *Let X and Y be Banach spaces, and $T : S_X \to S_Y$ a surjective isometry. Then, C is a maximal convex subset of S_X if and only if $T(C)$ is a maximal face of S_Y.*

The term of proper exposed point was coined by Tanaka in (2, Definition 3.2), and it is given below. This definition will inspire an original one of this work, and it will be introduced in the following chapter as *pre-maximal face* (see Definition 4.3.1).

Definition 3.2.10 (Proper exposed point). *Let X be a normed space and $A \subseteq B_X$, a proper face of the unit ball which is the intersection of all maximal faces containing A, that is,*

$$A = \bigcap \{C \subseteq B_X : \ C \text{ is a maximal face such that } A \subseteq C\}. \tag{3.2.3}$$

If $A = \{x\}$ is a singleton, x is called a proper exposed point, and the set of all proper exposed points of B_X is denoted by $\operatorname{pexp}(B_X)$.

The following chain of inclusions is trivially verified

$$\operatorname{rot}(B_X) \subseteq \operatorname{pexp}(B_X) \subseteq \operatorname{ext}(B_X),$$

and, by relaying on Remark 3.2.5 or on (2, Proposition 3.4), one can see that $\operatorname{pexp}(B_X) \subseteq$

$\mathrm{ext}(B_X)$, and the above equation is generalized as

$$\mathrm{rot}(B_X) \subseteq \mathrm{pexp}(B_X) \subseteq \exp(B_X) \subseteq \mathrm{ext}(B_X). \tag{3.2.4}$$

As we mentioned before, in (5), the term of facet used by Tingley is what we call maximal faces of the unit ball. Next, we give the notion of facet that is the same as the one given in (34, Definiton 1.2)

Definition 3.2.11 (Facet)**.** *If X is a normed space, a proper face A of B_X is called a facet if $\mathrm{int}_{S_X}(A) \neq \emptyset$, and we will the denote the set of all facets of the unit ball as*

$$\mathscr{C}_X = \{C \subset S_X : \ C \text{ is a facet}\}.$$

Remark 3.2.12. *It is clear that surjective isometries $T : S_X \to S_Y$ which we work with along this work are homeomorphisms. As a consequence, if X, Y are Banach spaces, $T : S_X \to S_Y$ is an onto isometry, and $C \subset S_X$ is a facet, then so it is its image. Indeed, since C is a facet, $\mathrm{int}_{S_X}(C) \neq \emptyset$, and, since T is an homeomorphism, we claim that $\mathrm{int}_{S_Y}(T(C)) \neq \emptyset$, which is the condition for $T(C)$ to be a facet.*

We recover some geometric properties of facets (we refer the reader to (34, Theorem 2.8) for more detailed proofs).

Theorem 3.2.13. *Let X be a Banach space and C a convex subset of S_X. The following conditions are equivalent:*

(1) *C is a facet of B_X.*

(2) *C is a face of B_X satisfying $\mathrm{int}_{S_X}(C) \neq \emptyset$.*

(3) *C is a convex component satisfying* $\mathrm{int}_{S_X}(C) \neq \varnothing$.

The third statement of theorem above makes clear that facets of the unit ball are maximal faces. Using Theorem 3.2.9 and Remark 3.2.12, its image will be also a facet. We put the invariance of facets under onto isometries of the unit sphere in the following result.

Corollary 3.2.14. *Let X, Y be Banach spaces and $T : S_X \to S_Y$ a surjective isometry. If $C \subseteq S_X$ is a facet of B_X, then $T(C)$ is a facet of S_Y, meaning that $T(\mathscr{C}_X) = \mathscr{C}_Y$.*

3.2.2 **Starlike structure**

Starlike sets were originally introduced (for finite dimensional vector spaces) in (5), and they were characterized by Tingley in the same paper, see (5, Lemma 4 and Corollary 5). In this subsection, we recall some known terms and properties about starlike sets.

Definition 3.2.15 (Starlike set)**.** *Let X be a normed space and $x \in S_X$. The starlike set of x is defined as*

$$\mathrm{st}(x, B_X) = \{y \in B_X : \ \|x + y\| = 2\}. \tag{3.2.5}$$

We have to do some observations about starlike sets. The first one is that $\mathrm{st}(x, B_X) \subseteq S_X$ for an arbitrary $x \in S_X$, and $\mathrm{st}(-x, B_x) = -\mathrm{st}(x, B_X)$. The other one is starlike sets are easier to work with when we use metric space notation, more precisely, the starlike set of x is the sphere of center $-x$ and radius 2 in the metric space given by the unit sphere, that is, $S_{S_X}(-x, 2)$. Finally, it is easy to check using Proposition 3.2.7 that the following characterization of starlike sets holds:

$$\mathrm{st}(x, BX) = \{y \in S_X : [y, x] \subseteq S_X\} = \bigcup \{C \subseteq S_X : \ C \text{ is a maximal face containing } x\}. \tag{3.2.6}$$

Remark 3.2.16. *It is clear that if $T : X \to Y$ is an isometry between metric spaces, for all $x \in X$ and $r > 0$, the following contentions hold: $T(B_X(x,r)) \subseteq B_Y(T(x),r)$, $T(U_X(x,r)) \subseteq U_Y(T(x),r)$, and $T(S_X(x,r)) \subseteq X_Y(T(x),r)$. Furthermore, if T is an onto isometry, the equalities hold in every previous contentions, that is, $T(B_X(x,r)) = B_Y(T(x),r)$, $T(U_X(x,r)) = U_Y(T(x),r)$, and $T(S_X(x,r)) = S_Y(T(x),r)$.*

In view of previous remark, we have the following infinite dimensional version of (5, Lemma 10 and Corollary 11).

Remark 3.2.17. *Let $T : S_X \to S_Y$ be a surjective isometry between metric spaces X, Y. Then, for every $x \in S_X$*

$$T(\mathrm{st}(x, B_X)) = T(S_{S_X}(-x, 2)) = S_{S_Y}(T(-x), 2) = S_{S_Y}(-(-T(-x)), 2) = \mathrm{st}(-T(-x), B_Y).$$

3.2.3 **Smoothness**

In this section, we give a few strokes about smoothness and we recommend (32; 33; 35) for a deeper understanding. Firstly, we recall the definition of smoothness.

Definition 3.2.18 (Smooth point, smooth space)**.** *Let X be a normed space. We say that $x \in S_X$ is a smooth point of B_X if there exists a unique functional $f \in S_{X^*}$ attaining its norm at x, that means, $f(x) = 1$. We will denote the set of all smooth points of B_X as* $\mathrm{smo}(B_X)$. *In case* *that* $\operatorname{smo}(B_X) = S_X$, *we will call* X *a smooth space.*

A smooth point $x \in S_X$ is, geometrically speaking, an element of the unit ball which has a unique hyperplane at x. This fact allows us to notice that in smooth spaces, the exposed faces are pairwise disjoint.

We recall some terms related to smoothness that will be useful along this memory.

Definition 3.2.19 (Duality mapping, spherical image map). *Let* X *be a normed space. The duality mapping of* X *is defined as follows*

$$\begin{aligned} J : X &\to \mathscr{P}(X^*) \\ x &\mapsto J(x) := \{x^* \in X^* : \ x^*(x) = \|x^*\|\|x\|\}. \end{aligned}$$

This mapping induces the definition of spherical image map:

$$\begin{aligned} \nu : S_X &\to \mathscr{P}(S_{X^*}) \\ x &\mapsto \nu(x) := \{x^* \in S_{X^*} : \ x^*(x) = 1\}. \end{aligned}$$

For each $x \in S_X$, the reader have to notice that $\nu(x) = J(x) \cap S_{X^*}$, in other words, $\nu(x)$ is the subset of the unit sphere S_{X^*} whose members are the supporting functionals of the unit ball B_X at x. More than that, $\nu(x) = F(x)$ when x is seen as an element of X^{**}.

Remark 3.2.20. *If a point is smooth,* $x \in \operatorname{smo}(B_X)$*, then* $\nu(x)$ *is a singleton, and we will identify* $\nu(x)$ *with its only element. Under this statements,* $F(\nu(x))$ *is the only exposed face of* B_X *containing* x*, thus* $F(\nu(x))$ *is the only maximal face of* B_X *containing* x.

By using the spherical image map, we provide a novel reformulation of the frame of the unit ball of a Banach space X. The known formulation for the frame could be found in (4, Equation before Theorem 1.1) and (1, Section 2).

Definition 3.2.21 (Frame). *Let X be a normed space. The frame of the unit ball B_X is defined as follows*

$$\operatorname{frm}(B_X) := \bigcup \left\{ E(f) : f \in \cup_{x \in S_X} \nu(x) \right\}.$$

With this definition, it is easy to see that the frame of the unit ball is formed, geometrically speaking, by all the edges related to a supporting functional. The invariance of the frame of Banach spaces was proved in (1, Theorem 3.7):

Theorem 3.2.22. *Let $T : S_X \to S_Y$ be a surjective isometry between the unit spheres of Banach spaces X, Y. Then $T(\operatorname{frm}(B_X)) = \operatorname{frm}(B_Y)$.*

3.2.4 **Inner structure**

Even if inner structure appears implicitly for convex sets in (36; 37), it was not introduced until (20, Definition 1.2) for non-convex sets. For a wider perspective on inner structure, we refer the reader to (20; 21; 38). Concerning this manuscript, we will only make use of inner structure for convex sets. Below, in this subsection we will make a brief review about inner points and some results about them.

In vector spaces, we will write a closed segment as

$$[x, y] := \{tx + (1-t)y : t \in [0,1]\},$$

an open segment as

$$(x, y) := \{tx + (1-t)y : t \in (0,1)\},$$

and a half-open segment (or half-closed) segment as

$$[x, y) := \{tx + (1-t)y : t \in (0, 1]\}.$$

The following definition can be also given for non-necessary convex sets, as it could be seen in (20, Definition 1.2). However, as far as what is concerned, we will focus in the study of inner points for convex sets.

Definition 3.2.23 (Inner points)**.** *Let X be a vector space and $M \subseteq X$ a convex subset with at least two points. We define the set of inner points of M as follows:*

$$\operatorname{inn}(M) := \{x \in X : \forall m \in M \setminus \{x\}, \exists n \in M \setminus \{m, x\} \textit{ such that } x \in (m, n)\}.$$

The particularity of those inner points is that they are independent of the ambient space.

The set of inner points of a convex subset in a finite dimensional vector space is what Tingley called "relative interior" of convex sets in $\mathbb{R}^n$ in (5). In this article, Tingley pointed out that "the relative interior of convex sets may be empty" for infinite dimensional Banach spaces. In fact, in (20, Theorem 5.1) it is proved that every non-singleton convex subset of a finite dimensional vector space has inner points:

Theorem 3.2.24. *If X is a finite dimensional vector space and M a convex subset which is not a singleton, then* $\operatorname{inn}(M) \neq \varnothing$.

For infinite dimensional vector spaces (20, Corollary 5.3), it is always possible to find a non-singleton convex subset free of inner points. For sake of showing with detail the convex set free of inner points, we have to make a digression to introduce some concepts.

A sequence $(x_n)_{n\in\mathbb{N}}$ of a Banach space X is a *basic sequence* if it is a Schauder basis of $\mathrm{cl}(\mathrm{span}(x_n : n \in \mathbb{N}))$, where the latter is the closure of the vector subspace generated by $(x_n)_{n\in\mathbb{N}}$. There exists separable Banach spaces with no basis, as it is shown in (39), whereas the existence of basic sequences will be always guaranteed in our framework, as it is shown in the following result, which is originally proved in the literature for complete normed spaces, and here we provided an original proof for general normed spaces:

Theorem 3.2.25. *Every infinite dimensional normed space has a basic sequence.*

Proof. Let Y be an infinite dimensional normed space and let X denote the completion of Y. Since X is Banach, by the Basic Sequence Existence Theorem (40, Corollary 1.5.3), there exists a basic sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$. Denote by $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ the sequence of coordinate functionals associated to $(x_n)_{n\in\mathbb{N}}$. Fix an arbitrary $0 < \rho < 1$. Since Y is dense in X, for every $n \in \mathbb{N}$, there exists $y_n \in Y$ with $\|x_n - y_n\| < \frac{\rho}{2^n \|x_n^*\|}$. Then

$$\sum_{n=1}^{\infty} \|x_n - y_n\| \|x_n^*\| \leq \sum_{n=1}^{\infty} \frac{\rho}{2^n} = \rho < 1.$$

By the Basic Sequence Perturbation Theorem (41, Theorem 12.2.6), $(y_n)_{n\in\mathbb{N}}$ is a basic sequence in X equivalent to $(x_n)_{n\in\mathbb{N}}$. Finally, $(y_n)_{n\in\mathbb{N}}$ is a basic sequence in Y because $\overline{\mathrm{span}}^Y\{y_n : n \in \mathbb{N}\} = Y \cap \overline{\mathrm{span}}^X\{y_n : n \in \mathbb{N}\}$. □

Finally, we can properly shows the example of a convex subset with no inner points, borrowed from (20, Theorem 5.2):

Example 3.2.26. *if X is an infinite dimensional normed space, the convex subset of the unit ball free of inner points $C \subseteq B_X$ is given by*

$$C = \left\{ \sum_{n=1}^{\infty} t_n e_n : \ t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1, t_n \in c_{00} \right\},$$

where $(e_n)_{n\in\mathbb{N}} \subseteq S_X$ *is a basic sequence and* c_{00} *is the vector space of eventually zero sequences.*

Remark 3.2.27. *We convey that* $\operatorname{inn}(M) = \varnothing$ *when* M *is a singleton in Definition 3.2.23. Also, it is trivial that* $\operatorname{inn}(M) \subseteq M$*. In (21, Remark 1.1) is claimed that if* $x \in \operatorname{inn}(M)$ *then* $[x,m) \subseteq \operatorname{inn}(M)$ *for all* $m \in M$*, concluding another direct property for the set of inner points, which is that* $\operatorname{inn}(M)$ *is also convex, and* $\operatorname{cl}(\operatorname{inn}(M)) = \operatorname{cl}(M)$*. In the same article, in particular, in (21, Lemma 2.1), it is proved that if* F *is an extremal subset of* M*, then* $F \cap \operatorname{inn}(M) = \varnothing$*. More precisely, what the lemma sais is that for a convex subset* $M \subseteq X$ *of a vector space,* $C \subseteq M$ *a convex subset, and* D *a face of* M *satisfying that* $\operatorname{inn}(C) \cap D \neq \varnothing$*, then* $C \subseteq D$*. Besides, in this reference it is also proved that for a topological vector space* X *and* E *a extremal subset of a convex set* $C \subseteq X$ *satisfying* $E \cap \operatorname{int}(C) \neq \varnothing$*, then* $E = C$*.*

The following remark we will be made use of later on and it will turn out tu be crucial.

Remark 3.2.28. *Let* X *be a vector space and* $x, y, z \in X$ *three different points not aligned. Then*

$$\begin{aligned} \operatorname{inn}(\operatorname{co}(\{x,y,x\})) &= \operatorname{co}(\{x,y,z\}) \setminus ([x,y] \cup [y,z] \cup [x,z]) \\ &= \{rx + sy + tz : r,s,t \in (0,1), r+s+t = 1\}. \end{aligned}$$

Let us show a few results about inner points that, besides giving us a wider perspective about inner structure, will be used along the proofs of the following chapter.

We recall that an element $x \in X$ of a vector space X is called an *internal point* of a non-empty subset $M \subseteq X$ if every line of X passing through x contains a segment entirely contained in M, with x laying on its interior (we suggest (42, TVS II.26) for a wider perspective about internal points). The set of all the internal points of M will be denote by $\mathrm{inter}(M)$. We refer the reader to (43; 44) for an example of an internal point which is not in the topological interior. Next result is borrowed from Lemma 5 (22) and it shows the relations between internal, interior and inner points.

Lemma 3.2.29. *Let X be a topological vector space and M a non-empty subset of X. Then:*

1. $\mathrm{int}(M) \subseteq \mathrm{inter}(M) \subseteq \mathrm{inn}(M) \subseteq M$.

2. *If M is open, then* $\mathrm{int}(M) = \mathrm{inter}(M) = \mathrm{inn}(M) = M$.

3. *If M is convex and* $\mathrm{inter}(M) \neq \varnothing$ *then* $\mathrm{inter}(M) = \mathrm{inn}(M)$.

4. *If M is convex and* $\mathrm{int}(M) \neq \varnothing$*, then* $\mathrm{int}(M) = \mathrm{inn}(M)$.

5. *If M is convex,* $\mathrm{int}(M) \neq \varnothing$ *and* $M = \mathrm{inter}(M)$*, then M is open.*

The following theorem, whose proof can be found in Theorem 6 (22), will be necessary to prove Theorem 4.2.2 in next chapter.

Theorem 3.2.30. *Let $M \subseteq X$ be a non-empty convex and absorbing subset of a vector space X. Then M is a neighborhood of 0 in the finest locally convex vector topology of X. More generally, if M is convex satisfying that* $\mathrm{inter}(M) \neq \varnothing$*, then* $\mathrm{inter}(M)$ *is the interior of M in the finest locally convex vector topology of X.*

We also want the reader to notice that isomorphisms preserves inner points, as it is pointed out in (20, Proposition 1.3).

Proposition 3.2.31. *Isomorphisms of vector spaces map inner points to inner points.*

Next result gives a relation between the set of internal and inner points for an arbitrary set.

For the details about its proof, we refer the reader to (20, Theorem 4.1).

Theorem 3.2.32. *Let M be a non-empty subset of a vector space X. Then,* $\operatorname{inter}(M) \subseteq \operatorname{inn}(M)$*. Moreover, if M is also convex and* $\operatorname{inter}(M) \neq \varnothing$*, then* $\operatorname{inter}(M) = \operatorname{inn}(M)$*.*

Theorem 3.2.32 is very accurate as we can see in the following examples original from (20): consider $M := ((-1,1) \times (-1,1)) \cup ((2,3) \times \{0\})$ in $\mathbb{R}^2$ (see Figure 3.1). Note that $M = \operatorname{inn}(M) \supsetneq \operatorname{inter}(M) = ((-1,1)\times(-1,1))$. On the other side, consider now $M := (-1,1)\times\{0\}$ in $\mathbb{R}^2$ (see Figure 3.2), then $M = \operatorname{inn}(M)$ but $\operatorname{inter}(M) \neq \varnothing$.

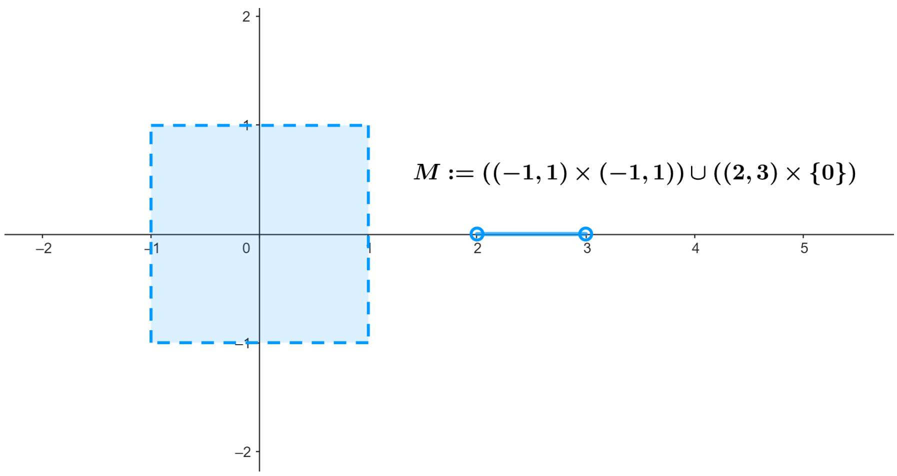

Figure 3.1: Example of inner points which are not internal

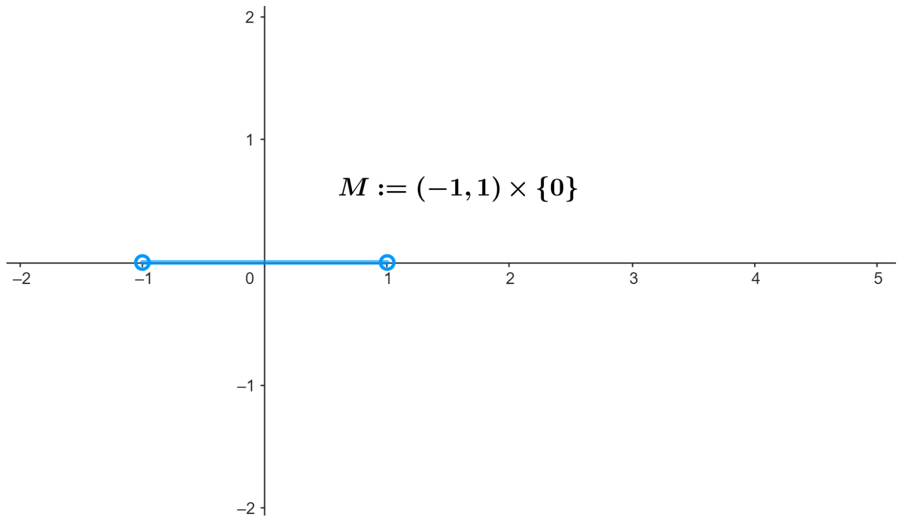

Figure 3.2: Example of the need to the condition $\mathrm{inter}(M) \neq \varnothing$

For next two theorems, originally from (20, Theorem 2.3) and (20, Theorem 4.2), it is necessary the definition of strongly adjacent point: for a non-empty subset M of a vector space X we say that an element $x \in X$ is *adjacent* to M if there exists $m \in \mathrm{inn}(M) \setminus \{x\}$ such that $(x, m] \subset \mathrm{inn}(M)$. Under the same hypotheses, we say that $x \in X$ is *strongly adjacent* to M if the same condition holds for all $m \in \mathrm{inn}(M) \setminus \{x\}$. We denote them by $\mathrm{adj}(M)$ and $\mathrm{sadj}(M)$ respectively.

Theorem 3.2.33. *Let $\varnothing \neq M \subseteq X$ be a non-empty subset of a vector space X such that* $\mathrm{inn}(M) \neq \varnothing$.

1. *If $M \subseteq \mathrm{sadj}(M)$, then $\mathrm{inn}(\mathrm{inn}(M)) = \mathrm{inn}(M)$.*

2. *If M is convex, then $\mathrm{inn}(M)$ is convex and $M \subseteq \mathrm{sadj}(M) = \mathrm{adj}(M)$.*

3. *If $\mathrm{inn}(M)$ is convex and $M = \mathrm{adj}(M)$, then M is convex.*

Theorem 3.2.34. *Let M be a non-empty subset of a vector space X. The following conditions hold:*

- *If M is absorbing, then $0 \in \operatorname{inn}(M)$ and $\operatorname{span}(M) = X$.*

- *If $0 \in \operatorname{inn}(M)$, $\operatorname{span}(M) = X$ and $M \subseteq \operatorname{sadj}(M)$, then $\operatorname{inn}(M)$ is absorbing.*

We recall the notion of non-support point:

Definition 3.2.35 (Non-support point)**.** *Let A be a non-empty subset of a vector space X, and let $h(A) := \{f \in X^* : f$ attains its sup at A and it is not constant on $A\}$. The set of non-support points of A is defined as $\operatorname{nsupp}(A) := A \setminus \bigcup_{f \in h(A)} F(f, A)$.*

We have to make the reader note that $h(A)$ could be empty. For example, if A is the open unit ball of a normed space, then $h(A) = \varnothing$.

3.2.5 **Minkowski functional**

We give a briefly survey about some parts of the folklore of the literature of Geometry in Topological Vector Spaces. For a deeper perspective, we refer the reader to (42; 45; 46).

Definition 3.2.36 (Balanced, absorbing, absolutely convex, linearly bounded)**.** *For a subset $A \subseteq X$ on a vector spaces, A is said to be:*

- *Balanced if $[-1, 1]A \subseteq A$.*

- *Absorbing if for all $x \in X$, there exists $\delta > 0$ such that $[-\delta, \delta]x \subseteq A$.*

- *Absolutely convex if A is balanced and convex.*

- *Linearly bounded if A does not contain straight lines or rays.*

We recall the definition of the Minkowski functional, also called gauge function. For a wider perspective on this functional see (41, Chapter 10.4)

Definition 3.2.37 (Minkowski functional). *Let A be a non-empty subset of a vector space X. Suppose that A is absorbent, balanced and convex. The Minkowski functional of A is the mapping $\mu_A : X \to \mathbb{R}$ given by*

$$\mu_A(x) = \inf\{\lambda > 0 : \ x \in \lambda A\} = \inf\left\{\lambda > 0 : \ \frac{x}{\lambda} \in A\right\}, \ x \in X.$$

The Minkowski functional assures that every absorbing absolutely convex subset $A \subseteq X$ defines a seminorm on X:

$$\|x\|_A := \inf\{\lambda > 0 : x \in \lambda A\}, \ x \in X.$$

If $U_{X,A}, B_{X,A}$ are the open and closed unit balls of $(X, \|\cdot\|_A)$, it is easy to check that $U_{X,A} = \mathrm{int}(A)$ and $B_{X,A} = \mathrm{cl}(A)$. Another fact about this seminorm is that $\|\cdot\|_A$ is a norm on X if and only if A is linearly bounded.

Let X be a Banach space and A a bounded, closed, absolutely convex subset of X with non-empty interior. The Minkowski functional of A defines an equivalent norm in X since every absolutely convex subset with non-empty interior in a topological vector space is a neighborhood of 0.

In view of the Krein-Milman Theorem (see C.3.1), for a Hausdorff locally convex topological vector space X and a non-empty compact subset $K \subseteq X$, then $\mathrm{ext}(K) \neq \emptyset$. Moreover, if K is also closed and convex, then $K = \overline{\mathrm{co}}(\mathrm{ext}(K))$. As a consequence, we notice the following:

Remark 3.2.38. *If X is a reflexive Banach space, and $C \subseteq X$ is a closed, convex and bounded subset, then* $\mathrm{ext}(C) \neq \emptyset$*, since C is weakly closed and bounded, thus it is weakly compact.*

3.3 Operator Theory

This section is devoted to summarize the necessary background about projections and supporting vectors that will be used in Chapter 6.

It is well known that if X is a vector space and M, N are subspaces of X such that $X = M + N$ and $M \cap N = \{0\}$, then X is said to be the direct sum of M and N, and we use the notation $X = M \oplus N$. In this case, it is possible to write every vector $x \in X$ as $x = m + n$ with $m \in M, n \in N$, and this form is unique. We say that M and N are complementary subspaces and both are the algebraic complement to each other. Since every Hamel basis of a particular subspace can be extended to a Hamel basis of the whole space, the existence of the algebraic complement is always guaranteed for any subspace M, but, it has not to be unique, for example, in $\mathbb{R}^3$, the algebraic complement of every plane is any line not contained in the plane.

Let us make a brief introduction about projections on vector spaces and Banach spaces in order to use it in Chapter 6. We recommend (47, Chapter 4) for a wider perspective about projections.

First of all, we recall the definition. Some texts establish the definition for vector spaces and non-continuous projections, for instance, Proposition 3.3.3 remains true if the space considered is a vector space and P is just linear and idempotent. For what it may concern to us, X will be, at least, a normed space and our projections will be continuous.

Definition 3.3.1 (Projection). *For a normed space X, a linear operator $P : X \to X$ is called a projection if it is continuous and idempotent, that is, $P(P(x)) = P(x)$ for all $x \in X$, or equivalently, $P^2 = P$.*

Examples of famous projections are the orthogonal projections in Hilbert spaces or the L^p-projections, with $1 \leq p \leq \infty$. We recall their definitions in sake of completeness.

Definition 3.3.2 (L^p-projection)**.** *Let $P : X \to X$ be a projection in a normed space X. Then, P is called an L^p-projection if it satisfies*

$$\|x\|_X = \|(\|P(x)\|_X, \|(I-P)(x)\|_X)\|_p\,, \quad \forall x \in X, \;\; 1 \leq p \leq \infty.$$

Particular studied examples of L^p-projections are M-projections and L^2-projection (also called orthogonal projections), whose conditions to verify are $\|x\| = \max\{\|P(x)\|, \|(I-P)(x)\|\}$, $\|x\|^2 = \|P(x)\|^2 + \|(I-P)(x)\|^2$, for all $x \in X$, respectively.

The proof of the following result can be found in (47, Proposition 4.8) and it shows how to split a vector space into a direct sum by using projections.

Proposition 3.3.3. *Let X be a normed space and $P : X \to X$ a linear mapping. Let M, N be closed subspaces of X.*

i) *P is a projection if and only if $I - P$ is a projection. In such case, it holds that*

$$P(X) = \ker(I-P), \;\; \ker(P) = (I-P)(X), \quad \textit{and}\; X = \ker(P) \oplus P(X).$$

ii) *If $X = M \oplus N$, then, there exists a projection $P : X \to X$ such that $P(X) = M$ and* $\ker(P) = N$. *In this case, P is called the projection of X onto M with kernel N.*

In the above proposition, the projection $I - P$ is named as the complementary projection of P. Another important projection related to P is its dual operator $P^* : X^* \to X^*$, which is also a projection, where $P^* : X^* \to X^*$ is defined as $f \longmapsto P^*(f) := (f \circ P)(x) = f(P(x))$ with

$x \in X$ and for each $f \in X^*$.

We also refer the reader to (33, Chapter 3) to delve into the study of projections. For example, next results are borrowed from this reference, where the proofs are detailed.

Theorem 3.3.4. *Let X be a normed space and $M \subseteq X$ a subspace. Then, M is topologically complemented in X if either of the following statements holds:*

1. *M is closed and finite-codimensional.*

2. *M is finite-dimensional.*

Remark 3.3.5. *If $P : X \to X$ is a projection in a normed space X, then $P(X) = \{x \in X : P(x) = x\}$.*

Let us see an interesting property about projections, but first, we need to recall a few notions about supporting vectors. As we have mentioned in the state of the art, supporting vectors are connected with the Extremal Theory when the operators considered are functionals, besides the fact that they are very useful not just in optimization problems, but also if we work with projections, as it can be seen in (26). Next, let us recall the definition of supporting vectors for a continuous linear operator. The geometrical nature and topological structure of the set of supporting vectors is discussed in (26; 48) and we recommend these references for a wider perspective on this topic.

Definition 3.3.6 (Supporting vector)**.** *Let $T : X \to Y$ be a linear operator between normed spaces. The set of supporting vectors of T is defined as*

$$\operatorname{suppv}(T) := \{x \in S_X : \|T(x)\| = \|T\|\}.$$

The reader has to pay special attention to the closed convex set $\mathrm{suppv}_1(x^*) := \{x \in S_X : x^*(x) = \|x^*\|\}$, with $x^* \in X^* \setminus \{0\}$, which is also known by the Banach space geometers as exposed faces of B_X (see Definition 3.2.4 with $x^* \in S_{X^*}$).

Supporting vectors have plenty of applications, not just in functional analysis but also in engineering problems (24; 25). Besides, supporting vectors serve to characterize certain types of operator, such as isometries: a norm-one operator T is an isometry if and only if $S_X = \mathrm{suppv}(T)$. In our case, they will be a useful tool in Chapter 6 to study the extremal structure under 1-complementation. And now, let us make the following note about the norm of a projection.

Remark 3.3.7. *Notice that the idempotence of P gives an inequality about the norm of the operator, that is, every non-zero projection has norm greater than or equal to* 1. *Indeed, if $P : X \to X$ is a projection, in particular, it is linear an idempotent, then*

$$\|P\| = \|P \circ P\| \leq \|P\|\|P\|.$$

Looking at this equality, if $P \neq 0$, then $\|P\| \neq 0$, hence

$$1 = \cancel{\|P\|} \leq \cancel{\|P\|}\|P\| \iff 1 \leq \|P\|.$$

Related to this comment, we have the following definition.

Definition 3.3.8 (1-Projection)**.** *A projection with norm equal to* 1 *it is called a* 1*-projection, norm-one projection or contractive projection, and if its complementary projection is also a* 1*-projection, then it is called a* $(1,1)$*-projection or bicontractive.*

Well-known examples of norm-one projections are the orthogonal projections in the setting of Hilbert spaces (49, Theorem 3.1). We recommend (49) for a deep perspective about the study of 1-projections.

Example 3.3.9. *Notice that for* M*-projections the condition* $\|x\| = \max\{\|P(x)\|, \|(I-P)(x)\}$ *and* $\|P\| \geq 1$*,* $\|(I-P)\| \geq 1$*, implies that* M*-projections have norm one. Even more, in (26, Proposition 3.1),* M*-projections are characterized as those* $(1,1)$*-projections for which* $S_X = \text{suppv}(P) \cup \text{suppv}(I-P)$.

We end this summary about projections by talking about complemented subspaces. Again, we refer the reader to (47, Chapter 4) for more details about this topic, such that the motivation of the following definition.

Definition 3.3.10 (Complemented subspace)**.** *Let* X *be a normed space and* M*, a closed subspace of* X*. We said that* M *is complemented in* X *if there exists a projection (that is, linear, continuous and idempotent)* $P : X \to X$ *such that* $P(X) = M$.

In definition above, if we call $N := \ker(P) = (I-P)(X)$, it holds that $X = M \oplus N$, and it is said that X is the topological direct sum of M and N. Besides, M and N are known as the topological complement of each other.

Next result is borrowed from (47, Corollary 4.7), where the proof can be found on detail. As a consequence of the following result, Proposition 3.3.12 holds.

Proposition 3.3.11. *Let* $T : X \to Y$ *be a linear operator between normed spaces.*

$a)$ *If* X *is finite dimensional, then* T *is continuous.*

$b)$ *If* $T(X)$ *has finite dimension, then* T *is continuous if and only if* $\ker(T)$ *is closed.*

$c)$ *If* Y *has finite dimension, then* T *is an open mapping if and only if* T *is surjective.*

We include the proof of the following proposition in order to know how to split a normed space in a direct sum, but the proof is original from (47, Proof of Proposition 4.10)

Proposition 3.3.12. *Let X be a normed space and $M \subseteq X$ a closed subspace with finite codimension. Then every algebraic complement of M is a topological complement.*

Proof. Let N be an algebraic complement of M, that is, $X = M \oplus N$. Let us show that the projection $P : X \to X$ verifying $P(X) = N$ is continuous. Indeed, just observe that $\ker(P) = M$, which is closed by hypothesis, and $P(X) = N$, which has finite dimension, hence, by Proposition 3.3.11, P is continuous. $\square$

Let us finally mention that finite codimensionality does not imply closedness. Indeed, if X is any infinite-dimensional normed space, then there exists a non-continuous linear functional whose kernel is obviously not closed but of codimension 1.

CHAPTER

4

Geometric structure of the unit ball

Contents

The aim of this chapter is to study the behaviour of the geometrical notions we have introduced; by this behaviour we mean how they interact between them or original characterizations of those concepts that we will present. First section is a compilation of some original technical tools that we have needed during the develop of the dissertation. We have leaved these results aside just in order to facilitate the reading. We carry the chapter on with a wide study on Inner Structure: we underline Theorem 4.2.2, where we prove the equivalence between the set of inner points and non-support points for convex sets with inner points, and Theorem 4.2.5, which gives an approach in the affirmative to the question about the existence of non-trivial compact and convex subsets lacking inner points, result borrowed from the submitted paper (23). The section about facets and frames starts with some original definitions: what a pre-maximal face is (Definition 4.3.1) and what we understand by Property P (Definition 4.3.2), proving in Corollary 4.3.7 that there are several examples of Banach spaces failing this P-property. Noteworthy results in this section are Proposition 4.3.10, which shows one of the cases when faces and facets coincide, and Theorem 4.3.12. This last result presents a new reformulation of the frame of the unit ball that will be crucial to prove the invariance of this geometrical concept in Chapter 5. The last section of the current chapter is devoted to the study of a “new” part of the unit ball: the *flat* section. Again, we present new original definitions, in this case, motivated by the concept of starlike set, which are the starlike envelope, almost flat/flat set or the starlike-compatible/generated set (see Definition 4.4.1). As we will see in the following chapter, these geometrical subsets of the unit sphere will be invariants under surjective isometries. We highlight Example 4.4.3 as an example of an almost flat set which is not flat and a non-convex-starlike generated set. This example motivates the Flat Property (Definition 4.4.5), which states a convenient environment considering the hypotheses of Lemmas 4.4.8 and 4.4.9. This last result establishes a generalization of Tingley’s results (5, Lemma 2 and Corollary 3), as well as it gives a deeper perspective about starlike sets. In particular, it will be used in order to prove Theorems 4.4.10 and 4.4.13. These theorems show that starlike sets are the only maximal faces containing its center when the starlike is convex and it shows conditions for which the convexity holds for starlike sets.

4.1 Some technical tools

This section is a compilation of technical lemmas, theorem and results about the behaviour of the unit ball and its elements. We will also include some statements which works in the unit ball but are related to general cases in normed spaces.

The following remark will be crucial towards finding geometric invariants under surjective isometries, which is the main purpose of the following chapter.

Remark 4.1.1. *Let X be a normed space. For every non-zero $x, y \in X$, $\|x + y\| = \|x\| + \|y\|$ if and only if the closed segment $\left[\frac{x}{\|x\|}, \frac{y}{\|y\|}\right] \subseteq S_X$. Indeed,*

⇒) *if $\|x + y\| = \|x\| + \|y\|$, then*

$$\frac{\|x\|}{\|x+y\|}\frac{x}{\|x\|} + \frac{\|y\|}{\|x+y\|}\frac{y}{\|y\|} = \frac{x+y}{\|x+y\|}$$

is a strict convex combination which is in the unit sphere S_X, therefore it should be contained in a face of the sphere, which means that the whole closed segment $\left[\frac{x}{\|x\|}, \frac{y}{\|y\|}\right] \subseteq S_X$.

⇐) *Conversely, if $\left[\frac{x}{\|x\|}, \frac{y}{\|y\|}\right] \subseteq S_X$, then using a similar reasoning as above*

$$\frac{x+y}{\|x+y\|} = \frac{\|x\|}{\|x+y\|}\frac{x}{\|x\|} + \frac{\|y\|}{\|x+y\|}\frac{y}{\|y\|} \in \left[\frac{x}{\|x\|}, \frac{y}{\|y\|}\right] \subseteq S_X,$$

then $\|\frac{x+y}{\|x+y\|}\| = 1$, which means than $\|x + y\| = \|x\| + \|y\|$.

We recall to the reader that a normed space X is said to be rotund (or strictly convex) if its unit sphere has no non-trivial segments, which is equivalent to $\mathrm{ext}(B_X) = S_X$. We suggest

(33, Chapter 5) for a deeper study about rotundity.

Remark 4.1.2. *If $C \subseteq X$ is a convex set of a normed spaces containing at least three points which are not aligned, then $C \setminus \{c\}$ is connected for all $c \in C$.*

The following remark will be useful to construct a new unit ball in $\mathbb{R}^3$, that will serve as a counterexample.

Remark 4.1.3. *Let K be a compact convex subset of a finite dimensional Banach space X, satisfying that K has non-empty interior. Suppose that there exists a functional $x^* \in S_{X^*}$ such that $x^*(x) > 0$ for all $x \in K$. Take $A := \mathrm{co}(K \cup -K)$. Then A is compact, absolutely convex and a neighborhood of 0. Therefore, A defines an equivalent norm in X.*

Lemma 4.1.4. *Under the settings of Remark 4.1.3,* $\mathrm{ext}(A) \subseteq \mathrm{ext}(K) \cup \mathrm{ext}(-K)$.

Proof. Let $a \in \mathrm{ext}(A) = \mathrm{ext}(\mathrm{co}(K \cup -K))$ be an arbitrary extreme point of A. Notice that there exist $t \in [0,1]$, $k_1, k_2 \in K$ such that $a = tk_1 + (1-t)(-k_2)$. Since a is an extreme point, it holds that $a = k_1$ or $a = -k_2$. Suppose with no loss of generality that $a = k_1$. Then, $a \in \mathrm{ext}(A) \cap K \subseteq \mathrm{ext}(K)$. $\square$

Now, we go through several technical remarks and lemmas that are going to made use of with the goal of finding an example of a non-trivial convex and compact set free of inner points in Section 4.2 of this chapter.

Remark 4.1.5. *Let X be an infinite-dimensional Banach space and $(e_n)_{n\in\mathbb{N}} \subseteq X$ a bounded* *basic sequence. Let us consider the following operator:*

$$\begin{aligned} T : \ell_1 &\longrightarrow X \\ (t_n)_{n\in\mathbb{N}} &\longmapsto T\left((t_n)_{n\in\mathbb{N}}\right) := \sum_{n=1}^{\infty} t_n e_n \end{aligned}$$

The previous operator verifies some properties: T is well-defined. Also, $\ker(T) = 0$ since $(e_n)_{n\in\mathbb{N}}$ is a basic sequence, thus, T is one-to-one or injective. Besides, the operator is linear by its definition. Let us show that T is also bounded. Indeed, $(e_n)_{n\in\mathbb{N}}$ is bounded, which means that $\left\|(\|e_n\|_X)_{n\in\mathbb{N}}\right\|_\infty < \infty$. Then

$$\begin{aligned} \|T\|_\infty &= \sup\left\{\|T((t_n)_{n\in\mathbb{N}})\|_X : \ \ \|(t_n)_{n\in\mathbb{N}}\|_1 = 1\right\} \\ &\le \sup\left\{\sum_{n=1}^{\infty} |t_n|\|e_n\|_X : \ \ \|(t_n)_{n\in\mathbb{N}}\|_1 = 1\right\} \\ &\le \sup\left\{\|e_n\|_X : n \in \mathbb{N}\right\} = \left\|(\|e_n\|_X)_{n\in\mathbb{N}}\right\|_\infty . \end{aligned}$$

The above inequalities show that T is bounded. Even more, we can prove that $\|T\| = \left\|(\|e_n\|_X)_{n\in\mathbb{N}}\right\|_\infty$. Let $(u_n)_{n\in\mathbb{N}} \subseteq \ell_1$ be the canonical basis, then $\|u_n\|_1 = 1$ and $T(u_n) = e_n$ for each $n \in \mathbb{N}$, hence, $\|T(u_n)\|_X = \|e_n\|_X$ for all $n \in \mathbb{N}$. Now, consider a subsequence $(e_{n_k})_{k\in\mathbb{N}}$ satisfying $\|e_{n_k}\|_X \to \left\|(\|e_n\|_X)_{n\in\mathbb{N}}\right\|_\infty$ as $k \to \infty$. This is equivalent to $\|T(u_{n_k})\|_X \to \left\|(\|e_n\|_X)_{n\in\mathbb{N}}\right\|_\infty$ as $k \to \infty$, meaning that $\left\|(\|e_n\|_X)_{n\in\mathbb{N}}\right\|_\infty \le \sup_{x\in B_{\ell_1}} \|T(x)\|_X = \|T\|$.

Observe that this supremum is attained: let $\varepsilon > 0$, then, there exists $k \in \mathbb{N}$ such that

$$M - \varepsilon < \|e_k\| \le M, \tag{4.1.1}$$

where $M = \sup_{n\in\mathbb{N}}\{\|e_n\|\}$. Now, let $(u_n)_{n\in\mathbb{N}} \subseteq \ell_1$ be the canonical basis. Notice that $(u_n)_{n\in\mathbb{N}} \subseteq B_{\ell_1}$ and the image of this basis is precisely our bounded sequence, that is, $T(u_n) = e_n$ for each

$n \in \mathbb{N}$, hence, $\|T(u_n)\|_X = \|e_n\|_X$ for all $n \in \mathbb{N}$. Putting this together with Equation 4.1.1, it holds that

$$M - \varepsilon < \|T(u_k)\| \leq \|T\| = \sup\{\|T(t_n)\| : (t_n)_{n\in\mathbb{N}} \subseteq B_{\ell_1}\}.$$

If we make $\varepsilon \to 0$, the next inequality holds $M \leq \|T\| \leq M$, meaning that $\|T\| = \sup_{n\in\mathbb{N}}\{\|e_n\|\}$.

As a consequence, T is continuous. In particular, T is $w-w$ continuous, so T maps w-compact subsets of ℓ_1 to w-compact subsets of X. Also, T is an isomorphism of vector spaces over its image, which means that T maps convex sets free of inner points of ℓ_1 to convex sets of X free of inner point, in view of (20, Section 1).

We refer the reader to (41, Chapter 6, Section 4) for more details about weak-$*$ and weak convergence. We recover the following observation.

Remark 4.1.6. *Let X be a normed space. If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence of X^* such that there exists $x_0^* \in X^*$ verifying that $(x_n^*(y))_{n\in\mathbb{N}}$ is convergent to $x_0^*(y)$ for all $y \in Y$, where Y is a dense subspace of X, then $(x_n^*)_{n\in\mathbb{N}}$ is w^*-convergent to x_0^*.*

Lemma 4.1.7. *Consider $(x_k)_{k\in\mathbb{N}}$ a bounded sequence of ℓ_1 such that $(x_k(n))_{k\in\mathbb{N}}$ is convergent for all $n \in \mathbb{N}$. Then, there exists $x_0 \in \ell_1$ such that $(x_k)_{k\in\mathbb{N}}$ is w^*-convergent to x_0.*

Proof. We pick x_0 defined as the sequence whose n^{th}-term is given by $x_0(n) := \lim_{k\to\infty} x_k(n)$, and let $M > 0$ such that $\|x_k\|_1 \leq M$ for all $k \in \mathbb{N}$. We will follow some steps through this proof:

1) Let us see that $x_0 \in \ell_1$. For this, we will show that for every $p \in \mathbb{N}$, $\sum_{n=1}^{p} |x_0(n)| \leq M$.

   Indeed, fir an arbitrary $p \in \mathbb{N}$ and $\varepsilon > 0$. By hypothesis, we know that $(x_k)_{k\in\mathbb{N}}$ is coordinate-convergent, thus, for every $n \in \{1, \dots, p\}$ there exists $k_n \in \mathbb{N}$ such that

$$|x_k(n) - x_0(n)| < \frac{\varepsilon}{p}, \quad \forall k \geq k_n.$$

Take $k_0 := \max\{k_1, \dots, k_p\}$, then

$$\sum_{n=1}^{p} |x_0(n)| \geq \sum_{n=1}^{p} |x_0(n) - x_{k_0}(n)| + \sum_{n=1}^{p} |x_{k_0}(n)| \leq \sum_{n=1}^{p} \frac{\varepsilon}{p} + \|x_{k_0}\|_1 \geq \varepsilon + M.$$

The arbitrariness of $\varepsilon > 0$ leads to $\sum_{n=1}^{p} |x_0(n)| \leq M$, and the arbitrariness of $p \in \mathbb{N}$ gives us that $x_0 \in \ell_1$.

2) Let us show that $(x_k)_{k\in\mathbb{N}}$ w^*-converges to x_0 using the previous remark. Notice that c_{00} is dense in c_0, and $(x_k)_{k\in\mathbb{N}}$ is coordinate-convergent to x_0. In particular, $(x_k(y))_{k\in\mathbb{N}} \to x_0(y)$, for all $y \in c_{00}$, where $x_k(y)$ means x_k acting on y in the dual $\ell_1 = (c_0)^*$, that is $x_k(y) := \sum_{n=1}^{\infty} x_k(n)y(n)$. Then, by Remark 4.1.6, $(x_k)_{k\in\mathbb{N}}$ is w^*-convergent to x_0.

□

The lemma above can also be tackled in other way a bit different from the one used in (50, Lemma 2.3). For a wider perspective about nets and subnets, we refer the reader to (33). In particular, following Remark is proved in (e) of (33, Proposition 2.1.31).

Remark 4.1.8. *Let X be a topological space, $\gamma : D \to X$ a net in X, where D is a directed set and $x \in X$. If every subnet λ of γ has a further subnet δ with $x \in \lim(\delta)$, then $x \in \lim(\gamma)$.*

Next lemma generalizes Remark 4.1.6 and Lemma 4.1.7 at once.

Lemma 4.1.9. *Let $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ be a bounded and pointwise convergent sequence of a normed space. Then, there exists $x_0^* \in X^*$ such that $(x_n^*)_{n\in\mathbb{N}}$ is w^*-convergent to x_0^*.*

Proof. By the Banach-Alaoglu Theorem C.2.3, $(x_n^*)_{n\in\mathbb{N}}$ has a w^*-convergent subnet to some x_0^* in X^*. The pointwise converges assures that $x_0^*(x) = \lim_{n\to\infty} x_n^*(x)$ for all $x \in X$. We will prove that $(x_n^*)_{n\in\mathbb{N}}$ is w^*-convergent to x_0^* using Remark 4.1.8. Indeed, let $\left(x_{n_\alpha}^*\right)_{\alpha\in A}$ be any subnet of $(x_n^*)_{n\in\mathbb{N}}$. Since $\left(x_{n_\alpha}^*\right)_{\alpha\in A}$ is bounded, the Banach-Alaoglu Theorem assures again that it is possible to extract a further subnet $\left(x_{n_{\alpha_\beta}}^*\right)_{\beta\in B}$ which is w^*-convergent to some $x_1^* \in X$. For every $x \in X$, $\left(x_{n_{\alpha_\beta}}^*(x)\right)_{\beta\in B}$ is a subnet of $(x_n^*(x))_{n\in\mathbb{N}}$, then

$$x_1^*(x) = \lim_{\beta\in B}\left(x_{n_{\alpha_\beta}}^*(x)\right)_{\beta\in B} = \lim_{n\to\infty} x_n^*(x) = x_0^*(x).$$

This equality implies that $x_1^* = x_0^*$ and $\left(x_{n_{\alpha_\beta}}^*\right)_{\beta\in B}$ is w^*-convergent to x_0^*. By using Remark 4.1.8, $(x_n^*)_{n\in\mathbb{N}}$ is w^*-convergent to x_0^*. □

The next result is a refinement of (50, Lemma 2.3) with a simpler proof. This lemma will be crucial to show the example of a weakly compact convex subset with no inner points (the example will be studied in Theorem 4.2.5).

Lemma 4.1.10. *Let $(e_n)_{n\in\mathbb{N}}$ be a basic sequence of an infinite-dimensional Banach space X, and let T the operator given in Remark 4.1.5. Then, if A is a w^*-closed bounded subset of ℓ_1, then $T(A)$ is w-sequentially closed in X. If A is also convex, then $T(A)$ is w-closed.*

Proof. Let $\left((t_n^k)_{n\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ be a sequence of A satisfying that its image via T is weakly-convergent, that is, $\left(T\left((t_n^k)_{n\in\mathbb{N}}\right)\right)_{k\in\mathbb{N}}$ is w-convergent to some $x \in X$. For every $n \in \mathbb{N}$, we will denote by $e_n^* \in X^*$ the Hahn-Banach extension of the coordinate functional respect to e_n. We notice that $\left(e_n^*\left(T\left((t_n^k)_{n\in\mathbb{N}}\right)\right)\right)_{k\in\mathbb{N}}$ converges to $(e_n^*(x))$ for each $n \in \mathbb{N}$. Besides, $\left(e_n^*\left(T\left((t_n^k)_{n\in\mathbb{N}}\right)\right)\right)_{k\in\mathbb{N}} = t_n^k$, for all $n, k \in \mathbb{N}$. This means that $\left((t_n^k)_{n\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ is a bounded sequence of ℓ_1 which is pointwise convergent to $(e_n^*(x))_{n\in\mathbb{N}}$, therefore, we are under the hypotheses of Lemma 4.1.7 or Lemma 4.1.9, concluding that $(e_n^*(x))_{n\in\mathbb{N}} \in \ell_1$ and $\left((t_n^k)_{n\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ is w^*-convergent to $(e_n^*(x))_{n\in\mathbb{N}}$. We recall that $\left((t_n^k)_{n\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ is a sequence of A, which is w^*-closed, then, $(e_n^*(x))_{n\in\mathbb{N}} \in A$. We conclude that $x \in T(A)$ noticing that

$$x = \sum_{n=1}^{\infty} e_n^*(x) e_n = T\left((e_n^*(x))_{n\in\mathbb{N}}\right) \in T(A).$$

To conclude the proof, assume A is convex. Then, $T(A)$ is convex. Besides, $T(A)$ is w-sequentially closed, so, $T(A)$ is closed (respect to the norm) and convex, then, by the Hahn-Banach Separation Theorem, $T(A)$ is w-closed. □

We conclude this section about technical necessary remarks in order to give an example of a non-trivial weakly compact convex set free of inner points with the following remark.

Remark 4.1.11. *The set $\{(t_n)_{n\in\mathbb{N}}) \in \ell_1 : \ t_n \geq 0\}$ is w^*-closed in ℓ_1. This assertion is clear just considering that w^*-convergence implies pointwise convergence in ℓ_1, since ℓ_1 is a dual space, $\ell_1 = c_0^*$, and in a dual space the pointwise convergence is equivalent to weak*-convergence (see Proposition C.2.16.1).*

4.2 **Inner Structure**

Along this section, we will profoundly develop the inner structure introduced in the previous chapter and give new results about concepts related to this study. Notice that for a generalized view of the inner structure, some parts of the work are not into the unit ball, but it still suits our purpose in the next results.

We start this study giving the relation between the set of inner points and non-support points of a convex subset under particular conditions. This relation belongs to (3, Theorem 5).

Theorem 4.2.1. *Let C be a convex subset of a vector space X, satisfying that $h(C) \neq \varnothing$. Then,* $\mathrm{inn}(C) \subseteq \mathrm{nsupp}(C)$.

Proof. Fix an arbitrary $f \in h(C)$, where

$$h(C) = \{f \in X^* : f \text{ attains it sup at } C \text{ and it is not constant on } C\}.$$

We know that $F(f, C)$ is extremal in C in view of Definition 3.2.2. Next, we call Remark 3.2.27 to conclude that $F(f, C) \cap \mathrm{inn}(C) = \varnothing$, which means that $\mathrm{inn}(C) \subseteq C \setminus F(f, C)$. Keep in mind that $\mathrm{nsupp}(C) = C \setminus \cup_{f \in h(C)} F(f, C)$ and the arbitrariness of $f \in h(C)$, it holds

$$\mathrm{inn}(C) \subseteq \bigcap_{f \in h(C)} C \setminus F(f, C) = C \setminus \bigcup_{f \in h(C)} F(f, C) = \mathrm{nsupp}(C).$$

□

We complete result above in (23, Theorem 3.3), which is exposed in next theorem.

Theorem 4.2.2. *Let X be a vector space and $C \subseteq X$ a convex subset of X. If $h(C) \neq \varnothing$ and* $\mathrm{inn}(C) \neq \varnothing$*, then* $\mathrm{inn}(C) = \mathrm{nsupp}(C)$.

Proof. $\subseteq$ This inclusion has been already proved in Theorem 4.2.1.

- $\supseteq$ First, we assume that $0 \in \mathrm{inn}(C)$ and consider $Y := \mathrm{span}(C)$. We claim that C is absorbing in Y by Theorem 3.2.34, see (2) of (20, Theorem 4.2). Indeed, by hypotheses, $0 \in \mathrm{inn}(C)$ and $\mathrm{span}(C) = Y$. The condition $C \subseteq \mathrm{sadj}(C)$ is due to the convexity of C and Theorem 3.2.33(2), see also (20, Theorem 2.3), then $\mathrm{inn}(C)$ is absorbing, implying that C is also absorbing, since $\mathrm{inn}(C) \subseteq C$. By Theorem 3.2.4 (with (22, Theorem 6) as the original reference), it holds that C is a neighborhood of 0 in

Y endowed with the finest locally convex vector topology. Keeping in mind Lemma 3.2.29(6), we have that $\mathrm{inn}(C) = \mathrm{int}_Y(C)$. Now, we are in the right position to prove the other inclusion. Assume on the contrary that there exists $x \in \mathrm{nsupp}(C) \setminus \mathrm{inn}(C)$. Thus, $x \notin \mathrm{int}_Y(C)$. Since the finest locally convex vector topology is Hausdorff and locally convex, the Hahn-Banach Theorem guarantees the existence of a functional $f \in X^*$ such that $f(c) < f(x)$ for all $c \in \mathrm{int}_Y(C)$, therefore, $\sup f(\mathrm{int}_Y(C)) = f(x)$ and f is not constant in C. If we denote the linearly extension of f to the whole space X by f, we have that $f \in h(C)$ and $x \in F(f, C)$, which contradicts the fact that $x \in \mathrm{nsupp}(C)$. Now, consider that $0 \notin \mathrm{inn}(C)$ and let $c \in \mathrm{inn}(C)$. Now we follow the same reasoning for the translation $D := C - c$. Notice that $\mathrm{inn}(D) = \mathrm{inn}(C) - c$, $\mathrm{nsupp}(D) = \mathrm{nsupp}(C) - c$, and $0 = c - c \in \mathrm{inn}(D)$. We finish the proof using that $\mathrm{nsupp}(C) = c + \mathrm{nsupp}(D) \subseteq c + \mathrm{inn}(D) = \mathrm{inn}(C)$.

□

We recall that ℓ_1 is the notation for the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}$ such that its series is absolutely convergent, that is, $\ell_1 = \left\{(x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty\right\}$.

In (20), it was showed that it is not an easy task to find non-trivial convex sets lacking inner points (see Example 3.2.26) when we work with infinite dimensional vector spaces, and it is even harder to find closed convex sets which are non-trivial and satisfying the same property, as we can see in next result taken from (20, Theorem 5.4).

Theorem 4.2.3. *There exists a closed convex subset D of S_{ℓ_1} free of inner points. In particular,*

$$D = \left\{(t_n)_{n\in\mathbb{N}} \in S_{\ell_1} : t_n \geq 0 \text{ for all } n \in \mathbb{N}\right\}.$$

Therefore, the next natural step is to find a non-singleton compact and convex subset $K \subseteq X$ free of inner points, with X a Hausdorff locally convex topological vector space. Let us see that the example given above does not fit our purpose.

Proposition 4.2.4. *The closed convex subset* $D := \left\{(t_n)_{n\in\mathbb{N}} \in S_{\ell_1} : t_n \geq 0 \text{ for all } n \in \mathbb{N}\right\}$ *is not weakly compact in* ℓ_1.

Proof. Suppose on the contrary that D is weakly compact. Then, by the Eberlein-Smulian Theorem C.3.2, D is weakly sequentially compact. As D is in ℓ_1, sequential weak convergence and norm convergence are equivalent by Schur Theorem C.3. Thus, D is sequentially compact, which implies that D is compact, as ℓ_1 is a metric space. Finally, consider the sequence of canonical vectors $(e_n)_{n\in\mathbb{N}}$ of ℓ_1. This sequence is contained in D. However, notice that $(e_n)_{n\in\mathbb{N}}$ satisfies that $\|e_i - e_j\|_1 = 2$ whenever $i \neq j$, which means that $(e_n)_{n\in\mathbb{N}}$ does not have convergent subsequences, reaching the contradiction. □

Taking into consideration all these previous results and technical remarks and lemmas given in Section 4.1, we are in the right position to provide a compact and convex subset free of inner points in an infinite-dimensional Banach space endowed with the weak topology, as we have done in the pre-printed article (23, Theorem 4.8).

Theorem 4.2.5. *Let X be an infinite-dimensional Banach space. Then, there exists a non-trivial weakly compact and convex subset $K \subseteq X$ lacking inner points.*

Proof. Let us construct $K \subseteq X$ and prove the assertions given in the theorem in a few steps. Consider $(u_n)_{n\in\mathbb{N}} \subseteq S_X$ a normalized basic sequence, and let $(e_n)_{n\in\mathbb{N}}$ be the scaled basic sequence based on $(u_n)_{n\in\mathbb{N}}$, that is $e_n := \dfrac{u_n}{n}$ for each $n \in \mathbb{N}$. The reader has to notice that since $(e_n)_{n\in\mathbb{N}}$ converges to 0, $\{0\}\cup\{e_n : n\in\mathbb{N}\}$ is compact. Now, let us define K,

$$K := \left\{\sum_{n=1}^{\infty} t_n e_n : \; t_n \geq 0, \; \sum_{n=1}^{\infty} t_n \leq 1\right\},$$

and we will follow three steps:

- K is closed and convex. The convexity of K is clear by how it is defined. Notice that $K = T(A)$, where T is the operator given in Remark 4.1.5, where A is the w^*-closed and bounded subset of ℓ_1 defined by $A := B_{\ell_1} \cap \{(t_n)_{n\in\mathbb{N}} \in \ell_1 : \; t_n \geq 0\}$ (see Remark 4.1.11). Then, we only need to take into consideration Lemma 4.1.10 to conclude that K is closed.

- K is weakly compact. Indeed, by bearing in mind the Krein-Smulian Theorem (see Theorem C.3.3), it will be sufficient to show that $K = \overline{\text{co}}(\{0\}\cup\{e_n : n\in\mathbb{N}\})$. Let us see it by double inclusion. On the one hand, $\{0\}\cup\{e_n \in \mathbb{N}\} \subseteq K$, then $\overline{\text{co}}(\{0\}\cup\{e_n : n \in \mathbb{N}\}) \subseteq K$, since K is closed and convex, as it is shown in the previous item. Besides, observe that every element with the form $\sum_{n=1}^{\infty} t_n e_n \in K$ is the limit of a sequence of convex combinations $\left(\sum_{n=1}^{p} t_n e_n\right)_{p\in\mathbb{N}} \subseteq \text{co}(\{0\}\cup\{e_n : n\in\mathbb{N}\})$, which means that $K \subseteq \overline{\text{co}}(\{0\}\cup\{e_n : n\in\mathbb{N}\})$.

- $\text{inn}(K) = \varnothing$. To prove it, we will follow a similar idea as in (20, Theorem 5.4). Let us suppose on the contrary that there exists $(t_n)_{n\in\mathbb{N}} \in B_{\ell_1}$, $t_n \geq 0$ for all $n\in\mathbb{N}$, such that $\sum_{n=1}^{\infty} t_n e_n \in \text{inn}(K)$. Let us see first that $t_n > 0$ for all $n\in\mathbb{N}$. Suppose again on the contrary that $t_n = 0$ for some $k\in\mathbb{N}$. Then, since $\sum_{n=1}^{\infty} t_n e_n \in \text{inn}(K)$, there must exist $s_k < 0$ such that $s_k e_k + (1-s_k)\sum_{n=1}^{\infty} t_n e_n \in K$, that is, $s_k e_k + (1-s_k)\sum_{n=1}^{\infty} t_n e_n = \sum_{n=1}^{\infty} \alpha_n e_n$ with $\alpha_n \geq 0$ for all $n\in\mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n \leq 1$. By taking this into consideration, it is possible to reach to the following contradiction:

$$s_k = e_k^*\left(s_k e_k + (1-s_k)\sum_{n=1}^{\infty} t_n e_n\right) = e_k^*\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \alpha_n \geq 0.$$

Consequently, $t_n > 0$ for all $n \in \mathbb{N}$. Next, since $(t_n)_{n\in\mathbb{N}}$ converges to 0, it is possible to inductively construct a strict increasing sequence $(n_k)_{k\geq 0}$ verifying that $n_0 = 0$, and, for every $k \in \mathbb{N}$,

$$\frac{t_{n_k}}{t_{n_{k-1}+1}} < \frac{1}{k}.$$

So, for every $n \in \mathbb{N}$, let us define

$$s_n := \begin{cases} 0, & n \neq n_k \text{ for all } k \geq 1, \\ t_{n_{k-1}+1} + t_{n_{k-1}+2} + \cdots + t_{n_k-1} + t_{n_k}, & n = n_k \text{ for some } k \geq 1. \end{cases}$$

Notice that $s_n \geq 0$ for all $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} s_n = \sum_{k=1}^{\infty} s_{n_k} = \sum_{n=1}^{\infty} t_n \geq 1,$$

meaning that $\sum_{n=1}^{\infty} s_n e_n \in K$. Since $\sum_{n=1}^{\infty} t_n e_n \in \operatorname{inn}(K)$, then there exists $\lambda > 1$ such that

$$(1-\lambda)\sum_{n=1}^{\infty} s_n e_n + \lambda \sum_{n=1}^{\infty} t_n e_n \in K.$$

Besides, by construction of K, $(1-\lambda)s_n + \lambda t_n \geq 0$, for all $n \in \mathbb{N}$. In particular, for $k \in \mathbb{N}$, $(1-\lambda)s_{n_k} + \lambda t_{n_k} \geq 0$, which means

$$\begin{aligned}
\lambda \geq \frac{s_{n_k}}{s_{n_k} - t_{n_k}} &= \frac{t_{n_{k-1}+1} + t_{n_{k-1}+2} + \cdots t_{n_k-1} + t_{n_k}}{t_{n_{k-1}+1} + t_{n_{k-1}+2} + \cdots t_{n_k-1}} \\
&= 1 + \frac{t_{n_k}}{t_{n_{k-1}+1} + t_{n_{k-1}+2} + \cdots t_{n_k-1}} < 1 + \frac{t_{n_k}}{t_{n_{k-1}+1}} \\
&< 1 + \frac{1}{k},
\end{aligned}$$

which gives us the contradiction $\lambda \geq 1$.

□

Finally, a slight modification to the set D given in Proposition 4.2.4 gives an example of a w^*-compact convex subset in ℓ_1 with no inner points.

Proposition 4.2.6. *The closed convex subset* $L := \{(t_n)_{n\in\mathbb{N}} \in S_{\ell_1} : t_n \geq 0, \forall n \in \mathbb{N}\}$ *is* w^*-*compact in* ℓ_1 *lacking inner points,* $\mathrm{inn}(L) = \varnothing$.

Proof. We claim that L is w^*-closed in B_{ℓ_1} by using pointwise convergence. This makes L w^*-compact. A similar proof as in Theorem 4.2.5 applies to assure that $\mathrm{inn}(L) = \varnothing$. □

4.3 Structure of Facets and Frames

Some results of this section appear in a light version and in a spread manner throughout the literature of the Geometry of Banach spaces. We generalize some of them and provide all the proofs in sake of completeness. Likewise, we refer the reader to (32; 33; 34; 51; 52) for more information about facets and frames.

We start this section presenting a new property related to the extremal structure of the unit ball which will be a useful tool in order to prove the preservation of particular subsets of the unit ball under surjective isometries (see Theorem 5.2.4). This interesting property is strongly motivated by (2, Definition 3.2). This reference also inspires the next definition, which is original from this work:

Definition 4.3.1 (Pre-maximal face). *For a normed space X, we will called a proper face A a pre-maximal face if it is the intersection of all the maximal faces containing A.*

Definition 4.3.2 (Property P). *We say that a normed space X has Property P or the P-property (Pp) if every proper face of B_X is a pre-maximal face.*

The following result provides an example of sufficient condition for Banach spaces lacking Pp.

Lemma 4.3.3. *If X is a smooth Banach space with extreme points which are not rotund, that is* $\mathrm{ext}(B_X) \setminus \mathrm{rot}(B_X) \neq \emptyset$*, then X fails Pp.*

Proof. Let $e \in \mathrm{ext}(B_X) \setminus \mathrm{rot}(B_X)$. We will prove that such point is a face of B_X which is not the intersection of all maximal faces containing it, then X fails Property P. Indeed, there is a maximal face $C \subseteq S_X$ which contains e. If there exists another maximal face D containing e, then there exists $x^* \in S_{X^*}$ such that $D = \{x \in S_X : x^*(x) = 1\}$. The same reasoning shows us that there exists $y^* \in S_{X^*}$ such that $C = \{x \in S_X : y^*(x) = 1\}$. Therefore, since $e \in D$ and $e \in C$, $x^*(e) = 1 = y^*(e)$, concluding that $x^* = y^*$ because of the smoothness of X. As a consequence, $C = D$ is the only maximal face of S_X containing e, but $C \neq \{e\}$, since $e \notin \mathrm{rot}(B_X)$.

□

It is easy to check that if Y is a subspace of a normed subspace X, then $\mathrm{rot}(B_X) \cap B_Y \subseteq \mathrm{rot}(B_Y)$ and $\mathrm{ext}(B_X) \cap B_Y \subseteq \mathrm{ext}(B_Y)$. Following results shows the behaviour of extremal points under direct sums.

Lemma 4.3.4. *Let $X := Y \oplus_2 Z$, where Y and Z are Banach spaces. Then,* $\mathrm{ext}(B_Y) \subseteq \mathrm{ext}(B_X)$.

Proof. Fix an arbitrary $y \in \mathrm{ext}(B_Y)$. Let us show that y satisfies the extremal condition in B_X. Consider $x_1, x_2 \in B_X$, thus $\|x_1\| \leq 1$, $\|x_2\| \leq 1$. Let $t \in (0,1)$ such that $y = tx_1 + (1-t)x_2 \in B_Y$. Since $X = Y \oplus_2 Z$, $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ where $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ with $Y \cap Z = \{0\}$. Then, $1 \geq \|x_1\|^2 = \|y_1\|^2 + \|z_1\|^2$, where we have used again that $X = Y \oplus_2 Z$. Notice that $\|z_1\|^2 \geq 0$, so $\|y_1\|^2 \leq 1$. A similar reasoning shows that $\|y_2\|^2 \leq 1$. On the other hand,

$$\begin{aligned} Y \ni y &= tx_1 + (1-t)x_2 \\ &= t(y_1 + z_1) + (1-t)(y_2 + z_2) \\ &= \underbrace{ty_1 + (1-t)y_2}_{\in Y} + \underbrace{tz_1 + (1-t)z_2}_{\in Z}, \end{aligned}$$

so, the only possibility is that $tz_1 + (1-t)z_2 = 0$. We conclude that $y = ty_1 + (1-t)y_2$, but $y, y_1, y_2 \in B_Y$ and $y \in \mathrm{ext}(B_Y)$, therefore, $y = y_1 = y_2$. Besides, $y \in \mathrm{ext}(B_Y) \subseteq S_X$, then $\|y\| = 1$ concluding that $\|y_1\| = \|y_2\| = 1$. This last equality forces $\|z_1\| = 0 = \|z_2\|$, which is equivalent to $z_1 = z_2 = 0$. This means that $x_1 = y_1 = y_2 = x_2 = y$. □

Lemma 4.3.5. *If Y is a 2-dimensional Banach space, then it is isomorphic to a smooth space Y' satisfying the condition* $\mathrm{ext}(B_{Y'}) \setminus \mathrm{rot}(B_{Y'}) \neq \varnothing$. *Moreover, if Z is another Banach space, then* $\mathrm{ext}(B_{Y'}) \setminus \mathrm{rot}(B_{Y'}) \subseteq \mathrm{ext}(B_{X'}) \setminus \mathrm{rot}(B_{X'})$*, where $X' := Y' \oplus_2 Z$.*

Proof. It is enough to smoothen the corners of $B_{\ell_\infty^2}$ and take Y' as $\mathbb{R}^2$ endowed with the norm defined by this unit ball. Therefore, Y' is not strictly convex and its unit ball is compact, so any non-singleton maximal face of $B_{Y'}$ contains extreme points by the Krein-Milman Theorem C.3.1, and these extreme points are not rotund. Let us see the last part of the lemma. Consider $x \in \mathrm{ext}(B_{Y'}) \setminus \mathrm{rot}(B_{Y'})$. Notice that if $x \notin \mathrm{rot}(B_{Y'})$ then $x \notin \mathrm{rot}(B_{X'})$. Indeed, if $x \notin \mathrm{rot}(B_{Y'})$ there exists a non-singleton maximal face of $B_{Y'}$, C, such that $x \in C$. If C is a maximal face of $B_{X'}$, then $x \notin \operatorname{rot}(B_{X'})$. If C is not a maximal face of $B_{X'}$, since $X' := Y' \oplus_2 Z$, there exists a maximal face D of $B_{X'}$ such that $\{x\} \subsetneq C \subseteq D$, thus, $x \notin \operatorname{rot}(B_{X'})$. Finally, by using Lemma 4.3.4 we end the proof. □

We can now state and prove an important result about Banach spaces lacking Property P.

Theorem 4.3.6. *Any Banach space X with dimension greater than or equal to 2 which admits a smooth equivalent norm can be equivalently renormed to be smooth and to satisfy* $\operatorname{ext}(B_X) \setminus \operatorname{rot}(B_X) \neq \emptyset$. *As a consequence, X fails Pp with this equivalent norm by Lemma 4.3.3.*

Proof. Let X be a Banach space with dimension greater than or equal to 2 endowed with a smooth norm, and fix a 2-dimensional subspace $Y \subseteq X$. According to Lemma 4.3.5, Y is isomorphic to a smooth non-strictly convex Banach space Y with dimension 2. Let Z be a closed subspace of X such that $X = Y \oplus Z$ (the existence of such space is guaranteed by Theorem 3.3.4). Then, X is isomorphic to $X' := Y' \oplus_2 Z$, and notice that $Y' \oplus Z$ is smooth. Thanks to Lemma 4.3.5, it is possible to find $e \in \operatorname{ext}(B_{Y'}) \setminus \operatorname{rot}(B_{Y'})$. Lemma 4.3.5 also assures that $e \in \operatorname{ext}(B_{X'}) \setminus \operatorname{rot}(B_{X'})$ □

As a consequence, we present a sufficient condition for a Banach space to fail Property P.

Corollary 4.3.7. *Every reflexive or separable Banach space whose dimension is greater than or equal to 2 can be equivalently renormed to fail Property P.*

Proof. It is a direct consequence of the fact the reflexive Banach spaces and separable Banach spaces admit an equivalent smooth renorming, see (53, Corollary 4) and (35, Corollary 4.3) respectively. □

Now, we will go through the study of faces, facets and frames. But first, let us see the following useful note about maximal segments containing a certain segment.

Remark 4.3.8. *Let X be a normed space, and consider the unit ball centered in $x \in X$ with radius $r < 0$, $B_X(x,r)$, and $x \neq y \in B_X(x,r)$. Let us compute the maximal segment of $B_X(x,r)$ containing $[x,y]$.*

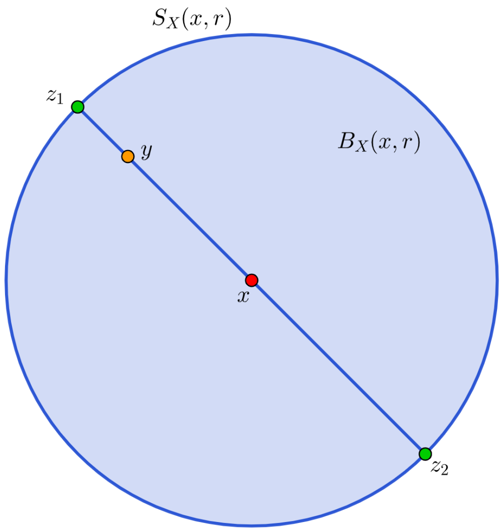

Figure 4.1: $B_X(x,r)$

Let $[z_1,z_2]$ be the maximal segment of $B_X(x,r)$ containing $[x,y]$. It is clear that $z_1,z_2 \in S_X(x,r)$, then $\|z_1-x\| = \|z_2-x\| = r$. On the other hand, it is known that $z_1 = (1-s)x+sy$, $z_2 = (1+s)x-sy$, with $s > 1$ (see Figure 4.1). Let us see the reasoning for z_1 and for z_2 is analogous:

$$\|z_1-x\| = r \iff \|(1-s)x+sy-x\| = r \iff \|-sx+sy\| = r \iff s = \frac{r}{\|y-x\|}.$$

Then, we conclude that the extremes of the maximal segment in $B_X(x,r)$ are given by

$$\left(1+\frac{r}{\|x-y\|}\right)x-\frac{r}{\|x-y\|}y \quad \textit{and} \quad \left(1-\frac{r}{\|x-y\|}\right)x+\frac{r}{\|x-y\|}.$$

Notice that these two points are the only ones of the segment

$$\left[\left(1+\frac{r}{\|x-y\|}\right)x-\frac{r}{\|x-y\|}y,\left(1-\frac{r}{\|x-y\|}\right)x+\frac{r}{\|x-y\|}\right]$$

lying on the sphere $S_X(x,r)$.

Lemma 4.3.9. *Let X be a normed space with $\dim(X)\geq 2$, $x\in S_X$, $f\in S_{X^*}$ such that $f(x)=1$, and $r>0$. Then:*

1. *If $B_X(x,r)\cap S_X\subseteq f^{-1}(\{1\})$, then $B_X(x,\frac{r}{2})\cap f^{-1}(\{1\})\subseteq S_X$.*

2. *If $y\in S_X\setminus\{x\}$ satisfies $[x,y]\subseteq S_X\cap f^{-1}(\{1\})$, and $[u,v]$ is the maximal segment of S_X containing $[x,y]$, then $u,v\in \mathrm{cl}(S_X\setminus f^{-1}(\{1\}))$.*

3. *If $B_X(x,r)\cap S_X\subseteq f^{-1}(\{1\})$, then $B_X(x,r)\cap S_X$ is convex and $r<1$. Even more, if $y\in(B_x(x,r)\cap S_X)\setminus\{x\}$, then*

$$\left[\left(1+\frac{r}{\|x-y\|}\right)x-\frac{r}{\|x-y\|}y,\left(1-\frac{r}{\|x-y\|}\right)x+\frac{r}{\|x-y\|}\right]\subseteq S_X.$$

4. *If $B_X(x,r)\cap f^{-1}(\{1\})\subseteq S_X$, then $r\leq 1$ and there exists $0<s<r$ such that $B_X(x,s)\cap S_X\subseteq f^{-1}(\{1\})$.*

Proof. 1. Let $y\in B_X(x,\frac{r}{2})\cap f^{-1}(\{1\})$ be an arbitrary point, and let us see that $y\in S_X$.

Notice that $1 = f(y) \leq \|y\|$. Next,

$$\begin{aligned}
\left\| x - \frac{y}{\|y\|} \right\| &\leq \|x-y\| + \left\| y - \frac{y}{\|y\|} \right\| \\
&= \|x-y\| + \frac{\| \|y\| y - y\|}{\|y\|} \\
&= \|x-y\| + \frac{\|y\| - 1 \mid \|y\|}{\|y\|} \\
&= \|x-y\| + |\|y\| - 1| \\
&= \|x-y\| + |\|y\| - \|x\|| \\
&\leq 2\|y-x\| \\
&\leq r.
\end{aligned}$$

Then, $\frac{y}{\|y\|} \in B_X(x,r) \cap S_X \subseteq f^{-1}(\{1\})$. That means that $f\left(\frac{y}{\|y\|}\right) = 1 = f(y)$, concluding that $\|y\| = 1$.

2. It suffices to show that $u, v \notin \mathrm{int}_{S_X}\left(S_X \cap f^{-1}(\{1\})\right)$. Suppose to the contrary that, for example, $u \in \mathrm{int}\left(S_X \cap f^{-1}(\{1\})\right)$. Then, there exists $\alpha > 0$ such that $u \in B_X(u,\alpha) \cap S_X \subseteq S_X \cap f^{-1}(\{1\})$. In virtue of the previous item, $B_X(u, \frac{\alpha}{2}) \cap f^{-1}(\{1\}) \subseteq S_X$, then, we can find $s < 0$ sufficiently small such that $(1-s)u + sv \in B_X(u, \frac{\alpha}{2})$. Notice that $[x,y] \subseteq [u,v]$ and $[x,y] \subseteq f^{-1}(\{1\})$, then $[u,v] \subseteq f^{-1}(\{1\})$, which clearly implies that $(1-s)u+sv \in f^{-1}(\{1\})$. As a consequence, $(1-s)u+sv \in B_X(u, \frac{\alpha}{2}) \cap f^{-1}(\{1\}) \subseteq S_X$. In particular, $[(1-s)u+sv, v] \subseteq B_X \cap f^{-1}(\{1\}) \subseteq S_X$. We conclude the proof by noticing that $[u,v] \subsetneq [(1-s)u+sv, v] \subseteq S_X$, which contradicts the maximality of $[u,v]$.

3. First, let us see that $B_X(x,r) \cap S_X$ is convex. Let $y,z \in B_X(x,r) \cap S_X$, and $t \in [0,1]$. Notice that $ty + (1-t)z \in B_X(x,r) \cap B_X$, then $\|ty + (1-t)z\| \leq 1$. On the other hand, since $y,z \in B_X(x,r) \cap S_X \subseteq f^{-1}(\{1\})$, then $ty + (1-t)z \in f^{-1}(\{1\})$, therefore,

$1 = |f(ty + (1-t)z)| \leq \|ty + (1-t)z\|$. So, we conclude that $\|ty + (1-t)z\| = 1$, which shows that $B_X(x,r) \cap S_X$ is convex under our hypotheses. Now, let us prove that $r \leq 1$ by contradiction. Assume that $r > 1$. Notice that $x \in S_X = \mathrm{cl}(S_X \setminus \{x\})$, because $\dim(X) \geq 2$. Then, there exists $y \in (B_X(x,r) \cap S_X) \setminus \{x\}$. By the convexity of $B_X(x,r) \cap S_X$, $[x,y] \subseteq B_X(x,r) \cap S_X$. By using Remark 4.3.8, the maximal segment of $B_X(x,r)$ containing $[x,y]$ is

$$\left[\left(1 + \frac{r}{\|x-y\|}\right)x - \frac{r}{\|x-y\|}y, \left(1 - \frac{r}{\|x-y\|}\right)x + \frac{r}{\|x-y\|}\right] =: [z_1, z_2].$$

We denote by $[u,v]$ the maximal segment of S_X containing $[x,y]$. Let us distinguish between two options:

- $[z_1,z_2] \subseteq [u,v]$. Since $[u,v] \subseteq S_X$, then $z_1, z_2 \in S_X$, which take us to the contradiction

$$\begin{aligned}
2 = \mathrm{diam}(B_X) &\geq \|z_2 - z_1\| = \\
&= \left\| \left(\left(1 - \frac{r}{\|x-y\|}\right)x + \frac{r}{\|x-y\|}y \right) - \left(\left(1 + \frac{r}{\|x-y\|}\right)x - \frac{r}{\|x-y}y \right) \right\| \\
&= \left\| \cancel{x} - \frac{r}{\|x-y\|}x + \frac{r}{\|x-y\|}y \cancel{-x} - \frac{r}{\|x-y\|}x + \frac{r}{\|x-y\|}y \right\| \\
&= \left\| 2\frac{r}{\|x-y\|}(y-x) \right\| = 2r > 2.
\end{aligned}$$

- $[z_1,z_2] \nsubseteq [u,v]$. In this case, either u or v are in the interior of the segment, $u \in (z_1,z_2)$ or $v \in (z_1,z_2)$. Keeping in mind Remark 4.3.8, z_1, z_2 are the only ones laying in $S_X(x,r)$, which means that $(z_1,z_2) \subseteq U_X(x,r)$. We suppose with no loss of generality that $u \in U_X(x,r)$. Let $\delta > 0$ such that $B_X(u,\delta) \subseteq U_X(x,r)$. Besides, since $[x,y] \subseteq [u,v]$ and $[x,y] \subseteq B_X(x,r) \cap S_X \subseteq f^{-1}(\{1\})$, then $[u,v] \subseteq f^{-1}(\{1\})$. By the second item of this lemma, $u \in \mathrm{cl}\big(S_X \setminus f^{-1}(\{1\})\big)$, which allows us to find $u' \in \big(S_X \setminus f^{-1}(\{1\})\big) \cap B_X(u,\delta) \subseteq U_X(x,r) \subseteq B_X(x,r)$. Then, $u' \in (B_X(x,r) \cap S_X) \setminus f^{-1}(\{1\})$, which contradicts the hypothesis of $B_X(x,r) \cap S_X \setminus f^{-1}(\{1\})$.

Therefore, $r \leq 1$. As a consequence, the case $[z_1,z_2] \subseteq [u,v]$ is possible, but not the second one, which implies that $[z_1,z_2] \subseteq S_X$, completing this proof.

4. Suppose to the contrary that $r > 1$. Since $\dim(X) \geq 2$, there exists $y \in \big(B_X(x,r) \cap f^{-1}(\{1\})\big) \setminus \{x\}$. Let us distinguish two cases:

   - Suppose that $\|y-x\| > 1$. Notice that $2x - y \in B_X(x,r) \cap f^{-1}(\{1\})$. Therefore, we reach to the contradiction

$$2 = \mathrm{diam}\,(B_X) \geq \|(2x-y)-y\| = 2\|x-y\| > 2.$$

   - Suppose now that $\|y-x\| \leq 1$. We denote by $z_2 := \frac{r}{\|y-x\|}y + \left(1 - \frac{r}{\|y-x\|}\right)x \in B_X(x,r) \cap f^{-1}(\{1\})$, then $\|z_2 - x\| = r > 1$. Using the same reasoning as above, we reach to the contradiction that $\mathrm{diam}(B_X) > 2$.

Therefore, $r \leq 1$. To conclude the proof, let us show that there exists s with $0 < s < r$ such that $B_X(x,s) \cap S_X \subseteq f^{-1}(\{1\})$. Take $s \in (0,r)$ satisfying that $s + \frac{s}{1-s} \leq r$, and $y \in B_X(x,s) \cap S_X$. On the one side, $f(y) = f(x) - f(x-y) = 1 - f(x-y) \geq$

$1-\|x-y\|\geq 1-s$. On the other side,

$$\begin{aligned}\left\|x-\frac{y}{f(y)}\right\| &\leq \|x-y\|+\left\|y-\frac{y}{f(y)}\right\| \\ &= \|x-y\|+\frac{\|f(y)y-y\|}{|f(y)|} \\ &= \|x-y\|+\frac{|f(y)-1|\|y\|}{|f(y)|} \\ &= \|x-y\|+\frac{|f(y)-f(x)|}{|f(y)|} \\ &\leq \|x-y\|+\frac{\|y-x\|}{|f(y)|} \\ &\leq s+\frac{s}{1-s} \\ &\leq r.\end{aligned}$$

Therefore, $\frac{y}{f(y)}\in B_X(x,r)\cap f^{-1}(\{1\})\subseteq S_X$. As a consequence, $\|\frac{y}{f(y)}\|=1$, concluding that $f(y)=1$.

□

The next proposition is an extension of (34, Lemma 2.1). Also, it will show us one of the cases when faces and facets coincide.

Proposition 4.3.10. *Let X be a normed space and $C\subsetneq S_X$ a convex subset of the unit sphere. If $f\in S_{X^*}$ satisfies that $C\subseteq F(f)$, then* $\mathrm{int}_{S_X}(C)=\mathrm{int}_{f^{-1}(\{1\})}(C)$ *and* $\mathrm{bd}_{S_X}(C)=\mathrm{bd}_{f^{-1}(\{1\})}(C)$. *In particular,* $E(f)=\mathrm{bd}_{S_X}(F(f))$.

Proof. $\subseteq$) Let $x\in\mathrm{int}_{S_X}(C)\neq\varnothing$, then, there exists $r>0$ such that

$$B_X(x,r)\cap S_X\subseteq C\subseteq F(f)\subseteq f^{-1}(\{1\}). \tag{4.3.1}$$

We will show that $B_X(x, \frac{r}{2}) \cap f^{-1}(\{1\}) \subseteq C$, which implies that $x \in \mathrm{int}_{f^{-1}(\{1\})}(C)$. Notice that, by Lemma (4.3.9)(1) and the previous contention (4.3.1), we have that $B_X(x, \frac{r}{2}) \cap f^{-1}(\{1\}) \subseteq S_X$. Taking into consideration this with (4.3.1) again, we have that $B_X(x, \frac{r}{2}) \cap f^{-1}(\{1\}) \subseteq C$, which was our target.

$\supseteq$) Conversely, fix $x \in \mathrm{int}_{f^{-1}(\{1\})}(C) \neq \varnothing$. Then,

$$\exists r > 0 : \; B_X(x, r) \cap f^{-1}(\{1\}) \subseteq C \subseteq S_X. \tag{4.3.2}$$

This allows us to use Lemma 4.3.9(4), which asserts that $r \leq 1$ and there exists s such that $0 < s < r$ and $B_X(x, s) \cap S_X \subseteq f^{-1}(\{1\})$. In view of the contention (4.3.2), it only suffices to show that $B_X(x, s) \cap S_X \subseteq f^{-1}(\{1\})$ to reach to $B_X(x, s) \cap S_X \subseteq C$, which means that $x \in \mathrm{int}_{S_X}(C)$.

Finally, notice that either S_X and $f^{-1}(\{1\})$ are closed in X, so we have
$\mathrm{cl}(C) = \mathrm{cl}_{S_X}(C) = \mathrm{cl}_{f^{-1}(\{1\})}(C)$, then

$$\mathrm{bd}_{S_X}(C) = \mathrm{cl}_{S_X}(C) \setminus \mathrm{int}_{S_X}(C) = \mathrm{cl}_{f^{-1}(\{1\})}(C) \setminus \mathrm{int}_{f^{-1}(\{1\})}(C) = \mathrm{bd}_{f^{-1}(\{1\})}(C).$$

□

The next lemma shows some properties and notions about the behaviour of facets. Notice that the first item is known thanks to (34, Theorem 2.8) as we have mentioned in the previous chapter, but we give here its proof in sake of completeness. Besides, the last point gives an equality about the inner points of a facet.

Lemma 4.3.11. *For a normed space X and a facet $C \subset S_X$, we have the following assertions:*

1. *C is a convex component of S_X.*

2. *There exists a unique functional $f \in S_{X^*}$ such that $C \subseteq F(f)$. In fact, $C = F(f)$.*

3. *If $c \in \operatorname{int}_{S_X}(C)$ and $f \in \nu(c)$, then $C = F(f)$. This is equivalent to say that if $f(c) = 1$ for some $c \in \operatorname{int}_{S_X}(C)$ and $f \in S_{X^*}$, then $f(c) = 1$ for all point $c \in C$ in the facet.*

4. *$\operatorname{int}_{S_X}(C) \subseteq \operatorname{smo}(B_X)$.*

5. *If $c \in \operatorname{int}_{S_X}(C)$ and $f \in \nu(c)$, then $C - c$ is a closed convex neighborhood of 0 in $\ker(f)$. Moreover, $\operatorname{int}_{S_X}(C)$ is dense in C and $\operatorname{inn}(C) = \operatorname{int}_{S_X}(C)$.*

Proof. 1. Let us see the maximality of C by contradiction. Let $D \subseteq S_X$ be a convex subset strictly containing C. By Hahn-Banach Theorem, there exists $f \in S_{X^*}$ such that $C \subseteq D \subseteq F(f)$. Let $d \in D \setminus C \neq \varnothing$ and $c \in \operatorname{int}_{S_X}(C)$. By relying on Proposition 4.3.10, $c \in \operatorname{int}_{S_X}(C) = \operatorname{int}_{f^{-1}(\{1\})}(C)$, then, there exists $r > 0$ such that $B_X(c, r) \cap f^{-1}(\{1\}) \subseteq C$. We have to notice that it is possible to find $t > 1$ sufficient closed to 1 such that $tc + (1-t)d \in B_X(c, r)$. On the other hand, $f(tc + (1-t)d) = 1$, so $tc + (1-t)d \in B_X(c, r) \cap f^{-1}(\{1\})$. Since C is a facet, in particular, is a face, concluding that $d \in C$, which is a contradiction.

2. The maximality of C implies the existence of $f \in S_{X^*}$ such that $C = F(f)$. Suppose on the contrary that there exists $g \in S_{X^*}$ verifying $C = F(g)$. By Proposition 4.3.10, $\operatorname{int}_{S_X}(C) = \operatorname{int}_{f^{-1}(1)}(C) = \operatorname{int}_{g^{-1}(\{1\})}(C)$. Fix an arbitrary $c \in \operatorname{int}_{S_X}(C)$. Notice that $\operatorname{int}_{\ker(f))}(C - c) = \operatorname{int}_{\ker(g)}(C - c)$. Besides, observe that $C \subseteq f^{-1}(\{1\}) \cap g^{-1}(\{1\})$, therefore $C - c \subseteq \ker(f) \cap \ker(g)$. Thus, $\ker(f) \cap \ker(g)$ is a subspace of $\ker(f)$ with non-empty interior, so $\ker(f) \cap \ker(g) = \ker(f)$. The same reasoning serves to see that $\ker(f) \cap \ker(g) = \ker(f)$, implying that $\ker(f) = \ker(g)$, then they are proportionals.

Since the functionals coincide in every element of C, we have that $f = g$.

3. Let us find a contradiction. Let $y \in C \setminus F(f)$, then $f(y) < 1$. Take $g \in S_{X^*}$ such that $C = F(g)$. By Proposition 4.3.10, $c \in \operatorname{int}_{S_X}(C) = \operatorname{int}_{g^{-1}(\{1\})}$, thus there exists $r > 0$ such that $B_X(c,r) \cap g^{-1}(\{1\}) \subseteq C$. This means that we can find $t < 0$ sufficient closed to 0 such that $ty + (1-t)c \in B_X(c,r)$. We notice that since $y, c \in g^{-1}(\{1\})$, then $sy + (1-s)c \in g^{-1}(\{1\})$, for all $s \in \mathbb{R}$, in particular, it is true for $s = t$, that is, $ty + (1-t)c \in B_X(c,r) \cap g^{-1}(\{1\}) \subseteq C \subseteq S_X$. Therefore, we reach to the contradiction $1 \geq f(ty + (1-t)c) = tf(y) + (1-t)f(c) = 1 - t(1 - f(y)) > 1$.

4. If $c \in \operatorname{int}_{S_X}(C)$, and $f, g \in \nu(c)$ then, by previous item $C = F(f) = F(g)$. By (2), it holds that $f = g$, which is the definition of smooth point.

5. First of all, notice that, $c \in \operatorname{int}_{S_X}(C) = \operatorname{int}_{f^{-1}}(\{1\})$, so C is a closed $(C = F(f) = f^{-1}(\{1\}) \cap B_X)$ convex neighborhood of c in $f^{-1}(\{1\})$. Now, consider the translation

$$X \to X$$

$$x \mapsto x - c$$

which is an homeomorphism mapping $f^{-1}(\{1\})$ to $\ker(f)$, C to $C - c$, $\operatorname{int}_{f^{-1}(\{1\})}$ to $\operatorname{int}_{\ker(f)}(C - c)$ and c to 0. Then, $C - c$ is a closed convex neighborhood of 0 in $\ker(f)$. To see the density of $\operatorname{int}_{S_X}(C)$ in C, it is well-known that a convex subset C with non empty interior in a normed space X satisfies $\operatorname{cl}(C) = \operatorname{cl}(\operatorname{int}(C))$, in our case, $C - c$ is a convex subset of $\ker(f)$ with non-empty interior, then

$$\operatorname{cl}_{\ker(f)}(\operatorname{int}_{\ker(f)}(C - c)) = \operatorname{cl}_{\ker(f)}(C - c).$$

If we undo the translation and take into consideration that C and $f^{-1}(C)$ are closed, we obtain

$$\mathrm{cl}\left(\mathrm{int}_{S_X}(C)\right) = \mathrm{cl}\left(\mathrm{int}_{f^{-1}(\{1\})}(C)\right) = \mathrm{cl}_{f^{-1}(\{1\})}\left(\mathrm{int}_{f^{-1}(\{1\})}(C)\right) = \mathrm{cl}_{f^{-1}}(C) = \mathrm{cl}(C) = C,$$

so $\mathrm{int}_{S_X}(C)$ is dense in C. Finally, to see that $\mathrm{inn}(C) = \mathrm{int}_{S_X}(C)$, we use again the fact that $C - c$ is a convex set with non-empty interior in $\ker(f)$, then $\mathrm{int}_{\ker(f)}(C - c) = \mathrm{inn}(C)$ by Lemma 3.2.29(3). To conclude that $\mathrm{inn}(C) = \mathrm{int}_{f^{-1}(\{1\})}(C) = \mathrm{int}_{S_X}(C)$ it is enough to take into consideration that translations preserve inner points by Proposition 3.2.31.

□

The next result present a new reformulation of the frame of the unit ball. This characterization will be very useful to provide an immediate proof of its invariance in the following chapter (see Theorem 5.2.1). This invariance was already proved by Tanaka in (1, Theorem 3.7).

Theorem 4.3.12. *Let X be a normed space. Then*

$$\mathrm{frm}(B_X) = S_X \setminus \bigcup_{C \in \mathscr{C}_X} \mathrm{int}_{S_X}(C),$$

where $\mathscr{C}_X$ is the notation given in Definition 3.2.1 for the set of all the facets in the unit ball.

Proof. $\supseteq$) Let $x \in S_X \setminus \cup_{C \in \mathscr{C}_{\mathscr{X}}} \mathrm{int}_{S_X} C$. By Hahn-Banach Theorem, there exists $f \in S_{X^*}$ such that $x \in F(f)$. Let us distinguish cases:

- If $\mathrm{int}_{S_X}(F(f)) = \varnothing$, by Proposition 4.3.10, $F(f) = E(f)$, then $x \in E(f) \subseteq \mathrm{frm}(B_X)$, because of the definition of the frame.

- If $\mathrm{int}_{S_X}(F(f)) \neq \varnothing$, then $F(f)$ is a facet, $F(f) \in \mathscr{C}_X$, since $x \notin \mathrm{int}_{S_X}(F(f))$, therefore $x \in F(f) \setminus \mathrm{int}_{S_X}(F(f)) = \mathrm{bd}_{S_X}(F(f)) = E(f) \subseteq \mathrm{frm}(B_X)$.

$\subseteq$) Conversely, let $x \in \mathrm{frm}(B_X) = \bigcup \left\{E(f) : f \in \cup_{x \in S_X} \nu(x)\right\}$. Suppose to the contrary that $\exists C \in \mathscr{C}_X$ such that $x \in \mathrm{int}_{S_X}(C)$. By Lemma 4.3.11(4), $x \in \mathrm{int}_{S_X}(C) \subseteq \mathrm{smo}(B_X)$, then, there exists a unique functional $f \in S_{X^*}$ such that $x \in C = F(f)$. By hypotheses and the definition of the frame, there exists $g \in S_{X^*}$ such that $x \in E(g) \subseteq F(f)$, but f was unique, so $g = f$. Thus, on one hand, $x \in \mathrm{int}_{S_X}(C) = \mathrm{int}_{S_X}(F(f))$ and $x \in E(f) = \mathrm{bd}_{S_X}(F(f))$, which is a contradiction.

□

We obtain the following result as a direct consequence of Theorem 4.3.12.

Corollary 4.3.13. *Let X be a normed space, the following assertions are equivalent:*

- $\mathrm{int}_{S_X}(\mathrm{frm}(B_X)) = \varnothing$.
- $\cup_{C \in \mathscr{C}_X}(C)$ *is dense in* S_X.

4.4 Flatness

The concept of starlike hull was introduced for the first time in (6, Definition 11) and it was studied for general starlike sets. In this section, we present new and original terms related to starlike sets which fits our purposes much better. Also, we will widely develop these notions and study their invariance in the following chapter.

Definition 4.4.1 (Starlike envelope, almost flat, flat, starlike compatible, starlike generated sets). *Let X be a normed space and $E \subseteq S_X$. The starlike envelope of E is defined as follows*

$$\mathrm{st}(E) := \bigcap_{e \in E} \mathrm{st}(e, B_X).$$

Furthermore, we will say that

- *E is almost flat if $[e, f] \subseteq S_X$ for every $e, f \in E$.*
- *E is flat if $\mathrm{co}(E) \subseteq S_X$.*
- *E is starlike compatible if $E \subseteq \mathrm{st}(E)$.*
- *E is starlike generated if $E = \mathrm{st}(E)$.*

It is clear that every flat set is almost flat and it is easy to check that every subset of the unit sphere verifies that is almost flat if and only if is starlike compatible. Notice also that every convex subset of the unit sphere is trivially flat. In general, this assertion holds for every subset of the unit sphere contained in a convex subset of S_X. As a consequence of that and the Hahn-Banach Separation Theorem, a subset of the unit sphere is flat if and only if it is contained in an exposed face of the unit ball. Let us study the relations between the concepts provided in Definition 4.4.1 in the next lemma.

Lemma 4.4.2. *Let X be a normed space and $E \subseteq S_X$, then:*

1. *If E is convex, then E is flat and starlike compatible.*
2. *E is almost flat if and only if E is starlike compatible.*
3. *If E is flat, then it is almost flat.*
4. *E is flat if and only if $\mathrm{co}(E) \subseteq \mathrm{st}(E)$.*

5. *If E is flat and D is a convex component of the unit sphere containing E, then $D \subseteq \mathrm{st}(E)$.*

6. *If E is a convex component of S_X, then E is starlike generated.*

7. *If E is convex and starlike generated, then E is a convex component.*

Proof. 1. If E is convex, then E trivially satisfies the condition of flatness. Now, notice that since E is convex, for every $e, f \in E \subseteq S_X$, then $[e, f] \subseteq S_X$, which means that $f \in \mathrm{st}(e, B_X)$. This implies that $E \subseteq \mathrm{st}(e, B_X)$ for all e. As a consequence, $E \subseteq \cap_{e \in E} \mathrm{st}(e, B_X) = \mathrm{st}(E)$, meaning that E is starlike compatible.

2. It is direct by taking into consideration the Equality 3.2.6.

3. It is also trivial just by bearing in mind the definitions given in 4.4.1.

4. $\Rightarrow$) Suppose that E is flat, then $\mathrm{co}(E) \subseteq S_X$. Thus, $[x, e] \subseteq \mathrm{co}(E) \subseteq S_X$ for all $x \in \mathrm{co}(E)$, which means that $x \in \mathrm{st}(e, B_X)$ for all $x \in \mathrm{co}(E)$ and $e \in E$ arbitrary. This implies that $\mathrm{co}(E) \subseteq \cap_{e \in E} \mathrm{st}(e, B_X) = \mathrm{st}(E)$.

   $\Leftarrow$) Conversely, if $\mathrm{co}(E) \subseteq \mathrm{st}(E) \subseteq S_X$, we have that E is flat.

5. Let $d \in D$. Since E is flat and D is a maximal convex subset of S_X, it holds that for every $e \in E$, $[e, d] \subseteq D \subseteq S_X$, then $d \in \mathrm{st}(E)$. Since d is arbitrary, $D \subseteq \mathrm{st}(E)$.

6. By 1. of this lemma, we know that $E \subseteq \mathrm{st}(E)$ since E is convex. Let us see the other contention. Fix an arbitrary $x \in \mathrm{st}(E)$ and notice that

$$E \subseteq \mathrm{co}(E \cup \{x\}) = \bigcup_{e \in E} [x, e] \subseteq S_X,$$

   where the last contention is given by $x \in \mathrm{st}(E)$, then $[x, e] \subseteq S_X$ for all $e \in E$. The maximality of E implies that $E = \mathrm{co}(E \cup \{x\})$, thus $x \in E$. Therefore, we have proved that $E = \mathrm{st}(E)$, which is the definition of starlike generated.

7. suppose that D is a convex subset of S_X containing C. Let $d \in D$ and $e \in E$. The convexity of D shows that $[e, d] \subseteq D \subseteq S_X$, then $d \in \mathrm{st}(e, B_X)$. The arbitrariness of $e \in E$ gives as the next
$$d \in \bigcap_{e \in E} \mathrm{st}(e, B_X) = \mathrm{st}(E) = E,$$
then $D \subseteq C$ because $d \in D$ was arbitrary. This proves the maximality of C, concluding that C is a maximal component of S_X.

$\square$

Now we present a pair of examples of almost flat sets which are not flat.

Example 4.4.3. *The construction of this example is strongly based on Remark 4.1.3 and Lemma 4.1.4. The convex polyhedron displayed in Figure 4.2 is a unit ball whose facets equilateral triangles and diamonds; an easy way to construct it is by taking a regular octahedron and placing a regular tetrahedron (with the same faces or triangles) on the top and the opposite one on the bottom. If E is formed by the four vertices of the top tetrahedron, then E is almost flat but it is clearly not flat, since* $\mathrm{co}(E)$ *is the whole regular tetrahedron which is not contained in the boundary of this unit ball. We also want to notice that this unit ball is a convex polyhedron, then,* $\mathbb{R}^3$ *endowed with this unit ball satisfies the MUp by (11, Theorem 4.5). Also, any of the diamonds that compose the boundary of this unit ball is a maximal convex component, then, starlike generated by Lemma 4.4.2(6).*

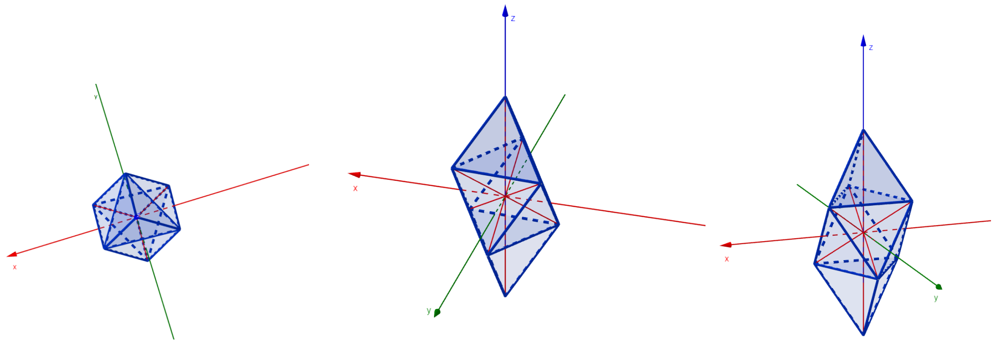

Figure 4.2: Pacheco-Campos unit ball

The key of failing the flat condition is to consider a set formed by the four vertices: three of them of the same face and the other one in an adjacent face. Keeping that in mind, it is easy to give a simpler example of an almost flat set which is not flat, specifically, it is enough to consider the set $E = \{A, B, C, D\}$ in the Figure 4.3. As we can see, all the possible segments whose extremes bellowing to E lay in the unit sphere, but the convex hull of E has points of the unit ball which are not in the sphere, implying that E is not flat.

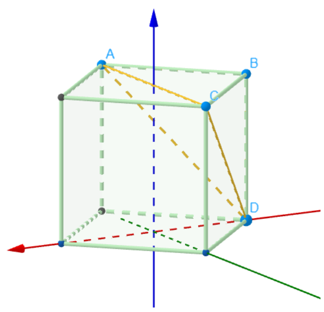

Figure 4.3: Another example of an almost flat set which is not flat

We will make use of the original Example 4.4.3 to construct a non-convex starlike generated set (see the green set in Figure 4.4).

Theorem 4.4.4. *Let us consider the upper regular tetrahedron of the unit ball given in Example*

4.4.3. If we denote by a the upper vertex and by b, c, d the ones forming the base of that regular upper tetrahedron, then $E := [a, b] \cup [a, c] \cup [a, d]$ is a non-convex starlike generated set.

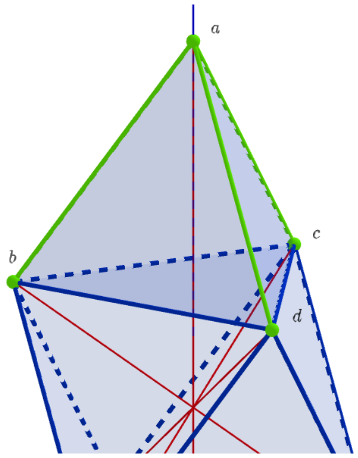

Figure 4.4: An example of a non-convex starlike generated set

Proof. It is clear that E is not convex. Notice that E is almost flat by definition, then, by Lemma 4.4.2(2), E is starlike compatible, meaning that $E \subseteq \mathrm{st}(E)$. It only remains to show that $\mathrm{st}(E) \subseteq E$. Suppose to the contrary that there exists $x \in \mathrm{st}(E) \setminus E$. First, observe that $\mathrm{st}(E) = \cap_{e \in E} \mathrm{st}(e, B_X) \subseteq \mathrm{st}(a, B_X) = \mathrm{co}(\{a, b, c\}) \cup \mathrm{co}(\{a, c, d\}) \cup \mathrm{co}(\{a, b, d\})$. Then, x is in the interior of one of these three faces $\mathrm{co}(\{a, b, c\}), \mathrm{co}(\{a, c, d\}), \mathrm{co}(\{a, b, d\})$. Suppose, for example and without any loss of generality, that $x \in \mathrm{co}(\{a, b, c\})$. Then, $(x, d) \subset U_X$, which contradicts the fact that $x \in \mathrm{st}(E) \subseteq \mathrm{st}(d, B_X)$. $\square$

The examples given in Figures 4.2 and 4.3 motivate the following property.

Definition 4.4.5 (Flat property)**.** *We will say that a normed space has the flat property or the F-property (Fp) if every almost flat of its unit ball is flat.*

Remark 4.4.6. *Note that $\mathbb{R}^3$ endowed with the unit ball constructed in Example 4.4.3 satisfied the MUp. However, it does not satisfied the Flat property, as we have shown in Theorem 4.4.4.*

Examples above shows the existence of Banach spaces lacking the F-property. In fact, let us show that it is possible to equivalently renormed a Banach space (with dimension greater or equal to 3) to fail Fp.

Theorem 4.4.7. *Let X be a Banach space with dimension* $\dim(X) \geq 3$. *Then, there exists an equivalent norm on X for which X fails Fp.*

Proof. Let us consider a 3-dimensional Banach space $Y \subset X$. Then, there exists a closed subspace $Z \subseteq X$ satisfying $X = Y \oplus Z$. It holds that Y is isomorphic to the 3-dimensional Banach space given in Example 4.4.3, so it is possible to endow Y with the equivalent norm given by the unit ball of Example 4.4.3, and we call the space with this norm by Y'. If we keep the norm considered in Z, then X is isomorphic to $X' := Y' \oplus_2 Z$. Finally, notice that X' fails Pp since so does Y'. □

The following lemma gives a sufficient condition for an almost flat set to be flat. It also provides conditions for a convex subset of the unit ball to be contained in the unit sphere, in particular, it will be enough the convex subset to contain inner points which are also in the unit sphere.

Lemma 4.4.8. *Let $D \subseteq B_X$ be a convex subset of the unit ball of a normed space X. Then:*

1. *If* $\operatorname{inn}(D) \cap S_X \neq \emptyset$, *then $D \subseteq S_X$.*
2. *If $D \subseteq S_X$ is an almost flat subset satisfying that $\exists d \in D$ such that $d \in$* $\operatorname{inn}(\operatorname{co}(D))$, *then D is flat.*

Proof. 1. Fix an arbitrary $d_0 \in \operatorname{inn}(D) \cap S_X$ and take $d \in D \setminus \{d_0\}$. Since d is an inner points, by definition, there exists $e \in D \setminus \{d, d_0\}$ such that $d_0 \in (d, e)$. As $d_0 \in S_X$ and

$d, e \in D \subseteq B_X$, the only possibility is that $[d, e] \subseteq S_X$.

2. We strongly based in the assertion previously proved: because $d \in \operatorname{inn}(\operatorname{co}(D)) \cap S_X$, then, $\operatorname{co}(D) \subseteq S_X$, which is the definition of flatness for D.

□

Next lemma presents some generalization of the results belonging to (5). In particular, Lemma 4.4.9(1) is a generalization to infinite dimensional normed spaces of (5, Lemma 2), and Lemma 4.4.9(2,5) constitutes the infinite dimensional version of (5, Corollary 3). This result was firstly proved in (3, Lemma 8) and it gives a deeper perspective about starlike sets.

Lemma 4.4.9. *For a normed space X and $E \subseteq S_X$, the following holds:*

1. *Let E be a face, $e \in \operatorname{inn}(E)$ and $y \in S_X$ satisfying that $[e, y] \subseteq S_X$, then $\operatorname{co}(E \cup \{y\}) \subseteq S_X$ and $E \subseteq \operatorname{st}(y, B_X)$.*

2. *If E is a convex component of the unit sphere S_X and $\exists e \in E$ for which E is the only convex component of S_X containing e, then it has to be its starlike set, $E = \operatorname{st}(e, B_X)$.*

3. *If E is a facet, then for every $e \in \operatorname{int}_{S_X}(E)$, E in the only convex component of S_X containing e.*

4. *If E is a maximal convex subset of S_X such that there exists a dense sequence $(e_n)_{n\in\mathbb{N}}$ in E satisfying that $\sum_{n=1}^{\infty} \frac{e_n}{2^n}$ is convergent, then E is the only convex component containing $e := \sum_{n=1}^{\infty} \frac{e_n}{2^n}$.*

5. *If E is a maximal face of B_X with inner points, $\operatorname{inn}(E) \neq \emptyset$, then E is the only convex component of S_X containing its inner points $e \in \operatorname{inn}(E)$.*

Proof. 1. Notice that $\mathrm{co}(E \cup \{y\}) = \cup_{d \in E}[d, y]$. Fix $d \in E \setminus \{e\}$ and $[e, y] \subseteq S_X$ by hypothesis. Since $e \in \mathrm{inn}(E)$, there exists $c \in E \setminus \{e, d\}$ such that $e \in (d, c)$.

- If c, d, y are aligned, then, the extremal condition of E guarantees that $y \in E$. Besides, the convexity of E implies that $\mathrm{co}(E \cup \{y\}) = E \subseteq S_X$. Finally, keeping in mind the definition of starlike set given in Equation 3.2.6, $E \subseteq \mathrm{st}(y, B_X)$.

- If c, d, y are not aligned, by Remark 3.2.28, $(e, y) \subseteq \mathrm{inn}(\mathrm{co}(\{d, c, y\})) = \{rd + sc + ty : r, s, t \in (0, 1), r + s + t = 1\}$, and $(e, y) \subseteq [e, y] \subseteq S_X$, thus, by Lemma 4.4.8(1) $\mathrm{co}(\{d, c, y\}) \subseteq S_X$. In particular, $[d, y] \subseteq S_X$. The arbitrariness of $d \in E \setminus \{e\}$ implies that $\mathrm{co}(e \cup \{y\}) = \cup_{d \in E}[d, y] \subseteq S_X$. This reasoning together Equation 3.2.6 implies that $E \subseteq \mathrm{st}(y, B_X) = \{x \in S_X : [x, y] \subseteq S_X\}$.

2. $\subseteq$) Since E is convex, by Lemma 4.4.2(1), E is starlike compatible, which means that $E \subseteq \mathrm{st}(E) = \cup_{d \in E} \mathrm{st}(d, B_X) \subseteq \mathrm{st}(e, B_X)$.

   $\supseteq$) Let $x \in \mathrm{st}(e, B_X)$. By Equation 3.2.6, $[x, e] \subseteq S_X$. Let D be the convex component of S_X containing $[x, e]$, in particular $e \in [x, e] \subseteq D$. But, by hypothesis, $D = E$, concluding that $x \in E$.

3. It is sufficient to take into consideration Lemma 4.3.11(4) to claim that $e \in \mathrm{int}_{S_X}(E)$ is a smooth point of B_X, then E is the only maximal convex subset of S_X satisfying $e \in E$.

4. Id $D \subseteq S_X$ is the maximal convex subset of S_X containing e, we know that there exists $g \in S_{X^*}$ such that $D = F(g)$. Notice that $g(e) = 1$ since $e \in D$. Therefore, $g(e_n) = 1$ for all $n \in \mathbb{N}$, and the density of $(e_n)_{n \in \mathbb{N}}$ implies that $g(E) = \{1\}$, implying that $E \subseteq F(g) = D$, which contradicts the maximality of E.

5. We will follow a similar reasoning to the first item proved in this lemma. Suppose on

the contrary that there exists D another convex component of S_X containing $e \in \text{inn}(E)$, and fix $d \in E \setminus \{e\}$. It holds that there exists $c \in E \setminus \{e, d\}$ such that $e \in (d, c)$. The extremal condition satisfied by D implies that $d, c \in D$, but the arbitrariness of $d \in E \setminus \{e\}$ concludes that $E \subseteq D$. Finally, the maximality of E forces that $E = D$.

□

Next result is borrowed from (3, Theorem 9). This theorem gives us two important properties about starlike sets: the first one is that every starlike set satisfies the extremal condition. The second one is about the uniqueness respect to the maximal faces containing the centre of the starlike set when we asked for convexity.

Theorem 4.4.10. *For every point $x \in S_X$ of the unit sphere of a normed space X, its related starlike set $\text{st}(x, B_X)$ satisfies the extremal condition respect to B_X. Moreover, if $\text{st}(x, B_X)$ is convex, then $\text{st}(x, B_X)$ is the only maximal face of B_X containing x.*

Proof. Let us check the extremal condition for $\text{st}(x, B_X)$: let $y, z \in B_X$ and $t \in (0, 1)$ such that $ty + (1-t)z \in \text{st}(x, B_X)$. By the equality given in 3.2.6,

$$\text{st}(x, B_X) = \bigcup \{C \subseteq S_X : C \text{ is a maximal face of containing } x\},$$

we know that there exists a maximal face C such that $ty + (1-t)z \in C$. Since C satisfies the extremal condition, $y, z \in C \subseteq \text{st}(x, B_X)$.Now suppose that $\text{st}(x, B_X)$ is also convex. Then, $\text{st}(x, B_X)$ verifies the definition of face, and let D be a maximal face of B_X containing x. Again, by Equation 3.2.6, $D \subseteq \text{st}(x, B_X)$, concluding the uniqueness of $\text{st}(x, B_X)$. □

The set of rotund points of the unit ball can be described in terms of the starlike sets as follows.

Remark 4.4.11. *For a normed space X,*

$$\operatorname{rot}(B_X) = \{x \in S_X : \operatorname{st}(x, B_X) = \{x\}\}.$$

This characterization of rotund points through singleton starlike sets allows us to express rotund points in another form.

Remark 4.4.12. *A rotund point $x \in \operatorname{rot}(B_X)$, where X is a normed space, is a singleton maximal face $\{x\}$ of S_X. This is equivalent to next condition: for all $y \in B_X$ such that $\left\|\frac{x+y}{2}\right\| = 1$, then $y = x$. Indeed, suppose on the first place that $x \in \operatorname{rot}(B_X)$, and let $y \in B_Y$ such that $\left\|\frac{x+y}{2}\right\| = 1$, equivalently, $\|x + y\| = 2$. By Equation 3.2.6 and Remark 4.4.11, $y \in \operatorname{st}(x, B_X) = \{x\}$, then $x = y$. Conversely, assume that for all $y \in B_X$ satisfying $\|y + x\| = 2$, then $y = x$. This fact forces that $\operatorname{st}(x, B_X) = \{y \in B_X : \|x + y\| = 2\} = \{x\}$. Again, in view of Remark 4.4.11, $x \in \operatorname{rot}(B_X)$.*

Next result combined with Theorem 4.4.10 constituted a generalization of (54, Lemma 2.7). It also gives a sufficient condition for a starlike set to be convex.

Theorem 4.4.13. *Let $x \in \operatorname{smo}(B_X)$ be a smooth point of the unit ball of a normed space X. Then, $\operatorname{st}(x, B_X) = F(\nu(x))$. In particular, $\operatorname{st}(x, B_X)$ is convex.*

Proof. The smoothness of x means that there exists a unique functional, let us denote it by $\nu(x)$, attaining its norm at x. This implies that there is only one exposed face of B_X containing x, which is $F(\nu(x))$. This exposed face is also the only maximal face of B_X containing x. By Equation 3.2.6, $\operatorname{st}(x, B_X)$ is forced to be that maximal face, implying the convexity of the starlike set. □

CHAPTER

5

Geometry of the unit ball under surjective isometries of the unit sphere

Contents

The main topic of this chapter is the study of the invariants of the unit sphere under surjective isometries. During the investigation of the unit ball, we have realized that the proofs of some already known invariants could be simplified by using the our own investigation. This is the goal of the first section, where we present different proofs for the invariance of the starlike sets (see Theorem 5.1.1) or the equation $T(-\mathrm{st}(x,B_X)) = -T(\mathrm{st}(x,B_X))$, with T our surjective isometry (see Corollary 5.1.2). The second section of this chapter shows, on the one hand, that the new geometrical terms defined in the flatness study are also invariants under surjective isometries (see Theorem 5.2.3) and, on the other hand, the invariance of faces in arbitrary Banach spaces by relying on the P-Property (Theorem 5.2.4). As far as we know, faces have not been proved as invariants under surjective isometries. In this section, we also proved in a simpler way the invariance of the frame by using the reformulation given in Theorem 4.3.12 and the invariance of the set of rotund points (Theorem 5.2.8). As a consequence of this result, we conclude the preservation of antipodal rotund points, that is, $T(-x) = -T(x)$ for all rotund point x and T, the surjective isometry under the same hypotheses as always, and we also deduce the invariance of the strict convexity of the large Banach space X. The latter invariance motivates to define the Inner Property (see Definition 5.2.10). Spaces with this property are under the hypotheses of Theorem 5.2.15 (as we have gathered in Corollary 5.2.16), which is a generalization of Tingley's results (5, Lemma 12 and 13) to infinite dimensions. We end this chapter by proving the invariance of segments under certain circumstances in Theorem 5.3.1.

5.1 New simpler proofs for already proved invariants

Along the study of the invariants under surjective isometries and the geometry of the unit ball, we have given new (simpler) proofs for already known results. The first we present here is the invariance of the starlike sets under surjective isometries defined between unit spheres. This invariance for the starlike sets was stated and proved in (9, Corollary 2.2), and so it does (7, Corollary 5.2), where the proof was omitted. Our different and more direct proof is based on Theorem 3.2.9.

Theorem 5.1.1. *Let $T : S_X \to S_Y$ be a surjective isometry defined between the unit spheres of two Banach spaces X, Y. Then, $T(\mathrm{st}(x, B_X)) = \mathrm{st}(T(x), B_Y)$, for every $x \in S_X$.*

Proof. Relying on Theorem 3.2.9 and taking into consideration Definition 3.2.5, we have the following chain of equalities:

$$\begin{aligned} T(\mathrm{st}(x, B_X)) &= T\left(\bigcup\{C \subseteq S_X : C \text{ is a maximal face containing } x\}\right) \\ &= \bigcup\{T(C) : T(C) \text{ is a maximal face of } B_X \text{ containing } x\} \\ &= \bigcup\{D \subseteq S_Y : D \text{ is a maximal face of } B_Y \text{ containing } T(x)\} \\ &= \mathrm{st}(T(x), B_Y). \end{aligned}$$

□

In Corollary 5.2.16(2), we provide a different proof of Theorem 5.1.1 for a wide class of Banach spaces containing the finite-dimensional Banach spaces. Besides, Theorem 5.1.1 together with Remark 3.2.17 lead us to the following original result.

Corollary 5.1.2. *Let X, Y be Banach spaces and $T : S_X \to S_Y$ be a surjective isometry between the unit spheres of those Banach spaces. Then, $T(-\mathrm{st}(x, B_X)) = -T(\mathrm{st}(x, B_X))$ for all $x \in S_X$.*

Proof. In virtue of Theorem 5.1.1 and Remark 3.2.17 the next holds:

$$T(-\mathrm{st}(x, B_X)) = T(\mathrm{st}(-x, B_X)) = \mathrm{st}(T(-x), B_Y) = -\mathrm{st}(-T(-x), B_Y) = -T(\mathrm{st}(x, B_X)).$$

□

5.2 Preservation of Flatness and Faces under surjective isometries

We will begin this section by giving a simpler proof of (1, Theorem 3.7) using some results given in the previous study, in detail, Theorem 3.2.9, Remark 3.2.12, and Theorem 4.3.12. The following result can be found for the first time in (3, Theorem 11), and shows that the frame is an invariant under surjective isometries defined between unit spheres.

Theorem 5.2.1. *Let $T : S_X \to S_Y$ be a surjective isometry defined between the unit spheres of Banach spaces X, Y. Then, $T(\operatorname{frm}(B_X)) = \operatorname{frm}(B_Y)$.*

Proof. Notice that $T(\mathscr{C}_X) = \mathscr{C}_Y$ (see Theorem 3.2.9 together with Remark 3.2.12). By bearing in mind that T is an homeomorphism (see Remark 3.2.12), and the reformulation given in Theorem 4.3.12, we have that

$$\begin{aligned} T(\operatorname{frm}(B_X)) = T\left(S_X \setminus \bigcup_{C\in\mathscr{C}_X} \operatorname{int}_{S_X}(C)\right) &= T(S_X) \setminus \bigcup_{C\in\mathscr{C}_X} T(\operatorname{int}_{S_X}(C)) \\ &= T(S_X) \setminus \bigcup_{C\in\mathscr{C}_X} \operatorname{int}_{S_Y}(T(C)) = S_Y \setminus \bigcup_{D\in\mathscr{C}_Y} \operatorname{int}_{S_Y}(D) \\ &= \operatorname{frm}(B_Y). \end{aligned}$$

□

The following example shows that the interior of the frame is a topological invariant under surjective isometries.

Example 5.2.2. *Let X be a Banach space such that $\operatorname{int}_{S_X}(\operatorname{frm}(B_X)) = \varnothing$, and let Y be another Banach space satisfying that there exists a surjective isometry $T : S_X \to S_Y$. According to the previous Theorem 5.2.1, we have that $T(\operatorname{frm}(B_X)) = \operatorname{frm}(B_Y)$. Even more, since T is an* *homeomorphism, it holds*

$$\mathrm{int}_{S_Y}(\mathrm{frm}(B_Y)) = \mathrm{int}_{S_Y}(T(\mathrm{frm}(B_X))) = T(\mathrm{int}_{S_X}(\mathrm{frm}(B_X)) = T(\varnothing) = \varnothing.$$

We would like to make the reader notice that some items of the following result, Theorem 5.2.3(2) and Theorem 5.2.3(5), state the same assertion according to Lemma 4.4.2(2). We have made the differentiation because Theorem 5.2.3(2) relies on Theorem 5.1.1, whereas Theorem 5.2.3(5) does not. The following result shows the invariance of other terms of the unit ball under surjective isometries, for instance, the starlike envelope, and the original proof can be found in (3, Theorem 12).

Theorem 5.2.3. *Let $T : S_X \to S_Y$ be a surjective isometry between the unit spheres of Banach spaces X, Y, and let $E \subseteq S_X$. Then:*

1. $T(\mathrm{st}(E)) = \mathrm{st}(T(E))$.

2. *If E is starlike compatible, so it is $T(E)$.*

3. *If E is starlike generated, then $T(E)$ is starlike generated too.*

4. *If E is flat, then, $T(E)$ is flat.*

5. *If E is almost flat, then $T(E)$ is almost flat.*

Proof. 1. By bearing in mind the invariance for starlike sets given in Theorem 5.1.1, it is easy to observe that

$$
\begin{aligned}
T(\mathrm{st}(E)) = T\left(\bigcap_{e\in E}\mathrm{st}(e,B_X)\right) &= \bigcap_{e\in E} T\left(\mathrm{st}(e,B_X)\right) \\
&= \bigcap_{e\in E}\mathrm{st}(T(e),B_Y) = \bigcap_{d\in T(E)}\mathrm{st}(d,B_Y) = \mathrm{st}(T(E)).
\end{aligned}
$$

2. By hypothesis, E is starlike compatible, so $E\subseteq \mathrm{st}(E)$, then $T(E)\subseteq T(\mathrm{st}(E)) = \mathrm{st}(T(E))$, which means that $T(E)$ is starlike compatible.

3. The proof is very similar to the previous one.

4. By hypothesis, $\mathrm{co}(E)\subseteq S_X$, and let D be a convex maximal subset of S_X, D is a convex component, containing $\mathrm{co}(E)\subseteq D$. The invariance of maximal convex subsets of S_X (Theorem 3.2.9) assures that $T(D)$ is a convex component of S_Y. Besides, since $E\subseteq \mathrm{co}(E)\subseteq D$, then $T(E)\subseteq T(\mathrm{co}(E))\subseteq T(D)$. The convexity of $T(D)$ gives us $\mathrm{co}(T(E))\subseteq T(D)\subseteq S_Y$, which is the condition for $T(E)$ of being flat.

5. We start by fixing two arbitrary elements of E, $e,f\in E$, and we want to prove that $[T(e),T(f)]\subseteq S_Y$ for $T(E)$ to satisfy the almost flat condition. Since E is almost flat by hypothesis, then $[e,f]\subseteq S_X$. Also, there exists a convex component $F\subseteq S_X$ such that $[e,f]\subseteq F$. Again, by Theorem 3.2.9, $T(F)$ is a maximal face of B_Y. Notice also that, since $[e,f]\subseteq F$, then $T([e,f])\in\subseteq T(F)$. This means that $T(e),T(f)\in T(F)$, and the convexity of $T(F)$ assures that the whole segment is contained, this is $[T(e),T(f)]\subseteq T(F)\subseteq S_Y$.

□

Let C be a flat subset of S_X, where X is a normed space. We will denote

$$M_C := \{D \subseteq S_X : \ D \text{ is a maximal face of } B_X \text{ containing } C\}.$$

When $C = \{c\}$ is a singleton, we will simply write M_c. Observe that $M_C = M_{\mathrm{co}(C)}$.

Next result presents the invariance of faces under particular circumstances: the hypothesis for the Banach space X of having the Property P. This theorem is originally presented in (3, Theorem 13), and it serves as an example of the interest in Pp. The proof of this result strongly relies on the invariance of maximal faces under surjective isometries (Theorem 3.2.9).

Theorem 5.2.4. *Let X, Y be two Banach spaces, $T : S_X \to S_Y$, a surjective isometry, and $E \subseteq S_X$, a flat subset. Then:*

1. *$T(M_E) = M_{T(E)}$.*
2. *If X has also Pp and E is a face of B_X, then $T(E)$ is a face of B_Y.*

Proof. 1. Let us fix an arbitrary $D \in M_E$. By Theorem 3.2.9, we know that $T(D)$ is also a maximal face of B_Y which contains $T(E)$, that is $T(D) \in M_{T(E)}$. This shows that $T(M_E) \subseteq M_{T(E)}$. The other inclusion is analogous by using T^{-1}.

2. Since X satisfies property P, every (proper) face is the intersection of all maximal faces containing it, that is $E = \cap_{D \in M_E} D$ by hypothesis. Besides, by Theorem 5.2.3(4), $T(E)$ is flat, and $T(M_E) = M_{T(E)}$ by previous item on this result. Then, keeping in mind that T is an homeomorphism,

$$T(E) = T\left(\bigcap_{D \in M_E} D\right) = \bigcap_{D \in M_E} T(D) = \bigcap_{C \in M_{T(E)}} C.$$

Observe that the equality above shows that $T(E)$ is the intersection of (maximal) faces of B_Y, therefore, $T(E)$ is a face of B_Y.

□

In order to prove Corollary 5.2.7, the following lemma will be necessary.

Lemma 5.2.5. *Let X be a normed space and $x, y \in S_X$. Then:*

1. *If $\nu(x) \subseteq \nu(y)$, then $M_x \subseteq M_y$, where $\nu(x) := \{x^* \in S_{X^*} : x^*(x) = 1\}$.*

2. *If $x, y \in \operatorname{smo}(B_X)$, and $M_x \subseteq M_y$, then $M_x = M_y$ and $\nu(x) = \nu(y)$.*

3. *$\operatorname{st}(x, B_X) \subseteq \operatorname{st}(y, B_X)$ if and only if $M_x \subseteq M_y$.*

4. *The equality $\operatorname{st}(x, B_X) = \operatorname{st}(y, B_X)$ holds if and only if $M_x = M_y$.*

Proof. 1. Let $D \in M_x$ be an arbitrary maximal face containing $x \in S_X$. Since maximal faces are exposed faces, there exists $f \in S_{X^*}$ such that $x \in D = F(f)$, which implies $f \in \nu(x)$. by hypothesis, $\nu(x) \subseteq \nu(y)$, then $f(y) = 1$, thus $y \in F(f) = D$, concluding that $D \in M_y$.

2. Since $x, y \in \operatorname{smo}(B_X)$, we call on Theorem 4.4.13 to assert that $\operatorname{st}(x, B_X)$ and $\operatorname{st}(y, B_X)$ are both convex. Also, Theorem 4.4.10 states that those starlike sets are the only maximal faces containing each center, that means, $M_x = \{\operatorname{st}(x, B_X\}$ and $M_y = \{\operatorname{st}(y, B_X)\}$. Thus, M_x and M_y are both singleton satisfying $M_x \subseteq M_y$, the only possibility is that $M_x = M_y$. Finally, by using again Theorem 4.4.13, $F(\nu(x)) = \operatorname{st}(x, B_X) = \operatorname{st}(y, B_X) = F(\nu(y))$. The smoothness of x and y forces $\nu(x) = \nu(y)$.

3. $\Rightarrow$ Suppose on the first place that $\operatorname{st}(x, B_X) \subseteq \operatorname{st}(y, B_X)$ and let $D \in M_x$ be an arbitrary

maximal face containing x. Observe that $D \subseteq \mathrm{st}(x, B_X)$ by using Equation 3.2.6. If we show that $\mathrm{co}(D \cup \{y\}) \subseteq S_X$, then the maximality of D implies that $\mathrm{co}(D \cup \{y\}) \subseteq D$, thus $y \in D$, which means that $D \in M_y$. The arbitrariness of D shows the contention $M_x \subseteq M_y$. Let us proof that $\mathrm{co}(D \cup \{y\}) \subseteq S_X$. Indeed, since $\mathrm{co}(D \cup \{y\}) = \cup_{d \in D}[d, y]$ and, by hypothesis, $D \subseteq \mathrm{st}(x, B_X) \subseteq \mathrm{st}(y, B_X)$, thus, keeping in mind again Equation 3.2.6, we claim that $[d, y] \subseteq S_X$ for all $d \in D$, hence $\mathrm{co}(D \cup \{y\}) = \cup_{d \in D}[d, y] \subseteq S_X$, concluding this part of the proof.

$\Leftarrow$ Conversely, suppose that $M_x \subseteq M_y$, therefore, by using again the Equality 3.2.6,

$$\mathrm{st}(x, B_X) = \bigcup_{D \in M_x} D \subseteq \bigcup_{D \in M_y} D = \mathrm{st}(y, B_X).$$

- This fourth statement is a direct consequence of the item proved above.

$\square$

Next example shows that the converse of Lemma 5.2.5(1) does not hold in general.

Example 5.2.6. *Let us consider $\mathbb{R}^2$ endowed with the norm given by the unit ball resulting from the intersection of the Euclidean ball $B_{\ell_2^2}$ with the band $\left\{(x_1, x_2) \in \mathbb{R}^2 : -\frac{1}{2} \leq x_2 \leq \frac{1}{2}\right\}$, denoted by X, and take $x := \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $y := \left(0, \frac{1}{2}\right)$ (see Figure 5.1). In this case, note that $M_x = M_y = \left\{\left[\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)\right]\right\}$, but $v(x) \nsubseteq v(y)$ since $y \in \mathrm{smo}(B_X)$, then $v(y)$ is a singleton, while $v(x)$ does not, since $x \notin \mathrm{smo}(B_X)$.*

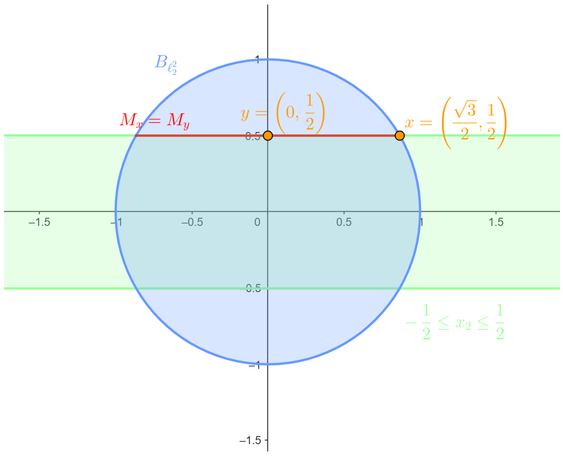

Figure 5.1: A counter-example of Lemma 5.2.5(1)

A direct consequence of Lemma 5.2.5 is the following.

Corollary 5.2.7. *Let X and Y be two Banach spaces, $T : S_X \to S_Y$, a surjective isometry and Y, smooth. Then, $\nu(T(-x)) = -\nu(T(x))$.*

Proof. First, Remark 3.2.17 and Theorem 5.1.1 establish that $T(\mathrm{st}(x, B_X)) = \mathrm{st}(-T(-x), B_Y)$ and $T(\mathrm{st}(x, B_X)) = \mathrm{st}(T(x), B_Y)$ respectively. Then, the next equality for starlike sets holds

$$T(\mathrm{st}(x, B_X)) = \mathrm{st}(-T(-x), B_y) = \mathrm{st}(T(x), B_Y).$$

The equality above allows us to use Lemma 5.2.5(4) and claim that $M_{-T(-x)} = M_{T(x)}$. Finally, since Y is smooth, Lemma 5.2.5(2) assures that $-\nu(T(-x)) = \nu(-T(-x)) = \nu(T(x))$. $\square$

We recall that singleton maximal faces of the unit ball are precisely the rotund points. Keeping this in mind, the following result is a particular case of Theorem 3.2.9 for singleton maximal faces, which allows us to provide a very simple proof of this invariance (the original proof was shown in (3, Theorem 14), and it is not the same as the one presented here, which is even simpler).

Theorem 5.2.8. *Let X and Y be Banach spaces and $T : S_X \to S_Y$, a surjective isometry. Then, $T(\mathrm{rot}(B_X)) = \mathrm{rot}(B_Y)$. Moreover, $T(-x) = -T(x)$ for all rotund point $x \in \mathrm{rot}(B_X)$.*

Proof. Let $x \in \mathrm{rot}(B_X)$, in particular, $\{x\}$ is a maximal face. Keeping in mind Equation 3.2.6, $\{x\} = \mathrm{st}(x, B_X)$. Besides, Remark 3.2.17 gives the next equality

$$T(\{x\}) = T(\mathrm{st}(x, B_X)) = \mathrm{st}(-T(-x), B_Y).$$

Notice that $T(\{x\}) = \{T(x)\}$ is convex, hence $\mathrm{st}(-T(-x), B_Y)$ is convex. In view of Theorem 4.4.10, $\mathrm{st}(-T(-x), B_Y) = \{T(x)\}$ is the only maximal face containing $\{-T(-x)\}$, in particular, $\{T(x)\}$ is a maximal face, hence, $T(x) \in \mathrm{rot}(B_Y)$.

□

As a direct consequence of the previous theorem, we have the following result, which shows that the strict convexity is invariant under surjective isometries defined between the unit spheres.

Corollary 5.2.9. *Let X and Y be Banach spaces and $T : S_X \to S_Y$ a surjective isometry. Then, if X is strictly convex, Y is also strictly convex.*

Last result motivates the following definition. This property will be a useful tool when we prove the invariance of some kind of sets and properties, as well as a generalization of Lemmas 12 and 13 in (5) to the infinite dimensional case for maximal faces of the unit sphere with inner points.

Definition 5.2.10. *A Banach space X is said to satisfy the inner property or the I-property (Ip)* *if it is strictly convex or every non-singleton maximal face of B_X has inner points.*

In (20, Theorem 5.1) it was proved that every finite dimensional Banach space has the Ip. The following examples show the existence of Banach spaces lacking the Ip for the infinite dimensional case.

Example 5.2.11. *We call a Banach space X transitive if for every $x, y \in S_X$, there exists a surjective linear isometry $T : X \to X$ such that $T(x) = y$. Keeping in mind (55, Corollary 2.21), where it was proved that every Banach space can be isometrically regarded as a subspace of a transitive Banach space, we can take a non-strictly convex Banach space Y and consider a transitive Banach space X such that it contains a subspace isometrically isomorphic to Y. As X contains a non-strictly convex subspace, X is not strictly convex. Besides, in (56, Theorem 3.2) it was proved that every non-singleton maximal face of a non-strictly convex and transitive space is free of inner points, then X has not the Ip.*

Example 5.2.12. *Let us consider ℓ_1, the space of all absolute summable sequences*

$$\ell_1 = \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

where the norm is given by

$$\|(x_n)_{n\in\mathbb{N}}\|_1 := \sum_{n=1}^{\infty} |x_n|.$$

In (20, Theorem 5.4) it was presented an example of a maximal face free of inner points. In particular,

$$D := \left\{ (x_n)_{n\in\mathbb{N}} \in S_{\ell_1} : x_n \geq 0 \textit{ for all } n \in \mathbb{N} \right\}$$

is a maximal face of B_{ℓ_1} satisfying $\operatorname{inn}(D) = \varnothing$. As a consequence, ℓ_1 does not satisfy Ip.

Example 5.2.13. *In Example 5.2.11, we have presented a non-strictly convex and transitive*

Banach space X lacking the Ip. Notice that B_X is also free of rotund points, otherwise, the transitivity forces that $\operatorname{rot}(B_X) = S_X$*, contradicting the non-strict convexity. Now, if we consider another Banach space Y and a surjective isometry $T : S_X \to S_Y$, and a maximal face $C \subseteq S_X$ with* $\operatorname{inn}(C) = \varnothing$ *(the existence of such maximal face is guaranteed by the condition of lacking the Ip), then $T(C)$ is also a maximal face in view of Theorem 3.2.9 and* $\operatorname{rot}(B_Y) = \varnothing$ *by using Theorem 5.2.8, then, Y also fails Ip.*

We recall that a topological space is said to be homogeneous if for any two points x, y in the topological space, there exists an homeomorphism (on the whole space) mapping one to another.

Theorem 5.2.14. *Let X and Y be Banach spaces and $T : S_X \to S_Y$, a surjective isometry. If X is transitive, then S_Y is homogeneous. If X is also separable, then Y is strictly convex.*

Proof. Let $y_1, y_2 \in S_Y$. The transitivity of X guarantees the existence of a surjective linear isometry $S : X \to X$ such that $S(T^{-1}(y_1)) = T^{-1}(y_2)$. Now, the wanted homeomorphism is given by the surjective isometry $T \circ S \circ T^{-1} : S_Y \to S_Y$, which clearly maps y_1 to y_2. To conclude the proof, if we ask X to be separable, then it is also rotund by (57, Theorem 28). Finally, by Corollary 5.2.9, Y is also strictly convex. □

Next theorem constitutes a generalization of Lemmas 12 and 13 proved by Tingley in (5). The result we are going to present was originally proved in (3, Theorem 15).

Theorem 5.2.15. *Let X and Y be Banach spaces and $T : S_X \to S_Y$ a surjective isometry. Consider a maximal face $F \subseteq S_X$ such that* $\operatorname{inn}(F) \neq \varnothing$*. Then,*

1. $T(-F) = -T(F)$.

2. *If there exists $x \in$* $\operatorname{inn}(F)$ *for which there exists $E \in M_{T(x)}$ satisfying* $\operatorname{inn}(E) \neq \varnothing$*, then*

$T(F) \subseteq E$.

Proof. 1. Firstly, let us see that $T(-F) \subseteq -T(F)$. Let $x \in \operatorname{inn}(F)$. Then, by Lemma 4.4.9(2,5), we know that $F = \operatorname{st}(x, B_X)$. Additionally, $T(F) = T(\operatorname{st}(x, B_X)) = \operatorname{st}(-T(-x), B_Y)$ by Lemma 4.4.2. Then, $-T(-x) \in T(F)$, so $T(-x) \in -T(F)$. Since $x \in \operatorname{inn}(F)$ is arbitrary, the contention $T(-\operatorname{inn}(F)) \subseteq -T(F)$ holds. Keeping in mind Remark 3.2.27, $\operatorname{inn}(F)$ is dense in F (since F is also closed). On the other hand, T is continuous, so $T(F)$ is closed in S_Y. Putting both consequences together, we conclude that $T(-F) \subseteq -T(F)$. For the other inclusion, notice that $-F$ is also a maximal face with $\operatorname{inn}(-F) \neq \varnothing$, since $\operatorname{inn}(-F) \neq \varnothing$ if and only if $\operatorname{inn}(F) \neq \varnothing$. Then, making the same reasoning, we reach to $T(F) \subseteq -T(-F)$.

2. Let us consider $e \in \operatorname{inn}(E)$. Hence, by Lemma 4.4.9(2,5), $E = \operatorname{st}(e, B_Y)$. By applying Remark 3.2.17 to $T^{-1} : S_Y \to S_X$, we have that $T^{-1}(\operatorname{st}(e, B_Y)) = \operatorname{st}(-T^{-1}(e), B_X)$, thus $T^{-1}(E) = \operatorname{st}(-T^{-1}(-e, B_X)$. By hypotheses, $E \in M_{T(x)}$, then $x \in T^{-1}(E)$, so $x \in \operatorname{st}(-T^{-1}(-e), B_X)$. This last equality guarantees that $[x, -T^{-1}(-e)] \subseteq S_X$. So, F is under the conditions of Lemma 4.4.9(1), concluding that $F \subseteq \operatorname{st}(-T^{-1}(-e), B_X) = T^{-1}(E)$, which is equivalent to $T(F) \subseteq E$.

□

We recall to the reader that (5, Lemma 13) has already been generalize to infinite dimensions in (7, Lemma 5.1) and in (1, Lemma 3.5) (see Theorem 3.2.9). The purpose of the Corollary 5.2.16(1) is to provide a simpler proof when the Banach space has the Ip. Also, in Corollary 5.2.16(2), we prove the invariance of the starlike sets showed in Theorem 5.1.1 without relying on Theorem 3.2.9.

Corollary 5.2.16. *Let X and Y be Banach spaces and consider $T : S_X \to S_Y$. Suppose that X*

and Y satisfy the Ip. Then,

1. *If F is a maximal face of B_X, $T(F)$ is a maximal face of B_Y.*

2. *$T(\mathrm{st}(x,B_X)) = \mathrm{st}(T(x),B_Y)$ for all $x \in S_X$.*

Proof. 1. Let us assume that F is not a singleton (if $F = \{x\}$, then $x \in \mathrm{rot}(B_X)$ and the proof is a particular case of Theorem 5.2.15). By hypothesis, $\mathrm{inn}(F) \neq \varnothing$, so we can fix $x \in \mathrm{inn}(F)$. The same reasoning allows us to consider $e \in \mathrm{inn}(E) \neq \varnothing$, where $E \in M_{T(x)}$. Then, by Theorem 5.2.15(2), it holds that $T(F) \subseteq E$. Now, consider any $D \in M_{T^{-1}(e)}$, since $\mathrm{inn}(D) \neq \varnothing$, by Theorem 5.2.15(2) again, we have that $T^{-1}(E) \subseteq D$, hence $F \subseteq T^{-1}(E) \subseteq D$. The maximality of F forces that $T(F) = E$, which was already maximal.

2. By Remark 3.2.17, we have the equality $T(\mathrm{st}(x,B_X)) = \mathrm{st}(-T(-x),B_Y)$, so it only suffices to prove that $\mathrm{st}(-T(-x),B_Y) = \mathrm{st}(T(x),B_Y)$, which is equivalent to prove that $M_{T(x)} = M_{-T(-x)}$ by Lemma 5.2.5(4). Let us see that $M_{T(x)} \subseteq M_{-T(-x)}$: fix an arbitrary $C \in M_{T(x)}$, then $x \in T^{-1}(C)$, thus $-x \in -T^{-1}(C) = T^{-1}(-C)$, where the latter equality is given by Theorem 5.2.15(1). So, we have that $T(-x) \in -C$, hence $-T(-x) \in C$. This shows that $C \in M_{-T(-x)}$. The arbitrariness of C guarantees that $M_{T(x)} \subseteq M_{-T(-x)}$. The other inclusion follows a similar reasoning: Let $D \in M_{-T(-x)}$, this means that $-T(-x) \in D$, so $T(-x) \in -D$, then $-x \in T^{-1}(-D) = -T^{-1}(D)$, where the latter equality is due to Theorem 5.2.15(1). Hence, $x \in T^{-1}(D)$, meaning that $T(x) \in D$, thus $D \in M_{T(x)}$ concluding the proof.

□

5.3 Invariance of Segments

The aim of this section is the study of the invariance of segments under surjective isometries and it is motivated as a refinement of Tanaka's result about the invariance of maximal faces proved in (1, Lemma 3.5). Meanwhile, affine properties of surjectives isometries on maximal-face segments is accomplished, reaching one step further in the achievement of linearity for surjective isometries.

The following result was proved in (3, Theorem 17) and it presents the invariance of segments under certain hypotheses.

Theorem 5.3.1. *Let X and Y be Banach spaces and $T : S_X \to S_Y$ a surjective isometry. Let $x, y \in S_X$ be two different points such that $[x, y] \subseteq S_X$. If we ask $T([x, y])$ to be convex, then $T([x, y]) = [T(x), T(y)]$ and T is affine in the whole segment $[x, y]$, that is, $T(tx+(1-t)y) = tT(x) + (1-t)T(y)$ for all $t \in [0, 1]$.*

Proof. The proof uses strongly the fact that $T : S_X \to S_Y$ is an homeomorphism. Keeping this in hand, it is clear that $T([x, y])$ is compact and convex (by hypothesis). Moreover, $T([x, y])$ is homeomorphic to $[x, y]$. Let us prove in the first place that $T([x, y])$ is a segment. If we suppose on the contrary that $T([x, y])$ is not a segment, since $T([x, y])$ is convex, it has to contain at least three points not aligned. Besides, notice that $[x, y] \setminus \left\{\frac{x+y}{2}\right\}$ is not connected, then $T\left([x, y] \setminus \left\{\frac{x+y}{2}\right\}\right) = T([x, y]) \setminus \left\{T\left(\frac{x+y}{2}\right)\right\}$ is not connected either. But Remark 4.1.2 claims that $\left\{T\left(\frac{x+y}{2}\right)\right\}$ has to be connected. Then, $T([x, y])$ is a segment, that is $T([x, y]) = [a, b]$ with $a, b \in S_Y$.

Now, consider $t, s \in [0, 1]$ such that $T(tx + (1-t)y) = a$ and $T(sx + (1-s)y) = b$. Keeping in mind that isometries preserves diameters, the following holds

$$\begin{aligned}\|x-y\| &= \operatorname{diam}([x,y])\\ &= \operatorname{diam}(T([x,y]))\\ &= \|a-b\|\\ &= \|T(tx+(1-t)y)-T(sx+(1-s)y)\|\\ &= \|tx+(1-t)y-(sx+(1-s)y)\|\\ &= |t-s|\|x-y\|,\end{aligned}$$

therefore, the only possibility is that $t=0$ and $s=1$ or $t=1$ and $s=0$. In any case, $T([x,y])=[T(x),T(y)]$. Finally, let see that T is affine on the whole segment $[x,y]$. Indeed, fix an arbitrary $t\in(0,1)$. Then, there exists $s\in(0,1)$ such that $T(tx+(1-t)y)=sT(x)+(1-s)T(y)$. Following a similar reasoning as before,

$$\begin{aligned}(1-s)\|x-y\| &= (1-s)\|T(x)-T(y)\|\\ &= \|(sT(x)+(1-s)T(y))-T(x)\|\\ &= \|T(tx+(1-t)y)-T(x)\|\\ &= \|tx+(1-t)y-x\|\\ &= (1-t)\|x-y\|,\end{aligned}$$

thus, $t=s$, concluding the proof. □

Theorem 3.2.9 together with the previous result give us the next direct consequence.

Corollary 5.3.2. *Let X and Y be Banach spaces and $T:S_X\to S_Y$, a surjective isometry. Consider two different points $x,y\in S_X$, $x\neq y$ such that $[x,y]\subseteq S_X$. If $[x,y]$ is a maximal face of B_X, then T is affine on $[x,y]$, that is $T(tx+(1-t)y)=tT(x)+(1-t)T(y)$ for all*

$t \in [0,1]$.

Proof. Theorem 3.2.9 guarantees that $T([x,y])$ is also a maximal face of B_Y, in particular, it is convex. Then, we are under the hypotheses of Theorem 5.3.1. □

Example 5.3.3. *An interesting example was presented in (57, Example 3.8). Here, a* 3-*dimensional Banach space was constructed whose unit ball is formed by rotund points except for two maximal segments which are maximal faces, so its unit ball satisfies the conditions of Corollary 5.3.2.*

CHAPTER

6

Geometry of the unit ball under projections

Contents

In Section 3.3 we have recalled the concepts of projections and supporting vectors in order to use them in the study of the extremal structure in this section. This chapter features the results presented in (23, Section 5) and it could be seen as a continuation of the research done in (26). In particular, we present a new type of norm-one projections called S-projections (Definition 6.1.2) motivated by the characterization given in (26, Proposition 3.1). This projections $P : X \to X$ are just those 1-projections satisfying that $S_{P(X)} = \text{suppv}(P)$. As well, we characterize the supporting vector of an L^{∞}-projection (also called M-projections) and the S-projections (Lemma 6.1.6). Notice that M-projections are not S-projection in general (see Proposition 6.1.4). This section ends with Theorem 6.1.7 by showing that the extremal condition in the projected unit ball $B_{P(X)}$ is kept if we consider the extremal subset respect to the whole unit ball B_X. In the last section of this chapter we introduce the concept of strongly maximal face. As the reader expects, every strongly maximal face is a maximal face (see Lemma 6.2.2) and they are invariants under L^2-projections, that is, if F is a strongly maximal face of the projected unit ball $B_{P(X)}$, the same condition holds in B_X (see Theorem 6.2.4). As a consequence, the same statement holds for (non-necessary strongly) maximal faces and the contention $\mathrm{rot}(B_{P(X)}) \subseteq \mathrm{rot}(B_X)$.

6.1 A new class of norm-one projections: S-projections

First of all, we want the reader to notice the following relation.

Remark 6.1.1. *Notice that if $P : X \to X$ is a 1-projection, then $S_{P(X)} \subseteq \mathrm{suppv}(P)$. Indeed, let $P(x) \in S_{P(X)} = S_X \cap P(X)$, that is, $\|P(x)\| = 1$. In this case, $\mathrm{suppv}(P) = \{x \in S_X : \ \|P(x)\| = 1\}$. Then, $P(x) \in \mathrm{suppv}(P)$ if $\|P(P(x))\| = 1$. But this equality holds by using the idempotence of P*

$$\|P(P(x))\| = \|P(x)\| = 1 = \|P\|.$$

Next original definition is motivated by this previous observation.

Definition 6.1.2. *For a Banach space X, a 1-projection $P : X \to X$ is called and S-projection provided that $S_{P(X)} = \mathrm{suppv}(P)$.*

Let us show some examples of S-projections.

Proposition 6.1.3. *Every L^p-projection is an S-projection, with $1 \le p < \infty$.*

Proof. Indeed, since every projection satisfies $\|P\| \ge 1$, let us show that $\|P\| \le 1$. The next holds for every L^p-projection

$$\|x\|^p = \|P(x)\|^p + \underbrace{\|(I-P)(x)\|^p}_{\ge 0} \ge \|P(x)\|^p,$$

then, $\|x\|^p \le \|P(x)\|^p$ for every $x \in X$. Take the p-th root, which is strictly increasing, and the supremum in S_X, hence,

$$1 = \sup_{x \in S_X} \|x\| \le \sup_{x \in S_X} \|P(x)\| = \|P\|.$$

A similar reasoning for $I-P$ reach us to the fact that every L^p-projections is a $(1,1)$-projection. Finally, let us see that $S_{P(X)} = \mathrm{suppv}(P)$. The contention $S_{P(X)} \subseteq \mathrm{suppv}(P)$ remains true for every norm-one projection. Since $\mathrm{suppv}(P) \subseteq S_X$, it only suffices to prove that $\mathrm{suppv}(P) \subseteq P(X)$. Let $x \in \mathrm{suppv}(P)$, then $\|x\| = 1 = \|P(x)\|$, besides, using that P is an L^p-projection,

$$1 = \|x\|^p = \underbrace{\|P(x)\|^p}_{=1} + \|(I-P)(x)\|^p,$$

which forces that $(I-P)(x) = 0$, meaning that $x \in P(X)$. □

Next result proves the existence of projections which are not S-projections.

Proposition 6.1.4. *Let $P : X \to X$ be a non-trivial M-projection defined in a Banach space X. Then, P does not satisfy the condition of being an S-projection.*

Proof. Let $M := P(X)$ and $N := \ker(P)$, then $X = M \oplus_\infty N$, that is, every element $x \in X$ has an expression $x = m + n$, with $m \in M$ and $n \in N$. In order to see that $S_{P(X)} \neq \mathrm{suppv}(P)$, it suffices to show that $A := \{m+n \in S_X : \ \ \|m\| = 1, \|n\| = 1\} \subseteq \text{suppv}(P)$, since $\{m+n \in X : \ \ \|m\| = 1, \|n\| = 1\} \nsubseteq S_{P(X)}$. Since every M-projections is a norm-one projection, $\text{suppv}(P) = \{x \in S_X : \ \ \|P(x)\| = 1 = \|P\|\}$. Fix an arbitrary $x \in A$, then, since P is an M-projection, $\|x\| = \max\{\|m\|, \|n\|\} = 1$, then $x \in S_X$. Notice that $\|P(x)\| = \|P(m) + P(n)\| = \|P(m)\| = \|m\| = 1$, because $m \in P(X)$, then $P(m) = m$. This proves that $x \in \text{suppv}(P)$. □

The reader have to notice that this proof motivates the following characterization of the set of supporting vectors for an arbitrary M-projection.

Theorem 6.1.5. *Let $P : X \to X$ be an M-projection. Then,*

$$\text{suppv}(P) = \{m+n \in S_X : \ \ \|m\| = 1, \|n\| \leq 1\},$$

and

$$\text{suppv}(I-P) = \{m+n \in S_X : \ \|n\| = 1, \|m\| \leq 1\}.$$

Proof. As in the proof of Proposition 6.1.4, $X = M \oplus_\infty N$. Let $x \in \{m+n \in S_X : \ \|m\| = 1, \|n\| \leq 1\}$, then $\|x\| = \max\{\|m\|, \|n\|\} = 1$ and $P(x) = P(m)+P(n) = P(m)$, then $\|P(x)\| = 1$, implying that $x \in \text{suppv}(P)$. Conversely, consider $x \in \text{suppv}(P) \subseteq X$, thus $x = m+n$, and $\|x\| = 1 = \|P(x)\| = \|P(m) + \cancel{P(n)}\| = \|m\|$, since $m \in P(X) = M$. By using again that P is an M-projection, then $\|x\| = \max\{\|m\|, \|n\|\}$, which forces that $\|n\| \leq 1$. Finally, (26, Proposition 3.1) claims that M-projections are $(1,1)$-projections, then $\|I-P\| = 1$, and the same reasoning as before shows that

$$\text{suppv}(I-P) = \{m+n \in S_X : \ \|n\| = 1, \|m\| \leq 1\}.$$

□

Next lemma characterizes the those norm-one projections which are S-projections.

Lemma 6.1.6. *Let X be a Banach space. A 1-projection $P : X \to X$ is an S-projection if and only if for every $x \in X$ the condition $\|x\| = \|P(x)\|$ implies $x = P(x)$.*

Proof. $\Rightarrow$) On the first place, suppose that $P : X \to X$ is an S-projection, that is $S_{P(X)} = \mathrm{suppv}(P)$. Let $x \in X$ be an arbitrary element such that $\|x\| = \|P(x)\|$. Let us assume that $x \neq 0$, otherwise, $P(0) = 0$ by linearity and there is nothing to prove. Now define $y := \frac{x}{\|x\|}$. Notice that $y \in \mathrm{suppv}(P)$. Indeed,

$$\|P(y)\| = \left\| P\left(\frac{x}{\|x\|}\right) \right\| = \frac{1}{\|x\|} \|P(x)\| = 1 = \|P\|.$$

By hypothesis, P is an S-projection, thus, $y \in \mathrm{suppv}(P) = S_{P(X)} = S_X \cap P(X)$, meaning that there exists $z \in X$ such that $z = P(y)$. Using the idempotence of P, it holds that $P(y) = P(P(z)) = P(z) = y$, then

$$x = \|x\| y = \|x\| P(y) = P(\|x\| y) = P\left(\|x\| \frac{x}{\|x\|}\right) = P(x).$$

$\Leftarrow$) Conversely, suppose that for every $x \in X$, the equality $\|x\| = \|P(x)\|$ implies that $x = P(x)$. We already know that $S_{P(X)} \subseteq \mathrm{suppv}(P)$ for an arbitrary norm-one projection P. Let us see the other inclusion. Take $y \in \mathrm{suppv}(P) = \{x \in S_X : \|P(x)| = \|P\| = 1\}$. Notice that $\|y\| = 1 = \|P(y)\|$. By hypothesis, $y = P(y) \in P(X)$. Then, $y \in S_{P(X)} = S_X \cap P(X)$, concluding the proof.

$\square$

Lemma 6.1.6 shows that orthogonal projections (L^p-projections for $p = 2$) are trivial examples of S-projections.

Next result generalizes Lemma 4.3.5 to the extent of extremal subsets. This theorem was originally proved in (23, Theorem 5.3).

Theorem 6.1.7. *For a Banach space X and an S-projection $P : X \to X$, if $F \subseteq S_{P(X)}$ is extremal in $B_{P(X)}$, then F is extremal in B_X. In particular,* $\operatorname{ext}(B_{P(X)}) \subseteq \operatorname{ext}(B_X)$.

Proof. Let $y \in F \subseteq S_{P(X)} = S_X \cap P(X)$. Take $u, v \in S_X$ such that $y \in (u, v)$ and let us prove that $u, v \in F$. Since $y \in P(X)$, by Remark 3.3.5, $P(y) = y$ and $y \in (u, v)$, then $y = P(y) \in (P(u), P(v))$. Using that P is a norm-one projection, $\|P(u)\| \leq \|P\|\|u\| = 1$, then $P(u) \in B_{P(X)}$. The same reasoning gives $P(v) \in B_{P(X)}$. But F is extremal in $B_{P(X)}$, then $P(u), P(v) \in F \subseteq S_{P(X)}$, so $\|P(u)\| = 1 = \|u\|$ and $\|P(v)\| = 1 = \|v\|$. In view of Lemma 6.1.6, we have that $u = P(u) \in F$ and $v = P(v) \in F$, that is $u, v \in F$, which proves the extremal condition for F in B_X. $\square$

Since M-projections are not S-projections, the previous theorem does not remain true for M-projections, as it is shown in the following example.

Example 6.1.8. *In $\ell_{\infty^2} := \left(\mathbb{R}^2, \|\cdot\|_\infty\right)$, the point $(1,0)$ is extreme in the unit ball $B_{P(X)} = \{(x,0) : \ x \in \mathbb{R}\}$, but is not an extreme point in $B_{\ell^2_\infty}$.*

6.2 Extremal Structure under 1-Complementation

Before proving Theorem 6.2.4, we will introduce a novel definition in Extremal Theory. We want the reader to notice that this definition has some similarities with the one given in (6, Definition 11), but the one we present here is a generalization whose name is linked to the term of maximal face. As a matter of fact, this relation will be proved in Lemma 6.2.2.

Definition 6.2.1 (Strongly maximal face). *For a convex subset in the unit sphere $F \subseteq S_X$ of a normed space, we say that F is a strongly maximal face of B_X if $F = \cup_{e\in F} \mathrm{st}(e, B_X)$.*

As it is expected, every strongly maximal face is, in fact, a maximal face of B_X. We feature this fact in next result.

Lemma 6.2.2. *If $F \subseteq S_X$ is a strongly maximal face of B_X, in a real vector space X, then, F is a maximal face of B_X.*

Proof. Let $D \subseteq S_X$ be a convex component containing $F = \cup_{e\in F} \mathrm{st}(e, B_X)$, and take $d \in D$, $f \in F$. Observe that $f \in F \subseteq D$, which is convex, then $[d, f] \subseteq D \subseteq S_X$. By using Equation 3.2.6, $d \in \mathrm{st}(f, B_X) \subseteq F$, hence $D = F$. □

However, there exist examples of maximal faces which are not strongly maximal faces.

Example 6.2.3. *Any of the maximal faces of the unit ball $B_{\ell^2_\infty}$ is not a strongly maximal face, where $\ell^2_\infty := (\mathbb{R}^2, \|\cdot\|_\infty)$.*

An easy example of strongly maximal faces are rotund points of the unit ball.

Theorem 6.2.4. *Let $P : X \to X$ be an L^2-projection defined in a Banach space. If F is a (strongly) maximal face of $B_{P(X)}$, then F is a (strongly) maximal face of B_X. In particular,* $\mathrm{rot}(B_{P(X)}) \subseteq \mathrm{rot}(B_X)$.

Proof. We will only prove when F is a strongly maximal face of B_X. The other case follows a similar proof. This inclusion $F \subseteq \cup_{f\in F} \mathrm{st}(f, B_X)$ holds by bearing in mind Equation 3.2.6. Let us see the other one. Fix an arbitrary $x \in \cup_{f\in F} \mathrm{st}(f, B_X)$, in particular, there exists $f \in F$ such that $x \in \mathrm{st}(f, B_X)$. Then, by using again Equation 3.2.6, $\|x + f\| = 2$, that is, $\|x + f\|^2 = 4$.

Besides, notice that the linearity of P gives $P(x+f)+(I-P)(x) = P(x)+P(f)+x-P(x) = f+x$, where we have also used the fact that $P(f) = f$ since $f \in F \subseteq P(X)$. Putting this together with $\|x+f\|^2 = 4$, the following chain of equalities holds

$$
\begin{aligned}
4 &= \|x+f\|^2 \\
&= \|P(x+f)+(I-P)(x)\|^2 \\
&= \|P(x+f)\|^2 + \|(I-P)(x)\|^2 \\
&\leq (\|P(x)\| + \|f\|)^2 + \|(I-P)(x)\|^2 \\
&= \|P(x)\|^2 + \|f\|^1 + 2\|P(x)\|\|f\| + \|(I-P)(x)\|^2 \\
&= \|x\|^2 + \|f\|^2 + 2\|P(x)\|\|f\| \\
&= 1+1+2\|P(x)\| \\
&\leq 2+2\|P\|\|x\| = 4.
\end{aligned}
$$

The chain above forces that $\|P(x)\| = 1$. Notice that $\|x\|^2 = \|P(x)\|^2 + \|(I-P)(x)\|^2$ and $\|x\| = 1$, implying that $\|x-P(x)\| = 0$, hence $x = P(x) \in S_{P(X)}$. This means that $x \in= \mathrm{st}(f, B_X) \cap S_{P(X)} = \mathrm{st}(f, B_{P(X)}) \subseteq F$. This proves that $\mathrm{st}(f, B_X) \subseteq F$, concluding the proof.

□

Theorem above does not remains true for general S-projections.

Example 6.2.5. *In $\ell_1^2 := (\mathbb{R}^2, \|\cdot\|_1)$, the point $(1,0)$ is a rotund point of the unit ball $\{(x,0) : x \in \mathbb{R}\}$ which is not a rotund point in $B_{\ell_1^2}$.*

APPENDIX

A

Conclusions

We have revisited the geometric aspects of real Banach spaces related to the Mazur-Ulam property and Tingley's problem providing new original results, which, in some cases, indicate that Tingley's problem might be solved in the affirmative. For instance, one of our most remarkable result assures that if a maximal face of the unit ball of a Banach space has inner points, then any surjective isometry preserves opposite (or antipodal) faces (Theorem 5.2.15). Another strong result that we have obtained is the fact that if a unit sphere contains a maximal face consisting of a segment, then any surjective isometry is affine on such segment (Theorem 5.3.1). This result has strong consequences on the behavior of surjective isometries after equivalent renormings since every strictly convex Banach space can be equivalently renormed so that the new unit sphere contains only extreme points and maximal faces consisting of segments (Objective 5). All of these results are strong generalizations of classical results of Tingley and are aimed towards a positive solution of Tingley's problem, namely, Theorems 5.2.4, 5.2.15 and Lemma 4.4.9.

Also, these results, among others in the same chapter, indicate that we have accomplished the first five objectives of this dissertation. For instance, Objective 3 is studied throughout Chapter 5, where Theorems 5.2.8, 5.2.1, 5.1.1, and 5.2.3 show the invariance of some parts of the facial structure (which also serves to achieve Objective 6). Besides, one of the consequences of determining the invariance of facets under surjective isometries (Objective 4) is Theorem 4.3.12. As well, we want to underline that strict convexity is an invariant under surjective isometries as a consequence of the invariance of rotund points as it is shown in Corollary 5.2.9. Another important achievement we want to highlight in the study of the geometric structure is the equivalence between starlike sets and maximal faces when the starlike set is convex (Theorem 4.4.10) in contrast with Tanaka's result, who asked for the Banach space to be separable (1, Lemma 3.3).

On the other hand, with respect to the final objective of this dissertation, we have introduced a new definition in the literature of the Mazur-Ulam property and Tingley's problem. This definition drove us to provide a positive partial solution to Tingley's problem, accomplishing also the sixth objective of this dissertation.

We will denote by $\mathscr{B}$ to the class of all real Banach spaces. A subclass $\mathscr{C} \subseteq \mathscr{B}$ is said to be isometric (isomorphic) if $\mathscr{C}$ is invariant under surjective linear isometries (isomorphisms), that is, if $X \in \mathscr{C}$, $Y \in \mathscr{B}$, and $T : X \to Y$ is a surjective linear isometry (isomorphism), then $Y \in \mathscr{C}$.

Definition A.0.1 (MUp class)**.** *A subclass $\mathscr{C} \subseteq \mathscr{B}$ is said to be an MUp class if $\mathscr{C}$ is invariant under surjective isometries between unit spheres, that is, if $X \in \mathscr{C}$, $Y \in \mathscr{B}$, and $T : \mathsf{S}_X \to \mathsf{S}_Y$ is a surjective isometry, then $Y \in \mathscr{C}$.*

Notice that every MUp class is an isometric class. According to classical results in the literature, the class of strictly convex Banach spaces is an MUp class (Objective 2). In view of our Theorem 5.2.8, the class of all Banach spaces whose unit sphere has a dense amount of rotund points is also an MUp class. Finally, the subclass of all Banach spaces whose unit sphere is made of extreme points except for two maximal segments (opposite to each other) is also an MUp class.

APPENDIX

B

Notation

Next, we will proceed to explain the main notation followed in this dissertation:

B_X .... the closed unit ball in X

U_X .... the open unit ball in X

S_X .... the unit sphere in X

$B_X(x,r)$ .... the closed ball of center x and radius r in X

$U_X(x,r)$ .... the open ball of center x and radius r in X

$S_X(x,r)$ .... the sphere of center x and radius r in X

$\mathrm{int}(M)$ .... the topological interior of M

$\mathrm{int}_A(M)$ .... the topological interior of M relative to A

$\mathrm{cl}(M)$ .... the topological closure of M

$\mathrm{cl}_A(M)$ .... the topological closure of M relative to A

$\mathrm{bd}(M)$ .... the topological boundary of M

$\mathrm{bd}_A(M)$ .... the topological boundary of M relative to A

$\mathrm{ext}(M)$ .... the set of extreme points F

X^* .... the topological dual of X

X^{**} .... the topological bidual of X

$F(f,A)$ .... the supporting hyperplane relative to $f \in X^*$ in A

$F(f)$ .... the supporting hyperplane relative to f in B_X

$\exp(B_X)$ .... the set of exposed points of B_X

$E(f)$ .... is the edge of the unit ball with respect to $f \in S_{X^*}$

$\mathrm{rot}(B_X)$ .... the set of rotund points of B_X

$\mathrm{pexp}(B_X)$ .... the set of proper exposed points of B_X

$\mathscr{C}_X$ .... the set of facets of B_X

$\mathrm{st}(x, B_X)$ .... the starlike set of center x

$\mathrm{smo}(B_X)$ .... the set of the smooth points of B_X

$\mathscr{P}(\mathscr{X})$ .... the power set of $\mathscr{X}$

ν .... the spherical image map from S_X to S_{X^*}

$\mathrm{frm}(B_X)$ .... the frame of the unit ball

$\mathrm{inn}(M)$ .... the set of inner points of M

$\mathrm{span}(M)$ .... the linear span of M

$\overline{\mathrm{span}}(M)$ .... the closed linear span of M

$\mathrm{co}(M)$ .... the convex hull of M

$\overline{\mathrm{co}}(M)$ .... the closed convex hull of M

$\mathrm{inter}(M)$ .... the set of the internal points of M

$\mathrm{adj}(M)$ .... the set of the adjacent elements of M

$\mathrm{sadj}(M)$ .... the set of the strongly adjacent elements of M

$\mathrm{suppv}(T)$ .... the set of supporting vectors of the operator T

$\mathrm{suppv}_1(x^*)$ .... the set of 1-supporting vectors of the functional x^*

nsupp(M) .... the set of non-support points of M

μ_A .... the Minkowski functional of A

MUp .... the Mazur-Ulman property

Pp .... the P-property or property P

Ip .... the I-property or inner property

Fp .... the F-property or flat property

APPENDIX

C

Pillars of Functional Analysis

Along the development of the papers backing up the dissertation, it was necessary the usage of some well-known results, most of them in Functional Analysis, which are mentioned below in pursuit to establish a shared framed between the reader and the author. We suggest (47) and (41) to the reader as a complete bibliography on this area.

C.1 Basic background

Next, we list several versions about one of the greater results in Mathematical Analysis.

Theorem C.1.1 (Hahn-Banach Theorems). 1) *Analytic Version. Let X be a vector space and p a sublinear functional in X. If M is a proper subspace of X and g is a linear functional in M such that* $\operatorname{Re} g(m) \leq p(m)$*,* $\forall m \in M$*, then there exists a linear functional f in X whose restriction to M is g and verifying* $\operatorname{Re} f(x) \leq p(x)$*, for all $x \in X$. In other words,* *every linear functional dominated by p could be extended to a linear functional in the whole space such that it continues being dominated by p. Even more, if p is a seminorm, it holds $|f(x)| \leq p(x)$, for all $x \in X$.*

2) *Geometric Versions.*

Let A and B be non empty convex disjoint subsets of a vector space X and suppose that there exists $a_0 \in A$ such that $A - a_0$ is absorbent. Then, there exists a non zero linear functional, and $\alpha \in \mathbb{R}$ such that $\operatorname{Re} f(a) \leq \alpha \leq \operatorname{Re} f(b)$, for each $a \in A, b \in B$.

3) *Extension Hahn-Banach Theorem for normed spaces.*

Let X be a normed space, Y a subspace of X, and $g \in Y^*$. Then, there exists $f \in X^*$ such that $\|f\| = \|g\|$ and $f(y) = g(y)$ for all $y \in Y$. As a consequence, if X is a normed space, for each $0 \neq x \in X$ there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$, therefore, we have the following expression for the norm in X

$$\|x\| = \max\{|f(x)| : \quad f \in S_{X^*}\}, \quad \textit{for each } x \in X.$$

4) *Separation Theorems for normed spaces.*

   $i)$ *Let X be a real normed space, A and B non empty convex and disjoint subsets, such that B is open. Then, there exists a continuous linear functional $f \in X^*$ such that $f(a) > f(b)$, if $a \in A, b \in B$.*

   $ii)$ *Let X be a normed space, A, B non empty convex subsets such that $\operatorname{int}(A) \neq \varnothing$ and $B \cap \operatorname{int}(A) = \varnothing$. Then, there exists a continuous linear functional $f \in X^*$ (even, we can* *take f with $\|f\| = 1$), and $\alpha \in \mathbb{R}$ satisfying*

$$\operatorname{Re} f(a) \leq \alpha \leq \operatorname{Re} f(b), \quad \textit{with } a \in A, b \in B.$$

In fact, it holds that $\operatorname{Re} f(a) < \alpha$ *for all* $a \in \operatorname{int}(A)$.

In the particular case that A is closed with non empty interior, and $B = \{x_0\}$, where $x_0 \in \operatorname{bd}(A)$, we obtain a continuous linear functional $f \in X^*$ verifying $\operatorname{Re}(f)(x) \leq \operatorname{Re} f(x_0)$*, for all $x \in A$, which is equivalent to* $\max \operatorname{Re} f(A) = \operatorname{Re} f(x_0)$*. In that case, the real affine hyperplane $H = \{x \in X : \operatorname{Re} f(x) = \operatorname{Re} f(x_0)\}$ touches A in x_0, leaving A on one side of the hyperplane. This is used to say that f is a supporting hyperplane of A at x_0, or H is a supporting hyperplane for A at x_0. To sum up, if X is a normed space and A is a non empty convex closed subset with non empty interior, for each point* $x_0 \in \operatorname{bd}(A)$ *there exists $f \in S_{X^*}$ such that* $\max\{\operatorname{Re} f(x) : x \in A\} = \operatorname{Re} f(x_0)$.

$iii)$ *For a normed space X and $A \subset X$ a non empty convex open subset, let V be an affine variety such that $V \cap A = \emptyset$. Then, there exists an affine closed hyperplane H such that $V \subset H$ and $H \cap A = \emptyset$.*

$iv)$ *Strong separation for normed spaces. Let A and B be non empty convex subsets of a normed space X, and suppose that* $\operatorname{dist}(A, B) = d > 0$*. Then, there exists a continuous linear functional $f \in S_{X^*}$ such that* $\sup \operatorname{Re} f(A) + d \leq \inf \operatorname{Re} f(B)$*, in which case it is said that f strongly separate the sets A and B.*

As a consequence, we have the following expression for the closed convex hull in a normed space. Let $A \subset X$ be a non empty subset of a normed space X. Then,

$$\overline{co}(A) = \bigcap_{f \in X^*} \{x \in X : \operatorname{Re} f(x) \leq \sup \operatorname{Re} f(A)\}.$$

Theorem C.1.2 (Riesz Theorem). $a)$ *Let X be an infinite dimensional normed space, then, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in the unit sphere such that $\|x_n - x_m\| \geq 1$ for every $n \neq m$. As a consequence, both the unit sphere, S_X, and the unit ball, B_X, are not compact subsets. Even more, none ball with positive radius is compact, though, every compact set has empty interior.*

$b)$ *For each normed space X the following sentences are equivalent*

$i)$ *The norm topology on X is locally compact.*

$ii)$ *Every closed-bounded subset is compact.*

$iii)$ *The unit ball, B_X, is compact.*

$iv)$ *The unit sphere, S_X, is compact.*

$v)$ *X is infinite dimensional.*

The following result is used to give a natural (isometric) identification between a Hilbert space and its dual.

Theorem C.1.3 (Riesz-Frèchet Theorem). *Let H be a Hilbert space and $\phi : H \to H^*$ the map given by $y \mapsto \phi(y)$, where $\phi(y) : H \to \mathbb{K}$ is the functional defined by $\phi(y)(x) := \langle x, y\rangle$, for each $x \in X$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product. Then, phi is an isometric conjugated-linear bijection from H onto H^*.*

We recall to the reader that if $A \subset E$, where E is a topological space, A is said to be first category in E if A is contained in a countable union of closed subsets in E, each of them with non-empty interior. Otherwise, we said A is second category in E.

Theorem C.1.4 (Banach-Steinhaus Theorem). *Let X be a Banach space, Y a normed space* *and $A \subset L(X,Y)$, where $L(X,Y)$ is the set of all linear and continuous operators from X onto Y. Consider $B = \{x \in X : \sup\{\|Tx\| : T \in A\} < \infty\}$. The following assertions are equivalent:*

a) B is second category in X,

b) $B = X$, which means A in punctually bounded.

c) A is uniformly bounded, that means, there exists $M > 0$ such that $\|T\| \leq M$ for all $T \in A$.

C.2 **Weak and Weak* Topologies**

We recall the reader that if X is a normed space and $x \in X$, then

$$\|x\| = \max\{|x^*(x)| : \|x^*\| \leq 1\} = \max\{|\hat{x}(x^*)| : \|x^*\| \leq 1\}.$$

This equality shows that the functional $\hat{x} \in X^{**}$ attains its norm in B_{X^*}, where X^* and X^{**} are the dual and bidual of X, and $\hat{x} : X^* \to \mathbb{R}$ is the evaluate functional $\hat{x}(x^*) := x^*(x)$. On the other side, if $x^* \in X^*$, then there exists $x^{**} \in X^{**}$ such that $\|x^{**}\| = 1$ and $x^{**}(x^*) = \|x^*\|$. Moreover, if X is reflexive, the functional is actually of the form $x^{**} = \hat{x}$ for some $x \in X$, with $\|x\| = 1$, therefore

$$\|x^*\| = \max\{|x^*(x)| : \|x\| \leq 1\}, \quad \text{with } x^* \in X^*,\ X \text{ reflexive.}$$

In other words, x^* attains its norm in B_X. These equalities shows that the supremums are attained, and it has to be proved as consequences of the Hahn-Banach Theorem, and not through the compacity, since even the unit ball of an infinite dimensional normed space is not compact, neither its dual's unit ball. As a matter of fact, this lacking of compact subsets in the infinite dimensional case is a direct consequence of the Riesz Theorem, because every compact subset has empty interior for those cases.

Since the compacity is a very useful property, in stead of the norm topology, mathematicians are pointed to consider smaller topologies, which make continuous the elements of the dual space and provide compact subsets, solving the lacking of compact sets for the case of infinite dimensional Banach spaces with the norm topology. These are the weak and weak* topologies, the first one considered for every normed space, and the latter, for its duals. These two topologies are essential to proof important results in functional analysis.

C.2.1 **Initial topology**

We refer the reader to (47) for all the missed proofs and details about this section.

Let X be a non-empty set, (Y,τ) a topological space and $\mathscr{F}$ a family of maps from X onto Y. We can consider the smallest topology in X for which every element of $\mathscr{F}$ is continuous (by *smallest topology* we mean that the topology is minimal respect to the number of open sets). This is called **the initial topology** in X for the family $\mathscr{F}$, and we shall denote it by $\tau_{\mathscr{F}}$. A basis of open sets for this topology is formed by

$$\bigcap_{i=1}^{q} f_i^{-1}(\mathscr{O}_i): \ \ f_i \in \mathscr{F}, \ \ \mathscr{O}_i \in \tau, \ \ 1 \leq i \leq 1,$$

and, for each $x \in X$, if we take $\mathscr{O}_i$ as a neighborhood of $f_i(x)$ in Y, we have a basis of neighborhoods for $x \in X$. Next, we recall some basic properties of this topology (check (47, chapter 9) for detailed proofs):

Proposition C.2.1. *1. The convergence $\{x_n\} \subset X$ verifies that $\{x_n\} \to x$ in $(X, \tau_{\mathscr{F}})$ if, and only if, $\{f(x_n)\} \to f(x)$ for every $f \in \mathscr{F}$,*

2. If Z is a topological space, a map $T : Z \to X$ is continuous if, and only if, $f \circ T : Z \to Y$ is continuous for every $f \in \mathscr{F}$.

C.2.2 **Weak topology for a normed space**

We keep following the lecture notes (47) in this section, where the reader could check the proofs and details missed.

Let X be a normed space and X^* is dual, that is, the set of continuous and linear functionals. If we consider the particular case of the initial topology in X for the family X^*, we have what is called the **weak topology**. The usual notations for this topology is $\sigma(X,X^*)$, $w(X)$, τ_w or simply w, and every concept in this topology shall usually be preceded by w or weak (weak-compactness, w-convergence, etc.). In case it is possible to confuse the topologies, the terms for the norm topology shall be ahead of $\|\cdot\|$. For a point $x_0 \in X$, a neighborhood basis in the weak topology is formed by

$$\begin{aligned} V(x_0, f_1, \ldots, f_n, \varepsilon) &= \{x \in X : \ |f_i(x) - f_i(x_0)| < \varepsilon, \ i = 1, \ldots, n\} \\ &= \bigcap_{i=1}^{n} \{x \in X : \ |f_i(x) - f_i(x_0)| < \varepsilon\} = x_0 + \bigcap_{i=1}^{n} \{x \in X : \ |f_i(x)| < \varepsilon\} \\ &= x_0 + V(0, f_1, \ldots, f_n, \varepsilon), \quad \text{with } \varepsilon > 0, \ n \in \mathbb{N}, \ f_1, \ldots, f_n \in X^*. \end{aligned}$$

Some properties for the weak topology are

1. The weak topology is Hausdorff, and (X, τ_w) is a **topological vector space**, which means that the addition and the scalar multiplication are continuous. Therefore, the translations and homotecies are homeomorphism in X respect to the weak topology.

2. The weak closure of a subspace or a convex set is also a subspace or convex.

Let us see some examples for specific normed spaces with the weak topology.

Example C.2.2. • *If H is a Hilbert space, as a consequence of Riesz-Frèchet Theorem, see Theorem* (C.1.3)*, the basic weak-neighborhoods for zero are*

$$V(0, x_1, x_2, \dots, x_m, \varepsilon) = \bigcap_{i=1}^{m} \{x \in H : \ |\langle x, x_i\rangle| < \varepsilon\}, \quad \textit{with } \varepsilon > 0, x_i \in H, \forall i = 1, \dots, m.$$

• *Let $p, q \in \mathbb{N}$ such that $1/p + 1/q = 1$, or $p = 1, q = \infty$ (conjugated indices). We consider the sequence space ℓ_p, and its dual $\ell_p^* \cong \ell_q$ (see (47, Chapter 2) for a wider study of sequence spaces), then, in ℓ_p with the weak topology, the basic w-neighborhoods of zero have the form*

$$V(0, y_1, y_2, \dots, y_m, \varepsilon) = \bigcap_{i=1}^{m} \left\{ x \in \ell_p : \ \left| \sum_{k=1}^{\infty} y_i(k)x(k) \right| < \varepsilon \right\},$$

with $\varepsilon > 0,\ y_i \in \ell_q,\ i = 1, \dots, m$.

• *If Ω is a set with positive Lebesgue measure, for the quotient functions space $L_p(\Omega)$ with the weak topology, we know that $L_q(\Omega) \cong L_p(\Omega)^*$ and p, q are under the hypotheses of the above example, the w-neighborhoods of zero are*

$$V(0, g_1, g_2, \dots, g_m, \varepsilon) = \bigcap_{i=1}^{m} \left\{ f \in L_p(\Omega) : \ \left| \int_{\Omega} f(x) g_i(x) dx \right| < \varepsilon \right\},$$

with $\varepsilon > 0, g_i \in L_q(\Omega), i = 1, \dots, m.$

The next result will be a useful tool in proofs along this memoire.

Proposition C.2.3. *Let* $\{f_1, f_2, \dots, f_n\}$ *be a family of linear functionals on a vector space* X. *The following assertions are equivalent*

1. $f \in \operatorname{span}\{f_1, f_2, \dots, f_n\} := Lin(\{f_1, \dots, f_n\})$.
2. *There exist* $C \geq 0$ *such that* $|f(x)| \leq C \max\{|f_i(x)| : \ 1 \leq i \leq n\}$ *for all* $x \in X$.
3. f *is bounded, (majorated or minorated) in* $\bigcap_{i=1}^{n} \ker(f_i)$.
4. $\bigcap_{i=1}^{n} \ker(f_i) \subset \ker(f)$.

Remark C.2.4. *Since* $f \in X^*$ *is continuous respect to the norm topology, it verifies* $\tau_w \subseteq \tau_{\|\cdot\|}$. *Even more, the topological dual characterizes the weak continuity, that is, if* $f : X \to \mathbb{K}$ *is a linear form,* f *is w-continuous, if and only if,* $f \in X^*$.

As a consequence of the proposition above, if $\{f_1, \dots, f_n\}$ are linear forms in X such that $\bigcup_{i=1}^{n} \ker(f_i) = \{0\}$, then $\bigcup_{i=1}^{n} \ker(f_i) \subset \ker(f)$, and $f \in \langle\{f_1, \dots, f_n\}\rangle$. We conclude that $X^{\#}$ is finite dimensional, where $X^{\#}$ denotes the algebraic dual of X, that is, the set of linear forms in X. Therefore, X is also finite dimensional. On the other hand, if X has infinite dimension, the vector subspace $\bigcap_{i=1}^{n} \ker(f_i) \neq \{0\}$. Then, a w-neighborhood of zero $\bigcap_{i=1}^{n} \{x \in X : |f_i(x)| < \varepsilon\}$ contains a non-zero vector subspace $\bigcap_{i=1}^{n} \ker(f_i)$. Thus, if X is an infinite dimensional normed space, every w-neighborhood of zero is not bounded respect to the norm, in particular, the unit ball is not a weak neighborhood of zero, concluding that when the environment is the infinite dimensional case, the contention seen in $C.2.4$ is strict. In turn, through translations, we gather that every weak neighborhood contains, at least, an affine line, and so these w-

neighborhood are not norm-bounded.

Proposition C.2.5. *The weak and the norm topologies coincide if and only if the normed space is finite dimensional.*

The following result establishes the behaviour of convergence respect to the weak topology.

Proposition C.2.6. *Let $\{x_n\}$ be a sequence of a normed space X and $x \in X$. It holds*

1. $\{x_n\} \to x$ *if and only if* $\{f(x_n)\} \to f(x)$*, for all* $f \in X^*$.
2. *Norm convergence implies weak convergence.*
3. *If* $\{x_n\} \xrightarrow{w} x$*, then, the sequence* $\{x_n\}$ *is bounded and* $\|x\| \leq \liminf\{\|x_n\|\}$.
4. *If* $\{x_n\} \xrightarrow{w} x$ *and* $\{f_n\} \xrightarrow{\|\cdot\|} f$ *in* X^**, then* $\{f_n(x_n)\} \to f(x)$.

Corollary C.2.7. *The weak topology is not metrizable for infinite dimensional normed spaces.*

As a consequence of this result, some terms and properties used in metric space are not true in the weak topology environment. This assertion is a direct consequence of (41, Theorem 6.1.6)

The weak and norm topologies are pretty different for infinite dimensional spaces, as we show in the following result.

Proposition C.2.8. *If X is an infinite dimensional normed space, the $\overline{S_X}^w = B_X$. As a consequence, the mapping $x \to \|x\|$ is w-lower semicontinuous but not weak-continuous.*

The following well-known result is a consequence of the Separation Theorems previously recalled for convex sets, and it will be useful in the study of the convex and inner structure of the unit ball in this dissertation.

Theorem C.2.9 (Mazur's Theorem). *The norm-closure and the weak-closure are the same for a convex set C of a normed space X.*

The next result gives a sufficient condition for the weak convergence.

Proposition C.2.10. *Let X be a normed space and $F \subset X^*$ such that $\overline{\langle F \rangle} = X^*$. If a sequence $\{x_n\} \subset X$ is norm-bounded and there exists $x \in X$ such that* $\lim\{f(x_n)\} = f(x)$ *for all $f \in F$, then $\{x_n\} \xrightarrow{w} x$.*

The following example holds as a particular case for this proposition.

Example C.2.11.

- *If $\{x_n\}$ is a bounded sequence in c_0, the space of sequences convergent to 0, and there exists $x \in c_0$ such that $\lim\limits_{n\to\infty} x_n(k) = x(k)$ for all $k \in \mathbb{N}$, then $\{x_n\}$ is weakly convergent to x.*

- *A bounded sequence $\{x_n\}$ in ℓ_p, with $p < 1$, satisfying that there exists $x \in \ell_p$ such that $\lim\limits_{n\to\infty} x_n(k) = x(k)$ for all $k \in \mathbb{N}$, satisfies that $\{x_n\} \xrightarrow{w} x$.*

In these cases, the hypotheses are sufficient but also necessary, that is, the weak convergence in the spaces ℓ_p and c_0 is equivalent to require both pointwise convergence and norm-bounded.

- *Another consequence is when H is a Hilbert space: let $\{e_i : i \in I\}$ be an orthonormal basis of H, then, the weak convergence $\{x_n\} \xrightarrow{w} x$ is equivalent to ask $\{x_n\}$ to be bounded and $\{\langle x_n, e_i \rangle\} \to \langle x, e_i \rangle$ for all $i \in I$.*

C.2.3 Weak* topology for the dual of a normed space

This section is borrowed from (47), and we omit the proofs for a simpler reading.

Let X be a normed space and X^*, its dual. As we have seen, we can consider the weak topology in every normed space, in particular, in X^*. This topology is the initial topology for the linear forms of X^{**}. Besides, we can take the family formed by the evaluation mappings, $\hat{x} : X^* \to \mathbb{K}$, $\hat{x}(x^*) = x^*(x)$, for all $x^* \in X^*$, in other words, the linear forms defined by the elements of X in X^*, which are continuous. What is known as the **weak* topology** of X^* is precisely this initial topology, which is denoted by $\sigma(X^*, X)$. All the topological notions will be preceded by the prefix w^* when the topology involved is the weak* topology. Note that this topology is coarser that the weak topology $\sigma(X^*, X^{**})$.

For an element $x_0^* \in X^*$, a neighborhood basis in the weak* topology has the following form

$$\begin{aligned} V(x_0^*, x_1, x_2, \ldots, x_n, \varepsilon) &= \{x^* \in X^* : \ |x^*(x_i) - x_0^*(x_i)| < \varepsilon, \ 1 \le i \le n\} \\ &= \bigcap_{i=1}^{n} \{x^* \in X^* : |(x^* - x_0^*)(x_i)| < \varepsilon\} = x_0^* + \bigcap_{i=1}^{n} \{x^* \in X^* : \ |x^*(x_i)| < \varepsilon\}, \end{aligned}$$

with $\varepsilon > 0$, $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in X$.

The following result is a consequence of (C.2.1).

Proposition C.2.12. *Let (Y, τ) be a topological space. A mapping $T : (Y, \tau) \to (X^*, w*)$ is continuous if, and only if, for all $x \in X$ the mapping $\hat{x} \circ T : (Y, \tau) \to \mathbb{K}$ is continuous, where*

$$(\hat{x} \circ T)(y) = (T(y))(x), \ \forall y \in Y.$$

Some properties for the weak* topology are gathered in the next result.

Proposition C.2.13. *1. (X^*, w^*) is Hausdorff and a topological vector space. As a consequence, translations and homotecies are homeomorphisms of that topological space.*

2. *The weak* * *closure of a subspace or a convex subset are also a subspace or a convex subset, respectively.*

Remark C.2.14. *Following a similar reasoning as in the weak topology, every* w^**-open subset contains an affine subspace which is not a singleton and, therefore, it is not norm-bounded. Then, in the infinite dimensional case, the open balls are not bounded for the weak* * *topology.*

In summery, in a dual space, we could consider three topologies: the topology of the norm, the weak topology and the weak* topology, each one contained in the previous one.

Proposition C.2.15. *Let* X *be a normed space. A linear functional* $f : X^* \to \mathbb{K}$ *is* w^**-continuous if, and only if, there exists* $x \in X$ *such that* $f = \hat{x}$*, a evaluation functional. That means, the weak and the weak* * *topologies for* X^* *are the same when* X *is reflexive.*

As a consequence, the norm, the weak and the weak * *topologies coincide if, and only if,* X *is finite dimensional.*

Proposition C.2.16. *Let* $\{x_n^*\}$ *be a sequence in* X^**, the dual space of a normed space* X*, and* $x^* \in X^*$*, then:*

1. $\{x_n^*\} \xrightarrow{w^*} x^*$ *if, and only if,* $\{x_n^*(x)\} \to x^*(x)$ *for all* $x \in X$.

2. *If* X *is a Banach space, and* $\{x_n^*\} \xrightarrow{w^*} x^*$*, then* $\{x_n^*\}$ *is norm-bounded and* $\|x^*\| \leq \liminf\{\|x_n^*\|\}$.

The following example displays a w^*-convergent sequence which is not bounded.

Example C.2.17. *Let* $X := (c_{00}, \|\cdot\|_\infty)$*. Notice that* X^* *is linearly isometric to* $(\ell_1, \|\cdot\|_1)$*. For every* $n \in \mathbb{N}$*, let*

$$x_n^* := \frac{1}{n}\sum_{k=1}^{n} \frac{1}{2^k} e_k + \frac{n}{\sum_{k=n+1}^{\infty} \frac{1}{2^k}} \sum_{k=n+1}^{\infty} \frac{1}{2^k} e_k.$$

For each $k \in \mathbb{N}$, $\left(x_n^*(e_k)\right)_{n\in\mathbb{N}}$ converges to 0 because if $n > k$, then $x_n^*(e_k) = \dfrac{1/n}{2^k}$. As a consequence, $\left(x_n^*(x)\right)_{n\in\mathbb{N}}$ converges to 0 for all $x \in c_{00}$ due to the fact that $c_{00} = \operatorname{span}\{e_n : n \in \mathbb{N}\}$. In other words, $\{x_n^*\} \xrightarrow{w^*} 0$. However, notice that

$$\begin{aligned}\left\|x_n^*\right\|_1 &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k} + \frac{n}{\sum_{k=n+1}^{\infty}\frac{1}{2^k}}\sum_{k=n+1}^{\infty}\frac{1}{2^k}\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k} + n\\ &\geq n\\ &\to \infty \text{ as } n \to \infty.\end{aligned}$$

Under the settings of Example C.2.17, notice that, if now $X := (c_0, \|\cdot\|_\infty)$, then $\{x_n^*\} \not\xrightarrow{w^*} 0$. Indeed, if $\{x_n^*\} \xrightarrow{w^*} 0$, the completeness of c_0 forces that $\{x_n^*\}$ is norm-bounded in view of Proposition C.2.16(2), contradicting that $\left\|x_n^*\right\|_1 \to \infty$ as $n \to \infty$.

For a dual normed space, it is interesting how to split w^*-closed convex subsets via w^*-continuous functionals.

Theorem C.2.18 (Separation Theorem for convex subsets with the w^* topology)**.** *If X is a normed space, $A \subset X^*$ a non-empty subset, w^*-closed and convex, and $x_0^* \in X \setminus A$, then, there exists $x \in X$ such that*

$$\sup\{\operatorname{Re} a^*(x) : \ a^* \in A\} < \operatorname{Re} x_0^*(x).$$

Note that under the hypotheses of theorem above, if $\alpha \in \mathbb{R}$ is an arbitrary number satisfying that

$$\sup\{\operatorname{Re} a^*(x) : \ a^* \in A\} < \alpha < \operatorname{Re} x_0^*(x),$$

then, the w^*-closed hyperplane $H = \{x^* \in X : \ \operatorname{Re} x^*(x) = \alpha\}$ strictly splits A and x_0^*, since

$A \subset \{x^* \in X^* : \operatorname{Re} x^*(x) < \alpha\}$ and $x_0^* \in \{x^* \in X^* : \operatorname{Re} x^*(x) > \alpha\}$.

The Mazur Theorem is not true in the weak* environment (it is sufficient to consider a non reflexive normed space X, and the hyperplane $H = \{x^* \in X^* : f(x^*) = 0\}$ with $f \in X^{**} \setminus X$).

Proposition C.2.19. *If X is an infinite dimensional normed space, then $\overline{S_{X^*}}^{w^*} = B_{X^*}$. As a consequence, the mapping $x^* \mapsto \|x^*\|$ is w^*-lower semicontinuous but not w^*-continuous.*

Proposition C.2.20. *Let X be a normed space and $F \subset X$ such that $\overline{\operatorname{span}\{F\}} = X$. If $\{x_n^*\} \subset X^*$ is a norm-bounded sequence, and there exists $x^* \in X^*$ such that $\lim\{x_n^*(y)\} = x^*(y)$, for each $y \in F$, then $\{x_n^*\} \xrightarrow{w^*} x^*$.*

One of the main advantages of building up the weak* topology is appearance of a somehow "large amount" of compact subset For this, we present an important theorem. Firstly, let us consider a non-empty set X and a family of non-empty sets $\{Y_x : x \in X\}$. The cartesian product of such family is denoted by $\prod_{x\in X} Y_x$, and it is the family of all the mappings $f : X \to \cup_{x\in X} Y_x$ satisfying $f(x) \in Y_x$, for all $x \in X$. For each $x \in X$, it is also considered the projection mappings $\pi_x : \prod Y_x \to Y_x$, defined by $\pi_x(f) = f(x)$, for all $f \in \prod_{x\in X} Y_x$, and suppose that each (Y_x, τ_x) is a topological space. In this case, the *product topology* in $\prod_{x\in X} Y_x$ is the initial topology for the family $\{\pi_x : x \in X\}$, that is, it is the smaller topology in $\prod_{x\in X} Y_x$ which make the projections continuous. The sets which form the basis of this topology are

$$\bigcap_{x\in J} \pi_x^{-1}(U_x) = \bigcap_{x\in J} \left\{ f \in \prod_{x\in X} Y_x : f(x) \in U_x \right\}, \quad U_x \in B_x,\ J \subset X, J \text{ finite},\ B_x \text{ basis of } \tau_x.$$

Theorem C.2.21 (Tychonoff Theorem). *The product of compacts topological spaces, with the* *product topology, is a compact topological space.*

In the particular case $Y_x = \mathbb{K}$ for all $x \in X$, the product $\prod_{x\in X} Y_x = \mathbb{K}^X$ are all the mappings from X to $\mathbb{K}$. This reasoning is key for the proof of the following well-known result (it is detailed in the reference given at the beginning of this section).

Theorem C.2.22 (Banach-Alaoglu Theorem)**.** *The closed unit ball of a dual space is w^*-compact. As a consequence, every subset of the dual of a normed space verifying that is w^*-closed and norm-bounded, is w^*-compact.*

The following results shows the behaviour of a normed space seen into its bidual.

Theorem C.2.23 (Goldstein Theorem)**.** *Let X be a normed space, and X^{**} its bidual. Then, the unit ball B_X seen into X^{**} is dense in $B_{X^{*}*}$ respect to the weak$*$ topology in X^{**}. As a consequence, $X \subset X^{**}$ is dense in X^{**} respect to its w^*-topology.*

Corollary C.2.24. *If X is an infinite dimensional normed space, then $S_X \subset X^{**}$ is w^*-dense in $B_{X^{**}}$.*

Nest results solves the lacking of compact sets for the weak topology when the environment normed space is reflexive.

Theorem C.2.25 (Dieudonné Theorem)**.** *A normed space is reflexive if, and only if, its unit ball is weakly compact. Therefore, every w-closed subset which is norm-bounded is w-compact for a reflexive normed space.*

The following consequence shows how assorted are the $C(K)$ type spaces.

Corollary C.2.26. *Let X be a normed space. Then, there exists a Hausdorff compact topological space, K, such that X is isometrically isomorphic to a subspace of $C(K)$.*

C.3 Another important results in Functional Analysis

The purpose of this section is to give a compilation of some important results in functional analysis that will be used along this memory. For details about the proofs or a wider perspective on this topics, we refer the reader to (41).

We start by recalling a so-called result related to the Extreme Theory in topological vector spaces:

Theorem C.3.1 (Krein-Milman Theorem). *For a Hausdorff locally convex topological vector space X and a non-empty compact subset $K \subseteq X$, it holds that $K \subseteq \overline{\text{co}}(\text{ext}(K))$. In particular, if K is closed and convex, then it is the closed convex hull of its extreme points.*

Another important results in the study of weakly compacity.

Theorem C.3.2 (Eberlein-Smulian Theorem). *Let X be a normed space. A subset $K \subseteq X$ is relatively weakly compact if and only if every sequence of K has a weakly convergent subsequence.*

Theorem C.3.3 (Krein-Smulian Theorem). *If K is a weakly compact subset of a Banach space X, then, the closed convex hull of K, denoted by $\overline{\text{co}}(K)$, is weakly compact.*

Following result established an equivalence between w-convergence and $\|\cdot\|$-convergence in ℓ_1.

Theorem C.3.4 (Schur Theorem). *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of ℓ_1. Then, (x_n) is weakly convergent if and only if $(x_n)_{n\in\mathbb{N}}$ is convergent respect to the norm. In this case,*
$w\lim_{n\to\infty}(x_n)_{n\in\mathbb{N}} = \lim_{n\to\infty}(x_n)_{n\in\mathbb{N}}$.

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