Lie models of classifying fibrations

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Contents

Introduction

Broadly speaking, Rational Homotopy Theory deals with the homotopical behavior of the non-torsion part of topological spaces. For it, and in general terms, one first associates to any reasonable space, say a CW-complex, another space (call it rational) which keeps just the rational information of the homotopy invariants of the original space. Then, one finds algebraic models which faithfully determines the homotopy type of any rational space. This last task can be accomplished in two different way. One is the *Sullivan approach* [50] by which the homotopy type of any rational (nilpotent) complex of finite type can be functorially encoded by a commutative differential graded algebra (cdga henceforth). The other one is the *Quillen approach* [42] based on that any simply connected rational complex can be uniquely determined up to homotopy, and also in a functorial manner, by a differential graded Lie algebra (dgl from now on). More precisely, Quillen constructed a pair of functors between the categories of positively graded dgl's and 1-reduced simplicial sets which induce equivalences when passing to the respective homotopy categories.

The main disadvantage of this approach lies in the restriction, imposed somehow ad hoc by the construction of these functors, of considering "simply connected" simplicial sets on one side and dgl's graded over the positive integers on the other. However, this drawback has been recently overcome by U. Buijs, Y. Félix, A. Murillo and D. Tanré in [14]. In very brief terms, they construct a pair of adjoint functors, "model" and "realization",

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl} \qquad (1)$$

between the categories of simplicial sets (under no connectivity assumptions) and that of *complete* differential graded Lie algebras (cdgl henceforward). These are dgl's endowed with a filtration of differential Lie ideals which determines a complete topology. In other terms, the inverse limit of the quotients by these ideals recovers the original dgl. There is a model category structure on **cdgl** which extends the traditional one on positively graded dgl's for which these functors become a Quillen pair. Moreover, they both extend, up to homotopy, the original Quillen functors [11].

At this point there are many interesting classes of topological objects, whose usefulness has been proved in many instances, which are now susceptible of being studied from the rational point of view within this new general framework.

One of these classes, preeminent in this thesis, consists on the so called *universal fibrations*. These are fibration sequences which classify, by means of homotopy classes of maps into their base spaces, certain types of fibrations with prescribed fiber.

The first, original, and more general result in this direction was proved by J. Stasheff in [48] where homotopy types of fibrations sequences $X \to E \to B$, with fixed fibre X, were classified by homotopy classes of maps $[\![B, B\operatorname{aut}(X)]\!]$, where $B\operatorname{aut}(X)$ denotes the classifying space, i.e., the geometrical bar construction of the topological monoid $\operatorname{aut}(X)$ of "homotopy automorphism", that is, self homotopy equivalences of X. Explicitly, this bijection assigns to each homotopy class of a map $f\colon B \to B\operatorname{aut}(X)$ the pullback fibration over f of the *universal fibration sequence*

$$X \longrightarrow B\operatorname{aut}^*(X) \longrightarrow B\operatorname{aut}(X) \tag{2}$$

resulting of applying the geometrical bar construction to the inclusion $\operatorname{aut}^*(X) \hookrightarrow \operatorname{aut}(X)$, being $\operatorname{aut}^*(X)$ the submonoid of $\operatorname{aut}(X)$ consisting of pointed self homotopy equivalences.

One readily observes that, even if X is simply connected, $B\operatorname{aut}^*(X)$ and $B\operatorname{aut}(X)$ are not. Far from that, their fundamental groups $\pi_1 B\operatorname{aut}^*(X) = \pi_0\operatorname{aut}^*(X)$ and $\pi_1 B\operatorname{aut}(X) = \pi_0\operatorname{aut}(X)$ are, respectively, the groups $\mathcal{E}^*(X)$ and $\mathcal{E}(X)$ of pointed and free homotopy classes of pointed and free self homotopy equivalences. As the reader is aware (see the general reference [43]) these groups, and the monoids of self equivalences from which they emerge, are shown to be of vital importance in many topological contexts.

Nevertheless, and as long as we start with a simply connected complex X, the classical Quillen approach let us describe in algebraic terms the simply connected cover

$$X \longrightarrow B\widetilde{\operatorname{aut}^*(X)} \longrightarrow B\widetilde{\operatorname{aut}(X)} \tag{3}$$

of the universal fibration (2). Indeed, let L be the minimal Quillen model of the simply connected complex X of finite type and consider the dgl $\operatorname{Der} L$ of derivations of L with the usual Lie bracket and differential. We then truncate this dgl to obtain its *simply connected cover* $\widetilde{\operatorname{Der} L}$ consisting of all derivations of degree bigger than or equal to 2 and the kernel of the differential in degree 1. Finally consider the dgl sequence

$$L \xrightarrow{\operatorname{ad}} \widetilde{\operatorname{Der} L} \longrightarrow \widetilde{\operatorname{Der} L}\,\tilde{\times}\, sL, \tag{4}$$

where ad is the usual adjoint operator and, in the "twisted product" $\widetilde{\operatorname{Der} L}\,\tilde{\times}\, sL$:

- sL denotes the suspension of L, it is a sub abelian Lie algebra and $Dsx = -sdx + \operatorname{ad}_x$ for any $x \in L$.
- $\widetilde{\operatorname{Der} L}$ is a sub dgl and $[\theta, sx] = (-1)^{|\theta|} s\theta(x)$ for any $\theta \in \widetilde{\operatorname{Der} L}$ and any $x \in L$.

Then, the following was proved in [51, Corollary VII.4.(4)] (cf. [45]):

Theorem A. *This dgl sequence is a Quillen model of the fibration sequence (3).*

The situation drastically changes in the general scenario. Indeed, $B\operatorname{aut}(X)$ is a complicated space and, even if X is simply connected and rational, its rational behaviour may not be easily described. In fact, we show in Example 7.1 that, for any $n \geq 1$, the classifying space $B\operatorname{aut}(S^n_{\mathbb{Q}})$ does not lie within the image of the realization functor $\langle - \rangle$ in (1).

Nevertheless, we find in this work a large class of universal fibrations whose rational homotopy type can be determined by Lie models, always involving particular cdgl's of derivations.

Let X be a CW-complex and let $\mathcal{H}$ be a subgroup of $\mathcal{E}(X) = \pi_0 \operatorname{aut}(X)$. We denote by $\operatorname{aut}_{\mathcal{H}}(X)$ the topological monoid of those self homotopy equivalences of X such that their homotopy classes lie in $\mathcal{H} \subset \mathcal{E}(X)$. Similarly we can define $\operatorname{aut}^*_{\mathcal{H}}(X)$ as those pointed self homotopy equivalences of X whose free homotopy classes lie in $\mathcal{H}$.

Applying the geometric-bar construction, we obtain new spaces $B\operatorname{aut}_{\mathcal{H}}(X)$ and $B\operatorname{aut}^*_{\mathcal{H}}(X)$ which can be shown to be covering spaces (up to homotopy) of the classifying spaces $B\operatorname{aut}(X)$ and $B\operatorname{aut}^*(X)$ respectively. In particular, their higher homotopy groups agree while $\pi_1 B\operatorname{aut}_{\mathcal{H}}(X) = \pi_0 \operatorname{aut}_{\mathcal{H}}(X) = \mathcal{H}$. The inclusion $\operatorname{aut}^*_{\mathcal{H}}(X) \hookrightarrow \operatorname{aut}_{\mathcal{H}}(X)$ induces a fibration sequence

$$X \to B\operatorname{aut}^*_{\mathcal{H}}(X) \to B\operatorname{aut}_{\mathcal{H}}(X) \tag{5}$$

which classifies a certain type of fibrations: recall that an arbitrary fibration sequence $X \to E \to B$ determines an action of $\pi_1(B)$ on the fiber X, the *holonomy action*, which in turn defines a group homomorphism

$$\pi_1(B) \longrightarrow \mathcal{E}(X). \tag{6}$$

Then, fibration sequences with fiber X can be cataloged depending on the image of this homomorphism. If its image lies in $\mathcal{H} \subset \mathcal{E}(X)$, we say that it is an $\mathcal{H}$-fibration sequence. Then, in Theorem 2.17 we prove

Theorem B. *Given a subgroup $\mathcal{H} \subset \mathcal{E}(X)$, for any CW-complex B, the set of equivalence classes of $\mathcal{H}$-fibrations over B with fiber X is naturally isomorphic to the set $[\![B, B\operatorname{aut}_{\mathcal{H}}(X)]\!]$ of homotopy classes of maps from B to $B\operatorname{aut}_{\mathcal{H}}(X)$. This bijection assigns to each homotopy class of a given map $B \to B\operatorname{aut}_{\mathcal{H}}(X)$ the pullback over this map of the universal $\mathcal{H}$-fibration sequence*

$$X \to B\operatorname{aut}^*_{\mathcal{H}}(X) \to B\operatorname{aut}_{\mathcal{H}}(X).$$

We plan to find cdgl models of certain classes of these universal fibrations. For it we fix X a nilpotent finite complex and $\mathcal{H} \subset \mathcal{E}(X)$ a group of homotopy classes of self homotopy equivalences which acts nilpotently on the homology of X. Then, [9, Theorem D] affirms that both $B\operatorname{aut}_{\mathcal{H}}(X)$ and $B\operatorname{aut}^*_{\mathcal{H}}(X)$ are nilpotent spaces and a straightforward argument shows that the rationalization of (5) has the homotopy type of the universal rational fibration sequence

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

Let L be a cdgl model of X and denote by $\mathcal{E}^*(L)$ the group of homotopy classes of cdgl automorphism of L. The group $H_0(L)$ endowed with the Baker-Campbell-Hausdorff product acts on $\mathcal{E}^*(L)$ via the exponential map. Explicitly, the map

$$H_0(L) \longrightarrow \mathcal{E}^*(L), \quad [x] \mapsto e^{\mathrm{ad}_x},$$

is a well defined group morphism. We then define $\mathcal{E}(L) = \mathcal{E}^*(L)/H_0(L)$ so that there is an exact sequence of group morphisms

$$H_0(L) \longrightarrow \mathcal{E}^*(L) \longrightarrow \mathcal{E}(L) \to 0.$$

This mimics at the cdgl level the topological context: recall the surjective map $\mathcal{E}^*(X) \xrightarrow{\zeta} \mathcal{E}(X)$ which just forgets about the basepoint and the action of $\pi_1(X)$ on $\mathcal{E}^*(X)$ whose orbit set is precisely $\mathcal{E}(X)$. This translates to an exact sequence of group morphisms

$$\pi_1(X) \longrightarrow \mathcal{E}^*(X) \xrightarrow{\zeta} \mathcal{E}(X) \to 0$$

which turns out to be equivalent to the above one in the rational setting. Indeed, there is a commutative diagram (Remark 4.11)

$$\begin{tikzcd}
\pi_1(X_{\mathbb{Q}}) \arrow[r] \arrow[d, "\cong"] & \mathcal{E}^*(X_{\mathbb{Q}}) \arrow[r, "\zeta"] \arrow[d, "\cong"] & \mathcal{E}(X_{\mathbb{Q}}) \arrow[r] \arrow[d, "\cong"] & 0 \\
H_0(L) \arrow[r] & \mathcal{E}^*(L) \arrow[r] & \mathcal{E}(L) \arrow[r] & 0.
\end{tikzcd} \tag{7}$$

Therefore, for the fixed subgroup $\mathcal{H} \subset \mathcal{E}(X)$ acting nilpotently on the homology, we can consider its rationalization $\mathcal{H}_{\mathbb{Q}} \subset \mathcal{E}(X_{\mathbb{Q}})$, take its preimage $\zeta^{-1}(\mathcal{H}_{\mathbb{Q}}) \subset \mathcal{E}^*(X_{\mathbb{Q}})$ and identify it with a subgroup $\mathcal{h} \subset \mathcal{E}^*(L)$. We then define $\operatorname{Der}^{\mathcal{h}} L \subset \operatorname{Der} L$ as

$$\operatorname{Der}^{\mathcal{h}}_{\geq 1} L = \operatorname{Der}_{\geq 1} L, \qquad \operatorname{Der}^{\mathcal{h}}_0 L = \{\theta \in \operatorname{Der}_0 L \text{ such that } D\theta = 0 \text{ and } [e^{\theta}] \in \mathcal{h}\}.$$

This means that we consider all the derivations in positive degree and those derivations whose exponentials are cdgl automorphisms whose homotopy classes lie in $\mathcal{h}$. Then, the following, which is Theorem 7.13, constitutes one of the main results of this text.

Theorem C. *Let L be a Lie model of a nilpotent space X and $\mathcal{H} \subset \mathcal{E}(X)$ a subgroup acting nilpotently on the homology of X. Then, the analogue of (4)*

$$L \xrightarrow{\mathrm{ad}} \operatorname{Der}^{\mathcal{h}} L \to \operatorname{Der}^{\mathcal{h}} L \widetilde{\times} sL \tag{8}$$

is a cdgl sequence whose realization is homotopy equivalent to the universal $\mathcal{H}_{\mathbb{Q}}$-fibration sequence

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

The proof of this result have, roughly speaking, two cornerstones.

The first one, although it may seem technical at first sight, lies at the very foundation of the general theory: only dgl's which are complete are susceptible of being geometrically realized. However, connected sub dgl's of $\operatorname{Der} L$ are not complete in general, even if L is simply connected. The same applies to twisted product of dgl's. Hence we first need to check the highly non trivial fact that (8) is a cdgl sequence. In fact, showing simply that $\operatorname{Der}^{\mathcal{H}} L$ is closed by the operations of sum and the Lie bracket requires a deep study of the Malcev equivalence between Malcev $\mathbb{Q}$-complete groups and complete (ungraded) Lie algebras. On the other hand, the completeness of the $\operatorname{Der}^{\mathcal{H}} L$ relies in a specific filtration of L depending on the lower central series of the action of $\mathcal{H} \subset \mathcal{E}(X)$ on $H_*(X)$. The same applies to the twisted product $\operatorname{Der}^{\mathcal{H}} L \widetilde{\times} sL$.

Once this has been sorted out, the fact that the geometrical realization of (8) has the asserted homotopy type is deduced from a more ambitious statement. Indeed, the sequence (8) of connected cdgl's fits in a larger one

$$\operatorname{Hom}(\overline{\mathscr{C}}(L), L) \to \operatorname{Hom}(\mathscr{C}(L), L) \to L \stackrel{\mathrm{ad}}{\to} \operatorname{Der}^{\mathcal{H}} L \to \operatorname{Der}^{\mathcal{H}} L \widetilde{\times} sL, \tag{9}$$

where $\mathscr{C}(L)$ is the cocommutative differential graded coalgebra provided by the classical Quillen *chain functor*, $\overline{\mathscr{C}}(L)$ denotes the augmentation ideal of $\mathscr{C}(L)$ and both, $\operatorname{Hom}(\overline{\mathscr{C}}(L), L)$ and $\operatorname{Hom}(\mathscr{C}(L), L)$, are considered as cdgl's with the usual differential and the convolution Lie bracket. These two highly non connected cdgl's are, respectively, Lie models of the pointed and free mapping spaces

$$\operatorname{map}^*(X, X) \quad \text{and} \quad \operatorname{map}(X, X).$$

Then, by restricting these models to the algebraic components representing pointed and free automorphisms of $X_{\mathbb{Q}}$, and leaving untouched the last part of (9), we find a cdgl sequence whose geometrical realization is the fibration sequence

$$\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \hookrightarrow \operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \xrightarrow{\mathrm{ev}} X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$$

where ev denotes the evaluation at the base point.

Theorems B and C have a "homotopical version" which we now describe. For a fixed pointed CW-complex X we may consider fibration sequence of pointed spaces $X \to E \to B$ endowed with a homotopy section (pointed fibrations from now on). Then, given $\Pi \subset \mathcal{E}^*(X)$ a subgroup of pointed homotopy classes of pointed self equivalences, we may consider the class of Π-fibrations consisting of pointed fibrations as above where the image of the holonomy action, this time understood as $\pi_1(X) \to \mathcal{E}^*(X)$, lies in Π. Then, Theorem 2.39 for A a point, reads

Theorem D. *Given a subgroup $\Pi \subset \mathcal{E}^*(X)$, for any CW-complex B, the set of equivalence classes of Π-fibrations over B with fiber X is naturally isomorphic to the set of homotopy classes $[\![B, B\operatorname{aut}^*_{\Pi}(X)]\!]$. This bijection assigns to each homotopy classes of a given map $B \to B\operatorname{aut}^*_{\Pi}(X)$ the pullback over this map of a universal fibration of the form*

$$X \longrightarrow Z \longrightarrow B\operatorname{aut}^*_{\Pi}(X). \tag{10}$$

Here, Z is the general geometrical bar construction applied to the triple $(*, \operatorname{aut}^*_{\Pi}(X), X)$.

Now, let X be a nilpotent finite complex with Lie model L and let $\Pi \subset \mathcal{E}^*(X)$ a subgroup which acts nilpotently on $\pi_*(X)$. Then, Π is a nilpotent group and $B\operatorname{aut}^*_\Pi(X)$ is a nilpotent space so that (10) is a nilpotent fibration whose rationalization is the universal fibration

$$X_\mathbb{Q} \longrightarrow Z_\mathbb{Q} \longrightarrow B\operatorname{aut}^*_{\Pi_\mathbb{Q}}(X_\mathbb{Q}).$$

As in the "homological setting", $\Pi_\mathbb{Q}$ is isomorphic to a subgroup $\pi \subset \mathcal{E}^*(L)$ of homotopy classes of automorphisms of L and again we can consider $\operatorname{Der}^\pi L \subset \operatorname{Der} L$ with

$$\operatorname{Der}^\pi_{\geq 1} L = \operatorname{Der}_{\geq 1} L, \quad \operatorname{Der}^\pi_0 L = \{\theta \in \operatorname{Der}_0 L \mid D\theta = 0, [e^\theta] \in \pi\}.$$

Then, Theorem 7.18 reads:

Theorem E. *Let X be a finite nilpotent complex with Lie model L. Let $\Pi \subset \mathcal{E}^*(X)$ a subgroup which acts nilpotently on $\pi_*(X)$ and which is invariant under the action of $\pi_1(X)$. Then, the universal fibration*

$$X_\mathbb{Q} \longrightarrow Z_\mathbb{Q} \longrightarrow B\operatorname{aut}^*_{\Pi_\mathbb{Q}}(X_\mathbb{Q})$$

has the homotopy type of the realization of the cdgl sequence

$$L \to L \widetilde{\times} \operatorname{Der}^\pi L \to \operatorname{Der}^\pi L.$$

Theorems C and E, together with the auxiliary results leading to them, have interesting applications. The first immediate consequence, for a given finite nilpotent complex X, is the algebraic description of the rationalization of given subgroups $\mathcal{H} \subset \mathcal{E}(X)$ and $\Pi \subset \mathcal{E}^*(X)$ which act nilpotently on the homology and homotopy groups of X respectively. Their classifying spaces can also be formulated in terms of derivations. Indeed, with the notation used above:

Corollary A. $\Pi_\mathbb{Q} \cong H_0(\operatorname{Der}^\pi L)$ *and* $\mathcal{H}_\mathbb{Q} \cong H_0(\operatorname{Der}^\hbar L)/\operatorname{Im} H_0(\operatorname{ad})$.

Corollary B. *$B\mathcal{H}$ and $B\Pi$ have the rational homotopy type of the realization of the cdgl's $\operatorname{Der}^\hbar_0 L \oplus R$ and $\operatorname{Der}^\pi_0 L \oplus S$ where R and S denote a complement of the 1-cycles of $\operatorname{Der}^\hbar L \widetilde{\times} sL$ and $\operatorname{Der}^\pi L$ respectively.*

With the same notation and within the same context, we may also compute the homotopy nilpotency index of $\operatorname{aut}^*_{\Pi_\mathbb{Q}}(X_\mathbb{Q})$ and $\operatorname{aut}_{\mathcal{H}_\mathbb{Q}}(X_\mathbb{Q})$ in a purely algebraic way. Recall that given an H-group Y, its homotopy nilpotency index $\operatorname{nil} Y$, is the least integer $n \leq \infty$ for which the $(n+1)$th homotopy commutator of Y is homotopically trivial. On the other hand, for any dgl L we denote by $\operatorname{nil} L$ the usual nilpotency index. Then:

Corollary C. $\operatorname{nil}\operatorname{aut}^*_{\Pi_\mathbb{Q}}(X_\mathbb{Q}) = \operatorname{nil} H(\operatorname{Der}^\pi L)$ *and* $\operatorname{nil}\operatorname{aut}_{\mathcal{H}_\mathbb{Q}}(X_\mathbb{Q}) = \operatorname{nil} H(\operatorname{Der}^\hbar L \widetilde{\times} sL)$.

We also show in Proposition 8.6 that any finitely generated nilpotent rational group is isomorphic to a subgroup $\mathcal{H} \subset \mathcal{E}(X)$ which acts nilpotently on the homology of X for some space X. In the same way, in Proposition 8.7 we prove that such a group is also isomorphic to a subgroup $\Pi \subset \mathcal{E}^*(Y)$ which acts nilpotently on the homotopy groups of Y for some space Y.

Finally, we use Theorems C and E to obtain models of others interesting fibration sequences involving classifying spaces of topological monoids of self homotopy equivalences, (see Theorem 8.8).

Structure of the thesis

All the material sketched above is presented throughout 8 chapters, dealing with different topics.

Chapter 1 is primarily devoted to an introduction and presentation of the foundations of the homotopy theory of complete differential graded Lie algebras, the main algebraic object of this thesis, whose main reference is [14], where this theory is exhaustively developed. Other preliminary material, as the basics of simplicial sets, the model structure of some model categories and some topics about nilpotency and rationalization, are also outlined in this section.

The reader is supposed to be familiar with the main aspects of classical rational homotopy theory (see for example [12] or [51]). Nevertheless, some of its basics facts are introduced in this first section and some properties can be deduced by applying the new homotopy theory of cdgl's to simply connected spaces.

In §2 we study the classification of fibrations and pointed fibrations with prescribed holonomy actions. In particular, we state and prove Theorem B, which is Theorem 2.17 in the text, and Theorem D, which is Theorem 2.39 whenever A is a point. This chapter is, somehow, independent to the rest of the text, in the sense that the results presented are not bounded to the rational context: they are valid for arbitrary CW-complexes and arbitrary subgroups of $\mathcal{E}(X)$ or $\mathcal{E}^*(X)$. The main reference for this chapter is [32].

Chapter 3 starts with the statement of the Malcev equivalence on its more general form, a equivalence between the categories of complete Lie algebras and Malcev complete groups (see [16, §8]). We include here Theorem 3.5, a useful characterization of a Malcev complete group as pronilpotent and 0-local.

Then we focus on subgroups of $\operatorname{aut}(L)$ and their corresponding subgroups of $\mathcal{E}^*(L)$ by choosing homotopy classes. We show in Theorem 3.9 that, for a given connected minimal cdgl L, if $K \subset \mathcal{E}^*(L)$ is a nilpotent 0-local group, then $\operatorname{aut}_K(L)$ is Malcev complete, and therefore equivalent to a complete Lie algebra. We also prove in this chapter that the subgroup $\operatorname{aut}_1(L)$ of automorphisms of L homotopic to the identity are precisely the *inner automorphisms*, i.e., the exponential $e^{D(\operatorname{Der}_1 L)}$ of the boundary of derivations of degree 0 (see Theorem 3.7).

The presence of non trivial fundamental groups makes it necessary to translate elementary topological facts to the new homotopy theory of cdgl's. We do that in §4 where we present a detailed description of the "pointed" $[\![L, L']\!]^* = \operatorname{Hom}_{\mathbf{cdgl}}(L, L')/\sim$ and "free" $[\![L, L']\!] = [\![L, L']\!]^*/H_0(L')$ homotopy classes of cdgl's morphisms. We also exhibit how this mimics perfectly the topological context. In particular the commutative diagram (7) is readily obtained.

We also introduced in this chapter the *algebraic holonomy action* for a given cdgl fibration sequence,

$$0 \to L \longrightarrow M \longrightarrow N \to 0,$$

which is given by the cdgl morphism

$$H_0(N) \longrightarrow \mathcal{E}(L), \quad [x] \mapsto \overline{[e^{\operatorname{ad}_x}]},$$

and also mimics the topological counterpart (see Remark 4.11).

In chapter 5 we make a very detailed study of Lie models of the evaluation fibration sequence

$$\mathrm{map}^*(X,Y) \to \mathrm{map}(X,Y) \xrightarrow{\mathrm{ev}} Y$$

and its restriction to some connected components by means of convolutions cdgl's and cdgl's of derivations. These Lie models will prove to be essential for the proof of our main results.

As we stressed above, $\mathrm{Der}\, L$ may not be complete even if L is. In the same way, the twisted product of two cdgl's may also fail to be complete. In §6, which the reader may consider technical, we present some connected sub dgl of derivations of a given cdgl which can be endowed with particular filtrations so that they become complete. This is a highly non-trivial problem, since the natural attempts may fail. Explicitly, we will see how a filtration on the homology of a finite complex induces a filtration on its Lie model L and, in turn, it determines a filtration in a specific connected sub dgl of $\mathrm{Der}\, L$ which makes it complete while keeping all positive derivations. Analogous result are proved for some twisted products of certain cdgl's. As a result, we get the desired 'closeness' of the category of cdgl's under these operations.

Chapter 7 contains Theorems C and E, and their detailed proofs.

Finally, in §8 we include all the consequence of Theorems C and E sketched above. Here, we also see how Theorem C effectively generalizes the classical Theorem A .

We finally remark that the content of this thesis is an expanded and detailed version of the results presented in the references [10] and [18].

Resumen en Español

La Teoría de Homotopía Racional trata el comportamiento homotópico de la parte sin torsión de los espacios topológicos. Para ello, y en términos generales, primero se asocia a un espacio razonable, digamos un CW-complejo, otro espacio (llamado racional) que mantiene solo la información racional de los invariantes homotópicos del espacio original. Entonces, se pueden encontrar modelos algebraicos que determinan de forma fiel el tipo de homotopía de cualquier espacio racional. Esta última tarea puede lograrse de dos maneras diferentes. Una es el *método de Sullivan* [50] por el cual el tipo de homotopía de cualquier complejo (nilpotente) racional de tipo finito puede ser funtorialmente codificado por una algebra graduada diferencial conmutativa (o cdga por sus siglas en inglés). El otro enfoque es el *método de Quillen* [42] basado en que cualquier complejo racional simplemente conexo puede ser únicamente determinado (salvo homotopía), y de forma funtorial también, por un álgebra de Lie graduada diferencial (o dgl por sus siglas en inglés). Más precisamente, Quillen construyó un par de funtores entre la categoría de dgls positivamente graduadas y la de conjuntos simpliciales 1-reducidos, que inducen una equivalencia en las respectivas categorías homotópicas.

La principal desventaja de esta aproximación recae en la restricción, impuesta por la construcción ad hoc de estos funtores, de considerar conjuntos simpliciales “simplemente conexos” en un lado y dgls positivamente graduadas en el otro.

Sin embargo, este inconveniente ha sido recientemente resuelto por U. Buijs, Y. Félix, A. Murillo and D. Tanré en [14]. En términos breves, en este libro, construyen un par de funtores adjuntos, denominados “modelo” y “realización”.

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl} \tag{1}$$

entre las categorías de conjuntos simpliciales (sin restricciones de conexión) y la de álgebras de Lie graduadas diferenciales *completas* (o cdgl). Estas son dgls junto a una filtración de ideales diferenciales y de Lie que determina una topología completa. Dicho de otra manera, el límite inverso de los cocientes por dichos ideales recupera la dgl original. Existe una estructura de categoría de modelo en **cdgl** que extiende a la tradicional en la categoría de dgls positivamente graduadas, de forma que estos funtores resultan ser un par de Quillen. Además, ambos funtores extienden, salvo homotopía, los funtores originales de Quillen [11].

En este punto, hay muchas clases interesantes de objetos topológicos cuya utilidad ha sido probada en diversas ocasiones, que ahora son susceptibles de ser estudiados desde el punto de vista racional con esta nueva maquinaria.

Una de estas clases, preeminente en esta tesis, consiste en las llamadas *fibraciones universales*. Estas son sucesiones fibrantes que clasifican, mediante clases de homotopía de aplicaciones hacia los espacios base, ciertos tipos de fibraciones con una fibra dada.

El primer resultado original en esta dirección fue probado por J. Stasheff en [48], donde los tipos de homotopía de las sucesiones fibrantes $X \to E \to B$, con fibra X fijada, fueron clasificados por las clases de homotopía de aplicaciones $[\![B, B\operatorname{aut}(X)]\!]$ donde $B\operatorname{aut}(X)$ denota el espacio clasificante, es decir, la construcción barra geométrica del monoide topológico $\operatorname{aut}(X)$ de "automorfismos homotópicos", esto es, las autoequivalencias homotópicas de X. Explícitamente, esta biyección asigna a cada clase de homotopía de una aplicación $f\colon B \to B\operatorname{aut}(X)$ la fibración pullback sobre f de la *sucesión fibrante universal*

$$X \longrightarrow B\operatorname{aut}^*(X) \longrightarrow B\operatorname{aut}(X) \tag{2}$$

que es el resultado de aplicar la construcción barra geométrica a la inclusión $\operatorname{aut}^*(X) \hookrightarrow \operatorname{aut}(X)$, siendo $\operatorname{aut}^*(X)$ el submonoide de $\operatorname{aut}(X)$ consistiendo en autoequivalencias homotópicas punteadas.

De forma directa se observa que, incluso si X es simplemente conexo $B\operatorname{aut}^*(X)$ y $B\operatorname{aut}(X)$ pueden no serlo. De hecho, sus grupos fundamentales $\pi_1 B\operatorname{aut}^*(X) = \pi_0\operatorname{aut}^*(X)$ y $\pi_1 B\operatorname{aut}(X) = \pi_0\operatorname{aut}(X)$ son, respectivamente, los grupos $\mathcal{E}^*(X)$ y $\mathcal{E}(X)$ de clases de homotopía, libre y punteada, de autoequivalencias, libres y punteadas, homotópicas de X. Estos grupos, y los monoides de autoequivalencias de los que surgen, han probado ser de vital importancia en muchos contextos topológicos (véase la referencia general [43]).

No obstante, siempre que partamos de un complejo simplemente conexo X, el método de Quillen clásico nos permite describir en términos algebraicos el recubridor universal

$$X \longrightarrow B\widetilde{\operatorname{aut}^*(X)} \longrightarrow B\widetilde{\operatorname{aut}(X)} \tag{3}$$

de la fibración universal (2). Ciertamente, sea L el modelo de Quillen minimal de un complejo simplemente conexo X de tipo finito y considera la dgl $\operatorname{Der} L$ de derivaciones de L con el corchete de Lie y la diferencial usuales.

A continuación, truncamos esta dgl para obtener su *recubrimiento simplemente conexo* $\widetilde{\operatorname{Der} L}$ consistiendo en aquellas derivaciones de grado mayor o igual que 2 y el kernel de la diferencial en grado 1. Finalmente, considera la sucesión de dgls

$$L \xrightarrow{\operatorname{ad}} \widetilde{\operatorname{Der} L} \longrightarrow \widetilde{\operatorname{Der} L} \tilde{\times} sL, \tag{4}$$

donde ad es el operador adjunto usual y, en el "producto torcido" $\widetilde{\operatorname{Der} L} \tilde{\times} sL$:

- sL denota la suspensión de L, es una sub algebra de Lie abeliana y $Dsx = -sdx + \operatorname{ad}_x$ para todo $x \in L$.

- $\widetilde{\operatorname{Der} L}$ es una sub dgl y $[\theta, sx] = (-1)^{|\theta|} s\theta(x)$ para cualquier $\theta \in \widetilde{\operatorname{Der} L}$ y cualquier $x \in L$.

Entonces, el siguiente teorema fue probado en [51, Corollary VII.4.(4)] (cf. [45]):

Teorema A. *Esta sucesión de dgls es un modelo de Quillen de la sucesión fibrante (3).*

La situación cambia dramáticamente en el caso general. Ciertamente $B\operatorname{aut}(X)$ es un espacio complicado e incluso si X es simplemente conexo y racional, su comportamiento homotópico puede no ser fácilmente descrito. De hecho, mostraremos en el ejemplo 7.1 que, para cada $n \geq 1$, el espacio clasificante $B\operatorname{aut}(S^n_{\mathbb{Q}})$ no pertenece a la imagen del funtor realización $\langle - \rangle$ en (1).

No obstante, encontramos en este trabajo una amplia clase de fibraciones universales cuyos tipos de homotopía racional pueden ser determinados por modelos de Lie, siempre utilizando ciertos tipos de cdgls de derivaciones.

Sea X un CW-complejo y $\mathcal{H}$ un subgrupo de $\mathcal{E}(X) = \pi_0 \operatorname{aut}(X)$. Denotamos por $\operatorname{aut}_{\mathcal{H}}(X)$ el monoide topológico conformado por aquellas autoequivalencias homotópicas de X tales que sus clases de homotopía pertenecen a $\mathcal{H} \subset \mathcal{E}(X)$. Similarmente, podemos definir $\operatorname{aut}^*_{\mathcal{H}}(X)$ como aquellas autoequivalencias homotópicas punteadas de X cuya clase de homotopía libre pertenece a $\mathcal{H}$.

Aplicando la construcción barra geométrica obtenemos nuevos espacios $B\operatorname{aut}_{\mathcal{H}}(X)$ y $B\operatorname{aut}^*_{\mathcal{H}}(X)$ y se puede probar que son recubridores (salvo homotopía) de los espacios clasificantes $B\operatorname{aut}(X)$ y $B\operatorname{aut}^*(X)$ respectivamente. En particular, sus grupos de homotopía superiores coinciden mientras que $\pi_1 B\operatorname{aut}_{\mathcal{H}}(X) = \pi_0 \operatorname{aut}_{\mathcal{H}}(X) = \mathcal{H}$. La inclusión $\operatorname{aut}^*_{\mathcal{H}}(X) \hookrightarrow \operatorname{aut}_{\mathcal{H}}(X)$ induce una sucesión fibrante

$$X \to B\operatorname{aut}^*_{\mathcal{H}}(X) \to B\operatorname{aut}_{\mathcal{H}}(X) \tag{5}$$

la cual clasifica ciertos tipos de fibraciones: obsérvese que una fibración arbitraria $X \to E \to B$ determina una acción de $\pi_1(B)$ en la fibra X, llamada la *acción de holonomía*, la cual a su vez define un homomorfismo de grupos

$$\pi_1(B) \longrightarrow \mathcal{E}(X). \tag{6}$$

Entonces, una sucesión fibrante con fibra X puede ser catalogada dependiendo de la imagen de este homomorfismo. Si su imagen está contenida en $\mathcal{H} \subset \mathcal{E}(X)$ diremos que es una sucesión $\mathcal{H}$-fibrante. Entonces, en el Teorema 2.17 probamos que

Teorema B. *Dado un subgrupo $\mathcal{H} \subset \mathcal{E}(X)$, para cualquier CW-complejo B, el conjunto de clases de equivalencia de $\mathcal{H}$-fibraciones sobre B con fibra X es naturalmente isomorfo a el conjunto $[\![B, B\operatorname{aut}_{\mathcal{H}}(X)]\!]$ de clases de homotopía de aplicaciones de B a $B\operatorname{aut}_{\mathcal{H}}(X)$. Esta biyección asigna a cada clase de homotopía de una aplicación $B \to B\operatorname{aut}_{\mathcal{H}}(X)$ el pullback a lo largo de dicha aplicación de la sucesión $\mathcal{H}$-fibrante universal*

$$X \to B\operatorname{aut}^*_{\mathcal{H}}(X) \to B\operatorname{aut}_{\mathcal{H}}(X).$$

Pretendemos encontrar cdgls como modelos de ciertas clases de estas fibraciones universales. Para ello, fijamos X un complejo finito nilpotente y $\mathcal{H} \subset \mathcal{E}(X)$ un grupo de clases de homotopía de autoequivalencias homotópicas que actúa nilpotentemente en la homología de X. Entonces [9, Theorem D] afirma que $B\operatorname{aut}_{\mathcal{H}}(X)$ y $B\operatorname{aut}^*_{\mathcal{H}}(X)$ son ambos espacios nilpotentes y una argumentación directa muestra que la racionalización de (5) tiene el tipo de homotopía de la sucesión fibrante universal racional

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

Sea L una cdgl modelo de X y denota por $\mathcal{E}^*(L)$ el grupo de clases de homotopía de automorfismos de cdgls de L. El grupo $H_0(L)$ con el producto Baker-Campbell-Hausdorff actúa en $\mathcal{E}^*(L)$ vía la aplicación exponencial. Explícitamente, la aplicación

$$H_0(L) \longrightarrow \mathcal{E}^*(L), \quad [x] \mapsto e^{\operatorname{ad}_x},$$

es un morfismo de grupos bien definido. A continuación, definimos $\mathcal{E}(L) = \mathcal{E}^*(L)/H_0(L)$, de forma que hay una sucesión exacta de morfismos de grupos

$$H_0(L) \to \mathcal{E}^*(L) \to \mathcal{E}(L) \to 0.$$

Esto es el análogo a nivel de cdgls de la situación en el contexto topológico: considera la aplicación sobreyectiva $\mathcal{E}^*(X) \xrightarrow{\zeta} \mathcal{E}(X)$ que consiste simplemente en olvidar el punto base y la acción de $\pi_1(X)$ en $\mathcal{E}^*(X)$ cuyo espacio de órbitas es precisamente $\mathcal{E}(X)$. Esto se traduce en una sucesión exacta de morfismos de grupos

$$\pi_1(X) \longrightarrow \mathcal{E}^*(X) \xrightarrow{\zeta} \mathcal{E}(X) \to 0$$

que resulta ser equivalente a la anterior sucesión en el mundo racional. Efectivamente, hay un diagrama conmutativo (observación 4.11)

$$\begin{tikzcd}
\pi_1(X_{\mathbb{Q}}) \arrow[r] \arrow[d, "\cong"] & \mathcal{E}^*(X_{\mathbb{Q}}) \arrow[r, "\zeta"] \arrow[d, "\cong"] & \mathcal{E}(X_{\mathbb{Q}}) \arrow[r] \arrow[d, "\cong"] & 0 \\
H_0(L) \arrow[r] & \mathcal{E}^*(L) \arrow[r] & \mathcal{E}(L) \arrow[r] & 0.
\end{tikzcd} \tag{7}$$

Por tanto, una vez fijado el subgrupo $\mathcal{H} \subset \mathcal{E}(X)$ actuando nilpotentemente en la homología, podemos considerar su racionalización $\mathcal{H}_{\mathbb{Q}} \subset \mathcal{E}(X_{\mathbb{Q}})$, tomar su preimagen $\zeta^{-1}(\mathcal{H}_{\mathbb{Q}}) \subset \mathcal{E}^*(X_{\mathbb{Q}})$ e identificarlo con un subgrupo $\hbar \subset \mathcal{E}^*(L)$. Entonces, definimos $\operatorname{Der}^{\hbar} L \subset \operatorname{Der} L$ como

$$\operatorname{Der}^{\hbar}_{\geq 1} L = \operatorname{Der}_{\geq 1} L, \qquad \operatorname{Der}^{\hbar}_0 L = \{\theta \in \operatorname{Der}_0 L \text{ tales que } D\theta = 0 \text{ y } [e^{\theta}] \in \hbar\}.$$

Esto significa que consideramos todas las derivaciones en grados positivos y aquellas, en grado 0, tales que sus exponenciales son automorfismos de cdgls cuya clase de homotopía está en $\hbar$. El Teorema 7.13, constituye uno de los resultados principales de este texto.

Teorema C. *Sea L un modelo de Lie de un espacio nilpotente X y $\mathcal{H} \subset \mathcal{E}(X)$ un subgrupo actuando nilpotentemente en la homología de X. Entonces, el análogo de (4)*

$$L \xrightarrow{\text{ad}} \operatorname{Der}^{\hbar} L \to \operatorname{Der}^{\hbar} L \widetilde{\times} sL \tag{8}$$

es una sucesión de cdgls cuya realización es del tipo de homotopía de la sucesión $\mathcal{H}_{\mathbb{Q}}$-fibrante universal

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

La demostración de este resultado tiene, de forma general, dos puntos claves.

El primero, de carácter aparentemente técnico a primera vista, recae en principios fundamentales de la teoría general: solo aquellas dgls que son completas son susceptibles de ser realizadas geométricamente. Sin embargo, sub dgls conexas de $\operatorname{Der} L$ no son necesariamente completas en general, incluso si L es simplemente conexa. Y lo mismo ocurre para los productos torcidos de dgls. Por tanto, inicialmente debemos comprobar el hecho altamente no trivial de que (8) es una sucesión de cdgls. De hecho, demostrar simplemente que $\operatorname{Der}^{\hbar} L$ es cerrado bajo las operaciones de suma y el corchete de Lie, requiere un profundo estudio de la equivalencia de Malcev entre grupos Malcev $\mathbb{Q}$-completos y algebras de Lie (no graduadas) completas. Por otro lado, la completitud de $\operatorname{Der}^{\hbar} L$ se basa en una filtración específica de L que depende de la serie central descendente de la acción de $\mathcal{H} \subset \mathcal{E}(X)$ en $H_*(X)$. Lo mismo ocurre para el producto torcido $\operatorname{Der}^{\hbar} L \widetilde{\times} sL$.

Una vez resulta esta cuestión, el hecho de que la realización geométrica de (8) tiene el tipo de homotopía deseado se deduce de un resultado más ambicioso. Ciertamente, la sucesión (8) de cdgls conexas encaja en una sucesión más larga

$$\operatorname{Hom}(\overline{\mathscr{C}}(L), L) \to \operatorname{Hom}(\mathscr{C}(L), L) \to L \overset{\text{ad}}{\to} \operatorname{Der}^{\hbar} L \to \operatorname{Der}^{\hbar} L \widetilde{\times} sL, \tag{9}$$

donde $\mathscr{C}(L)$ es la coálgebra graduada diferencial coconmutativa proporcionada por el clásico *funtor de cadenas* de Quillen, $\overline{\mathscr{C}}(L)$ denota el ideal de los elementos de $\mathscr{C}(L)$ de longitud no cero y consideramos a $\operatorname{Hom}(\overline{\mathscr{C}}(L), L)$ y $\operatorname{Hom}(\mathscr{C}(L), L)$ como cdgls con la diferencial usual y el corchete de Lie convolución. Estas dos cdgls, altamente no conexas, son modelos de Lie de los espacios de aplicaciones punteadas y libres

$$\operatorname{map}^*(X, X) \quad \text{y} \quad \operatorname{map}(X, X)$$

respectivamente.

Entonces, si restringimos estos modelos a las componentes algebraicas que representan automorfismos punteados y libres de $X_{\mathbb{Q}}$, y dejando inalterada la parte final de (9), encontramos una sucesión de cdgls cuya realización geométrica es la sucesión fibrante

$$\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \hookrightarrow \operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \xrightarrow{\text{ev}} X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$$

donde ev denota la evaluación en el punto base.

Los teoremas B y C tienen una “versión homotópica” que procedemos a describir. Para un CW-complejo punteado fijo, podemos considerar aquellas sucesiones fibrantes de espacios punteados $X \to E \to B$ junto con una sección homotópica (que denominaremos fibraciones punteadas a partir de ahora). Entonces, dado $\Pi \subset \mathcal{E}^*(X)$ un subgrupo de las clases de homotopía punteada de autoequivalencias punteadas, consideramos la clase de Π-fibraciones, que consiste en aquellas fibraciones punteadas tales que la imagen de la acción de holonomía está contenida en Π. En esta ocasión entendemos que la acción de holonomía está definida como $\pi_1(X) \to \mathcal{E}^*(X)$. Entonces el Teorema 2.39, especializado para A un punto, pasa a ser

Teorema D. *Dado un subgrupo $\Pi \subset \mathcal{E}^*(X)$, para cualquier CW-complejo B, el conjunto de clases de equivalencias de Π-fibraciones sobre B con fibra X es naturalmente isomorfo al conjunto de clases de homotopía $[\![B, B\operatorname{aut}^*_\Pi(X)]\!]$. Esta biyección asigna a cada clase de homotopía de una aplicación dada $B \to B\operatorname{aut}\Pi^*(X)$ el pullback sobre esta aplicación de una fibración universal de la forma*

$$X \longrightarrow Z \longrightarrow B\operatorname{aut}^*_\Pi(X). \tag{10}$$

Aquí, Z denota la construcción barra geométrica general aplicada a la tupla $(*, \operatorname{aut}^*_\Pi(X), X)$.

Ahora, sea X un complejo nilpotente finito con modelo de Lie L y sea $\Pi \subset \mathcal{E}^*(X)$ un subgrupo que actúa nilpotentemente en $\pi^*(X)$. Entonces, Π es un grupo nilpotente y $B\operatorname{aut}^*_\Pi(X)$ es un espacio nilpotente tal que (10) es una fibración de espacios nilpotentes cuya racionalización es la fibración universal

$$X_{\mathbb{Q}} \longrightarrow Z_{\mathbb{Q}} \longrightarrow B\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

Al igual que en la "versión homológica", $\Pi_{\mathbb{Q}}$ es isomorfo a un subgrupo $\pi \subset \mathcal{E}^*(L)$ de clases de homotopía de automorfismos de L y de nuevo podemos considerar $\operatorname{Der}^\pi L \subset \operatorname{Der} L$ con

$$\operatorname{Der}^\pi_{\geq 1} L = \operatorname{Der}_{\geq 1} L, \quad \operatorname{Der}^\pi_0 L = \{\theta \in \operatorname{Der}_0 L \mid D\theta = 0, [e^\theta] \in \pi\}.$$

Entonces, el teorema 7.18 dice:

Teorema E. *Sea X un complejo nilpotente finito con modelo de Lie L. Sea $\Pi \subset \mathcal{E}^*(X)$ un subgrupo que actúa nilpotentemente en $\pi_*(X)$ y que es invariante bajo la acción de $\pi_1(X)$. Entonces, la fibración universal*

$$X_{\mathbb{Q}} \longrightarrow Z_{\mathbb{Q}} \longrightarrow B\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})$$

tiene el tipo de homotopía de la realización de la sucesión de cdgls

$$L \to L \widetilde{\times} \operatorname{Der}^\pi L \to \operatorname{Der}^\pi L.$$

Los teoremas C y D, junto los resultados auxiliares que nos han llevado a su demostración, tienen aplicaciones interesantes. La primera consecuencia inmediata es, para un complejo finito nilpotente X dado, la descripción de la racionalización de ciertos subgrupos $\mathcal{H} \subset \mathcal{E}(X)$ y $\Pi \subset \mathcal{E}^*(X)$ actuando nilpotentemente en la homología y los grupos de homotopía de X respectivamente. Sus espacios clasificantes también pueden ser formulados en términos de derivaciones. Efectivamente, con la notación usada anteriormente,

Corolario A. $\Pi_{\mathbb{Q}} \cong H_0(\operatorname{Der}^{\pi} L)$ *y* $\mathcal{H}_{\mathbb{Q}} \cong H_0(\operatorname{Der}^{\hbar} L)/\operatorname{Im} H_0(\mathrm{ad})$.

Corolario B. *$B\mathcal{H}$ y $B\Pi$ tienen el tipo de homotopía racional de la realización de cdgls $\operatorname{Der}_0^{\hbar} L \oplus R$ y $\operatorname{Der}_0^{\pi} L \oplus S$ donde R y S denotan un complemento de los 1-ciclos de $\operatorname{Der}^{\hbar} L \widetilde{\times} sL$ y $\operatorname{Der}^{\pi} L$ respectivamente.*

Con la misma notación y en el mismo contexto, podemos también calcular el índice de nilpotencia homotópica de $\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})$ y $\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$ de una forma puramente algebraica. Dado un H-grupo Y su índice de nilpotencia homotópica $\operatorname{nil} Y$, es el menor entero $n \leq \infty$ para el cual el $(n+1)$-enésimo conmutador homotópico de Y is homotópicamente trivial. Por otro lado, para una dgl L denotamos por $\operatorname{nil} L$ su índice de nilpotencia habitual. Entonces:

Corolario C. $\operatorname{nil} \operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}}) = \operatorname{nil} H(\operatorname{Der}^{\pi} L)$ *y* $\operatorname{nil} \operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) = \operatorname{nil} H(\operatorname{Der}^{\hbar} L \widetilde{\times} sL)$.

También demostramos en la proposición 8.6 que cualquier grupo racional nilpotente finitamente generado es isomorfo a un subgrupo $\mathcal{H} \subset \mathcal{E}(X)$ que actúa nilpotentemente en la homología de X para algún espacio X. De la misma manera, en la proposición 8.7 se prueba que tal grupo es siempre isomorfo a un subgrupo $\Pi \subset \mathcal{E}^*(Y)$ que actúa nilpotentemente en los grupos de homotopía de Y para algún espacio Y.

Finalmente utilizamos los teoremas C y E para obtener modelos de otras sucesiones fibrantes interesantes que involucran a los espacios clasificantes de monoides topológicos de autoequivalencias homotópicas (véase el teorema 8.8).

Estructura de la tesis

Todo el material resumido previamente se presenta a lo largo de 8 capítulos, que versan sobre distintos temas.

El capítulo 1 está principalmente dedicado a introducir y presentar los fundamentos de la teoría de homotopía de las álgebras de Lie graduadas diferenciales y completas, el objeto algebraico principal de esta tesis. La principal referencia al respecto es [14], donde esta teoría es desarrollada de forma exhaustiva. También se expone otro material preliminar como propiedades básicas de los conjuntos simpliciales, la estructura de modelos de ciertas categorías y aspectos de la nilpotencia y la racionalización.

Se asume que el lector está familiarizado con aspectos principales de la teoría de homotopía racional clásica (véase, por ejemplo, [12] o [51]). No obstante, algunos datos básicos son introducidos en esta primera sección y algunas propiedades pueden ser deducidas al aplicar la nueva teoría homotópica de cdgls a espacios simplemente conexos.

En §2 estudiamos la clasificación de las fibraciones y fibraciones punteadas con una acción de holonomía prescrita. En particular, establecemos y probamos el teorema B, el cual es el teorema 2.17 en el texto, así como el teorema D que es el teorema 2.39, especializado para A un punto, en el texto. Este capítulo, de alguna manera, es independiente del resto del texto, en el sentido de que los resultados presentados no están acotados al mundo racional: son válidos para CW-complejos y subgrupos de $\mathcal{E}^*(X)$ y $\mathcal{E}(X)$ arbitrarios. La principal referencia para este capítulo es [32].

El capítulo 3 comienza con el enunciado de la equivalencia de Malcev en su forma más general: como una equivalencia de categorías entre álgebras de Lie completas y grupos Malcev completos (véase [16, §8]). Se incluye también aquí el teorema 3.5, una caracterización útil de un grupo Malcev completo como aquellos que son pronilpotentes y 0-locales.

A continuación, nos centramos en los subgrupos de $\operatorname{aut}(L)$ y sus correspondientes subgrupos de $\mathcal{E}^*(L)$ seleccionando clases de homotopía. Se prueba en el teorema 3.9, que, para una cdgl minimal conexa L, si $K \subset \mathcal{E}^*(L)$ es nilpotente y 0-local, entonces $\operatorname{aut}_K(L)$ es Malcev completo y, por tanto, equivalente a un álgebra de Lie completa. También demostramos en este capítulo que el subgrupo $\operatorname{aut}_1(L)$ de automorfismos de L homotópicos a la identidad son precisamente los *automorfismos internos*, es decir, la exponencial $e^{D(\operatorname{Der}_1 L)}$ de las fronteras de derivaciones de grado 0 (véase el teorema 3.7).

La presencia de grupos fundamentales no triviales hace necesario traducir ciertos hechos topológicos elementales a la nueva teoría homotópica de cdgls. Esto se realiza en §4, donde presentamos una descripción detallada de las clases de homotopía "punteadas" $[\![L, L']\!]^* = \operatorname{Hom}_{\mathbf{cdgl}}(L, L')/\sim$ y "libres" $[\![L, L']\!] = [\![L, L']\!]^*/H_0(L')$ de morfismos de cdgls. También exponemos como esto replica perfectamente la situación en el contexto topológico. En particular, obtenemos el diagrama conmutativo (7).

También introducimos en este capítulo la *acción de holonomía algebraica* para una sucesión de cdgls

$$0 \to L \longrightarrow M \longrightarrow N \to 0,$$

que viene dada por el morfismo de cdgls

$$H_0(N) \longrightarrow \mathcal{E}(L), \quad [x] \mapsto \overline{[e^{\operatorname{ad}_x}]},$$

que, de nuevo, replica a la versión topológica (véase la observación 4.11).

En el capítulo 5 se realiza un estudio detallado de los modelos de Lie de la sucesión fibrante de evaluación

$$\operatorname{map}^*(X, Y) \to \operatorname{map}(X, Y) \xrightarrow{\operatorname{ev}} Y$$

y su restricción a ciertas componentes conexas, por medio de cdgls de convolución y cdgls de derivaciones. Estos modelos de Lie resultarán ser esenciales para la demostración de los resultados principales.

Como hemos remarcado anteriormente, $\operatorname{Der} L$ puede no ser completo incluso cuando L lo es. De la misma manera, el producto torcido de dos cdgls pudiera igualmente ser no completo. En el capítulo técnico §6, se presentan ciertas sub dgl's de derivaciones de una cdgl dada, las cuales pueden ser dotadas de filtraciones particulares que las hagan completas. Este hecho es altamente no trivial ya que los intentos naturales de filtraciones no funcionan. Explícitamente, veremos como una filtración en la homología de un complejo finito, induce una filtración en su modelo de Lie L y, consecuentemente, determina una filtración de una sub dgl conexa de $\operatorname{Der} L$ que la hace completa mientras mantiene todas las derivaciones de grado positivo. Una situación análoga se da para ciertos productos torcidos de algunas cdgls. Como resultado, en la categoría de cdgls, conseguimos la deseada propiedad de clausura bajo estas operaciones.

El capítulo 7 contiene los teoremas C y E así como sus demostraciones detallas.

Finalmente, en §8 incluimos todas las consecuencias de los teoremas C y E que hemos presentado previamente. También se prueba que el teorema C efectivamente generaliza el teorema clásico A.

El contenido de esta tesis es una versión extendida y detallada de los resultados presentados en los artículos [10] y [18].

Notations and conventions

The following general notation will be used throughout the text.

- The underlying field is $\mathbb{Q}$, the field of rational numbers. For example, a vector space is understood to be a $\mathbb{Q}$-vector space, or the homology of a space $H_*(X)$ with no explicit coefficient is understood to be $H_*(X;\mathbb{Q})$.
- In this text by a topological space we mean a weak Hausdorff compactly generated space, and we denote by **top** the category whose objects are topological spaces and the morphisms are continuous maps. Working with this category instead of a more general category has several advantages: for example, it is closed under pushouts and their mapping spaces have a good behavior.
- Categories are denoted by bold letters: for example **set** is the category of sets, **top** is defined above, **CW** is the full subcategory of topological spaces of the homotopy type of a CW-complex and **sset** is the category of simplicial sets.
- The adorned categories $\mathbf{set}^*$, $\mathbf{top}^*$, $\mathbf{CW}^*$ and $\mathbf{sset}^*$ denote the pointed versions of the categories above.
- By abuse of language, we say that an element belongs to a category if it is an object of such category.
- Unless otherwise specified, maps and arrows are morphisms in the underlying category. For example, a map $X \to Y$ between topological spaces is understood to be continuous.
- The standard notation to denote the homotopy classes of maps between two spaces X, Y (or more generally, in the context of model categories) is $[X, Y]$. However, in this text, Lie algebras play a central role, and there could be confusion with the Lie bracket denoted by $[-,-]$. Therefore, the set of equivalence classes of homotopy classes of morphisms will be denoted by
$$[\![X, Y]\!].$$

- Given X, Y spaces or simplicial sets, the unadorned $[\![X, Y]\!]$ denote (free) homotopy classes of (free) maps, as stated above. To indicate pointed homotopy classes of pointed maps, we use $[\![X, Y]\!]^*$.
- The class of a map f, in the set $[\![X, Y]\!]$ is denoted by $[f]$; however, by abuse of notation, we often do not distinguish between the map and its homotopy class.
- When an index appears with no explicit domain, for example $\sum_n x_n$, the index is understood to run in $\mathbb{Z}$.
- If a pair of functors F, G are adjoint we write

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \ ,$$

where the upper functor is left adjoint to the bottom functor.

- Through this text, several differential objects will appear (for example, differential Lie algebras) and we will use the following symbols to denote the differentials: ∂, d, D. The specific symbol used will be indicated at the moment, however the following general rules will be applied. ∂ will denote the differential of a Lie algebra or a vector space, d the differential of commutative algebra and D the inherited differential in derivations or homomorphisms.
- In the categories **top** or **sset**, we use the following symbols: $\cong$ to denote homeomorphisms, $\simeq$ homotopy equivalences and $\simeq_w$ weak homotopy equivalences.
- In this text we use different but similar "triangular" symbols to denote simplices in different contexts: $\triangle$ will denote the simplex category, $\triangle^n$ the simplicial set, Δ^n the chain complex and $\boldsymbol{\Delta}^n$ the topological n-simplex. See §1.4 for more details.
- In a simplicial set X we do not distinguish between an n-simplex $x \in X_n$ and the subsimplicial set generated by that element.
- For X, Y topological spaces and $y \in Y$, the map $c_y \colon X \to Y$ denotes the constant map $c_y(x) = y$ for all $x \in X$. Similarly for X, Y in **sset**.
- The *Koszul convention* is applied throughout the text: whenever the position of two elements of degrees n and m is permuted in a expression, the sign $(-1)^{nm}$ appears.
- To make a coherent choice of signs, we fix that $ss^{-1} = \mathrm{id}$ and $s^{-1}s = -\mathrm{id}$ (this is the choice done in [51, §0.1]), where s is the suspension (see §1.1).
- Between two linear graded objects (for example Lie algebras, algebras or coalgebras), not necessarily in the same category, the unadorned $\mathrm{Hom}(-,-)$ will denote the graded vector space of linear maps between them. As usual, the adorned $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denotes the set of morphism in the category $\mathcal{C}$. See §1.4.1 for more details.

CHAPTER 1

Homotopy theory of complete differential graded Lie algebras

The goal of this section is to introduce the definitions and basic facts about the theory of Lie models in topology. The main reference for this chapter is the book [14]: this chapter tries to be a brief summary of some results of this book and we refer to it as a general reference for a more complete and detailed explanation of the concepts presented below.

1.1 Lie algebras

A *graded vector space* is a direct sum

$$V = \oplus_n V_n$$

of vector spaces V_n. An element $v \in V$ is *homogeneous* if $v \in V_n$ for some n and we say that n is the degree of v or $|v| = n$. Whenever $|-|$ is applied to some element of V, this element is assumed to be homogeneous. A linear map $f\colon V \to V'$ between two graded vector spaces is said to have degree k if

$$f(V_n) \subset V'_{n+k}$$

for all n. The category of graded vector spaces and 0-degree linear maps is denoted by **gvect**.

The kth suspension of V is the graded vector space $s^k V$ where $(s^k V)_n = V_{n-k}$.

A *differential* is a linear map ∂ of degree -1 such that $\partial^2 = 0$. Given a differential graded vector space (V, ∂), we denote by $H_*(V)$ its homology, which is a graded vector space. A morphism of differential graded vector spaces is a linear map of degree 0 which commutes with the differential. Note that such morphism induces a map in the homology of the vector spaces. A morphism $f\colon (V, \partial) \to (V', \partial')$ is a *quasi-isomorphism* if $H_*(f)\colon H(V) \to H_*(W)$ is an isomorphism. We write $V = (V, \partial)$ whenever there is not ambiguity about which the differential is. We denote by **dgvect** the category of differential graded vector spaces (recall that a differential graded vector space is equal to a chain complex).

Given an object (V, ∂) in **dgvect**, we define the kth *suspension* as $(s^k V, \partial)$ where

$$\partial(s^k v) = (-1)^{|v|} s^k(\partial v).$$

A *graded Lie algebra* is a graded vector space L with a Lie bracket, this is, a linear map

$$[-,-]\colon L \otimes L \to L$$

of degree 0 which satisfies antisymmetry

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

and the Jacobi identity

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$$

for any $x, y, z \in L$.

A *differential graded Lie algebra*, or dgl, is a graded Lie algebra with a differential ∂ in the underlying vector space and such that

$$\partial[x,y] = [\partial x, y] + (-1)^{|x|}[x, \partial y]$$

for all $x, y \in L$.

We denote by **dgl** the category whose objects are dgl's and the morphisms are linear maps of degree 0

$$f\colon (L, \partial) \to (L', \partial')$$

such that

$$f[x,y] = [f(x), f(y)], \quad f(\partial x) = \partial(f(x))$$

for all $x, y \in L$.

We say that a dgl $L = (L, \partial)$ is *k-connected* if $L = L_{\geq k}$. By connected we means 0-connected and the *k-connected cover* of L is the k-connected dgl $L^{(k)}$ which is degreewise defined by

$$L_n^{(k)} = \begin{cases} \ker(\partial\colon L_k \to L_{k-1}), & \text{if } n = k \\ L_n, & \text{if } n > k \end{cases}.$$

Similarly to the case of a topological space, we write $\tilde{L} = L^{(1)}$ for the 1-connected cover.

A *derivation of degree k* is a linear map $\theta\colon L \to L$ of degree k such that:

$$\theta[x,y] = [\theta(x), y] + (-1)^{k|x|}[x, \theta(y)]$$

for all $x, y \in L$. We denote by $\operatorname{Der}_k L$ the space of derivations of degree k.

Note that the space of derivations $\operatorname{Der} L = \oplus_k \operatorname{Der}_k L$ is a dgl with the following Lie bracket and differential:

$$[\theta,\eta] = \theta\circ\eta - (-1)^{|\theta||\eta|}\eta\circ\theta, \quad D\theta = [\partial,\theta] = \partial\circ\theta - (-1)^{|\theta|}\theta\circ\partial.$$

More generally, if $f\colon L\to L'$ is a dgl morphism, we denote by $\operatorname{Der}_f(L,L')$ the differential graded vector space of linear maps $\theta\colon L\to L'$ such that

$$\theta[x,y] = [\theta(x), f(y)] + (-1)^{|\theta||x|}[f(x),\theta(y)]$$

and with the differential given as above.

The following lemma [7, Lemma 6] will be used in the text.

Lemma 1.1. *A dgl quasi-isomorphism $g\colon L''\to L$ induces a quasi-isomorphism of differential graded vector spaces*

$$g^*\colon \operatorname{Der}_f(L,L') \to \operatorname{Der}_{f\circ g}(L'',L'), \quad \theta\mapsto\theta\circ g$$

if both L and L'' are free.

Given a commutative differential graded algebra (A,d) with d a differential of degree -1 (see §1.9), the tensor product $A\otimes L$ inherits a dgl structure given by

$$[a\otimes x, b\otimes y] = (-1)^{|b||x|}ab\otimes[x,y], \qquad \partial(a\otimes x) = da\otimes x + (-1)^{|a|}a\otimes\partial x$$

for $a,b\in A$ and $x,y\in L$.

1.2 Complete Lie algebras

A *filtration* of a dgl L is a sequence of vector subspaces $\{F^n\}_{n\geq 0}$ of L such that

$$L = F^1 \supset \cdots \supset F^n \supset F^{n+1} \supset \cdots$$

$$[F^n, F^m] \subset F^{n+m}, \quad \partial F^n \subset F^n$$

for all $n,m\geq 1$. Given a subspace F^n we write

$$F^n_k = F^n \cap L_k$$

for any k.

For any dgl L, the lower central series of L, $\{L^n\}_{n\geq 1}$, where $L^1 = L$ and $L^n = [L, L^{n-1}]$ for $n>1$, is a filtration of L.

A filtered dgl $(L,\{F^n\}_{n\geq 1})$ is a *complete differential graded Lie algebra*, cdgl henceforth, if the natural dgl map

$$L \xrightarrow{\cong} \varprojlim_n L/F^n$$

is an isomorphism. An element in L can be written as a formal series

$$\sum_{n\geq 1} x_n, \quad x_n\in F^n$$

where two such series $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ represent the same element in the limit if and only if

$$\sum_{n=1}^{q}(x_n - y_n) \in F^{q+1}$$

for all $q \geq 1$.

Let **cdgl** be the category whose objects are cdgl's and the morphisms are dgl maps which preserve the filtrations. This category is complete and cocomplete [14, Proposition 3.5].

Given a filtered dgl $(L, \{F^n\}_{n\geq 1})$, its *completion* is

$$\hat{L} = \varprojlim_n L/F^n,$$

which is complete with respect to the filtration

$$\hat{F}^n = \ker(\hat{L} \to L/F^n).$$

If (A, d) is a commutative differential graded algebra and L is a cdgl with filtration $\{F^n\}_{n\geq 1}$, we define the *complete tensor product* as the cdgl

$$A \mathbin{\widehat{\otimes}} L = \varprojlim_n A \otimes (L/F^n).$$

1.2.1 Free Lie algebras

The free Lie algebras and their completion play a central role in the theory of Lie models. Given a graded vector space V, we denote by $\mathbb{L}(V)$ the free Lie algebra generated by V. The functor

$$\mathbb{L}\colon \textbf{vect} \to \textbf{dgl}$$

sending a vector space to its associated free Lie algebra is left adjoint to the forgetful functor.

An element $x \in \mathbb{L}(V)$ is said to have *bracket length* n if it can be written as a sum of elements of the form

$$[v_1, [v_2, [\dots, [v_{n-1}, v_n], \dots]]$$

with $v_1, \dots, v_n \in V$. Let $\mathbb{L}^n(V)$ be the subspace of elements whose bracket length is n. Then

$$(\mathbb{L}(V))^n = \mathbb{L}^{\geq n}(V) = \oplus_{m\geq n}\mathbb{L}^m(V).$$

Note that the dgl $\mathbb{L}(V)$ is not complete in general, so we define $\widehat{\mathbb{L}}(V)$ as the completion of $\mathbb{L}(V)$ with respect to its lower central series,

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}^{\geq n}(V),$$

which is complete with respect to the filtration

$$\widehat{\mathbb{L}}^{\geq n}(V) = \prod_{m\geq n} \mathbb{L}^m(V).$$

A free cdgl $(\widehat{\mathbb{L}}(V), \partial)$ is said to be *minimal* if $\partial V \subset \widehat{\mathbb{L}}^{\geq 2}(V)$.

1.2.2 Maurer-Cartan elements

A *Maurer-Cartan element* or MC element, in a dgl L, is an element $a \in L_{-1}$ satisfying

$$\partial a + \frac{1}{2}[a, a] = 0.$$

We denote by $\mathrm{MC}(L)$ the set of MC elements of L, which always contains 0.

For $a \in \mathrm{MC}(L)$ the *perturbed differential by* a, $\partial_a = \partial + \mathrm{ad}_a$ is again a differential on L, where

$$\mathrm{ad}_a(x) = [a, x]$$

is the adjoint operator.

For $a \in \mathrm{MC}(L)$, the *component of* L *at* a or (L^a, ∂_a) is the connected dgl defined degreewise as

$$L^a_n = \begin{cases} \ker \partial_a, & \text{if } n = 0, \\ L_n, & \text{if } n > 0. \end{cases}$$

In particular, $L^a = (L, \partial_a)^0$.

1.2.3 Baker-Campbell-Hausdorff product

We describe in this subsection the construction of the Baker-Campbell-Hausdorff product on a cdgl L; more details can be found in [14, §4.2]. See also [16, §7 and §8] for a detailed treatment using Hopf algebras. This section can be thought of as an introduction of the more general Malcev equivalence which will be studied at §3.

Given a complete graded Lie algebra, denote by TL the tensor algebra on the graded vector space L. Then, the universal enveloping algebra of L, UL, is the quotient of TL by the ideal generated by the elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$$

for $x, y \in L$.

UL_0 has a filtration given by $\{I^n\}_{n\geq 0}$ where $I^0 = UL_0$, $I^1 = I$ is the ideal generated by L_0 and $I^n = I^{n-1}I$ for $n > 1$. Completing UL_0 and I we get

$$\widehat{UL_0} = \varprojlim_{n\geq 0} UL_0/I^n, \qquad \widehat{I} = \varprojlim_{n\geq 1} I/I^n.$$

Then, we have the bijections

$$\widehat{I} \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + \widehat{I} \quad (\cong)$$

given by

$$\exp(x) = e^x = \sum_{n\geq 0} \frac{x^n}{n!}, \qquad \log(1 + x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n}.$$

In a general setting, given two filtered graded vector spaces $(V, \{F^n\}_{n\geq 0})$ and $(W, \{G^n\}_{n\geq 0})$ we define the *complete tensor product* of V and W as

$$V \mathbin{\widehat{\otimes}} W = \varprojlim_n (V \otimes W)/(\oplus_{i+j=n} F^i \otimes G^j).$$

Denote by $v \mathbin{\widehat{\otimes}} w$ the image of $v \otimes w$ under the natural map $V \otimes W \to V \mathbin{\widehat{\otimes}} W$, for any $v \in V$, $w \in W$.

We can extend the *diagonal map*

$$\Delta \colon L_0 \to L_0 \times L_0, \quad x \mapsto (x, x),$$

to $\Delta \colon UL_0 \to U(L_0 \times L_0) = UL_0 \otimes UL_0$, and, using the definition above and noting that Δ is filtration-preserving, we extend it again to

$$\Delta \colon \widehat{UL_0} \to \widehat{UL_0} \mathbin{\widehat{\otimes}} \widehat{UL_0}.$$

We consider the group of *grouplike elements*

$$\mathcal{G} = \{x \in \widehat{UL_0} \mid \Delta(x) = x \mathbin{\widehat{\otimes}} x\}$$

and the set of *primitive elements*

$$\mathcal{P} = \{x \in \widehat{I} \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

Whenever L_0 is complete with respect to its lower central series (in particular, whenever L itself is complete), the injection $L_0 \to \mathcal{P}$ is an isomorphism and the bijections above restrict to

$$L_0 \underset{\log}{\overset{\exp}{\underset{\cong}{\rightleftarrows}}} \mathcal{G} \ . \tag{1.1}$$

Therefore, the group structure on $\mathcal{G}$ induces a multiplication in L_0 which is called the *Baker-Campbell-Hausdorff product* or BCH product, defined by

$$x * y = \log(\exp(x) \cdot \exp(y)) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x - y, [x, y]] + \cdots .$$

In the group $(L_0, *)$, 0 is the identity element of this product and $x^{-1} = -x$ is the inverse of x. Furthermore, given $n \in \mathbb{Z}$, $x^n = x * x * \cdots * x = nx$. The group structure of L_0 is deeply studied in Chapter 3.

This group acts on the set of MC elements via the *gauge action* defined as

$$x \mathcal{G} a = \sum_{n\geq 0} \frac{\mathrm{ad}_x^n(a)}{n!} - \sum_{n\geq 0} \frac{\mathrm{ad}_x^n(\partial x)}{(n+1)!}.$$

This action generates an equivalence relation in $\mathrm{MC}(L)$ and we denote by $\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\mathcal{G}$ the orbit set. Furthermore, $\widetilde{\mathrm{MC}}$ is a functor from **cdgl** to the category of pointed sets.

1.2.4 Exponential of derivations

In this subsection we recall some facts about the exponentiation of a derivation; the main reference is [14, §4.2]. The main purpose is to construct properly the concept of exponential of a derivation. Of course this can be done directly by the usual formula of the exponential series; however, from such formula deducing some general properties can be difficult. Instead, we use the bijection between $\widehat{I}$ and $1+\widehat{I}$ to give a more general approach to the exponential of a derivation.

Let L be a cdgl with respect to the filtration $\{F^n\}_{n\geq 1}$. Consider the following sub Lie algebra of $\operatorname{Der}_0 L \subset \operatorname{Der} L$:

$$\mathcal{D}\mathrm{er}_0 L = \{\theta \in \operatorname{Der}_0 L \mid D\theta = 0,\ \ \theta(F^n) \subset F^{n+1}, n \geq 1\}.$$

This (ungraded) Lie algebra is complete with respect to the filtration

$$\mathcal{F}^n = \{\theta \in \mathcal{D}\mathrm{er}_0 L \mid \theta(F^i) \subset F^{i+n}, i \geq 1\}$$

for $n \geq 1$. Note that, in particular, for any $x \in L_0$, $\mathrm{ad}_x \in \mathcal{D}\mathrm{er}_0 L$.

Let $M \subset \mathcal{D}\mathrm{er}_0 L$ be a sub Lie algebra; in particular, it is complete. In Example 6.1 we will show that, taking sub Lie algebras of $\operatorname{Der}_0 L$ is necessary if we expect getting a complete Lie algebra.

We apply the result of §1.2.3 to the complete Lie algebra M. Note that there is an injection

$$M \hookrightarrow \operatorname{Hom}(L, L)$$

where $\operatorname{Hom}(L,L)$ denotes graded linear maps (not necessarily commuting with the Lie bracket or the differential). We can consider $\operatorname{Hom}(L,L)$ as an algebra with the composition as operation. Thus, the universal property of UM implies that there is an algebra morphism

$$UM \to \operatorname{Hom}(L, L).$$

Moreover, since the derivations in M increases the filtration degree, we have morphisms of the form

$$I^n \to \operatorname{Hom}(L, F^n)$$

for any $n \geq 1$. These expressions induce the following morphism

$$\widehat{I} \to \varprojlim_{n\geq 1} \operatorname{Hom}(L,L)/\operatorname{Hom}(L,F^n) \cong \varprojlim_{n\geq 1} \operatorname{Hom}(L, L/F^n) \cong \operatorname{Hom}(L, \varprojlim_{n\geq 1} L/F^n) \cong \operatorname{Hom}(L,L)$$

which can be extended to

$$\xi\colon 1+\widehat{I} \to \operatorname{Hom}(L,L)$$

by sending 1 to id_L. Note that ξ sends grouplike elements of $1+\widehat{I}$ to invertible elements in $\operatorname{Hom}(L,L)$ i.e. automorphisms (again, as graded vector spaces). Now, given $\theta \in M$ we can compose the bijection $\exp\colon M \to \mathcal{G}$ of (1.1) with ξ, to obtain a (graded vector space) automorphism $\xi(\exp(\theta)) \in \operatorname{Hom}(L,L)$. This linear map acts on each element of L as

$$\sum_{n\geq 0} \frac{\theta^n}{n!}$$

so we abuse of notation and simply write this map as e^{θ}.

By the definition of the BCH product and the fact that ξ is a algebra morphism we deduce that

$$e^{\theta * \eta} = e^{\theta} \circ e^{\eta} \tag{1.2}$$

for any $\theta, \eta \in M$.

Finally, a direct computation [14, Proposition 4.10] using the formula above shows that e^{θ} commutes with the Lie bracket, and, if $D\theta = 0$, then e^{θ} commutes with the differential, thus it is a cdgl morphism. We conclude that there is a morphism of groups

$$\exp : M \cap \ker D \to \operatorname{aut}(L)$$

with the BCH product and the composition as the respective products.

1.3 Twisted products of Lie algebras

Let $(L, [-,-], \partial)$ and $(L', [-,-]', \partial')$ be two dgl's. A *twisted product* of L and L' is a dgl structure on the graded vector space $L \times L'$, that we denote by $(L \widetilde{\times} L', [-,-]^{\sim}, \widetilde{\partial})$ such that

a) L is a sub dgl of $L \widetilde{\times} L'$, i.e. $[x,y]^{\sim} = [x,y]$ and $\widetilde{\partial} x = \partial x$ for $x, y \in L$.

b) Given $x' \in L'$, then $\widetilde{\partial} x' - \partial' x' \in L$.

c) Given $x \in L$ and $x' \in L'$, then $[x, x']^{\sim} \in L$.

d) $[x', y']^{\sim} = [x', y']'$ for $x', y' \in L'$, this means that L' is a (non-differential) sub Lie algebra of $L \widetilde{\times} L'$.

The previous conditions can be summarized by saying that there is a dgl exact sequence

$$L \to L \widetilde{\times} L' \to L'$$

which splits as graded Lie algebra (forgetting about the differential).

Remark 1.2. We could have defined a twisted product just as a dgl exact sequence which splits as graded vector spaces (i.e. removing condition d)). This could be a natural choice, since any surjective map of dgl's would induce a (non-canonical) twisted product. However we follow [51, Definition VII.2. (9)] and include the requirement d) to define a twisted product. At all the twisted products in the text, condition d) will hold, so it does not matter which definition we choose.

However, in the category **cdgl** we do not have such a good behavior. The twisted product of two cdgl's is not necessarily a cdgl as the following example shows.

Example 1.3. Consider the two abelian Lie algebras $L = \mathbb{L}(x)$ and $L' = \mathbb{L}(y)$ with $|x| = 0$ and $|y|$ is a positive even number. The differentials are mandatorily zero. Since these Lie algebras are abelian, they are clearly complete.

Now consider the twisted product $L\widetilde{\times}L' = \langle x, y\rangle$ where L and L' are sub dgl's and $[x, y]^{\sim} = y$. This can be checked to be a Lie algebra. However, it is clear that $\langle y\rangle$ belongs to any term of the lower central series $(L\widetilde{\times}L')^n$ so, in particular, it is not complete with respect to any possible filtration. Then we have a exact sequence

$$L \to L\widetilde{\times}L' \to L',$$

where the first and last objects are cdgl's but the middle one is not.

Remark 1.4. Suppose that $L\widetilde{\times}L'$ is a cdgl. In this case, any Maurer-Cartan element of L will also be a Maurer-Cartan element in $L\widetilde{\times}L'$. Furthermore, if $a \in \mathrm{MC}(L)$ and $x \in L_0$, then the restult of the gauge action $x\mathcal{G}a$ is the same element if it is performed in L or in $L\widetilde{\times}L'$.

1.4 Simplicial sets and simplicial objects

Denote by $\triangle$ the *simplicial category*, whose objects are the sets $[n] = \{0, \dots, n\}$ and the morphism are non-decreasing maps between these sets. The morphisms in this category are generated by two classes: the *cofaces* $\delta^i\colon [n-1] \to [n]$ for $i = 0, \dots n$ and $n \geq 1$, and the *codegeneracies* $\sigma^i\colon [n+1] \to [n]$, for $i = 0, \dots, n$ and $n \geq 0$, defined as follows:

$$\delta^i = \left[\begin{array}{ccccccccc} 0 & 1 & \dots & i-1 & i & & \dots & n-1 & \\ \downarrow & \downarrow & & \downarrow & & \searrow & & & \searrow \\ 0 & 1 & \dots & i-1 & i & i+1 & & \dots & n \end{array}\right]$$

and

$$\sigma^i = \left[\begin{array}{ccccccccc} 0 & 1 & \dots & i & & i+1 & & i+2 & \dots & n+1 \\ \downarrow & \downarrow & & \downarrow & \swarrow & & \swarrow & & \swarrow & \\ 0 & 1 & \dots & i & & i+1 & \dots & n & & \end{array}\right]$$

A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor

$$X\colon \triangle^{\mathrm{op}} \to \mathcal{C},$$

while a *cosimplicial object* is a covariant functor

$$X\colon \triangle \to \mathcal{C}.$$

The image of the cofaces and codegeneracies under a simplicial object are called the face and degeneracy operators respectively:

$$d_i = X(\delta^i)\colon X_n \to X_{n-1}, \qquad s_i = X(\sigma^i)\colon X_n \to X_{n+1},$$

where $X_n = X([n])$.

The morphisms between simplicial objects are natural transformation between two functors $X, Y\colon \triangle^{\mathrm{op}} \to \mathcal{C}$. Equivalently, such a natural transformation is a sequence of morphisms $\{f_n\colon X_n \to Y_n\}_{n\geq 0}$ in $\mathcal{C}$ compatible with the face and degeneracy operators. We denote the category of simplicial objects as $\mathbf{s}\mathcal{C}$ for a given category $\mathcal{C}$.

Of special importance is the case $\mathcal{C} = \mathbf{set}$. In that case $\mathbf{sset}$ denotes the category of *simplicial sets*. If X is a simplicial set, we call the elements x of X_n simplices of degree n. We say that a simplex $x \in X_n$ of a simplicial set X is *degenerated* if $x = s_i(y)$ for some $y \in X_{n-1}$ and some i. In other case, we say that x is *non-degenerated*. A simplicial set is *finite* if it has finitely many non-degenerated simplices, and it is *of finite type* if, for each $n \geq 0$, it has finitely many non-degenerated n-simplices.

For $n \geq 0$ we denote by $\triangle^n$ the simplicial set defined as

$$\triangle^n = \mathrm{Hom}_{\triangle}(-, [n])\colon \triangle^{\mathrm{op}} \to \mathbf{set}\,.$$

An arbitrary element in $\triangle^n_m$ is a map sending $j \in [m]$ to $i_j \in [n]$, with $0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n$ and it is denoted by

$$(i_1, \dots i_m).$$

In particular, the 0-simplices of $\triangle^n$ are $(0), (1), \dots, (n)$.

Note that we can think of $\triangle^\bullet$ as a cosimplicial simplicial set. For $0 \leq i \leq n$, the ith *horn* is the subsimplicial set $\wedge^n_i \subset \triangle^n$ generated by all non-degenerate simplices except $(0, \dots n)$ and $(0, \dots, \hat{i}, \dots, n)$.

1.4.1 Mapping spaces and closed categories

A category $\mathcal{C}$ with a terminal object and finite products is said to be *(cartesian) closed* (see [39] and [16, §0.13 and §0.14] as general references) if there is functor

$$\mathrm{map}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that for all $Y \in \mathcal{C}$ the following functors are adjoint

$$\mathcal{C} \underset{\mathrm{map}(Y,-)}{\overset{-\times Y}{\rightleftarrows}} \mathcal{C}\ . \tag{1.3}$$

The bijection

$$\mathrm{Hom}_{\mathcal{C}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{map}(Y, Z)),$$

which is natural in X and Z is called the *exponential law*.

We easily deduce some properties from the exponential law. The counit of this adjunction is called the *evaluation morphism*

$$\mathrm{ev}\colon\ \mathrm{map}(X, Y) \times X \to Y.$$

If $*$ is the terminal object in $\mathcal{C}$, then $- \times *\colon \mathcal{C} \to \mathcal{C}$ is a full and faithful functor, so the unit is a natural isomorphism

$$X \cong (*, X).$$

Applying the Yoneda lemma, it can be easily deduced that the bijection (1.3) extends to an isomorphism in $\mathcal{C}$:

$$\mathrm{map}(X \times Y, Z) \cong \mathrm{map}(X, \mathrm{map}(Y, Z)).$$

Both categories **top** and **sset** are cartesian closed with the following functors as mapping spaces. In the category **top**, the *(topological) mapping space* is

$$\mathrm{map}(X, Y) = \mathrm{Hom}_{\mathbf{top}}(X, Y)$$

where the set of continuous map is topologized with the compact-open topology. Working in the category of weak Hausdorff compactly generated spaces, instead of with arbitrary topological spaces, ensures that the exponential law holds.

In the category **sset**, the *(simplicial) mapping space* is given by the simplicial set

$$\mathrm{map}(X, Y)_\bullet = \mathrm{Hom}_{\mathbf{sset}}(X \times \triangle^\bullet, Y)$$

where the face and degeneracy operators of the simplicial set $\mathrm{map}(X, Y)$ are induced by those of $\triangle^\bullet$.

If (X, x_0) and (Y, y_0) are pointed topological spaces or pointed simplicial sets, then write

$$\mathrm{map}^*(X, Y) = (\mathrm{ev}_{x_0})^{-1}(y_0) \subset \mathrm{map}(X, Y)$$

for the pointed mapping space, which is a subspace or sub simplicial set of the mapping space.

A final example of closed categories is given by the category **dgvect** of differential graded vector spaces. If V, W are elements in **dgvect** we construct its mapping space as

$$\mathrm{Hom}(V, W) = \bigoplus_k \mathrm{Hom}_k(V, W)$$

where $\mathrm{Hom}_k(V, W)$ is the vector space of degree k-linear maps from V to W. The differential is given by the formula

$$Df = \partial \circ f - (-1)^{|f|} f \circ \partial$$

for $f \in \mathrm{Hom}(V, W)$. Note that $\mathrm{Hom}_{\mathbf{dgvect}}(V, W) = \mathrm{Hom}_0(V, W)$.

In this case instead of taking the monoidal operation of the product we use the tensor product of vector space. With this operation, we have an exponential law

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

for U, V and W in **dgvect** which is natural in U and V. See [38, §2] for more details.

1.4.2 Chain complexes and simplicial sets

Recall that a chain complex is the same that an object in **dgvect**. Given a simplicial set X , we can construct an associated chain complex in the following way:

$$C_*(X) = \bigoplus_{n\geq 0} C_n(X)$$

where $C_n(X)$ is the vector space generated by the elements of X_n. The differential is defined as

$$\partial\colon C_q(X) \to C_{q-1}(X), \quad \partial x = \sum_{i=0}^{q} (-1)^i d_i x.$$

Similarly, let $D_*(X)$ be the chain complex generated by degenerated simplices. Then $(D_*(X), \partial)$ is a sub chain complex of $(C_*(X), \partial)$, and we define the *non-degenerate chain complex associated to* X as

$$(N_*(X), \partial) = \left(\frac{C_*(X)}{D_*(X)}, \bar{\partial}\right),$$

whose homology is equal to the homology of $C_*(X)$.

For $n \geq 0$, denote by Δ^n the non-degenerate chain complex associated to $\triangle^n$, this is

$$(\Delta^n, \partial) = (N_*(\triangle^n), \partial)$$

and an explicit description of this chain complex is given as follows: for each $q \geq 1$ the generators of Δ^n_q are the elements $a_{i_0,\dots,i_q}$ with $0 \leq i_0 < i_1 < \dots i_q \leq n$ and ∂ the boundary operator is

$$\partial a_{i_0,\dots,i_q} = \sum_{j=0}^{q} (-1)^j a_{i_0,\dots,\widehat{i_j},\dots,i_q}.$$

We can think of $\Delta^\bullet$ as a cosimplicial chain complex.

1.5 Model category structures

We will use the definitions and conventions of [14, §1.3] about model categories. In this section we will briefly describe the model structures that we will use on **top**, **sset** and **cdgl**.

1.5.1 Model structure on topological spaces

The (standard) model category of **top** is defined as follows: a continuous map $f\colon X \to Y$ is

- a fibration if it is a Serre fibration.
- a weak equivalence if it is a weak homotopy equivalence.
- a cofibration if it has the left lifting property with respect to trivial Serre fibrations.

In this model category all objects are fibrant and the CW-complexes are cofibrant.

1.5.2 Model structure on simplicial sets

The (standard) model category of **sset** is defined as follows: a simplicial map $f\colon X \to Y$ is

- a cofibration if $f_n\colon X_n \to Y_n$ is injective for all $n \geq 0$.
- a weak equivalence if its geometric realization is a weak equivalence.
- a fibration if f is a Kan fibration.

Recall that a *Kan fibration* is a map $f\colon X \to Y$ with the lifting property for all horn inclusions, i.e. for all $0 \leq n$, $0 \leq i \leq n$ and a commutative diagram

$$\begin{tikzcd} \wedge_i^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \triangle^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a lifting making the diagram commutative.

In this model category each object is cofibrant and the fibrant objects are called *Kan complexes*; in other words X is a Kan complex if $X \to * = \triangle^0$ is a Kan fibration.

There is an adjunction between **sset** and **top**, which turns out to be a Quillen equivalence with these model structures:

$$\mathbf{sset} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathbf{top}\,. \tag{1.4}$$

This pair of functors are defined using the *topological simplices*. For $n \geq 0$, define the topological space

$$\boldsymbol{\Delta}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1 \text{ and } t_i \geq 0\}.$$

By the usual formulas, $\boldsymbol{\Delta}^\bullet$ can be thought of as a cosimplicial topological space. Then for $X \in \mathbf{top}$, define the simplicial set $\mathrm{Sing}\, X$ as

$$(\mathrm{Sing}\, X)_\bullet = \mathrm{Hom}_{\mathbf{top}}(\boldsymbol{\Delta}^n, X)$$

where the faces and degeneracies are induced by those of $\boldsymbol{\Delta}^\bullet$. On the other hand, given $X \in \mathbf{sset}$ the topological space $|X|$ is called the *geometric realization* of X and it is the quotient

$$|X| = \bigsqcup_{n \geq 0} (X_n \times \boldsymbol{\Delta}^n)/\sim,$$

where each X_n is discrete and the equivalence relation is defined by

$$(d_i x, u) \sim (x, \delta^i u), \quad \text{for } x \in X_{n+1}, u \in \boldsymbol{\Delta}^n,$$

$$(s_j x, u) \sim (x, \sigma^j u), \quad \text{for } x \in X_{n-1}, u \in \boldsymbol{\Delta}^n,$$

for $0 \leq i \leq n+1$, $0 \leq j \leq n-1$ and $n \geq 0$.

From this equivalence we deduce some important facts that will be used along the text. The realization of a simplicial set $|X|$ is a CW-complex with one n-cell for each non-degenerate n-simplex. And by [17, Proposition 4.3.3 (i)] any point in $|X|$ has a unique representation by a pair (x, u) with x a non-degenerate simplex and u an interior point of $\mathbf{\Delta}^{|x|}$.

The unit of the adjunction $X \xrightarrow{\simeq_w} \operatorname{Sing}|X|$ is a natural weak homotopy equivalence. Since each object in **top** is fibrant, $\operatorname{Sing}|X|$ is a Kan complex. We can use the unit as functorial fibrant replacement.

If $x_0 \in X_0$ is a basepoint in a simplicial set, the point $(x_0, 0) \in X_0 \times \mathbf{\Delta}^0 \subset |X|$ can be taken as the basepoint in the geometric realization. Conversely, if $y_0 \in Y$ is a basepoint in a topological space Y, then the constant map $c_{y_0}\colon * = \mathbf{\Delta}^0 \to Y$ is a 0-simplex in $\operatorname{Sing} Y$, and it can be taken as the basepoint in the simplicial set. Unless otherwise specified we take these points as canonical basepoints along the adjoint functors and by abuse of notation we write $|x_0|$ and $\operatorname{Sing}(y_0)$ to denote them.

1.5.3 Model category on cdgl's

On the other hand, the category **cdgl** admits a model category (see [14, Theorem 8.1]) given in the following way: a morphism $f\colon L \to M$ is

- a fibration if it is surjective in non-negative degrees.
- a weak equivalence if
$$\widetilde{\mathrm{MC}}(f)\colon \widetilde{\mathrm{MC}}(L) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(M)$$
is a bijection and for all $a \in \widetilde{\mathrm{MC}}(L)$,
$$f^a\colon L^a \xrightarrow{\simeq} M^{f(a)}$$
is a quasi-isomorphism.
- a cofibration if it has the left lifting property with respect to trivial fibrations.

In this model category we can find a path object, which allows us to describe (right) homotopies between cdgl morphisms. For a given cdgl L, filtered by $\{F^n\}_{n\geq 1}$, its path object is the cdgl
$$L^I = \wedge(t, dt) \widehat{\otimes} L = \varprojlim_n (\wedge(t, dt) \otimes L/F^n)$$
where $\wedge(t, dt)$ is the tensor algebra generated by an element of degree 0 and its differential.

For $i = 0, 1$, define the cdgl morphism
$$\varepsilon_i\colon L^I = \wedge(t, dt) \widehat{\otimes} L \to L, \quad 1 \otimes x \mapsto x, \quad t^n \otimes x \mapsto ix, \quad t^m dt \otimes x \mapsto 0,$$
for $x \in L$, $n \geq 1$ and $m \geq 0$. In other words, ε_i sends t to i and dt to zero.

Then, a (right) homotopy in the category **cdgl** between $f, g\colon M \to L$ is a morphism
$$\Phi\colon M \to L^I$$

such that $\varepsilon_0 \circ \Phi = f$ and $\varepsilon_1 \circ \Phi = g$, and we write $f \sim g$. We denote by

$$[\![M, L]\!]$$

the set of equivalences classes of cdgl morphisms from M to L.

1.6 Nilpotent groups and spaces

Given a group G we denote its commutator by curved brackets (in order to differentiate it from the Lie bracket):

$$(x, y) = xyx^{-1}y^{-1}, \quad \text{for } x, y \in G.$$

We denote by $\{G^n\}_{n\geq 1}$ its lower central series, this is

$$G = G^1 \supset G^2 \supset \cdots \supset G^i \supset G^{i+1} \supset \ldots$$

where $G^1 = G$ and $G^i = (G^{i-1}, G)$ for $i \geq 2$.

A group is *nilpotent* if $G^q = \{1\}$ for some $q \geq 1$. For example, abelian groups are nilpotent groups with $q = 2$.

A group G acts *nilpotently* on an abelian group Γ if the lower central series of the action

$$\Gamma = \Gamma^0 \supset \Gamma^1 \supset \Gamma^2 \supset \ldots$$

is finite, i.e. $\Gamma^q = 0$ for some q, where Γ^i is the subgroup generated by $\{g\gamma - \gamma, g \in G, \gamma \in \Gamma^{i-1}\}$ for $i \geq 1$ and $\Gamma^0 = \Gamma$.

Let X be a pointed connected CW-complex or simplicial set. Then the fundamental group $\pi_1(X)$ acts on the higher homotopy groups $\pi_i(X)$ for $i \geq 2$. We call this action the *fundamental action*. An space X is *nilpotent* if its fundamental group is nilpotent and it acts nilpotently on $\pi_i(X)$ for $i \geq 2$. For example, simply connected spaces and the circle S^1 are nilpotent spaces.

1.7 Rationalization of spaces

For this section, we use [12, §9 (c)] and [23] as general references.

The *rationalization* of an abelian group G is just its tensor product by $\mathbb{Q}$

$$G \otimes \mathbb{Q}.$$

An abelian group G is *rational* if, for all $k \geq 1$, the map

$$G \to G, \quad g \mapsto g^k$$

is an isomorphism. Clearly, the rationalization of any abelian group is a rational group.

Let X be a CW-complex or a simplicial set, which is simply connected. Its homotopy groups are abelian group, so we say that it is a *rational space* if its homotopy groups are rational. There exists a functor, called the rationalization and a natural morphism

$\mu_X \colon X \to X_{\mathbb{Q}}$ from a space to its rationalization, such that $X_{\mathbb{Q}}$ is a rational space, and the induced morphisms

$$\pi_i(X) \otimes \mathbb{Q} \to \pi_i(X_{\mathbb{Q}})$$

are isomorphisms for all $i \geq 2$. The rationalizations of a space are characterized by the following universal property: given a map $f \colon X \to Y$ from a simply connected space X to a rational space Y there exits a (unique up to homotopy) map $X_{\mathbb{Q}} \to Y$ making the following homotopy diagram commutative

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ X_{\mathbb{Q}} \arrow[ur, dashed] & \end{tikzcd}.$$

In particular

$$[\![X, Y]\!] \cong [\![X_{\mathbb{Q}}, Y]\!].$$

These construction and statements can be generalized to nilpotent spaces rather than simply connected spaces (see §3 for a precise definition of nilpotent spaces).

In [5] a more general construction is given. The $\mathbb{Q}$-completion functor

$$\mathbb{Q}_\infty \colon \textbf{sset} \to \textbf{sset}$$

is characterized by the following property: a map $f \colon X \to Y$ induces an isomorphism in the rational reduced homology

$$\tilde{H}(f;\mathbb{Q}) \colon \tilde{H}(X;\mathbb{Q}) \xrightarrow{\cong} \tilde{H}(Y;\mathbb{Q})$$

if and only if the induced map

$$\mathbb{Q}_\infty f \colon \mathbb{Q}_\infty X \xrightarrow{\simeq} \mathbb{Q}_\infty Y$$

is a homotopy equivalence. For each simplicial set X there is also a natural map $X \to \mathbb{Q}_\infty X$.

For X a simply connected simplicial set, both construction agree: $X_{\mathbb{Q}}$ and $\mathbb{Q}_\infty X$ are homotopy equivalent (see [5, Chapter V]).

1.8 Model and realization functors

In [14, §6] it is defined a cosimplicial cdgl

$$\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0}$$

where,

$$\mathfrak{L}_n = (\widehat{\mathbb{L}}(s^{-1}\Delta^n), \partial)$$

and the cofaces and codegeneracies in $\mathfrak{L}_\bullet$ are induced by those of the cosimplicial chain complex $s^{-1}\Delta^\bullet$. The differential ∂ in $\mathfrak{L}_n$ is the only one (up to cdgl isomorphism) satisfying:

i) The elements a_i for $0 \leq i \leq n$ of degree -1 are MC elements.

ii) The linear part of ∂ is equal to the differential of the chain complex $s^{-1}\Delta^n$.

iii) The cofaces and codegeneracies are cdgl morphism.

In particular, following these rules, we have that $\mathfrak{L}_0$ is the free Lie algebra $\mathbb{L}(a)$ with $\partial a = -\frac{1}{2}[a,a]$, i.e. a is a MC-element. $\mathfrak{L}_1$ is the *Lawrence-Sullivan interval* described in [14, §5] but originally constructed in [25].

This cosimplicial cdgl plays a central role in the pair of adjoint functors

$$\mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl}$$

constructed and studied in [14, §7].

The *realization* of a cdgl L is the simplicial set

$$\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

In particular, the set of 0-simplices $\langle L \rangle_0$ is equal to the set of MC elements $\mathrm{MC}(L)$, so given a 0-simplex $a \in \langle L \rangle_0$, we denote by the same symbol $a \in L_{-1}$ the associated MC element, and vice versa. By [14, §7.2], there are homotopy equivalences

$$\langle L \rangle^a \simeq \langle L^a \rangle, \qquad \langle L \rangle \simeq \coprod_{a \in \widetilde{\mathrm{MC}}(L)} \langle L^a \rangle$$

where $\langle L \rangle^a$ is the path connected component of $\langle L \rangle$ containing the 0-simplex a. A direct consequence of this fact is that for any MC element a,

$$\langle (L, \partial) \rangle \simeq \langle (L, \partial_a) \rangle$$

since the sets $\widetilde{\mathrm{MC}}(L,\partial)$ and $\widetilde{\mathrm{MC}}(L,\partial_a)$ are bijective (see [14, Proposition 4.28]).

If L is connected, then for any $n \geq 1$, the map

$$\rho_n \colon \pi_n \langle L \rangle \xrightarrow{\cong} H_{n-1}(L), \quad \rho_n[\varphi] = [\varphi(a_{0,\dots,n})]$$

is a group isomorphism [14, Theorem 7.18]. For $n \geq 1$, the (abelian) group structure in $H_n(L)$ is the inherited by the vector space L; however, in $H_0(L)$ the group structure is the inherited by the BCH product.

On the other hand, the global model of a simplicial set has the following properties [14, Proposition 7.8]: $\mathfrak{L}_X = \widehat{\mathbb{L}}(s^{-1}N_*(X))$ and its differential ∂ is uniquely determined by

i) The 0-simplices of X are MC elements.

ii) The linear part of ∂ is equal to the differential of the chain complex $s^{-1}N_*(X)$.

iii) If $j \colon X \to Y$ is a subsimplicial set inclusion, then $\mathfrak{L}_j = \widehat{\mathbb{L}}(s^{-1}N_*(j))$.

If X is a connected simplicial set, a is a 0-simplex of X, and there is a quasi-isomorphism of connected cdgl's $(\mathfrak{L}(V),\partial) \xrightarrow{\simeq} \mathfrak{L}_X^a$ with $(\widehat{\mathbb{L}}(V),\partial)$ a (minimal) free cdgl, then $(\widehat{\mathbb{L}}(V),\partial)$ is called the *(minimal) model* of X. A minimal model of X is unique up to cdgl isomorphism and by [14, Proposition 8.35] there are isomorphisms

$$sV \cong \tilde{H}_*(X;\mathbb{Q}), \qquad sH_*(\widehat{\mathbb{L}}(V),\partial) \cong \pi_*\langle \mathfrak{L}_X^a\rangle.$$

To sum up, we have Quillen pairs relating the model categories of **top**, **sset** and **cdgl**

$$\mathbf{top} \underset{\mathrm{Sing}}{\overset{|-|}{\leftrightarrows}} \mathbf{sset} \underset{\langle - \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \mathbf{cdgl}\ , \tag{1.5}$$

where the upper functors are left adjoint to the bottom functors, and the first pair is a Quillen equivalence.

1.9 Algebras and coalgebras

As explained before, the cdgl's are the central algebraic objects of this text. However, we also need other algebraic structures as auxiliary objects.

We denote by **cdga** the category of *commutative differential graded algebras*, these are graded vector spaces with an associative linear product of degree 0 (that we denote just by juxtaposition), with a unit and a differential d (of degree $+1$), such that

$$xy = (-yx)^{|x||y|}, \quad d(xy) = (dx)y + (-1)^{|x|}xdy.$$

We assume that our cdga's are *augmented*, i.e. there exists a cdga morphism $\varepsilon\colon A \to \mathbb{Q}$ and cdga morphisms respect the augmentations.

The *free commutative graded algebra* on a vector space V is the following quotient

$$\wedge V = T(V)/I$$

where $T(V)$ is the (non-commutative) tensor algebra on V and I is the ideal generated by the elements of the form $x\otimes y - (-1)^{|x||y|}y\otimes x$.

There is a rational homotopy theory that models spaces on the category **cdga**. This is a well-known field (see [12], [13], [51], for some standard references), so we do not introduce here its basic aspects.

For example, the simplicial set $\triangle^1$ has as cdga model, the free cdga

$$\wedge(t, dt)$$

where $|t| = 0$, $|dt| = 1$ and $d(t) = dt$. This particular example is useful to construct homotopies.

The dual concept of a cdga is that of *cocommutative differential graded coalgebra*, whose category we denote by **cdgc**. A cdgc C is a graded vector space with: a degree zero map $\Delta\colon C \to C\otimes C$ which is coassociative

$$(\Delta\otimes \mathrm{id}_C)\circ\Delta = (\mathrm{id}_C \otimes \Delta)\circ\Delta$$

and cocommutative

$$\tau \circ \Delta = \Delta$$

where τ is the graded permutation of the factors; a differential such that

$$\Delta \circ d = (d \otimes \mathrm{id}_C + \tau \circ (d \otimes \mathrm{id}_C) \circ \tau) \circ \Delta;$$

and a counit $\varepsilon \colon C \to \mathbb{Q}$ and a coaugmentation $\eta \colon \mathbb{Q} \to C$ such that

$$d \circ \eta = 0, \quad \varepsilon \circ \eta = \mathrm{id}_{\mathbb{Q}}, \quad \Delta\eta(1) = \eta(1) \otimes \eta(1).$$

See [51, §0.3] for more details. We define $\bar{C} = \ker(\varepsilon)$ and the *reduced comultiplication* by

$$\bar{\Delta} \colon \bar{C} \to \bar{C} \otimes \bar{C}, \quad \bar{\Delta}c = \Delta c - (c \otimes \eta(1) + \eta(1) \otimes c).$$

The *cofree cocommutative graded coalgebra* on V is the graded vector space $\wedge V$ with the comultiplication defined by

$$\Delta(v) = v \otimes 1 + 1 \otimes v,$$

on the generators and extended to $\wedge V$ declaring $\Delta : \wedge V \to \wedge V \otimes \wedge V$ an algebra morphism.

The dual $C^\sharp = \mathrm{Hom}(C, \mathbb{Q})$ of a cdgc C is a cdga with the product

$$(\alpha\beta)(c) = (\alpha \otimes \beta)(\Delta c)$$

for $\alpha, \beta \in C^\sharp$, $c \in C$.

Finally recall that there are a pair of adjoint functors

$$\mathbf{cdgc} \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{dgl}$$

given as follows: if C is a cdgc, then $\mathscr{L}(C)$ is $(\mathbb{L}(s^{-1}\bar{C}), d)$ where $d = d_1 + d_2$ and

$$d_1(s^{-1}dc) = -s^{-1}dc, \quad d_2(s^{-1}c) = \frac{1}{2}[-,-] \circ (s^{-1} \otimes s^{-1}) \circ \bar{\Delta}(c)$$

for $c \in C$, and $\mathscr{C}(L)$ is $\wedge(sL), d)$ where $d = d_1 + d_2$ and

$$d_1(sv_1 \wedge \cdots \wedge sv_n) = -\sum_{i=1}^{n} (-1)^{n_i} sv_i \wedge \cdots \wedge s(dv_i) \wedge \cdots \wedge sv_n$$

$$d_2(sv_1 \wedge \cdots \wedge sv_n) = \sum_{1 \le i < j \le n} (-1)^{|sv_i|} \rho_{ij} s[v_i, v_j] \wedge sv_1 \wedge \cdots \wedge \widehat{sv_i} \wedge \cdots \wedge \widehat{sv_j} \wedge \cdots \wedge sv_n$$

where $n_i = \sum_{j<i} |sv_j|$ and ρ_{ij} is the Koszul sign of the permutation

$$sv_1 \wedge \cdots \wedge sv_n \mapsto sv_i \wedge sv_j \wedge sv_i \wedge \cdots \wedge \widehat{sv_i} \wedge \cdots \wedge \widehat{sv_j} \wedge \cdots \wedge sv_n.$$

In particular, we have the expressions

$$d_1(sv) = -sdv, \quad d_2(sv \wedge sw) = -(-1)^{|v|} s[v, w].$$

For a dgl L, the counit of the adjunction is the dgl morphism

$$\alpha_L \colon \mathscr{L}\mathscr{C}(L) = (\mathbb{L}(s^{-1} \wedge^+ sL), d) \to L$$

defined by

$$\alpha_L(s^{-1}sx) = -x, \quad \alpha_L(s^{-1} \wedge^{\ge 2} sL) = 0.$$

For any dgl L, α_L is a quasi-isomorphism (see [14, Proposition 2.3]).

CHAPTER 2

$\mathcal{H}$-FIBRATION SEQUENCES AND CLASSIFYING SPACES

Fibrations with a given fiber were initially classified in the paper of Stasheff [48]. There are several references about this topic, extending the initial theorem (see [4, 20, 32]) or the notion of classifying fibrations of a certain type (see [19, 53]). Through these texts we find a classifying space, namely $B\operatorname{aut}(F)$, such that the set of homotopy classes of maps $[\![B, B\operatorname{aut}(F)]\!]$ is in bijective correspondence with the set of equivalence classes of fibrations over B with fiber F, where both B and F are of the homotopy type of a CW-complex. Here $\operatorname{aut}(F)$ is the monoid of self-homotopy equivalences of F and $B\operatorname{aut}(F)$ the geometric-bar construction associated to this monoid.

Here, and broadly speaking, we classify fibrations with a prescribed holonomy action: more concretely, given a subgroup $\mathcal{H}$ of the group $\mathcal{E}(F) = \pi_0(\operatorname{aut}(F))$ we are interested in the monoid $\operatorname{aut}_{\mathcal{H}}(F) \subset \operatorname{aut}(F)$ consisting on the connected components of maps whose homotopy classes live in $\mathcal{H}$. From this monoid, we get a new space $B\operatorname{aut}_{\mathcal{H}}(F)$. The main purpose of this chapter is to show that this is, again, a classifying space: concretely, it classifies fibrations with fiber F such that the action fundamental group of the base on the fiber is contained in $\mathcal{H} \leqslant \mathcal{E}(F)$.

Though this fact may be known (see [9, §4]), there is no formal treatment of this topic in the literature. In the first section of this chapter, the concepts introduced above will be rigorously presented along with some necessary properties. In section 2, the classification theorem will be proved. Finally, in section 3 the relative case will be treated by considering homotopy automorphisms which keep unaltered a given subspace of the fiber.

Throughout this chapter we are going to use the results and notation of [32].

2.1 Definition and properties

In this section we recall some known facts about fibrations and the geometric-bar construction, and we introduce the new concept of $\mathcal{H}$-fibration sequences.

2.1.1 Fibrations

A (Hurewicz) fibration is a continuous map $\pi\colon E \to B$ such that it satisfies the homotopy lifting property with respect to any space X.

The following conventions are going to be used: we assume that B is an object in **CW** and is connected with basepoint b_0. In particular, $\pi^{-1}(b) \simeq \pi^{-1}(b_0)$ for all $b \in B$. Whenever we have a path $\beta\colon [0,1] \to B$, the map $\beta\colon X \times [0,1] \to B$ will denote $\beta(x,t) = \beta(t)$ and a map between a subspace and the total space without any label will denote the inclusion. Similarly $X \to X \times [0,1]$ will denote the inclusion at 0.

For F an object in **CW**, we say that a fibration $\pi\colon E \to B$ has fiber weakly equivalent to F if $\pi^{-1}(b)$ is weakly equivalent to F for some (every) point $b \in B$.

A *fibration sequence* is a sequence

$$F \xrightarrow{\omega} E \xrightarrow{\pi} B$$

where π is a fibration and $\omega\colon F \to \pi^{-1}(b_0)$ is a weak homotopy equivalence. A map between two fibration sequences $F \xrightarrow{\omega_1} E_1 \xrightarrow{\pi_1} B$ and $F \xrightarrow{\omega_2} E_2 \xrightarrow{\pi_2} B$ is a homotopy commutative diagram of the form,

$$\begin{tikzcd} & E_1 \arrow[dd, "f"] \arrow[dr, "\pi_1"] & \\ F \arrow[ur, "\omega_1"] \arrow[dr, "\omega_2"'] & & B \\ & E_2 \arrow[ur, "\pi_2"'] & \end{tikzcd} \tag{2.1}$$

In particular, f is a weak homotopy equivalence and there is no loss of generality if the right triangle is imposed to be strictly commutative, due to the lifting properties of π_1 and π_2. The maps of fibration sequences generate (by imposing symmetry and transitivity) an equivalence relation and we denote by $\mathcal{F}\mathrm{ib}(B,F)$ the quotient set.

Remark 2.1. There is an important clarification that has to be made: this definition of $\mathcal{F}\mathrm{ib}(B,F)$, which is the one considered in [20], differs with the one considered in [48] or [32]. There, they consider fibrations $\pi\colon E \to B$ whose fiber is weakly equivalent to F, but no explicit weak equivalence is given. Both approaches give rise to different equivalence classes and, therefore, to different classifying spaces. However, in [33, Explanation below Theorem 1.2] it is proved that both classifying spaces are weakly equivalent. Since the base spaces are always considered to be in **CW**, both approaches coincide in our setting.

For a given fibration $\pi\colon E \to B$, let us recall the action of the fundamental group of the base on the homotopy automorphisms of the fiber.

Given a path $\beta\colon [0,1] \to B$ with $\beta(0) = b_0$ and $\beta(1) = b_1$, let $F_0 = \pi^{-1}(b_0)$ and $F_1 = \pi^{-1}(b_1)$ be the fibers at those points. Apply the homotopy lifting property to the following diagram

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ F_0 \times [0,1] \arrow[r, "\beta"] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

to obtain a map $h\colon F_0 \times [0,1] \to E$. Then, define

$$\bar{\beta} = h(-,1)\colon F_0 \to F_1$$

Since the choice of the lifting is not unique, different choices would give different but homotopic maps. In other words, $\bar{\beta}$ is unique up to homotopy. This is proved more generally in the following proposition.

Proposition 2.2. *Given a fibration $\pi\colon E \to B$, a path $\beta\colon [0,1] \to B$, a space X and a map $f\colon X \to \pi^{-1}(\beta(0)) \subset E$. Then, two maps h_0, h_1 making the following diagram commutative,*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & E \arrow[d, "\pi"] \\ X \times [0,1] \arrow[r, "\beta"] \arrow[ur, dashed, "{h_0,h_1}"] & B \end{tikzcd}$$

are fiberwise homotopic. This means that there exists a homotopy between h_0 and h_1

$$h'\colon X \times [0,1] \times [0,1] \to E$$

such that $h'(x,t,s) \in \pi^{-1}(\beta(s))$ for all $s \in [0,1]$.

Proof. Consider

$$C = [0,1] \times \{0\} \cup \{0,1\} \times [0,1] \subset [0,1]^2$$

$$f'\colon X \times C \to E,\ \ f'(x,t,0) = f(x,t),\ \ f'(x,0,t) = h_0(x,t),\ \ f'(x,1,t) = h_1(x,t)$$

which make the following diagram commutative

$$\begin{tikzcd} X \times C \arrow[r, "f'"] \arrow[d] & E \arrow[d, "\pi"] \\ X \times [0,1] \times [0,1] \arrow[r, "\beta"] \arrow[ur, dashed, "h'"] & B \end{tikzcd}.$$

Since the pair $([0,1] \times [0,1], C)$ is homeomorphic to $([0,1] \times [0,1], [0,1] \times \{0\})$ we can apply the homotopy lifting property to obtain a map $h' = X \times [0,1] \times [0,1] \to E$. Since $h'(x,0,t) = h_0(x,t)$ and $h'(x,1,t) = h_1(x,t)$, this implies that h_0 and h_1 are fiberwise homotopic.

□

Similarly, using the homotopy lifting property, and the uniqueness (up to homotopy) of the lifting described in the Proposition 2.2, other properties of the map $\bar{\beta}$ can be deduced:

- If $\beta_1 \simeq_{\{0,1\}} \beta_2$ are homotopic paths (with respect to their endpoints) then $\bar{\beta}_1 \simeq \bar{\beta}_2$.
- The map $\bar{\beta}\colon \pi^{-1}(\beta(0)) \to \pi^{-1}(\beta(1))$ is a homotopy equivalence.
- The composition of paths $\beta_1\beta_2$ gives a map $\overline{\beta_1\beta_2} \simeq \bar{\beta}_2 \circ \bar{\beta}_1$.

When $\beta(0) = \beta(1) = b_0$, we get a homotopy equivalence $\bar{\beta}\colon F_0 \to F_0 = \pi^{-1}(b_0)$ for each element $[\beta] \in \pi_1(B, b_0)$. In what follows we refer to $\bar{\beta}$ as the lifting of β in the given fibration sequence.

Definition 2.3. For a map $\pi\colon E \to B$, we define

$$\Gamma E = \{(x, \beta) \in E \times B^I \mid \beta(0) = \pi(x)\}$$

$$\Gamma\pi\colon \Gamma E \to B, \quad (x, \beta) \mapsto \beta(1).$$

For an arbitrary map $\pi\colon E \to B$, $\Gamma\pi\colon \Gamma E \to B$ is a fibration, whose fiber is

$$\mathcal{F} = (\Gamma\pi)^{-1}(b_0) = \{(x, \beta) \in E \times B^I \mid \beta(0) = \pi(x),\ \beta(1) = b_0\}.$$

The space $\mathcal{F}$ is called the *homotopy fiber* of π.

There is a natural homotopy equivalence given by the inclusion

$$i\colon E \to \Gamma E, \qquad x \mapsto (x, c_{\pi(x)})$$

where $c_{\pi(x)}$ is the constant path at $\pi(x)$.

A map $\pi\colon E \to B$ is called a *quasi-fibration* if the inclusion $i\colon E \to \Gamma E$ induces a weak homotopy equivalence

$$i\colon \pi^{-1}(b) \xrightarrow{\simeq_w} (\Gamma\pi)^{-1}(b)$$

for all $b \in B$.

Definition 2.4. A *quasi-fibration sequence* is a sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ with $\omega(E) \subset \pi^{-1}(b_0)$, where π is a quasi-fibration and the map

$$i \circ \omega\colon F \to \mathcal{F},$$

given by the composition of ω and the inclusion $E \to \Gamma E$, is a weak homotopy equivalence.

By the *associated fibration sequence* of a quasi-fibration sequence, we mean the fiber sequence

$$F \xrightarrow{i\circ\omega} \Gamma E \xrightarrow{\Gamma\pi} B,$$

We say that two quasi-fibration sequences are equivalent if their associated fibration sequences are equivalent.

2.1.2 $\mathcal{H}$-fibration sequences

Consider the topological monoid:

$$G = \mathrm{aut}(F) = \{\varphi\colon F \to F \mid \varphi \text{ homotopy equivalence}\}$$

with the compact-open topology; it is clearly grouplike as $\pi_0(G)$ is the group of homotopy classes of self homotopy equivalences of F, with the operation given by the composition. Henceforth, we denote $\pi_0(G)$ by $\mathcal{E}(F)$.

Definition 2.5. Let $\mathcal{H}$ be a subgroup of $\mathcal{E}(F)$, then, define

$$H = \mathrm{aut}_{\mathcal{H}}(F) = \{\varphi \colon F \to F \mid \varphi \in \mathrm{aut}(F), \varphi \in \mathcal{H}\},$$

Note that H is a topological submonoid of G, consisting of the union of some of its connected components.

Example 2.6. One way of constructing subgroups of $\mathcal{E}(F)$ is via a functor $\mathscr{H} \colon \mathrm{Ho}\,\mathbf{top} \to \mathcal{C}$ from the homotopy category of the category of topological spaces to an arbitrary category $\mathcal{C}$. Consider

$$H = \{\varphi \in \mathrm{aut}(F) \mid \mathscr{H}(\varphi) = \mathrm{id} \colon \mathscr{H}(F) \to \mathscr{H}(F)\}$$

and $\mathcal{H} = \pi_0(H)$. In other words,

$$H = \ker(\mathrm{aut}(F) \xrightarrow{\mathscr{H}} \mathrm{aut}(\mathscr{H}(F))).$$

Thus $\mathcal{H} = \pi_0(H)$ is the subgroup of $\mathcal{E}(F)$ obtained as the kernel of the natural morphism

$$\mathcal{E}(F) \to \mathrm{aut}(\mathscr{H}(F)).$$

These subgroups have proved to be important in many situations and have been deeply studied (see for example [52], [29] or [1]). Just to name a few notable instances, observe that choosing $\mathscr{H}$ to be the homotopy or homology groups of a given space, $\mathcal{H}$ became the well known subgroups $\mathcal{E}_\pi(F)$ or $\mathcal{E}_H(F)$ of $\mathcal{E}(F)$. These consist of homotopy classes of homotopy automorphisms of F inducing the identity on homotopy or homology groups. Note that, for $\mathcal{E}_\pi(F)$, one has to think of $\mathscr{H}$ as defined on the pointed homotopy category $\mathrm{Ho}\,\mathbf{top}^*$.

Also, choosing $\mathscr{H}$ to be the loop or suspension function (again, in the first case, this is defined on $\mathrm{Ho}\,\mathbf{top}^*$), one finds $\mathcal{H}$ to be the subgroups $\mathcal{E}_\Omega(F)$ and $\mathcal{E}_\Sigma(F)$ of $\mathcal{E}_\pi(F)$ and $\mathcal{E}_H(F)$ respectively, consisting of homotopy classes of homotopy automorphism that become the identity once you loop or suspend them respectively (see [15] or [40]).

Recall from the "notation and conventions" section that $[\![X, Y]\!]$ stands for (free) homotopy classes of maps from X to Y. By a classical result [36, Theorem 3] given the weak homotopy equivalence $\omega \colon F \to F_0$, with F a CW-complex, the map ω_* is a bijection:,

$$\begin{tikzcd} {[\![F, F]\!]} \arrow[r, shift left, "\omega_*"] & {[\![F, F_0]\!]} \arrow[l, shift left, dashed, "\Psi_\omega"] \end{tikzcd} .$$

We denote by Ψ_ω the inverse function.

Definition 2.7. Given a fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$, define the holonomy action of $\pi_1(B, b_0)$ on the fiber as the map

$$\begin{array}{ccc} \pi_1(B, b_0) & \to & \mathcal{E}(F) \\ \beta & \mapsto & \hat{\beta} = \Psi_\omega[\bar{\beta} \circ \omega]. \end{array}$$

Note that this map is well defined as any representative of $\hat{\beta}$ is a weak equivalence and F is a CW-complex. Note also that a map g is in the class of $\hat{\beta}$ if and only if $\omega \circ g \simeq \bar{\beta} \circ \omega$.

Now, we introduce the main concept of this section.

Definition 2.8. Given a subgroup $\mathcal{H}$ of $\mathcal{E}(F)$, we say that $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ is an $\mathcal{H}$-*fibration sequence* if the image of the holonomy action is contained in $\mathcal{H}$. In other words, if for all $\beta \in \pi_1(B, b_0)$, $\hat{\beta} \in \mathcal{H}$.

Proposition 2.9. *Given a map of fibration sequences over B with fiber F, if one of the fibration sequence is an $\mathcal{H}$-fibration sequence, the other one also is.*

Proof. As warned, we do not distinguish here a map from its homotopy class. Consider a diagram as (2.1), and let $\bar{\beta}_i$ be the corresponding lifting of a path β for $i = 1, 2$. Then, by Proposition 2.2,

$$f \circ \bar{\beta}_1 \simeq \bar{\beta}_2 \circ f.$$

Composing with ω_1, and using that $f \circ \omega_1 = \omega_2$, we obtain

$$\omega_2 \circ \hat{\beta}_1 \simeq \omega_2 \circ \hat{\beta}_2.$$

Hence $\hat{\beta}_1 \simeq \hat{\beta}_2$ and we get the result. □

Recall that, given a fiber sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ and a based map $f\colon (A, a_0) \to (B, b_0)$, the pullback fibration sequence is defined as

$$F \xrightarrow{f^*\omega} f^*E = \{(x, a) \in E \times A \mid f(a) = \pi(x)\} \xrightarrow{f^*\pi} A$$

where $f^*\omega(x) = (\omega(x), a_0)$ and $f^*\pi(x, a) = a$.

Definition 2.10. Given $\mathcal{H}$ a subgroup of $\mathcal{E}(F)$, we denote by $\mathcal{F}\mathrm{ib}_{\mathcal{H}}(B, F) \subset \mathcal{F}\mathrm{ib}(B, F)$ the set of equivalence classes of $\mathcal{H}$-fibration sequences over B with fiber F.

By Proposition 2.9, this is a well defined set.

It is immediate to check that the pullback fibration sequence of an $\mathcal{H}$-fibration sequence is an $\mathcal{H}$-fibration sequence , hence we can define a contravariant functor

$$\mathcal{F}\mathrm{ib}_{\mathcal{H}}(-, F)\colon \mathbf{CW} \to \mathbf{set}$$

in the obvious way.

Furthermore, by [34, Chapter 7, Section 5] and [32, Proposition 2.5], homotopic maps induce equivalent pullback fibration sequences. Therefore, $\mathcal{F}\mathrm{ib}_{\mathcal{H}}(-, F)$ defines in fact a contravariant functor from the homotopy category of $\mathbf{CW}$:

$$\mathcal{F}\mathrm{ib}_{\mathcal{H}}(-, F)\colon \operatorname{Ho}\mathbf{CW} \to \mathbf{set}\,.$$

The goal of the rest of the chapter is constructing a classifying object for this functor, that is, an object B_∞ such that $\mathcal{F}\mathrm{ib}_{\mathcal{H}}(-, F)$ is naturally isomorphic to $[\![-, B_\infty]\!]$.

Definition 2.11. An $\mathcal{H}$-*quasi-fibration sequence* is a quasi-fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ whose associated fibration sequence is an $\mathcal{H}$-fibration sequence, given $\mathcal{H}$ a subgroup of $\mathcal{E}(F)$.

2.1.3 Geometric bar construction

In this section, let G be an arbitrary topological monoid such that its identity element e is a strongly nondegenerate basepoint and let X and Y be left and right G-spaces respectively. The *geometric bar construction* is defined as follows.

Definition 2.12. The simplicial topological space $B_*(Y,G,X)$ has as j-simplices the space $Y \times G^j \times X$. The face and degeneracy maps are defined by:

$$d_i(y, g_1, \dots, g_j, x) = \begin{cases} (y \cdot g_1, g_2, \dots, g_j, x) & \text{if } i = 0 \\ (y, g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_j, x) & \text{if } 1 \leq i < j \\ (y, g_1, \dots, g_{j-1}, g_j \cdot x) & \text{if } i = j \end{cases}$$

$$s_i(y, g_1, \dots, g_j, x) = (y, g_1, \dots, g_i, e, g_{i+1}, \dots, g_j, x)$$

Define $B(Y,G,X)$ as the geometric realization of $B_*(Y,G,X)$. A triple consisting of a morphism $f\colon G \to G'$ of topological monoids and f-equivariant maps $g\colon Y \to Y'$ and $h\colon X \to X'$ induces a map

$$B(g,f,h)\colon B(Y,G,X) \to B(Y',G',X').$$

Hence B can be thought of as a functor on any of the variables. We define

$$BG = B(*,G,*), \quad EG = B(*,G,G),$$

and choose the basepoint of BG as the only 0-simplex.

In particular, the trivial map $X \to *$ is a G-map and it induces the map

$$p_{G,X}\colon B(*,G,X) \to B(*,G,*) = BG.$$

We write $p = p_{G,X}$ if there is no possible confusion. If G is a grouplike topological monoid, i.e. $\pi_0(G)$ is a group, then [32, Theorem 7.6] asserts that p is a quasi-fibration with fiber X. Note that $X \hookrightarrow B(*,G,X)$ is the inclusion of the 0-simplices of $B(*,G,X)$. In particular, this implies that the inclusion

$$i\colon X \to \mathcal{F} = (\Gamma p)^{-1}(*) = \{(z,\beta) \mid z \in B(*,G,X), \beta\colon [0,1] \to BG, p(z) = \beta(0), \beta(1) = *\}$$

given by $i(x) = (x, c_*)$ is a weak homotopy equivalence, where $* \in BG$, as chosen above, is the unique 0-simplex and $x \in B(*,G,X)$ lies in the 0-simplices of $B(*,G,X)$.

Consider now, the 1-skeleton of these spaces. From the definition of $B_*(*,G,*)$, the 1-skeleton $\hat{B}G$ of BG is

$$\hat{B}G = \mathrm{sk}_1(BG) = \frac{G \times [0,1]}{(g,0) \sim (g',1) \sim (e,t) \sim *},$$

where $g, g' \in G$, e is the identity element in G and $t \in [0,1]$. Similarly,

$$\hat{B}(*,G,X) = \mathrm{sk}_1(B(*,G,X)) = \frac{G \times X \times [0,1]}{(g,x,0) \sim (g',x',1) \sim (e,x,t), \ \text{for } g' \cdot x' = x}.$$

When G is grouplike, by [32, Proposition 8.7], there is a weak homotopy equivalence

$$\xi \colon G \xrightarrow{\simeq_w} \Omega BG$$

given by $g \mapsto \xi_g$ where

$$\xi_g \colon [0,1] \to \hat{B}G \subset BG, \quad \xi_g(t) = (g,t).$$

Some lemmas are necessary for proving the main theorem. The first one allows us to get information about the quasi-fibration p.

Lemma 2.13. *Let G be a grouplike topological monoid with e a strongly nondegenerate basepoint and X a left G-space. Given a loop $\gamma \colon [0,1] \to BG$, $\gamma(0) = \gamma(1) = *$, the lifting $\bar{\gamma}$ in the fibration $\Gamma p \colon \Gamma B(*,G,X) \to BG$, is given by the juxtaposition*

$$\bar{\gamma} \colon \mathcal{F} \to \mathcal{F}, \quad (z,\beta) \mapsto (z, \beta\gamma)$$

where $\mathcal{F} = (\Gamma p)^{-1}(*)$ is the fiber described above.

Furthermore, for each $g \in G$, the following diagram, where the upper map is the multiplication by g, commutes up to homotopy:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i", "\simeq_w"'] & X \arrow[d, "i", "\simeq_w"'] \\ \mathcal{F} \arrow[r, "\bar{\xi}_g"] & \mathcal{F} \end{tikzcd}.$$

Proof. The map $\bar{\gamma}$ is well-defined since $p(z) = \beta(0)$, $\beta(1) = \gamma(0) = \gamma(1) = *$. Recall from Section 2.1 that we need to find a map h making this diagram commutative:

$$\begin{tikzcd} \mathcal{F} \arrow[rr] \arrow[d] & & \Gamma B(*,G,X) \arrow[d, "\Gamma p"] \\ \mathcal{F} \times [0,1] \arrow[rr, "\gamma"] \arrow[urr, dashed, "h"] & & BG \end{tikzcd}.$$

Here the upper horizontal arrow is the inclusion of the fiber $\mathcal{F}$. Defining h as $h(z,\beta,t) = (z, \beta\gamma\mid_{[0,t]})$, the previous diagram clearly commutes and, since $h(z,\beta,1) = (z,\beta\gamma) = \bar{\gamma}(z,\beta)$, we have proved the first claim.

For the second claim, fix $g \in G$. We need to see that the two maps $i \circ g$ and $\bar{\xi}_g \circ i \colon X \to \mathcal{F}$ are homotopic. These maps are,

$$i \circ g \colon X \to \mathcal{F}, \quad x \mapsto ((e, g \cdot x, 0), c_*) \in \hat{\mathcal{F}},$$

$$\bar{\xi}_g \circ i \colon X \to \mathcal{F}, \quad x \mapsto ((e, x, 0), c_* \xi_g) \in \hat{\mathcal{F}},$$

where $\hat{\mathcal{F}} \subset \mathcal{F}$ is the following subset of $\mathcal{F}$:

$$\hat{\mathcal{F}} = \{(z,\beta) \mid z \in \hat{B}(*,G,X), \ \beta \colon [0,1] \to BG, \ p(z) = \beta(0), \ \beta(1) = *\}.$$

Consider the following map

$$\phi\colon X \times [0,1] \to \hat{\mathcal{F}} \subset \mathcal{F}, \quad \phi(x,s) = ((g,x,s), \xi_g\mid_{[s,1]}).$$

This map is well-defined as for given $x \in X$ and $s \in [0,1]$, $\xi_g\mid_{[s,1]}(0) = \xi_g(s) = (g,s) = p(g,x,s)$ and $\xi_g\mid_{[s,1]}(1) = \xi_g(1) = (g,1) \sim *$. Evaluating at $s = 0$ and $s = 1$, we get:

$$\phi(x,0) = ((g,x,0),\xi_g) = ((e,x,0), c_*\xi_g)$$

$$\phi(x,1) = ((g,x,1),\xi_g\mid_{[1,1]}) = ((e, g\cdot x, 0), c_*)$$

since $c_*\xi_g = \xi_g, \xi_g\mid_{[1,1]} = c_*$, $(g,x,0) \sim (e,x,0)$ and $(g,x,1) \sim (e, g\cdot x, 0)$. Therefore, ϕ is a homotopy between $i \circ g$ and $\bar{\xi}_g \circ i$, which completes the proof.

□

The following is proved by a straightforward verification.

Lemma 2.14. *Let H and G be grouplike topological monoids where identity elements are strongly nondegenerate basepoints in both cases. Let X be a left H-space and a left G-space and let $f\colon H \to G$ be a map of grouplike topological monoid such that*

$$g \cdot x = f(g) \cdot x$$

for any $g \in H$ and $x \in X$. Write $\mu = Bf\colon BH \to BG$, $\mu' = B(*, f, \mathrm{id})\colon B(*,H,X) \to B(*,G,X)$ and

$$\tilde{\mu}\colon \Gamma B(*,H,X) \to \Gamma B(*,G,X), \quad \hat{\mu}(z,\beta) = (\mu'(z), \mu\circ\beta).$$

Then, $\hat{\mu}$ is well-defined, and the following diagram is commutative.

$$\begin{tikzcd}
B(*,H,X) \arrow[r, "\mu'"] \arrow[d, "i"] \arrow[dd, bend right=60, "p"'] & B(*,G,X) \arrow[d, "i"] \arrow[dd, bend left=60, "p"] \\
\Gamma B(*,H,X) \arrow[r, "\tilde{\mu}"] \arrow[d, "\Gamma p"] & \Gamma B(*,G,X) \arrow[d, "\Gamma p"] \\
BH \arrow[r, "\mu"] & BG
\end{tikzcd}.$$

Proof. □

By [32, Proposition 7.8] the map μ' is a weak homotopy equivalence when is restricted to the fibers and so are the maps i, since the maps p are quasi-fibrations. We deduce that $\tilde{\mu}$ is a weak homotopy equivalence when restricted to the fibers.

Finally, we need the following result about fibrations with a ‘weakly discrete’ fiber.

Lemma 2.15. *Let $\rho' \colon X' \to Y'$ be a quasi-fibration between connected spaces, with fiber weakly equivalent to a discrete set. Then, there is a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\kappa", "\simeq_w"'] \arrow[d, "\rho"] & X' \arrow[d, "\rho'"] \\ Y \arrow[r, "\lambda", "\simeq_w"'] & Y' \end{tikzcd}$$

where X and Y are CW-complexes, κ and λ are weak homotopy equivalences, and ρ is the covering map corresponding to the subgroup $\pi_1(X')$ of $\pi_1(Y')$.

Proof. It is a just a straightforward application of the CW approximation theorem and the properties of covering maps.

□

2.2 Classifying theorem

From now on, let F be an object in **CW** and $G = \operatorname{aut}(F)$. Fix $\mathcal{H}$ a subgroup of $\mathcal{E}(F)$, write $H = \operatorname{aut}_{\mathcal{H}}(F)$ and denote by $j \colon H \to G$ the inclusion of the monoids. Note that F can be considered as a left G- and H-space with the obvious action $\varphi \cdot x = \varphi(x)$ for $x \in F$.

Remark 2.16. The condition of the unity e being a strongly nondegenerate basepoint could not be satisfied. In that case we will use the 'whiskering construction' (see [35, A.8]): let $\tilde{G} = G \vee [0,1]$ be the topological space obtained by growing a whisker from e. It can be made a topological monoid by the formula

$$g \cdot t = g = t \cdot g$$

for $g \in G$ and $t \in [0,1]$, the usual multiplication in $[0,1]$ and the given product in G. Note that 1 is the new identity element in $\tilde{G}$. The deformation retract $\tilde{G} \to G$ is a morphism of monoids, which, in particular, implies that $\pi_0(\tilde{G}) = \pi_0(G)$. Define $\tilde{H}$ analogously and $\tilde{H}$. Let $\tilde{j} \colon \tilde{H} \to \tilde{G}$ be the inclusion of topological monoids.

Note that in both cases, j and $\tilde{j}$ are inclusion of connected components, which implies that:

$$\pi_n(j) \colon \pi_n(H) \to \pi_n(G), \quad \pi_n(\tilde{j}) \colon \pi_n(\tilde{H}) \to \pi_n(\tilde{G})$$

are isomorphisms for $n \geq 1$. Note also that both $\tilde{G}$ and $\tilde{H}$ act on F, defining $t \cdot x = x$ for $t \in [0,1]$.

Whenever $e \in G = \operatorname{aut}(F)$ is not a strongly nondegenerate basepoint, replace G by $\tilde{G}$ and H by $\tilde{H}$. This change does not affect the results to follow (see [32, §9.3] for more details).

We present now the main theorem of this chapter.

Theorem 2.17. *For any CW-complex B, the set of equivalence classes of $\mathcal{H}$-fibrations over B with fiber F is naturally isomorphic to $[\![B, BH]\!]$. The bijection is given explicitly by*

$$\Lambda_{\mathcal{H}} \colon [\![B, BH]\!] \to \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B, F)$$

which sends a map $f \colon B \to BH$ to the $\mathcal{H}$-fibration sequence

$$F \xrightarrow{f^* i} f^*\Gamma B(*, H, F) \xrightarrow{f^*\Gamma p} B.$$

We say that

$$F \longrightarrow B(*, H, F) \xrightarrow{p} BH$$

is the *universal $\mathcal{H}$-quasi-fibration sequence.*

The the rest of the section is devoted to prove this theorem.

Proposition 2.18. *The sequence $F \longrightarrow B(*, H, F) \xrightarrow{p} BH$ is an $\mathcal{H}$-quasi-fibration sequence.*

Proof. Fix $* \in BH$, the unique 0-simplex as the basepoint. We need to check that for any $\gamma \in \pi_1(BH, *)$, the class $\hat{\gamma} \in \mathcal{E}(F)$, produced by the holonomy action of the quasi-fibration p (see Definition 2.7), actually lives in $\mathcal{H}$.

Since $H \xrightarrow{\xi} \Omega BH$ is a weak homotopy equivalence, $\pi_1(BH) \cong \pi_0(H)$ and we only need to check that $\hat{\xi}_g \in \mathcal{H}$ for $g \in H$. By the Lemma 2.13, the following diagram commutes up to homotopy:

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "i", "\simeq_w"'] & F \arrow[d, "i", "\simeq_w"'] \\ \mathcal{F} \arrow[r, "\bar{\xi}_g"] & \mathcal{F} \end{tikzcd} .$$

Here $\mathcal{F} = (\Gamma p)^{-1}(*)$ and $i \colon F \to \mathcal{F}$ is a weak homotopy equivalence. Then, by definition,

$$\bar{\xi}_g \circ i \simeq i \circ g,$$

which implies that $g = \hat{\xi}_g$ in $\mathcal{E}(F)$. In particular $\hat{\xi}_g \in \mathcal{H}$, which concludes the proof. □

As a consequence of this proposition and [32, Proposition 2.5], given an object $B \in$ **CW**, we get a map

$$\Lambda_{\mathcal{H}} \colon [\![B, BH]\!] \to \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B, F)$$

which sends the homotopy class of a map $f \colon B \to BH$ to the fiber sequence

$$F \xrightarrow{f^* i} f^*\Gamma B(*, H, F) \xrightarrow{f^*\Gamma p} B.$$

Remark 2.19. Without loss of generality we can assume f to be basepoint preserving, for a given non degenerate basepoint of B. This means that the set of (free) homotopy classes of maps and the set of (free) homotopy classes of based maps are equal. Do not confuse with the, generally different, set of based homotopy classes of based maps.

Our next goal is to 'transform' the map $Bj\colon BH \to BG$ into a covering map $\rho\colon X \to Y$.

Proposition 2.20. *There exists a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\kappa", "\simeq_w"'] \arrow[d, "\rho"] & BH \arrow[d, "Bj"] \\ Y \arrow[r, "\lambda", "\simeq_w"'] & BG \end{tikzcd}$$

where X and Y are CW-complexes, λ and κ are weak equivalences and ρ is the covering map given by the subgroup $\mathcal{H} \subset \mathcal{E}(F)$.

Proof. Our goal is to check that $Bj\colon BH \to BG$ is a quasi-fibration with a weakly discrete fiber, in order to apply lemma 2.15.

Write $H\backslash G = B(*, H, G)$. Then, by [32, Remark 8.9] there is a commutative diagram

$$\begin{tikzcd} BH \arrow[d, "Bj"] & B(H\backslash G, G, *) \arrow[l] \arrow[r] \arrow[d] & B(H\backslash G, G, *) \arrow[d, "q"] \\ BG & B(G\backslash G, G, *) \arrow[l] \arrow[r] & BG \end{tikzcd}$$

where all the horizontal maps are weak homotopy equivalences and $q\colon B(H\backslash G, G, *) \to B(*, G, *)$ is induced by $H\backslash G \to *$, which is a quasi-fibration with fiber $H\backslash G$ by [32, Theorem 7.6]. In particular, this exhibits Bj as a quasi-fibration.

With this information, we get a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_n(H\backslash G) \longrightarrow \pi_n(BH) \xrightarrow{\pi_n(Bj)} \pi_n(BG) \longrightarrow \pi_{n-1}(H\backslash G) \longrightarrow \cdots$$

Since $H = \mathrm{aut}_{\mathcal{H}}(F) \subset G = \mathrm{aut}(F)$ is defined as a union of connected components, we have that

$$\pi_n(j)\colon \pi_n(H) \to \pi_n(G)$$

is an isomorphism for $n \geq 1$ and $\pi_0(j)$ is the inclusion of $\mathcal{H}$ in $\mathcal{E}(F)$. Then, by the natural weak homotopy equivalence $G \simeq_w \Omega BG$, we have that

$$\pi_n(Bj)\colon \pi_n(BH) \to \pi_n(BG)$$

is a isomorphism for $n \geq 2$. In addition note that $H\backslash G$, BH and BG are connected. Hence, we deduce that,

$$\pi_n(H\backslash G) = 0, \text{ for } n \geq 1 \text{ and } \pi_0(H\backslash G) = \mathcal{E}(F)/\mathcal{H}.$$

In other words, $H\backslash G$ is weakly homotopic to a discrete space and applying Lemma 2.15 we get the desired result.

□

In particular, for $B \in \mathbf{CW}$, we get a commutative diagram

$$\begin{tikzcd} \llbracket B, X \rrbracket \arrow[r, "\kappa\circ -", "\cong"'] \arrow[d, "\rho\circ -"] & \llbracket B, BH \rrbracket \arrow[d, "Bj\circ -"] \\ \llbracket B, Y \rrbracket \arrow[r, "\lambda\circ -", "\cong"'] & \llbracket B, BG \rrbracket. \end{tikzcd}$$

By the lifting properties of the covering maps, the map

$$\rho \circ - \colon \llbracket B, X \rrbracket \to \llbracket B, Y \rrbracket$$

is injective, so we conclude that $Bj \circ -$ is also injective.

The situation can be summarized in the diagram of sets

$$\begin{tikzcd} \llbracket B, Y \rrbracket \arrow[r, "\lambda\circ -", "\cong"'] & \llbracket B, BG \rrbracket \arrow[r, "\Lambda", "\cong"'] & \mathcal{F}\mathrm{ib}(B, F) \\ \llbracket B, X \rrbracket \arrow[r, "\kappa\circ -", "\cong"'] \arrow[u, hook, "\rho\circ -"] & \llbracket B, BH \rrbracket \arrow[r, "\Lambda_{\mathcal{H}}"] \arrow[u, hook, "Bj\circ -"] & \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B, F) \arrow[u, hook] \end{tikzcd} ,$$

where the vertical arrows are inclusions of subsets, the map Λ is the bijection given by [32, Theorem 7.5] and $\Lambda_{\mathcal{H}}$ is the map described above.

Lemma 2.21. *The diagram above is commutative.*

Proof. We have seen that the left square is commutative. We see now that the right square also is.

Since the pullbacks of equivalent fibration sequence are equivalent, note that it is enough to prove that

$$\Gamma p_{H,F} \colon \Gamma B(*, H, F) \to BH$$

and

$$(Bj)^*\Gamma p_{G,F} \colon (Bj)^*\Gamma B(*, G, F) \to BH$$

are equivalent fibrations.

By the properties of the pullback we have a map τ fitting in this commutative diagram

$$\begin{tikzcd} \Gamma B(*, H, F) \arrow[dr, dashed, "\tau"] \arrow[drr, bend left, "\tilde{\mu}"] \arrow[ddr, bend right, "\Gamma p_{H,F}"'] & & \\ & (Bj)^*\Gamma B(*, G, F) \arrow[r] \arrow[d, "(Bj)^*\Gamma p_{G,F}"] & \Gamma B(*, G, F) \arrow[d, "\Gamma p_{G,F}"] \\ & BH \arrow[r, "Bj"] & BG \end{tikzcd}$$

where $\tilde{\mu}$ is defined and studied in Lemma 2.14. In that lemma, we saw that, when restricted to fibers, $\tilde{\mu}$ is a weak homotopy equivalence, so we deduce that τ also is. Therefore, we conclude that both fibrations are equivalent.

□

Recall that we have proved that the diagram

$$\begin{tikzcd}
\llbracket B, BG \rrbracket \arrow[r, "\Lambda", "\cong"'] & \mathcal{F}\mathrm{ib}(B,F) \\
\llbracket B, BH \rrbracket \arrow[u, hook] \arrow[r, "\Lambda_{\mathcal{H}}"] & \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B,F) \arrow[u, hook]
\end{tikzcd}$$

is commutative. In particular, the map $\Lambda_{\mathcal{H}}$ is injective. We will prove that $\Lambda_{\mathcal{H}}$ is also surjective. We first need the following.

Lemma 2.22. *Let $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ be an $\mathcal{H}$-fibration sequence which , via Λ^{-1}, it corresponds with a map $f\colon B \to BG$. Then, the image of*

$$\pi_1(f)\colon \pi_1(B) \to \pi_1(BG) \cong \mathcal{E}(F)$$

is contained in $\mathcal{H}$.

Proof. Recall from [32, Theorem 7.5] how the bijection $\Lambda\colon \llbracket B, BG \rrbracket \to \mathcal{F}\mathrm{ib}(B,F)$ is constructed. Given a fibration $\pi\colon E \to B$ with fiber weakly equivalent to F, consider the commutative diagram

$$\begin{tikzcd}
PE \arrow[d, "P\pi"] & B(PE,G,G) \arrow[l] \arrow[r, "q"] \arrow[d, "p"] & EG \arrow[d, "p"] \\
B \arrow[r, dashed, shift right, "\varphi"'] & B(PE,G,*) \arrow[l] \arrow[r, "q"] & BG .
\end{tikzcd}$$

Then, the inverse of Λ is given by:

$$\Lambda^{-1}\colon \mathcal{F}\mathrm{ib}(B,F) \to \llbracket B, BG \rrbracket, \quad (\pi\colon E \to B) \mapsto (f = q \circ \varphi\colon B \to BG).$$

In the diagram above, the arrows pointing left are weak homotopy equivalences, so we can find the map φ, a right homotopy inverse.

The space PE is the subspace of $\hom_{\mathbf{top}}(F,E)$ of maps $\psi\colon F \to E$ such that $\psi(F) \subset \pi^{-1}(b)$ for some $b \in B$ and $\psi\colon F \to \pi^{-1}(b)$ is a weak homotopy equivalence. See [32, Definition 4.3] for more details. The map $P\pi\colon PE \to B$, which sends ψ to $\psi(F)$, is a fibration with fiber weakly equivalent to G.

To choose basepoints in the spaces of the diagram it is enough to take a basepoint in $B(PE,G,G)$ and declare all maps to be pointed. The 0-simplices of $B(PE,G,G)$ is $PE \times G$ and we choose as basepoint (ω, e) with e the identity element of G and $\omega\colon F \to F_0 = \pi^{-1}(b_0) \subset E$ the weak homotopy equivalence given in the input.

We now recall some facts about the fibration $P\pi\colon PE \to B$ whose fiber is

$$\mathcal{F} = (P\pi)^{-1}(b_0) = \{\psi\colon F \to F_0 \mid \psi \text{ is a weak homotopy equivalence}\}.$$

Given a loop β at b_0, we first compute $\dot{\beta}\colon \mathcal{F} \to \mathcal{F}$, the lifting of β in $P\pi : PE \to B$. By the homotopy lifting property we have a commutative diagram

$$\begin{tikzcd}
F_0 \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
F_0 \times [0,1] \arrow[r, "\beta"] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

where $\bar{\beta} = h(-,1)$ is the lifting of β in the fibration. Consider the map

$$\dot{h}\colon \mathcal{F} \times [0,1] \to \mathcal{F}, \quad (\psi, t) \mapsto h(-,t) \circ \psi\colon F \to \pi^{-1}(\beta(t)).$$

It is immediate to check that this map fits in the following diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & PE \arrow[d, "P\pi"] \\
\mathcal{F} \times [0,1] \arrow[r, "\beta"] \arrow[ur, "\dot{h}"] & B
\end{tikzcd} .$$

Therefore

$$\dot{\beta} = \dot{h}(-,1)\colon \mathcal{F} \to \mathcal{F}, \quad \psi \mapsto \bar{\beta} \circ \psi\colon F \to F_0.$$

By [32, Propositions 7.5 and 7.8] the first diagram in the proof represents maps of quasi-fibrations with fiber weakly equivalent to G. Hence, we have the following diagram, where the columns are quasi-fibration sequences,

$$\begin{tikzcd}
\mathcal{F} \arrow[d] & G \arrow[l, "\omega\circ-"'] \arrow[r, "\mathrm{id}"] \arrow[d] & G \arrow[d] \\
PE \arrow[d, "P\pi"] & B(PE,G,G) \arrow[l] \arrow[r, "q"] \arrow[d, "p"] & EG \arrow[d, "p"] \\
B \arrow[r, dashed, shift right, "\varphi"'] & B(PE,G,*) \arrow[l, shift right, "\simeq_w"'] \arrow[r, "q"] & BG
\end{tikzcd} .$$

Consider the basepoints previously described and $e \in G, \omega \in \mathcal{F}$ as basepoints in the fibers. Then, we have the following commutative diagram, where the columns are exact sequences:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\pi_1(PE) \arrow[d, "\pi_1(P\pi)"] & \pi_1(B(PE,G,G)) \arrow[l] \arrow[r, "\pi_1(q)"] \arrow[d, "\pi_1(p)"] & 0 \arrow[d] \\
\pi_1(B) \arrow[d, "\delta"] & \pi_1(B(PE,G,*)) \arrow[l, "\cong"'] \arrow[r, "\pi_1(q)"] \arrow[d] & \pi_1(BG) \arrow[d, "\cong"] \\
\pi_0(\mathcal{F}) \arrow[d] & \pi_0(G) \arrow[l, "\omega\circ-"'] \arrow[r, "\mathrm{id}"] \arrow[d] & \pi_0(G) \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

Recall that $\omega\circ -\colon \pi_0(\mathcal{F}) \to \pi_0(G)$ is a bijection, whose inverse map is Ψ_ω. Therefore, we can compute $\pi_1(f)$ using the commutativity of the diagram above.

$$\pi_1(f) = \Psi_\omega \circ \delta\colon \pi_1(B) \to \pi_0(G) \cong \pi_1(BG).$$

In particular, given a loop $\beta\colon [0,1] \to B$ with $\beta(0) = \beta(1) = b_0$,

$$\pi_1(f)([\beta]) \in \pi_0(H)$$

if and only if there exists $g \in H$ such that $\omega \circ g \simeq \delta(\beta)$.

On the other hand, observe that $\delta\colon \pi_1(B) \to \pi_0(\mathcal{F})$ is given by evaluating $\hat{\beta}\colon \mathcal{F} \to \mathcal{F}$ at the basepoint. That is,

$$\delta(\beta) = \omega \circ \hat{\beta} = \bar{\beta} \circ \omega\colon F \to F_0.$$

Therefore, if we suppose that $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ is an $\mathcal{H}$-fibration sequence, that is, $\hat{\beta} \in \mathcal{H}$, we conclude that the image of $\pi_1(f)\colon \pi_1(B) \to \pi_1(BG) \cong \pi_0(G)$ is also contained in $\pi_0(H) = \mathcal{H}$.

□

Proposition 2.23. *The map $\Lambda_{\mathcal{H}}\colon [\![B, BH]\!] \to \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B,F)$ is surjective.*

Proof. Along this section we have constructed a commutative diagram:

$$\begin{tikzcd}
{[\![B,Y]\!]} \arrow[r, "\lambda\circ-", "\cong"'] & {[\![B,BG]\!]} \arrow[r, "\Lambda", "\cong"'] & \mathcal{F}\mathrm{ib}(B,F) \\
{[\![B,X]\!]} \arrow[u, "\rho\circ-"] \arrow[r, "\kappa\circ-", "\cong"'] & {[\![B,BH]\!]} \arrow[u, "Bj\circ-"] \arrow[r, "\Lambda_{\mathcal{H}}"] & \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B,F) \arrow[u, hook]
\end{tikzcd} .$$

Suppose that we start with an $\mathcal{H}$-fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$. Via Λ^{-1} we get a map $f\colon B \to BG$. Using the upper left bijection, we get a map $f'\colon B \to Y$ such that $\lambda \circ f' \simeq f$. Consider now the morphism

$$\pi_1(f') = \pi_1(\lambda)^{-1} \circ \pi_1(f)\colon \pi_1(B) \to \pi_1(Y).$$

By Lemma 2.22 the image of $\pi_1(f)$ is contained in $\mathcal{H}$, so we conclude that

$$\pi_1(f')(\pi_1(B)) \subset \pi_1(\rho)(\pi_1(X)).$$

This is the condition needed to lift the function f' to a map $\tilde{f}\colon B \to X$ such that $\rho \circ \tilde{f} = f'$. Then, by the commutativity of the first square, we have

$$Bj \circ \kappa \circ \tilde{f} \simeq f$$

and by the commutativity of the second one

$$\Lambda_{\mathcal{H}}(\kappa \circ \tilde{f}) = \Lambda(Bj \circ \kappa \circ \tilde{f}) = \Lambda(f)$$

which is equivalent to the initial fibration sequence. Therefore $\Lambda_{\mathcal{H}}$ is surjective.

□

With this, we have finished the proof of Theorem 2.17 which classifies $\mathcal{H}$-fibration sequences.

2.2.1 Reformulation and extension of the universal fibration

We give here a more convenient expression of the universal $\mathcal{H}$-quasi-fibration sequence. For it, fix a basepoint $x_0 \in F$ and consider the grouplike monoid

$$G^* = \mathrm{aut}^*(F) = \{\varphi\colon F \to F \mid \varphi \text{ homotopy equivalence and } \varphi(x_0) = x_0\}.$$

Similarly, define

$$H^* = \mathrm{aut}^*_{\mathcal{H}}(F) = G^* \cap H.$$

The evaluation map

$$\mathrm{ev}\colon G \to F, \quad \mathrm{ev}(g) = g(x_0)$$

is a fibration with fiber G^* (see [47, §II.8, Theorem 2]). When we restrict the space G^* to H^*, which is a collection of connected components of G^*, we get a fibration sequence

$$H^* \to H \to F.$$

Recall that there is a natural homeomorphism $B(X, *, *) \cong X$. The following proposition and its proof are analogous to the ones presented in [3, Lemma 2.2].

Proposition 2.24. *The map $B(H, H^*, *) \to B(F, *, *) \cong F$ induced by $(\mathrm{ev}, *, \mathrm{id})$, is a weak homotopy equivalence.*

Proof. By [32, Proposition 7.9],

$$H^* \to H = B(H, *, *) \to B(H, H^*, *)$$

is a quasi-fibration sequence, where the second map is induced by $(\mathrm{id}, c_{\mathrm{id}}, \mathrm{id})$, in which $c_{\mathrm{id}}\colon * \to H^*$ sends a point to the identity in H^*. In addition, we have the fibration sequence

$$H^* \to H \to F$$

and we can consider the following diagram

$$\begin{tikzcd}
H^* \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & H \cong B(H,*,*) \arrow[r, "B(\mathrm{id},c_{\mathrm{id}},\mathrm{id})"] \arrow[d, "\mathrm{id}"] & B(H,H^*,*) \arrow[d, "B(\mathrm{ev},*,\mathrm{id})"] \\
H^* \arrow[r, "i"] & H \cong B(H,*,*) \arrow[r, "B(\mathrm{ev},\mathrm{id},\mathrm{id})"] & B(F,*,*)
\end{tikzcd}$$

which is clearly commutative. Applying the five lemma we get that

$$B(\mathrm{ev},*,\mathrm{id})\colon B(H,H^*,*) \to B(F,*,*) \cong F$$

is a weak homotopy equivalence.

□

Theorem 2.25. *The inclusion $i\colon H^* \to H$ induces a quasi-fibration sequence*

$$F \to BH^* \to BH$$

which is equivalent to the universal $\mathcal{H}$-quasi-fibration sequence

$$F \to B(*,H,F) \to BH.$$

Proof. By [32, Remark 8.9], the maps $Bi\colon BH^* \to BH$ and $p\colon B(*,H,H/H^*) \to BH$ are equivalent. Furthermore, by the previous proposition, the weak homotopy equivalence $B(H,H^*,*) \simeq_w F$ gives us as an equivalence of maps $p\colon B(*,H,H/H^*) \to BH$ and $p\colon B(*,H,F_0) \to BH$. □

As in the ordinary case, we can extend the universal quasi-fibration sequence by using the Puppe Sequence. Given a quasi-fibration sequence $F \to E \to B$, then the Puppe sequence

$$\Omega B \to \mathcal{F} \to \Gamma E$$

has an associated long exact sequence of homotopy groups (as if it were a fibration sequence, but it is not in general). The map $\Omega B \to \mathcal{F}$ sends a loop β in B to (y_0, β) in $\mathcal{F}$, where y_0 is the basepoint of E. In our specific setting the translation of this result is the following proposition.

Proposition 2.26. *The sequence*

$$\mathrm{aut}_{\mathcal{H}}(F) \xrightarrow{\mathrm{ev}_{x_0}} F \to B\,\mathrm{aut}^*_{\mathcal{H}}(F)$$

has an associated long exact sequence of homotopy groups, where ev_{x_0} is the evaluation at a fixed point and the second map is the one given in Theorem 2.25.

Proof. Apply the Puppe sequence to the sequence given in Theorem 2.25 to get the sequence

$$\Omega B\,\mathrm{aut}_{\mathcal{H}}(F) \to \mathfrak{F} \to B\,\mathrm{aut}^*_{\mathcal{H}}(F)$$

where the first map is given by $\beta \mapsto (\mathrm{id}_F, \beta)$. Consider the following diagram, where the vertical arrows are weak homotopy equivalences:

$$\begin{tikzcd} \mathrm{aut}_{\mathcal{H}}(F) \arrow[r, "\mathrm{ev}"] \arrow[d, "\simeq_w"', "\xi"] & F \arrow[d, "\simeq_w"', "i"] \\ \Omega B\,\mathrm{aut}_{\mathcal{H}}(F) \arrow[r] & \mathfrak{F} \end{tikzcd}\ .$$

Using Lemma 2.13 it can be checked that this diagram commutes up to homotopy, which concludes the proof.

□

2.3 The relative case

Consider A an arbitrary space.

Definition 2.27. Given a map $\pi\colon E \to B$, an A-*section* is a map $\sigma\colon A \times B \to E$ such that:

i)

$$\pi \circ \sigma = \mathrm{proj}_B\colon A \times B \to B.$$

ii) For all $b \in B$, the following map is a homeomorphism,

$$\sigma_b = \sigma\mid_{A\times\{b\}}\colon A \xrightarrow{\cong} \sigma_b(A) \subset \pi^{-1}(b) \subset E.$$

Note that if we take $A = *$, the condition *ii*) trivially holds and we recover the usual notion of a section.

Given a pair $A \subset F$, using CW approximation, we can assume that they are both CW complexes and the inclusion is a cofibration.

Definition 2.28. An (F, A)-*fibration sequence* is a fibration-sequence

$$F \xrightarrow{\omega} E \xrightarrow{\pi} B$$

with an A-section $\sigma\colon A \times B \to E$ such that $\omega(A) \subset \sigma_{b_0}(A)$ and

$$\omega\mid_A = \sigma_{b_0}\colon A \to \sigma_{b_0}(A).$$

We write $A_0 = \sigma_{b_0}(A) \subset F_0 = \pi^{-1}(b_0)$.

Again, if we take $A = *$, we recover the usual notion of based fibration (see [32, §5]).

Remark 2.29. In [32, Definition 5.2], the map σ is required to be a fiberwise cofibration. However in [32, Addenda] a way to avoid this requirement is explained, via growing whiskers on fibers. We are going to use this construction, so the condition of being a fiberwise cofibration is omitted.

Definition 2.30. A map of (F, A)-fibration sequences over B is a map of fibration sequences f, such that the A-sections make the following diagram commutative:

$$\begin{tikzcd} & E_1 \arrow[dd, "f"] \\ A \times B \arrow[ur, "\sigma_1"] \arrow[dr, "\sigma_2"'] & \\ & E_2 \end{tikzcd}$$

The maps of (F, A)-fibration sequences generate a equivalence relation, and we denote by $\mathcal{F}\mathrm{ib}(B, (F, A))$ the set of equivalence classes.

Remark 2.31. In [32] only the case $A = *$ is considered and classified. However, in [22, Appendix B] the results of [32] are extended and generally in order to cover this case.

In particular, [32, Theorem 9.2] generalizes to the following statement:

Theorem 2.32. *There is a bijection of sets*

$$\Lambda\colon [\![B, B\,\mathrm{aut}^A(F)]\!] \to \mathcal{F}\mathrm{ib}^A(B, (F, A))$$

given by taking the pullback with respect to the universal (F, A)*-quasi-fibration sequence*

$$F \longrightarrow B(*, \mathrm{aut}^A(F), F) \xrightarrow{p} B\,\mathrm{aut}^A(F).$$

Here $\mathrm{aut}^A(F)$ is the following topological monoid

$$\mathrm{aut}^A(F) = \{\varphi\colon F \to F \mid \varphi \text{ homotopy equivalence and } \varphi\,|_A = \mathrm{id}_A\}.$$

Note that, since A is a trivial $\mathrm{aut}^A(F)$-space, the map $p\colon B(*, \mathrm{aut}^A(F), F) \to B\,\mathrm{aut}^A(F)$ has an A-section given by

$$\sigma\colon B\,\mathrm{aut}^A(F) \times A = B(*, \mathrm{aut}^A(F), A) \to B(*, \mathrm{aut}^A(F), F).$$

We consider now the whisker construction, which has been mentioned above.

Definition 2.33. Given a (F, A)-fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$, with A-section σ, define $\tilde{F} \xrightarrow{\tilde{\omega}} \tilde{E} \xrightarrow{\tilde{\pi}} B$, where

$$\tilde{E} = \frac{A \times B \times [0,1] \sqcup E}{(a, b, 0) \sim \sigma(a, b), \forall a \in A, b \in B}, \qquad \tilde{F} = \frac{A \times [0,1] \sqcup F}{(a, 0) \sim a, \forall a \in A},$$

$$\tilde{\omega}(x) = \omega(x), \quad \tilde{\omega}(a, t) = (a, b_0, t), \quad \tilde{\pi}(y) = \pi(y), \quad \tilde{\pi}(a, b, t) = b,$$

for any $x \in F, y \in E, a \in A, b \in B$ and $t \in [0,1]$.

Note that $\tilde{F} \xrightarrow{\tilde{\omega}} \tilde{E} \xrightarrow{\tilde{\pi}} B$ is a fibration sequence and $\tilde{\omega}\colon \tilde{F} \to \tilde{F}_0 = \tilde{\pi}^{-1}(b_0)$ is a weak homotopy equivalence.

We next construct a holonomy action which respects the subspace A.

Proposition 2.34. *Given an (F, A)-fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ and a path $\beta\colon [0,1] \to B$, $\beta(0) = b_0$, there is a lifting h in the following diagram*

$$\begin{tikzcd} \tilde{F}_0 \arrow[r] \arrow[d] & \tilde{E} \arrow[d, "\tilde{\pi}"] \\ \tilde{F}_0 \times [0,1] \arrow[r, "\beta"] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

such that $h((a, b_0, 1), s) = (a, \beta(s), 1)$, for any $a \in A$ and $s \in [0,1]$.

Proof. Consider the maps

$$\begin{aligned} i'\colon\ & \tilde{F}_0 \to E, & \beta'\colon\ & \tilde{F}_0 \times [0,1] \to B, \\ & x \mapsto x, & & (x, s) \mapsto \beta(s), \\ & (a, b_0, t) \mapsto \sigma(a, \beta(t)), & & ((a, b_0, t), s) \mapsto \beta(t+s), \end{aligned}$$

where $x \in F_0$, $a \in A$, $t \in [0,1] \subset \tilde{F}_0$ and $s \in [0,1]$. Here, we are using the convention $\beta(t) = \beta(1)$ if $t > 1$. Then, the following diagram commutes

$$\begin{tikzcd} \tilde{F}_0 \arrow[r, "i'"] \arrow[d] & E \arrow[d, "\pi"] \\ \tilde{F}_0 \times [0,1] \arrow[r, "\beta'"] \arrow[ur, dashed, "\phi"] & B \end{tikzcd} ,$$

and, therefore, there is a lifting $\phi\colon \tilde{F}_0 \times [0,1] \to E$. Define the following map $h\colon \tilde{F}_0 \times [0,1] \to \tilde{E}$:

$$(x, s) \mapsto \phi(x, s), \quad ((a, b_0, t), s) \mapsto \begin{cases} (a, \beta(s), \frac{2t-s}{2-s}), & \text{if } s \leq 2t \\ \phi(a, 2t, s-2t), & \text{if } s \geq 2t \end{cases}$$

where $x \in F_0$, $a \in A$ and $t \in [0,1] \subset \tilde{F}$. Note that $h(1, s) = (\beta(s), 1)$. It easy to check that this map is continuous, well defined and that it fits in the diagram,

$$\begin{tikzcd} \tilde{F} \arrow[r] \arrow[d] & \tilde{E} \arrow[d, "\tilde{\pi}"] \\ \tilde{F} \times [0,1] \arrow[r, "\beta"] \arrow[ur, dashed, "h"] & B \end{tikzcd} ,$$

where the upper map is the inclusion and, as usual, $\beta(-, s) = \beta(s)$. Then h is the required map. □

Observe that h plays the role of the usual lifting of a given path β of B. In this context define $\tilde{\beta} = h(-, 1)\colon \tilde{F}_0 \to \tilde{\pi}^{-1}(\beta(1))$. In particular, if $[\beta] \in \pi_1(B, b_0)$, then

$$\tilde{\beta}\colon \tilde{F}_0 \to \tilde{F}_0$$

is a homotopy equivalence such that $\tilde{\beta}(a, b_0, 1) = (a, b_0, 1)$ for any $a \in A$. Remark that $\tilde{\beta}$ is the analog of $\bar{\beta}$ in the usual context.

In the current setting, we write

$$[\![\tilde{F}, \tilde{F}_0]\!]^* = \{\varphi\colon \tilde{F} \to \tilde{F}_0 \text{ such that } \varphi\,|_{A\times\{1\}} = \tilde{\omega}\,|_{A\times\{1\}}\}/\sim_{A\times\{1\}}$$

$$[\![\tilde{F}, \tilde{F}]\!]^* = \{\varphi\colon \tilde{F} \to \tilde{F} \text{ such that } \varphi\,|_{A\times\{1\}} = \mathrm{id}\}/\sim_{A\times\{1\}}$$

$$[\![F, F]\!]^* = \{\varphi\colon F \to F \text{ such that } \varphi\,|_{A} = \mathrm{id}\}/\sim_{A}\ .$$

Note that when A is a point, we recover the usual notions of classes of equivalences of basepoint-preserving maps.

Note that $\tilde{F}$ is of the homotopy type of a CW-complex and $\tilde{\omega}$ is a weak homotopy equivalence. By an analogous result to [21, Proposition 4.22] we can see that

$$[\![\tilde{F}, \tilde{F}_0]\!]^* \xleftarrow[\cong]{\tilde{\omega}_*} [\![\tilde{F}, \tilde{F}]\!]^*$$

is a bijection.

On the other hand, between the spaces F and $\tilde{F}$ there is a map $\chi\colon \tilde{F} \to F$ given by the projection, which sends $A \times \{1\}$ to A. However, the inclusion map $i\colon F \to \tilde{F}$ does not send A to $A \times \{1\}$, so we need a different map. Since $A \to F$ is a cofibration, there is a lifting in the following diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & \tilde{F}^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ F \arrow[r, "i"] \arrow[ur, dashed, "h"] & \tilde{F} \end{tikzcd}$$

where $\alpha(a, t) = (a, t)$. Then define $\iota = h(-, 1)\colon F \to \tilde{F}$, and note that $\iota(a) = (a, 1)$ for $a \in A$. These two maps define the following bijection

$$[\![\tilde{F}, \tilde{F}]\!]^* \xleftarrow[\cong]{\iota\circ - \circ\chi} [\![F, F]\!]^*.$$

Definition 2.35. Define the *relative holonomy action* with respect to A of $\pi_1(B, b_0)$ on the fibre as the map

$$\begin{array}{ccc} \pi_1(B, b_0) & \to & [\![F, F]\!]^* \\ \beta & \mapsto & \check{\beta} \end{array}$$

in which $\check{\beta}$ is the preimage of $\tilde{\beta} \circ \tilde{\omega}$ in the bijection

$$[\![F, F]\!]^* \xrightarrow[\cong]{\iota\circ - \circ\chi} [\![\tilde{F}, \tilde{F}]\!]^* \xrightarrow[\cong]{\tilde{\omega}_*} [\![\tilde{F}, \tilde{F}_0]\!]^*\ .$$

That is, $\check{\beta}$ is the only element in $[\![F, F]\!]^*$ for which

$$\tilde{\omega} \circ \iota \circ \check{\beta} \circ \chi \simeq_{A\times\{1\}} \tilde{\beta} \circ \tilde{\omega}$$

We have previously introduced the topological monoid

$$G^A = \mathrm{aut}^A(F) = \{\varphi\colon F \to F \text{ such that } \varphi\,|_A = \mathrm{id}_A\}.$$

Since $A \to F$ is a cofibration, it is easy to check that $\pi_0(G^A)$, which will be denoted by $\mathcal{E}^A(F)$ henceforth, is a group. Given a subgroup $\mathcal{H}$ of $\mathcal{G}^A$, define

$$H^A = \mathrm{aut}^A_{\mathcal{H}}(F) = \{\varphi\colon F \to F \text{ such that } \varphi \in \mathrm{aut}^A(F), [\varphi] \in \mathcal{H}\}.$$

Definition 2.36. Given $\mathcal{H}$ a subgroup of $\mathcal{E}^A(F)$, an (F,A)-fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ is an $\mathcal{H}$-(F,A)-*fibration sequence* if $\tilde{\Theta}(\beta) \in \mathcal{H}$ for all $\beta \in \pi_1(B, b_0)$.

The good behaviour of the $\mathcal{H}$-(F,A)-fibration sequences with respect to maps of (F,A)-fibrations sequences and pullbacks can be proved in analogous way that in the ordinary case.

Definition 2.37. Denote by $\mathcal{F}\mathrm{ib}^A_{\mathcal{H}}(B,F)$ the set of equivalence classes of $\mathcal{H}$-(F,A)-fibration sequences over B under the equivalence relation generated by maps of (F,A)-fibrations sequences.

Remark 2.38. Similarly, we can talk about (F,A)-*quasi-fibration sequences* or $\mathcal{H}$-(F,A)-*quasi-fibration sequences*: if $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ is a quasi-fibration sequence and $\sigma\colon A \times B \to E$ is an A-section, then

$$\Gamma\sigma = i \circ \sigma\colon A \times B \to E \to \Gamma E$$

is an A-section for the fibration sequence $F \xrightarrow{i\circ\omega} \Gamma E \xrightarrow{\Gamma\pi} B$.

As in the non-relative case, we have a universal $\mathcal{H}$-(F,A)-quasi-fibration sequence

$$F \longrightarrow B(*, H^A, F) \xrightarrow{p} BH^A$$

with the A-section

$$\sigma\colon BH^A \times A = B(*, H^A, A) \to B(*, H^A, F)$$

induced by the inclusion of A in F, where the first equality comes from the fact that A is a trivial H^A-space, as it was explained above.

A meticulous revision of Theorem 2.17, by requiring that all the maps in the proof preserve the subspace A, gives rise to a relative version of that Theorem:

Theorem 2.39. *Let F and B be objects in $\boldsymbol{CW}$, let A be a subcomplex of F and let $\mathcal{H}$ be a subgroup of $\mathcal{E}^A(F)$. Then the set of equivalence classes of $\mathcal{H}$-(F,A)-fibrations over B is naturally isomorphic to $[\![B, BH^A]\!]$. The bijection is given explicitly by*

$$\Lambda_{\mathcal{H}}\colon [\![B, BH^A]\!] \to \mathcal{F}\mathrm{ib}^A_{\mathcal{H}}(B,F)$$

which sends a map $f\colon B \to BH$ to the $\mathcal{H}$-(F,A)-fibration sequence

$$F \xrightarrow{f^*i} f^*\Gamma B(*, H^A, F) \xrightarrow{f^*\Gamma p} BH^A.$$

CHAPTER 3

Malcev complete groups

The relationship between rational nilpotent groups and nilpotent Lie algebras is well known from the original work [28]. Several reformulations and generalizations of the Malcev equivalence can be found in the literature [13, 16, 26, 42, 49]. The purpose of the first section of this chapter is to formulate the 'generalized' Malcev equivalence in a suitable form for our goals and to compile useful facts from the references above.

In the second section, we prove that a certain class of subgroups of the automorphisms of a cdgl are Malcev complete.

3.1 Malcev equivalence

Recall from §1.6 that a group G is nilpotent if its lower central series $\{G^n\}_{n\geq 1}$ terminates. Let us introduce some related concepts.

Definition 3.1. A group is *pronilpotent* if the natural map $G \xrightarrow{\cong} \varprojlim_n G/G^n$ is an isomorphism. A group is *0-local* if, for all $n \geq 1$, the map

$$G \to G, \quad g \mapsto g^n$$

is a bijection.

Remark 3.2. In a 0-local group G, the expression g^λ for any $g \in G$, $\lambda \in \mathbb{Q}$ makes sense: if $\lambda = m/n$, g^λ is the unique element in G such that

$$(g^\lambda)^n = g^m.$$

This is why these groups are also known as *uniquely divisible* groups.

Definition 3.3. A group G is *Malcev $\mathbb{Q}$-complete* (or simply Malcev complete in our setting) if it is pronilpotent and for each $n \geq 1$ the abelian group G^n/G^{n+1} is a $\mathbb{Q}$-vector space.

Remark 3.4. In the references cited in the introduction of this chapter, the definitions of Malcev complete groups may differ. For example, Quillen asks an extra condition ([41, A.3, Definition 3.1]) for a group being Malcev complete: in addition to the conditions in definition 3.3, the associated graded Lie algebra is required to be generated by G/G^2.

On the other hand in [16, §8.2] a Malcev complete group is defined as the grouplike elements of a complete Hopf algebra. This approach, using Hopf algebras and their grouplike and primitive elements is extremely useful (in fact the whole section §1.2.3 can be formulated in such terms). However, we are not interested on a deep study of these objects, but only on some properties. Thus, we present a direct approach, which relies in [16, §7 and §8].

These definitions of a Malcev $\mathbb{Q}$-complete group (the one given in definition 3.3, that of [41, A.3, Definition 3.1] and that of [16, §8.2]) are equivalent by [16, Proposition 8.2.3]. Note that in our case the ground field is $\mathbb{Q}$, and that we are using the lower central series as filtration.

We denote by **groups** the category of groups and **cgroups** the full subcategory of Malcev complete groups. There is a functor

$$\mathbf{groups} \to \mathbf{cgroups}$$

called the Malcev completion functor, which is left adjoint to the forgetful functor (see [16, §I.8.3]). We denote by $\hat{G}$ the image of this functor, for a given group G. Furthermore, there is a natural transformation μ, from the identity functor to the Malcev completion functor. This means, that there is a natural morphism

$$\mu_G \colon G \to \hat{G}$$

for all G in **groups**. It is easy to check that if G is in **cgroups**, $\mu_G \colon G \xrightarrow{\cong} \hat{G}$ is an isomorphism.

On the other hand, denote by **cl** the category of complete ungraded Lie algebras (or equivalently, concentrated in degree 0). Clearly if L is a cdgl, then L_0 is in **cl**.

The Malcev equivalence [42, A.3] or [16, §8.2.8] can be formulated as the existence of an isomorphism of categories

$$\begin{tikzcd} \mathbf{cl} \arrow[r, bend left, "\varpi"] & \mathbf{cgroups} \arrow[l, bend left, "\vartheta"] \end{tikzcd}$$

with the following properties:

- The underlying set is not altered by any of the functors.
- Given L a complete Lie algebra, $\varpi(L)$ is the group $(L, *)$ where $*$ is the BCH product. There are also explicit formulas for the Lie algebra structure of $\vartheta(G)$ for any Malcev complete group G (see [49, §2.3]).
- If $G = \varpi(L)$, then for each $n \geq 1$, there is an equality of sets $L^n = G^n$ (see [13, Theorem 2.2]).

- When restricted to nilpotent Lie algebras and 0-local nilpotent groups, this is the original equivalence given in [28] and [26].

For a given complete Lie algebra L, L^n and L/L^n are also complete Lie algebras for any $n \geq 1$. Thus we deduce that G^n and G/G^n are Malcev complete groups if G is a Malcev complete group. The following result is due to Y. Félix.

Theorem 3.5. *A group G is Malcev complete if and only if it is pronilpotent and 0-local.*

Proof. The Malcev equivalence implies that a Malcev complete group is equivalent to a Lie algebra: in particular, the multiplication by scalars in the Lie algebra implies that the group is 0-local.

For the other implication, note that an abelian group is a $\mathbb{Q}$-vector space if and only if it is 0-local. So we only have to prove that G^n/G^{n+1} is 0-local for all $n \geq 1$.

Firstly, we see that the group G/G^n is torsion free for $n \geq 1$. Suppose that there is $x \in G$ such that

$$x^k \equiv 0 \pmod{G^n}.$$

This means that $x^k \in G^n$, so it can be written as

$$x^k = \prod_{i=1}^{m}(a_i, b_i)$$

where $a_i \in G$ and $b_i \in G^{n-1}$. Since G is 0-local, for each $i = 1, \dots m$ we can find $c_i \in G$ such that $c_i^k = a_i$. Note that in G/G^{n+1} the elements of $G^n = (G, G^{n-1})$ commute with any other element, so we can check that

$$(a_i, b_i) \equiv (c_i^k, b_i) \equiv (c_i, b_i)^k \pmod{G^{n+1}}.$$

Then write

$$y = \prod_{i=1}^{m}(c_i, b_i)$$

and note that it is a product of elements of G^n so that, in G/G^{n+1}, we have the following equation

$$y^k = \left(\prod_{i=1}^{m}(c_i, b_i)\right)^k \equiv \prod_{i=1}^{m}(c_i, b_i)^k \equiv \prod_{i=1}^{m}(a_i, b_i) = x^k \pmod{G^{n+1}}.$$

In particular, since $y \in G^n$,

$$(xy^{-1})^k \equiv 0 \pmod{G^{n+1}}.$$

Then write $x_n = x$, $x_{n+1} = xy^{-1}$ and repeat this process to obtain a sequence of elements $(x_j)_{j \geq n}$ in G such that

$$x_{j+1} \equiv x_j \pmod{G^j}$$

and $x_j^k \in G^j$ for all $j \geq n$. Since G is a pronilpotent group, we can find an element $x_0 = \lim x_j$, such that

$$x_0 \equiv x_i \pmod{G^j}$$

for all $i \geq j$. In particular, $x_0^k \in G^j$ for all $j \geq n$, which implies that $x_0^k = 0$. Since G is 0-local, we deduce that $x_0 = 0$. Therefore, the sequence

$$(x_j) \in \varprojlim_j G/G^j$$

is identically zero, so, in particular, $x \in G^n$, and therefore G/G^n is torsion-free.

Since G/G^n is nilpotent and torsion free, by [23, Theorem 2.2] the map $x \mapsto x^k$ is injective. On the other hand, it is clear that the map $x \mapsto x^k$ is surjective in G/G^n, so we conclude that G/G^n is 0-local, for any $n \geq 1$.

To finish, consider the short exact sequence

$$G^n/G^{n+1} \to G/G^{n+1} \to G/G^n$$

where the two right groups are 0-local and all of them are nilpotent. Then [23, Corollary 2.5] implies that G^n/G^{n+1} is 0-local for any $n \geq 1$.

□

3.2 Nilpotent groups of $\mathcal{E}^*(L)$

Let $L = (\widehat{\mathbb{L}}(V), \partial)$ be a minimal free cdgl with $V = V_{\geq 0}$. Denote by $\operatorname{aut}(L)$ the groups of automorphism of L (or equivalently, by [14, Proposition 3.20], the group of self-quasi-isomorphisms of L).

Recall from [14, Definition 8.18] that a (right) homotopy in the category **cdgl** between $f, g\colon L' \to L$ is a morphism

$$\Phi\colon L' \to \wedge(t, dt) \,\widehat{\otimes}\, L = \varprojlim_n (\wedge(t, dt)) \otimes L/F^n)$$

such that $\varepsilon_0 \circ \Phi = f$ and $\varepsilon_1 \circ \Phi = g$, where $\{F^n\}_{n \geq 1}$ is the filtration of L, $\wedge(t, dt)$ is the free algebra generated by an element of degree 0 and its differential and $\varepsilon_i\colon \wedge(t, dt) \,\widehat{\otimes}\, L \to L$ is the cdgl morphism which sends t to i for $i = 0, 1$.

If we denote by $\sim$ the equivalence relation given by homotopic maps, then we can define a new group

$$\mathcal{E}^*(L) = \operatorname{aut}(L)/\sim.$$

Remark 3.6. The notation $\mathcal{E}^*(L)$ instead of $\mathcal{E}(L)$ may appear redundant, since each automorphism sends 0 to 0, so it is automatically 'pointed'. However this notation will reveal to be meaningful, when we compare with the topological case (see §4.1).

Given a subgroup $K \subset \mathcal{E}^*(L)$, we write

$$\operatorname{aut}_K(L) = \{\varphi \in \operatorname{aut}(L) \mid [\varphi] \in K\},$$

which is a subgroup of $\operatorname{aut}(L)$. Analyzing these subgroups will be the goal of the rest of this chapter. The following is the "Eckmann-Hilton" dual of [50, Proposition 6.3 and 6.5]

Theorem 3.7. *Given $L = (\widehat{\mathbb{L}}(V), \partial)$ a free minimal cdgl, we have the following equality of groups*
$$e^{D(\mathrm{Der}_1 L)} = \mathrm{aut}_1(L).$$

Proof. Since L is minimal, $D\eta$ increases the filtration and $D^2\eta = 0$, so $e^{D\eta}$ is a well-defined automorphism of L. Furthermore by (1.2), $e^{D\eta_1} \circ e^{D\eta_2} = e^{D\eta_1 * D\eta_2}$, so the group structure is identical in both sets. We only have to prove that both sets are equal.

Firstly, we prove that
$$e^{D(\mathrm{Der}_1 L)} \subset \mathrm{aut}_1(L).$$

Given $\eta \in \mathrm{Der}_1 L$, define a new derivation $\tilde{\eta}$ in $\wedge(t, dt) \,\widehat{\otimes}\, L$ by the formula
$$\tilde{\eta}(t^n \otimes x) = t^{n+1} \otimes \eta(x), \qquad \tilde{\eta}(t^n dt \otimes x) = -t^{n+1} dt \otimes \eta(x), \qquad \text{for } n \geq 0, x \in L.$$

From this we construct the homotopy
$$\Phi = e^{D\tilde{\eta}} \circ \iota \colon L \to \wedge(t, dt) \,\widehat{\otimes}\, L$$
where $\iota\colon L \to \wedge(t, dt) \,\widehat{\otimes}\, L$ is the inclusion $x \mapsto 1 \otimes x$. This is a cdgl morphism and a straightforward computation shows that
$$\varepsilon_0 \circ \Phi = \mathrm{id}_L, \qquad \varepsilon_1 \circ \Phi = e^{D\eta}.$$

Therefore, $e^{D\eta} \sim \mathrm{id}_L$, and $e^{D(\mathrm{Der}_1 L)} \subset \mathrm{aut}_1(L)$.

We see now that $\mathrm{aut}_1(L) \subset e^{D(\mathrm{Der}_1 L)}$. Consider a homotopy
$$\Phi\colon L \to \wedge(t, dt) \,\widehat{\otimes}\, L$$
between $\mathrm{id}_L = \varepsilon_0 \circ \Phi$ and $\varphi = \varepsilon_1 \circ \Phi$, being φ an arbitrary cdgl automorphism homotopic to the identity.

Recall that a generic element in $\wedge(t, dt) \,\widehat{\otimes}\, L = \varprojlim_n \wedge(t, dt) \otimes L/F^n$ can be written as a formal series
$$\sum_{n\geq 0} t^n \otimes x_n + \sum_{n \geq 0} t^n dt \otimes y_n$$
for some $x_n, y_n \in L$ with the following "convergence criterion": for each $n \geq 1$ only finitely many elements x_i, y_i do not belong to F^n.

Extend the previous homotopy to $\tilde{\Phi}\colon \wedge(t, dt) \,\widehat{\otimes}\, L \to \wedge(t, dt) \,\widehat{\otimes}\, L$ by
$$\tilde{\Phi}(t^n \otimes x) = t^n \Phi(x), \qquad \tilde{\Phi}(t^n dt \otimes x) = t^n dt \Phi(x), \qquad \text{for } n \geq 0, x \in L,$$
which can be checked to be a cdgl morphism. Now construct a new map $\theta\colon \wedge(t, dt) \,\widehat{\otimes}\, L \to \wedge(t, dt) \,\widehat{\otimes}\, L$ using the following formulas:
$$\theta(1 \otimes x) = \log \tilde{\Phi}(1 \otimes x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\tilde{\Phi} - \mathrm{id})^n (1 \otimes x), \qquad \theta(t^i dt^j \otimes x) = t^i dt^j \theta(1 \otimes x),$$
for $x \in L, i \geq 0$ and $j = 0, 1$. Note that, since $\varepsilon_0 \circ \Phi = \mathrm{id}$, the formal series above satisfies the convergence criterion, so it is a well-defined element in $\wedge(t, dt) \,\widehat{\otimes}\, L$. Furthermore,

by the properties of the logarithm, θ is an element in $\operatorname{Der}_0(\wedge(t,dt)\widehat{\otimes} L)$ and, since Φ commutes with the differential, $D\theta = 0$.

Given $x \in L$, the image of $1\otimes x$ by the derivation θ constructed above, will be of the form

$$\theta(1\otimes x) = \sum_{n=0}^{\infty} t^n \otimes \alpha_n(x) + \sum_{n=0}^{\infty} t^n dt \otimes \beta_n(x)$$

for some maps $\alpha_n, \beta_n \colon L \to L$ of degrees 0 and 1 respectively. Each one of these maps is necessarily a derivation since θ is a derivation. Then the map

$$\eta\colon L \to L, \quad \eta(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\beta_n(x),$$

is a derivation at L of degree 1. This specific choice of coefficients in the series above makes $D\eta(x)$ equals to $\alpha(x) = \sum_{n\geq 0}\alpha_n(x)$. Finally, a careful inspection shows that $\alpha\circ\varepsilon_1 = \varepsilon_1\circ\theta\colon \wedge(t,dt)\widehat{\otimes} L \to L$. In particular $\varepsilon_1\circ e^{\theta} = e^{\alpha}\circ\varepsilon_1$. By the way that θ was constructed, $e^{\theta}(1\otimes x) = \tilde{\Phi}(1\otimes x) = \Phi(x)$ for any $x\in L$. Hence, we deduce that

$$\varphi(x) = \varepsilon_1(\Phi(x)) = \varepsilon_1(e^{\theta}(1\otimes x)) = e^{\alpha}(x) = e^{D\eta}(x),$$

for any $x \in L$, which implies $\operatorname{aut}_1(L) \subset e^{D(\operatorname{Der}_1 L)}$.

□

As an immediate consequence we get the following.

Corollary 3.8. *The group* $\operatorname{aut}_1(L)$ *is Malcev* $\mathbb{Q}$*-complete.*

Proof. If M is a cdgl with respect to the filtration $\{F^n\}_{n\geq 1}$, then ∂M_1 is a complete Lie algebra concentrated at degree 0, with the filtration $\{\partial F_1^n\}_{n\geq 1}$. In particular, $D(\operatorname{Der}_1 L)$ is complete since $\operatorname{Der}_{\geq 1} L$ is. Finally, apply the Malcev equivalence (see §3.1) to deduce that $D(\operatorname{Der}_1 L)$ is a Malcev complete group with the BCH product. □

Theorem 3.9. *Given a subgroup* $K \subset \mathcal{E}^*(L)$*, which is nilpotent and 0-local, then* $\operatorname{aut}_K(L)$ *is a Malcev complete group.*

This theorem is obtained by applying the lemma below to the short exact sequence

$$\operatorname{aut}_1(L) \to \operatorname{aut}_K(L) \to K.$$

Lemma 3.10. *Let*

$$H \hookrightarrow G \overset{p}{\twoheadrightarrow} K$$

be a short exact sequence of groups. If H is Malcev complete and K is nilpotent and 0-local, then G is Malcev complete.

Proof. There exists $c \geq 1$ such that $K^c = 1$. Then, $p(G^c) \subset K^c = 1$. This implies that $G^c \subset H$. In particular, for any $n \geq 1$, $H^{nc} \subset G^{nc} \subset H^n$. Then,

$$\bigcap_{n\geq 1} G^n \subset \bigcap_{n\geq 1} H^n = 1$$

since H is pronilpotent. This implies that the map $G \to \varprojlim_n G/G^n$ is injective. To see that it is surjective, we choose a sequence $(g_n)_{n\geq 1}$ with $g_n \in G^n$ and check that the formal product

$$\prod_{n\geq 1} g_n$$

converges in G. Note that for each $n \geq 1$, $g_{nc}g_{nc+1}\dots g_{nc+c-1} \in G^{nc} \subset H^n$, so the product

$$\prod_{n\geq c} g_n = h \in H$$

converges in H, since H is pronilpotent. Then

$$g_1 g_2 \dots g_{c-1} h \in G$$

is an element such that is sent to $(g_n)_{n\geq 1}$ under the map $G \to \varprojlim_n G/G^n$. This proves that G is pronilpotent.

By Theorem 3.5, since K is 0-local and pronilpotent, it is Malcev complete.

Applying the Malcev completion functor to the short exact sequence we obtain a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\mu_H", "\cong"'] & G \arrow[r, "p"] \arrow[d, "\mu_G"] & K \arrow[d, "\mu_K", "\cong"'] \\ \hat{H} \arrow[r] & \hat{G} \arrow[r, "\hat{p}"] & \hat{K} \end{tikzcd}$$

where the upper row is exact. We show that the bottom row is also exact: this would imply that, since μ_H and μ_K are isomorphisms, μ_G is also an isomorphism.

Since the Malcev completion functor is left adjoint to the forgetful functor, as remarked at the beginning of the chapter, it preserves cokernels, so the right side of the bottom sequence is exact. Thus, we only have to prove that $\hat{H} \to \hat{G}$ is an injective morphism.

Consider an element $h \in H \subset G$. Since G is pronilpotent, we can write h as $(\pi_n(h))_{n\geq 1} \in \varprojlim_n G/G^n$ where $\pi_n \colon G \to G/G^n$ is the projection.

Suppose that $\mu_G((\pi_n(h))_{n\geq 1}) = 0$. This implies by [23, Corollary 2.3] that $\pi_n(h)$ is a torsion element in G/G^n. This means that there exists $k_n \geq 1$ such that $h^{k_n} \in G^n$. In particular, $h^{k_{nc}} \in G^{nc} \subset H^n$ for all $n \geq 1$. Since each H^n is 0-local, we deduce that $h \in H^n$ for all $n \geq 1$, so $h = 1$, which concludes the proof. □

We state a final corollary of general nature which allows to identify homotopic automorphisms.

Corollary 3.11. *Let $G \subset \operatorname{aut}(L)$ be a complete subgroup. Then, two automorphisms $f, g \in G$ are homotopic if and only if $\log(f) * (-\log(g)) = D\eta$ for some derivation $\eta \in \operatorname{Der}_1 L$.*

Proof. Firstly note that $\log(f) * (-\log(g))$ is a well-defined element in $\operatorname{Der}_0 L$ since G is complete. Then we have that $f \sim g$ if and only if $f \circ g^{-1} \sim \mathrm{id}_L$. By Theorem 3.7 this is equivalent to the existence of $\eta \in D(\mathrm{Der}_1 L)$ such that $e^{D\eta} = f \circ g^{-1}$ which amounts to say that

$$\log\Big(f \circ g^{-1}\Big) = \log f * (-\log(g)) = D\eta.$$

$\square$

CHAPTER 4

Homotopy classes of maps and actions of the fundamental group

Given two pointed simplicial sets X and Y, consider the sets

$$[\![X, Y]\!], \text{ and } [\![X, Y]\!]^*$$

of free and pointed homotopy classes of maps from X to Y. Our first goal in this chapter is to model these sets in **cdgl**. Afterwards, we model the action of the fundamental group of an space on itself and the holonomy action of a fibration sequence.

4.1 Free and pointed homotopy classes of maps

Let X be a connected simplicial set whose basepoint is the 0-simplex $b \in X_0$ and let L' be a connected cdgl. In particular $\mathrm{MC}(L') = \{0\}$ and $\langle L' \rangle$ is connected.

Proposition 4.1. *Denote by* (b) *the Lie ideal generated by* $b \in X_0$. *Then there are bijections of sets:*

$$[\![X, \langle L' \rangle]\!] \cong [\![\mathfrak{L}_X, L']\!], \qquad [\![X, \langle L' \rangle]\!]^* \cong [\![\mathfrak{L}_X/(b), L']\!].$$

Proof. The first identity comes from the fact that adjoint functors $\mathfrak{L}$ and $\langle - \rangle$ constitute a Quillen pair.

Recall from [14, §7.1] that the bijection

$$\mathrm{Hom}_{\mathbf{sset}}(X, \langle L' \rangle) \cong \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_X, L')$$

associates to a simplicial map $f\colon X \to \langle L' \rangle$, the cdgl morphism $\varphi_f\colon \mathfrak{L}_X \to L'$ defined in the following way: given $\sigma \in X_q$ a non-degenerate simplex, it corresponds to a generator of degree $q-1$ in $\mathfrak{L}_X$. Then $\varphi_f(\sigma) = f(\sigma)(a_{0,\dots q})$ where $a_{0,\dots q} \in s^{-1}\Delta^q$ is the top generator. In particular, since L is connected, $\varphi_f(b) = 0$ so there is an induced map

$$\bar{\varphi}_f \colon \mathfrak{L}_X/(b) \to L'.$$

We check that two homotopic maps $f \sim g \colon X \to \langle L' \rangle$ give rise to two homotopic maps $\varphi_f \sim \varphi_g \colon \mathfrak{L}_X \to L'$. Let $h \colon X \times \triangle^1 \to \langle L' \rangle$ be the homotopy between f and g, then

$$h\,|_{X\times(0)} = f, \quad h\,|_{X\times(1)} = g$$

where (0) and (1) are the subsimplicial sets of $\triangle^1$ generated by the two 0-simplices. Equivalently, by the exponential law (1.3), we can think of h as a simplicial map

$$h \colon X \to \operatorname{map}(\triangle^1, \langle L' \rangle)$$

where $\operatorname{map}(-,-)$ is the mapping space defined in §1.4.1. By [14, Theorem 12.18], since $\wedge(t,dt)$ is a cdga model of $\triangle^1$,

$$\langle \wedge(t,dt) \mathbin{\widehat{\otimes}} L' \rangle \simeq \operatorname{map}(\triangle^1, \langle L' \rangle).$$

Therefore h can be identified with an element in

$$\operatorname{Hom}_{\mathbf{sset}}(X, \langle \wedge(t,dt) \mathbin{\widehat{\otimes}} L' \rangle)$$

which, by the adjunction of the model and realization functors, is in bijective correspondence with $\operatorname{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_X, \wedge(t,dt) \mathbin{\widehat{\otimes}} L')$. Through this bijection, the simplicial map h is sent to a cdgl homotopy

$$\Phi \colon \mathfrak{L}_X \to \wedge(t,dt) \mathbin{\widehat{\otimes}} L'.$$

By the naturality of the adjunctions we get that $\varepsilon_0 \circ \Phi = \varphi_f$ and $\varepsilon_1 \circ \Phi = \varphi_g$, so we deduce that $\varphi_f \sim \varphi_g$. A reverse argument using the same bijections also shows that $\varphi_f \sim \varphi_g$ implies that $f \sim g$, so we conclude that

$$[\![X, \langle L' \rangle]\!] \cong [\![\mathfrak{L}_X, L']\!].$$

In the pointed case, consider a pointed homotopy h between the pointed maps $f, g \colon X \to \langle L' \rangle$. Then $h \colon X \times \triangle^1 \to \langle L' \rangle$ is constant to the only 0-simplex along the subsimplicial set $(b) \times \triangle^1$. As above, identify h with a cdgl homotopy Φ. By the naturality of the bijections, $\Phi(b) = 0 \in \wedge(t,dt) \mathbin{\widehat{\otimes}} L'$. Thus, it induces a homotopy $\bar{\Phi} \colon \mathfrak{L}_X/(b) \to L' \mathbin{\widehat{\otimes}} \wedge(t,dt)$ between $\bar{\varphi}_f$ and $\bar{\varphi}_g$. Reversing the argument we conclude that

$$[\![X, \langle L' \rangle]\!]^* \cong [\![\mathfrak{L}_X/(b), L']\!].$$

$\square$

Corollary 4.2. *Let L' be a connected cdgl and let X be a connected simplicial set whose minimal Lie model is L. Then*

$$[\![X, \langle L' \rangle]\!]^* \cong [\![L, L']\!].$$

In particular if L' is a Lie model of a simplicial set Y of finite type, then

$$[\![X, \mathbb{Q}_\infty Y]\!]^* \cong [\![L, L']\!].$$

Proof. By [14, Proposition 8.7] the following composition of quasi-isomorphisms is a weak equivalence of cdgl's

$$\mathfrak{L}_X^b \xrightarrow{\simeq} (\mathfrak{L}_X, \partial_b) \xrightarrow{\simeq} \mathfrak{L}_X/(b).$$

Therefore

$$[\![L, L']\!] \cong [\![\mathfrak{L}_X/(b), L']\!] \cong [\![X, \langle L' \rangle]\!]^*.$$

If L' is a Lie model of a finite type simplicial set Y, then $\langle L' \rangle \simeq \mathbb{Q}_\infty Y$ (see §1.8). □

Remark 4.3. Suppose that both X and Y are nilpotent simplicial sets of finite type, then $\langle L \rangle \simeq \mathbb{Q}_\infty X \simeq X_\mathbb{Q}$ and analogously for Y. The bijection above becomes along with the universal property of the rational spaces (see §1.8) gives bijections

$$[\![L, L']\!] \cong [\![X, Y_\mathbb{Q}]\!]^* \cong [\![X_\mathbb{Q}, Y_\mathbb{Q}]\!]^*$$

Using that the unit of the adjunction between $\mathfrak{L}$ and $\langle - \rangle$ is homotopy equivalent to the rationalization map $X \to X_\mathbb{Q}$ (up to an extra point, see [14, Corollary 11.18]) we conclude that the bijection $[\![L, L']\!] \cong [\![X_\mathbb{Q}, Y_\mathbb{Q}]\!]^*$ is given by the map $\varphi \mapsto \langle \varphi \rangle$

Remark 4.4. In the previous remark, if we take $X = Y$ and $L = L'$, we have not only a bijection but an isomorphism of monoids $[\![L, L]\!] \cong [\![X_\mathbb{Q}, X_\mathbb{Q}]\!]^*$. Thus, their respective groups of invertible elements are also isomorphic. We denote the homotopy classes of automorphisms $\operatorname{aut}(L)/\sim$ as $\mathcal{E}^*(L)$, so this isomorphism becomes

$$\mathcal{E}^*(L) \cong \mathcal{E}^*(X_\mathbb{Q}). \tag{4.1}$$

Of course, any cdgl morphism is pointed (since it sends 0 to 0), but this notation makes the comparison between cdgl's and simplicial sets simpler.

4.2 Fundamental action at the cdgl level

Let L' be a connected cdgl. For any other cdgl L, the group $(L'_0, *)$ acts on $\operatorname{Hom}_{\mathbf{cdgl}}(L, L')$ by

$$x \bullet \varphi = e^{\operatorname{ad}_x} \circ \varphi.$$

This is a well defined action as $e^{\operatorname{ad}_0} = \operatorname{id}'_L$ and taking into account (1.2),

$$x \bullet (x' \bullet \varphi) = e^{\operatorname{ad}_x} \circ e^{\operatorname{ad}_{x'}} \circ \varphi = e^{\operatorname{ad}_{x * x'}} \circ \varphi = (x * x') \bullet \varphi.$$

Proposition 4.5. *This action induces an action of $(H_0(L'), *)$ on $[\![L, L']\!]$ in the obvious way*

$$[x] \bullet [\varphi] = [x \bullet \varphi] = [e^{\operatorname{ad}_x} \circ \varphi]$$

Proof. First, notice that the BCH product $*$ on L_0 induces also a group structure on $H_0(L')$. Now, if $x, y \in L'_0$ with $[x] = [y]$ and $\varphi, \psi \in \operatorname{Hom}_{\mathbf{cdgl}}(L, L')$ with $[\varphi] = [\psi]$, write $x * (-y) = \partial z$ so that

$$e^{D \operatorname{ad}_z} = e^{\operatorname{ad}_{\partial z}} = e^{\operatorname{ad}_{x*(-y)}} = e^{\operatorname{ad}_x} \circ e^{\operatorname{ad}_{-y}}$$

By Theorem 3.7 we deduce that $e^{\mathrm{ad}_x} \circ e^{\mathrm{ad}_{-y}} \sim \mathrm{id}_L$ and therefore, $e^{\mathrm{ad}_x} \sim e^{\mathrm{ad}_y}$ by composing with e^{ad_y} on both sides. Hence

$$e^{\mathrm{ad}_x} \circ \varphi \sim e^{\mathrm{ad}_y} \circ \psi$$

which translates to

$$[x] \bullet [\varphi] = [y] \bullet [\psi]$$

and the proposition follows.

□

Definition 4.6. We call this action of $(H_0(L'), *)$ on $[\![L, L']\!]$ the fundamental action and denote by $[\![L, L']\!]/H_0(L')$ the orbit set.

Remark 4.7. Observe that, choosing $L' = L$, $[\varphi] \in \mathcal{E}^*(L)$ and $[x] \in H_0(L)$, $[x] \bullet [\varphi] \in \mathcal{E}^*(L)$, so the fundamental action restricts to an action of $(H_0(L), *)$ on the group $\mathcal{E}^*(L)$. We denote by $\mathcal{E}(L) = \mathcal{E}^*(L)/H_0(L)$ the orbit group which is a quotient group, as $H_0(L)$ maps into $\mathcal{E}^*(L)$ via

$$\begin{array}{ccc} H_0(L) & \to & \mathcal{E}^*(L) \\ [x] & \mapsto & [x] \bullet \mathrm{id}_L = [e^{\mathrm{ad}_x}]. \end{array}$$

Again, this notation for the groups

$$\mathcal{E}^*(L) = \mathrm{aut}(L)/\sim, \quad \text{and }, \quad \mathcal{E}(L) = \mathcal{E}^*(L)/H_0(L)$$

will become clear when we compare with their analogous version for simplicial sets (see remark 4.11).

Let's see the topological counterpart of this action. Let (X, x_0) and (Y, y_0) be two pointed CW-complexes, and $\beta \in \pi_1(Y, y_0)$ a loop. Since the inclusion of the basepoint is a cofibration, we have a lift in the following diagram

$$\begin{tikzcd} \{x_0\} \arrow[r, "\beta"] \arrow[d] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "h"] & Y \end{tikzcd} \quad . \tag{4.2}$$

Thus $h(-,1)\colon X \to Y$ is a map sending x_0 to $\beta(1) = y_0$ and which is (free) homotopic to f. However, they are not necessarily pointed homotopic. Using similar arguments to those of §2.1.1 we can check that the pointed homotopic class of $h(-,1)$ depends only on the pointed homotopy class of f and β. Therefore, we have a map

$$\pi_1(Y, y_0) \times [\![X, Y]\!]^* \to [\![X, Y]\!]^*, \quad ([\beta], [f]^*) \mapsto [\beta] \bullet [f]^* = [h(-,1)]^*$$

Furthermore, the composition of paths $\beta\gamma$ can be checked to act on $[f]$ as $[\beta] \bullet ([\gamma] \bullet [f]^*)$. Therefore, we have a group action of the fundamental group.

Definition 4.8. We call the action of the group $\pi_1(Y, y_0)$ on $[\![X, Y]\!]^*$ the (topological) fundamental action.

Using the Quillen equivalence between the categories **top** and **sset** we can extend this action to simplicial sets. Let X and Y be pointed simplicial sets. Using the adjunction, we have the bijection

$$[\![X,Y]\!] \cong [\![X, \operatorname{Sing}|Y|]\!] \cong [\![|X|,|Y|]\!], \quad f \mapsto |f|$$

which, using the canonical basepoints (see §1.5.2), induces a bijection

$$[\![X,Y]\!]^* \cong [\![|X|,|Y|]\!]^*$$

of the pointed homotopy classes of maps.

Then the group $\pi_1(Y) \cong \pi_1(|Y|)$ acts on $[\![X,Y]\!]^*$ as

$$\pi_1(Y) \times [\![X,Y]\!]^* \cong \pi_1(|Y|) \times [\![|X|,|Y|]\!]^* \xrightarrow{\bullet} [\![|X|,|Y|]\!]^* \cong [\![X,Y]\!]^*.$$

The theorem below proves that both fundamental actions are compatible via the bijection of Corollary 4.2.

Theorem 4.9. *Let L' be a connected cdgl and let X be a connected simplicial set whose minimal Lie model is L, which is free. Then, the following diagram commutes*

$$\begin{tikzcd} H_0(L') \times [\![L,L']\!] \arrow[r] \arrow[d, "\cong"] & [\![L,L']\!] \arrow[d, "\cong"] \\ \pi_1(\langle L' \rangle) \times [\![X, \langle L' \rangle]\!]^* \arrow[r] & [\![X, \langle L' \rangle]\!]^* \end{tikzcd}$$

where the horizontal arrows are the fundamental actions and the vertical arrows are induced by the bijection $[\![L,L']\!] \cong [\![X,\langle L'\rangle]\!]^*$ of Corollary 4.2 and the group isomorphism $\rho_1 \colon \pi_1(\langle L' \rangle) \to H_0(L')$ (see §1.8).

Proof. Let $\varphi\colon L \to L'$ be a cdgl morphism and let x be an element in L'_0. Recall that $\rho_1\colon \pi_1(\langle L'\rangle) \to H_0(L')$ sends a cdgl morphism $\beta\colon \mathfrak{L}_1 \to L'$ to $\beta(a_{01})$. Consider the 1-simplex $\beta \in \langle L' \rangle_1$ defined as

$$\beta\colon \mathfrak{L}_1 \to L', \quad a_{01} \mapsto x, \quad a_0 \mapsto 0, \quad a_1 \mapsto 0$$

which can be identified with a loop $\beta\colon \triangle^1 \to \langle L'\rangle$ of $\pi_1(\langle L' \rangle)$. Then, clearly $\rho_1(\beta) = [x]$. Furthermore $|\beta|\colon |\triangle^1| = [0,1] \to |\langle L'\rangle|$ can be thought as a loop with $|\beta|(0) = |\beta|(1) = y_0$, where y_0 is the canonical basepoint of $|\langle L' \rangle|$.

On the other hand, the cdgl morphism φ corresponds to a pointed map $f\colon X \to \langle L' \rangle$. We need to construct a lift in the following diagram

$$\begin{tikzcd} \{x_0\} \arrow[r, "|\beta|"] \arrow[d] & |\langle L' \rangle|^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ |X| \arrow[r, "|f|"] & |\langle L' \rangle| \end{tikzcd} .$$

Our goal is to construct a cdgl diagram such that, when realized, we obtain the diagram above.

Recall (see §1.5.3) that a path object for L' is given by

$$\wedge(t, dt) \mathbin{\widehat{\otimes}} L'.$$

By [14, Theorem 12.18], there is a homotopy equivalence

$$\langle \wedge(t, dt) \mathbin{\widehat{\otimes}} L' \rangle \simeq \operatorname{map}(\triangle^1, \langle L' \rangle).$$

Furthermore, checking carefully this homotopy equivalence (see [14, Proposition 11.1]), the path $\beta\colon \triangle^1 \to \langle L' \rangle$, which is a 0-simplex in $\operatorname{map}(\triangle^1, \langle L' \rangle)$, is sent to the realization of the Maurer-Cartan element $-dt \otimes x \in \wedge(t, dt) \mathbin{\widehat{\otimes}} L'$.

We need to find a cdgl morphism, such that, when realized, we obtain $\beta : * \to \operatorname{map}(\triangle^1, \langle L' \rangle)$. The problem is that any cdgl sends 0 to 0, so we cannot expect to send the basepoint to a non-trivial point, as it happens with $\beta : * \to \operatorname{map}(\triangle^1, \langle L' \rangle)$. We do the following technical modification: we are going to add a basepoint to $*$ and $\langle L \rangle$. This is analogous to add a Maurer-Cartan element (see [14, Corollary 7.17]); therefore, for $L = (\widehat{\mathbb{L}}(V), \partial)$, consider the new cdgl

$$\tilde{L} = (\widehat{\mathbb{L}}(V \oplus \langle a \rangle), \tilde{\partial} = \partial_a)$$

where $\partial a = -\frac{1}{2}[a, a]$. In this cdgl there are two MC elements 0 and $-a$ (note that the differential has been perturbed). Then we have that $\tilde{L}^0$ is homologically trivial and $\tilde{L}^{-a} \cong L$. Similarly, we consider the cdgl $(\mathbb{L}(a), \partial_a)$ whose MC elements are 0 and $-a$. Then, the cdgl morphism

$$\psi\colon (\mathbb{L}(a), \partial_a) \mapsto \wedge(t, dt) \mathbin{\widehat{\otimes}} L', \quad -a \mapsto -dt \otimes x$$

has as realization the simplicial map

$$\langle \psi \rangle = c \sqcup \beta\colon\ * \sqcup * \to \operatorname{map}(\triangle^1, \langle L' \rangle)$$

sending the first point to the constant map $c\colon \triangle^1 \to \langle L' \rangle$ and the second point to β. Similarly, if we realize it as a topological map we get the disjoint union of the constant map and the loop $|\beta|$.

Consider the diagram

$$\begin{tikzcd}
(\mathbb{L}(a), \partial_a) \arrow[r, "\psi"] \arrow[d] & \wedge(t, dt) \mathbin{\widehat{\otimes}} L' \arrow[d, "\varepsilon_0"] \\
(\mathbb{L}(V \oplus \langle a \rangle), \partial_a) \arrow[r, "\varphi \circ \mathrm{proj}"] & L'
\end{tikzcd}$$

where $\mathrm{proj}\colon \mathbb{L}(V \oplus \langle a \rangle) \to \mathbb{L}(V) = L$ is the projection. This diagram is clearly commutative.

We construct a lift as

$$h\colon (\widehat{\mathbb{L}}(V \oplus \langle a \rangle), \partial_a) \to \wedge(t, dt) \mathbin{\widehat{\otimes}} L', \quad a \mapsto dt \otimes x, \quad y \mapsto e^{\operatorname{ad}_{t \otimes x}}(\varphi(y)) = \sum_{n \geq 0} \frac{t^n}{n!} \otimes \operatorname{ad}_x^n(\varphi(y))$$

for $y \in \widehat{\mathbb{L}}(V) = L$.

By the properties of the exponential of derivations, it is well-defined and it is compatible with the Lie brackets. Fix $y \in \widehat{\mathbb{L}}(V)$, then

$$h(\partial_a y) = \sum_{n\geq 0} \frac{t^n}{n!} \otimes \mathrm{ad}_x^n(\partial\varphi(y)) + [dt \otimes x, h(y)]$$

and

$$\partial h(y) = \sum_{n\geq 1} \frac{t^{n-1}dt}{(n-1)!} \otimes \mathrm{ad}_x^n(\varphi(y)) + \sum_{n\geq 0} \frac{t^n}{n!} \otimes \mathrm{ad}_x^n(\partial\varphi(y)).$$

Both expressions coincide since

$$[dt\otimes x, h(y)] = [dt\otimes x, \sum_{n\geq 0} \frac{t^n}{n!}\otimes \mathrm{ad}_x^n(\varphi(y))] = \sum_{n\geq 0} \frac{t^n dt}{n!}\otimes[x, \mathrm{ad}_x^n(\varphi(y))] = \sum_{n\geq 1} \frac{t^{n-1}dt}{(n-1)!}\otimes \mathrm{ad}_x^n(\varphi(y)).$$

Therefore h is a cdgl morphism and it is clear that it fits in the diagram above. Now, apply the adjunction between $\mathfrak{L}_\bullet$ and $\langle - \rangle$ and realize the diagram in **top** to obtain the commutative diagram

$$\begin{tikzcd} * \sqcup * \arrow[r, "c\sqcup\beta"] \arrow[d] & |\langle L'\rangle|^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ * \sqcup |X| \arrow[r, "|f|"] \arrow[ur, dashed, "\tilde{h}"] & |\langle L'\rangle| \end{tikzcd}$$

where $\tilde{h} = |\langle h \rangle|$. Ignoring the extra points we added, we get that $|\beta| \bullet |f| = \tilde{h}(-,1)$. In particular, by the lower path of the diagram of the statement, (x, φ) is sent to

$$\langle h(-,1)\rangle = \langle \varepsilon_1 \circ h\rangle.$$

By the formula of h, $\varepsilon_1 \circ h = e^{\mathrm{ad}_x} \circ \varphi$.

On the other hand, the upper path of the diagram, sends (x, φ) to the realization of the fundamental action. This is $\langle e^{\mathrm{ad}_x} \circ \varphi\rangle$. We conclude that the diagram is commutative. □

Corollary 4.10. *In the same situation than in the theorem, we have a bijection*

$$[\![X, \langle L'\rangle]\!] \cong [\![L, L']\!]/H_0(L')$$

Proof. In the topological setting it is easy to check that two pointed homotopy classes $[f]^*$ and $[g]^*$ in $[\![X,Y]\!]^*$ are freely homotopic if and only if there is a loop $\beta \in \pi_1(Y)$ such that $[\beta] \bullet [f]^* = [g]^*$. Thus we conclude that the orbit space $[\![X,Y]\!]^*/\pi_1(Y)$ is bijective with $[\![X,Y]\!]$.

By the compatibility of the fundamental actions we have the bijections of sets

$$[\![X, \langle L'\rangle]\!] \cong [\![X, \langle L'\rangle]\!]^*/\pi_1(\langle L'\rangle) \cong [\![L, L']\!]/H_0(L').$$

□

Remark 4.11. Let L be the minimal Lie model of a nilpotent simplicial set of finite type X and take $L = L'$. Then the fundamental action restricts to $\mathcal{E}^*(L)$ and to $\mathcal{E}^*(X_{\mathbb{Q}})$ respectively. Theorem 4.9 then gives a commutative diagram of the form

$$\begin{tikzcd} H_0(L) \arrow[r] \arrow[d, "\cong"] & \mathcal{E}^*(L) \arrow[d, "\cong"] \\ \pi_1(X_{\mathbb{Q}}) \arrow[r] & \mathcal{E}^*(X_{\mathbb{Q}}) \end{tikzcd}$$

where the horizontal arrows are the group action on the identity and the vertical right arrow is given by Remark (4.1). Furthermore, Corollary 4.10 becomes

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}^*(X_{\mathbb{Q}})/\pi_1(X_{\mathbb{Q}}) \cong \mathcal{E}^*(L)/H_0(L) \cong \mathcal{E}(L).$$

4.3 Fibration sequences in sset and cdgl

In Chapter 2, we describe the (topological) fibration sequences as Hurewicz fibrations and a weak equivalence to the fiber of the fibration. We describe now analogous versions of this construction in the model categories **sset** and **cdgl**. Furthermore, we study the behavior of these fibration sequences under the Quillen pair given in (1.5).

Definition 4.12. A *Kan fibration sequence* is a sequence in **sset**

$$F \xrightarrow{\omega} E \xrightarrow{\pi} B$$

where π is a Kan fibration and $\omega\colon F \to \pi^{-1}(b_0)$ is a weak equivalence, with b_0 a 0-simplex in B.

Proposition 4.13. *The Quillen equivalence*

$$\mathbf{sset} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathbf{top}$$

sends Kan fibration sequences to (topological) fibration sequences and vice versa.

Proof. Consider a fibration sequence in **top**

$$F \xrightarrow{\omega} E \xrightarrow{\pi} B.$$

Then, since $\mathrm{Sing}(-)$ is a right adjoint functor in a Quillen pair and π is a Hurewicz fibration (in particular, a Serre fibration), $\mathrm{Sing}(\pi)\colon \mathrm{Sing}(E) \to \mathrm{Sing}(B)$ is a Kan fibration. Furthermore, the fiber $\pi^{-1}(b_0)$ can be considered as the equalizer $\mathrm{Eq}(\pi, c_{b_0})$ of the maps π and $c_{b_0}\colon E \to B$. As the right adjoint functor $\mathrm{Sing}(-)$ preserves limits, we have

$$\mathrm{Sing}(\pi^{-1}(b_0)) \cong \mathrm{Eq}(\mathrm{Sing}(\pi), \mathrm{Sing}(c_{b_0})).$$

Noting that c_{b_0} factors through the terminal topological space $* = \boldsymbol{\Delta}^0$, we deduce that $\mathrm{Sing}(c_{b_0}) = c_{\mathrm{Sing}(b_0)}$ where we are identifying the basepoints of B and $\mathrm{Sing}(B)$ in the way described in §1.5.1. Therefore,

$$\mathrm{Sing}(\pi^{-1}(b_0)) \cong (\mathrm{Sing}(\pi))^{-1}(\mathrm{Sing}(b_0)).$$

Finally, since the functor $\mathrm{Sing}(-)$ preserves weak equivalences, we conclude that

$$\mathrm{Sing}(F) \xrightarrow{\mathrm{Sing}(\omega)} \mathrm{Sing}(E) \xrightarrow{\mathrm{Sing}(\pi)} \mathrm{Sing}(B)$$

is a Kan fibration sequence.

The converse is also true but can not be deduced from the general properties of a Quillen equivalence. However, this particular Quillen equivalence has some very good properties that we will exploit.

Consider a Kan fibration sequence in **sset**

$$F \xrightarrow{\omega} E \xrightarrow{\pi} B.$$

By a result of Quillen [41] the geometric realization of a Kan fibration is a Serre fibration. Furthermore, in the category **top** of weak Hausdorff compactly generated spaces we also have that the geometric realization of a Kan fibration is a Hurewicz fibration [17, Theorem 4.5.25].

Therefore $|\pi|\colon |E| \to |B|$ is a Hurewicz fibration. As in the previous case, we can compute $\pi^{-1}(b_0)$ as the equalizer of π and c_{b_0}. By [17, Proposition 4.3.13] the geometric realization preserves equalizers and the constant map is sent by $|-|$ to a constant map. Thus, we deduce that

$$|\pi|^{-1}(|b_0|) \cong \left|\pi^{-1}(b_0)\right|.$$

Finally, since $|\omega|\colon |F| \to \left|\pi^{-1}(b_0)\right|$ is a weak equivalence, we conclude that

$$|F| \xrightarrow{|\omega|} |E| \xrightarrow{|\pi|} |B|$$

is a fibration sequence.

□

Finally, we introduce fibration sequences in **cdgl**.

Definition 4.14. A *cdgl fibration sequence* is a short exact sequence in **cdgl**

$$0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0.$$

When L, M and N are connected we say that this is a connected cdgl sequence.

In particular, the twisted products defined at §1.3 are examples of cdgl fibration sequences.

Since the realization functor $\langle - \rangle\colon$ **cdgl** $\to$ **sset** is right adjoint to the model functor and $\langle p\rangle^{-1}(0) = \langle i\rangle\langle L\rangle$, we conclude that the realization of a cdgl fibration sequence is a Kan fibration sequence.

4.4 Holonomy action of cdgl fibration sequences

Consider a connected cdgl sequence

$$L \to M \xrightarrow{p} N$$

in which L is free.

Given $x \in N_0$ choose $y \in p^{-1}(x)$ and observe that the automorphism e^{ad_y} of M restricts to an automorphism of L which we denote in the same way

$$e^{\mathrm{ad}_y} : L \xrightarrow{\cong} L.$$

Take its homotopy class $[e^{\mathrm{ad}_y}] \in \mathcal{E}^*(L)$ and its class $\overline{[e^{\mathrm{ad}_y}]}$ in the quotient $\mathcal{E}(L) = \mathcal{E}^*(L)/H_0(L)$.

Definition 4.15. The cdgl holonomy action of the given cdgl fibration sequence is the map

$$\begin{array}{ccc} H_0(N) & \to & \mathcal{E}(L) \\ [x] & \mapsto & \overline{[e^{\mathrm{ad}_y}]} \end{array}$$

Proposition 4.16. *This is a well defined morphism*

Proof. To avoid excessive notation, we identify L with $\ker(p)$.

Let $x, x' \in N_0$ be two elements with $[x] = [x']$ and choose $y \in p^{-1}(x)$ and $y' \in p^{-1}(x')$. As $[x] = [x']$ write $x * (-x') = \partial b$ and choose $a \in p^{-1}(b)$. Then

$$p(\partial a) = x * (-x') = p(y * (-y')).$$

Hence $\partial a * (y * (-y')) = \partial a * y' * (-y) \in L_0$. We then have the following identities, in which we use Theorem 3.7:

$$[e^{\mathrm{ad}_{y'}}][e^{\mathrm{ad}_y}]^{-1} = [e^{\mathrm{ad}_{y'}} \circ e^{\mathrm{ad}_{-y}}] = [\mathrm{id}_L \circ e^{\mathrm{ad}_{y'}} \circ e^{\mathrm{ad}_{-y}}] = [e^{D\,\mathrm{ad}_a} \circ e^{\mathrm{ad}_{y'}} \circ e^{\mathrm{ad}_{-y}}] = [e^{\mathrm{ad}_{\partial a * y' * (-y)}}]$$

which lives in the image of $H_0(L)$ inside $\mathcal{E}^*(L)$.

In other words

$$\overline{[e^{\mathrm{ad}_y}]} = \overline{[e^{\mathrm{ad}_{y'}}]}.$$

This shows that the holonomy action does not depend on any possible choices and, thus, it is a well defined map. □

We next see that Definition 4.15 provides the cdgl analogue of the topological holonomy action.

Theorem 4.17. *Let $L \to M \xrightarrow{p} N$ be a connected cdgl fibration sequence with L free. Apply $|\langle - \rangle|$ to obtain the topological fibration sequence (see §4.3)*

$$F = |\langle L \rangle| \xrightarrow{\omega} E = |\langle M \rangle| \xrightarrow{\pi = |\langle p \rangle|} B = |\langle N \rangle|$$

Then the following diagram commutes

$$\begin{tikzcd} H_0(N) \arrow[r] \arrow[d, "\cong"] & \mathcal{E}(L) \arrow[d, "|\langle - \rangle|"] \\ \pi_1(B) \arrow[r] & \mathcal{E}(F) \end{tikzcd}$$

where the horizontal arrows are the holonomy actions and the left vertical arrows are the canonical isomorphisms of groups.

Proof. By the results of §4.3 the sequence

$$F = |\langle L\rangle| \xrightarrow{\omega} E = |\langle M\rangle| \xrightarrow{\pi=|\langle p\rangle|} B = |\langle N\rangle|$$

is a topological fibration sequence, so its holonomy action is well-defined. Without loss of generality we can assume that F is strictly the fiber and ω is the inclusion.

On the other hand, given $\varphi \in \mathcal{E}(L)$, $|\langle\varphi\rangle|$ is an automorphism of F. By Theorem 4.9, the image of the fundamental action is sent to a map (freely) homotopic to the identity, so the map $|\langle - \rangle|\colon \mathcal{E}(L)/H_0(L) \to \mathcal{E}(F)$ is well-defined.

Fix $\beta \in \pi_1(\langle N\rangle)$, which is associated to an element $x \in N_0$, and take $y \in M_0$ such that $p(y) = x$.

We proceed as in the proof of Theorem 4.9: in $L = (\widehat{\mathbb{L}}(V), \partial)$, we add extra points (adding the a MC element and perturbing the differential), in order to get flexibility in the choice of where the basepoint is sent, and we construct the following commutative diagram in **cdgl**:

$$\begin{tikzcd} (\mathbb{L}(a), \partial_a) \arrow[r, "\varphi"] \arrow[d] & \wedge(t,dt)\,\widehat{\otimes}\, M \arrow[d, "\mathrm{id}\,\widehat{\otimes}\, p"] \\ (\widehat{\mathbb{L}}(V \oplus \langle a\rangle), \partial_a) \arrow[r, "\psi"] \arrow[ur, dashed, "h"] & \wedge(t,dt)\,\widehat{\otimes}\, N \end{tikzcd}$$

where $\varphi(a) = dt \otimes y$, $h(a) = dt \otimes y$, $h(z) = e^{\mathrm{ad}_{t\otimes y}}(z)$ for $z \in \widehat{\mathbb{L}}(V)$ and $\psi(a) = dt \otimes x$, $\psi(z) = 0$ for $z \in \widehat{\mathbb{L}}(V)$. This diagram is clearly commutative, and checking that h is a cdgl morphism is analogous to the proof of Theorem 4.9.

Realizing the diagram above (and removing the extra points) we get:

$$\begin{tikzcd} * \arrow[r, "\beta'"] \arrow[d] & E^{[0,1]} \arrow[d, "\pi\circ -"] \\ F \arrow[r, "\beta"] \arrow[ur, "\tilde{h}"] & B^{[0,1]} \end{tikzcd}$$

where the map $\beta\colon F \to B^{[0,1]}$ sends any point to the path β, the map $\beta'\colon [0,1] \to E$ is the path associated to $y \in M_0$ and it is such that $\pi \circ \beta' = \beta$ and $\tilde{h} = |\langle h\rangle|$. Furthermore, by the definition of the cdgl morphism h, we have that $\varepsilon_0 \circ h\colon L \to M$ is the inclusion, so $\tilde{h}(-,1)\colon F \to E$ is the inclusion of the fiber. Then the following diagram commutes:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ F \times [0,1] \arrow[r, "\beta"] \arrow[ur, "\tilde{h}"] & B \end{tikzcd}$$

By definition, $\tilde{h}(-,1) \in \mathcal{E}(F)$ is the holonomy action of the topological fibration sequence, which concludes the proof.

$\square$

CHAPTER 5

Evaluation fibration sequences

In this chapter we study the evaluation fibration associated to the evaluation of a mapping space at some point, and model it in **cdgl**.

We two connected simplicial sets X and Y. Recall from §1.4.1 the properties of the simplicial mapping space $\operatorname{map}(X,Y)$. Without loss of generality we can assume that X and Y are Kan complexes taking their fibrant replacement if necessary. Then $\operatorname{map}(X,Y)$ is a Kan complex by [31, Theorem 6.9]. Note that $\pi_0(\operatorname{map}(X,Y)) = [\![X,Y]\!]$.

By [31, Theorem 7.13] if x_0 is a vertex of X, then

$$\operatorname{ev}_{x_0}\colon \operatorname{map}(X,Y) \to \operatorname{map}(*,Y) \cong Y$$

is a Kan fibration whose fiber is $\operatorname{map}^*(X,Y)$. Then, the *evaluation fibration sequence* is the Kan fibration sequence

$$\operatorname{map}^*(X,Y) \to \operatorname{map}(X,Y) \xrightarrow{\operatorname{ev}_{x_0}} Y \tag{5.1}$$

and the goal of this chapter is to model it.

Denote by $\zeta\colon \operatorname{map}^*(X,Y) \to \operatorname{map}(X,Y)$ the inclusion of the fiber. In particular if $f\colon X \to Y$ is A based simplicial map, then $\zeta(f)$ is the simplicial map f forgetting about the basepoints. Remark that

$$\pi_0(\zeta)\colon [\![X,Y]\!]^* \twoheadrightarrow [\![X,Y]\!]$$

is the natural map which assigns to each pointed homotopy class of a pointed map its free homotopy class.

If we have a subset $S \subset [\![X,Y]\!]$ we write

$$\operatorname{map}_S(X,Y) = \bigsqcup_{[f]\in S} \operatorname{map}_f(X,Y)$$

where $\mathrm{map}_f(X,Y)$ is the connected component of $\mathrm{map}(X,Y)$ containing f. Analogously if $S \subset [\![X,Y]\!]^*$, then we write

$$\mathrm{map}_S(X,Y) = \bigsqcup_{[f]\in S} \mathrm{map}^*_f(X,Y).$$

In particular, if $f\colon X \to Y$ is a pointed map, then the fiber of

$$\mathrm{ev}_{x_0}\colon\ \mathrm{map}_f(X,Y) \to Y$$

is the non-connected simplicial set

$$\mathrm{map}^*_{\zeta^{-1}(f)}(X,Y) = \bigsqcup_{\substack{[g]^*\in[\![X,Y]\!]^*\\ g \text{ freely homotopic to } f}} \mathrm{map}^*_g(X,Y) = \bigsqcup_{[g]^*\in\pi_0(\zeta)^{-1}[f]} \mathrm{map}^*_g(X,Y).$$

As recall in section 4.1 note that the number of components equals the cardinal of $\pi_1(Y)\bullet[f]^*$.

5.1 Modeling the fibration sequence

Before proceeding with the main subject of this section we need some preliminary results of purely algebraic nature.

For any cdgc C and any dgl L there is a well known dgl structure on the graded vector space $\mathrm{Hom}(C,L)$ (see §1.4.1) given by the usual differential and the *convolution Lie bracket*:

$$Df = d\circ f - (-1)^{|f|} f\circ d, \qquad [f,g] = [-,-]\circ(f\otimes g)\circ\Delta$$

for $f,g\in\mathrm{Hom}(C,L)$.

Proposition 5.1. *If L is a cdgl, then* $\mathrm{Hom}(C,L)$ *is also complete.*

Proof. Let $\{F^n\}_{n\geq 1}$ be the filtration of L. Then, for $n\geq 1$, the projection $L\to L/F^n$ induces a map

$$\mathrm{Hom}(C,L)\to\mathrm{Hom}(C,L/F^n)$$

which can be checked to be a dgl morphism. Thus, we have a dgl morphism

$$\mathrm{Hom}(C,L)\to\varprojlim_n \mathrm{Hom}(C,L/F^n)$$

which, omitting the Lie bracket and regarding them as graded vector spaces, is a bijection, since $\mathrm{Hom}(V,-)$ is right adjoint to $-\otimes V$, for any vector space V, and therefore it commutes with limits (see §1.4.1). Hence, the natural map above is a bijective dgl morphism, so it is an isomorphism in **dgl**:

$$\mathrm{Hom}(C,L)\cong\varprojlim_n \mathrm{Hom}(C,L/F^n).$$

On the other hand, for $n \geq 1$, the map $\mathrm{Hom}(C, L) \to \mathrm{Hom}(C, L/F^n)$ is clearly surjective and has $J^n = \mathrm{Hom}(C, L^n)$ as kernel, therefore, we conclude that, as cdgl's

$$\mathrm{Hom}(C, L) \cong \varprojlim_n \mathrm{Hom}(C, L/F^n) \cong \varprojlim_n \mathrm{Hom}(C, L)/J^n$$

which proves that $\mathrm{Hom}(C, L)$ is complete with respect to $\{J^n\}_{n\geq 1}$.

$\square$

Proposition 5.2. $\mathrm{Hom}(\overline{C}, L)$ *is a complete sub dgl of* $\mathrm{Hom}(C, L)$*. Furthermore there is an isomorphism*

$$\mathrm{Hom}(C, L) \cong \mathrm{Hom}(\overline{C}, L)\widetilde{\times} L$$

where both L *and* $\mathrm{Hom}(\overline{C}, L)$ *are sub cdgl's of the twisted product and* $[x, \varphi] = \mathrm{ad}_x \circ \varphi$ *for* $x \in L$ *and* $\varphi \in \mathrm{Hom}(\overline{C}, L)$*.*

Proof. Checking that $\mathrm{Hom}(\overline{C}, L)$ is complete is analogous to the proof of the previous proposition.

For the second part, construct a map

$$\mathrm{Hom}(C, L) \to \mathrm{Hom}(\overline{C}, L)\widetilde{\times} L, \quad \varphi \mapsto (\varphi|_{\overline{C}}\,, \varphi(1))$$

which can be checked to be bijective and a dgl morphism with the dgl structure in the twisted product defined in the proposition. $\square$

It is important to notice that, via this result and unlike in the general case, the twisted product $\mathrm{Hom}(\overline{C}, L)\widetilde{\times} L$ is a complete dgl.

Let L' be a complete connected cdgl and X a connected simplicial set of finite type. We plan to a give a particular cdgl model of the evaluation fibration

$$\mathrm{map}^*(X, \langle L' \rangle) \to \mathrm{map}(X, \langle L' \rangle) \xrightarrow{\mathrm{ev}} \langle L' \rangle$$

specially adapted to our purposes.

For it let L be the usual model of X. The projection $L/L^n \to L/L^{n-1}$ induces a cdgl morphism $\mathrm{Hom}(\mathscr{C}(L/L^{n-1}), L') \to \mathrm{Hom}(\mathscr{C}(L/L^n), L')$, where $\mathscr{C}$ is the functor defined in §1.9. Therefore, we can consider the colimit in **cdgl**:

$$\varinjlim_n \mathrm{Hom}(\mathscr{C}(L/L^n), L'),$$

and similarly for $\varinjlim_n \mathrm{Hom}(\overline{\mathscr{C}}(L/L^n), L')$. The projection $\wedge V \to \wedge V/\wedge^0 V = \wedge^+ V$ for $V = sL/L^n$, induces a morphism $\mathscr{C}(L/L^n) \to \overline{\mathscr{C}}(L/L^n)$ for all n, which in turn induces a morphism

$$\varinjlim_n \mathrm{Hom}(\overline{\mathscr{C}}(L/L^n), L') \to \varinjlim_n \mathrm{Hom}(\mathscr{C}(L/L^n), L').$$

On the other hand, evaluating at $1 \in \mathbb{Q}$ gives a map $\mathrm{ev}_1 \colon \mathrm{Hom}(\mathscr{C}(L/L^n), L') \to L'$ which can be checked to be a cdgl morphism. Then we have a cdgl sequence

$$\varinjlim_n \mathrm{Hom}(\overline{\mathscr{C}}(L/L^n), L') \to \varinjlim_n \mathrm{Hom}(\mathscr{C}(L/L^n), L') \to L'$$

which, after a direct inspection, is a short exact sequence. The following theorem exhibits its geometrical realization.

Theorem 5.3. *There is commutative diagram*

$$\begin{tikzcd} \operatorname{map}^*(X, \langle L' \rangle) \arrow[r] & \operatorname{map}(X, \langle L' \rangle) \arrow[r, "\mathrm{ev}"] & \langle L' \rangle \\ \langle \varinjlim_n \operatorname{Hom}(\overline{\mathscr{C}}(L/L^n), L') \rangle \arrow[r] \arrow[u, "\simeq"] & \langle \varinjlim_n \operatorname{Hom}(\mathscr{C}(L/L^n), L') \rangle \arrow[r] \arrow[u, "\simeq"] & \langle L' \rangle \arrow[u, "\simeq"] \end{tikzcd}$$

where the vertical maps are homotopy equivalences.

Proof. Let A be a cdga model of the simplicial set X, then by [14, Proposition 12.25] there is a commutative diagram

$$\begin{tikzcd} \operatorname{map}^*(X, \langle L' \rangle) \arrow[r] & \operatorname{map}(X, \langle L' \rangle) \arrow[r, "\mathrm{ev}"] & \langle L' \rangle \\ \langle A^+ \widehat{\otimes} L' \rangle \arrow[r] \arrow[u, "\simeq"] & \langle A \widehat{\otimes} L' \rangle \arrow[r] \arrow[u, "\simeq"] & \langle L' \rangle \arrow[u, "\simeq"] \end{tikzcd}$$

where the evaluation is at the basepoint of X and the bottom row is the realization of the short exact sequence $A^+ \widehat{\otimes} L' \to A \widehat{\otimes} L' \to L'$.

Let $\mathcal{A}$ be a a connected cdga of finite type. Then $\mathcal{A}^\sharp$ is a cdgc and we have the following isomorphisms of cdgl's:

$$\mathcal{A} \widehat{\otimes} L' = \varprojlim_n (\mathcal{A} \otimes L'/F^n) \cong \varprojlim_n \operatorname{Hom}(\mathcal{A}^\sharp, L'/F^n) \cong \operatorname{Hom}(\mathcal{A}^\sharp, \varprojlim_n L'/F^n) \cong \operatorname{Hom}(\mathcal{A}^\sharp, L')$$

By [14, Theorem 10.8] the following cdga is a cdga model of X:

$$A = \varinjlim_n \mathscr{C}(L/L^n)^\sharp.$$

Since, for each n, $\mathscr{C}(L, L^n)^\sharp$ is a connected cdga of finite type, we use all of the above to rewrite the objects in the first diagram of the proof, using that the complete tensor product commutes with direct limits:

$$A \widehat{\otimes} L' \cong \varinjlim_n \mathscr{C}(L/L^n)^\sharp \widehat{\otimes} L' \cong \varinjlim_n \operatorname{Hom}(\mathscr{C}(L/L^n), L')$$

and, similarly, if we consider the augmentation ideal A^+:

$$A^+ \widehat{\otimes} L' \cong \varinjlim_n (\mathscr{C}(L/L^n)^\sharp)^+ \widehat{\otimes} L' \cong \varinjlim_n \operatorname{Hom}(\overline{\mathscr{C}}(L/L^n), L').$$

□

The following remark allows us to remove the direct limits in the theorem above in a special case.

Remark 5.4. Neisendorfer defined in [37, §7], a model for a nilpotent complex X of finite type, as any dgl quasi-isomorphic to $\mathscr{L}(A^\sharp)$ where A is a finite type Sullivan model of X. We call such model a *Neisendorfer model*. Any other minimal cdgl model of X will be quasi-isomorphic to $\mathscr{L}(A^\sharp)$ [14, Theorem 10.2]. Moreover, the cdga realization of $\mathscr{C}(\mathscr{L}(A^\sharp))^\sharp$ is of the homotopy type of $X_{\mathbb{Q}}$.

Corollary 5.5. *Let L be a Lie model of a nilpotent complex X of finite type, and L' a cdgl. Then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{map}^*(X,\langle L'\rangle) \arrow[r] & \mathrm{map}(X,\langle L'\rangle) \arrow[r, "\mathrm{ev}"] & \langle L'\rangle \\ \langle \mathrm{Hom}(\overline{\mathscr{C}}(L),L')\rangle \arrow[r] \arrow[u, "\simeq"] & \langle \mathrm{Hom}(\overline{\mathscr{C}}(L),L')\widetilde{\times} L'\rangle \arrow[r] \arrow[u, "\simeq"] & \langle L'\rangle \arrow[u, "\simeq"] \end{tikzcd}$$

where the twisted product is the one given in Proposition 5.2.

Proof. Let A be a Sullivan minimal model of X. By [14, Proposition 12.25] $A^+\,\widehat{\otimes}\,L' \to A\,\widehat{\otimes}\,L' \to L'$ is a short exact sequence whose realization is the upper row. Since A is of finite type, this short exact sequence becomes

$$\mathrm{Hom}(A^\sharp_+, L') \to \mathrm{Hom}(A^\sharp_+, L')\widetilde{\times} L' \to L'.$$

As previously remarked

$$\mathscr{L}(A^\sharp) \simeq L,$$

and, since $\mathscr{C}$ preserves quasi-isomorphisms,

$$\mathscr{C}\mathscr{L}(A^\sharp) \simeq \mathscr{C}(L).$$

Moreover, as $\mathscr{C}\mathscr{L}(A^\sharp)$ is quasi-isomorphic to $A^\sharp$ we deduce that

$$A^\sharp \simeq \mathscr{C}\mathscr{L}(A^\sharp) \simeq \mathscr{C}(L).$$

Therefore $A^\sharp_+ \simeq \overline{\mathscr{C}}(L)$ and the corollary follows. □

5.2 Components of the mapping space

Here we keep the same notation and assumptions that in the previous section, where L' is assumed to be a connected cdgl and X is a nilpotent simplicial set of finite type with Lie model L.

Note that restricting the evaluation fibration

$$\mathrm{map}(X,\langle L'\rangle) \xrightarrow{\mathrm{ev}} \langle L'\rangle$$

to the path component of $\mathrm{map}(X,\langle L'\rangle)$ containing a given map $f\colon X \to \langle L'\rangle$ produces a new fibration sequence

$$\mathrm{map}^*_{\zeta^{-1}(f)}(X,\langle L'\rangle) \to \mathrm{map}_f(X,\langle L'\rangle) \xrightarrow{\mathrm{ev}} \langle L'\rangle, \tag{5.2}$$

where, as remarked in the introduction of this chapter, $\mathrm{map}^*_{\zeta^{-1}(f)}(X,\langle L\rangle)$ is the non-connected simplicial set of pointed homotopy classes of pointed maps which are freely homotopic to f.

Combining the results of corollaries 4.2, 4.10 and 5.5 and recalling that the path components of the geometric realization of a cdgl are bijective with the set of classes of MC elements (see §1.8), we get a commutative diagram of sets

$$\begin{tikzcd}
\llbracket L, L' \rrbracket \arrow[r, two heads] \arrow[d, "\cong"] & \llbracket L, L' \rrbracket / H_0(L') \arrow[d, "\cong"] \\
\pi_0 \operatorname{map}^*(X, \langle L' \rangle) \arrow[r, two heads] \arrow[d, "\cong"] & \pi_0 \operatorname{map}(X, \langle L' \rangle) \arrow[d, "\cong"] \\
\widetilde{\mathrm{MC}}(\mathrm{Hom}(\overline{\mathscr{C}}(L), L')) \arrow[r, two heads] & \widetilde{\mathrm{MC}}(\mathrm{Hom}(\mathscr{C}(L), L')).
\end{tikzcd} \tag{5.3}$$

Let's give a explicit expression of the vertical bijections. Define a degree -1 linear map

$$q \colon \overline{\mathscr{C}}(L) \to L, \quad q(sx) = -x, \quad q(\wedge^{\geq 2} sL) = 0$$

where $x \in L$. The sign comes from the fact $s^{-1}s = -\operatorname{id}$.

Proposition 5.6. *In the diagram (5.3), the bijection* $\llbracket L, L' \rrbracket \to \widetilde{\mathrm{MC}}(\mathrm{Hom}(\overline{\mathscr{C}}(L), L'))$ *is given by*

$$\varphi \mapsto \bar{\varphi} = \varphi \circ q.$$

As a consequence, the bijection $\llbracket L, L' \rrbracket / H_0(L') \to \widetilde{\mathrm{MC}}(\mathrm{Hom}(\mathscr{C}(L), L'))$ is given by

$$\varphi \mapsto \bar{\varphi} = \varphi \circ q$$

where $q \colon \mathscr{C}(L) \to L$ is extended by defining $q(1) = 0$.

Proof. We first check that it is well defined, this is, that $\bar{\varphi}$ is a MC-element. The differential is $D\bar{\varphi} = \partial' \circ \bar{\varphi} + \bar{\varphi} \circ (d_1 + d_2)$. For $x, y \in L$ we have:

$$(D\bar{\varphi})(sx) = \partial'(\bar{\varphi}(sx)) + \bar{\varphi}(d_1 sx) = -\partial'\varphi(x) + \varphi(\partial x) = 0$$

$$(D\bar{\varphi})(sx \wedge sy) = \bar{\varphi}(d_2(sx \wedge sy)) = (-1)^{|x|}\varphi[x, y]$$

where the first expression is zero because φ is a cdgl morphism. The terms of length 3 or larger are sent to zero.

On the other hand, $[\bar{\varphi}, \bar{\varphi}]$ sends any element to zero except elements of the form $sx \wedge sy$, which are sent to

$$[-,-]\circ(\bar{\varphi}\otimes\bar{\varphi})\circ\bar{\Delta}(sx\wedge sy) = (-1)^{|x|+1}[\varphi(x), \varphi(y)] + (-1)^{|x||y|+|x|}[\varphi(y), \varphi(x)] = -2(-1)^{|x|}\varphi[x, y].$$

Therefore we deduce that $D\bar{\varphi} = -1/2[\bar{\varphi}, \bar{\varphi}]$ so it is a Maurer-Cartan element.

Finally from the bijection given in [14, Proposition 11.1] and checking carefully the bijections involved in our construction, we deduce that φ is sent to $\bar{\varphi}$.

□

Recall from §1.2.2 that $\mathrm{Hom}(\mathscr{C}(L), L')^{\bar{\varphi}}$ is a connected sub cdgl of $\mathrm{Hom}(\mathscr{C}(L), L')$ with the perturbed differential $D_{\bar{\varphi}}$. The following proposition gives us a model of the fibration in (5.2).

Proposition 5.7. *If $f\colon X \to \langle L\rangle$ corresponds with $\bar\varphi \in \widetilde{\mathrm{MC}}(\mathrm{Hom}(\overline{\mathscr{C}}(L), L'))$ through the bijection of (5.3), then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{map}_f(X,\langle L'\rangle) \arrow[r, "\mathrm{ev}"] & \langle L'\rangle \\ \langle \mathrm{Hom}(\mathscr{C}(L), L')^{\bar\varphi}\rangle \arrow[u, "\simeq"] \arrow[r, "\langle \mathrm{ev}_1\rangle"] & \langle L'\rangle . \arrow[u, "\simeq"'] \end{tikzcd}$$

Proof. Recall from §1.8 that $\langle \mathrm{Hom}(\mathscr{C}(L), L')^{\bar\varphi}\rangle \simeq \langle \mathrm{Hom}(\mathscr{C}(L), L')\rangle^{\bar\varphi}$. Then, restricting the diagram in Corollary 5.5 to the connected component $\mathrm{map}_f(X,\langle L'\rangle)$ gives the result. $\square$

Example 5.8. The cdgl morphism $\mathrm{ev}_1\colon \mathrm{Hom}(\mathscr{C}(L), L')^{\bar\varphi} \to L'$ of the proposition above is not, in general, a cdgl fibration. However, when it is realized, it is equivalent to a fibration of simplicial sets.

Consider the cdgl's $L = \mathbb{L}(x)$, $L' = \widehat{\mathbb{L}}(y,z)$ with $|x| = |y| = |z| = 0$ and no differential and the cdgl morphism $\varphi\colon L \to L', \varphi(x) = y$. Suppose that $\psi \in \mathrm{Hom}_0(\mathscr{C}(L), L')$ is such that $\psi(1) = z$. By degree reasons, since $\mathscr{C}(L) = \wedge(sx)$, then $\psi(sx) = 0$. This implies that ψ is the only preimage of z under the morphism

$$\mathrm{ev}_1\colon \mathrm{Hom}(\mathscr{C}(L), L') \to L'.$$

However, it can be checked that

$$D_{\bar\varphi}(\psi) = D\psi + [\varphi\circ q, \psi] = [\varphi \circ q, \psi]$$

is not zero, since it sends the element $sx \in \mathscr{C}(L)$ to $[z,y] \in L'$. Therefore ψ is not an element of $\mathrm{Hom}(\mathscr{C}(L), L')^{\bar\varphi}$ and we deduce that $\mathrm{ev}_1\colon \mathrm{Hom}(\mathscr{C}(L), L')^{\bar\varphi} \to L'$ is not surjective.

Remark 5.9. Recall that the fiber of $\mathrm{ev}\colon \mathrm{map}_f(X,\langle L'\rangle) \to \langle L'\rangle$ is $\mathrm{map}^*_{\zeta^{-1}(f)}(X,\langle L'\rangle)$ which consists of those pointed maps which are freely homotopic to f. Then, in view of (5.3), we deduce that the fiber of

$$\langle \mathrm{ev}_1\rangle\colon \langle \mathrm{Hom}(\mathscr{C}(L), L')\rangle^{\bar\varphi} \to \langle L'\rangle$$

is the non-connected simplicial set

$$\bigsqcup_{[\psi]} \langle \mathrm{Hom}(\overline{\mathscr{C}}(L'), L)\rangle^{\bar\psi}$$

where the index $[\psi]$ runs through the homotopy classes of morphisms $[\psi] \in [\![L, L']\!]$ such that $[\psi] = [\varphi]$ in $[\![L, L']\!]/H_0(L')$.

5.3 Modeling evaluation fibration sequences with derivations

Fix X a nilpotent complex with L its Lie model, L' a connected cdgl, and $\varphi\colon L \to L'$ a cdgl morphism. The goal of this section is to rewrite the model given in Proposition 5.7 in terms of derivations.

Recall from §1.9 that there is a quasi-isomorphism of dgl's $\alpha_L\colon \mathscr{L}\mathscr{C}(L) \to L$. Then we denote by $\tilde{\varphi}$ the dgl morphism

$$\tilde{\varphi} = \varphi \circ \alpha_L\colon \mathscr{L}\mathscr{C}(L) \to L'.$$

Recall from §1.1 that $\mathrm{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L), L')$ is a differential graded vector space. Then its desuspension $s^{-1}\mathrm{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L), L')$ has differential $D(s^{-1}\theta) = -s^{-1}D\theta$.

Consider the map

$$\Gamma\colon (s^{-1}\mathrm{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L), L'), D) \to (\mathrm{Hom}(\overline{\mathscr{C}}(L), L'), D_{\bar{\varphi}})$$

defined as

$$\Gamma(s^{-1}\theta)(c) = (-1)^{|\theta|}\theta(sc)$$

for $\theta \in \mathrm{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L), L')$ and $c \in \overline{\mathscr{C}}(L)$.

Proposition 5.10. Γ *is a isomorphism between differential graded vector spaces.*

Proof. Since $\mathscr{L}\mathscr{C}(L) = \mathbb{L}(s^{-1}\overline{\mathscr{C}}(L))$, then a $\tilde{\varphi}$-derivation is uniquely determined by its values at $s^{-1}\overline{\mathscr{C}}(L)$. From this observation it is clear that Γ is bijective. Furthermore it is a graded linear map. Let's check that it commutes with the differentials.

Let c be an element in $\overline{\mathscr{C}}(L)$ and let θ be a $\tilde{\varphi}$-derivation. For

$$\bar{\Delta}(c) = \sum_i c_i \otimes c_i',$$

the formula of the differential in $\mathscr{L}\mathscr{C}(L) = (\mathbb{L}(s^{-1}\overline{\mathscr{C}}(L)), d_1 + d_2)$ gives

$$(d_1 + d_2)s^{-1}c = -s^{-1}dc + \frac{1}{2}\sum_i (-1)^{|c_i|}[s^{-1}c_i, s^{-1}c_i'].$$

Let's check that $\Gamma(Ds^{-1}\theta)(c) = D_{\bar{\varphi}}\Gamma(s^{-1})(c)$ in L':

$$\begin{aligned}
\Gamma(Ds^{-1}\theta) &= -\Gamma(s^{-1}D\theta)(c) = (-1)^{|\theta|}(D\theta)(s^{-1}c) = \\
&= (-1)^{|\theta|}\partial'\theta(s^{-1}c) - \theta(d_1 s^{-1}c) - \theta(d_2 s^{-1}c) = \\
&= (-1)^{|\theta|}\partial'\theta(s^{-1}c) + \theta(s^{-1}dc) - \frac{1}{2}\sum_i (-1)^{|c_i|}\theta[s^{-1}c_i, s^{-1}c_i'].
\end{aligned}$$

$$\begin{aligned}
D_{\bar{\varphi}}\Gamma(s^{-1}\theta)(c) &= \partial'\Gamma(s^{-1}\theta)(c) + (-1)^{|\theta|}\Gamma(s^{-1}\theta)(dc) + [\bar{\varphi}, \Gamma(s^{-1}\theta)](c) = \\
&= (-1)^{|\theta|}\partial'\theta(s^{-1}c) + \theta(s^{-1}dc) + (-1)^{|\theta|}[\bar{\varphi}, \theta s^{-1}](c).
\end{aligned}$$

These two expressions agree since $\tilde{\varphi}(s^{-1}c) = \bar{\varphi}(c)$ and because of the commutativity of $\bar{\Delta}$. $\square$

As a consequence of this proposition $s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')$ inherits a Lie bracket

$$[s^{-1}\theta, s^{-1}\eta] = \Gamma^{-1}[\Gamma(s^{-1}\theta), \Gamma(s^{-1}\eta)]$$

which makes $s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')$ isomorphic to $\operatorname{Hom}(\overline{\mathscr{C}}(L),L')$ as cdgl's. A more explicit expression of this Lie bracket is given by

$$[s^{-1}\theta, s^{-1}\eta] = -s^{-1}[-,-]\circ(\theta s^{-1}\otimes \eta s^{-1})\circ\bar\Delta\circ s.$$

Recall that in Proposition 5.2 is constructed a twisted product $\operatorname{Hom}(\overline{C},L')\widetilde{\times}L'$. Using the isomorphism Γ, we can substitute $\operatorname{Hom}(\overline{\mathscr{C}}(L),L')$ by $s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')$ in the twisted product to obtain:

$$(s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}L', D)$$

where

$$[x, s^{-1}\theta] = (-1)^{|x|}s^{-1}(\operatorname{ad}_x\circ\theta), \quad Dx = \partial' x - s^{-1}(\operatorname{ad}_x\circ\bar\varphi\circ s)$$

for any $x\in L'$ and $\theta\in\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')$. By $\operatorname{ad}_x\circ\theta$ and $\operatorname{ad}_x\circ\bar\varphi\circ s$ we mean the $\tilde\varphi$-derivations that act on $s^{-1}\overline{\mathscr{C}}(L)$ in this way. With these definitions is direct to check that

$$\Gamma\times\operatorname{id}_{L'}\colon (s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}L', D)\to(\operatorname{Hom}(\overline{\mathscr{C}}(L),L')\widetilde{\times}L', D_{\bar\varphi})$$

is a cdgl isomorphism. As a consequence we get the following proposition.

Proposition 5.11. *The realization of the short exact cdgl sequence*

$$0\to s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')\to s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}L'\to L'\to 0$$

has the homotopy type of the fibration sequence

$$\operatorname{map}^*(X,\langle L'\rangle)\to\operatorname{map}(X,\langle L'\rangle)\xrightarrow{\mathrm{ev}}\langle L'\rangle.$$

Proof. We have the following diagram of isomorphic cdgl sequences

$$\begin{tikzcd}
s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L') \arrow[r] \arrow[d, "\Gamma", "\cong"'] & s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}L' \arrow[r] \arrow[d, "\Gamma\times\operatorname{id}_{L'}", "\cong"'] & L' \arrow[d, "\operatorname{id}_{L'}", "\cong"'] \\
\operatorname{Hom}(\overline{\mathscr{C}}(L),L') \arrow[r] & \operatorname{Hom}(\overline{\mathscr{C}}(L),L')\widetilde{\times}L' \arrow[r] & L'
\end{tikzcd}$$

where in the bottom sequence the differential is the perturbed differential $D_{\bar\varphi}$. However perturbing the differential does not affect the realization of a cdgl (see §1.8), so using Corollary 5.5, we deduce the result. □

Similarly we can rewrite Proposition 5.7 in terms of derivations. From the cdgl isomorphisms

$$(s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}L', D)^0\cong(\operatorname{Hom}(\overline{\mathscr{C}}(L),L')\widetilde{\times}L', D_{\bar\varphi})^0\cong(\operatorname{Hom}(\overline{\mathscr{C}}(L),L')\widetilde{\times}L', D)^{\bar\varphi}$$

we deduce the following corollary.

Corollary 5.12. *The realization of the cdgl morphism*

$$(s^{-1}\operatorname{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}L',D)^0\to L'$$

has the homotopy type of the fibration

$$\operatorname{map}_f(X,\langle L\rangle)\xrightarrow{\mathrm{ev}}\langle L'\rangle.$$

5.3.1 Chain complexes of derivations

In order to obtain more comfortable expressions of these results, we would want to rewrite the cdgl $s^{-1}\operatorname{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L),L')$ as $s^{-1}\operatorname{Der}_{\varphi}(L,L')$ and removing the functors $\mathscr{L}$ and $\mathscr{C}$. However, these two differential graded vector spaces are not isomorphic and we can not endow $s^{-1}\operatorname{Der}_{\varphi}(L,L')$ with a Lie bracket. Nevertheless, they are quasi-isomorphic as differential graded vector spaces, so we can recover information about the homotopy groups of $\operatorname{map}^*(X,\langle L'\rangle)$ and $\operatorname{map}(X,\langle L'\rangle)$.

Consider the map

$$\alpha_L^*\colon \operatorname{Der}_{\varphi}(L,L')\to\operatorname{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L),L'),\quad \theta\mapsto\theta\circ\alpha_L.$$

Since $\mathscr{L}\mathscr{C}(L)$ and L are both cofibrant (because $\mathscr{L}\mathscr{C}(L)$ is free and L is a Lie model) and α_L is a quasi-isomorphism, Lemma 1.1 implies that it is a quasi-isomorphism of differential graded vector spaces.

Consider the twisted chain complex

$$\operatorname{Der}_{\varphi}(L,L')\widetilde{\times}sL'$$

where $\operatorname{Der}_{\varphi}(L,L')$ is a sub chain complex and the differential at sL' is

$$D(sx)=-s\partial'x+\operatorname{ad}_x\circ\varphi,\quad x\in L'.$$

A short computation shows that the map $\alpha_L^*\times\operatorname{id}_{sL'}$

$$\alpha_L^*\times\operatorname{id}_{sL'}\colon\operatorname{Der}_{\varphi}(L,L')\widetilde{\times}sL'\to\operatorname{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}sL'=s(s^{-1}\operatorname{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}L')$$

is a morphism of differential graded vector spaces, i.e. it commutes with the differentials. The differential at $\operatorname{Der}_{\tilde{\varphi}}(\mathscr{L}\mathscr{C}(L),L')\widetilde{\times}sL'$ is the suspension of the differential of the twisted product given before Proposition 5.11.

We finally get that the homology of these chain complexes gives the homotopy long exact sequence of the evaluation fibration.

Theorem 5.13. *The homology long exact sequence of the differential graded vector space sequence*

$$0\to\operatorname{Der}_{\varphi}(L,L')^{(1)}\to(\operatorname{Der}_{\varphi}(L,L')\widetilde{\times}sL')^{(1)}\to sL'\to 0$$

is isomorphic to the homotopy long exact sequence of the fibration sequence

$$\operatorname{map}_f^*(X,\langle L'\rangle)\to\operatorname{map}_f(X,\langle L'\rangle)\xrightarrow{\mathrm{ev}}\langle L'\rangle$$

In particular, for $n\geq 1$ we have the group isomorphisms

$$\pi_n(\operatorname{map}_f^*(X,\langle L'\rangle))\cong H_n(\operatorname{Der}_{\varphi}(L,L')),\qquad \pi_n(\operatorname{map}_f(X,\langle L'\rangle)\cong H_n(\operatorname{Der}_{\varphi}(L,L')\widetilde{\times}sL').$$

This theorem generalizes the original result of G. Lupton and S.B. Smith [27, Theorem 4.1].

Remark 5.14. In the homotopy groups of the theorem, the basepoint is $f \in \operatorname{map}^*_f(X, \langle L'\rangle) \subset \operatorname{map}_f(X, \langle L'\rangle)$. Note that for $n = 1$, the group structure in $H_1(\operatorname{Der}_\varphi(L, L'))$ and $H_1(\operatorname{Der}_\varphi(L, L')\widetilde{\times} sL')$ is the one induced for the BCH product of $H_0(s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L), L'))$ and $H_0(s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L), L')\widetilde{\times} L')$ respectively.

Proof. Consider the following diagram of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Der}_\varphi(L, L') \arrow[r] \arrow[d, "\alpha^*_L"] & \operatorname{Der}_\varphi(L, L')\widetilde{\times} sL' \arrow[r] \arrow[d, "\alpha^*_L\times \mathrm{id}_{sL'}"] & sL' \arrow[r] \arrow[d, "\mathrm{id}_{sL'}"] & 0 \\
0 \arrow[r] & \operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L), L') \arrow[r] & \operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L), L')\widetilde{\times} sL' \arrow[r] & sL' \arrow[r] & 0
\end{tikzcd}$$

where both rows are short exact sequences and the bottom one is the suspension of the cdgl sequence of Proposition 5.11. Clearly this diagram is commutative, and since α^*_L and $\mathrm{id}_{sL'}$ are quasi-isomorphisms, then applying the five lemma to the long exact sequence of homology groups, we deduce that $\alpha^*_L \times \mathrm{id}_{sL'}$ is also a quasi-isomorphism.

Then take the 1-connected covers of the differential graded vector spaces of the diagram above. At the top row we get the sequence of the statement, while at the bottom row we get the suspension of the 0-connected cover of the sequence of Proposition 5.11. Because of Corollary 5.12, the realization of

$$(s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L), L')\widetilde{\times} L', D)^0 \to L'$$

is the fibration

$$\operatorname{map}_f(X, \langle L'\rangle) \xrightarrow{\mathrm{ev}} \langle L'\rangle.$$

So we deduce from Remark 5.9 that the realization of the connected cdgl $s^{-1}\operatorname{Der}_{\tilde\varphi}(\mathscr{L}\mathscr{C}(L), L')^0$ is of the homotopy type of $\operatorname{map}^*_f(X, \langle L'\rangle)$, which is one of the connected components of $\operatorname{map}^*_{\zeta^{-1}(f)}(X, \langle L'\rangle))$.

Finally, recall from §1.8 that, for any connected cdgl M, there is a natural isomorphism

$$\pi_n\langle M\rangle \cong H_{n-1}(M) \cong H_n(sM)$$

for any $n \geq 1$, so we deduce that both long exact sequences of the statement agree.

□

CHAPTER 6

Filtrations of derivations and twisted products

Our main goal in the rest of the text is to obtain homotopical information about $B\operatorname{aut}(X)$ and $B\operatorname{aut}^*(X)$. In order to do that, we are interested on finding cdgl's M and M', depending on a Lie model L of X, such that

$$\langle M\rangle \simeq B\operatorname{aut}(X), \quad \langle M'\rangle \simeq B\operatorname{aut}^*(X).$$

Note that because of Theorem 5.13, for $\langle L\rangle = X$ we have the group isomorphisms

$$H_n(\operatorname{Der}_{\mathrm{id}}(L,L)) \cong \pi_n(\operatorname{map}^*_{\mathrm{id}}(X,X)) \cong \pi_{n+1}(B\operatorname{aut}^*(X)),$$

$$H_n(\operatorname{Der}_{\mathrm{id}}(L,L)\widetilde{\times} sL) \cong \pi_n(\operatorname{map}_{\mathrm{id}}(X,X)) \cong \pi_{n+1}(B\operatorname{aut}(X)).$$

In this particular case $\operatorname{Der}_{\mathrm{id}}(L,L) = \operatorname{Der} L$ is not only a chain complex but a dgl, so it is a promising candidate to be a model for $B\operatorname{aut}^*(X)$. However, we will see that even when L is a cdgl, $\operatorname{Der} L$ is not necessarily complete. A similar unpleasant situation happens with the twisted product of cdgl's: it may fail to be complete.

Furthermore, we will prove that, in general, there are no cdgl's M and M' such that when realized gives $B\operatorname{aut}(X)$ and $B\operatorname{aut}^*(X)$. However, an slight modification of the dgl $\operatorname{Der}(L)$ will give a complete dgl, which allows us to obtain information about the spaces above.

The purpose of this technical chapter is to provide examples of these affirmations and to study the filtrations of the derivations and twisted products in order to produce useful cdgl's.

6.1 Complete Lie algebras of derivations

We start with an example that $\operatorname{Der} L$ is not complete even for L complete.

Example 6.1. Consider the cdgl $L = (\widehat{\mathbb{L}}(x,y), \partial = 0)$ with $|x| = |y| = 2$, which is immediately complete since it is positive graded. Then consider the following elements η, θ_1 and θ_2 in $\operatorname{Der}_0 L$:

$$\eta(x) = x,\ \eta(y) = -y, \qquad \theta_1(x) = y,\ \theta_1(y) = 0, \qquad \theta_2(x) = 0,\ \theta_2(y) = x.$$

The Lie brackets are

$$[\eta, \theta_1] = -2\theta_1, \quad [\eta, \theta_2] = 2\theta_2, \quad [\theta_1, \theta_2] = -\eta,$$

so we deduce that for any possible filtration $\{F^n\}_{n\geq 1}$ of $\operatorname{Der} L$ these three derivations belong to F^n for any n. Therefore $\cap_{n\geq 1} F^n \neq 0$ and

$$L \to \varprojlim_{n\geq 1} \operatorname{Der} L / F^n$$

is not an isomorphism.

Through this section fix $L = (\widehat{\mathbb{L}}(V), \partial)$ a connected minimal free cdgl and consider that the graded vector space V is finite dimensional and filtered by a finite sequence of graded vector subspaces:

$$V = V^0 \supset V^1 \supset \cdots \supset V^{q-1} \supset V^q = 0.$$

We refine the usual filtration in the free algebra L by considering, for $n \geq 1$ and $p \geq 0$,

$$\widehat{\mathbb{L}}^{n,p}(V) = \operatorname{Span}\{[v_1, [v_2, [\dots, [v_{n-1}, v_n]\dots]]] \in \widehat{\mathbb{L}}^n(V), v_i \in V^{\alpha_i} \text{ and } \sum_{i=1}^n \alpha_i = p\}.$$

Obverse that $\widehat{\mathbb{L}}^{n,p}(V) = 0$ for $p \geq nq$. For $n \geq 1$ and $0 \leq p \leq nq - 1$, define

$$F^{n,p} = \widehat{\mathbb{L}}^{n,p}(V) \oplus \widehat{\mathbb{L}}^{\geq n+1}(V)$$

and we have the following sequence

$$\begin{aligned}
\widehat{\mathbb{L}}(V) = F^{1,0} &\supset F^{1,1} \supset \cdots \supset F^{1,q-1} \\
&\supset F^{2,0} \supset F^{2,1} \supset \cdots \supset F^{2,2q-1} \\
&\dots \\
&\supset F^{n,0} \supset F^{n,1} \supset \cdots \supset F^{n,nq-1} \\
&\dots \qquad\qquad\qquad\qquad\qquad .
\end{aligned}$$

Note that $F^{n,p}$ ranks

$$t = q + \cdots + (n-1)q + p + 1 = \frac{(n-1)nq}{2} + p + 1 \tag{6.1}$$

in the order given by this chain of inclusions. If n, p and t are as in the formula (6.1), then we write $(n,p) \equiv t$. Note also that $F^{n,0} = \widehat{\mathbb{L}}^{\geq n}(V)$.

Definition 6.2. For $(n,p)\equiv t$, we write

$$F^t = F^{n,p}.$$

Proposition 6.3. *For $(n_1,p_1)\equiv t_1$ and $(n_2,p_2)\equiv t_2$, then*

$$[F^{t_1},F^{t_2}]\subset F^{t_1+t_2}.$$

Proof. An easy inspection shows that $[F^{n_1,p_1},F^{n_2,p_2}]\subset F^{n_1+n_2,p_1+p_2}$. Then note that

$$(n_1+n_2,p_1+p_2)\equiv \frac{(n_1+n_2-1)(n_1+n_2)q}{2}+p_1+p_2+1=$$
$$=\frac{(n_1-1)n_1q}{2}+\frac{(n_2-1)n_2q}{2}+n_1n_2q+p_1+p_2+1\geq$$
$$\geq \frac{(n_1-1)n_1q}{2}+\frac{(n_2-1)n_2q}{2}+p_1+p_2+2=t_1+t_2.$$

Since $\{F^t\}_{t\geq 1}$ is a decreasing sequence, we have $F^{n_1+n_2,p_1+p_2}\subset F^{t_1+t_2}$.

□

Proposition 6.4. *The sequence $\{F^t\}_{t\geq 1}$ is a filtration of L and L is complete with respect to this filtration.*

Proof. Note that, since L is minimal, $\partial F^t\subset F^{t+1}$ for $t\geq 1$. This fact along with Proposition 6.3 imply that $\{F^t\}_{t\geq 1}$ is a filtration.

Furthermore, since $\cap_t F^t=\cap_{n,p}F^{n,p}=0$, the natural map

$$L\to \varprojlim_t L/F^t=\varprojlim_{n,p} L/F^{n,p}$$

is injective. To see that it is surjective, consider an element in $x\in \varprojlim_{n,p} L/F^{n,p}$ which can be written as a formal series

$$\sum_{n,p} x_{n,p}$$

where $x_{n,p}\in F^{n,p}$. Note that for each $m\geq 1$, this series restricted to $\widehat{\mathbb{L}}^m(V)$ contains only a finite sum. Therefore $\sum_{n,p}x_{n,p}$ is a well-defined element in L, which proves the surjectivity of the natural map.

□

Furthermore, since for each $t\geq 1$, $\widehat{\mathbb{L}}^t(V)=F^{t,0}\subset F^t$, the identity

$$\mathrm{id}\colon (L,\{\widehat{\mathbb{L}}^{\geq t}(V)\}_{t\geq 1})\to (L,\{F^t\}_{t\geq 1})$$

is a cdgl isomorphism. So we can replace the usual filtration with the new filtration $\{F^t\}_{t\geq 1}$.

Definition 6.5. For $t\geq 1$ define the following subspaces of $\operatorname{Der}_{\geq 0} L$

$$\mathcal{F}^t=\{\theta\in \operatorname{Der}_{\geq 0} L, \theta(F^r)\subset F^{t+r},\ \text{for all } r\geq 1\}.$$

The following lemma gives a characterization of the derivations belonging to $\mathcal{F}^t$.

Lemma 6.6. *A derivation $\theta \in \operatorname{Der}_{\geq 0} L$ is such that $\theta(V^i) \subset F^{i+t+1}$ for all $0 \leq i < q$ if and only if $\theta \in \mathcal{F}^t$.*

Proof. Since $V^i \subset F^{i+1}$, it is clear that, if $\theta \in \mathcal{F}^t$, then $\theta(V^i) \subset F^{i+t+1}$. The converse implication can be proved inductively.

Suppose that $\theta(F^{r'}) \subset F^{t+r'}$ for any $r' \equiv (n', p)$ with $n' \leq n$. Then an element in $F^{n+1,p} = F^r$ can be written as a sum of elements of the form $x = [a, b]$ with $a \in F^{1,\alpha}$ and $b \in F^{n,\beta}$, where $\alpha + \beta = p$, $(1, \alpha) \equiv r_1 = \alpha + 1$ and $(n, \beta) \equiv r_2$.

Then $\theta(a) \in F^{1+\alpha+t} = F^{n_1,p_1}$, where $(n_1, p_1) \equiv 1 + \alpha + t$. So

$$[\theta(a), b] \in [F^{n_1,p_1}, F^{n,\beta}] \subset F^{n_1+n,p_1+\beta}.$$

Then the following computations are straightforward:

$$\begin{aligned}
(n_1 + n, p_1 + \beta) &\equiv \frac{(n_1 + n - 1)(n_1 + n)q}{2} + p_1 + \beta + 1 = \\
&= \frac{(n_1 - 1)n_1 q}{2} + \frac{(n_1 - 1)nq}{2} + \frac{n(n + n_1)q}{2} + p_1 + \beta + 1 \geq \\
&\geq 1 + \alpha + \beta + t + \frac{n(n + 1)}{2} q = 1 + p + t + \frac{n(n + 1)q}{2} = t + r.
\end{aligned}$$

Therefore $[\theta(a), b] \in F^{t+r}$. On the other hand, by the induction hypothesis $\theta(b) \in F^{r_2+t}$, where $r_2 + t \equiv (n_2, p_2)$ with $n_2 \geq n$, thus

$$[a, \theta(b)] \in [F^{1,\alpha}, F^{n_2,p_2}] \subset [F^{1+n_2,\alpha+p_2}]$$

and

$$(1 + n_2, \alpha + p_2) \equiv \frac{n_2(n_2 + 1)q}{2} + \alpha + p_2 + 1 = \frac{n_2(n_2 - 1)q}{2} + n_2 q + \alpha + p_2 + 1 =$$

$$= r_2 + t + n_2 q + \alpha = \frac{n(n-1)}{2} q + \beta + 1 + t + n_2 q + \alpha \geq \frac{n(n-1)}{2} q + p + 1 + t + nq = r + t.$$

Finally we deduce that $\theta[a, b] \in F^{r+t}$, which concludes the proof. $\square$

Definition 6.7. For $L = (\widehat{\mathbb{L}}(V), \partial)$ a free minimal cdgl and a fixed finite filtration of V, we define $\mathcal{D}\mathrm{er} L$ as the connected sub dgl of $\operatorname{Der} L$ defined as

$$\mathcal{D}\mathrm{er}_k L = \begin{cases} \{\theta \in \operatorname{Der}_0 L \mid \theta(V^i) \subset V^{i+1} \oplus \widehat{\mathbb{L}}^{\geq 2}(V), \text{ for all } i\}, & \text{if } k = 0 \\ \operatorname{Der}_k L, & \text{if } k > 0 \end{cases}$$

In particular, Lemma 6.6 implies that $\mathcal{D}\mathrm{er}_0 L = \mathcal{F}^1_0$.

Remark 6.8. Given a finite filtration of V as before, we can make a new filtration from the former one:

$$\begin{aligned} V^0 &\supset V_0^1 \oplus V_{\geq 1} \supset V_0^2 \oplus V_{\geq 1} \supset \cdots \supset V_0^{q-1} \oplus V_{\geq 1} \supset V_{\geq 1} \supset \\ &\supset V_1^1 \oplus V_{\geq 2} \supset V_1^2 \oplus V_{\geq 2} \supset \cdots \supset V_1^{q-1} \oplus V_{\geq 2} \supset V_{\geq 2} \supset \\ &\dots \\ &\supset V_m^1 \supset V_m^2 \supset \dots V_m^{q-1} \supset 0 \end{aligned}$$

where m is such that $V_{>m} = 0$. Note that a linear map of degree 0 preserves the former filtration if and only if it preserves the new one. Therefore, the dgl $\mathcal{D}\mathrm{er}_0 L$ is not altered if we replace the former filtration by the new one presented above.

Then, without loss of generality, we can assume that our given filtration has the following property: if $0 \neq v \in V_k^i$, then $V_{>k} \subset V^{i+1}$. We say that a filtration with such property is *degree-respectful*.

If our filtration is as in Remark 6.8, then any derivation θ of positive degree is such that $\theta(V^i) \subset V^{i+1} \oplus \widehat{\mathbb{L}}^{\geq 2}(V)$. So we conclude that

$$\mathcal{D}\mathrm{er}_k L = \mathrm{Der}_k\, L = \mathcal{F}_k^1, \quad \text{for } k \geq 1.$$

Proposition 6.9. *The sequence $\{\mathcal{F}^t\}_{t\geq 1}$ is a filtration of $\mathcal{D}\mathrm{er}L$ and $\mathcal{D}\mathrm{er}L$ is complete with respect to this filtration.*

Proof. If $\theta_1 \in \mathcal{F}^{t_1}$ and $\theta_2 \in \mathcal{F}^{t_2}$ then

$$[\theta_1, \theta_2](F^r) \subset F^{t_1+t_2+r}$$

so $[\theta_1, \theta_2] \in \mathcal{F}^{t_1+t_2}$ which implies that $[\mathcal{F}^{t_1}, \mathcal{F}^{t_2}] \subset \mathcal{F}^{t_1+t_2}$. Furthermore, since L is minimal, then

$$D\mathcal{F}^t \subset \mathcal{F}^{t+1}.$$

We conclude that $\{\mathcal{F}^t\}$ is a filtration of $\mathcal{D}\mathrm{er}L$. The proof that

$$\mathcal{D}\mathrm{er}L \to \varprojlim_t \mathcal{D}\mathrm{er}L/\mathcal{F}^t$$

is an isomorphism is analogous to that of Proposition 6.4: if there is a derivation θ in the intersection $\cap_t \mathcal{F}^t$ then, $\theta(V) \subset F^t$ for all $t \geq 1$, which implies that $\theta(V) = 0$. Thus, the natural map to the inverse limit is injective. Finally, take a formal series $\sum_t \theta_t$ with $\theta_t \in \mathcal{F}^t$. Note that for each $v \in V$ and for each number m, the series

$$\sum_t \theta_t(v)$$

restricted to $\widehat{\mathbb{L}}^m(V)$ contains only a finite sum. Therefore, the formal series constitutes a well-defined derivation in $\mathcal{D}\mathrm{er}L$, which proves that the natural map above is surjective. $\square$

We finalize this section by computing the exponential of $\mathcal{D}\mathrm{er}_0 L$. Recall from §1.2.4 that if a derivation $\theta \in \operatorname{Der}_0 L$ is such that it increases the filtration degree, then e^θ is a well defined cdgl automorphism. In particular $e^{\mathcal{D}\mathrm{er}_0 L}$ is a subset of $\operatorname{aut}(L)$ which is explicitly described in the following proposition.

Proposition 6.10.

$$e^{\mathcal{D}\mathrm{er}_0 L} = \{\varphi \in \operatorname{aut}(L) \mid (\varphi_* - \mathrm{id}_V)(V^i) \subset V^{i+1} \textit{ for all } i\}.$$

The notation $\varphi_* \colon V \to V$ means the linear part of the cdgl morphism.

Proof. By the exponential formula,

$$(e^\theta)_* = e^{\theta_*} \implies (e^\theta)_* - \mathrm{id}_V = \sum_{n\geq 1} \frac{\theta_*^n}{n!}$$

where $\theta_* \colon V \to V$ is the linear part of the derivation. Thus if $\theta \in \mathcal{D}\mathrm{er}_0 L$, then

$$\theta(V^i) \subset V^{i+1} \oplus \widehat{\mathbb{L}}^{\geq 2}(V) \implies \theta_*(V^i) \subset V^{i+1} \implies \theta_*^n(V^i) \subset V^{i+1}$$

for all i and $n \geq 1$, so $\varphi = e^\theta$ belongs to the right hand side.

Conversely, if $\varphi \in \operatorname{aut}(L)$ belongs to the right hand side, define a linear morphism

$$\theta \colon V \to L, \quad \theta(v) = \sum_{n\geq 1} (-1)^n \frac{(\varphi - \mathrm{id}_L)^n(v)}{n}$$

for $v \in V$. For a fixed $v \in V$ and $m \geq 1$, since $(\varphi_* - \mathrm{id}_V)(V^i) \subset V^{i+1}$, we can check that the component of $(\varphi - \mathrm{id}_L)^n(v)$ in $\widehat{\mathbb{L}}^m(V)$ is zero for n large enough. Then θ is well defined and it can be extended to a derivation $\theta \in \operatorname{Der}_0 L$ which, furthermore, belongs to $\mathcal{D}\mathrm{er}_0 L$. Finally, since the formula that defines θ is that of $\log(\varphi)$, we have that

$$e^\theta = \varphi$$

and we have checked that both sets are equal. □

6.2 Complete twisted products

As we have seen, twisted products play a central role in the modelization of fibration sequences. However, recall from Example 1.3 that the twisted product of cdgl's is not necessarily a cdgl. In this section we will prove that, using the new cdgl $\mathcal{D}\mathrm{er} L$, the twisted products that will appear in the following chapters are complete.

Fix $L = (\widehat{\mathbb{L}}(V), \partial)$ a connected minimal free cdgl with V finite and filtered by a degree-respectful filtration (see Remark 6.8) and C a cdgc. Recall from Proposition 5.1 that $\operatorname{Hom}(C, L)$ and $\operatorname{Hom}(\overline{C}, L)$ are cdgl's. We can replace the usual filtration $\{\widehat{\mathbb{L}}^{\geq n}(V)\}_{n\geq 1}$ by the new one $\{F^n\}_{n\geq 1}$ in the filtration of $\operatorname{Hom}(C, L)$ and $\operatorname{Hom}(\overline{C}, L)$.

Consider the twisted product

$$\mathrm{Hom}(C,L)\widetilde{\times}\mathcal{D}\mathrm{er}L \tag{6.2}$$

where $\mathrm{Hom}(C,L)$ and $\mathcal{D}\mathrm{er}L$ are sub dgl's and for $\theta\in\mathcal{D}\mathrm{er}L$ and $\varphi\in\mathrm{Hom}(C,L)$

$$[\theta,\varphi]=\theta\circ\varphi\colon C\to L.$$

Proposition 6.11. *The twisted product* $\mathrm{Hom}(C,L)\widetilde{\times}\mathcal{D}\mathrm{er}L$ *is a dgl. Furthermore it is complete with respect to the filtration*

$$\mathcal{J}^n=\mathrm{Hom}(C,F^n)\times\mathcal{F}^n$$

for $n\geq 1$*.*

Proof. A direct calculation shows that this definition of Lie bracket makes $\mathrm{Hom}(C,L)\widetilde{\times}\mathcal{D}\mathrm{er}L$ a dgl. Note that if $\varphi\in\mathrm{Hom}(C,F^n)$ and $\theta\in\mathcal{F}^m$ then

$$[\theta,\varphi]=\theta\circ\varphi\in\mathrm{Hom}(C,F^{n+m}).$$

Then, since both $\{\mathrm{Hom}(C,F^n)\}_{n\geq 1}$ and $\{\mathcal{F}^n\}_{n\geq1}$ are filtrations, we deduce that $[\mathcal{J}^n,\mathcal{J}^m]\subset\mathcal{J}^{n+m}$ for all $n,m\geq 1$. In addition, since the differential does not mix terms in the product, $D\mathcal{J}^n\subset\mathcal{J}^n$ for all $n\geq 1$, so it is a filtration.

Finally, the natural map

$$\mathrm{Hom}(C,L)\widetilde{\times}\mathcal{D}\mathrm{er}L\to\varprojlim_n(\mathrm{Hom}(C,L)\widetilde{\times}\mathcal{D}\mathrm{er}L)/\mathcal{J}^n$$

is bijective, since, as vector spaces, we have isomorphisms

$$(\mathrm{Hom}(C,L)\times\mathcal{D}\mathrm{er}L)/\mathcal{J}^n\cong\mathrm{Hom}(C,L)/\mathrm{Hom}(C,F^n)\times\mathcal{D}\mathrm{er}L/\mathcal{F}^n$$

for each $n\geq 1$. Therefore, the natural map is a cdgl isomorphism and we conclude that the twisted product is complete.

□

An analogous procedure shows that the twisted product

$$\mathcal{D}\mathrm{er}L\widetilde{\times}sL \tag{6.3}$$

with

$$Dsx=-s\partial x+\mathrm{ad}_x,\qquad[\theta,sx]=(-1)^{|\theta|}s\theta(x)$$

for any $x\in L$ and $\theta\in\mathcal{D}\mathrm{er}L$ is a cdgl with respect to the filtration $\{\mathcal{F}^n\times sF^n\}_{n\geq1}$.

And similarly

$$L\widetilde{\times}\mathcal{D}\mathrm{er}L \tag{6.4}$$

with L and $\mathcal{D}\mathrm{er}L$ sub dgl's and

$$[\theta,x]=\theta(x)$$

for any $x\in L$ and $\theta\in\mathcal{D}\mathrm{er}L$ is a cdgl with respect to the filtration $\{F^n\times\mathcal{F}^n\}_{n\geq1}$.

CHAPTER 7

Models for classifying fibrations

Recall from §2 that for each CW-complex X there is a universal quasi-fibration sequence

$$X \to B\operatorname{aut}^*(X) \to B\operatorname{aut}(X).$$

The goal of this chapter is to obtain rational homotopical information from this sequence. In the classical setting (this means, working only with simply-connected spaces), this is a well-known object in the rational homotopy theory (see for example [51, VII.4.(4)] or [45, §9]). If L is the minimal model of a finite simply connected space then

$$L \xrightarrow{\mathrm{ad}} \widetilde{\operatorname{Der} L} \to \widetilde{\operatorname{Der} L} \widetilde{\times} sL$$

is a dgl fibration sequence which models the simply-connected cover of the universal quasi-fibration sequence above: this is

$$X \to B\operatorname{aut}^*_{\mathrm{id}}(X) \to B\operatorname{aut}_{\mathrm{id}}(X)$$

using the notation of §5. Here $\widetilde{\operatorname{Der} L} \widetilde{\times} sL$ is the restriction of the twisted product (6.3), to its 1-connected cover.

This result has been extended and generalized, for example, to the relative case [3] or to the fiberwise context [2].

However this result has two restrictions: 1) It can be applied only to simply-connected spaces X. 2) It models the 1-connected cover of the universal quasi-fibration sequence, but no information is obtained about the fundamental groups of $B\operatorname{aut}^*(X)$ or $B\operatorname{aut}(X)$. We will try to partly solve these issues with a new approach. The results given so far, have been applied to nilpotent spaces rather than simply-connected ones, so we can weaken restriction 1).

Restriction 2) can also be weakened, but only partially. We will see that there is no cdgl such that when realized is of the type of homotopy of $B\operatorname{aut}(X_{\mathbb{Q}})$ or $B\operatorname{aut}^*(X_{\mathbb{Q}})$.

Nevertheless given some class of groups $\mathcal{H} \subset \mathcal{E}(X_{\mathbb{Q}})$ we will be able to model the universal $\mathcal{H}$-quasi-fibration sequence (see §2):

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}}(X_{\mathbb{Q}}).$$

Through this chapter, and in order to simplify the notation, the term *fibration sequence* will mean *quasi-fibration sequence*. Since we are working in the homotopy category, this change does not affect the results.

7.1 Classifying fibrations

We start with an example that shows that, in general, $B\operatorname{aut}^*(X_{\mathbb{Q}})$ and $B\operatorname{aut}(X_{\mathbb{Q}})$ does not lie in the image of the realization functor, even for X simply connected.

Example 7.1. Consider the space $S^n_{\mathbb{Q}}$ for $n \geq 2$. Since it is simply connected we have that

$$\pi_1(B\operatorname{aut}(S^n_{\mathbb{Q}})) \cong \pi_1(B\operatorname{aut}^*(S^n_{\mathbb{Q}})) \cong \pi_0\operatorname{aut}^*(S^n_{\mathbb{Q}}) = \mathcal{E}^*(S^n_{\mathbb{Q}}).$$

Suppose that there exits a connected cdgl L such that $|\langle L\rangle| \simeq B\operatorname{aut}^*(S^n_{\mathbb{Q}})$. In particular, we have a group isomorphism

$$H_0(L) \cong \mathcal{E}^*(S^n_{\mathbb{Q}}).$$

Since $\mathbb{L}(x)$ with $|x| = n-1$ is a minimal Lie model for S^n, we can apply Remark 4.4 to deduce that

$$\mathcal{E}^*(S^n_{\mathbb{Q}}) \cong \mathcal{E}^*(\mathbb{L}(x)) \cong \mathbb{Q} \setminus \{0\} = \mathbb{Q}^*.$$

Therefore, we have a group isomorphism $\mathbb{Q}^* \cong H_0(L)$ where the group structure $H_0(L)$ is the one given by the BCH product. In particular, for any $a \in H_0(L)$ and $\lambda, \mu \in \mathbb{Q}$, we have that

$$\lambda a * \nu a = (\lambda + \nu)a$$

because $[a, a] = 0$. It is a divisible group: given $a \in H_0(L)$ there exists $b = a/2 \in H_0(L)$ such that $b * b = a$. However, there is no element $x \in \mathbb{Q}^*$ such that $x^2 = 2$ so there is a contradiction. We conclude that there was no cdgl L such that $\langle L\rangle \simeq B\operatorname{aut}^*(S^n_{\mathbb{Q}})$. An analogous procedure shows that there is no cdgl L such that $\langle L\rangle \simeq B\operatorname{aut}(S^n_{\mathbb{Q}})$.

Remark 7.2. Before stating which classifying spaces can be realized, we make the following general observation. The geometric-bar construction (see §2.1.3) is only defined for topological spaces, and gives a topological space as result. However, the Lie Models theory and results obtained so far, work in the category **sset**. We will use the Quillen equivalence given by the singular and geometric realization functors to deal with this issue. If X is a connected Kan complex, we write $B\operatorname{aut}(X)$ and $B\operatorname{aut}^*(X)$ to denote

$$\operatorname{Sing}(B\operatorname{aut}(|X|)), \qquad \operatorname{Sing}(B\operatorname{aut}^*(|X|))$$

respectively. And similarly, if $\mathcal{H}$ is a subgroup of $\mathcal{E}(X)$ or $\mathcal{E}^*(X)$ we use the isomorphisms $\mathcal{E}(X) \cong \mathcal{E}(|X|)$ to define

$$B\operatorname{aut}_{\mathcal{H}}(X), \qquad B\operatorname{aut}^*_{\mathcal{H}}(X).$$

These definitions allow a simplification of the notation, and these new Kan complexes have all the desired properties of their topological versions. They classify fibration sequences in **sset** (see in §4.3 that the Quillen equivalence is well-behaved with respect to fibration sequences) and their homotopy groups are given in the following proposition.

Proposition 7.3. *Let $\mathcal{H}$ be a subgroup of $\mathcal{E}(X)$ for X a connected Kan complex. Then*

$$\pi_1(B\,\mathrm{aut}_{\mathcal{H}}(X)) \cong \mathcal{H}, \quad \pi_n(B\,\mathrm{aut}_{\mathcal{H}}(X)) \cong \pi_{n-1}(\mathrm{map}_{\mathrm{id}}(X,X)), \ \textit{for } n \geq 2$$

and

$$\pi_1(B\,\mathrm{aut}^*_{\mathcal{H}}(X)) \cong \zeta^{-1}(\mathcal{H}), \quad \pi_n(B\,\mathrm{aut}^*_{\mathcal{H}}(X)) \cong \pi_{n-1}(\mathrm{map}^*_{\mathrm{id}}(X,X)), \ \textit{for } n \geq 2.$$

Proof. Since the homotopy groups of Y and $\mathrm{Sing}(Y)$ are isomorphic we have that

$$\pi_n(\mathrm{Sing}(B\,\mathrm{aut}_{\mathcal{H}}(|X|))) \cong \pi_n(B\,\mathrm{aut}_{\mathcal{H}}(|X|)) \cong \pi_{n-1}(\mathrm{aut}_{\mathcal{H}}(|X|))$$

from where we deduce that $\pi_1(B\,\mathrm{aut}_{\mathcal{H}}(X)) \cong \mathcal{H}$ and that

$$\pi_n(B\,\mathrm{aut}_{\mathcal{H}}(X)) \cong \pi_{n-1}\,\mathrm{map}_{\mathrm{id}}(|X|,|X|).$$

We finalize the proof invoking the weak homotopy equivalence (see [6, Theorem 2.1] and [8, §1])

$$|\mathrm{map}(X,Y)| \simeq_w \mathrm{map}(|X|,|Y|) \tag{7.1}$$

whenever Y is a Kan complex (this an immediate consequence of the Quillen equivalence between **top** and **sset** and the fact that the geometric realization commutes with products). Then, the homotopy groups of $\mathrm{map}_{\mathrm{id}}(|X|,|X|)$ and $|\mathrm{map}_{\mathrm{id}}(X,X)|$ are isomorphic. The pointed version is analogous.

□

Let X be a connected Kan complex and $\mathcal{H}$ a subgroup of $\mathcal{E}(X)$. From the previous remark, we deduce that

$$X \to B\,\mathrm{aut}^*_{\mathcal{H}}(X) \to B\,\mathrm{aut}_{\mathcal{H}}(X)$$

is a simplicial fibration sequence, which is universal with respect to simplicial fibration sequences with fiber X and such that, when realized, they are $\mathcal{H}$-fibration sequences in the sense of §2.

From now, we fix X a nilpotent Kan complex, and $\mathcal{H}$ a subgroup of $\mathcal{E}(X)$ acting nilpotently on $H_*(X)$ (recall from §1.6 the definition of nilpotent action). Then, [9, Theorem D] affirms that both $B\,\mathrm{aut}_{\mathcal{H}}(X)$ and $B\,\mathrm{aut}^*_{\mathcal{H}}(X)$ are nilpotent spaces and, in particular, the groups $\mathcal{H} \cong \pi_1 B\,\mathrm{aut}_{\mathcal{H}}(X)$ and $\zeta^{-1}(\mathcal{H}) \cong \pi_1 B\,\mathrm{aut}^*_{\mathcal{H}}(X)$ are nilpotent. Denote by

$$H_*(X) = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^q = 0$$

the finite lower central series of the action of $\mathcal{H}$ on $H_*(X)$.

Definition 7.4. Given a series as above, we define $\mathcal{K}$ as the subgroup of $\mathcal{E}(X)$ which 'stabilizes' the series. This mean:

$$\mathcal{K} = \{[f] \in \mathcal{E}(X) \mid H_*(f)\colon \Gamma^i/\Gamma^{i+1} \to \Gamma^i/\Gamma^{i+1} \text{ is the identity for all } i\}.$$

In particular, $\mathcal{H} \subset \mathcal{K} \subset \mathcal{E}(X)$ and $\zeta^{-1}(\mathcal{H}) \subset \zeta^{-1}(\mathcal{K}) \subset \mathcal{E}^*(X)$. Note that $\mathcal{K}$ acts nilpotently on $H_*(X)$ so [9, Theorem D] claims that $\mathcal{K}$ is a nilpotent group.

We can compute the rationalization of this nilpotent group. By an analogous result of [30, Theorem 3.3] but using homology instead of homotopy groups, we deduce the following characterization of the rationalization of $\mathcal{K}$.

Theorem 7.5. *The rationalization $\mathcal{K}_{\mathbb{Q}}$ of $\mathcal{K}$ is the subgroup of $\mathcal{E}(X_{\mathbb{Q}})$ which stabilizes the series*

$$H_*(X_{\mathbb{Q}}) = H_*(X)_{\mathbb{Q}} = \Gamma^0_{\mathbb{Q}} \supset \Gamma^1_{\mathbb{Q}} \supset \cdots \supset \Gamma^q_{\mathbb{Q}} = 0.$$

Then a map $[f] \in \mathcal{E}(X_{\mathbb{Q}})$ belongs to $\mathcal{K}_{\mathbb{Q}}$ if and only if $H_*(f)(\Gamma^i_{\mathbb{Q}}) \subset \Gamma^i_{\mathbb{Q}}$ for all i. Moreover, the map

$$\mathcal{K} \to \mathcal{K}_{\mathbb{Q}}, \quad [f] \mapsto [f_{\mathbb{Q}}]$$

is the rationalization.

In particular, since $\mathcal{H}$ is a subgroup of $\mathcal{K}$, the naturality and exactness of the rationalization morphism implies that $\mathcal{H}_{\mathbb{Q}}$ is a subgroup of $\mathcal{K}_{\mathbb{Q}}$ and the rationalization morphism is given by

$$\mathcal{H} \to \mathcal{H}_{\mathbb{Q}}, \quad [f] \mapsto [f_{\mathbb{Q}}].$$

Example 7.6. Consider $\mathcal{H} = \{\mathrm{id}\} \subset \mathcal{E}(X)$ the trivial subgroup for a nilpotent Kan complex X. It acts trivially (in particular, nilpotently) on $H_*(X)$, so the lower central series of the action is

$$H_*(X) = \Gamma^0 \supset \Gamma^1 = 0.$$

Thus, $\mathcal{K} \subset \mathcal{E}(X)$ is given by

$$\mathcal{K} = \{[f] \in \mathcal{E}(X) \mid H_*(f) = \mathrm{id}\colon H_*(X) \to H_*(X)\}.$$

Their rationalizations are given by

$$\mathcal{H}_{\mathbb{Q}} = \{\mathrm{id}\} \subset \mathcal{E}(X_{\mathbb{Q}}), \quad \mathcal{K}_{\mathbb{Q}} = \{[f] \in \mathcal{E}(X_{\mathbb{Q}}) \mid H_*(f) = \mathrm{id}\colon H_*(X_{\mathbb{Q}}) \to H_*(X_{\mathbb{Q}})\}.$$

Remark 7.7. We recall some useful facts about the rationalization of the mapping spaces. Firstly, by the weak homotopy equivalence (7.1), the following results are valid for simplicial sets and for CW-complexes. Let X be a finite CW-complex and Y a nilpotent space, then [23, Theorem 2.5] affirms that $\mathrm{map}_f(X,Y)$ and $\mathrm{map}^*_f(X,Y)$ are nilpotent spaces, where f is an arbitrary map from X to Y (pointed in the second case).

If $\mu_Y\colon Y \to Y_{\mathbb{Q}}$ is the rationalization, then [23, Theorem 3.11] establishes that

$$(\mu_Y)_*\colon \mathrm{map}_f(X,Y) \to \mathrm{map}_{\mu_Y\circ f}(X,Y_{\mathbb{Q}}), \quad (\mu_Y)_*\colon \mathrm{map}^*_f(X,Y) \to \mathrm{map}^*_{\mu_Y\circ f}(X,Y_{\mathbb{Q}})$$

are rationalizations. Next, [33, Prop 4.2] for the pointed case, and the nilpotent generalization of [46, Theorem 2.3] for the non-pointed case, claim that there are weak homotopy equivalences

$$(\mu_X)^*\colon \mathrm{map}(X_{\mathbb{Q}},Y_{\mathbb{Q}}) \to \mathrm{map}(X,Y_{\mathbb{Q}}), \quad (\mu_X)^*\colon \mathrm{map}^*(X_{\mathbb{Q}},Y_{\mathbb{Q}}) \to \mathrm{map}^*(X,Y_{\mathbb{Q}}).$$

Finally, using that $\mu_Y \circ f = f_{\mathbb{Q}} \circ \mu_X$, we conclude that

$$(-)_{\mathbb{Q}} \colon \operatorname{map}_f(X,Y) \to \operatorname{map}_{f_{\mathbb{Q}}}(X_{\mathbb{Q}}, Y_{\mathbb{Q}})$$

given by $g \mapsto g_{\mathbb{Q}}$ is, up to weak homotopy equivalence, a rationalization. And similarly for the pointed case.

Applying the previous remark, we get that the maps of monoids,

$$\operatorname{aut}_{\mathcal{H}}(X) \to \operatorname{aut}_{\mathcal{H}}(X_{\mathbb{Q}}), \quad \operatorname{aut}^*_{\mathcal{H}}(X) \to \operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}),$$

where $g \mapsto g_{\mathbb{Q}}$ are, up to weak equivalence, the componentwise rationalizations. Consequently, applying the functor $B(-)$, these maps induce the following commutative diagram

$$\begin{tikzcd}
\operatorname{aut}^*_{\mathcal{H}}(X) \arrow[r] \arrow[d] & \operatorname{aut}_{\mathcal{H}}(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & B\operatorname{aut}^*_{\mathcal{H}}(X) \arrow[r] \arrow[d] & B\operatorname{aut}_{\mathcal{H}}(X) \arrow[d] \\
\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r] & \operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r] & X_{\mathbb{Q}} \arrow[r] & B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r] & B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})
\end{tikzcd} \tag{7.2}$$

where the first three vertical maps are rationalizations and on the right hand side there are the universal $\mathcal{H}-$ and $\mathcal{H}_{\mathbb{Q}}-$fibrations respectively (see §2.2.1). By Proposition 7.3, both spaces $B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$ and $B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$ are rational spaces, and applying the five lemma to the diagram, we conclude that the right vertical maps are weak rational equivalences. Therefore, we have proved the following proposition.

Proposition 7.8. *The vertical maps in (7.2) are rationalizations. In other words, the rationalization of the universal $\mathcal{H}$-fibration sequence*

$$X \to B\operatorname{aut}^*_{\mathcal{H}}(X) \to B\operatorname{aut}_{\mathcal{H}}(X)$$

is given by

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

As being formed by rational spaces, this fibration sequence can be realized by a cdgl fibration sequence. Finding these cdgl's is the goal of the rest of the chapter.

7.2 Modeling the universal $\mathcal{H}$-fibration sequence

Fix X a nilpotent finite complex, $L = (\widehat{\mathbb{L}}(V), \partial)$ its minimal Lie model and $\mathcal{H} \subset \mathcal{E}(X)$ a subgroup acting nilpotently on $H_*(X)$. Then $\mathcal{H}_{\mathbb{Q}} \subset \mathcal{E}(X_{\mathbb{Q}})$ and $\zeta^{-1}(\mathcal{H}_{\mathbb{Q}}) \subset \mathcal{E}^*(X)$.

Recall from Remark 4.4 that there is a isomorphism of groups

$$\mathcal{E}^*(X_{\mathbb{Q}}) \cong \mathcal{E}^*(L).$$

Under this isomorphism, the subgroup $\zeta^{-1}(\mathcal{H}_{\mathbb{Q}})$ becomes $\hbar$, a subgroup of $\mathcal{E}(L)$. By Remark 4.11 $H_0(L)$ acts on $\hbar$ by the fundamental action and

$$\hbar / H_0(L) \cong \zeta^{-1}(\mathcal{H}_{\mathbb{Q}})/\pi_1(X_{\mathbb{Q}}) \cong \mathcal{H}_{\mathbb{Q}}.$$

Recall that in a connected cdgl the quasi-isomorphisms are isomorphisms [14, Proposition 3.20], so $\operatorname{aut}(L)$ is not only a monoid but a group. Define

$$\operatorname{aut}_{\mathcal{h}}(L) = \{\varphi \in \operatorname{aut}(L) \mid [\varphi] \in \mathcal{h}\}$$

which is clearly a subgroup of $\operatorname{aut}(L)$.

Analogously, consider $\mathcal{K} \subset \mathcal{E}(X)$ the subgroup that stabilizes the lower central series of the action of $\mathcal{H}$ on $H_*(X)$. Then, $\zeta^{-1}(\mathcal{K}_{\mathbb{Q}})$ corresponds with a subgroup $\mathcal{k} \subset \mathcal{E}(L)$ via the isomorphism above. And similarly define

$$\operatorname{aut}_{\mathcal{k}}(L) = \{\varphi \in \operatorname{aut}(L) \mid [\varphi] \in \mathcal{k}\}.$$

We now construct derivation Lie algebras $\operatorname{Der}^{\mathcal{h}} L$ and $\operatorname{Der}^{\mathcal{k}} L$ associated to these subgroups of $\mathcal{E}(L)$. First, note that via the isomorphism $sV \cong \tilde{H}_*(X_{\mathbb{Q}})$ (see §1.8), the lower central series of the action of $\mathcal{H}_{\mathbb{Q}}$ on $H_*(X_{\mathbb{Q}})$

$$H_*(X_{\mathbb{Q}}) = \Gamma^0_{\mathbb{Q}} \supset \Gamma^1_{\mathbb{Q}} \supset \dots \Gamma^q_{\mathbb{Q}} = 0$$

corresponds to a finite filtration of V

$$V = V^0 \supset V^1 \supset \cdots \supset V^q = 0.$$

We define $\operatorname{Der}^{\mathcal{k}} L$ as $\mathcal{D}\mathrm{er} L$ (see Definition 6.7) for this filtration of V. Use Proposition 6.10 to compute the exponential of $\operatorname{Der}^{\mathcal{k}}_0 L$:

$$e^{\operatorname{Der}^{\mathcal{k}}_0 L} = \{\varphi \in \operatorname{aut}(L) \mid (\varphi_* - \mathrm{id}_V)(V^i) \subset V^{i+1} \text{ for all } i\}.$$

The natural isomorphism $sV \cong \tilde{H}_*(X_{\mathbb{Q}})$ implies that for $\varphi \in \operatorname{aut}(L)$,

$$(\varphi_* - \mathrm{id}_V)(V^i) \subset V^{i+1} \Leftrightarrow (H_*\langle\varphi\rangle - \mathrm{id})(\Gamma^i) \subset \Gamma^{i+1}$$

and this last condition for all i is equivalent to say that

$$H_*\langle\varphi\rangle \colon \Gamma^i/\Gamma^{i+1} \to \Gamma^i/\Gamma^{i+1}$$

is the identity for all i. Therefore, we have proved that

$$e^{\operatorname{Der}^{\mathcal{k}}_0 L} = \operatorname{aut}_{\mathcal{k}}(L).$$

Using this expression we can now define $\operatorname{Der}^{\mathcal{h}} L$ as the subset of $\operatorname{Der}^{\mathcal{k}} L$ defined by

$$\operatorname{Der}^{\mathcal{h}}_{\geq 1} L = \operatorname{Der}_{\geq 1} L, \qquad \operatorname{Der}^{\mathcal{h}}_0 L = \{\theta \in \mathcal{D}\mathrm{er}_0 L \text{ such that } D\theta = 0 \text{ and } e^\theta \in \operatorname{aut}_{\mathcal{h}}(L)\}.$$

Proposition 7.9. $\operatorname{Der}^{\mathcal{h}} L$ *is a sub cdgl of* $\operatorname{Der}^{\mathcal{k}} L$*.*

Proof. We first prove that this construction is a dgl. This amounts to prove that the sum, the scalar multiplication, the differential and the Lie bracket of elements of $\operatorname{Der}^{\mathcal{h}} L$ lie in $\operatorname{Der}^{\mathcal{h}} L$. By Theorem 3.7 the exponential of the differential of a degree 1 derivation lies in $\operatorname{aut}_1(L)$, which is a subgroup of $\operatorname{aut}_{\mathfrak{h}}(L)$. We conclude that in positive degree, all the operations are well-defined.

We have to see that $\operatorname{Der}_0^{\mathfrak{h}} L$ is a sub dgl of $\operatorname{Der}_0^{\mathfrak{k}} L$. We have seen that $\exp\colon \operatorname{Der}_0^{\mathfrak{k}} L \to \operatorname{aut}_{\mathfrak{k}}(L)$ is a bijection. So restricting it to the preimage of $\operatorname{aut}_{\mathfrak{h}}(L)$ we get that $\exp\colon \operatorname{Der}_0^{\mathfrak{h}} L \to \operatorname{aut}_{\mathfrak{h}}(L)$ is a bijection. Furthermore equation (1.2) affirms that it is a group isomorphism, where the product in $\operatorname{Der}_0^{\mathfrak{h}} L$ is given by the BCH product and in $\operatorname{aut}_{\mathfrak{h}}(L)$ by the composition.

The group $\mathfrak{h}$ is isomorphic to $\zeta^{-1}(\mathcal{H}_{\mathbb{Q}}) = (\zeta^{-1}(\mathcal{H}))_{\mathbb{Q}}$ so it is 0-local and we have seen in §7.1 that $\zeta^{-1}(\mathcal{H})$ is nilpotent so, we can apply Theorem 3.9 to conclude that $\operatorname{aut}_h(L)$ is a Malcev complete group, and so is $\operatorname{Der}_0^{\mathfrak{h}}(L)$ with the BCH product. Using the isomorphism between Malcev complete groups and complete ungraded Lie algebras (see §3.1) we conclude that the sum, multiplication and Lie brackets of elements in $\operatorname{Der}_0^{\mathfrak{h}} L$ are well defined and they agree with that operations in $\operatorname{Der}_0^{\mathfrak{k}} L$.

Therefore $\operatorname{Der}^{\mathfrak{h}} L$ is a sub dgl of the complete dgl $\operatorname{Der}^{\mathfrak{k}} L$, so we conclude that $\operatorname{Der}^{\mathfrak{h}} L$ is complete with respect to the induced filtration.

□

Given $x \in L_0$, ad_x is an element of $\mathcal{D}\mathrm{er}_0 L$ since it increases the filtration length. The proposition below shows that it also lies in $\operatorname{Der}_0^{\mathfrak{h}} L$.

Proposition 7.10. *The image of the map*

$$\operatorname{ad}\colon L \to \mathcal{D}\mathrm{er} L$$

lies in the sub cdgl $\operatorname{Der}^{\mathfrak{h}} L$.

Proof. At positive degree, the statement is trivial. For $x \in L_0$, let's see that $[e^{\operatorname{ad}_x}]$ is an element in $\mathfrak{h}$. Recall from §4.2 that this automorphism is the image of x under the holonomy action

$$H_0(L) \to \mathcal{E}(L).$$

But, by the definition of $\mathfrak{h}$, it is closed under the action of $H_0(L)$. So we conclude that $[e^{\operatorname{ad}_x}]$ is an element in $\mathfrak{h}$, which implies that $\operatorname{ad}_x \in \operatorname{Der}_0^{\mathfrak{h}} L$.

□

Consider the twisted product $\mathcal{D}\mathrm{er} L \widetilde{\times} sL$ defined at (6.3), then $\operatorname{Der}^{\mathfrak{h}} L \widetilde{\times} sL$ is a sub cdgl of $\mathcal{D}\mathrm{er} L \widetilde{\times} sL$. Thus, we have a well defined cdgl sequence

$$L \xrightarrow{\operatorname{ad}} \operatorname{Der}^{\mathfrak{h}} L \to \operatorname{Der}^{\mathfrak{h}} L \widetilde{\times} sL, \tag{7.3}$$

where the right map is the inclusion in the twisted product.

This is the central object of this section since, as we will see, its realization is going to be the universal fibration sequence of Proposition 7.8. Let's prove some results about this cdgl sequence.

Proposition 7.11. *The realization of (7.3) is a fibration sequence.*

Proof. The proof is analogous to the one given in [51, §VII.4(1)] for the simply connected case. We construct an auxiliary cdgl fibration sequence

$$L \to L\widetilde{\times}(\operatorname{Der}^{\hbar} L\widetilde{\times}sL) \to \operatorname{Der}^{\hbar} L\widetilde{\times}sL, \tag{7.4}$$

where the Lie bracket in the middle twisted product is given by

$$[\theta, x] = \theta(x), \quad [sy, x] = 0, \quad [\theta, sy] = (-1)^{|\theta|}s\theta(y),$$

and the differential $\widetilde{D}$ is given by

$$\widetilde{D}x = \partial x, \quad \widetilde{D}\theta = \theta, \quad \widetilde{D}sy = -y - s\partial y + \mathrm{ad}_y$$

for any $x \in L, sy \in sL, \theta \in \operatorname{Der}^{\hbar} L$. It can be checked that these operations define a dgl and by the techniques of §6.2 we can make it a cdgl.

Consider the maps

$$\iota\colon \operatorname{Der}^{\hbar} L \to L\widetilde{\times}(\operatorname{Der}^{\hbar} L\widetilde{\times}sL)$$

given by the inclusion and

$$\rho\colon L\widetilde{\times}(\operatorname{Der}^{\hbar} L\widetilde{\times}sL) \to \operatorname{Der}^{\hbar} L, \quad x \mapsto \mathrm{ad}_x, \quad sy \mapsto 0, \quad \theta \mapsto \theta$$

for $x \in L, sy \in sL, \theta \in \operatorname{Der}^{\hbar} L$.

They can be checked to be cdgl morphisms and $\rho \circ \iota$ is the identity on $\operatorname{Der}^{\hbar} L$. Furthermore, $\iota \circ \rho$ is homotopic to the identity (see [51, VII.4.(7)] for an explicit homotopy). Consider the following diagram

$$\begin{tikzcd}
L \arrow[r, "\mathrm{ad}"] \arrow[d, equal] & \operatorname{Der}^{\hbar} L \arrow[r] \arrow[d, shift left, "\iota"] & \operatorname{Der}^{\hbar} L\widetilde{\times}sL \arrow[d, equal] \\
L \arrow[r] & L\widetilde{\times}(\operatorname{Der}^{\hbar} L\widetilde{\times}sL) \arrow[r] \arrow[u, shift left, "\rho"] & \operatorname{Der}^{\hbar} L\widetilde{\times}sL
\end{tikzcd},$$

where the two squares are commutative and the central double arrow gives the retraction defined above. Therefore if we remove the map ρ, we get a homotopy commutative diagram (which is not strictly commutative in general), so realizing it we get

$$\begin{tikzcd}
\langle L\rangle \arrow[r, "\langle\mathrm{ad}\rangle"] \arrow[d, equal] & \langle\operatorname{Der}^{\hbar} L\rangle \arrow[r] \arrow[d, "\langle\iota\rangle"] & \langle\operatorname{Der}^{\hbar} L\widetilde{\times}sL\rangle \arrow[d, equal] \\
\langle L\rangle \arrow[r] & \langle L\widetilde{\times}(\operatorname{Der}^{\hbar} L\widetilde{\times}sL)\rangle \arrow[r] & \langle\operatorname{Der}^{\hbar} L\widetilde{\times}sL\rangle
\end{tikzcd},$$

which is commutative up to homotopy. Since the bottom row is the realization of a cdgl fibration sequence, it is a fibration sequence, which exhibits the realization of (7.3) as equivalent to a fibration sequence.

□

We can use the auxiliary cdgl fibration sequence (7.4) of the previous proof to compute the holonomy action of the realization of (7.3). Recall from §4.4 that the holonomy action is constructed by the exponential of degree 0 elements of the base. In (7.4), if $x \in L$ and $sy \in (sL)_0$, then $[sy, x] = 0$, so

$$e^{\mathrm{ad}_{sy}} = \mathrm{id}_L \colon L \to L.$$

On the other hand, if we take $\theta \in \operatorname{Der}_0^{\hbar} L$, then $[\theta, x] = \theta(x)$, thus

$$e^{\mathrm{ad}_\theta} = e^\theta \colon L \to L.$$

By definition of $\operatorname{Der}_0^{\hbar} L$, this is an automorphism in $\operatorname{aut}_{\hbar}(L)$. Finally use the following results:

- the realization of cdgl sequences (7.3) and (7.4) are equivalent (Proposition 7.12),
- the realization of the holonomy action is the holonomy action of the realization (Theorem 4.17),
- being $\mathcal{H}$-fibration sequence is invariant under equivalence of fibrations sequences (Proposition 2.9)

to conclude the following proposition.

Proposition 7.12. *The realization of the cdgl sequence (7.3) is an $\mathcal{H}_{\mathbb{Q}}$-fibration sequence.*

Proof. □

We can now present the central theorem.

Theorem 7.13. *Let L be the minimal Lie model of a nilpotent complex X and $\mathcal{H} \subset \mathcal{E}(X)$ a subgroup acting nilpotently on the homology of X. Then the realization of the cdgl sequence (7.3)*

$$L \xrightarrow{\mathrm{ad}} \operatorname{Der}^{\hbar} L \to \operatorname{Der}^{\hbar} L \widetilde{\times} sL$$

is homotopy equivalent to the universal $\mathcal{H}_{\mathbb{Q}}$-fibration sequence

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

The strategy of the proof is to extend both sequences, attach them via homotopy equivalences and apply the five lemma recursively to obtain weak homotopy equivalences. So the first step is to study an extension on the left of the cdgl sequence (7.3).

For C a cdgc, the twisted product the twisted product

$$\operatorname{Hom}(C, L) \widetilde{\times} \mathcal{D}\mathrm{er} L$$

is a cdgl (see §6.2). Then, we can consider its sub cdgl

$$\operatorname{Hom}(C, L) \widetilde{\times} \operatorname{Der}^{\hbar} L.$$

where both terms are sub dgl and the twisted Lie bracket is given by

$$[\theta,\varphi]=\theta\circ\varphi\colon C\to L.$$

Take $C=\mathscr{C}(L)$ and consider the cdgl fibration sequence

$$\operatorname{Hom}(\mathscr{C}(L),L)\to\operatorname{Hom}(\mathscr{C}(L),L)\,\widetilde{\times}\,\operatorname{Der}^{\hbar}L\to\operatorname{Der}^{\hbar}L. \tag{7.5}$$

Note that the first and second cdgl's in this sequence are not connected in general, so they may have several connected components. We want to fix a Maurer-Cartan element in $\operatorname{Hom}(\mathscr{C}(L),L)\,\widetilde{\times}\,\operatorname{Der}^{\hbar}L$. For it, recall from §5.2 that there is a Maurer-Cartan element in $\operatorname{Hom}(\mathscr{C}(L),L)$ (and, therefore, a Maurer-Cartan element in the twisted product by Remark 1.4) given by

$$q\colon\mathscr{C}(L)\to L,\quad q(sx)=-x,\quad q(1)=0,\quad q(\wedge^{\geq 2}sL)=0.$$

Lemma 7.14. *There is a quasi-isomorphism of connected cdgl's*

$$L\simeq(\operatorname{Hom}(\mathscr{C}(L),L)\,\widetilde{\times}\,\operatorname{Der}^{\hbar}L)^{q}.$$

Proof. We check that the map

$$\gamma\colon L\to(\operatorname{Hom}(\mathscr{C}(L),L)\,\widetilde{\times}\,\operatorname{Der}^{\hbar}L,D_q),\quad x\mapsto-\varphi_x+\operatorname{ad}_x,$$

is a cdgl morphism, where $\varphi_x\colon\mathscr{C}(L)\to L$ is the map sending 1 to x and the elements in $\wedge^{\geq 1}sL$ to zero. Commuting with the differential means that:

$$\varphi_{\partial x}+\operatorname{ad}_{\partial x}=-D\varphi_x+D\operatorname{ad}_x-[q,\varphi_x]+[q,\operatorname{ad}_x].$$

This equality is true since $\operatorname{ad}_{\partial x}=D\operatorname{ad}_x$ and the elements in $\operatorname{Hom}(\mathscr{C}(L),L)$ act as

$$\varphi_{\partial x}(1)=\partial x,\quad D\varphi_x(1)=\partial x,\quad [q,\varphi_x](sy)=(-1)^{|x|}[x,y],\quad [q,\operatorname{ad}_x](sy)=(-1)^{|x|}[x,y]$$

and zero on any other element. On the other hand, commuting with the Lie bracket means that

$$-\operatorname{ad}_x\circ\varphi_y+[\varphi_x,\varphi_y]+(-1)^{|x||y|}\operatorname{ad}_y\circ\varphi_x+[\operatorname{ad}_x,\operatorname{ad}_y]=-\varphi_{[x,y]}+\operatorname{ad}_{[x,y]}$$

which is true, since $[\operatorname{ad}_x,\operatorname{ad}_y]=\operatorname{ad}_{[x,y]}$ and the rest of the elements act trivially on any element different of 1 and 1 is sent to $-[x,y]$.

In particular, we can restrict γ to the connected component

$$\gamma\colon L\to(\operatorname{Hom}(\mathscr{C}(L),L)\,\widetilde{\times}\,\operatorname{Der}^{\hbar}L,D_q)^{0}.$$

This cdgl morphism will turn out to be a quasi-isomorphism. Recall from Proposition 5.2 that we can write $\operatorname{Hom}(\mathscr{C}(L),L)$ as $\operatorname{Hom}(\overline{\mathscr{C}}(L),L)\,\widetilde{\times}\,L$ with $[x,\varphi]=\operatorname{ad}_x\circ\varphi$. Under this isomorphism, the differential on $(\operatorname{Hom}(\overline{\mathscr{C}}(L),L)\,\widetilde{\times}\,L)\,\widetilde{\times}\,\operatorname{Der}^{\hbar}L$ becomes:

$$D_q\varphi=D\varphi+[q,\varphi],\quad D_qx=\partial x-(-1)^{|x|}\operatorname{ad}_x\circ q,\quad D_q\theta=D\theta-(-1)^{|\theta|}\theta\circ q$$

for $\varphi \in \operatorname{Hom}(\overline{\mathscr{C}}(L), L)), x \in L$ and $\theta \in \operatorname{Der}^{\hbar} L$.

Forget about the Lie brackets in the following spaces, and just consider them as differential graded vector spaces. Recall from §5.3 that

$$\Gamma\colon (s^{-1}\operatorname{Der}_{\alpha_L}(\mathscr{L}\mathscr{C}(L), L'), D) \to (\operatorname{Hom}(\overline{\mathscr{C}}(L), L), D_q)$$

is an isomorphism (the chosen morphism $L \to L$ is the identity). And from §5.3.1 that

$$\alpha_L^*\colon \operatorname{Der} L \to \operatorname{Der}_{\alpha_L}(\mathscr{L}\mathscr{C}(L), L)$$

is a quasi-isomorphism and so is the desuspension $s^{-1}\alpha_L^*$. Composing both maps we get a map

$$\Gamma \circ s^{-1}\alpha_L^*\colon s^{-1}\operatorname{Der}(L) \to (\operatorname{Hom}(\overline{\mathscr{C}}(L), L), D_q)$$

which is a quasi-isomorphism and which maps a derivation $s^{-1}\eta$ to $(-1)^{|\eta|}\eta \circ q$. Consider the map

$$\phi\colon ((s^{-1}\operatorname{Der} L \tilde{\times} L) \tilde{\times} \operatorname{Der}^{\hbar} L, \tilde{D}) \to ((\operatorname{Hom}(\mathscr{C}(L), L) \tilde{\times} L) \tilde{\times} \operatorname{Der}^{\hbar} L, D_q)$$

defined as $\Gamma \circ s^{-1}\alpha_L^*$ on $s^{-1}\operatorname{Der} L$ and as the identity on L and on $\operatorname{Der}^{\hbar} L$. On the right hand side, we have the twisted product of cdgl's constructed above. On the left hand side, we have a graded vector space and we can define a differential as

$$\tilde{D}s^{-1}\eta = -s^{-1}D\eta, \quad \tilde{D}x = \partial x - s^{-1}\operatorname{ad}_x, \quad \tilde{D}\theta = D\theta - s^{-1}\theta$$

for $s^{-1}\eta \in s^{-1}\operatorname{Der} L, x \in L$ and $\theta \in \operatorname{Der}^{\hbar} L$. It is a straightforward calculation to show that $\tilde{D}^2 = 0$ and that ϕ commutes with the differentials. Furthermore, since ϕ consists of a quasi-isomorphism extended by the identity along the twisted products, ϕ is also a quasi-isomorphism of differential graded vector spaces.

Therefore, our initial map γ factors through ϕ:

$$\begin{tikzcd}
L \arrow[d, "\gamma'"] \arrow[dr, "\gamma"] & \\
((s^{-1}\operatorname{Der} L \tilde{\times} L) \tilde{\times} \operatorname{Der}^{\hbar} L, \tilde{D}) \arrow[r, "\phi"] & ((\operatorname{Hom}(\overline{\mathscr{C}}(L), L) \tilde{\times} L) \tilde{\times} \operatorname{Der}^{\hbar} L, D_q)
\end{tikzcd}$$

where we define $\gamma'(x) = -x + \operatorname{ad}_x$. It is easy to check that it is a commutative diagram of differential graded vector spaces. We want to prove that γ' induces an isomorphism on the homology at non-negative degrees.

Suppose that a cycle $x \in L$ is such that $H_*(\gamma')[x] = 0$. Then there is an element

$$s^{-1}\eta + y + \theta \in (s^{-1}\operatorname{Der} L \tilde{\times} L) \tilde{\times} \operatorname{Der}^{\hbar} L \text{ such that } \tilde{D}(\eta + y + \theta) = \gamma'(x) = -x + \operatorname{ad}_x .$$

Comparing both sides of the equation we deduce that $\partial y = x$, so $[x] = 0 \in H_*(L)$ and we conclude that $H_*(\gamma')$ is injective. Conversely, we check that it is surjective at non-negative degrees. Take an arbitrary cycle

$$s^{-1}\eta + y + \theta \in ((s^{-1}\operatorname{Der} L \tilde{\times} L) \tilde{\times} \operatorname{Der}^{\hbar} L)_n$$

for $n \geq 0$. From the condition $\tilde{D}(s^{-1}\eta + y + \theta) = 0$ we deduce the following identities

$$\partial y = 0, \qquad D\theta = 0, \qquad D\eta + ad_y + \theta = 0.$$

Since η is a derivation of degree $n+1 > 0$ it belongs to $\mathrm{Der}^{\hbar} L$. Its differential is

$$\tilde{D}\eta = D\eta - s^{-1}\eta = -\operatorname{ad}_y - \theta - s^{-1}\eta,$$

thus in the homology $s^{-1}\eta + y + \theta$ is equivalent to

$$s^{-1}\eta + y + \theta + \tilde{D}\eta = y - \operatorname{ad}_y = \gamma'(-y).$$

We conclude that γ' is a quasi-isomorphism when restricted to the connected component at zero. The same applies to ϕ, so we have that the cdgl morphism

$$\gamma\colon L \to ((\mathrm{Hom}(\overline{\mathscr{C}}(L), L) \widetilde{\times} L) \widetilde{\times} \mathrm{Der}^{\hbar} L, D_q)^0$$

induces an isomorphism on the homology; this means that γ is a cdgl quasi-isomorphism. □

Remark 7.15. Though is not necessary for the previous proof, we can find a retraction of γ. Consider the projection

$$\tau\colon (\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^{\hbar} L)^q \cong ((\mathrm{Hom}(\overline{\mathscr{C}}(L), L) \widetilde{\times} L) \widetilde{\times} \mathrm{Der}^{\hbar} L)^q \to L.$$

Then $\tau \circ \gamma = \mathrm{id}_L$ and, therefore, τ is also a quasi-isomorphism.

In the cdgl fibration sequence (7.5) we have identified one of the component of the central cdgl. If we realize this cdgl fibration and focus on this component we will obtain a new fibration sequence

$$F \to \langle (\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^{\hbar} L)^q \rangle \to \langle \mathrm{Der}^{\hbar} L \rangle \tag{7.6}$$

where the fiber F consists of some connected components of the former fiber $\langle \mathrm{Hom}(\mathscr{C}(L), L) \rangle$. The lemma below identifies these connected components.

Lemma 7.16. *The connected components of F are in bijection with $\hbar/H_0(L)$, this means*

$$F = \bigsqcup_{[\varphi] \in \hbar/H_0(L)} \langle \mathrm{Hom}(\mathscr{C}(L), L)^{\bar{\varphi}} \rangle.$$

Proof. We have to identify which Maurer-Cartan elements of $\mathrm{Hom}(\mathscr{C}(L), L)$ are gauge related with q, when considered in $\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^{\hbar} L$. Recall from Proposition 5.6 that there is a bijection

$$[\![L, L]\!]/H_0(L) \to \widetilde{\mathrm{MC}}(\mathrm{Hom}(\mathscr{C}(L), L)), \quad \varphi \mapsto \bar{\varphi} = \varphi \circ q.$$

We need to compute the gauge action in $\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^{\hbar} L$. If $\theta \in \mathrm{Der}_0^{\hbar} L$, then $D\theta = 0$ and

$$\theta \mathscr{G} \psi = e^{\operatorname{ad}_\theta}(\psi) = e^{\theta} \circ \psi$$

for any $\psi \in \mathrm{MC}(\mathrm{Hom}(\mathscr{C}(L), L))$. Since $e^\theta \in \mathrm{aut}_\hbar(L)$ it is clear that $\theta\mathscr{G}-$ sends elements in $\hbar/H_0(L)$ to elements in $\hbar/H_0(L)$.

On the other hand, the gauge action by an element of $\mathrm{Hom}_0(\mathscr{C}(L), L)$ is the same that when is performed in the twisted product or in $\mathrm{Hom}(\mathscr{C}(L), L)$ (see Remark 1.4). This means that the gauge action by $\mathrm{Hom}_0(\mathscr{C}(L), L)$ relates elements which were already the same in $\widetilde{\mathrm{MC}}(\mathrm{Hom}(\mathscr{C}(L), L))$. In particular, by the bijection above, it sends a class in $\hbar/H_0(L)$ to the same class.

Therefore the gauge action of any BCH product of elements of $\mathrm{Hom}_0(\mathscr{C}(L), L)$ and $\mathrm{Der}_0^\hbar L$ will preserve the subset $\hbar/H_0(L)$. By the Malcev equivalence (see §3), a generic element in $(\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^\hbar L)_0$ could be written as such product. So we have proved that, for any gauge related element with q in $\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^\hbar L$, its class is in $\hbar/H_0(L)$.

Conversely, if $\varphi \in \mathrm{aut}_\hbar(L)$, then just take $\theta = \log\varphi \in \mathrm{Der}_0^\hbar L$ and one gets $\theta\mathscr{G}q = \bar{\varphi}$. Therefore we have proved that $\bar{\varphi}$ is gauge related with q if and only if $[\varphi] \in \hbar/H_0(L)$. □

The next step, before proving the theorem is to reformulate the fibration sequence (7.6) using the quasi-isomorphisms γ and τ. Consider the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[dr, bend right] & \langle (\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^\hbar L)^q \rangle \arrow[r] \arrow[d, shift right, "\langle\tau\rangle"'] & \langle \mathrm{Der}^\hbar L \rangle \\
& \langle L \rangle \arrow[u, shift right, "\langle\gamma\rangle"'] \arrow[ur, bend right] &
\end{tikzcd}$$

where the vertical arrows are homotopy equivalences and we want to identify the curved arrows, given by the compositions. Clearly the map γ composed with the projection gives $\mathrm{ad}\colon L \to \mathrm{Der}^\hbar L$, whose realization gives the right curved arrow. On the other hand, for each $\varphi \in \hbar/H_0(L)$, we have a commutative diagram of cdgl's:

$$\begin{tikzcd}
(\mathrm{Hom}(\mathscr{C}(L), L))^{\bar{\varphi}} \arrow[r] \arrow[dr, "\mathrm{ev}_1"'] & (\mathrm{Hom}(\mathscr{C}(L), L) \widetilde{\times} \mathrm{Der}^\hbar L)^q \arrow[d, "\tau"] \\
& L
\end{tikzcd}$$

so the left curved arrows is given by $\langle \mathrm{ev}_1 \rangle$. Therefore the fibration sequence (7.6) is equivalent to

$$F \xrightarrow{\langle \mathrm{ev}_1 \rangle} \langle L \rangle \xrightarrow{\langle \mathrm{ad} \rangle} \langle \mathrm{Der}^\hbar L \rangle. \tag{7.7}$$

Finally, apply Proposition 5.7 to each component of F. Since $\langle L \rangle \simeq X_{\mathbb{Q}}$ and $\hbar/H_0(L) \cong \mathcal{H}_{\mathbb{Q}}$, we get a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\langle \mathrm{ev}_1 \rangle"] \arrow[d, "\simeq"] & \langle L \rangle \arrow[d, "\simeq"] \\
\mathrm{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r, "\mathrm{ev}"] & X_{\mathbb{Q}}
\end{tikzcd} \; . \tag{7.8}$$

We have all the ingredients for proving the main theorem.

Proof of Theorem 7.13. From Proposition 7.12 the realization of the cdgl sequence (7.3) is an $\mathcal{H}_{\mathbb{Q}}$-fibration sequence. By Theorem 2.17 it can be obtained from the universal $\mathcal{H}_{\mathbb{Q}}$-fibration sequence, which was reformulated in Theorem 2.25. Therefore there is a homotopy commutative diagram

$$\begin{tikzcd} \langle L\rangle \arrow[r, "\langle \mathrm{ad}\rangle"] \arrow[d, "\simeq"] & \langle \operatorname{Der}^{\hbar} L\rangle \arrow[r] \arrow[d] & \langle \operatorname{Der}^{\hbar} L \widetilde{\times} sL\rangle \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r] & B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \ . \end{tikzcd}$$

Now attach the diagram (7.8) to obtain

$$\begin{tikzcd} F \arrow[r, "\langle \mathrm{ev}_1\rangle"] \arrow[d, "\simeq"] & \langle L\rangle \arrow[r, "\langle \mathrm{ad}\rangle"] \arrow[d, "\simeq"] & \langle \operatorname{Der}^{\hbar} L\rangle \arrow[r] \arrow[d] & \langle \operatorname{Der}^{\hbar} L \widetilde{\times} sL\rangle \arrow[d] \\ \operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r, "\mathrm{ev}"] & X_{\mathbb{Q}} \arrow[r] & B\operatorname{aut}^*_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r] & B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}). \end{tikzcd}$$

We focus on the left side of the diagram. The first triple in the upper row is (7.7), which is a fibration sequence. On the other hand the bottom row is the result of applying the Puppe sequence at the universal fibration as it was proved in Proposition 2.26.

In the left part of the diagram, we construct the long exact sequences of homotopy groups associated to those sequences and we apply the five lemma. Since the first two vertical maps are homotopy equivalences, we deduce that the third one is a weak homotopy equivalence. Now we focus on the right part of the diagram and apply the same argument to conclude that the last vertical arrow is also a weak homotopy equivalence.

Finally, since the complex X is finite, the involved mapping spaces are of the homotopy type of a CW-complex. So the weak homotopy equivalences are homotopy equivalences.

□

Before studying some consequences and generalizations of this result, we see that we can not weaken the hypothesis of our theorem. Suppose that we take a subgroup $\mathcal{H} \subset \mathcal{E}(X_{\mathbb{Q}})$ acting nilpotently on $H_*(X_{\mathbb{Q}})$, instead of rationalizing a subgroup of $\mathcal{E}(X)$. Then $B\operatorname{aut}_{\mathcal{H}}(X_{\mathbb{Q}})$ is not, in general, of the homotopy type of the realization of a cdgl.

Example 7.17. Consider $X = S^n \vee S^n$ with $n \geq 2$, a simply connected space whose minimal model is $L = (\mathbb{L}(x,y), 0)$ with $|x| = |y| = n-1$. Let $\hbar \subset \mathcal{E}(L)$ be the subgroup generated by the automorphism

$$\varphi\colon L \xrightarrow{\cong} L, \quad x \mapsto x+y, \quad y \mapsto y.$$

This subgroup corresponds with a subgroup $\mathcal{H} \subset \mathcal{E}(X_{\mathbb{Q}}) = \mathcal{E}^*(X_{\mathbb{Q}})$ which acts nilpotently on the homology. Suppose that there exists a cdgl M such that $\langle M\rangle \simeq B\operatorname{aut}^*_{\mathcal{H}}(X_{\mathbb{Q}})$. Then $H_0(M)$ with the BCH group structure has to be isomorphic to $\pi_1(B\operatorname{aut}_{\mathcal{H}}(X_{\mathbb{Q}})) \cong \mathcal{H} \cong \hbar$ which is isomorphic to $\mathbb{Z}$. However, this is not a 0-local group, while, $H_0(M)$ is 0-local (using the multiplication by scalar), so we get a contradiction.

7.3 The pointed case

In this section we extend the results of the previous section to the pointed case. Again consider X a finite nilpotent space with $L = (\widehat{\mathbb{L}}(V), \partial)$ its minimal Lie model. Now we fix $\Pi \subset \mathcal{E}^*(X)$ a subgroup which acts nilpotently on $\pi_*(X)$. Then [9, Theorem C] affirms that $B\operatorname{aut}^*_\Pi(X)$ is a nilpotent complex and, in consequence, Π is a nilpotent group.

Using §4.1, we can associate the subgroup $\Pi \subset \mathcal{E}^*(X)$ with a subgroup $\pi \subset \mathcal{E}(L)$ isomorphic to Π. Note that, in the non-pointed case, we worked with the preimage via ζ of a subgroup of (free) classes of automorphisms, so the associated group was invariant under the action of $\pi_1(X_\mathbb{Q})$ or equivalently of $H_0(L)$. In this case, the group $\pi \subset \mathcal{E}(L)$ is not necessarily invariant under the action of $H_0(L)$, so this is an extra requirement which should be included in the hypothesis.

By [24, Theorem 2.1] if $\Pi_\mathbb{Q}$ acts nilpotently on $\pi_*(X_\mathbb{Q})$ it also does on $H_*(X_\mathbb{Q})$. Thus, as in the non-pointed case, we construct the lower central series of the action on the homology, define $K_\mathbb{Q}$ as the group which stabilizes such series and we construct an associated filtration $\{V^n\}$ of the graded vector space V.

Analogously, define

$$\operatorname{aut}_\Pi(L) = \{\varphi \in \operatorname{aut}(L) \mid [\varphi] \in \pi\}$$

and $\operatorname{Der}^\pi L \subset \mathcal{D}\mathrm{er} L$ as

$$\operatorname{Der}^\pi_{\geq 1} L = \operatorname{Der}_{\geq 1} L, \quad \operatorname{Der}^\pi_0 L = \{\theta \in \mathcal{D}\mathrm{er}_0 L \mid D\theta = 0, e^\theta \in \operatorname{aut}_\Pi(L)\}.$$

Note that $\operatorname{Der}^\pi_0 L$ with the BCH product is a subgroup of $\mathcal{D}\mathrm{er} L$, so similarly to Proposition 7.9 we deduce $\operatorname{Der}^\pi L$ is a sub cdgl of $\mathcal{D}\mathrm{er} L = \operatorname{Der}^{\mathcal{k}} L$. The map $\operatorname{ad}\colon L \to \operatorname{Der}^\pi L$ is well defined since we have imposed Π to be invariant under the action of $H_0(L)$, so an analogous version of Proposition 7.10 applies.

Recall from Theorem 2.39 that

$$X \to B(*, \operatorname{aut}^*_\Pi(X), X) \xrightarrow{p} B\operatorname{aut}^*_\Pi(X)$$

is the universal Π-pointed fibration sequence. Unfortunately, we can not reformulate the middle term, as we did in the non-pointed case: for simplicity let's write $Z = B(*, \operatorname{aut}^*_\Pi(X), X)$. Using that the rationalization of $B\operatorname{aut}^*_\Pi(X)$ is of the homotopy type of $B\operatorname{aut}^*_{\Pi_\mathbb{Q}}(X_\mathbb{Q})$ (see Proposition 7.8) a five lemma argument shows that $Z_\mathbb{Q}$ is of the homotopy type of $B(*, \operatorname{aut}^*_{\Pi_\mathbb{Q}}(X_\mathbb{Q}), X_\mathbb{Q})$. In particular, the sequence

$$X_\mathbb{Q} \to Z_\mathbb{Q} \to B\operatorname{aut}^*_{\Pi_\mathbb{Q}}(X_\mathbb{Q})$$

is universal, since it classifies pointed fibration sequences, with fiber of the homotopy type of $X_\mathbb{Q}$ and whose pointed holonomy action lies in $\Pi_\mathbb{Q}$.

The following theorem allows to model such universal fibration sequence.

Theorem 7.18. *For X a finite nilpotent complex with $L = (\widehat{\mathbb{L}}(V), \partial)$ its minimal Lie model, and $\Pi \subset \mathcal{E}^*(X)$ a subgroup which acts nilpotently on $\pi_*(X)$ and which is invariant under the action of $\pi_1(X)$, the realization of the cdgl sequence*

$$L \to L \,\tilde{\times}\, \operatorname{Der}^\pi L \to \operatorname{Der}^\pi L$$

is homotopy equivalent to the universal $\Pi_{\mathbb{Q}}$*-pointed fibration sequence*

$$X_{\mathbb{Q}} \to Z_{\mathbb{Q}} \to B\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}}).$$

The twisted product of $L \tilde{\times} \operatorname{Der}^\pi L$ is the one inherited from (6.4).

Proof. Apply Theorem 7.13 with $\mathcal{H}_{\mathbb{Q}} = \zeta(\Pi_{\mathbb{Q}}) \subset \mathcal{E}(X_{\mathbb{Q}})$ to obtain the following homotopy commutative diagram (where we use that $\zeta^{-1}\zeta(\Pi_{\mathbb{Q}}) = \Pi_{\mathbb{Q}}$ because of being invariant under the action of $\pi_1(X_{\mathbb{Q}})$):

$$\begin{tikzcd}
\langle L \rangle \arrow[r, "\langle \mathrm{ad} \rangle"] \arrow[d, "\simeq"] & \langle \operatorname{Der}^\pi L \rangle \arrow[r] \arrow[d, "\simeq"] & \langle \operatorname{Der}^\pi L \tilde{\times} sL \rangle \arrow[d, "\simeq"] \\
X_{\mathbb{Q}} \arrow[r] & B\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}}) \arrow[r] & B\operatorname{aut}_{\zeta(\Pi_{\mathbb{Q}})}(X_{\mathbb{Q}})
\end{tikzcd}.$$

Write $G = \operatorname{aut}_{\zeta(\Pi_{\mathbb{Q}})}(X_{\mathbb{Q}})$ and $G^* = \operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})$. Then the inclusion $G^* \to G$ induces a diagram

$$\begin{tikzcd}
B(*, G^*, X_{\mathbb{Q}}) \arrow[r] \arrow[d] & B(*, G, X_{\mathbb{Q}}) \arrow[d, "p"] \\
BG^* \arrow[r] & BG
\end{tikzcd}.$$

By [32, Proposition 7.8], it is a pullback diagram. Recall from §2.2.1 that $p\colon B(*, G, X_{\mathbb{Q}}) \to BG$ is homotopy equivalent to the map $BG^* \to BG$ induced by the inclusion. On the other hand we have $Z_{\mathbb{Q}} \simeq B(*, G^*, X_{\mathbb{Q}})$, so there is a homotopy pullback

$$\begin{tikzcd}
Z_{\mathbb{Q}} \arrow[r] \arrow[d] & BG^* \arrow[d] \\
BG^* \arrow[r] & BG
\end{tikzcd}$$

where both maps $BG^* \to BG$ are induced by the inclusion $G^* \to G$. The realization functor preserves homotopy limits (since it is part of Quillen pair see §1.3), so we can compute $Z_{\mathbb{Q}}$ as the realization of the cdgl homotopy pullback of

$$\begin{tikzcd}
 & \operatorname{Der}^\pi L \arrow[d] \\
\operatorname{Der}^\pi L \arrow[r] & \operatorname{Der}^\pi L \tilde{\times} sL
\end{tikzcd}$$

where both maps are inclusions. Consider the diagram

$$\begin{tikzcd}
L \tilde{\times} \operatorname{Der}^\pi L \arrow[r] \arrow[d] & (L \tilde{\times} (\operatorname{Der}^\pi L \tilde{\times} sL), \tilde{D}) \arrow[d] & \operatorname{Der}^\pi L \arrow[l, "\iota"', "\simeq"] \arrow[dl] \\
\operatorname{Der}^\pi L \arrow[r] & \operatorname{Der}^\pi L \tilde{\times} sL &
\end{tikzcd}$$

where we are using the notation of the proof of Proposition 7.9 and all the maps are inclusions or projections. The square is clearly a pullback in the category **cdgl**, so when realized we obtain the following diagram

$$
\begin{tikzcd}
 & Z_{\mathbb{Q}} \arrow[rr] \arrow[dd] & & BG^* \arrow[dd] \\
\langle L \widetilde{\times} \operatorname{Der}^{\pi} L \rangle \arrow[ur] \arrow[rr] \arrow[dd] & & \langle \operatorname{Der}^{\pi} L \rangle \arrow[ur] \arrow[dd] & \\
 & BG^* \arrow[rr] & & BG \\
\langle \operatorname{Der}^{\pi} L \rangle \arrow[ur] \arrow[rr] & & \langle \operatorname{Der}^{\pi} L \widetilde{\times} sL \rangle \arrow[ur] &
\end{tikzcd}
$$

We want to study the properties of this diagram: in what follows, all such properties are considered up to homotopy. It is commutative and the front and the back faces are pullbacks by the arguments above. The diagonal maps arriving to BG^* or to BG are homotopy equivalences because of Theorem 7.13, and the other diagonal map is also a homotopy equivalence for the uniqueness of the pullback in the homotopy category. In addition the bottom side is also a pullback by the proof of Theorem 7.13. Then, a diagram chasing argument shows that the upper side is also a pullback. Then we can extend the upper face homotopy equivalences to theirs fibers, giving the desired result.

$\square$

CHAPTER 8

Examples and applications

We finally study some consequences of the central theorems 7.13 and 7.18. Through this chapter the notations and hypothesis of §7 are assumed.

8.1 Covering of the universal fibration sequence

Take X a simply-connected finite complex, and $L = L_{>0}$ its Lie model, which is positively graded. As it was explained at the introduction of §7, the realization of the dgl fibration sequence

$$L \xrightarrow{\text{ad}} \widetilde{\operatorname{Der} L} \to \widetilde{\operatorname{Der} L} \widetilde{\times} sL$$

is homotopy equivalent to

$$X \to B\,\widetilde{\operatorname{aut}^*(X)} \to B\,\widetilde{\operatorname{aut}(X)}.$$

This result can be obtained as a particular case of Theorem 7.13. Let X be a finite nilpotent complex with $L = (\widehat{\mathbb{L}}(V), \partial)$ its minimal model. We take $\mathcal{H} = \{[\mathrm{id}]\} \subset \mathcal{E}(X)$ the trivial subgroup, which clearly acts nilpotently on $H_*(X)$. Note that $\zeta^{-1}(\mathcal{H}) \subset \mathcal{E}^*(X)$ consists of those homotopy classes of pointed maps which are freely homotopic to the identity. If X is simply connected then $\zeta^{-1}(\mathcal{H})$ is the trivial subgroup, but is not trivial in general.

Then, the isomorphism $\mathcal{E}^*(X_{\mathbb{Q}}) \cong \mathcal{E}^*(L)$ gives a subgroup $\hbar \subset \mathcal{E}^*(L)$ isomorphic to $\zeta^{-1}(\mathcal{H}_{\mathbb{Q}})$. Since $\hbar/H_0(L) \cong \mathcal{H}_{\mathbb{Q}} = \{[\mathrm{id}]\}$, we conclude that $\hbar$ is equal to the action of $H_0(L)$, this means:

$$\hbar = \{[e^{\mathrm{ad}_x}] \mid x \in L_0\} \cong H_0(L).$$

The following proposition allows to compute the cdgl $\operatorname{Der}^{\hbar} L$.

Proposition 8.1. *We have the following equality of subsets of* $\operatorname{Der}_0 L$*:*

$$\operatorname{Der}_0^{\hbar} L = \mathrm{ad}(L_0) \oplus D \operatorname{Der}_1 L$$

Note that $[D\eta, \mathrm{ad}_x] = D[\eta, \mathrm{ad}_x]$, for $x \in L_0, \eta \in \operatorname{Der}_1 L$, so the right hand side is a sub dgl.

Proof. By the definition of $\hbar$, ad_x belongs to $\operatorname{Der}_0^{\hbar} L$ for any $x \in L_0$. Recall from Theorem 3.7 that for any $\eta \in \operatorname{Der}_1 L$, $e^{D\eta} \sim \mathrm{id}_L$ so $D\eta \in \operatorname{Der}_0^{\hbar} L$ and we have one of the inclusions.

Conversely suppose that $\theta \in \operatorname{Der}_0^{\hbar} L$, then $[e^{\theta}] = [e^{\mathrm{ad}_x}]$ so $\mathrm{id}_L \sim e^{\theta} \circ e^{-\mathrm{ad}_x}$ which implies that

$$e^{\theta} = e^{D\eta} \circ e^{\mathrm{ad}_x} = e^{D\eta * \mathrm{ad}_x}$$

for some $\eta \in \operatorname{Der}_1 L$. This shows that θ is an element of $\mathrm{ad}(L_0) \oplus D \operatorname{Der}_1 L$ (which is closed under BCH products, since it closed under Lie brackets).

□

Recall from §1.1 the concept of 1-connected cover of a dgl M, that we denote by $\widetilde{M}$.

Proposition 8.2. *The inclusion induces quasi-isomorphisms*

$$\widetilde{\operatorname{Der} L} \oplus \mathrm{ad}(L_0) \simeq \operatorname{Der}^{\hbar} L, \qquad \widetilde{\operatorname{Der} L \widetilde{\times} sL} \simeq \operatorname{Der}^{\hbar} L \widetilde{\times} sL$$

The twisted product $\operatorname{Der} L \widetilde{\times} sL$ is the usual one given by (6.3).

Proof. The first quasi-isomorphism comes from the previous proposition. We focus on the second one.

We only need to check that $H_0(-)$ applied to the inclusion is an isomorphism, since in positive degrees the cycles and boundaries of the cdgl's agree.

A degree 0 element in $\operatorname{Der}^{\hbar} L \widetilde{\times} sL$ has to be an element $\operatorname{Der}_0^{\hbar} L$ which is of the form $D\eta + \mathrm{ad}_x$ by the proposition above. Then $\eta + sx$ belongs to $(\operatorname{Der}^{\hbar} L \widetilde{\times} sL)_1$ and its differential is

$$D\eta - s\partial x + \mathrm{ad}_x = D\eta + \mathrm{ad}_x$$

which is the desired element of degree 0. This shows that $H_0(\operatorname{Der}^{\hbar} L \widetilde{\times} sL) = 0$. □

Then Theorem 7.13 affirms that the realization of

$$L \xrightarrow{\mathrm{ad}} \operatorname{Der}^{\hbar} L \to \widetilde{\operatorname{Der} L \widetilde{\times} sL}$$

is homotopy equivalent to

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\zeta^{-1}\{\mathrm{id}\}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\{\mathrm{id}\}}(X_{\mathbb{Q}})$$

Note that $B\operatorname{aut}_{\mathrm{id}}(X_{\mathbb{Q}})$ is homotopy equivalent to the universal cover of $B\operatorname{aut}(X_{\mathbb{Q}})$ by Proposition 2.20. In order to recover the classical case, take $X_{\mathbb{Q}}$ simply connected. In that case $\zeta^{-1}(\mathrm{id})$ is the trivial group and $B\operatorname{aut}^*_{\mathrm{id}}(X_{\mathbb{Q}})$ is also homotopy equivalent to the universal cover of $B\operatorname{aut}^*(X_{\mathbb{Q}})$. Finally, from Proposition 8.1 we deduce that the inclusion induces a quasi-isomorphism $\widetilde{\operatorname{Der} L} \simeq \operatorname{Der}^{\hbar} L$. So the classical result is obtained as particular example.

8.2 Description of the rationalizations $\mathcal{H}$ and Π

Giving an explicit expression of the rationalization of a nilpotent group can be a difficult issue in some cases. The theorems give such explicit description for $\Pi \subset \mathcal{E}^*(X)$ and $\mathcal{H} \subset \mathcal{E}(X)$ acting nilpotently on the homotopy groups and on the homology groups respectively, for X a finite nilpotent complex (and Π invariant under the action of $\pi_1(X)$).

Theorem 8.3. *There are isomorphisms of groups*

$$\Pi_{\mathbb{Q}} \cong H_0(\operatorname{Der}^{\pi} L), \qquad \mathcal{H}_{\mathbb{Q}} \cong \frac{H_0(\operatorname{Der}^{\hbar} L)}{H_0(\operatorname{ad}(L_0))}.$$

As usual, we are considering the BCH product on H_0 and Π and $\hbar$ are constructed as in §7.

Proof. The first isomorphism follows immediately from Theorem 7.18:

$$\Pi_{\mathbb{Q}} \cong \pi_1(B\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})) \cong \pi_1\langle \operatorname{Der}^{\pi} L\rangle \cong H_0(\operatorname{Der}^{\pi} L).$$

And by a similar argument now using Theorem 7.13 and the results about the holonomy action (see §4.4) we get the second isomorphism:

$$\mathcal{H}_{\mathbb{Q}} \cong \pi_1(B\operatorname{aut}_{\mathcal{H}_Q} X_{\mathbb{Q}}) \cong \frac{\pi_1(B\operatorname{aut}^*_{\zeta^{-1}(\mathcal{H}_{\mathbb{Q}})}(X_{\mathbb{Q}}))}{\pi_1(X_{\mathbb{Q}})} \cong \frac{H_0(\operatorname{Der}^{\hbar} L)}{H_0(\operatorname{ad}(L_0))}.$$

□

Example 8.4. Consider the following subgroups:

$$\mathcal{H} = \mathcal{E}_H(X_{\mathbb{Q}}) = \{[f] \mid H_*(f) = \mathrm{id}\} \subset \mathcal{E}(X_{\mathbb{Q}})$$

and

$$\Pi = \mathcal{E}^*_{\pi}(X_{\mathbb{Q}}) = \{[f] \mid \pi_*(f) = \mathrm{id}\} \subset \mathcal{E}^*(X_{\mathbb{Q}}).$$

Then the corresponding derivation algebras are defined by

$$\operatorname{Der}^{\hbar}_0 L = \{\theta \in \operatorname{Der}_0 L \mid D\theta = 0 \text{ and } \theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V)\}$$

and

$$\operatorname{Der}^{\pi}_0 L = \{\theta \in \operatorname{Der}_0 L \mid D\theta = 0 \text{ and } e^{\theta} \text{ induces the identity on } H_*(L)\}.$$

Theorem 8.3 in these particular cases recovers the result [44, Proposition 12].

Another direct consequence of the theorem above concerns the homotopy nilpotency of $\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})$ and $\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$. The homotopy nilpotency of an H-group is the least integer n for which the $(n+1)$th homotopy commutator is homotopically trivial. Similarly is defined the nilpotency index of a dgl. In both cases we write $\operatorname{nil}(-)$ to indicate this number.

Proposition 8.5. $\operatorname{nil}\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}}) = \operatorname{nil} H(\operatorname{Der}^{\pi} L)$ *and* $\operatorname{nil}\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) = \operatorname{nil} H(\operatorname{Der}^{\hbar} L \widetilde{\times} sL)$.

Proof. The numbers $\mathrm{nil}\,\mathrm{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})$ and $\mathrm{nil}\,\mathrm{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$ coincide by [44, Theorem 3] with the iterated Whitehead product length of $B\,\mathrm{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})$ and $B\,\mathrm{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$ respectively. For any connected cdgl M there is an isomorphism of Lie algebras $\pi_{*+1}\langle M\rangle \cong H_*(M)$ by [14, §12.5.2], so we can compute these iterated Whitehead product length using the models given by Theorems 7.13 and 7.18.

□

8.3 Nilpotent rational group as self homotopy equivalences

We will see through this section that any finitely generated nilpotent rational group can be realized as a subgroup of self homotopy equivalences of a finite complex; furthermore, the subgroup will act nilpotently on the homology (or the homotopy groups) of the complex with the same nilpotency index.

We use [49, §2] as general reference. The Malcev equivalence (see §3.1) can be described in simple terms, in the case of finitely generated rational groups of nilpotency index n. Such groups can be embedded in $T(n)$ the group of $n\times n$ *unitriangular matrices* over $\mathbb{Q}$, this means, the group of upper triangular matrices with 1 in the diagonal entries. On the other hand, any finitely generated nilpotent Lie algebra of nilpotency index n can be embedded in $U(n)$. This is the group of $n \times n$ *strictly triangular matrices* over $\mathbb{Q}$, this means upper triangular matrices 0 in the diagonal entries. We equip this group with the usual commutator bracket, which makes it a Lie algebra.

Then the logarithm and exponential series give a bijection

$$U(n) \underset{\log}{\overset{\exp}{\rightleftarrows}} T(n) \ .$$

Proposition 8.6. *Let H be a finitely generated nilpotent rational group with nilpotency index n. Then H is isomorphic to a subgroup of self homotopy equivalences of a rationalization of a finite nilpotent complex, which acts nilpotently on the homology.*

Proof. Let H be a finitely generated rational group of nilpotency index n. Without loss of generality we can consider that $H \subset T(n)$. Then $M = \log(H) \subset U(n)$ is a Lie algebra concentrated in degree 0 with nilpotency index n. Take $X = \bigvee_{j=1}^n S^m$ with $m > 1$, whose minimal Lie model is given by $L = (\mathbb{L}(x_1,\dots,x_n),0)$ with $|x_1| = \dots = |x_n| = m-1$.

A derivation $\theta \in \mathrm{Der}_0\, L$ sends a generator to a linear combination of generators, by degree reasons. So we can identify θ with a $n\times n$-matrix acting on $V = \mathrm{Span}\{x_1,\dots,x_n\}$ and we identify $U(n)$ as a subspace of $\mathrm{Der}_0(L)$. With this identification M is seen as a subspace of $\mathrm{Der}_0(L)$ and it determines a finite filtration of $V \cong s^{-1}\tilde{H}_*(X_{\mathbb{Q}})$

$$V = V^0 \supset V^1 \supset \dots \supset V^n = 0,$$

with $V^i = MV^{i-1}$ for $i \geq 1$.

On the other hand, an automorphism of L, $\varphi \in \mathrm{aut}(L)$ sends a generator to a linear combination of generators, also by degree reason. Furthermore, since the differential is zero, then $\mathrm{aut}(L) = \mathrm{aut}(L)/\sim$. So we can identify $\mathrm{aut}(L)$ with invertible matrices $n\times n$

and the group H can be identified with a subgroup $\mathcal{h} = \operatorname{aut}_{\mathcal{h}}(L)$ of $\operatorname{aut}(L) = \mathcal{E}(L)$. Then we deduce that

$$\operatorname{Der}_0^{\mathcal{h}}(L) = \{\theta \in \operatorname{Der}_0 L \mid e^\theta \in \mathcal{h}\} = M.$$

We finally use the isomorphism $\mathcal{E}(L) \cong \mathcal{E}^*(X_{\mathbb{Q}}) = \mathcal{E}(X_{\mathbb{Q}})$ to identify $\mathcal{h}$ with a subgroup $\mathcal{H}$ of $\mathcal{E}(X_{\mathbb{Q}})$. Then $\mathcal{H}$ acts nilpotently on $H_*(X_{\mathbb{Q}})$ giving as central series the suspension of the filtration $\{V^i\}$ via the isomorphism $V \cong s^{-1}\tilde{H}_*(X_{\mathbb{Q}})$. Therefore we can see $H \cong \mathcal{H}$ as a subgroup of $\mathcal{E}(X_{\mathbb{Q}})$ acting nilpotently on the homology of $X_{\mathbb{Q}}$. Furthermore the realization of

$$L \xrightarrow{\mathrm{ad}} \operatorname{Der}^{\mathcal{h}} L \to \operatorname{Der}^{\mathcal{h}} L \widetilde{\times} sL$$

is homotopy equivalent to

$$X_{\mathbb{Q}} \to B\operatorname{aut}^*_{\mathcal{H}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}}(X_{\mathbb{Q}}).$$

$\square$

A similar procedure can be performed in the dual setting: let π be a finitely generated nilpotent rational group, with nilpotency index n. We identify π with a subgroup of $T(n)$. Take $Y = \prod_{j=1}^n S^m$ for odd $m \geq 1$. Its Sullivan model is $A = (\wedge V, 0)$ with $V = \operatorname{Span}\{x_1, \dots, x_n\}$ with $|x_1| = \cdots = |x_n| = m$. A map $Y_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is modeled by an cdga morphism $\psi\colon A \to A$ which can be identified with a $n \times n$-matrix as above.

The differential being zero implies that $\operatorname{aut}(A) = \operatorname{aut}(A)/\sim$, so we identify π with a subgroup of $\operatorname{aut}(A)$ and with a subgroup $\Pi \subset \mathcal{E}(Y_{\mathbb{Q}})$ acting nilpotently on $\pi_*(Y_{\mathbb{Q}}) \cong V$ and thus acting nilpotently on $H_*(Y_{\mathbb{Q}}) \cong A^\sharp$. The action of $\pi_1(Y_{\mathbb{Q}})$ on $[\![Y_{\mathbb{Q}}, Y_{\mathbb{Q}}]\!]^*$ is trivial in both cases $m = 1$ or $m > 1$, so $\mathcal{E}^*(Y_{\mathbb{Q}}) = \mathcal{E}(Y_{\mathbb{Q}})$.

Then a minimal Lie model of Y can be constructed from A (see [14, §10]) and Theorem 7.18 gives the model for the universal fibration associated to such group. We conclude the proposition below.

Proposition 8.7. *Let π be a finitely generated nilpotent rational group with nilpotency index n. Then π is isomorphic to a subgroup of self pointed homotopy equivalences of a rationalization of a finite nilpotent complex, which acts nilpotently on the homotopy.*

8.4 Other fibration sequences

For X a finite nilpotent complex, and $\mathcal{H} \subset \mathcal{E}(X)$ and $\Pi \subset \mathcal{E}^*(X)$, there a fibration sequences

$$\operatorname{aut}_{\mathrm{id}}(X) \to \operatorname{aut}_{\mathcal{H}}(X) \to \mathcal{H}, \qquad \operatorname{aut}^*_{\mathrm{id}}(X) \to \operatorname{aut}^*_{\Pi}(X) \to \Pi.$$

Applying the functor $B(-)$ we obtain fibration sequences

$$B\operatorname{aut}_{\mathrm{id}}(X) \to B\operatorname{aut}_{\mathcal{H}}(X) \to B\mathcal{H}, \qquad B\operatorname{aut}^*_{\mathrm{id}}(X) \to B\operatorname{aut}^*_{\Pi}(X) \to B\Pi. \tag{8.1}$$

The rationalization of such fibration sequences can be modeled using derivation Lie algebras. As usual, let $L = (\widehat{\mathbb{L}}(V), \partial)$ be the minimal Lie model of X.

Theorem 8.8. *The realization of the cdgl fibration sequences*

$$\widetilde{\operatorname{Der} L\tilde{\times} sL} \to \operatorname{Der}^{\hbar} L\tilde{\times} sL \to (\operatorname{Der}^{\hbar} L\tilde{\times} sL)/(\widetilde{\operatorname{Der} L\tilde{\times} sL})$$

and

$$\widetilde{\operatorname{Der} L} \to \operatorname{Der}^{\pi} L \to \operatorname{Der}^{\pi} L/\widetilde{\operatorname{Der} L}$$

are homotopy equivalent to

$$B\operatorname{aut}_{\mathrm{id}}(X_{\mathbb{Q}}) \to B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\mathcal{H}_{\mathbb{Q}}, \quad \textit{and} \quad B\operatorname{aut}^*_{\mathrm{id}}(X_{\mathbb{Q}}) \to B\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}}) \to B\Pi_{\mathbb{Q}}$$

respectively.

Proof. Note that the fibration sequences of the theorem can be obtained as the fibration of $B\operatorname{aut}_{\mathcal{H}_{\mathbb{Q}}}(X_{\mathbb{Q}})$ and $B\operatorname{aut}^*_{\Pi_{\mathbb{Q}}}(X_{\mathbb{Q}})$ over their first Postnikov stage.

Given a connected cdgl M, denote by $Z_n \subset M_n$ the subspace of cycles. Then [14, Proposition 12.43] affirms that the realization of the cdgl fibration sequence

$$M_{>n} \oplus Z_n \to M \to M/(M_{>n} \oplus Z_n)$$

is homotopy equivalent to the fibration of $\langle M \rangle$ over its nth Postnikov stage. Then, take $n = 1$ and $M = \operatorname{Der}^{\hbar} L\tilde{\times} sL$ or $M = \operatorname{Der}^{\pi} L$. $\square$

Remark 8.9. A short computation let us observe that

$$(\operatorname{Der}^{\hbar} L\tilde{\times} sL)/(\widetilde{\operatorname{Der} L\tilde{\times} sL}) = \operatorname{Der}^{\hbar}_0 L \oplus R_1, \quad \text{and} \quad \operatorname{Der}^{\pi} L/\widetilde{\operatorname{Der} L} = \operatorname{Der}^{\pi}_0 L \oplus S_1,$$

where R_1 and S_1 denote a complement of the cycles of degree 1 of $\operatorname{Der}^{\hbar} L\tilde{\times} sL$ and $\operatorname{Der}^{\pi} L$ respectively. In particular, we deduce

$$\langle \operatorname{Der}^{\hbar}_0 L \oplus R_1 \rangle \simeq B\mathcal{H}_{\mathbb{Q}}, \quad \text{and} \quad \langle \operatorname{Der}^{\pi}_0 L \oplus S_1 \rangle \simeq B\Pi_{\mathbb{Q}}.$$

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