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Doctoral Thesis

Contributions to the theory of Large Cardinals through the method of Forcing

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“Els matemàtics són una mena de poetes fraudulents que, de fet, intenten l’única poesia possible.”

Joan Fuster

UNIVERSITAT DE BARCELONA

Abstract

Facultat de Matemàtiques i Informàtica

“Contributions to the theory of Large Cardinals through the method of Forcing”

by Alejandro Poveda Ruzafa

The present dissertation is a contribution to the field of Mathematical Logic and, more particularly, to the subfield of Set Theory. Within Set theory, we are mainly concerned with the interactions between the large-cardinal axioms and the method of Forcing. This is the line of research with a deeper impact in the subsequent configuration of modern Mathematics. This area has found many central applications in Topology [ST71][Tod89], Algebra [She74][MS94][DG85][Dug85], Analysis [Sol70] or Category Theory [AR94][Bag+15], among others. The dissertation is divided in two thematic blocks: In Block I we analyze the large-cardinal hierarchy between the first supercompact cardinal and Vopěnka’s Principle (Part I). In Block II we make a contribution to Singular Cardinal Combinatorics (Part II and Part III).

Specifically, in Part I we investigate the Identity Crisis phenomenon in the region comprised between the first supercompact cardinal and Vopěnka’s Principle. As a result, we settle all the questions that were left open in [Bag12, §5]. Afterwards, we present a general theory of preservation of $C^{(n)}$–extendible cardinals under class forcing iterations from which we derive many applications.

In Part II and Part III we analyse the relationship between the Singular Cardinal Hypothesis (SCH) and other combinatorial principles, such as the tree property or the reflection of stationary sets. In Part II we generalize the main theorems of [FHS18] and [Sin16] and manage to weaken the large-cardinal hypotheses necessary for Magidor-Shelah’s theorem [MS96]. Finally, in Part III we introduce the concept of Σ-Prikry forcing as a generalization of the classical notion of Prikry-type forcing. Subsequently we devise an abstract iteration scheme for this family of posets and, as an application, we prove the consistency of $\mathsf{ZFC} + \neg\mathsf{SCH}_\kappa + \mathrm{Refl}({<}\omega, \kappa^+)$, for a strong limit singular cardinal κ with $\mathrm{cof}(\kappa) = \omega$.

UNIVERSITAT DE BARCELONA

Resum

Facultat de Matemàtiques i Informàtica

"Contributions to the theory of Large Cardinals through the method of Forcing"

per Alejandro POVEDA RUZAFA

La present tesi és una contribució a l'estudi de la Lògica Matemàtica i més particularment a la Teoria de Conjunts. Dins de la Teoria de Conjunts, la nostra àrea de recerca s'emmarca dins l'estudi de les interaccions entre els Axiomes de Grans Cardinals i el mètode de Forcing. Aquestes dues eines han tigut un impacte molt profund en la configuració de la matemàtica contemporànea com a conseqüència de la resolució de qüestions centrals en Topologia [ST71][Tod89], Àlgebra [She74][MS94][DG85][Dug85], Anàlisi Matemàtica [Sol70] o Teoria de Categories [AR94][Bag+15], entre d'altres. La tesi s'articula entorn a dos blocs temàtics. Al Bloc I analitzem la jerarquia de Grans Cardinals compresa entre el primer cardinal supercompacte i el Principi de Vopěnka (Part I), mentre que al Bloc II estudiem alguns problemes de la Combinatòria Cardinal Singular (Part II i Part III).

Més precisament, a la Part I investiguem el fenòmen de Crisi d'Identitat en la regió compresa entre el primer cardinal supercompacte i el Principi de Vŏpenka. Com a conseqüència d'aquesta anàlisi resolem totes les preguntes obertes de [Bag12, §5]. Posteriorment presentem una teoria general de preservació de cardinals $C^{(n)}$–extensibles sota iteracions de longitud ORD, de la qual en derivem nombroses aplicacions.

A la Part II i Part III analitzem la relació entre la Hipòtesi dels Cardinals Singulars (SCH) i altres principis combinatoris, tals com la Propietat de l'Arbre o la reflexió de conjunts estacionaris. A la Part II obtenim sengles generalitzacions dels teoremes principals de [FHS18] i [Sin16] i afeblim les hipòtesis necessàries perquè el teorema de Magidor-Shelah [MS96] siga cert. Finalment, a la Part III, introduïm el concepte de forcing Σ-Prikry com a generalització de la noció clàssica de forcing del tipus Prikry. Posteriorment dissenyem un esquema d'iteracions abstracte per aquesta família de forcings i, com a aplicació, derivem la consistència de $\mathsf{ZFC} + \neg\mathsf{SCH}_\kappa + \mathrm{Refl}({<}\omega, \kappa^+)$, per a κ un cardinal fortament límit i singular amb $\mathrm{cof}(\kappa) = \omega$.

Contents

Als meus pares.

A Laura.
Al meu amic Juan.

Agraïments

Arribats a aquest punt hom s'adona que agraïr és sempre la tasca més feixuga de l'ofici d'escriure: o *l'ofici de viure* que diria Pavese. Ho feia vers Estellés quan deia *"Pense que ha arribat l'hora del teu cant a València. Temies el moment. Confessa-t'ho: temies"*. Ho sé d'antuvi: mai seré capaç d'escriure *el meu cant a Barcelona*. Si més no, el que ella, i els qui ella m'ha donat, mereixen. Han estat incontables aprenentatges, i algun que altre amor, els que aquest raconet de la Mediterrànea, què bramà amb els versos d'en March i Espriu i es sentí universal amb la prosa en Llull, m'ha ofert. Tanmateix, no puc defugir l'obligació de que aquestes línees servisquen per fer un just homenatge a tots aquells sense els quals res d'això no haguera estat possible.

Certament, Joan Bagaria ha estat la figura acadèmica més rellevant durant aquest període de temps. Ell fou qui fa quatre anys respongué la crida d'un estudiant poc versat en aquestes coses de la Lògica i qui va creure les meues promeses d'esforç i dedicació. Des d'aquests inicis precaris(sims) ell ha estat una de les poques figures acadèmiques que m'ha tramés el coratge i la confiança necessària per iniciar la meua carrera d'investigador. *"El talent no val res sense el treball"*: i, a sabendes de la meua manca del primer, vaig abraçar-me a l'esperança del segon.

Joan és qui sempre ha tingut la paciència necessària i els consells sempre adients perquè focalitzara els meus esforços en allò realment important. Sempre que alguna circumstància m'ha assetjat, Joan, més que el meu conseller, ha estat el meu salvavides. Amb independència dels entrebancs que ens depare el futur Joan será, per sempre més, peça fonamental d'aquesta etapa crucial de la meua maduresa. Mai més podré entendre'm com a matemàtic sense la participació de la seua figura.

No cal doncs apel·lar a protocols perquè ell siga la primera persona a la què ací esmente. Des d'aquesta xicoteta tribuna que se m'ofereix aprofite l'avinentesa per manifestar-li la meua estima i admiració. Ha estat senzillament un plaer aprendre de tu, Joan.

Aquesta tesi tampoc haguera estat possible sense el concurs d'altres figures matemàtiques prèvies a Joan. No puc pas oblidar-me del primer responsable de que iniciés aquesta aventura: Vicent Baeza, el meu mestre de secundaria. Gràcies a la seua vocació vaig descobrir la meua passió per les matemàtiques i sense el seu encoratjament mai m'haguera decidit a emprendre aquest camí. La llavor que ell conreà avui és més prop de ser arbre.

Altra de les figures imprescindibles per entendre la meua història és Aníbal Moltó. Aníbal, a banda de ser molt més que un mestre durant la meua època universitària, va ser qui em va introduir el cuquet de la Teoria de Conjunts. La meua vocació actual no es pot entendre sense la seua fascinació per aquesta última i sense la influència que la seua personalitat va tenir sobre mi.

Paradoxal sembla, però, que un analista haja sigut el responsable que algú haja abraçat la *fe dels Axiomes de Grans Cardinals*. Pensant-ho millor, potser d'aquesta forma hem perdut un analista mediocre i, si més no, hem guanyat un matemàtic feliç. No cal que et reitere, Aníbal, la teua importància en el meu desenvolupament personal i matemàtic.

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temps. Siga el que siga el que ens depare el futur aquesta menció restarà, per sempre més, com el meu homenatge al nostre amor.

Introduction

The subject matter of the present dissertation is Set Theory, the subfield of Mathematics devoted to the study of mathematical truth and infinity. More particularly, Set Theory is concerned with the study of the abstract infinite sets and the possible extensions of the standard axiomatization of Mathematics. These apparently different ambits of interest are actually reminiscent of the two souls that have coexisted in the heart of Set Theory since its birth.

The first of these souls is connected with the ontological status of infinity and, most specially, with its mathematical nature. This aspect of Set Theory – namely, as the mathematical theory of the actual infinity – arose from the pioneering work of G. Cantor and R. Dedekind [Fer08], and has evolved to the modern field of Infinitary Combinatorics [She94][Eis10][HSW10]. The second soul of Set Theory is connected with the problem of finding a solid and reliable foundation for Mathematics. The goal in this context is to find the right formal system from which all mathematical truths can be derived. These two aspects can be respectively termed the mathematical and the metamathematical conceptions of Set Theory, and both together have determined the current role of this area within Mathematics.

The standard axiomatization of Mathematics is provided by ZFC, namely the **Z**ermelo-**F**raenkel axioms plus the Axiom of **C**hoice. Starting with just ZFC we can derive most of the classical theorems of Mathematics: from Gauss's *Theorema Egregium* to Hahn-Banach's theorem. Nonentheless, since K. Gödel's discovering of incompleteness [Göd31], it is known that any recursively enumerable system of axioms such as ZFC, if consistent, is incomplete, i.e., there are statements expressed in the language of the system that are neither provable nor disprovable within the system. This sort of mathematical statements are called independent or undecidable (in the corresponding system).

Gödel's discovering, far from being a mere logical trick, bears on relevant mathematical questions; the most famous being the Continuum Hypothesis (CH). The CH states that every infinite set $A \subseteq \mathbb{R}$ is either countable (i.e. equipotent with $\mathbb{N}$) or equipotent with the whole set of real numbers. An equivalent formulation of the CH is that $2^{\aleph_0} = \aleph_1$. In 1983, G. Cantor conjectured that the CH was true and after stubborn and unfruitful attempts finally abandoned his hope to settle the problem. Cantor's conjecture experienced a renewed interest in 1902, when D. Hilbert put it in the first place of his famous list of twenty three unsolved mathematical problems [Hil02]. Nonetheless, the first (partial) satisfactory answer to Cantor's CH still had to wait for almost forty years.

In 1938, Gödel [Göd38] made a breakthrough by proving that $\neg$CH cannot be a theorem of ZFC, provided ZFC is consistent. He proved so by defining the so-called constructible universe of sets L and by showing that it satisfies all the axioms of ZFC plus the CH. Formally speaking, L is a model of ZFC plus the CH. An outright consequence of Gödel's theorem is that the consistency of ZFC yields the consistency of ZFC+CH. Gödel's work marked the birth of the future field of *Inner model theory*, one of the most prominent areas of research in modern Set Theory [Mit10].

Nevertheless, Gödel's answer did not provide a totally satisfactory solution to the *continuum problem*, as it left open the door for the CH to be a theorem of ZFC. In his seminal work twenty-five years after Gödel's breakthrough, P. Cohen [Coh64] introduced the method of Forcing as a means to prove the consistency of ZFC plus the negation of the CH from the consistency of ZFC. With this method, one starts with an arbitrary (countable transitive) model M of ZFC and a partial order $\mathbb{P} \in M$, and then pass to a *generic extension* $M[G]$ in which a new set G is adjoined. The model $M[G]$ is the smallest transitive model of ZFC that contains G and all the elements of M.

Both combined, Gödel and Cohen theorems show that the CH cannot be decided on the basis of ZFC. Therefore, it was not a lack of cleverness but a foundational issue which prevented Cantor to prove or refute his conjecture.

Seen in perspective, Cohen's legacy goes far beyond the work that led him to win the Fields Medal. The history of Mathematics of the last fifty years confirms that the real breakthrough was not the independence of the *continuum problem* but rather the discovering of Forcing.

Shortly after Cohen's method was announced, the set-theoretic community realized that Forcing was a more versatile tool than expected. A new powerful method to prove independence results in Mathematics had been discovered. As a result, Set Theory flourished in a series of spectacular applications of the method which established the independence of longstanding mathematical questions. This was the case, for instance, of *Suslin's Hypothesis* [ST71], the Lebesgue measurability of all projective sets of real numbers [Sol70] or the question of whether every Whitehead group is free [She74]. Since its discovery, Forcing has played a central role in the subsequent development of Set Theory [Kan09][Kan12].

Another central concept in Set Theory is the notion of *Large Cardinal* [Kan09]. Broadly speaking, a cardinal κ is a large cardinal if the κ-stratum of the universe of sets V (i.e., V_κ) is so large that it *resembles* the whole of V. For instance, an inaccessible cardinal κ is a regular cardinal such that $V_\kappa \models \mathsf{ZFC}$. In particular, by virtue of Gödel's Second Incompeteness Theorem (i.e., ZFC, if consistent, does not prove Con(ZFC)), the existence of inaccessible cardinals is not provable in ZFC. This phenomenon is also extensible to the rest of large cardinal notions, hence their existence can not be established on the basis of ZFC.

The degree of resemblance between V_κ and V depends on *how large* the cardinal κ is. Properly speaking, it depends on the *large cardinal strength* of κ. For example, if κ is a supercompact cardinal – a much stronger notion than inaccessibility – then $V_\kappa \models \mathsf{ZFC}$ and moreover $V_\kappa \prec_{\Sigma_2} V$.[1] Therefore, if κ is supercompact, then V_κ resembles more faithfully the universe of sets V than if it was just inaccessible.

A Large Cardinal axiom is a statement asserting the existence of a certain large cardinal. Even though the existence of large cardinals is not provable within ZFC, there is a wide consensus among the community that, together with ZFC, they are necessary for a right foundation of Mathematics [Mag12][Koe11][Koe14][Mad11]. One of the most relevant arguments is that large cardinals provide a natural generalization of Cantor's thesis about the indescribability of the universe of sets [Fer08][Mad11]. To be more specific, it has been shown that many important large cardinals (such as, inaccessible, supercompact or extendible) are equivalent to *Reflection Principles* extending the so-called *Reflection Theorem* [Kan09][Mag71][Bag12].

The hegemony of large cardinals in the current conception of Set Theory is explained by the following metamathematical phenomenon: Given any natural mathematical statement φ, either φ is (modulo ZFC) equiconsistent with ZFC, or equiconsistent with ZFC plus some large cardinal axiom. Another crucial feature of the axioms of large cardinals is that they form a hierarchy, which is linearly ordered in terms of consistency strength. This turns out to be very useful when studying the mutual relationship between undecidable mathematical statements; in particular, one may show that a statement φ does not imply a statement ψ by showing that ψ entails the existence of large cardinals that are consistency-wise stronger than those needed for the consistency of φ. Thus, large cardinals provide a unified framework to deal with mathematical independence.

The dominating role of large cardinals in Metamathematics is in good measure a heritage of Gödel's Platonism, whose main thesis can be summarized as follows: despite there is no hope for any *reasonable* formal system to reveal us all *mathematical truths*, we can still hope to use (canonical) strong axioms of infinity to *potentially* know about any of them [Koe11][Koe14][Mad11].

[1] I.e., for each Σ_2 formula $\varphi(x_0, \dots, x_{n-1})$ in the language of Set Theory and each $a_0, \dots, a_{n-1}$ finite collection of parameters in V_κ, $V_\kappa \models \varphi(a_0, \dots, a_{n-1})$ iff $\varphi(a_0, \dots, a_{n-1})$.

As our title announces, the present dissertation is a contribution to the theory of Large Cardinals with a special emphasis on the applications of the method of Forcing. The coupling Forcing/Large Cardinals has been a remarkable success from which not only set theorist have benefited. Indeed, in many other areas of pure Mathematics, such as Category Theory, Topology, Group Theory or Analysis, the use of large cardinals and Forcing has been remarkable. For some of the major applications see [Tod89][She94][She74] [AR94][EM02][Sol70].

There are also many natural questions in Set Theory itself that require the interplay between Forcing and Large Cardinals. This is the case of problems arising from the study of reflection/compactness principles [Jec10][Mag71] [MS89], from Singular Cardinal Combinatorics [She94][Git10][Eis10], or from the possible configurations of the large-cardinal hierarchy [Mag76][AG98] [AC00][AC01]. Another major area of research which mixes these two techniques is the preservation of large cardinals under Forcing [Lav78][GS89] [Cum10][Bag+16][BP18].

The present dissertation is a contribution to the four aforementioned fields. The novelties that we introduce here are of two types: on the one hand, we give the solution to some open questions and, on the other hand, we introduce new techniques for the further development of these areas.

The dissertation is conceptually divided in two thematic blocks. In the first one (consisting of Part I) we study the large cardinal hierarchy between the first supercompact cardinal and Vopěnka's Principle (VP). To this aim we study the large cardinal configurations at these scales and discuss the possible effects of Forcing.

In the second block (consisting of Part II and Part III) we study some important combinatorial principles in the context of singular cardinals, such as the *Singular Cardinal Hypothesis*, the *Tree property*, and the *Reflection of Stationary sets*. Our investigations here are a contribution to the major area of research known as Singular Cardinal Combinatorics.

Each of these parts will we preceded by a technical introduction motivating the corresponding problems. The notation we shall use will be either standard, as in [Kun14] or [Jec03], or will be properly explained. Finally, the necessary preliminaries are covered in **Chapter 1.**

We will now describe the main contributions of the present dissertation to each of these two blocks.

Block 1 (Part I): The Large cardinal hierarchy between the first supercompact cardinal and Vopěnka's Principle.

The interval comprised between the first supercompact cardinal an Vopěnka's Principle is one of the most important and fruifully exploited of the large-cardinal hierarchy. The large cardinals at these latitudes appear in the proofs of many important consistency results: for instance, the consistency of *Martin's Maximum* (MM) follows from the consistency of a supercompact cardinal [FMS88]. Besides, it is conjectured that the former is actually an equiconsistency result. There are also many relevant applications in other areas of pure Mathematics, such as Category theory [Bag+15][BBT13] or Algebra [EM02][MS94]. Other connections and applications have been also found in Model-Theoretic logics and the Philosophy of Mathematics [Mag71][Bar17] [MV11][KMV16].

A concrete aspect that we want to explore is the so-called ***S**tructural **R**eflection principle* (**SR**) and, more particularly, its large-cardinal counterparts. The **SR** principle, due to J. Bagaria [Bag12, Definition 4.1], yields a myriad of natural extensions of the Reflection theorem and is intimately tied with the architecture of V at these scales.

> **SR**: For any class of relational structures $\mathfrak{C}$ in the same language there is an ordinal θ such that θ reflects $\mathfrak{C}$, i.e., for every $\mathcal{M} \in \mathfrak{C}$ there is a structure $\mathcal{N} \in \mathfrak{C} \cap V_\theta$ and an elementary embedding $j : \mathcal{N} \to \mathcal{M}$.

Thus, **SR** asserts that the universe of sets V is saturated, in the sense that for each class of structures $\mathfrak{C}$ in the same language, there is an ordinal θ such that all the information about $\mathfrak{C}$ can be *coded* within V_θ. Since the above statement is too general it is natural to consider its restrictions to concrete degrees of definability. Namely, for $\Gamma \in \{\Sigma, \Pi\}$ and each degree of complexity n, the Γ_n-**SR** principle reads as follows:

> Γ_n**-SR:** For any Γ_n-definable class of relational structures $\mathfrak{C}$ in the same language there is an ordinal θ which reflects $\mathfrak{C}$.

While Σ_1-**SR** is provable in ZFC the analogous principle for Π_1-definable classes only holds under the presence of (very) large cardinals. In his paper from 1971 [Mag71], M. Magidor showed that the **SR** principle for the Π_1-definable class of structures $\{\langle V_\alpha, \in\rangle \mid \lambda \in \mathrm{ORD}\}$ is equivalent to the existence of a supercompact cardinal. Actually, θ is the least ordinal witnessing Π_1-**SR** –equivalently, Σ_2-**SR**– if and only if θ is the first supercompact cardinal [Bag12]. Thus, the large-cardinal counterpart of the Π_1-**SR** principle is supercompactness. In this regard it is worth to stress that the above is not an equiconsistency result but rather an equivalence.

A reformulation of the Π_1-**SR** principle which is more in the spirit of the Reflection Theorem [Kun14, Ch. II, §5] is the following: There is a cardinal θ such that for every ordinal $\lambda > \theta$ and $a \in V_\lambda$, there is $\mu < \theta$ and $b \in V_\mu$ and a non-trivial elementary embedding $j : \langle V_\mu, \in\rangle \to \langle V_\lambda, \in\rangle$ with $j(b) = a$. In a nutshell, if Π_1-**SR** holds then there is a stratum of the universe of sets V_θ which *captures* all the Σ_1-truths, modulo permutations of parameters.[2]

Following up on Magidor's work, in 2012 J. Bagaria found the large-cardinal counterparts of the principles Γ_n-**SR**, for $n \geq 2$ [Bag12]. Bagaria discovered that these large-cardinal *companions* were given by a strengthening of the classical notion of extendibility: the $C^{(n)}$–extendible cardinals. This family of cardinals was first introduced in [Bag+15], where the authors use them to obtain many applications in Category Theory and Algebraic Topology. In [Bag12] the following level-by-level equivalence is proved: for each n, the following holds:

$$\Sigma_{n+2}\text{-}\mathbf{SR} \Leftrightarrow \Pi_{n+1}\text{-}\mathbf{SR} \Leftrightarrow \text{There is a } C^{(n)}\text{–extendible cardinal.}$$

Moreover, Σ_{n+2}-**SR** is equivalent to VP(Π_{n+1}), namely Vopěnka's Principle (VP) restricted to Π_{n+1}-definable classes of structures.[3] Thus, Bagaria's result actually yields the equivalence between the principle VP(Π_{n+1}) and the existence of a $C^{(n)}$–extendible cardinal, hence the equivalence between VP and the existence of a $C^{(n)}$–extendible cardinal, for each n. It is in this sense that $C^{(n)}$–extendible cardinals can be conceived as canonical representatives of the large-cardinal hierarchy in the region between the first supercompact cardinal and VP. An outright consequence of the previous discussion is that $C^{(n)}$–extendibility yields a proper hierarchy, i.e., the first $C^{(n)}$–extendible cardinal is strictly smaller than the first $C^{(n+1)}$-extendible.

Due to the success achieved with $C^{(n)}$–extendibility, Bagaria [Bag12] also considered the $C^{(n)}$–forms of other classical large-cardinal notions, such as $C^{(n)}$–supercompactness, $C^{(n)}$–hugness or $C^{(n)}$–superhugness. The first natural question for these classes is if they form a proper hierarchy. Bagaria [Bag12] shows that this is the case for all of them, with the exception of $C^{(n)}$–supercompactness, leaving open in [Bag12, §5] the following questions: Is the first $C^{(1)}$-supercompact a Σ_3-correct cardinal or rather it coincides with the first supercompact? Does the class of $C^{(n)}$–supercompact cardinals form a strong hierarchy? What is the relationship between $C^{(n)}$–extendibility and $C^{(n)}$–supercompactness? Specifically, is any $C^{(n)}$–extendible cardinal also

[2] A similar result holds for Π_{n+1}-**SR** by considering Π_{n+1}-definable proper classes of structures of the form $\langle V_\lambda, \in, A\rangle$ [Bag12][BP18].

[3] For more about Vopěnka's Principle see [Kan09][SRK78][AR94][Bag12][Bag+15].

$C^{(n)}$–supercompact? Is the first $C^{(n)}$–supercompact strictly smaller than the first $C^{(n)}$–extendible?

In his Ph.D. thesis [Tsa12], K. Tsaprounis answered affirmatively one of these questions. Namely, any $C^{(n)}$–extendible cardinal is also $C^{(n)}$–supercompact. He managed to prove this by means of a nice characterization of the notion of $C^{(n)}$–extendibility: κ is $C^{(n)}$–extendible if and only if κ is $C^{(n)}$–supercompact and κ-superstrong [Tsa12, §2.6]. Nevertheless this characterization does not provide any information about the other questions raised by Bagaria.

In Part I of this dissertation we settle all the questions that were left open. Specifically:

1. In **Chapter 2** we show that the first $C^{(n)}$–supercompact is strictly smaller than the first $C^{(n)}$–extendible. To this aim we introduce a new class of $C^{(n)}$–cardinals that we have coined with the name of $\mathfrak{a}$-$C^{(n)}$–extendible cardinals. We prove that any $C^{(n)}$–extendible is a limit of $\mathfrak{a}$-$C^{(n)}$–extendible and also that any $\mathfrak{a}$-$C^{(n)}$–extendible is $C^{(n)}$–supercompact. We also study the consequences of Woodin's Extender Embedding Axiom (WEEA) for these large cardinals.

2. In **Chapter 3** we prove that the first $C^{(1)}$-supercompact cardinal can be strictly greater than the first supercompact. For this, we first show that even minor strengthenings of measurability are fragile under Prikry-type forcings, like Radin forcing. Then, we take advantage of this fragility to derive the desired consistency result. A different proof using iterated forcing was obtained joint with Y. Hayut and M. Magidor [HMP20].

3. In **Chapter 4** we show that the class of $C^{(n)}$–supercompact cardinals does not form a strong hierarchy. We prove this by establising the consistency of the first $C^{(n)}$–supercompact to be the first supercompact cardinal.[4] In particular, the first $C^{(n)}$–supercompact is not a Σ_3-correct cardinal. Furthermore, we show that the first $C^{(n)}$–supercompact can be collapsed to be the first supercompact, for each $n < \omega$. All these results provide the natural analogue of Magidor's Identity Crisis theorems [Mag76] for the class of $C^{(n)}$–supercompact cardinals. Observe that this shows that $C^{(n)}$–supercompactness does not entail more structural reflection than that provided by the Π_1-**SR** principle. These results have been obtained in collaboration with Y. Hayut and M. Magidor [HMP20].

Digressing a bit from the previous topic, we close this block by exploring the effects of Forcing upon the class of $C^{(n)}$–extendible cardinals. These

[4] Actually the first ω_1-strong compact cardinal.

cardinals have found relevant applications in Category Theory and Alegbraic Topology [Bag+15] and are also connected with one of the most important open problems in Set Theory, namely Woodin's HOD-conjecture [Woo10]. Thus, the investigation of the preservation of such cardinals under forcing is a worthwhile project.

In **Chapter 5** we present a general theory of preservation of $C^{(n)}$–extendible cardinals under class forcing iterations, which have been developed joint with J. Bagaria [BP18]. We first identify a wide family of class forcing iterations which we called *Suitable iterations* and prove that they preserve $C^{(n)}$–extendible cardinals. Afterwards, we use these preservation theorems to derive many consistency results for $C^{(n)}$–extendible cardinals (hence, also for $\mathrm{VP}(\Pi_{n+1})$ and VP):

1. In **Section 5.6.1** we prove a level-by-level version of Brooke-Taylor's theorem on the robustness of VP under Suitable iterations [BT11].

2. In **Section 5.6.2** we show that $C^{(n)}$–extendible cardinals are consistent with many different configurations of the function $\kappa \mapsto 2^\kappa$ at regular cardinals. Our results here extend Tsaprounis' result on the consistency of $C^{(n)}$–extendible cardinals with the GCH [Tsa18].

3. In **Section 5.6.3** we show that $C^{(n)}$–extendible cardinals are consistent with strong versions of $\diamondsuit$-principles.

4. In **Section 5.6.4** we prove the consistency of $C^{(n)}$–extendible cardinals with class many instances of $\square^*_\lambda$ at singular cardinals λ. This extends to the setting of $C^{(n)}$–extendible cardinals a previous result of Cummings, Foreman and Magidor [CFM01] for supercompact cardinals.

5. In **Section 5.6.5** we prove that $C^{(n)}$–extendible cardinals are consistent with strong disagreements between V and HOD on the computation of successors of regular cardinals. Specifically, we show that a $C^{(n)}$–extendible cardinal is consistent with $(\lambda^{+\omega})^{\mathrm{HOD}} < \lambda^+$, for every regular cardinal λ. This shows that even if Woodin's HOD-conjecture is true there may still be no agreement at all between V and HOD about successors of regular cardinals.

6. In **Section 5.7** we show that mild strengthenings of $C^{(n)}$–extendibility can be preserved under non-weakly homogeneous and non-definable class iterations. In Section 5.7.1 we use this to prove the consistency of $C^{(n)}$–extendible cardinals with "$\mathrm{V} = \mathrm{HOD}$" and with "$\mathrm{V} \neq \mathrm{HOD}+\mathsf{GA}$", where GA denotes the Ground Axiom [Reipt]. This latter extends a previous result due to J. Hamkins, J. Reitz and W. Woodin [HRW08].

In the last chapter of this block (**Chapter 6**) we explore briefly the extent of possible disagreements between V and HOD. In this regard we prove that it is consistent for a successor cardinal to be $C^{(n)}$–extendible in HOD. This result is a contribution to the program in Set Theory that studies the resemblance between V and HOD, which is currently of great interest for the community (see [GM18a][BNU17][CFG15][Cum+18][Woo10]).

Block 2 (Part II and Part III) : Singular Cardinal Combinatorics.

Singular Cardinal Combinatorics is the area of Set Theory devoted to the study of the combinatorial properties of singular cardinals and their successors. This field is intimately connected with one of the foundational problems of Set Theory, namely: *What is the value of* $2^{\aleph_0}$*?* Or more generally: *What are the rules describing cardinal exponentiation?*

In 1965, L. Bukovský [Buk65] showed that both the power-set function $\kappa \mapsto 2^\kappa$ and the exponential function $\lambda \mapsto \kappa^\lambda$ can be computed by means of the *Gimel function* $\gimel : \kappa \mapsto \kappa^{\mathrm{cof}(\kappa)}$. Thereby, any (non-trivial) question about infinite arithmetic can be translated into a question about the Gimel function, so that Cardinal Arithmetic is nothing but the study of $\gimel : \kappa \mapsto \kappa^{\mathrm{cof}(\kappa)}$.

With the discovering of Forcing, P. Cohen proved that $\gimel(\aleph_0)$ can be (consistently) any infinite cardinal with cofinality $> \aleph_0$. Thus, $\gimel(\aleph_0)$ cannot be $\aleph_\omega$ but it might be, e.g., $\aleph_1$ or $\aleph_{\omega_1}$. A subsequent result of Easton [Eas70] generalized this by showing that there are just two constraints ruling the behaviour of the $\gimel$-function restricted to the class of regular cardinals; namely, monotonicity (i.e. $\lambda \leq \kappa \to \gimel(\lambda) \leq \gimel(\kappa)$) and König's theorem (i.e. $\gimel(\kappa) > \kappa$).[5]

In the early years of Forcing the general belief was that an Easton-like result for singular cardinals should be true and that it was a matter of time that the result would follow from the subsequent development of the method. As we will see, this turned out not to be the case.

The **S**ingular **C**ardinal **H**ypothesis at a singular cardinal κ (SCH_κ) establishes that if $2^{\mathrm{cof}(\kappa)} < \kappa$ then $\gimel(\kappa) = \kappa^+$. The **S**ingular **C**ardinal **H**ypothesis (SCH) is the statement that SCH_κ holds, for each singular cardinal κ. This principle can be motivated as follows: An outright consequence of König's theorem is that $\gimel(\kappa) \geq \kappa^+$. Also, if κ is singular with $2^{\mathrm{cof}(\kappa)} \geq \kappa$ then elementary computations yield $2^{\mathrm{cof}(\kappa)} = \gimel(\kappa)$. In the remaining case where $2^{\mathrm{cof}(\kappa)} < \kappa$ the SCH_κ states that $\gimel(\kappa)$ takes its least possible value, i.e., κ^+.

Recall that the **G**eneralized **C**ontinuum **H**ypothesis at an infinite cardinal κ (GCH_κ) asserts that $2^\kappa = \kappa^+$. Similarly, the **G**eneralized **C**ontinuum **H**ypothesis (GCH) claims that the GCH_κ holds, for any infinite cardinal κ.

Notice that if κ is a strong limit singular cardinal then the SCH_κ is equivalent to the GCH_κ. Since the SCH follows from the GCH, and the latter holds

[5]For a more general formulation of the theorem see [HSW10, Theorem 1.6.7(b)]

in the constructible universe L, the former is consistent with ZFC. Thus the natural question was if $\neg\mathsf{SCH}$ is consistent with ZFC.

The first major progress in this direction was obtained by J. Silver. In his paper from 1975 [Sil75], Silver showed that if κ is a strong limit singular cardinal of uncountable cofinality and, for each $\lambda < \kappa$, $2^\lambda = \lambda^+$ then $\gimel(\kappa) = \kappa^+$.[6] In particular, the first instance for a failure of the SCH cannot be a strong limit singular cardinal of uncountable cofinality.

Another possible interpretation of Silver's theorem is as a *compactness principle*. Namely, as claiming the following property about any strong limit singular cardinal κ with uncountable cofinality: if the power set of *many* small cardinals λ is *not large* (i.e., $2^\lambda = \lambda^+$) then the power set of κ is not large either (i.e., $2^\kappa = \kappa^+$).

A somewhat similar situation occurs with strong limit singular cardinals of countable cofinality. This can be illustrated through an impressive result due to S. Shelah [She94] which is heir to a previous groundbreaking theorem of F. Galvin and A. Hajnal [GH75]: if $\aleph_\omega$ is a strong limit cardinal then $2^{\aleph_\omega} < \aleph_{\omega_4}$. Once again, if many small cardinals have not too large power set then the bigger one does not have it either. Observe however that Shelah's bound does not put any restriction for $\aleph_\omega$ to be the first cardinal where the SCH fails. Regarding all of these results, it is worth to emphasize that all of them are theorems of ZFC, hence do not rely on the existence of large-cardinals.

The compactness principles are also relevant outside the borders of Set Theory. A paradigmatic example takes place in Infinite Abelian Group Theory via Shelah's Singular Compactness Theorem [EM02, Theorem 3.5]. In this dissertation we will be interested in two important compactness principles in Singular Cardinal Combinatorics, namely the *tree property* and the *reflection of stationary sets*.

Although the above discussion already points out the existence of substantial differences between regular and singular cardinals, these became in time even more dramatic. With the publication of [DJ75], Jensen accomplished a major breakthrough in the study of the constructible hierarchy of sets by proving the so-called *Jensen's covering theorem* [Mit72, Theorem 1.1]: if $0^\sharp$ does not exist then for each set of ordinals x there is $y \in \mathrm{L}$ with $|y| = |x| + \aleph_1$ such that $x \subseteq y$. Thus, if $0^\sharp$ does not exist then V is *very close* to L, otherwise V is *very far* from L. An easy corollary of Jensen's theorem which clarifies the situation about the $\neg\mathsf{SCH}$ is the following: if $0^\sharp$ does not exist then the Singular Cardinal Hypothesis holds.

The assertion "$0^\sharp$ exists" follows from the existence of a measurable cardinal, and implies the existence of many large cardinals in L, e.g., every uncountable cardinal is weakly compact (and even more) in L [Kan09, §9]. In

[6] Actually, it suffices with $\{\lambda < \kappa \mid 2^\lambda = \lambda^+\}$ being a stationary subset of κ.

particular this assertion contradicts Gödel's axiom of constructibility $V = L$.

Jensen's theorem reveals that without large cardinals in the background there are no hopes to obtain the consistency of $\neg\mathsf{SCH} + \mathsf{ZFC}$. An alternative interpretation of this is the following: if large cardinals do not exist then V is so *close* to L that their respective theories of singular cardinals coincide.

The proof of the consistency of $\mathsf{ZFC} + \neg\mathsf{SCH}$ was obtained in 1970 by J. Silver and K. Prikry [Pri70]: Firstly, Silver showed if the GCH holds and κ is a supercompact, then one can force a generic extension where κ is supercompact and $2^\kappa = \kappa^{++}$ [Cum10, §12]. Secondly, Prikry defined a forcing notion (now called *Prikry forcing*) such that for a given measurable cardinal κ produces a cardinal-preserving generic extension where κ is a strong limit singular cardinal with $\mathrm{cof}(\kappa) = \omega$. Combining both things, the desired result follows.

Seven years later, in [Mag77a][Mag77b], M. Magidor proved an analogous consistency result for the more *down to earth* singular cardinal $\aleph_\omega$. Assuming enough large cardinals, and for each $1 \leq \alpha < \omega_1$, Magidor managed to produce a generic extension where "$\aleph_\omega$ is a strong limit cardinal$+\mathsf{GCH}_{<\aleph_\omega}+2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$" holds.[7] In particular, an *ultimate failure of the* SCH is consistent with ZFC: that is, the SCH can fail at the first singular cardinal (i.e., $\aleph_\omega$) while the GCH holds below it. This result was latter extended to $\aleph_{\omega_1}$ by Shelah [She83] who proved, for each $1 \leq \alpha < \omega_2$, that "$\aleph_{\omega_1}$ is a strong limit cardinal$+2^{\aleph_{\omega_1}} = \aleph_{\omega_1+\alpha+1}$" is consistent with ZFC. Unlike in Magidor's result, here Silver's theorem implies that $\mathsf{GCH}_{<\aleph_{\omega_1}}$ fails in Shelah's model. This result was later extended by M. Gitik [Git00] to any cardinal κ which is fixed point of the aleph function $\kappa \mapsto \aleph_\kappa$. Specifically, assuming the consistency of a supercompact cardinal, there is a generic extension of the universe with a cardinal $\kappa = \aleph_\kappa$ such that $\mathrm{GCH}_{<\kappa}$ and $2^\kappa = \lambda^+$, for any cardinal $\lambda \geq \kappa^+$.

The exact consistency strength for a failure of the SCH at a strong limit singular cardinal was later established by M. Gitik, that being the existence of a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ [Git89]. The need of very large cardinals is a recurrent theme in Singular Cardinal Combinatorics and, as we have already mentioned, it is related to fundamental issues arising from Inner Model theory. For more about this, see [Eis10][Mit10].

As a final word, let us say that the existence of very large cardinals also has an influence upon the global configuration of the function $\gimel$. Indeed, by a celebrated theorem of R. Solovay [Sol74] the SCH holds above the first strong compact cardinal κ. Thus, $\gimel(\lambda) = \lambda^+$ for each singular cardinal $\lambda > \kappa$ such that $2^{\mathrm{cof}(\lambda)} < \lambda$.

In the present dissertation we will be particularly interested in the connections between the failures of the SCH and other combinatorial principles

[7] Here $\mathsf{GCH}_{<\aleph_\omega}$ is a shorthand for "$\mathsf{GCH}_{\aleph_n}$, for all $n < \omega$"

at successors of strong limit singular cardinals. In Part II we focus our attention on the Tree Propery at the first two successors of a strong limit singular cardinal. As result we generalize the main theorems of [FHS18] and [Sin16]. In Part III we present a viable iteration scheme for Σ-Prikry forcings, an abstraction of the classical notion of Prikry-type forcing introduced in [PRS20]. As a first application we prove the consistency, modulo ω-many supercompact cardinals, of "$\mathsf{ZFC} + \neg\mathsf{SCH}_\kappa + \mathrm{Refl}({<}\omega, \kappa^+)$", when κ is a strong limit singular cardinal of countable cofinality. The principle $\mathrm{Refl}({<}\omega, \kappa^+)$ asserts that any finite family of stationary subsets of κ^+ *reflects*. That is, for each finite family $\langle S_n \mid n \leq m\rangle$ of stationary subsets of κ^+ there is an ordinal $\delta < \kappa^+$ with $\mathrm{cof}(\delta) > \omega$ such that $\langle S_n \cap \delta \mid n \leq m\rangle$ is a family of stationary subsets of δ.

Part II: The tree property

Given an infinite cardinal κ, a tree $\mathcal{T} = \langle T, \preceq\rangle$ is called a κ-tree if its height is κ and all of its levels are of size $<\kappa$. If moreover κ is a regular cardinal, a κ-tree $\mathcal{T}$ is called κ-Aronszajn if it has no cofinal branches, i.e., no $\preceq$-linearly order subsets of size κ. A regular cardinal κ is said to have the ***Tree Property*** (in symbols, $\mathrm{TP}(\kappa)$) if there are no κ-Aronszajn trees.

A classical result of J. König says that $\mathrm{TP}(\aleph_0)$ holds, so that there are no $\aleph_0$-Aronszajn trees. Our intuition would lead us to expect a similar result for larger cardinals but this turns out not to be the case. In 1934, N. Aronszajn showed that one can construct a $\aleph_1$-Aroszajn tree in ZFC, hence $\mathrm{TP}(\aleph_1)$ fails. This surprising finding prompts the following question:

(Υ) Does ZFC prove the existence of a κ-Aronszajn tree, for some regular cardinal $\kappa \geq \aleph_2$?

The first partial (negative) answer was given by W. Micthell [Mit72] and J. Silver who showed that "$\mathsf{ZFC} + \mathrm{TP}(\aleph_2)$" is equiconsistent with ZFC plus the existence of a weakly compact cardinal. Building on this, U. Abraham [Abr83] proved in 1983 the consistency of "$\mathsf{ZFC} + \mathrm{TP}(\aleph_2) + \mathrm{TP}(\aleph_3)$" from the consistency of ZFC plus the existence of two supercompact cardinals with a weakly compact cardinal above them. In the Mitchell and Abraham's models the GCH fails at all the relevant cardinals. This phenomenon does not happen by chance as, by virtue of a theorem of Specker [Spe90], if $\kappa^{<\kappa} = \kappa$ then there is a κ^{++}-Aronszajn tree.

A decade latter, J. Cummings and M. Foreman [CF98] generalized Abraham's result by deriving the consistency of "$\mathsf{ZFC} + \forall n < \omega\, \mathrm{TP}(\aleph_{n+2})$" from the consistency of ZFC plus the existence of ω-many supercompact cardinals. This result was later improved by I. Neeman who managed to force the $\mathrm{TP}(\aleph_{\omega+1})$ in the Cummings-Foreman model [Nee14]. Also in [FW91],

M. Foreman and W. Woodin proved the consistency of a global failure of the GCH, which is a necessary condition for a negative answer to (Υ).

The above consistency results provide a non-negligible evidence that the answer to (Υ) is negative: more precisely, that using large cardinals one may obtain a model of ZFC where $\mathrm{TP}(\kappa)$ holds, for each regular cardinal $\kappa \geq \aleph_2$. Nonetheless, this is just a conjecture, and actually one of the most important open problems in infinite combinatorics.

The above mentioned result of Specker reveals that the SCH has an impact upon the possible tree property configurations. Indeed, if κ is a singular cardinal and the SCH_κ holds then there is a κ^{++}-Aronszajn tree. Thus, under these conditions a failure of the SCH_κ is an essential requirement to have the $\mathrm{TP}(\kappa^{++})$. In [CF98] the authors proved that this scenario is consistent, modulo suitable large cardinals. More precisely, starting with a supercompact cardinal κ and a weakly compact cardinal above, Cummings and Foreman produced a generic extension where κ is a strong limit singular cardinal of countable cofinality and "$\mathsf{ZFC} + 2^\kappa = \kappa^{++} + \mathrm{TP}(\kappa^{++})$" holds.

A natural question is if in the Cummings-Foreman (**CF**) model one can also obtain more instances of the tree property: say, $\mathrm{TP}(\kappa^+)$ and $\mathrm{TP}(\kappa^{+3})$. Surprisingly, it turns out that each of these configurations lead to completely different problems. On the one hand, if one aims to get $\mathrm{TP}(\kappa^{+3})$ in the **CF**-model the first test question is if $\neg\mathsf{GCH}_{\kappa^+}$ can be forced there. Alternatively, one can ask whether 2^κ can be made arbitrarily large. In [FHS18], S. Friedman, R. Honzik and Š. Stejskalová answered this affirmatively starting from the same large-cardinal assumptions of [CF98].

In contrast, the situation for $\mathrm{TP}(\kappa^+)$ is much more involved. The reason for this being that this tree property configuration needs either a violation of weak covering [Mit10, Definition 1.9] or to begin with a model where weak square fails at a singular cardinal. In any of these two scenarios the necessary large-cardinal assumptions are much more stronger that those required for a failure of the SCH [Eis10, Theorem 2.6].

One of the most important questions in the field is related to this issue and was raised by W. H. Woodin [For05]: If κ is a strong limit singular cardinal with $\mathrm{cof}(\kappa) = \omega$, does $\neg\mathsf{SCH}_\kappa$ imply $\neg\mathrm{TP}(\kappa^+)$? For a historical motivation of this problem, see the introduction to Part II.

The first partial answer to Woodin's problem was given by M. Gitik and A. Sharon [GS08] who proved that it is consistent for a strong limit singular cardinal κ of countable cofinalitity to have $\neg\mathsf{SCH}_\kappa + \neg\square^*_\kappa$. For this purpose the authors assumed the consistency of ZFC with the existence of a $\kappa^{+\omega+2}$-supercompact cardinal κ. Nonentheless, this did not answer Woodin's question, since a failure of $\square^*_\kappa$ yields a weaker statement than $\mathrm{TP}(\kappa^+)$ [Jen72].

Following the previous work of Gitik and Sharon, I. Neeman [Nee09] subsequently obtained the consistency of $\mathsf{ZFC} + \neg\mathsf{SCH}_\kappa + \mathrm{TP}(\kappa^+)$ for a strong limit singular cardinal κ of countable cofinalitity. This finally settled negatively Woodin's problem. For this purpose Neeman assumed the consistency of ZFC with the existence of ω-many supercompact cardinals. Building upon all of these works, D. Sinapova [Sin16] finally arrived to the desired conclusion: namely, $\mathrm{TP}(\kappa^+)$ can be forced in the **CF**-model. More precisely, assuming the existence of ω-many measurable cardinals together with a weakly compact cardinal above them all, one can force a generic extension where $\mathsf{ZFC} + 2^\kappa = \kappa^{++} + \mathrm{TP}(\kappa^+) + \mathrm{TP}(\kappa^{++})$ holds for a strong limit singular cardinal κ with $\mathrm{cof}(\kappa) = \omega$.

Our aim in Part II is to obtain a simultaneous generalization of the main results of [FHS18] and [Sin16], also allowing arbitrary cofinalities.

1. In **Chapter 7** we prove the consistency of $\mathsf{ZFC} + 2^\kappa = \Theta + \mathrm{TP}(\kappa^{++})$, for a strong limit singular cardinal κ with arbitrary cofinality, $\Theta \geq \kappa^{++}$ with $\mathrm{cof}(\Theta) > \kappa$. This generalization of the main result of [FHS18] has been obtained joint with M. Golshani [GP20].

2. In **Chapter 8** (see also [Pov20]) we generalize the main theorem of **Chapter 7** and prove the consistency of $\mathsf{ZFC} + 2^\kappa = \Theta + \mathrm{TP}(\kappa^+) + \mathrm{TP}(\kappa^{++})$, for a strong limit singular cardinal κ with $\mathrm{cof}(\kappa) = \mu$ and $\Theta \geq \kappa^{++}$ with $\mathrm{cof}(\Theta) > \kappa$. This extends the main result of [Sin16] and [FHS18] in two ways: first, by allowing arbitrary failures of the SCH and, second, by allowing arbitrary cofinalities. As additional results we prove a criterion for genericity for Sinapova sequences and show how to define Sinapova generics by means of iterated ultrapowers. This extends a classical result of A. Mathias [Mat73] (resp. W. Mitchell [Mit82]) and R. Solovay [Kan09, Theorem 19.18(a)] (resp. G. Fuchs [Fuc14]) in the context of Prikry forcing (resp. Magidor forcing).

It is also known that the mere presence of very large cardinals has an impact upon the existence of Aronszajn trees. A celebrated result in this direction is due to M. Magidor and S. Shelah [MS96] and says that if $\langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of strong compact cardinals then $\mathrm{TP}(\kappa_\omega^+)$ holds, where $\kappa_\omega := \sup_{n<\omega} \kappa_n$. In the last chapter of Part II we revisit this result and manage to weaken the large-cardinal assumptions for this to hold.

3 In **Chapter 9** we prove that the Magidor-Shelah theorem can be proved by just assuming that $\langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of the first δ_n-strong compact cardinals, where $\delta_n \leq \kappa_n < \delta_{n+1}$, for $n < \omega$.

Part III: Σ-Prikry forcings and their iterations

Recursive definitions and iterative arguments are ubiquitous in Mathematics and, of course, Set Theory is not an exception.

Let us illustrate a very common example of iterative argument in the set-theoretic practice: Assume we want to prove the consistency of a proposition φ saying that "Every uncountable group having property p has property q as well". Suppose that M is a model of ZFC and that $\mathcal{G} \in M$ is an uncountable group in M. Without loss of generality we may assume that $\mathcal{G}$ is a counterexample to φ in M. A typical strategy is then to define a forcing notion $\mathbb{P}_{\mathcal{G}}$ such that for any $\mathbb{P}_{\mathcal{G}}$-generic set G, either $M[G]$ thinks that $\mathcal{G}$ has property q, or $M[G]$ thinks that $\mathcal{G}$ ceased to have property p. In other words, one defines a poset whose generic extension leads to a solution for the problem raised by $\mathcal{G}$.

Let us assume for a moment that the above attempt was successful. Of course this will solve the problem suggested by $\mathcal{G}$, but it is very likely that in our new model $M[G]$ there are other (possibly new) counterexamples to φ. This means that we need to fix yet another counterexample $\mathcal{H} \in M[G]$ and pass to a forcing extension $M[G][H]$ solving the problem witnessed by $\mathcal{H}$. Thus, we are lead to repeat this argument as many times as necessary until we manage to *catch our tail.*

To have a chance for the above iteration to succeed there is usually a need of a transfinite forcing iteration. However, such iterations need to be defined carefully as they may *collapse* cardinals, hence making countable all the uncountable groups from the intermediate models.

The first successful transfinite iteration was devised by R. Solovay and S. Tennenbaum [ST71] who applied it to prove the consistency of *Suslin's Hypothesis* (SH) with ZFC. Despite having been shown to be very versatile, the Solovay-Tennembaum iterations do not allow to tackle problems about cardinals $\geq \aleph_2$. The reason behind this is the poor behavior of the natural generalizations of the *countable chain condition* (ccc) to higher cardinals [Rin14; LHR18; Ros18]. Recall that the ccc – equivalently, $\aleph_1$-cc – of a forcing notion is the usual requirement to secure that the cardinal structure of the *ground model* has not been changed.

Still, for regular cardinals $\kappa \geq \aleph_2$ and forcings enjoying strong forms of the κ^+-cc there are available many successful iteration schemata [She78; She03a; RS01; Eis03], which contrasts with the dearth of techniques in the context of singular cardinals. Certainly, this entails a serious obstacle at the time of solving problems arising from Singular Cardinal Combinatorics. Among other reasons, this lack of results is consequence of the singular compactness phenomena mentioned in page **??**. Namely, singular cardinals are sensitive to changes made at smaller cardinals.

There are some few approaches to this problem with forcing iterations due to M. Džamonja and S. Shelah [DS03], and also to Džamonja and other authors [Cum+17]. The idea behind this type of iterations can be summarized as follows: Assume that κ is a very large cardinal (say supercompact) and that we want to establish the consistency of a strong limit singular cardinal to have certain combinatorial property $\varphi(x)$. To this aim first define $\mathbb{P}$ to be a forcing iteration which eliminates all the counterexamples to $\varphi(\kappa)$. In the resulting generic extension, look for a (Prikry-type) forcing $\mathbb{Q}$ aimed to make κ a strong limit singular cardinal of the desired cofinality, and force with the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$. If $\mathbb{P}$ has been devised to *anticipate* the further effect of $\mathbb{Q}$ then $\mathbb{P} * \dot{\mathbb{Q}}$ yields a model where κ is strong limit singular and $\varphi(\kappa)$ holds. For a concrete application of this iteration scheme see the above cited references.

A fundamental class of forcing notions in Singular Cardinal Combinatorics is the family of Prikry-type forcings [Git10]. These posets are mainly conceived to change cofinalities but certain sophisticated versions also allow to manipulate the size of the power set of a singular cardinal [GM94][Git19]. Besides, under certain conditions, some of them preserve regularity [CW], which has been fruitfully exploited in [FW91][Cum92][Mer07].

The first Prikry-type forcing discovered was the now called *Prikry forcing*. This poset was used by K. Prikry in his proof of the consistency of $\mathsf{ZFC}+\neg\mathsf{SCH}$ [Pri70]. Given a measurable cardinal κ and $\mathcal{U}$ a κ-complete normal ultrafilter over it, Prikry forcing $\mathbb{P}_{\mathcal{U}}$ yields a cardinal-preserving generic extension where κ becomes a strong limit singular cardinal of countable cofinality.[8] Further generalizations of Prikry forcing to the context of uncountable cofinalities were subsequently obtained by M. Magidor [Mag78] and L. Radin [Rad82]. Other more sophisticated constructions than Prikry forcing appear in [GM94][Git96][FW91][FHS18][Mer03][Sin08][Cum+18].

Prikry-type forcings have stood out by their prolific applications: many of them on Singular Cardinal Combintorics [Pri70][Mag77a][Mag77b][She83][GS08][Nee09], but other important ones have found its place in Algebra [MS94], Topology [Dow95] and other areas of Set Theory [Mag76][BM14a][GS89][Mit10][GM18a][BNU17]. On what we are concerned, the main contributions of this dissertation have been obtained by means of some of these forcings.

The above discussion leads us to the following conclusion: finding a *viable* procedure to iterate Prikry-type forcings is an essential enterprise for the development of Singular Cardinal Combinatorics. Here by *viable* we mean that the resulting forcing iterations are still of Prikry-type and also that they have a good chain condition.

The first viable iteration schemata for Prikry-type forcings were introduced by M. Magidor and M. Gitik [Git10, §6]. To explain how these work let us borrow the following example from [Mag76]: Assume that κ is a strong compact cardinal. Our goal is to prove the consistency of κ being both the first measurable and the first strong compact cardinal. Let $\mathbb{P}$ be a Magidor

[8] Actually, as shown by Devlin [Dev74], it is sufficient with $\mathcal{U}$ being a Ramsey ultrafilter.

iteration of Prikry forcings [Git10, §6] such that for each measurable cardinal $\lambda < \kappa$, $\mathbb{P}$ forces that λ ceases to be measurable. First, it is not hard to show that in the resulting model there are no measurable cardinals $\lambda < \kappa$. Second, by a result of Magidor, κ remains strong compact, hence measurable, after forcing with $\mathbb{P}$. This yields a generic extension where the first measurable cardinal coincides with the first strong compact cardinal, as wanted.

In Part III we propose a new type of iterations of Prikry-type forcings. As a preliminary step, we find an abstraction of the classical notion of Prikry-type forcing – called Σ-*Prikry forcings* – and show that it is *iterable*. Here by *iterable* we mean that the iteration of Σ-Prikry forcings is of Σ-Prikry-type, and thus it enjoys both the *Prikry property* [Git10, §1] and a good chain condition.

An illustrative – though informal – way to explain how these new iteration schemata work is the following: Assume that κ is a strong limit singular cardinal of countable cofinality. Our aim is to solve certain problem about κ or some of its small successors (say κ^+) by means of a forcing iteration $\mathbb{P}$. Assume that $\mathcal{P} := \langle z_\alpha \mid \alpha < \Theta \rangle$ is an enumeration anticipating all the problems that $\mathbb{P} := \mathbb{P}_\Theta$ aims to solve about κ or κ^+.[9] For each $\alpha < \Theta$, the α^{th}+1-stage of our iteration, $\mathbb{P}_{\alpha+1}$, is defined by means of a functor $\mathbb{A}(\cdot, \cdot)$ in the ground model which, given the Σ-Prikry forcing $\mathbb{P}_\alpha$ and the problem z_α, yields a Σ-Prikry forcing projecting onto $\mathbb{P}_\alpha$ and solving the problem z_α. At limit stages $\mathbb{P}_\alpha$ is defined as the $<\Theta$-support iteration of the previous stages $\mathbb{P}_\beta$, $\beta < \alpha$. See **Chapter 14** for a more accurate exposition.

This new style of iterations has also some advantage with respect to the previous ones. For a discussion about this see page 183.

The following is a summary of the main results presented in Part III, which are part of a broad common project with A. Rinot and D. Sinapova [PRS19][PRS20]:

1. In **Chapter 10** we introduce the notion of Σ-Prikry forcing and prove some of its general properties. Then we show that this class is abstract enough to include many classical examples of Prikry-type forcing centered on countable cofinalities, such as the Gitik-Sharon poset [GS08] or the Extender-Based Prikry Forcing [GM94].

2. In **Chapter 11** we introduce the key notion of *forking projection* and discuss their connections with the Σ-Prikry framework.

3. In **Chapter 12** we carry out a technical analysis of finite simultaneous reflection of stationary sets in Σ-Prikry generic extensions. Here we also discuss how to get a model of $\mathrm{Refl}(<\omega, \kappa^+)$, when κ is a strong limit singular cardinal of countable cofinality.

[9] More formally, $\mathcal{P}$ is a bookkeeping enumeration of all the problems we need to solve.

4. In **Chapter 13** we present a Σ-Prikry forcing which destroys certain non-reflecting stationary subsets of κ^+.

5. In **Chapter 14** we present our iteration scheme for Σ-Prikry posets.

6. In **Chapter 15** we exhibit the first application of our iterations scheme. Namely, we prove the consistency of $\mathsf{ZFC} + \neg\mathsf{SCH}_\kappa + \mathrm{Refl}({<}\omega, \kappa^+)$ for a strong limit cardinal κ of countable cofinality, modulo the existence of ω-supercompact cardinals. We also argue that this result is the best possible.

CHAPTER 1

Preliminaries

In this chapter we outline some set-theoretic concepts that will appear in subsequent chapters. Our notational conventions in this work are either standard or will be properly specified. We assume some basic background in Model Theory, Forcing and Large cardinals. For any non defined notion we refer the reader to [CK90][Kun14][Jec03][Kan09] or to the specific references of each section.

1.1 Measures and Extenders

In the subsequent sections we will show how measures and extenders can be used to define some of the most important large-cardinal notions.

1.1.1 Measures and large cardinals

We begin recalling the notion of *filter*.

Definition 1.1.1. A filter on a non-empty set X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ for which the following are true:

1. $\emptyset \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{F}$.

Example 1.1.2.

1. For any non-empty set X, the sets $\mathcal{F} := \{X\}$ and $\mathcal{G} := \mathcal{P}(X)$ are filters on X.
2. Let X be a non-empty set and $Y \subseteq X$. The set $\mathcal{F}_Y := \{A \subseteq X \mid Y \subseteq A\}$ forms a filter. This kind of filters are called *principal*.

3. Let κ be a regular cardinal. The set $\mathcal{F}_\kappa := \{A \subseteq \kappa \mid |\kappa \setminus A| < \kappa\}$ is the *Fréchet filter on κ.*

4. Let κ be a regular cardinal. The set

$$\mathrm{Club}(\kappa) := \{\mathrm{X} \subseteq \kappa \mid \mathrm{C} \subseteq \mathrm{X} \text{ for some club } C\}$$

is a filter on κ called the *Club filter on κ.*

5. Let $(X, \mathscr{T})$ be a topological space and $x \in X$. The set of $\mathscr{T}$-neighbourhoods of x forms a filter.

6. Let μ be the Lebesgue measure on the unit interval $[0, 1]$. The set all subsets of $A \subseteq [0, 1]$ with $\mu(A) = 1$ is a filter on $[0, 1]$.

Definition 1.1.3. A filter $\mathcal{F}$ on a set X is said to be an ultrafilter if for every $Y \subseteq X$, either $Y \in \mathcal{F}$ or $X \setminus Y \in \mathcal{F}$.

A filter $\mathcal{F}$ on a set X is called maximal if for every other filter $\mathcal{G}$ on X with $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}$. It is not hard to show that a filter $\mathcal{F}$ is an ultrafilter if and only if it is a maximal filter. A classical result of A. Tarski establishes, under the AC, that for every filter $\mathcal{F}$ on a set X there is an ultrafilter $\mathcal{U}$ which extends $\mathcal{F}$, i.e., $\mathcal{F} \subseteq \mathcal{U}$.

An important family of ultrafilters in the theory of Large Cardinals are the *κ-complete ultrafilters.*

Definition 1.1.4. Let κ be an infinite cardinal and $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ is *κ-complete* if it is closed under taking intersections of size less than κ, i.e., for every $\lambda < \kappa$ and every family of sets $\{A_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{F}$, $\bigcap_{\alpha<\lambda} A_\alpha \in \mathcal{F}$.

Notice that every filter is $\aleph_0$-complete and that every principal filter is κ-complete, for every cardinal κ. It is easy to prove that the filter of Example 1.1.2(6) is $\aleph_1$-complete while the filters $\mathcal{F}_\kappa$ and $\mathrm{Club}(\kappa)$ are κ-complete.

If κ is a singular cardinal then any κ-complete filter on κ is κ^+-complete. Thus, when speaking about filters on a cardinal one may restrict to regular degrees of completeness.

Definition 1.1.5. Let κ be a regular cardinal. An ultrafilter $\mathcal{U}$ on κ is called a *measure* if it is non-principal and κ-complete.

The above definition leads to the notion of *measurable cardinal*, one of the prominent concepts in the theory of Large Cardinals.

Definition 1.1.6. An uncountable cardinal κ is called *measurable* if there is a measure on κ.

A classical result of A. Tarski and L. Ulam establishes that any measurable cardinal κ is (a limit of) inaccessible(s) cardinal(s), hence $V_\kappa \models \mathsf{ZFC}$ [Kan09, Theorem 2.8]. In particular, by Gödel's Second Incompleteness Theorem (i.e. ZFC, if consistent, cannot prove $\mathrm{Con}(\mathsf{ZFC})$), the existence of measurable cardinals cannot be established in ZFC. Another important feature of measurable cardinals is that they contradict Gödel's Axiom of Constructibility $\mathrm{V} = \mathrm{L}$, hence they do not exist in L (see [Kan09, Corollary 5.5]).

Measurable cardinals appear in many fundamental questions in Mathematics and were crucial to establish the independence of the *Lebesgue measure problem*. This is precisely what motivates the adoption of this terminology. For more details and historical references see [Kan09, §2].

Another important class of filters are the *normal filters*. For that definition we previously need the notion of *diagonal intersection*.

Definition 1.1.7. Let κ be an uncountable regular cardinal and $\{A_\alpha \mid \alpha < \kappa\}$ be a family of subsets of κ. The *diagonal interesection of* $\{A_\alpha \mid \alpha < \kappa\}$, $\triangle\{A_\alpha \mid \alpha < \kappa\}$, is defined by

$$\bigtriangleup\{A_\alpha \mid \alpha < \kappa\} := \{\alpha < \kappa \mid \alpha \in \bigcap_{\beta<\alpha} A_\beta\}.$$

Definition 1.1.8. Let κ be an uncountable regular cardinal. A filter $\mathcal{F}$ on κ is called *normal* if it is closed under taking diagonal intersections, i.e., for every family $\{A_\alpha \mid \alpha < \kappa\}$ of elements of $\mathcal{F}$, $\triangle\{A_\alpha \mid \alpha < \kappa\} \in \mathcal{F}$.

A paradigmatic example of normal filter is the club filter $\mathrm{Club}(\kappa)$, which is contained in every κ-complete normal filter on κ [Kan09, p. 53]. In particular, the members of any κ-complete normal ultrafilter on κ are stationary sets.

Also, again under the AC, every measurable cardinal κ carries a normal measure on κ [Kan09, Exercise 5.12]. Thus, when we speak about these measures we will additionally assume that all of them are normal.

Other relevant type of measures in Set Theory are those on the set $\mathcal{P}_\kappa(\lambda)$, where κ is a regular uncountable cardinal and $\lambda \geq \kappa$. For the reader's benefit let us recall that $\mathcal{P}_\kappa(\lambda)$ is the set of all subsets of λ with cardinality less than κ, i.e., $\mathcal{P}_\kappa(\lambda) := \{A \subseteq \lambda \mid |A| < \kappa\}$.

A new concept arising in the context of $\mathcal{P}_\kappa(\lambda)$ is *fineness*.

Definition 1.1.9. Let κ be a regular uncountable cardinal, $\lambda \geq \kappa$ and $\mathcal{F}$ be a filter on $\mathcal{P}_\kappa(\lambda)$. The filter $\mathcal{F}$ is called *fine* if for every ordinal $\gamma < \lambda$, the set $\{x \in \mathcal{P}_\kappa(\lambda) \mid \gamma \in x\}$ is a member of $\mathcal{F}$.

Definition 1.1.10. Let κ be a regular uncountable cardinal, $\lambda \geq \kappa$ and $\mathcal{U}$ be an ultrafilter on $\mathcal{P}_\kappa(\lambda)$. The ultrafilter $\mathcal{U}$ is called a *strong compact measure* on $\mathcal{P}_\kappa(\lambda)$ if it is a fine and κ-complete ultrafilter.

Definition 1.1.11. Let κ be cardinal and $\lambda \geq \kappa$. The cardinal κ is called *λ-strong compact* if there is a *strong compact measure* on $\mathcal{P}_\kappa(\lambda)$. Similarly, κ is called *strong compact* if it is *λ-strong compact*, for each $\lambda \geq \kappa$.

The notion of strong compactness emerges from A. Tarski's work on the extensions of the *Compactness Theorem* of First Order Logic to infinitary logics [CK90]. Actually, κ is a strong compact cardinal if and only if the infinitary logic $\mathcal{L}_{\kappa,\kappa}$ satisfy the natural generalization of the Compactness Theorem of First Order Logic [Kan09, §4].

The following theorem provides a useful reformulation of this notion.

Theorem 1.1.12 (Keisler-Tarski [Kan09])**.** *A cardinal κ is strong compact if and only if, for every non-empty set X, every κ-complete filter $\mathcal{F}$ on X can be extended to a κ-complete ultrafilter, i.e., there is a κ-complete ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subseteq \mathcal{U}$.*

If κ is a cardinal satisfying the above property it is easy to prove that it must be regular. Thus, if κ is strong compact, $\mathcal{F} := \mathrm{Club}(\kappa)$ is a κ-complete filter which can be extended to a κ-complete ultrafilter. In particular, every strong compact cardinal is measurable. Nonetheless, not every strong compact cardinal is a limit of measurable cardinals, as the first strong compact cardinal can be the first measurable [Mag76].

Inspired by Theorem 1.1.12, J. Bagaria and M. Magidor considered the following weakening of strong compactness.

Definition 1.1.13. Let $\delta \leq \kappa$ be uncountable cardinals. We say that κ is *δ-strong compact* cardinal if, for every non-empty set X, every κ-complete filter $\mathcal{F}$ on X can be extended to a δ-complete ultrafilter on X. The cardinal κ is called *almost strong compact* if it is δ-strong compact, for every $\delta < \kappa$.

In the light of Theorem 1.1.12, a cardinal κ is strong compact if and only if it is κ-strong compact. Observe that if κ is δ-strong compact then it is η-strong compact, for every uncountable cardinal $\eta \leq \delta$. Besides, if κ is δ-strong compact and $\lambda > \kappa$ then λ is also δ-strong compact. Therefore, for each uncountable cardinal δ, the only δ-strong compact cardinal which will deserve our attention is the least δ-strong compact. Finally, it is not hard to show that every almost strong compact cardinal is regular.

In this dissertation we will devote a special attention to the first ω_1-strong compact cardinal. The first ω_1-strong compact cardinal is always a limit cardinal greater than or equal to the first measurable. Also, the former is not necessarily regular, but in any case its cofinality is at least the first measurable. For the proofs of these facts see [BM14b].

There are some natural questions about this notion which have not been answered yet. For instance,

Question 1.1.14. Is the first ω_1-strong compact a strong limit cardinal?

In Magidor's models of [Mag76] the first ω_1-strong compact cardinal coincides with the first (almost) strong compact cardinal, hence the former can be the first measurable. Alternatively, one can use Radin forcing to produce a generic extension where the first ω_1-strong compact cardinal is singular, and thus neither measurable nor (almost) strong compact [BM14a]. An open question about the latter is the following:

Question 1.1.15. Is the first strong compact cardinal the first almost strong compact?

A natural strengthening of strong compactness may be defined by requiring that the strong compact measures are also normal. The following gives the analogous version of normality in the context of $\mathcal{P}_\kappa(\lambda)$.

Definition 1.1.16. Let κ be a regular uncountable cardinal and $\lambda \geq \kappa$. Let also $\{A_\alpha \mid \alpha < \kappa\}$ be a family of subsets of $\mathcal{P}_\kappa(\lambda)$. The *diagonal interesection of* the family $\{A_\alpha \mid \alpha < \lambda\}$, $\triangle\{A_\alpha \mid \alpha < \lambda\}$ is defined by

$$\triangle\{A_\alpha \mid \alpha < \lambda\} := \{x \in \mathcal{P}_\kappa(\lambda) \mid x \in \bigcap_{\alpha \in x} A_\alpha\}.$$

Definition 1.1.17. Let κ be a regular uncountable cardinal and $\lambda \geq \kappa$. A filter $\mathcal{F}$ on $\mathcal{P}_\kappa(\lambda)$ is called *normal* if it is closed under taking diagonal intersections, i.e., for every family $\{A_\alpha \mid \alpha < \kappa\}$ of elements of $\mathcal{F}$,

$$\triangle\{A_\alpha \mid \alpha < \lambda\} \in \mathcal{F}.$$

Definition 1.1.18. Let κ be a regular uncountable cardinal, $\lambda \geq \kappa$ and let $\mathcal{U}$ be an ultrafilter on $\mathcal{P}_\kappa(\lambda)$. The ultrafilter $\mathcal{U}$ is called a *supercompact measure* on $\mathcal{P}_\kappa(\lambda)$ if it is a fine, κ-complete and normal ultrafilter.

The large-cardinal notion arising from supercompact measures are the so-called *supercompact cardinals.*

Definition 1.1.19. Let κ be cardinal and $\lambda \geq \kappa$. The cardinal κ is called λ-*supercompact* if there is a *supercompact measure* on $\mathcal{P}_\kappa(\lambda)$. We will say that κ is *supercompact* if it is λ-*supercompact*, for each $\lambda \geq \kappa$.

Clearly any supercompact cardinal κ is also strong compact and, in particular, measurable. Actually, any supercompact measure $\mathcal{U}$ on $\mathcal{P}_\kappa(2^\kappa)$ contains the set $\{x \in \mathcal{P}_\kappa(2^\kappa) \mid \text{“}\operatorname{otp}(x) \text{ is a measurable cardinal”}\}$, so that any supercompact cardinal is a limit of measurable cardinals. Nonetheless, not every supercompact cardinal is a limit of strong compact cardinals, as the first supercompact can be consistently the first (ω_1-)strong compact [Mag76].

In the introduction to this dissertation we have already discussed the extraordinary importance of supercompact cardinals both for Set Theory and for other areas of Mathematics. We refrain to provide more details about this and refer the reader to [Kan09] and [Jec03].

1.1.2 Measures and elementary embeddings

Fix a first order language $\mathcal{L}$ and let $\mathfrak{N}$ and $\mathfrak{M}$ be two $\mathcal{L}$-structures[1] with underlying universes N and M, respectively. A function $j : \mathfrak{N} \to \mathfrak{M}$ is called an $\mathcal{L}$-elementary embedding if for every $\mathcal{L}$-formula $\varphi(x_0, \dots, x_n)$ and every $a_0, \dots, a_n \in N$,

$$\mathfrak{N} \models \varphi(a_0, \dots, a_n) \Leftrightarrow \mathfrak{M} \models \varphi(j(a_0), \dots, j(a_n)).$$

A $\mathcal{L}$-elementary embedding $j : \mathfrak{N} \to \mathfrak{M}$ is called *non-trivial* if j is not the identity function. If there is no confusion, the natural tendency is to identify the structures $\mathfrak{N}$ and $\mathfrak{M}$ with their respective underlying universes, N and M. If the language is clear from the context it is customary to omit $\mathcal{L}$.

In this dissertation we are interested in the non-trivial $\mathcal{L}$-elementary embeddings where $\mathcal{L}$ is the language of Set Theory $\mathcal{L}_\in := \{\in\}$. More particularly, we will be mainly concerned with non-trivial elementary embeddings of the form $j : V \to M$, where V is the universe of sets and $M \subseteq V$ is a transitive proper class. For any such elementary embedding there is a least ordinal α such that $j(\alpha) \neq \alpha$. This ordinal is called the *critical point* of the embedding j and is typically denoted by $\mathrm{crit}(j)$. Standard arguments show that $\mathrm{crit}(j)$ is actually a cardinal [Kan09, Proposition 5.1]. If $j : V \to M$ is a non-trivial elementary embedding with $\mathrm{crit}(j) = \kappa$ it is customary to call $j(\kappa)$ the target of the embedding j. Unless otherwise specified, any elementary embedding that we consider will be non-trivial.

A classical tool in Model Theory are ultrapowers [CK90]. The first applications of this technique to the study of Large Cardinals came on the early sixties by the hand of D. Scott, J. Keisler and A. Tarski. As we will show, the ultrapower construction can be used to obtain equivalent formulations of the large cardinals of Section 1.1 in terms of elementary embeddings.

Let κ be cardinal and $\mathcal{U}$ be an ultrafilter on κ. Denote by ${}^\kappa V$ the class of all functions $f : \kappa \to V$. We shall consider the following equivalence relation on ${}^\kappa V$: for each $f, g \in {}^\kappa V$, set

$$f =_{\mathcal{U}} g \text{ if and only if } \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in \mathcal{U}.$$

For each function $f \in {}^\kappa V$ we denote by $[f]^*_{\mathcal{U}}$ the corresponding equivalence relation. Observe however that $[f]^*_{\mathcal{U}}$ is not necessarily a set. To fix this, for

[1] Here we allow class $\mathcal{L}$-structures with a proper class as universe.

each function $f \in {}^\kappa V$, we define $[f]_\mathcal{U}$ to be the collection of all the members of $[f]^*_\mathcal{U}$ with minimal rank. More formally,

$$[f]_\mathcal{U} := \{g \in {}^\kappa V \mid g =_\mathcal{U} f \wedge \forall h(h =_\mathcal{U} g \rightarrow \text{rank(g)} \leq \text{rank(h)}\}.^2$$

For economy of the notation, we shall suppress $\mathcal{U}$ when refer to $[f]_\mathcal{U}$.

Let ${}^\kappa V/\mathcal{U}$ be the class of all the sets $[f]$, where $f \in {}^\kappa V$. We want to regard ${}^\kappa V/\mathcal{U}$ as a $\mathcal{L}_\in$-structure so that we need to establish an interpretation $\in_\mathcal{U}$ of the symbol $\in$. For each $[f], [g] \in {}^\kappa V/\mathcal{U}$, set

$$[f] \in_\mathcal{U} [g] \text{ if and only if } \{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in \mathcal{U}.$$

Definition 1.1.20. Let κ be a cardinal and $\mathcal{U}$ be an ultrafilter on κ. The ultrapower by $\mathcal{U}$ is the $\mathcal{L}_\in$-structure $\text{Ult}(V, \mathcal{U}) := \langle {}^\kappa V/\mathcal{U}, \in_\mathcal{U}\rangle$.

The fundamental result about ultrapowers is Łos theorem [CK90].

Theorem 1.1.21. *For any $\mathcal{L}_\in$-formula $\varphi(x_0, \ldots, x_n)$ and any finite set of functions $f_0, \ldots, f_n \in {}^\kappa V$,*

$$\text{Ult}(V, \mathcal{U}) \models \varphi([f_0], \ldots, [f_n]) \textit{ if and only if } \{\alpha < \kappa \mid \varphi(f_0(\alpha), \ldots, f_n(\alpha))\} \in \mathcal{U}.$$

An outright consequence of Theorem 1.1.21 is that φ is true if and only if $\text{Ult}(V, \mathcal{U}) \models \varphi$, for each $\mathcal{L}_\in$-sentence φ. In model-theoretic terminology, $\text{Ult}(V, \mathcal{U})$ is $\mathcal{L}_\in$-elementary equivalent to V [CK90].

For each set x denote by c_x the member of ${}^\kappa V$ with constant value x. It is not hard to check that $j_\mathcal{U} \colon V \rightarrow \text{Ult}(V, \mathcal{U})$ defined as $x \mapsto [c_x]$ defines an elementary embedding.

From the set-theoretic perspective $\text{Ult}(V, \mathcal{U})$ can be very wild, as it might be ill-founded. To have more control on these ultrapowers one needs to consider ultrafilters with additional combinatorial properties. For instance, $\text{Ult}(V, \mathcal{U})$ is well-founded if and only if $\mathcal{U}$ is a $\aleph_1$-complete ultrafilter [Kan09, §5]. In this latter case, there is an isomorphism π between $\text{Ult}(V, \mathcal{U})$ and a transitive class $\langle M, \in\rangle$ [Kun14, §1.9]. The map π is called the *Mostowski collapsing map* and $\langle M, \in\rangle$ the *Mostowski collapse of* $\text{Ult}(V, \mathcal{U})$. It follows that $j \colon V \rightarrow M$ is an elementary embedding, where $j := \pi \circ j_\mathcal{U}$. In a mild abuse of notation we sometimes identity $j_\mathcal{U}$ with j.

The first important connection between ultrapowers and Large Cardinals was discovered by J. Keisler and D. Scott [Kan09, §5]: If κ is a measurable cardinal and $\mathcal{U}$ is a (normal) measure witnessing it one can show that the embedding $j_\mathcal{U} \colon V \rightarrow M$ has $\text{crit}(j_\mathcal{U}) = \kappa$. Conversely, let $j \colon V \rightarrow M$ be an elementary embedding, with M a transitive class and $\text{crit}(j) = \kappa$. Defining

$$X \in \mathcal{U} \text{ if and only if } X \subseteq \kappa \ \wedge \ \kappa \in j(X)$$

[2] This is known as the *Scott's trick* [Kan09, §19].

it is routine to check that $\mathcal{U}$ is a measure on κ. This measure is called the measure derived from the elementary embedding j.

The above leads to the following reformulation of measurability.

Definition 1.1.22 (Keisler-Scott)**.** A cardinal κ is measurable if and only if there is an elementary embedding $j : V \to M$, with M a transitive (proper) class and $\mathrm{crit}(j) = \kappa$.

This duality between measures and elementary embeddings is also present in the context of strong compactness and supercompactness. Arguing in a similar fashion one may arrive at the following reformulations.

Definition 1.1.23. Let κ be a cardinal and $\lambda \geq \kappa$. The cardinal κ is called λ-*strong compact* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, and $D \in M$ such that $j[\lambda] \subseteq D$ and $M \models |D| < j(\kappa)$. We say that κ is *strong compact* if it is λ-strong compact, for each $\lambda \geq \kappa$.

Definition 1.1.24. Let κ be a cardinal and $\lambda \geq \kappa$ be an ordinal. The cardinal κ is λ-*supercompact* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ such that $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. We say that κ is *supercompact* if it is λ-supercompact, for each ordinal $\lambda \geq \kappa$.

1.1.3 Extenders

So far we have shown that some important Large Cardinals can be defined in terms of the existence of certain ultrafilters. Nonetheless, the elementary embeddings arising from the latter have certain inherent restrictions. For instance, if $j\colon V \to M$ is the elementary embedding arising from a measure $\mathcal{U}$ on κ then $|j(\kappa)| = 2^{\kappa}$. In particular, none of such embeddings will map κ above $(2^{\kappa})^{+}$.

An important large-cardinal notion which cannot be defined in terms of the existence of a measure is *strongness*.

Definition 1.1.25. Let κ be a cardinal and $\lambda \geq \kappa$ be an ordinal. The cardinal κ is λ-*strong* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ such that $j(\kappa) > \lambda$ and $V_{\lambda} \subseteq M$. The cardinal κ is called *strong* if it is λ-strong, for each ordinal $\lambda \geq \kappa$.

By definition any cardinal κ which is κ-strong is also measurable. Moreover, κ is measurable if and only if κ is $\kappa + 1$-strong [Kan09, §5]. If κ is a $\kappa + 2$-strong cardinal then there is a stationary subset $S \subseteq \kappa$ formed by measurable cardinals, hence strongness yields a stronger notion than measurability [Kan09, §26].

However, it is not necessarily true that a λ-supercompact cardinal κ is also λ-strong: for instance, a κ^{+}-supercompact embedding with critical point

κ does not necessarily witness the κ^+-strongness of κ [Kan09, §22]. Nonetheless, any supercompact cardinal is strong and, furthermore, the first supercompact cardinal is strictly greater than the first strong cardinal [AC01, Lemma 2.1].

To get a characterization of strongness akin to those obtained in Section 1.1.1 we need the more sophisticated notion of *extender*.

Definition 1.1.26. Let κ be an infinite cardinal, $\lambda > \kappa$ and a sequence of sets $E := \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$. We say that E is a (κ, λ)-extender if, for some $\zeta \geq \kappa$, the following conditions are true:

1. (a) For all $a \in [\lambda]^{<\omega}$, E_a is a κ-complete ultrafilter on $[\zeta]^{|a|}$.
   (b) There is some $a \in [\lambda]^{<\omega}$ for which E_a is not κ^+-complete.
   (c) For all $\gamma < \zeta$, there is $a \in [\lambda]^{<\omega}$ such that $\{s \in [\zeta]^{|a|} \mid \gamma \in s\} \in E_a$.
2. (Coherence) For all $a, b \in [\lambda]^{<\omega}$, with $a \subseteq b$,
$$(\star)\ \ X \in E_a \iff \{s \in [\zeta]^{|b|} \mid \pi_{ba}(s) \in X\} \in E_b.$$
   Here $\pi_{ba}\colon [\zeta]^{|b|} \to [\zeta]^{|a|}$ is defined as follows. Say $b = \{\xi_1, \ldots, \xi_n\}$ and $a = \{\xi_{i_1}, \ldots, \xi_{i_m}\}$. Then, $\pi_{ba}(\{\alpha_1, \ldots, \alpha_n\}) := \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$.[3]
3. (Normality) If for some $a \in [\lambda]^{<\omega}$ and some $f : [\zeta]^{|a|} \to V$
$$\{s \in [\zeta]^{|a|} \mid f(s) \in \max(s)\} \in E_a,$$
   then there is $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that
$$\{s \in [\zeta]^{|b|} \mid f(\pi_{ba}(s)) \in s\} \in E_a.$$
4. (Well-foundedness) For any $\langle a_n \mid n < \omega\rangle \subseteq [\lambda]^{<\omega}$ and $\langle X_n \mid n < \omega\rangle \in \prod_{n<\omega} E_{a_n}$, there exists an order-preserving function $d : \bigcup_{n<\omega} a_n \to \zeta$ such that $d[a_n] \in X_n$, for all $n < \omega$.

Let $E := \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ be a (κ, λ)-extender. For each $a \in [\lambda]^{<\omega}$, let M_a be the Mostowski collapse of $\mathrm{Ult}(V, E_a)$ and $j_a\colon V \to M_a$ be the corresponding elementary embedding. Also, for each $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, set $j_{ba}([f]_{E_a}) := [f \circ \pi_{ba}]_{E_b}$. Standard arguments show that $j_{ba}\colon M_b \to M_a$ is a well-defined elementary embedding. Using the clauses of Definition 1.1.26 one can show that the family

$$\mathfrak{M} := \langle\langle M_a \mid a \in [\lambda]^{<\omega}\rangle, \langle j_{ba} \mid a, b \in [\lambda]^{<\omega}, a \subseteq b\rangle\rangle$$

[3] The reader familiar with ultrapowers would have noticed that π_{ba} is just the function representing the *seed* a in the ultrapower $\mathrm{Ult}(V, E_b)$. Notice that $(\star)$ is just saying that E_a is the Rudin-Keisler projection of E_b under π_{ba} [Kan09].

defines a *directed system* [Kan09, §0]. Thus, we can take $j_E^* : V \to M_E^*$, the direct limit of $\mathfrak{M}$. Finally, clause (4) of Definition 1.1.26 can be used to check that M_E^* is well-founded, hence we can compose with the corresponding Mostowski collapsing map to get an elementary embedding $j_E : V \to M_E$, where M_E is a transitive (proper) class [Kan09, §26]. This is customarily called the elementary embedding associated to E.

The following proposition summarizes the main properties of j_E and M_E.

Proposition 1.1.27. *Let E be a (κ, λ)-extender. Then the following hold:*

1. $\mathrm{crit}(j_E) = \kappa$ *and ζ is the least ordinal such that $j_E(\zeta) \geq \lambda$.*
2. $M_E := \{j_E(f)(a) \mid a \in [\lambda]^{<\omega},\ f\colon [\zeta]^{|a|} \to V,\ f \in V\}$.
3. *If $\mu > \lambda$ is a strong limit cardinal with* $\mathrm{cof}(\mu) > \zeta$*, then $j_E(\mu) = \mu$.*
4. *For any set X with $|X| > \zeta$, $j_E[X] \notin M_E$.*
5. *$E \notin M_E$.*

There is a duality between extenders and elementary embeddings which reminds of the analogous situation with measures: We have already shown that any (κ, λ)-extender E gives rise an elementary embedding $j_E\colon V \to M_E$, with M_E a transitive class and $\mathrm{crit}(j_E) = \kappa$. Conversely, let $j : V \to M$ be an elementary embedding, with M transitive, $\mathrm{crit}(j) = \kappa$ and $\lambda \geq \kappa$. Fix $\zeta \geq \kappa$ be the least ordinal for which $\lambda \geq j(\zeta)$. For each $a \in [\lambda]^{<\omega}$, define

$$X \in E_a \text{ if and only if } X \subseteq [\zeta]^{|a|} \wedge a \in j(X),$$

and set $E := \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$. It can be shown that E fulfils the clauses of Definition 1.1.26, hence E is a (κ, λ)-extender. This extender is called the (κ, λ)-extender derived from the embedding j.

Remark 1.1.28. Let E be a (κ, λ)-extender and form $j_E : V \to M_E$, the associated elementary embedding. Now use j_E to define E', the (κ, λ)-extender derived from j_E. It can be shown that $E = E'$ [Kan09, §26].

Finally, we are in conditions to present the desired characterization of strong cardinals.

Theorem 1.1.29 ([Kan09, §26])**.** *Let $\kappa \leq \lambda$.*

1. *A cardinal κ is λ-strong if and only if there is E a $(\kappa, |V_{\kappa+\lambda}|^+)$-extender such that $V_{\kappa+\lambda} \subseteq M_E$ and $j_E(\kappa) > \lambda$.*
2. *A cardinal κ is strong if and only if the above holds, for each $\lambda \geq \kappa$.*

Another important large-cardinal notion that can be naturally defined in terms of extenders is *superstrongness*.

Definition 1.1.30. A cardinal κ is *superstrong* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

Any superstrong cardinal is measurable and, actually, a limit of measurable cardinals. It is not true that a superstrong cardinal is necessarily strong or supercompact, nor viceversa. But, if κ is 2^κ-supercompact, then κ is a (stationary) limit of superstrong cardinals [Kan09, §26].

Theorem 1.1.31 ([Kan09, §26])**.** *A cardinal κ is superstrong if and only if there is $\lambda > \kappa$ and a (κ, λ)-extender E such that $V_{j_E(\kappa)} \subseteq M_E$.*

1.2 The $C^{(n)}$–large cardinals

Given a natural number n, an ordinal α and $\Gamma \in \{\Sigma, \Pi\}$, we will say that α is a Γ_n-correct ordinal if V_α is a Γ_n-elementary substructure of the universe of sets V (in symbols, $V_\alpha \prec_{\Gamma_n} V$). That is, for every Γ_n formula $\varphi(x_0, \dots, x_n)$ in the language of Set Theory and each finite set of parameters $a_0, \dots, a_n \in V_\alpha$,

$$(\star)\ \ V_\alpha \models \varphi(a_0, \dots, a_n) \ \Leftrightarrow\ \varphi(a_0, \dots, a_n).$$

It is easy to see that $V_\alpha \prec_{\Sigma_n} V$ if and only if $V_\alpha \prec_{\Pi_n} V$, hence there is no difference between Σ_n-correctness and Π_n-correctness. Following [Bag12], for each natural number n set

$$C^{(n)} := \{\alpha \in \mathrm{ORD} \mid V_\alpha \prec_{\Sigma_n} V\}.$$

The classical *Reflection Theorem* [Kun14, Theorem II.5.3] guarantees that each $C^{(n)}$ is a club proper class of ordinals, i.e., a proper class of ordinals which is closed under taking suprema. Clearly, $C^{(n+1)} \subseteq C^{(n)}$, for each natural number n.

Let $\alpha \in C^{(n)}$ and φ be a Γ_{n+1} formula. If $\Gamma = \Sigma$ and $V_\alpha \models \varphi$, then φ is true. Similarly, if $\Gamma = \Pi$ and φ is true, then $V_\alpha \models \varphi$. This easy observation will we used throughout the dissertation without any further comment.

Observe that $C^{(0)} = \mathrm{ORD}$, as Δ_0 formulae are absolute between transitive structures [Kun14, Ch. II]. In particular, $C^{(0)}$ is a Δ_0-definable proper class. With a bit more work one can prove the following

$$C^{(1)} = \{\alpha \in \mathrm{Card} \mid \alpha \geq \aleph_1 \wedge H_\alpha = V_\alpha\}.$$

Thus the class $C^{(1)}$ is Π_1-definable, but not Σ_1-definable. Similarly, for each $n \geq 2$, the classes $C^{(n)}$ are Π_n-definable but not Σ_n-definable. For this one uses the Σ_n-definability of the truth predicate $\models$ for Σ_n sentences.[4] An

[4] A combinatorial characterization akin to the case $n = 1$ is not possible. For more details, see [Tsa12, pag. 59]

outright consequence of this is that the first member of $C^{(n)}$ is below the corresponding member of $C^{(n+1)}$. Similarly, the first ordinal in $C^{(n)}$ is not a Σ_{n+1}-correct cardinal, hence $C^{(n+1)} \subsetneq C^{(n)}$.

The classes $C^{(n)}$- form a basis for the definable club proper classes. Specifically, if $\mathcal{C}$ is a Σ_n-definable proper class of ordinals then $C^{(n)} \subseteq \mathcal{C}$. For proofs and further information we refer the reader to [Bag12, §1].

A family of cardinals that we want to study is the class of $C^{(n)}$–supercompact and $C^{(n)}$–extendible cardinals. These classes of cardinals were introduced by J. Bagaria[5] in his study of the large-cardinal hierarchy comprised between the first supercompact cardinal and Vopěnka's Principle (VP). Hereafter let $n \geq 1$ be a fixed natural number.

Definition 1.2.1. Let κ be a cardinal and let $\lambda > \kappa$ be an ordinal. The cardinal κ is *λ-$C^{(n)}$–supercompact* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $j(\kappa) \in C^{(n)}$ and ${}^{\lambda}M \subseteq M$. The cardinal κ is *$C^{(n)}$–supercompact* if it is λ-$C^{(n)}$–supercompact, for each $\lambda \geq \kappa$.

Clearly, any $C^{(n)}$–supercompact cardinal is supercompact, but the converse is not necessarily true (cf. Theorem 3.1.1). For each $n \geq 1$, the statement "κ is λ-$C^{(n)}$–supercompact" is equivalent to a Σ_{n+1} property of κ and λ, hence $C^{(n)}$–supercompactness is a Π_{n+2}-definable property [Bag12, §5].

A natural question is if the class of $C^{(n)}$–supercompact cardinals forms a proper hierarchy, i.e., if the first $C^{(n)}$–supercompact cardinal is strictly below the first $C^{(n+1)}$-supercompact [Bag12, §5]. A test question for this is if the first $C^{(n)}$–supercompact cardinal is a member of $C^{(n+2)}$. In Chapter 3 and Chapter 4 we will answer these questions negatively.

The second notion that we want to study is $C^{(n)}$–extendibility, which is a strengthening of the classical notion of *extendible cardinal*:

Definition 1.2.2. Let κ be a cardinal and let $\lambda > \kappa$ be an ordinal. The cardinal κ is called *λ-extendible* if there is an ordinal θ and an elementary embedding $j : V_\lambda \to V_\theta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. The cardinal κ is called *extendible* if it is λ-extendible, for each ordinal $\lambda > \kappa$.

It should be mentioned that any extendible cardinal is also supercompact [Kan09, Proposition 23.6] and, actually, a (stationary) limit of supercompact cardinals. In particular, the first extendible cardinal is above the first supercompact and, as a result, far above the rest of large cardinals of Section 1.1.1 and Section 1.1.3.

Definition 1.2.3. Let κ be a cardinal and let $\lambda > \kappa$ be an ordinal. The cardinal κ is *λ-$C^{(n)}$–extendible* if there is an ordinal θ and an elementary

[5] Actually, $C^{(n)}$–extendibility first appeared in [Bag+15].

embedding $j : V_\lambda \to V_\theta$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. The cardinal κ is $C^{(n)}$*–extendible* if it is λ-$C^{(n)}$–extendible, for each $\lambda > \kappa$.

Evidently, any $C^{(n)}$–extendible cardinal is extendible and also any extendible cardinal is $C^{(1)}$-extendible. Besides, any $C^{(n)}$–extendible cardinal is $C^{(n)}$–supercompact, though this is not as straightforward as before.

Definition 1.2.4 (Tsaprounis [Tsa12])**.** A cardinal κ is $C^{(n)}$–supercompact and κ-superstrong if for every ordinal $\lambda \geq \kappa$ there is a λ-$C^{(n)}$–supercompact embedding $j : V \to M$ which moreover satisfies $V_{j(\kappa)} \subseteq M$.

Theorem 1.2.5 (Tsaprounis [Tsa12])**.** *A cardinal κ is $C^{(n)}$–extendible if and only if κ is $C^{(n)}$–supercompact and κ-superstrong.*

An outright consequence of Theorem 1.2.5 is that any $C^{(n)}$–extendible cardinal is also $C^{(n)}$–supercompact. Nonetheless, the converse is not true (cf. Theorem 2.0.6). Another important consequence of Theorem 1.2.5 is that the property "κ is $C^{(n)}$–extendible" is definable by means of a Π_{n+2} formula. Thus, if one combines this with the Σ_{n+2}-correctness of $C^{(n)}$–extendible cardinals [Bag12, Proposition 3.4] one arrives at the conclusion that the first $C^{(n+1)}$-extendible cardinal is above the first $C^{(n)}$–extendible, hence $C^{(n)}$–extendibility entails a proper hierarchy.

In [Bag12] the author considers the following apparently stronger form of $C^{(n)}$–extendibility:

Definition 1.2.6. Let κ be a cardinal and $\lambda \in C^{(n)} \setminus \kappa^+$. The cardinal κ is called λ-$C^{(n)+}$*-extendible* if there is a cardinal $\theta \in C^{(n)}$ and an elementary embedding $j : V_\lambda \to V_\theta$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. The cardinal κ is called *extendible* if it is λ-$C^{(n)+}$-extendible, for $\lambda \in C^{(n)} \setminus \kappa^+$.

In [Tsa18] Tsaprounis proved that any $C^{(n)}$–extendible cardinal is also $C^{(n)+}$-extendible, hence both notions are equivalent. In Theorem 5.2.3 (see also Remark 5.2.4) we also obtain an indirect proof of this when we prove our Magidor-like characterization of $C^{(n)}$–extendibility.

The class of $C^{(n)}$–extendible cardinals has found relevant applications in algebraic topology, homotopy theory and category theory [Bag+15][BBT13]. As shown in [Bag12, §4] and [Bag+15], there is a narrow relationship between *structural reflection principles* and $C^{(n)}$–extendible cardinals.

Definition 1.2.7 ([Bag12][Bag+15])**.** Let $n < \omega$ and $\Gamma \in \{\Sigma, \Pi\}$. We write $\mathrm{VP}(\Gamma_n)$ (resp. $\mathrm{VP}(\boldsymbol{\Gamma}_n)$) if for every Γ_n-definable (resp. Γ_n-definable with parameters) proper class $\mathfrak{C}$ of relational structures in the same language there are $\mathcal{M}, \mathcal{N} \in \mathfrak{C}$ with $\mathcal{M} \neq \mathcal{N}$, and an elementary embedding $j : \mathcal{M} \to \mathcal{N}$.

We say that Vopěnka's Principle (VP) holds if the schema of formulae $\mathrm{VP}(\Gamma_n)$ holds, for each $n < \omega$.

Definition 1.2.8 ([Bag12][Bag+15])**.** Let $n < \omega$, $\Gamma \in \{\mathbf{\Sigma}, \mathbf{\Pi}\}$ and κ be an infinite cardinal. We write $\mathrm{VP}(\kappa, \Gamma_n)$ if for every Γ_n-definable proper class $\mathfrak{C}$ of relational structures in the same language $\mathcal{L}$, such that both $\mathcal{L}$ and the parameters of some Γ-definition of $\mathfrak{C}$, if any, belong to H_κ, $\mathfrak{C}$ *reflects below* κ, i.e., for every $\mathcal{M} \in \mathfrak{C}$ there is $\mathcal{N} \in \mathfrak{C} \cap H_\kappa$ and an elementary embedding $j : \mathcal{N} \to \mathcal{M}$.

The following theorem establishes the equivalence of VP with the existence of $C^{(n)}$-extendible cardinals.

Theorem 1.2.9 ([Bag12][Bag+15])**.** *The following are equivalent:*

1. VP
2. *For every $n \geq 1$ there exists a $C^{(n)}$–extendible cardinal.*

This generalizes a previous theorem of Magidor [Mag71]: namely, VP implies that extendible cardinals form a stationary proper class. The exact level-by-level equivalence between VP and $C^{(n)}$-extendible cardinals is given by the following theorem:

Theorem 1.2.10 ([Bag12])**.** *Let $n \geq 1$. The following are equivalent:*

1. $\mathrm{VP}(\Pi_{n+1})$*;*
2. *There exists a $C^{(n)}$–extendible cardinal;*
3. $\mathrm{VP}(\kappa, \mathbf{\Sigma}_{n+2})$ *holds, for some cardinal κ.*

If $n = 0$, $\mathrm{VP}(\Pi_1)$ is equivalent to the existence of a supercompact cardinal, as well as to the existence of a cardinal κ such that $\mathrm{VP}(\kappa, \mathbf{\Sigma}_2)$ holds.

The above two theorems show that the $C^{(n)}$-extendible cardinals are arguably natural representatives of the large-cardinal hierarchy between the first supercompact cardinal and Vopěnka's Principle.

We close this section with a useful proposition collecting the complexities and degrees of definability of the large cardinals presented so far.

Proposition 1.2.11.

1. *Measurable cardinals are Σ_1-correct and Δ_2-definable.*
2. *Superstrong cardinals are Σ_1-correct and Σ_2-definable.*
3. *Strong cardinals are Σ_2-correct and Π_2-definable.*
4. *Supercompact cardinals are Σ_2-correct and Π_2-definable.*
5. *$C^{(n)}$–supercompact cardinals are Σ_2-correct and Π_{n+2}-definable.*
6. *$C^{(n)}$–extendible cardinals are Σ_{n+2}-correct and Π_{n+2}-definable.*

For the proofs of (1)-(4) see [Kan09], while for (5) and (6) see [Bag12][Tsa12].

1.3 Forcing

In this section we will recall some basic concepts from the theory of Forcing and discuss its effect upon large cardinals. Our exposition follows [Kun14] and [Cum10] where we refer the reader for further details.

1.3.1 Some generalities on Forcing

A *forcing notion*, or a simply a *forcing*, is a triple $\mathbb{P} := (P, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$, where $\leq_{\mathbb{P}}$ is a partial ordering on P and $\mathbb{1}_{\mathbb{P}}$ is the $\leq_{\mathbb{P}}$-greatest element of $\mathbb{P}$; i.e., $p \leq_{\mathbb{P}} \mathbb{1}_{\mathbb{P}}$, for each $p \in P$. The elements of P are usually called *conditions* of the forcing $\mathbb{P}$. In a mild abuse of notation it is customary to identify $\mathbb{P}$ with its underlying set of conditions. If $\mathbb{P}$ is clear from the context it is usual to omit the dependence of $\mathbb{P}$ both in $\leq_{\mathbb{P}}$ and $\mathbb{1}_{\mathbb{P}}$. Given two conditions $p, q \in \mathbb{P}$ we will say that p *is stronger than* q if $p \leq q$. The set of conditions in $\mathbb{P}$ stronger than a given condition $q \in \mathbb{P}$ is typically denoted by $\mathbb{P} \downarrow q$ or by $\mathbb{P}/q$. We say that p *and* q *are (*$\mathbb{P}$*-)compatible* if there is $r \in \mathbb{P}$ stronger than p and q. If two conditions p and q are not compatible it is said that they are *incompatible* and will be denoted by $p \perp q$. A forcing $\mathbb{P}$ is called *atomless* if it has no *atoms*, i.e., for all $r \in \mathbb{P}$ there are $p, q \in \mathbb{P} \downarrow r$ such that $p \perp q$. Finally, a forcing $\mathbb{P}$ is separative if for every $p, q \in \mathbb{P}$, $p \not\leq_{\mathbb{P}} q$ then there is $r \in \mathbb{P} \downarrow p$ such that $r \perp q$. Any forcing notion $\mathbb{P}$ can be identified with a separative forcing $\mathbb{Q}$, which is called the *separative quotient*[Jec03].

A set $A \subseteq \mathbb{P}$ is called an *antichain* if for each $p, q \in A$, $p \perp q$. Contrarily, a set $L \subseteq \mathbb{P}$ is called *linked* if every two conditions are compatible. Similarly, a set $C \subseteq \mathbb{P}$ is called *centered* if for every finite number of conditions $p_0, \dots, p_n \in C$ there is $q \in \mathbb{P}$ such that $q \leq p_i$, for each $0 \leq i \leq n$. Finally, a set $D \subseteq \mathbb{P}$ is called ($\mathbb{P}$-)*dense* (*below* q) if for every $p \in \mathbb{P}$ (resp. $p \in \mathbb{P} \downarrow q$) there is a condition $r \in D$ such that $r \leq p$.

Let $\mathfrak{M} := \langle M, \in \rangle$ be a countable transitive model of a big enough fragment of ZFC and $\mathbb{P} \in M$ be a forcing notion.[6] In this regard, we shall follow the convention established on page 6 and identify $\mathfrak{M}$ with its universe M. The set M is called the *ground model*.

A non-empty set G is called a $\mathbb{P}$-filter if $G \subseteq \mathbb{P}$, every two members of G are compatible and, if $p \in G$ and $p \leq q$, then $q \in G$. A filter G is called $\mathbb{P}$*-generic over* M if $G \cap D \neq \emptyset$, for every dense set $D \in M$. If both the ground model M and the forcing poset $\mathbb{P}$ are clear from the context we shall simply say that G is a *generic filter*. A classical result of H. Rasiowa and R. Sikorski says that for every ground model M and every $p \in \mathbb{P}$ there is a generic filter $G \subseteq \mathbb{P}$ such that $p \in G$ [Kun14, §III.3]. Besides, under the assumption that $\mathbb{P}$ is atomless, one can ensure that no generic filter G is a

[6] By big enough we mean that this fragment includes at least Kripke-Platek Set Theory KP [Bar17].

member of M [Kun14, §III.3]. Hereafter we will assume that M is our ground model, $\mathbb{P} \in M$ is a given forcing poset satisfying the said requirements and $G \subseteq \mathbb{P}$ is a generic filter.

Definition 1.3.1. The class $M^{\mathbb{P}}$ of *($\mathbb{P}$-)names* (over M) is defined in M by recursion on the ordinals as follows:

1. σ is a $\mathbb{P}$-name of rank 0 if $\sigma = \emptyset$.

2. σ is a $\mathbb{P}$-name of rank $\leq\alpha$ if all the elements of σ are pairs (π, p), where π is a $\mathbb{P}$-name of rank $<\alpha$ and $p \in \mathbb{P}$.

3. σ is a $\mathbb{P}$-name if it is a $\mathbb{P}$-name of some rank $\alpha \in \mathrm{ORD}$.

It is not hard to check that $M^{\mathbb{P}}$ is a Σ_1-definable proper class in M taking the forcing $\mathbb{P}$ as parameter. Actually, $M^{\mathbb{P}}$ is $\mathbf{\Delta}_1$-definable class in M, as the complement of $M^{\mathbb{P}}$ is also Σ_1-definable taking $\mathbb{P}$ as parameter.

Definition 1.3.2. For a set $x \in M$, the *standard name* of x is defined recursively as

$$\check{x} := \{(\check{y}, \mathbb{1}) \mid y \in x\}.$$

Similarly, the *standard name for the generic filter* G is defined recursively as

$$\dot{G} = \{(\check{p}, p) \mid p \in \mathbb{P}\}.$$

Definition 1.3.3. The *generic extension of* M *by* G is the set

$$M[G] := \{\sigma_G \mid \sigma \in M^{\mathbb{P}}\},$$

where σ_G is defined by recursion as follows:

1. If σ is a name of rank 0, then $\sigma_G = \emptyset$;

2. $\sigma_G := \{\tau_G \mid (\tau, p) \in \sigma,\ p \in G\}$.

For an ordinal α, $M[G]_\alpha := (M_\alpha)^{V[G]}$ and $M_\alpha[G] := \{\sigma_G \mid \sigma \in M_\alpha \cap M^{\mathbb{P}}\}$.

Remark 1.3.4.

- ($\star$) Observe that for each $x \in M$, $\check{x}_G = x$. Similarly, $\dot{G}_G = G$. Thus, $M \subseteq M[G]$ and $G \in M[G]$. Also, $M[G]$ is a transitive set.

- ($\star$) Under some assumptions on the forcing $\mathbb{P}$ one can guarantee that $M[G]_\alpha = M_\alpha[G]$, for relevant ordinals α. See, e.g., [Tsa12, §1.4.1].

An important feature of generic extensions is that the truth predicate in $M[G]$ can be coded within M by means of the so-called *forcing relation* $\Vdash_{\mathbb{P}}$.

Theorem 1.3.5 ([Kun14, §IV]). *Let $n \geq 1$ be a natural number. For each Σ_n formula $\varphi(x_0, \ldots, x_n)$ in the language $\mathcal{L}_{\mathbb{P}}$, $\sigma_0, \ldots, \sigma_n \in M^{\mathbb{P}}$ and $p \in \mathbb{P}$, the statement "$p \Vdash_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_n)$" is Σ_n-expressible in M taking $\mathbb{P}$, p and $\sigma_0, \ldots, \sigma_n$ as parameters. In particular,* the forcing relation $\Vdash_{\mathbb{P}}$ *restricted to Σ_n formulae in the language of Set Theory is Σ_n-definable in M with $\mathbb{P}$ as a parameter.*

Given a condition $p \in \mathbb{P}$ a formula $\varphi(x)$ and $\sigma \in M^{\mathbb{P}}$ we will say that p *forces* $\varphi(\sigma)$ if $p \Vdash_{\mathbb{P}} \varphi(\sigma)$. Similarly, we will say that p *decides* $\varphi(\sigma)$, in symbols $p \parallel_{\mathbb{P}} \varphi(\sigma)$, if either p forces $\varphi(\sigma)$ or p forces $\neg\varphi(\sigma)$. If the forcing $\mathbb{P}$ is clear from the context it is customary to suppress the mention of $\mathbb{P}$ both in $\Vdash_{\mathbb{P}}$ and in $\parallel_{\mathbb{P}}$.

The following is the so-called *Forcing Theorem.*

Theorem 1.3.6 ([Kun14, §IV]). *Let $\mathbb{P}$ be a forcing notion and $G \subseteq \mathbb{P}$ a generic filter. Then,*

$$M[G] \models \varphi(\sigma^1_G, \ldots, \sigma^n_G) \Leftrightarrow \exists p \in G \left(M \models p \Vdash_{\mathbb{P}} \varphi(\sigma^1, \ldots, \sigma^n)\right),$$

for each $\sigma^1, \ldots, \sigma^n \in M^{\mathbb{P}}$ and each formula $\varphi(x_0, \ldots, x_n)$.

Moreover, one can control the amount of ZFC that $M[G]$ satisfies. Namely,

Theorem 1.3.7 ([Kun14, §IV]). *If M is a model of ZFC then $M[G] \models \mathsf{ZFC}$.*

An easy consequence of the above theorems is the following:

Corollary 1.3.8. *Let M be a countable transitive model of ZFC, $\mathbb{P} \in M$ and $G \subseteq \mathbb{P}$ a generic filter. The generic extension $M[G]$ is the least transitive model N of ZFC such that $M \subseteq N$, $G \in N$ and* $\mathrm{ORD} \cap M = \mathrm{ORD} \cap N$.

It should be mentioned that many of the fundamental results in the theory of Forcing can be derived just from Kripke-Platek Set Theory (KP) or, at least, from mild extensions of it. For instance, if M is *an admissible set* (i.e., a transitive model of KP) satisfying, for some $n \geq 1$, Σ_n-Separation and Σ_n-Collection (see [Bar17]) then the forcing relation $\Vdash_{\mathbb{P}}$ restricted to Σ_n-formulae is Σ_n-definable. Besides, the equivalence of Theorem 1.3.6 holds for any Σ_n formula. If we moreover require that $M \models$ "Σ_n-recursion" it also follows that the generic extension $M[G]$ satisfies Σ_n-separation plus Σ_n-Collection.

Convention 1.3.9. A harmless practice in Forcing theory which we shall adopt is to consider the universe of sets V as our ground model. For a discussion of the metamathematical subtleties behind this see [Kun14, §IV.5.2].

An important class of maps between forcing notions are *projections.*

Definition 1.3.10. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions. A map $\pi : \mathbb{Q} \to \mathbb{P}$ is called a *projection* if π is order-preserving, $\pi(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$, and for all $q \in \mathbb{Q}$ and all $p \leq_{\mathbb{P}} \pi(q)$ there is $q' \leq_{\mathbb{Q}} q$ such that $\pi(q') \leq_{\mathbb{P}} p$.

Some standard facts about projections that we will use are the following

Lemma 1.3.11 ([Kun14, §IV])**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions and $\pi : \mathbb{Q} \to \mathbb{P}$ be a projection.*

1. *If $H \subseteq \mathbb{Q}$ is a $\mathbb{Q}$-generic filter over V then, $\{p \in \mathbb{P} \mid \exists q \in H\, \pi(q) \leq_{\mathbb{P}} p\}$ is $\mathbb{P}$-generic over V. This filter is called* the filter generated by $\pi[H]$.

2. *If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over V, let $\mathbb{Q}/G \in V[G]$ be defined as*

$$\mathbb{Q}/G := \{q \in \mathbb{Q} \mid \pi(q) \in G\}.$$

   Then, any $\mathbb{Q}/G$-generic filter over $V[G]$ is also $\mathbb{Q}$-generic over V.

3. *$V^{\mathbb{P}} \subseteq V^{\mathbb{Q}}$.*

Definition 1.3.12. Two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ are said to be *equivalent* if there are projections $\pi : \mathbb{Q} \to \mathbb{P}$ and $\sigma : \mathbb{P} \to \mathbb{Q}$.

If $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic forcings it is not hard to check that both yield the same generic extensions [Kun14, §IV].

We close this section stating an important theorem about forcing notions.

Theorem 1.3.13 ([Kun14, §III.4])**.** *For every poset $\mathbb{P}$ there exists a unique (up to isomorphism) Boolean algebra $\mathbb{B} := \mathrm{RO}(\mathbb{P})$ and a dense embedding $e : \mathbb{P} \to \mathbb{B} \setminus \{\mathbf{0}\}$ that preserves incompatibility. Moreover, if $\mathbb{P}$ is seperative, then e is injective.*

1.3.2 More about Forcing

In the previous section we have shown how to enlarge V to a transitive structure $V[G]$ satisfying the same axioms of ZFC and containing a given generic filter $G \subseteq \mathbb{P} \in V$. Nonetheless, unless $\mathbb{P}$ enjoys some additional properties, $V[G]$ will be typically very different to V. For instance, it can be the case that after adding the generic G the 2^{nd}-cardinal of V, $\aleph_2^V$, become the first infinite cardinal of $V[G]$, $\aleph_0^{V[G]}$. This is indeed an issue if, for instance, one aims that $V[G]$ contradicts the CH by adding $\aleph_2^V$-many Cohen reals. To prevent these unwanted situations one needs to require certain additional combinatorial assumptions on $\mathbb{P}$.

Definition 1.3.14. Let κ be an uncountable cardinal and let $\mathbb{P}$ be a forcing.

1. $\mathbb{P}$ has the *κ-chain condition* (κ-cc) if and only if every antichain of $\mathbb{P}$ has cardinality $<\kappa$. In the special case of $\kappa = \aleph_1$, it is customary to say that $\mathbb{P}$ has the *ccc* instead of the $\aleph_1$-cc.

2. $\mathbb{P}$ is *κ-Knaster* if and only if for every $X \subseteq \mathbb{P}$ with $|X| = \kappa$, there is a linked set $Y \subseteq X$ of the same cardinality. In the special case of $\kappa = \aleph_1$, it is customary to say that $\mathbb{P}$ is *Knaster* instead of $\aleph_1$-Knaster.

3. $\mathbb{P}$ has the *κ^+-linked property* if there is a function $c : \mathbb{P} \to \kappa$ such that, for all $\alpha < \kappa$, $c^{-1}(\{\alpha\})$ is linked.

It is not hard to check that (3) yields (2) and that the latter entails (1). These implications can not be reversed (see [Cum10, §5]).

A well-known fact in Forcing theory is that if $\mathbb{P}$ has the κ-cc then cofinalities – and thus also cardinals– greater or equal than κ are preserved [Kun14, §IV.3]. In particular, if $\mathbb{P}$ is ccc it preserves all cardinals, i.e., for every cardinal κ in V, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is a cardinal".

Definition 1.3.15. Let κ be an uncountable cardinal and $\mathbb{P}$ be a forcing.

1. $\mathbb{P}$ is *κ-distributive* if forcing with $\mathbb{P}$ does not add new sets of size $<\kappa$.

2. $\mathbb{P}$ is *κ-closed* if every $\leq$-decreasing sequence of conditions in $\mathbb{P}$ with length $<\kappa$ has a $\leq$-lower bound.

3. $\mathbb{P}$ is *κ-directed closed* if every directed set of conditions $D \subseteq \mathbb{P}$ of cardinality $<\kappa$ has a $\leq$-lower bound.

Clearly, (3) implies (2) and the latter yields (1). Once again, these implications can not be reversed (see [Cum10, §5]). It is not hard to show that if $\mathbb{P}$ is κ-distributive then it preserves all cardinals and cofinalities $\leq\kappa$.

There is an intermediate property between (2) and (1) in which we will be interested.

Definition 1.3.16. Let $\mathbb{P}$ be a forcing and θ be an ordinal. We denote by $\mathcal{G}(\mathbb{P}, \theta)$ the following two-players game: Player I ($\mathcal{P}_I$) and Player II ($\mathcal{P}_{II}$) play conditions in $\mathbb{P}$ during at most θ many rounds. $\mathcal{P}_{II}$ plays at odd stages, while $\mathcal{P}_I$ plays at even and limit stages. $\mathcal{P}_I$ must begin playing $\mathbb{1}_{\mathbb{P}}$. If p_α is the condition played at stage α then the player who played p_α loses automatically unless $p_\alpha \leq p_\beta$, for all $\beta < \alpha$. If no player loses at any round $\alpha < \theta$, then $\mathcal{P}_I$ wins.

Definition 1.3.17. Let κ be a regular cardinal and $\mathbb{P}$ be a forcing.

1. $\mathbb{P}$ is *$<\kappa$-strategically closed* if, for each $\theta < \kappa$, $\mathcal{P}_I$ has a winning strategy for $\mathcal{G}(\mathbb{P}, \theta)$.

2. $\mathbb{P}$ is *κ-strategically closed* if $\mathcal{P}_I$ has a winning strategy for $\mathcal{G}(\mathbb{P}, \kappa)$.

It is not hard to show that every κ-closed forcing is κ-strategically closed and that the latter are $<\kappa$-strategically closed. Besides, any $<\kappa$-strategically closed forcing is κ-distributive. Once again, these implications can not be reversed (see [Cum10, §5]).

An important result which will be used in subsequent sections is the so-called *Easton's Lemma* [Kun14, §IV]:

Lemma 1.3.18 (Easton's Lemma). *Let κ be a regular uncountable cardinal, let $\mathbb{P}$ be κ-cc and let $\mathbb{Q}$ be κ-closed. Then,*

1. $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ *is κ-cc".*
2. $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ *is κ-distributive".*

We close this section presenting some well-known forcing notions.

Definition 1.3.19. Let κ be a regular cardinal and $\lambda \geq \kappa$.

1. $\mathrm{Add}(\kappa, \lambda)$ is the forcing notion whose conditions are partial functions $p\colon \lambda \times \kappa \to 2$ with $|\mathrm{dom}(p)| < \kappa$. If $p, q \in \mathrm{Add}(\kappa, \lambda)$, we write $p \leq q$ if and only if $p \supseteq q$.
2. $\mathrm{Coll}(\kappa, \lambda)$ is the forcing notion whose conditions are partial functions $p\colon \kappa \to \lambda$ with $|\mathrm{dom}(p)| < \kappa$. If $p, q \in \mathrm{Coll}(\kappa, \lambda)$, we write $p \leq q$ if and only if $p \supseteq q$.
3. $\mathrm{Coll}(\kappa, <\lambda)$ is the forcing notion whose conditions are partial functions $p\colon \lambda \times \kappa \to \lambda$ with $|\mathrm{dom}(p)| < \kappa$ such that for all $(\alpha, \beta) \in \mathrm{dom}(p)$, $p(\alpha, \beta) \in \alpha$. If $p, q \in \mathrm{Coll}(\kappa, <\lambda)$, we write $p \leq q$ if and only if $p \supseteq q$.

Remark 1.3.20.

1. The forcing $\mathbb{A} := \mathrm{Add}(\kappa, \lambda)$ is the so-called *Cohen forcing* devised to add λ-many new subsets to κ. That is, $\mathbb{1}_{\mathbb{A}}$ forces that "$2^\kappa \geq \lambda$". Moreover, if λ is a cardinal and the GCH holds in V, $\mathbb{1}_{\mathbb{A}}$ forces "$2^\kappa = \lambda$". It is not hard to check that $\mathbb{A}$ has the $(2^{<\kappa})^+$-cc and that it is κ-directed closed. A special case of this forcing is when $\lambda = 1$ and $\kappa = \theta^+$, for some cardinal θ. In this latter case $\mathbb{1}_{\mathrm{Add}(\kappa,1)}$ forces the GCH_θ, i.e., "$2^\theta = \theta^+$".
2. The poset $\mathrm{Coll}(\kappa, <\lambda)$ is called the *Lévy collapse*.
   (a) $\mathbb{B} := \mathrm{Coll}(\kappa, \lambda)$ is a $(\lambda^{<\kappa})^+$-cc and κ-directed closed forcing. If moreover λ is a cardinal then $\mathbb{B}$ *collapses* λ to κ; that is, $\mathbb{1}_{\mathbb{B}}$ forces the existence of a surjection $f : \lambda \to \kappa$.
   (b) $\mathbb{C} := \mathrm{Coll}(\kappa, <\lambda)$ is κ-directed closed. Also, if λ is an inaccessible cardinal, then $\mathrm{Coll}(\kappa, <\lambda)$ is λ-cc. In this latter case $\mathbb{C}$ preserves all cardinals $\leq\kappa$ and $\geq\lambda$, and *collapses* all the cardinals in (κ, λ) to κ, i.e., for each $\theta \in (\kappa, \lambda)$, $\mathbb{1}_{\mathbb{C}}$ forces the existence of a surjection $f : \theta \to \kappa$. In particular, $\mathbb{1}_{\mathbb{C}} \Vdash_{\mathbb{C}}$ "$\lambda = \kappa^+$".

1.3.3 Forcing and elementary embeddings

A typical situation that we will encounter is the following: Let $\mathbb{P}$ be a forcing notion and let κ be some large cardinal in the ground model V. We aim to show that the said large-cardinal property of κ survives in certain generic extension $V[G]$ given by the forcing $\mathbb{P}$. To this aim we will need to check that in $V[G]$ there are elementary embeddings witnessing the said large-cardinal property of κ. The typical procedure to secure this is to *lift* the elementary embeddings from the ground model which witnessed that κ is a large cardinal.

Theorem 1.3.21 (J. Silver. See [Cum10, §9])**.** *Let $j : M \to N$ be an elementary embedding between two transitive models of* ZFC*. Let $\mathbb{P} \in M$ be a forcing notion, let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over M and let $H \subseteq j(\mathbb{P})$ be $j(\mathbb{P})$-generic over N. Then the following are equivalent:*

1. *$j[G] \subseteq H$.*
2. *There is an elementary embedding $j^* : M[G] \to N[H]$ such that $j^*(G) = H$ and $j^* \restriction M = j$.*

Remark 1.3.22. Let $j_E : V \to M$ be the elementary embedding derived from a (κ, λ)-extender $E \in V$. Suppose that the the hypothesis of the above theorem applies and that $H \in V[H]$. Let $j^* : V[G] \to M[H]$ be the unique elementary embedding with $j^* \restriction V = j_E$ and $j^*(G) = H$. Then there is a (κ, λ)-extender $E^* \in V[G]$ for which $j_{E^*} = j^*$. In particular, j^* is definable in $V[G]$ and $M[H] = \{j^*(f)(a) \mid f \colon [\kappa]^{|a|} \to V[G], f \in V[G], a \in [\lambda]^{<\omega}\}$. Thus, under the above conditions, the lifting of any extender embedding is the extender embedding derived by an extender in the generic extension $V[G]$. A similar result holds for any ultrapower embedding $j_{\mathcal{U}} : V \to M$ by a (supercompact) measure on κ (resp. on $\mathcal{P}_\kappa(\lambda)$).

An embedding $j^* \colon M[G] \to N[H]$ satisfying (2) of Theorem 1.3.21 is called a *lifting* of the elementary embedding j. If there is no confusion we shall tend to denote both embeddings by the same letter.

To illustrate how this result can be used to preserve large cardinals we will exhibit the proof of a classical theorem of A. Lévy and R. Solovay.

Theorem 1.3.23 (Lévy & Solovay)**.** *Let κ be a measurable cardinal and let $\mathbb{P}$ be a forcing poset with $|\mathbb{P}| < \kappa$. Then $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\kappa}$ is measurable".*

Proof. By Theorem 1.3.6 we are left with showing that κ is a measurable cardinal in $V[G]$, for each generic filter $G \subseteq \mathbb{P}$. Let $j : V \to M$ be the elementary embedding arising from a measure $\mathcal{U} \in V$ on κ. Since $|\mathbb{P}| < \kappa$ there is a forcing $\mathbb{Q} \in V_\kappa$ which is isomorphic to $\mathbb{P}$, hence yielding the same generic extension (cf. page 18). Thus, without loss of generality, we may assume that $\mathbb{P} \in V_\kappa$.

Since $\mathrm{crit}(j) = \kappa$, $j \upharpoonright \mathbb{P} = \mathrm{id}$ and $j(\mathbb{P}) = \mathbb{P}$. In particular, $j[G] = G$. Thus, G is $\mathbb{P}$-generic both over V and over M. By Theorem 1.3.21, j lifts to an elementary embedding $j : V[G] \to M[G]$ with $\mathrm{crit}(j) = \kappa$ and $M[G]$ being a transitive class. Finally, Remark 1.3.22 implies that j is definable in $V[G]$ and thus κ is a measurable cardinal in $V[G]$. $\square$

Similar arguments also work for the rest of large cardinals of Section 1.1 and Section 1.2.

The way to secure that clause (1) of Theorem 1.3.21 is fulfilled is by constructing a *master condition*.

Definition 1.3.24. Let $j : M \to N$ be an elementary embedding between two transitive models of ZFC and $\mathbb{P} \in M$ be a forcing notion. A *master condition* for j and $\mathbb{P}$ is a condition $q \in j(\mathbb{P})$ such that for every dense set $D \subseteq \mathbb{P}$ in M there is $p \in D$ such that $q \leq_{j(\mathbb{P})} j(p)$.

Assume that q is a master condition for an elementary embedding $j : M \to N$ and $\mathbb{P}$ is a forcing notion in M. Let H be a $j(\mathbb{P})$-generic filter over N with $q \in N$. It is easy to check that $G := \{p \in \mathbb{P} \mid \exists q \in H\, q \leq j(p)\}$ is a $\mathbb{P}$-generic filter over M which, by definition, satisfies $j[G] \subseteq H$. Thus, for every $j(\mathbb{P})$-generic filter H over N with $q \in H$, j lifts to $j : M[G] \to N[H]$.

1.4 $\square$-principles and scales

In this last section we give the definitions of $\square$-principles and scales, two of the central concepts in Cardinal Combinatorics. Here we only aim to cover the basic material necessary for the next chapters. For a complete treatment of these objects and their applications we refer the reader to [CFM01], [Eis10] and [Tod10a].

Definition 1.4.1. Let $\lambda \leq \kappa$ be two cardinals with $\kappa \geq \aleph_0$. A $\square_{\kappa,\lambda}$-sequence is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+,\ \alpha \in \mathrm{Lim}\rangle$ such that

1. $\mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha)$, $1 \leq |\mathcal{C}_\alpha| \leq \lambda$, and $\mathcal{C}_\alpha$ is a family of club subsets of α.
2. If $\mathrm{cof}(\alpha) < \kappa$, then $\forall C \in \mathcal{C}_\alpha\ \mathrm{otp}(C) < \kappa$.
3. $\forall C \in \mathcal{C}_\alpha\, \forall \beta \in \mathrm{Lim(C)}\, \mathrm{C} \cap \beta \in \mathcal{C}_\beta$.

The principle $\square_{\kappa,\lambda}$ asserts that there is a $\square_{\kappa,\lambda}$-sequence. Similarly, one can define the principle $\square_{\kappa,<\lambda}$ by exchanging in clause (1) "$\leq\lambda$" for "$<\lambda$". It is customary to denote the principles $\square_{\kappa,1}$ and $\square_{\kappa,\kappa}$ by $\square_\kappa$ and $\square^*_\kappa$, respectively.

Remark 1.4.2. The principle $\square_{\aleph_0}$ is not very interesting as any sequence of unbounded sets $\langle C_\alpha \mid \alpha < \aleph_1,\ \alpha \in \mathrm{Lim}\rangle$ with $\mathrm{otp}(C_\alpha) = \omega$ yields a $\square_{\aleph_0}$-sequence. Also notice that for each infinite cardinal κ and $\lambda \leq \mu \leq \kappa$, the principle $\square_{\kappa,\lambda}$ implies $\square_{\kappa,\mu}$. In particular, $\square_\kappa$ yields $\square^*_\kappa$.

Convention 1.4.3. If the principle $\square_\kappa$ (resp. $\square^*_\kappa$) holds it is customary to say that *(weak)square holds at κ.*

The principle $\square_\kappa$ was introduced by R. Jensen in [Jen72]. Jensen showed that the axiom of constructibility V = L entails the existence of a $\square_\kappa$-sequence for each uncountable cardinal κ and used this to construct a κ^+-Suslin tree in L. Jensen also introduced the weaker principle $\square^*_\kappa$ and showed that, for a singular cardinal κ, $\square^*_\kappa$ is equivalent to the existence of a (special) κ^+-Aronszajn tree. For details see [Jen72]. The intermediate principles $\square_{\kappa,\lambda}$ were introduced by Schimmerling in [Sch95].

Remark 1.4.4. If V and W are two inner models of ZFC such that $(\kappa^+)^V = (\kappa^+)^W$ and $V \models \square_\kappa$, then also $W \models \square_\kappa$. In other words, $\square_\kappa$ is absolute between transitive models of ZFC which agrees on the computation of κ^+. The same applies for the principle $\square^*_\kappa$.

In case $\kappa^{<\kappa} = \kappa$ it is always possible to manufacture a $\square^*_\kappa$-sequence. The study of $\square^*_\kappa$ becomes really interesting when κ is a singular cardinal. It is worth mentioning that obtaining a failure of weak square at a singular cardinal requires the existence of (very) large cardinals (see [Eis10, Theorem 2.3]). In particular, unless these large cardinals exist, it is not possible to obtain the tree property at the first successor of a singular cardinal κ (cf. Part II).

Square principles are prototypical examples of *non-compact* objects of size κ^+: namely, $\square_\kappa$ claims the existence of a *coherent* sequence of clubs $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \in \mathrm{Lim}\rangle$ at ordinals $\alpha < \kappa^+$ (cf. clause (3)) which cannot be *threaded* by any club $D \subseteq \kappa^+$: that is, there is no club $D \subseteq \kappa^+$ such that, for any $\alpha \in \mathrm{Lim(D)}$, $D \cap \alpha = C_\alpha$.

Actually, the typical application of $\square_\kappa$ is in the construction of non-compact objects of size κ^+. A paradigmatic example is the construction of a non-reflecting stationary subset of κ^+.

Theorem 1.4.5 (See [CFM01])**.** *If $\square_\kappa$ holds, then there is a stationary set $S \subseteq \kappa^+$ which does not reflect, i.e., for each ordinal $\lambda < \kappa^+$ of uncountable cofinality the set $S \cap \lambda$ is not stationary in λ.*

Under some combinatorial assumptions, similar results hold for the weaker principles $\square_{\kappa,<\lambda}$ [CFM01, Theorem 2.2]. Nonetheless, the principle $\square^*_\kappa$ is not strong enough for producing non-reflecting stationary sets. As a witness of this we have Theorem 15.0.3, where the generic extension constructed satisfies $\square^*_\kappa$ but every stationary subset of κ^+ reflects. For more applications of $\square_\kappa$ see any of the references mentioned at the beginning of this section.

In the light of the previous comments we should mention that large cardinals have an impact upon the $\square$-configurations. This does not happen by chance, for if $\square$-principles are prototypical non-compact objects then they should be natural antagonists of large cardinals entailing strong forms of compactness. The interplay between these large cardinals and the $\square$-principles have constituted a fruitful and important area of study in Set Theory [CFM01].

Theorem 1.4.6. *Assume that κ is a strong compact cardinal. Then the following hold:*

1. (Solovay) For each cardinal $\lambda \geq \kappa$, $\square_\lambda$ fails.

2. (Burke & Kanamori) For each cardinal $\lambda \geq \kappa$, $\square_{\lambda,<\mathrm{cof}(\lambda)}$ fails.

3. (Shelah) If κ is supercompact, then for each cardinal $\lambda \geq \kappa$ with $\mathrm{cof}(\lambda) < \kappa < \lambda$, $\square_{\lambda,\mathrm{cof}(\lambda)}$ fails.

4. (Bagaria & Magidor) If κ is ω_1-strong compact, then $\square_{\lambda,\omega}$ fails, for each $\lambda \geq \kappa$ with $\mathrm{cof}(\lambda) = \omega$.

The proofs of (1)-(3) can be found in [CFM01] and the proof of (4) corresponds to [BM14a, Theorem 3.1].

Remark 1.4.7. By a result of J. Cummings, M. Foreman and M. Magidor [CFM01, Theorem 9.1] it is consistent with a supercompact cardinal κ that $\square_{\lambda,\mathrm{cof}(\lambda)}$ holds, for some $\kappa \leq \mathrm{cof}(\lambda) < \lambda$.

The other key notion of this section is *scales*, one of the key concepts of Shelah's PCF theory [She94].[7]

Definition 1.4.8. Let κ be a singular cardinal and let $\vec{\kappa} := \langle \kappa_\xi \mid \xi < \mathrm{cof}(\kappa)\rangle$ be an increasing cofinal sequence in κ. A sequence of functions $\vec{f} := \langle f_\eta \mid \eta < \kappa^+\rangle$ is called a κ^+-*scale* if the following conditions are met:

1. $f_\xi \in \prod_{\xi<\mathrm{cof}(\kappa)} \kappa_\xi$, for each $\xi < \kappa^+$.

2. For all $\eta < \nu < \kappa^+$, $\{\mu < \mathrm{cof}(\kappa) \mid f_\eta(\mu) \geq f_\nu(\mu)\}$ is bounded in $\mathrm{cof}(\kappa)$.

3. For all $h \in \prod_{\xi<\mathrm{cof}(\kappa)} \kappa_\xi$ there is $\eta < \kappa^+$ such that
$$\{\mu < \mathrm{cof}(\kappa) \mid h(\mu) \geq f_\eta(\mu)\} \text{ is bounded in } \mathrm{cof}(\kappa).$$

A fundamental result around the existence of scales is the following:

Theorem 1.4.9 (Shelah [She94])**.** *If κ is a singular cardinal then there is a κ^+-scale.*

We will be interested in an important class of scales called (very) good.

[7]**PCF** stands for **P**ossible **C**o**F**inalities.

Definition 1.4.10. Let κ be a singular cardinal, $\vec{\kappa} := \langle \kappa_\xi \mid \xi < \mathrm{cof}(\kappa) \rangle$ be an increasing cofinal sequence in κ, and $\langle f_\eta \mid \eta < \kappa^+ \rangle$ be a κ^+-scale (with respect to $\vec{\kappa}$). An ordinal $\mu < \kappa^+$ with $\mathrm{cof}(\mu) > \mathrm{cof}(\kappa)$ is said to be a ***(very) good point*** if there is a **(club) unbounded set** $A \subseteq \alpha$ and $\mu < \mathrm{cof}(\kappa)$ such that the following holds:

$$\forall \eta, \nu \in A \, \forall \zeta > \mu \, (\eta < \nu \Longrightarrow f_\eta(\zeta) < f_\nu(\zeta)).$$

The scale $\langle f_\eta \mid \eta < \kappa^+ \rangle$ is said to be ***(very) good*** if there is a club set $C \subseteq \kappa^+$ such that for every $\mu \in C$ with $\mathrm{cof}(\mu) > \mathrm{cof}(\kappa)$, μ is a ***(very) good point*** for the scale. A scale which is not good is said to be **bad**.

Theorem 1.4.11 ([CFM01, §3])**.** *Let κ be a singular cardinal and $\lambda < \kappa$. Then, $\square_{\kappa,\lambda}$ yields the existence of a very good κ^+-scale.*

Despite $\square^*_\kappa$ has no effects on stationary reflection it has a deep influence upon the existence of good scales.

Theorem 1.4.12 ([CFM01, §6])**.** *If κ is a singular cardinal and $\square^*_\kappa$ holds then every κ^+-scale is good.*

Once again, (very) large cardinals have an impact on the properties of scales as it is shown by the following result of J. Bagaria and M. Magidor.

Theorem 1.4.13 ([BM14b])**.** *Suppose κ is a ω_1-strong compact cardinal. Then for every $\lambda > \kappa$ with $\mathrm{cof}(\lambda) = \omega$ all the λ^+-scales are bad.*

Remark 1.4.14. Despite not being mentioned in [BM14b], the above theorem can be easily adapted to any other degree of strong compactness.

Part I

On the large-cardinal hierarchy between supercompactness and Vopěnka's Principle

Preliminary words and notation

In this part we analyse the region of the large-cardinal hierarchy comprised between the first supercompact cardinal and Vopěnka's Principle. This is one of most important fragments of the large-cardinal hierarchy, with deep connections with many fundamental problems in Mathematics. For instance, supercompact and stronger large cardinals have found many applications both inside and outside Set theory. See [Kan09][Mag71][Bag+15][Bag12][Dug85][AR94][DG85].

In the following chapters we will study the structural properties and the effect of forcing upon the canonical large cardinal families of this region. Namely, the classes of supercompact, $C^{(n)}$–supercompact and $C^{(n)}$–extendible cardinals (see Section 1.1.2 and Section 1.2). For more details about the *canonicity* of these families we refer the reader to Section 1.2 where we discussed their connections with Vopěnka's Principle.

For the sake of readability we will use the following notation:

Notation 1.4.15.

1. $\mathfrak{M}$ denotes the class of all measurable cardinals;
2. $\mathfrak{M}_\infty$ denotes the class of all strong cardinals;
3. $\mathfrak{K}_{\omega_1}$ denotes the class of all ω_1-strong compact cardinals;
4. $\mathfrak{K}$ denotes the class of all strong compact cardinals;
5. $\mathfrak{S}$ denotes the class of all supercompact cardinals;
6. $\mathfrak{S}^{(n)}$ denotes the class of all $C^{(n)}$-supercompact cardinals;
7. $\mathfrak{a}$-$\mathfrak{E}^{(n)}$ denotes the class of all $\mathfrak{a}$-$C^{(n)}$-extendible cardinals;
8. $\mathfrak{E}^{(n)}$ denotes the class of all $C^{(n)}$-extendible cardinals;
9. $\mathfrak{S}^{(<\omega)}$ denotes the class of all $C^{(<\omega)}$-supercompact cardinals;

The notions corresponding to items (1)-(6) and (8) have already been defined in Section 1.1.2, Section 1.1.3 and Section 1.2. On the other hand, (7) will be defined in Definition 2.0.1 and (9) in page 50.

CHAPTER 2

Woodin's Extender Embedding Axiom and a question of Bagaria

This chapter is devoted to answering a question posed by Bagaria on the relative position between $C^{(n)}$–supercompactness and $C^{(n)}$–extendibility. In his study of the $C^{(n)}$–hierarchy, the said author asked whether any $C^{(n)}$–extendible cardinal is also $C^{(n)}$–supercompact and also if the first $C^{(n)}$–extendible cardinal is larger than the first $C^{(n)}$–supercompact [Bag12, §5].

The first of this question was affirmatively answered by Tsaprounis in his Ph.D. dissertation [Tsa12]. Tsaprounis proved that a cardinal κ is $C^{(n)}$–extendible if and only if it is $C^{(n)}$–supercompact and κ-superstrong (cf. Theorem 1.2.5). Nonetheless, Tsaprounis' characterization does not provide any clue for answering Bagaria's second question.

In this chapter we will answer Bagaria's question by showing that the first $C^{(n)}$–extendible cardinal is always a limit of $C^{(n)}$–supercompact cardinals. We prove so by introducing a new large-cardinal notion which we have called *almost $C^{(n)}$–extendibility*. This notion was discovered after a discussion of the author with W. H. Woodin and is related to Tsaprounis' characterization of $C^{(n)}$–extendibility. We would like to thank professor Woodin for his amiability and inspiring insights.

Finally, we close the chapter describing all the possible spatial configurations for these classes of $C^{(n)}$–cardinals. For this purpose we will present Woodin's Extender Embedding Axiom (WEEA) (cf. Definition 2.0.11) and explore some of its structural consequences for the $C^{(n)}$–hierarchy.

Definition 2.0.1 ($\mathfrak{a}$-$C^{(n)}$–extendible cardinal)**.** Let $n \geq 1$. A cardinal κ is called λ-$\mathfrak{a}$-$C^{(n)}$–extendible, for some $\lambda \geq \kappa$, if κ is λ-supercompact and there is a cardinal $\mu \in C^{(n)} \setminus \lambda^+$ with $\mathrm{cof}(\mu) > \lambda$, such that κ is superstrong with target μ; i.e. there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \mu$ and $V_{j(\kappa)} \subseteq M$. A cardinal κ is called *almost-$C^{(n)}$*–extendible, or shortly $\mathfrak{a}$-$C^{(n)}$–extendible, if it is λ-$\mathfrak{a}$-$C^{(n)}$–extendible for every $\lambda > \kappa$.

Any $C^{(n)}$–extendible cardinal is $\mathfrak{a}$-$C^{(n)}$–extendible by Theorem 1.2.5. However the converse is not true as the first $\mathfrak{a}$-$C^{(n)}$–extendible cardinal is not $C^{(n)}$–extendible (cf. Theorem 2.0.6).

It is worth to emphasize that in the definition of $\mathfrak{a}$-$C^{(n)}$–extendibility we do not require that the λ-supercompactness and superstrongness of κ are witnessed by the same elementary embedding. In that case $C^{(n)}$–extendibility and $\mathfrak{a}$-$C^{(n)}$–extendibility would be equivalent. This is precisely what motivates the adoption of the terminology almost-$C^{(n)}$–extendible.

Despite yielding different notions both $C^{(n)}$–extendibility and $\mathfrak{a}$-$C^{(n)}$–extendibility keep some similarities in terms of reflection:

Proposition 2.0.2. *Let $n \geq 1$. The following properties are true for a $\mathfrak{a}$-$C^{(n)}$–extendible cardinal κ:*

1. *"κ is $\mathfrak{a}$-$C^{(n)}$–extendible" is a Π_{n+2} expressible property of κ;*
2. *$\kappa \in C^{(n+2)}$.*

In particular, the first $\mathfrak{a}$-$C^{(n+1)}$-extendible cardinal is limit of $\mathfrak{a}$-$C^{(n)}$-extendible cardinals.

Proof. Observe that the last clause easily follows by combining (1) and (2). For (1) observe that κ is $\mathfrak{a}$-$C^{(n)}$–extendible iff the formula

$$\begin{aligned}\forall\lambda(\kappa < \lambda \to \kappa \text{ is } \lambda\text{-supercompact} \wedge \exists\mu\,\exists E\exists\theta\,(\mu \in C^{(n)} \setminus \lambda^+ \\ \wedge \operatorname{cof}(\mu) > \lambda \wedge \theta \in C^{(1)} \setminus \mu^+ \wedge \\ V_\theta \models \text{“}E \text{ is a } (\kappa,\mu)\text{-extender with } j_E(\kappa) = \mu \text{ and } V_\mu \subseteq M_E\text{”})),\end{aligned}$$

holds. Here $j_E : V \to M_E$ stands for the elementary embedding induced by the extender E (cf. Section 1.1.3).

For (2) let us simply show the argument for a $\mathfrak{a}$-$C^{(1)}$-extendible cardinal κ to be Σ_3-correct. The general case can be proved similarly by induction on $1 \leq n < \omega$. Let $\exists x\varphi(x, y_0, \ldots, y_{n-1})$ be a Σ_3 formula and a set of parameters $\langle a_0, \ldots, a_{n-1}\rangle \in V_\kappa$. Assume that $V_\kappa \models \exists x\varphi(x, a_0, \ldots, a_{n-1})$. Since κ is $\mathfrak{a}$-$C^{(1)}$-extendible, in particular supercompact and thus Σ_2-correct, it follows that $\exists x\varphi(x, a_0, \ldots, a_{n-1})$ is true.[1]

Conversely, assume that $\exists x\varphi(x, a_0, \ldots, a_{n-1})$ is true. Let a be some witness and $\mu \in C^{(1)} \setminus \kappa^+$ be such that κ is a superstrong cardinal with target μ and $a \in V_\mu$. Since $\varphi(x, a_0, \cdots, a_{n-1})$ is Π_2, $V_\mu \models \varphi(a, a_0, \ldots, a_{n-1})$, hence $V_\mu \models \exists x\varphi(x, a_0, \ldots, a_{n-1})$. If $j : V \to M$ is the corresponding superstrong embedding it follows that $M \models$ "$V_\mu \models \exists x\varphi(x, a_0, \ldots, a_{n-1})$" and thus, by elementary, $V_\kappa \models \exists x\varphi(x, a_0, \ldots, a_{n-1})$. ☐

[1] See Proposition 1.2.11.

The next proposition shows that $\mathfrak{a}$-$C^{(n)}$–extendibility entails a stronger notion than supercompactness. Actually we will later show that λ-$\mathfrak{a}$-$C^{(n)}$–extendibility entails the stronger notion of λ-$C^{(n)}$–supercompactness.

Proposition 2.0.3. *Let $n \geq 1$ and κ be a $\mathfrak{a}$-$C^{(n)}$–extendible cardinal. Let $\mathcal{U}$ be the measure derived from some elementary embedding witnessing the superstrongness of κ. Then,*

$$\{\eta < \kappa \mid \eta \textit{ is supercompact}\} \in \mathcal{U}.$$

In particular, any $\mathfrak{a}$-$C^{(n)}$–extendible cardinal is a (stationary) limit of supercompact cardinals.

Proof. Let κ be a $\mathfrak{a}$-$C^{(n)}$–extendible cardinal and let $j : V \to M$ be a superstrong embedding with $\mathrm{crit}(j) = \kappa$. Since κ is clearly supercompact, and supercompactness can be formulated as a Π_2 property, this is true in $V_{j(\kappa)}$. Thus, $M \models$ "$V_{j(\kappa)} \models \kappa$ is supercompact". By elementarity, $j(\kappa)$ is supercompact in M, hence $V_{j(\kappa)} \prec_{\Sigma_2} M$, so κ is supercompact in M. Again, by elementarity, $\{\eta < \kappa \mid \eta \text{ is supercompact}\} \in \mathcal{U}$. $\square$

Theorem 2.0.4. *If κ is a λ-$\mathfrak{a}$-$C^{(n)}$–extendible cardinal, for some $\lambda \geq \kappa$, then κ is λ-$C^{(n)}$–supercompact as well. In particular, any $\mathfrak{a}$-$C^{(n)}$–extendible cardinal is $C^{(n)}$–supercompact.*

Proof. Fix $\lambda \geq \kappa$ and let κ be a λ-$\mathfrak{a}$-$C^{(n)}$–extendible cardinal. Let $j : V \to M$ be a λ-supercompact embedding derived by some supercompact measure over $\mathcal{P}_\kappa(\lambda)$. Also let $\mu \in C^{(n)} \setminus \lambda^+$ be some cardinal with $\mathrm{cof}(\mu) > \lambda$ witnessing that κ is superstrong with target μ. Since μ is strong limit and $\mathrm{cof}(\mu) > \lambda$, standard arguments about ultrapowers yield $j(\mu) = \mu$ (see [Kan09, Lemma 22.12]). By elementarity,

$$M \models \text{“}j(\kappa) \text{ is superstrong with target } \mu\text{”}.$$

Let $E \in M$ be some $(j(\kappa), \mu)$-extender yielding a superstrong embedding $j_E : M \to N \cong \mathrm{Ult}(M, E)$ (cf. Proposition 1.1.31). We claim that $i := j_E \circ j$ defines a λ-$C^{(n)}$–supercompact embedding with $\mathrm{crit}(i) = \kappa$. Clearly i is an elementary embedding with $\mathrm{crit}(i) = \kappa$ and $\mu = i(\kappa) > \lambda$, so we are left with checking that ${}^\lambda N \cap V \subseteq N$. Let $\vec{x} = \langle x_\alpha \mid \alpha < \lambda \rangle \in {}^\lambda N \cap V$. Since N is definable in M, $\vec{x} \in {}^\lambda M \cap V \subseteq M$. Thus we are left with checking that N is closed under λ-sequences in M.

Claim 2.0.4.1. ${}^\lambda N \cap M \subseteq N$. *In particular, $\vec{x} \in N$.*

Proof of claim. Let $\langle a_\alpha \mid \alpha < \lambda \rangle \in M$ be a λ-sequence of elements in N. Since N arises from a $(j(\kappa), \mu)$-superstrong extender $E \in M$, for each $\alpha < \lambda$, $a_\alpha = j_E(f_\alpha)(s_\alpha)$, where $f_\alpha : V^M_{j(\kappa)} \to M$ and $s_\alpha \in V^M_\mu$. Observe that $j_E(\langle f_\alpha \mid \alpha < \lambda \rangle) = \langle j_E(f_\alpha) \mid \alpha < \lambda \rangle \in N$, as $\mathrm{crit}(j_E) > \lambda$. Also $\mathrm{cof}(\mu)^M > \lambda$, hence

V_μ^M is closed under λ-sequences. Thus $\langle s_\alpha \mid \alpha < \lambda\rangle \in V_\mu^M \subseteq N$. From this it is clear that $\langle a_\alpha \mid \alpha < \lambda\rangle$ is definable in N, hence a member of it. $\square$

Finally since the choice of $\vec{x}$ was arbitrary we infer that ${}^\lambda N \cap V \subseteq N$. $\square$

Remark 2.0.5. The converse implication is not necessarily true. For instance, it fails in the generic extension of Theorem 4.0.1. In that model the first $C^{(n)}$–supercompact cardinal is the first ω_1-strong compact and thus it can not be $\mathfrak{a}$-$C^{(n)}$–extendible. However, under WEEA (cf. Definition 2.0.11) both notions are equivalent.

The main result of the chapter is the following:

Theorem 2.0.6. *Let κ be a $\kappa+1$-$C^{(n)}$–extendible cardinal and let $\mathcal{U}$ be the measure on κ derived by some elementary embedding $j : V_{\kappa+1} \to V_{\theta+1}$ witnessing the $\kappa+1$-$C^{(n)}$–extendibility of κ. Then,*

$$\{\eta < \kappa \mid \eta \text{ is } \mathfrak{a}\text{-}C^{(n)}\text{–extendible}\} \in \mathcal{U}.$$

In particular, any $C^{(n)}$–extendible cardinal is a stationary limit of $\mathfrak{a}$-$C^{(n)}$–extendible cardinals and thus of $C^{(n)}$–supercompact cardinals.

Before addressing the proof of the theorem we need an auxiliary lemma.

Lemma 2.0.7. *Let $j : V \to M$ be a superstrong elementary embedding with* $\mathrm{crit}(j) = \kappa$ *and regular target. Then,*

$$C := \{\mu < j(\kappa) \mid \kappa \text{ is superstrong with target } \mu \text{ and } V_\mu \prec V_{j(\kappa)}\}^2$$

is a club subset of $j(\kappa)$.

Proof. We shall split the argument into a series of claims:

Claim 2.0.7.1. $\{\mu < j(\kappa) \mid V_\mu \prec V_{j(\kappa)}\}$ *is a club.*

Proof of claim. Closedness is clear so we will simply check unboundedness. Fix $\mu < j(\kappa)$ and define $\langle M_n \mid n < \omega\rangle$ a $\subseteq$-increasing sequence of transitive elementary substructures of $V_{j(\kappa)}$ and ordinals $\langle \eta_n \mid n < \omega\rangle$ as follows:

- $\eta_0 := \mu$.
- Let $M_n \prec V_{j(\kappa)}$ transitive with $V_{\eta_n} \cup \{\eta_n\} \subseteq M_n$ and $|M_n| = |V_{\eta_n}|$.
- $\eta_{n+1} := \min\{\alpha < j(\kappa) \mid M_n \subseteq V_\alpha\}$.

[2] This means that μ is the target of some elementary embedding witnessing the superstrongness of κ.

Observe that since $\beth_{j(\kappa)} = j(\kappa)$ we can carry out this recursive process by appealing to the Löwenheim-Skolem and Mostowski theorems [Kun14]. Set $\eta_\omega := \sup_{n<\omega} \eta_n$ and $M_\omega := \bigcup_{n<\omega} M_n$. Since $j(\kappa)$ is regular, $\eta_\omega < j(\kappa)$, hence $M_\omega = V_{\eta_\omega} \prec V_{j(\kappa)}$. Clearly, $\mu < \eta_\omega$, so that $\eta_\omega \in C \setminus (\mu + 1)$. □

Let us now check that $\overline{C} = \{\mu < j(\kappa) \mid \kappa \text{ is superstrong with target } \mu\}$ is also club. Since the arguments for unboundedness and closeness are essentially the same we only give the proof for unboundedness.

Fix $\mu < j(\kappa)$. Set $\mu_0 := \mu$ and $X_0 := \{j(f)(x) \mid f : V_\kappa \to V,\, x \in V_{\mu_0}\}$. By standard Löwenheim-Skolem-type arguments it is not hard to check that $X_0 \prec M$ (for details, see [Tsa12, Section 2.1]). Let π_0 be the Mostowski collapsing map on X_0 and set $M_0 := \pi_0[X_0]$, $j_0 := \pi_0 \circ j$. Clearly, $j_0 : V \to M_0$ is an elementary embedding. By recursion on $n < \omega$ define the following:

- $\mu_{n+1} := \sup(X_n \cap j(\kappa)) + \omega$.
- $X_{n+1} := \{j(f)(x) \mid f : V_\kappa \to V,\, x \in V_{\mu_{n+1}}\}$.[3]
- If π_{n+1} is the Mostowski collapsing map on X_{n+1}, set $M_{n+1} := \pi_{n+1}[X_{n+1}]$ and $j_{n+1} := \pi_{n+1} \circ j$.

Claim 2.0.7.2. *For each $n < \omega$, the following hold:*

1. $V_{\mu_n} \subseteq X_n$.
2. $\mu_0 < \mu_n$ *and* $\langle \mu_n \mid n \geq 1\rangle$ *is increasing.*
3. $\mu_n < j(\kappa)$.

Proof. Item (1) is straightforward since any $x \in V_{\mu_n}$ can be represented as $j(\mathrm{id})(x)$, where $\mathrm{id} : x \mapsto x$. Item (2) follows from item (1) and the definition of μ_n. For (3) we shall proceed by induction on $n < \omega$. By our initial assumption the base case is true; so it remains to discuss the successor step. Let $\alpha \in X_n \cap j(\kappa)$ and observe, by the definition of X_n, that we may choose $f : V_\kappa \to \kappa$ and $x \in V_{\mu_n}$ be such that $j(f)(x) = \alpha$. In particular, $|X_n \cap j(\kappa)| \leq |{}^\kappa V_\kappa| \cdot |V_{\mu_n}|$. Appealing to the induction hypothesis and the fact that $j(\kappa)$ is a $\beth$-fixed point, $\max(|{}^\kappa V_\kappa|, |V_{\mu_n}|) < j(\kappa)$, so that $|X_n \cap j(\kappa)| < j(\kappa)$. By regularity $\sup(X_n \cap j(\kappa)) < j(\kappa)$, hence $\mu_{n+1} < j(\kappa)$. □

The above recursion yields $\langle X_n \mid n < \omega\rangle$, an increasing chain of elementary substructures of M. Set $\mu_\omega := \sup_n \mu_n$ and $X_\omega := \bigcup_{n\in\omega} X_n$. Since $\mathrm{cof}(j(\kappa)) > \omega$, $\mu_\omega < j(\kappa)$, and clearly $X_\omega = \{j(f)(x) \mid f : V_\kappa \to M,\, x \in V_{\mu_\omega}\}$ is an elementary substructure of M. Again, if π_ω is the Mostowski collapsing map on X_ω, set $M_\omega := \pi_\omega[X_\omega]$ and $j_\omega := \pi_\omega \circ j$. We claim that $\mu_\omega \in \overline{C}$, as witnessed by the elementary embedding $j_\omega : V \to M_\omega$. Clearly j_ω is elementary and $\mathrm{crit}(j_\omega) = \kappa$, so that it remains to check that $\mu_\omega = j_\omega(\kappa)$ and $V_{j_\omega(\kappa)} \subseteq M_\omega$. To this end we shall need the next claim.

[3] As above, $X_{n+1} \prec M$.

Claim 2.0.7.3.

1. $X_\omega \cap j(\kappa) \in \mathrm{ORD}$.
2. $j_\omega(\kappa) = \sup(X_\omega \cap j(\kappa)) = \mu_\omega$.

Proof of claim. (1) It is enough to check that $X_\omega \cap j(\kappa)$ is transitive, so let $\alpha \in X_\omega \cap j(\kappa)$ and let us show that $\alpha \subseteq X_\omega \cap j(\kappa)$. By definition, there is $n < \omega$ such that $\alpha \in X_n \cap j(\kappa)$, hence $\alpha < \mu_{n+1}$. By virtue of Claim 2.0.7.2(1), $\alpha \subseteq V_{\mu_{n+1}} \subseteq X_{n+1} \subseteq X_\omega$, as wanted. For (2) observe that the second equality is immediate by definition of the recursion. On the other hand, $j_\omega(\kappa) = \pi_\omega(j(\kappa)) := \{\pi_\omega(\alpha)+1 \mid \alpha \in X_\omega \cap j(\kappa)\}$ and, by (1), this is the same as $\sup(X_\omega \cap j(\kappa))$. □

Clearly, $V_{\mu_\omega} \subseteq X_\omega$, hence $V_{j_\omega(\kappa)} \subseteq X_\omega$. Thus, $\pi_\omega \restriction V_{j_\omega(\kappa)} = \mathrm{id}$ which finally yields $V_{j_\omega(\kappa)} \subseteq M_\omega$. □

We are finally ready to prove Theorem 2.0.6.

Proof of Theorem 2.0.6. Fix $j : V_{\kappa+1} \to V_{j(\kappa)+1}$ a $\kappa+1$-$C^{(n)}$–extendible embedding. Evidently, κ is superstrong with regular target so we may appeal to the previous lemma to find a club $C \subseteq j(\kappa)$ of cardinals μ such that κ is superstrong with target μ and $V_\mu \prec V_{j(\kappa)}$. Recall that, for each $\mu < j(\kappa)$, the property "κ is superstrong with target μ" can be formalized via the existence of certain extender $E \in V_{j(\kappa)}$, hence this is expressible as a Σ_1 property of κ (cf. Theorem 1.1.31). Thus, $V_{j(\kappa)} \models$ "κ is superstrong with target μ", for each $\mu \in C$. Also, observe that $V_{j(\kappa)} \models$ "κ is supercompact", as $V_{j(\kappa)} \prec_{\Sigma_n} V$.

Claim 2.0.7.4. $V_{j(\kappa)} \models$ *"κ is $\mathfrak{a}$-$C^{(n)}$–extendible".*

Proof of claim. Observe that this boils down to prove that, in $V_{j(\kappa)}$, for each $\kappa < \lambda < j(\kappa)$ there is a cardinal $\mu \in C^{(n)} \setminus \lambda^+$ with $\mathrm{cof}(\mu) > \lambda$ such that κ is superstrong with target μ. Since C is a club in $j(\kappa)$ we may pick $\mu \in C \cap E^{j(\kappa)}_{\lambda^{++}}$. Since $V_\mu \prec V_{j(\kappa)} \prec_{\Sigma_n} V$ it follows that $V_\mu \prec_{\Sigma_n} V$. Since $V_{j(\kappa)} \prec_{\Sigma_n} V$, μ is a Σ_n-correct cardinal in $V_{j(\kappa)}$. Altogether,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is superstrong with target } \mu \in C^{(n)} \setminus \lambda^+ \text{ and } \mathrm{cof}(\mu) > \lambda\text{”},$$

as desired. □

Thus, $\{\eta < \kappa \mid V_\kappa \models \text{“}\eta \text{ is } \mathfrak{a}\text{-}C^{(n)}\text{–extendible”}\} \in \mathcal{U}$. Combining Proposition 2.0.2(1) with the Σ_{n+2}-correctness of $C^{(n)}$–extendible cardinals, this set is just $\{\eta < \kappa \mid \eta \text{ is } \mathfrak{a}\text{-}C^{(n)}\text{–extendible}\}$, which yields the desired result. □

We do not know if the existence of a $\mathfrak{a}$-$C^{(n+1)}$-extendible cardinal entails the existence of a $C^{(n)}$–extendible cardinal. However, if both classes are nonempty, then the least $C^{(n)}$–extendible is below the first $\mathfrak{a}$-$C^{(n+1)}$-extendible:

Proposition 2.0.8. *Assume that there is a* $\mathfrak{a}$*-*$C^{(n+1)}$*-extendible cardinal and a* $C^{(n)}$*–extendible cardinal. Then,* $\min \mathfrak{E}^{(n)} < \min \mathfrak{a}\text{-}\mathfrak{E}^{(n+1)}$.

Proof. Let $\kappa := \min \mathfrak{a}\text{-}\mathfrak{E}^{(n+1)}$ and $\lambda := \min \mathfrak{E}^{(n)}$ and assume that $\lambda \geq \kappa$. Let $\mu \in C^{(n+1)} \setminus \lambda$ be some cardinal such that κ is superstrong with target μ. Let $j : V \to M$ be a witness for this. Since being $C^{(n)}$–extendible is a Π_{n+2} property, this fact is witnessed by V_μ, hence

$$M \models \text{“}V_\mu \models \kappa \text{ is } C^{(n)}\text{–extendible”}.$$

By Proposition 2.0.2(2), $\kappa \in C^{(n+2)}$, hence $V_\mu \prec_{\Sigma_{n+2}} M$, and thus κ is $C^{(n)}$–extendible in M. By elementarity, there is some $C^{(n)}$–extendible cardinal below κ, which yields the desired contradiction. □

Remark 2.0.9. Actually, under the above conditions, the previous argument says more: namely, the first $\mathfrak{a}$-$C^{(n+1)}$-extendible cardinal is a stationary limit of $C^{(n)}$–extendible cardinals.

We thus arrive at the following corollary:

Corollary 2.0.10. *Let* $n \geq 1$.

1. $\min \mathfrak{S} < \min \mathfrak{a}\text{-}\mathfrak{E}^{(1)} < \min \mathfrak{E}$.
2. $\min \mathfrak{S} \leq \min \mathfrak{S}^{(n)} \leq \min \mathfrak{a}\text{-}\mathfrak{E}^{(n)} < \min \mathfrak{E}^{(n)} < \min \mathfrak{a}\text{-}\mathfrak{E}^{(n+1)}$.

Observe that the previous corollary is not informative about the relative position between, say, the first $C^{(n)}$–extendible and the first $C^{(n+1)}$-supercompact cardinal. In Chapter 4 we will prove the consistency of the first supercompact being the first $C^{(n)}$–supercompact, for each $n \geq 1$. In particular, this entails the consistency of the first extendible being larger than the first $C^{(n)}$–supercompact, for each $n \geq 1$. Nonetheless, it is possible to obtain a completely different scenario under an axiom suggested by Woodin.

Definition 2.0.11 (Woodin)**.** Woodin's Extender Embedding Axiom WEEA is the following assertion: Every elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa)$ a cardinal, and $M^\omega \subseteq M$, is superstrong.

Proposition 2.0.12 (WEEA)**.** *Let* $n \geq 1$ *and let* κ *be a* λ*-*$C^{(n)}$*–supercompact cardinal, for some* $\lambda \geq \kappa$*. Then,* κ *is* λ*-*$\mathfrak{a}$*-*$C^{(n)}$*–extendible. In particular, any* $C^{(n)}$*–supercompact cardinal is* $\mathfrak{a}$*-*$C^{(n)}$*–extendible.*

Proof. Let $j : V \to M$ be a λ-$C^{(n)}$–supercompact embedding with $\mathrm{crit}(j) = \kappa$. Clearly, κ is λ-supercompact. On the other hand, observe that $\mathrm{cof}(j(\kappa)) > \lambda$ and $j(\kappa) \in C^{(n)}$. Now since WEEA, holds $V_{j(\kappa)} \subseteq M$, hence κ is superstrong with target $j(\kappa)$. Altogether, κ is λ-$\mathfrak{a}$-$C^{(n)}$–extendible. □

The following results are evidently true under WEEA:

Proposition 2.0.13 (WEEA)**.** *For each $n \geq 1$, the following is true:*

1. *Every $C^{(n)}$–supercompact cardinal is a Σ_{n+2}-correct cardinal.*
2. *Every $C^{(n+1)}$-supercompact cardinal is a limit of $C^{(n)}$–supercompact cardinals, and the first $C^{(1)}$-supercompact cardinal is a limit of supercompact cardinals.*

Corollary 2.0.14. *Assume that* WEEA *is consistent.*[4] *Then, for each $n \geq 1$, the following large cardinal configuration is also consistent:*

$$\min \mathfrak{S} < \min \mathfrak{S}^{(n)} = \min \mathfrak{a}\text{-}\mathfrak{E}^{(n)} < \min \mathfrak{E}^{(n)} < \min \mathfrak{S}^{(n+1)} = \min \mathfrak{a}\text{-}\mathfrak{E}^{(n+1)}.$$

Prima facie it seems hard to prove the consistency of WEEA via a forcing argument. Instead, a more reasonable strategy would be to look at the corresponding canonical inner models. In this regard, Woodin has suggested that WEEA should hold in his canonical inner models for finite levels of supercompactness [Woo]. We have not managed to verify this so we leave it as an open question:

Question 2.0.15. Is WEEA consistent with ZFC plus suitable large cardinals?

Observe that since $\mathfrak{a}$-$C^{(n)}$–extendibility and $C^{(n)}$–extendibility form a proper large-cardinal hierarchy the above series of results yield all their possible spacial configurations within the universe of sets. Nonetheless, these theorems are not informative about the distribution of $C^{(n)}$–supercompact cardinals. In Chapter 3 and Chapter 4 we will address this problem and clarify the status of these cardinals.

[4]Namely, consistent with ZFC plus the necessary large cardinals.

CHAPTER 3

Distinguishing $C^{(1)}$-supercompactness and supercompactness

3.1 On another question of Bagaria

In this section we will show that the notions of supercompactness and $C^{(1)}$-supercompactness are different. The problem of whether these large-cardinal notions are the same or not was first raised in the work of Bagaria on $C^{(n)}$–cardinals [Bag12, §5]. Here we answer negatively Bagaria's question by proving the following:

Theorem 3.1.1. *Assume the* GCH *holds and let κ be a $C^{(1)}$-supercompact cardinal. Then there is cardinal-preserving generic extension of the universe where κ is (the first) supercompact cardinal but not $C^{(1)}$-supercompact. Moreover, in this model κ is the first ω_1-strong compact cardinal.*

An immediate corollary of the above is the following:

Corollary 3.1.2. *If the* GCH *is consistent with the existence of two $C^{(n)}$-supercompact cardinals, then "*$\mathsf{ZFC} + \min \mathfrak{S} < \min \mathfrak{S}^{(n)}$*" is also consistent.*

Another related question from [Bag12, §5] is if the first $C^{(1)}$-supercompact is necessarily a Σ_3-correct cardinal. Observe that the above theorem is not informative in this regard and thus we are still not in conditions to give a satisfactory answer. In Chapter 4, we will come back to this and give a negative answer.

Remark 3.1.3. Despite $C^{(1)}$-supercompact cardinals are consistent with the GCH[1] it is not clear to us if this can be achieved from the consistency of one $C^{(1)}$-supercompact. The reason being the troublesome interplay between $C^{(n)}$–supercompact cardinals and Forcing; specifically, it turns out that $C^{(n)}$–supercompact embeddings are hard to *lift*. For more details see Section 4.5.

[1] See Theorem 5.6.6

The proof idea behind Theorem 3.1.1 is to use Radin forcing $\mathbb{R}_w$ and show that in some generic extension there is a supercompact cardinal which is not $C^{(1)}$-supercompact. To show that κ remains supercompact in $V^{\mathbb{R}_w}$ we will make use of a result due to J. Cummings and W. H. Woodin [CW]. For the second claim we will prove that even a minor strengthening of the notion of measurability is fragile under Radin forcing. Roughly speaking, this fragility is a consequence of the fact that Prikry-type extensions tend to accommodate non-compact objects, such as $\square$-sequences (cf. Definition 1.4.1). All of these questions will be addressed at Section 3.1.1.

Since in the model of Theorem 3.1.1 the first supercompact is the first ω_1-strongly compact, both cardinals coincide with the first strong compact cardinal. This phenomenon is hardly avoidable if one uses Prikry-type forcings. As we will discuss in Section 3.1.1, the reason for this is the existence of unboundedly many instances of weak square sequences at cofinality ω.

This suggests the following question: Can we separate the first supercompact and the first strong compact cardinal in the model of Theorem 3.1.1? In light of the above it seems necessary to have control over the sort of $\square$-sequences that we are indirectly introducing by forcing. In Section 3.2 we answer affirmatively this question by proving the following result:

Theorem 3.1.4 (Hayut, Magidor, P.). *Let $\lambda < \kappa$ be two supercompact cardinals and μ be a $C^{(1)}$-supercompact above κ. Assume the $\mathsf{GCH}_{>\lambda}$ holds. Then there is a generic extension of the universe exhibiting the following large-cardinal configuration*

$$\min \mathfrak{M} < \min \mathfrak{M}_\infty = \min \mathfrak{K} < \min \mathfrak{S} < \min \mathfrak{S}^{(1)}.$$

The above theorem was obtained in collaboration with Y. Hayut and M. Magidor and corresponds to Theorem 1.4 of [HMP20]. Roughly, the proof idea is to force with an Easton support κ-iteration which adds $\square_{\theta,\mathrm{cof}(\theta)}$-sequences, for many cardinals $\theta < \kappa$ with $\mathrm{cof}(\theta) \geq \lambda$.

3.1.1 Radin Forcing and ω^*-measurable cardinals

An instructive way to introduce the incompactness phenomenon is via the $\square$-principles. In Section 1.4 we have shown that large cardinals, such as supercompact cardinals, have a strong impact on the possible $\square$-configurations of the universe of sets. Specifically, in Theorem 1.4.6 we have shown that a supercompact cardinal κ implies that $\square_{\lambda,\mathrm{cof}(\lambda)}$ fails, for each singular cardinal λ such that $\mathrm{cof}(\lambda) < \kappa < \lambda$. In the light of this it is natural to ask *how much square* can bear a supercompact cardinal below it. In [Apt05] it is showed that a supercompact cardinal κ is consistent with $\square_\lambda$, for a stationary subset of $\lambda < \kappa$. It is worth mentioning that all these λ of Apter's model are regular, though similar results can be obtained for stationary sets concentrating on any fixed cofinality $<\kappa$ (see e.g. Proposition 3.2.5). Later we will show that for $C^{(1)}$-supercompact cardinals this situation turns to be more restrictive, which actually is the key ingredient for the proof of Theorem 3.1.1.

Let us consider the following strengthening of measurability:

Definition 3.1.5. A cardinal κ is called ω^*-measurable if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa)$ a limit cardinal and ${}^{\omega}M \subseteq M$. For any regular cardinal μ one can similarly define the notion of μ^*-measurability.

An obvious remark is that not all the measurable elementary embeddings are witnesses for ω^*-measurability. For instance, no elementary embedding arising from a measure on κ (or on $\mathcal{P}_\kappa(\lambda)$) yields an ω^*-measurable embedding (cf. page 8). The same is true for any direct limit of an iteration of ultrapowers [Kan09, Lemma 19.5].

The results of this section will be formulated for ω^*-measurable cardinals but all of them can be straightforwardly adapted to μ^*-measurables. Note that if κ is $C^{(1)}$-supercompact, then it is μ^*-measurable, for every regular cardinal μ.

Proposition 3.1.6 (Some properties of ω^*-measurable cardinals)**.**

1. *"κ is ω^*-measurable" is a Σ_2-expressible property of κ.*
2. *Assume that there is a ω^*-measurable cardinal and a strong cardinal. Then the first strong cardinal is greater than the first ω^*-measurable cardinal.*
3. *If κ is strong and ω^*-measurable, $\{\lambda < \kappa \mid$ "λ is ω^*-measurable"$\} \in \mathcal{U}$, where $\mathcal{U}$ is the standard normal measure on κ derived by some strong embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$.*

Proof. (1) Mimicking the arguments of [Bag12, §5], one can check that κ is ω^*-measurable if and only if the following holds

$$\begin{aligned}\exists\lambda\exists E\exists Y\exists\zeta(\lambda \text{ is regular } \wedge\ \kappa, E, Y, \zeta \in V_\lambda\ \wedge\ Y \text{ is transitive}\\ \wedge\, [Y]^{\leq\aleph_0} \subseteq Y \wedge V_\lambda \models \text{“}E \text{ is an } (\kappa, Y)\text{-extender over } V_\zeta,\\ j_E[Y] \subseteq Y \text{ and } j_E(\kappa) \text{ is a limit cardinal”}).\end{aligned}$$

Here a (κ, Y)-extender is a generalization of the extenders defined in Section 1.1.3 called *Martin-Steel extenders*. For more about these see [Tsa12, A.3].

(2) follows from clause (3) of Proposition 1.2.11.

For (3) assume that κ is strong and ω^*-measurable. By (1) and strongness of κ we may find $E, Y, \zeta \in V_\lambda$, $\lambda \in C^{(1)}$ witnessing that κ is ω^*-measurable cardinal, and a λ-strong elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$. Clearly M thinks that κ is ω^*-measurable cardinal and thus the result follows by elementarity. $\square$

Proposition 3.1.7. *Assume the* GCH *holds. Let κ be a ω^*-measurable cardinal and assume that κ is also ω_1-strong compact. Then, there is no club $C \subseteq \kappa$ such that $\square_{\lambda,\omega}$ holds, for each $\lambda \in C \cap E^\kappa_\omega \cap \mathrm{Card}$.*

Proof. Aiming for a contradiction, assume that the result is false and let $C \subseteq \kappa$ be a club witnessing this. Since $M^\omega \subseteq M$, $\mathrm{cof}(j(\kappa)) \geq \aleph_1$ and thus $E^{j(\kappa)}_\omega$ is stationary. By elementarity, $M \models \mathsf{GCH}$ and for all cardinals $\lambda \in j(C) \cap E^{j(\kappa)}_\omega$, $M \models \square_{\lambda,\omega}$ holds. Thus, since $j(\kappa)$ is a limit cardinal, there is $\lambda \in j(C) \cap E^{j(\kappa)}_\omega \cap \mathrm{Card}$ above κ for which $\square_{\lambda,\omega}$ holds in M. Since ${}^\omega M \subseteq M$ and $M \models \mathsf{GCH}_\lambda$, $(\lambda^+)^M = (\lambda^{\aleph_0})^M$. By ω-closedness of M and the GCH_λ, $(\lambda^+)^V = (\lambda^+)^M$, hence $\square_{\lambda,\omega}$ holds in V. By Theorem 1.4.6(4), $\square_{\lambda,\omega}$ collides with the ω_1-strong compactness of κ, which yields the desired contradiction. $\square$

Remark 3.1.8. The previous argument naturally extends to any fixed cofinality $\mathrm{cof}(\mu) = \mu < \kappa$ by assuming that κ is μ^+-strongly compact and μ^*-measurable. The same is true if one replaces GCH by $\mathsf{GCH}_{>\lambda}$, for some cardinal $\lambda < \kappa$.

An immediate corollary of Proposition 3.1.7 is the following:

Corollary 3.1.9. *Assume the* GCH *holds. Let κ be regular and $\mathbb{P}$ be a forcing notion such that some condition $p \in \mathbb{P}$ forces the following:*

1. *"κ is ω_1-strongly compact and $\mathrm{cof}(\kappa) = \kappa$".*
2. *"$\exists \tau\, (\tau \subseteq \kappa$ club $\wedge\ \forall \lambda \in \tau \cap E^\kappa_\omega \cap \mathrm{Card}\, (\square_{\lambda,\omega}$ holds $))$"*
3. *"All cardinals and the* GCH *are preserved"*

Then, $p \Vdash_{\mathbb{P}}$ "κ is not ω^*-measurable". In particular, p forces that κ is not $C^{(1)}$-supercompact.

Remark 3.1.10. By Remark 3.1.8, the above corollary is true for any fixed cofinality $\mathrm{cof}(\mu) = \mu < \kappa$ by assuming that κ is μ^+-strong compact in (1). The same is true if one replaces GCH by $\mathsf{GCH}_{>\lambda}$, where $\lambda < \kappa$.

The following theorem due to M. Gitik [Git97] –and independently to M. Džamonja and S. Shelah [DS95]– will be crucial for our main result.

Theorem 3.1.11. *Suppose $V \subseteq W$ are two inner models of* ZFC*, κ is an inaccessible cardinal in V while singular of countable cofinality in W and $(\kappa^+)^V = (\kappa^+)^W$. Then, $W \models \square_{\kappa,\omega}$.*

The moral for this is that if one aims to change the cofinality of a large cardinal κ in a reasonable way (i.e. preserving κ^+) then non-compact objects will show up. Notice that in the particular case where κ is a measurable cardinal and W is a generic extension by Prikry forcing of V then (V, W) fulfills the assumptions of the theorem. In particular, $W \models \square_{\kappa,\omega}$.

Remark 3.1.12. The analogous result for uncountable cofinalities is false by virtue of a recent result of M. Levine and D. Sinapova [LS19].

Appealing to Theorem 3.1.11 one obtains the following:

Proposition 3.1.13. *Assume the* GCH *holds. Let κ be regular and $\mathbb{P}$ a forcing notion such that some condition $p \in \mathbb{P}$ forces the following:*

1. *"κ is ω_1-strongly compact and* $\mathrm{cof}(\kappa) = \kappa$*";*
2. *"$\exists \tau\, (\tau \subseteq \kappa \cap \mathrm{Inac}^{\check{V}} \wedge \tau$ is a club)"*
3. *"All cardinals and the* GCH *are preserved"*

Then, $\mathbb{P}$ and p satisfy the conditions of Corollary 3.1.9.

Proof. It suffices to check that p forces clause (2) of Corollary 3.1.9. Let $G \subseteq \mathbb{P}$ some generic filter with $p \in G$. By (2), in $V[G]$, there is a club subset $C \subseteq \kappa$ of V-inaccessible cardinals. Since κ is regular, $S := (E^\kappa_\omega)^{V[G]}$ is stationary and thus $C \cap S$ also. Observe that each $\lambda \in C \cap S$ is a V-inaccessible cardinal with countable cofinality in $V[G]$. Thus, since cardinals are preserved, Theorem 3.1.11 yields $V[G] \models \square_{\lambda,\omega}$, for each $\lambda \in C \cap S$. Since $G \subseteq \mathbb{P}$ was arbitrary it follows that $p \Vdash_{\mathbb{P}}$ "$\exists \tau\, (\tau \subseteq \kappa$ club $\wedge\, \forall \lambda \in \tau \cap E^\kappa_\omega \cap \mathrm{Card}\, (\square_{\lambda,\omega}$ holds $))$", as wanted. $\square$

Remark 3.1.14. The same is true if one replaces GCH by $\mathsf{GCH}_{>\lambda}$.

A relevant representative of the family of Prikry-type forcings fulfilling the above requirements is Radin forcing [CW][Git10]. We will devote the rest of this section to show this. Before that, let us recall the basics of this forcing. For our exposition we will follow [CW, §6] and [Git10, §5.1] where the reader is referred for more details.

Definition 3.1.15 (Constructing embeddings)**.** Let $j : V \to M$ be an elementary embedding into a transitive class M with $\mathrm{crit}(j) = \kappa$. The measure sequence $u_j = \langle u_j(\alpha) \mid \alpha < \ell(u) \rangle$ constructed by the embedding j is defined by recursion as follows:

- $u_j(0) := \langle \kappa \rangle$,
- if $u_j \restriction \alpha \in M$ and $\alpha > 0$, $u_j(\alpha) := \{X \subseteq V_\kappa \mid u_j \restriction \alpha \in j(X)\}$,

where $\ell(u) := \min\{\alpha \in \mathrm{ORD} \mid u_j \restriction \alpha \notin M\}$. We shall typically denote the cardinal κ by κ_u and refer to it as the critical point of u.

Observe that for each $0 < \alpha < \ell(u_j)$, $u_j(\alpha)$ yields a non-principal κ-complete ultrafilter over V_κ. For different ordinals α and β it is not hard to show that $u_j(\alpha)$ and $u_j(\beta)$ concentrate on disjoint sets.

Definition 3.1.16 (Measure sequences & good pairs)**.** The class of measure sequences $\mathcal{U}_\infty$ is defined as follows:

- $\mathcal{U}_0 := \{u \mid \exists \alpha \exists E (E$ is an extender $\wedge\ \alpha \leq \ell(u_{j_E}) \wedge\ u = u_{j_E} \restriction \alpha)\}$;[2]

[2] Here E is an extender means that E is a (κ, λ)-extender, for some $\kappa < \lambda$.

- $\mathcal{U}_{n+1} := \{u \in \mathcal{U}_n \mid \forall \alpha \in (0, \ell(u))\, (u(\alpha) \cap V_{\kappa_u} \in u(\alpha))\}$;
- $\mathcal{U}_\infty := \bigcap_{n<\omega} \mathcal{U}_n$.

Given a measure sequence $u \in \mathcal{U}_\infty$, the filter of u-large sets $\mathcal{F}(u)$ is defined as follows:

$$\mathcal{F}(u) := \begin{cases} \{\emptyset\}, & \text{if } \ell(u) = 1; \\ \bigcap_{0<\alpha<\ell(u)} u(\alpha), & \text{otherwise.} \end{cases}$$

A good pair is a pair (u, A), where $u \in \mathcal{U}_\infty$, $A \in \mathcal{F}(u)$ and $A \subseteq \mathcal{U}_\infty$.

It is not hard to check that $\mathcal{F}(u)$ defines a κ-complete filter over V_κ. Also observe that all the measure sequences $u \in \mathcal{U}_\infty$ concentrate on a set formed by measure sequences: formally, for each $\alpha \in (0, \ell(u))$, $\mathcal{U}_\infty \cap V_{u(0)} \in u(\alpha)$.

It is a worth to remark that under relatively modest large cardinal assumptions, such as κ being $(\kappa+2)$-strong, one can construct long measure sequences; i.e. $u \in \mathcal{U}_\infty$ with $\kappa_u = \kappa$ and $\ell(u) \geq (2^{\kappa_u})^+$. For details see [CW, Theorem 6.1.5].

We are already in conditions to give the definition of Radin forcing:

Definition 3.1.17 (Radin forcing)**.** Let $u \in \mathcal{U}_\infty$. Radin forcing $\mathbb{R}_u$ is the partial order whose conditions $p \in \mathbb{R}_u$ are of the form

$$p := \langle (u_0^p, A_0^p), \dots, (u_{n-1}^p, A_{n-1}^p), (u, A_n^p) \rangle,$$

where

1. for each $0 \leq i \leq n$, (u_i^p, A_i^p) is a good pair,
2. for each $0 \leq i \leq n-1$, $(u_i^p, A_i^p) \in V_{\kappa_{u_{i+1}}}$.

Given two conditions $p, q \in \mathbb{R}_u$, we shall write $p \leq q$ if, provided

$$p := \langle (u_0^p, A_0^p), \dots, (u_{n-1}^p, A_{n-1}^p), (u, A_n^p) \rangle$$
$$q := \langle (w_0^q, B_0^p), \dots, (w_{m-1}^q, B_{m-1}^p), (u, B_m^q) \rangle,$$

then the following are true:

1. $m \leq n$, and for each $j \leq m$, there is $i \leq n$ such that $w_j^q = u_i^p$,
2. for each $i \leq n$,
   (ℵ) if there is $j \leq m$ such that $w_j^q = u_i^p$, then $A_i^p \subseteq B_j^q$,
   (ℶ) and, otherwise, letting $j(i) := \min\{j \leq m \mid \kappa_{u_i^p} < \kappa_{w_j^q}\}$,

$$u_i^p \in B_j^q \text{ and } A_i^p \subseteq B_j^q \cap V_{\kappa_{u_i^p}}.$$

In the above conditions, we shall write $p \leq^* q$ in case that $p \leq q$ and $n = m$. Given a condition $p \in \mathbb{R}_u$ and $w \in \mathcal{U}_\infty$, we shall say that w appears in p if there is $i \leq n$ such that $w = u_i^p$.

Proposition 3.1.18 (Main properties of $\mathbb{R}_u$)**.** *Let $u \in \mathcal{U}_\infty$ with $\ell(u) \geq 2$ and $\mathbb{R}_u$ be Radin forcing. Let $G \subseteq \mathbb{R}_u$ a generic filter. Then, the following statements hold:*

1. *$\mathbb{R}_u$ is κ_u-cc, hence it preserves cofinalities $\geq \kappa_u$;*
2. *$\mathbb{R}_u$ has the Prikry property; namely, for each sentence φ in the language of forcing $\mathcal{L}_{\mathbb{R}_u}$ and $p \in \mathbb{R}_u$, there is $q \leq^* p$ such that $q \parallel \varphi$.*
3. *Set $\mathcal{M}_G := \{w \in \mathcal{U}_\infty \mid w \neq u \wedge \exists p \in G\,(w \text{ appears in } p)\}$ and let $\langle w_\alpha \mid \alpha < \Theta\rangle$ be an enumeration of $\mathcal{M}_G$ in such a way that, if $\alpha < \beta < \Theta$, $\kappa_{w_\alpha} < \kappa_{w_\beta}$. Then $C_G := \langle \kappa_{w_\alpha} \mid \alpha < \Theta\rangle$ is a club subset of κ_u. Conversely, G can be resembled from C_G in the following way: G is the set of all $p \in \mathbb{R}_u$ such that*

   (a) if w appears in p, $w \in C_G$,

   (b) if $w \in C_G$, there is $q \leq p$ such that w appears in q.
4. *In the above terminology,*
   $$\Theta := \begin{cases} \omega^{\ell(u)-1}, & \text{if } \ell(u) < \omega; \\ \omega^{\ell(u)}, & \text{if } \omega \leq \ell(u) < \kappa_u. \end{cases}$$
   In particular, if $\ell(u) < \kappa_u$, either $V[G] \models \mathrm{cof}(\kappa) = \omega$, or $V[G] \models \mathrm{cof}(\kappa_u) = \mathrm{cof}(\ell(u))^V$. Otherwise, if $\kappa_u \leq \ell(u)$,
   $$\mathrm{cof}(\kappa_u)^{V[G]} := \begin{cases} \mathrm{cof}(\kappa_u)^V, & \text{if } \mathrm{cof}^V(\ell(u)) > \kappa_u; \\ \mathrm{cof}(\ell(u))^V, & \text{if } \ell(u) \text{ is limit ordinal and } \mathrm{cof}^V(\ell(u)) < \kappa_u; \\ \aleph_0, & \text{if } \ell(u) \text{ is successor ordinal or } \mathrm{cof}^V(\ell(u)) = \kappa_u. \end{cases}$$
5. *For each $\beta < \Theta$, $C_\beta := C_G \restriction \beta$ yields a generic filter G_β for $\mathbb{R}_{w_\beta}$.*
6. *For each $\theta < \kappa_u$, set $\alpha_\theta := \min\{\alpha < \Theta \mid \kappa_{w_\alpha} \leq \theta < \kappa_{w_{\alpha+1}}\}$ and $G_{\alpha_\theta} := G \restriction \alpha_\theta$.[3] Then, $\mathcal{P}(\theta) \cap V[G] = \mathcal{P}(\theta) \cap V[G_{\alpha_\theta}]$. In particular, in the generic extension $V[G]$ all cardinals $<\kappa_u$ are preserved and κ_u is strong limit.*

For the proof of the previous facts we refer the reader to [Git10, §5].

One of the key advantages of Radin forcing over Magidor forcing (see Section 7.1) is that it may be prepared to preserve large cardinals. This

[3]Since C_G is a club α_θ always exists.

feature of Radin forcing has been fruitfully exploited in Cardinal Arithmetic [FW91][Cum92][Mer07] and in the study of the combinatorics of HOD [CFG15]. The said preparation requires of the following key concept:

Definition 3.1.19 (Repeat point)**.** Let $u \in \mathcal{U}_\infty$. An ordinal $\alpha < \ell(u)$ is said to be a *repeat point* of u if for each $X \in u(\alpha)$ there is $\beta < \alpha$ such that $X \in u(\beta)$.

Since $|V_{\kappa_u+1}| = 2^{\kappa_u}$ any measure sequence u with $\ell(u) \geq (2^{\kappa_u})^+$ has a repeat point $\alpha < (2^{\kappa_u})^+$ with $\mathrm{cof}(\alpha) > \kappa_u$. Thus, observe that if κ is a supercompact cardinal then there are measure sequences $u \in \mathcal{U}_\infty$ with $\kappa_u = \kappa$ having many repeat points.

The following is an unpublished result due to J. Cummings and W. H. Woodin [CW]:

Theorem 3.1.20 (Cummings & Woodin)**.** *Let $\kappa \leq \lambda$ and let $u \in \mathcal{U}_\infty$ be a measure sequence constructed by some λ-supercompact embedding $j : V \to M$ with* $\mathrm{crit}(j) = \kappa$. *Assume that u has a repeat point $\alpha < j(\kappa_u)$ and set $w := u \restriction \alpha$. Then, after forcing with $\mathbb{R}_w$, the embedding j lifts to a λ-supercompact embedding in $V^{\mathbb{R}_w}$.*

Corollary 3.1.21. *Let κ be a supercompact cardinal. Then there is a measure sequence $w \in \mathcal{U}_\infty$ with $\kappa_w = \kappa$ such that $\mathbb{1} \Vdash_{\mathbb{R}_w}$ "κ is supercompact".*

Proof. Let κ be a supercompact cardinal and set $\chi := (2^\kappa)^+$. For each cardinal $\theta \geq \chi$ let $j_\theta : V \to M$ be a θ-supercompact embedding with $\mathrm{crit}(j) = \kappa$ and set $u_\theta := u_{j_\theta} \restriction \chi$. Clearly, for each $\theta \geq \chi$, u_θ has a repeat point α_χ. By Theorem 3.1.20, for each $\theta \geq \chi$, $\mathbb{1} \Vdash_{\mathbb{R}_{u_\theta \restriction \alpha_\theta}}$ "κ is θ-supercompact". Observe that for a class of ordinals $\theta \in \mathfrak{C}$, $u_\theta = u$ and $\alpha_\theta = \alpha$. Set $w := u \restriction \alpha$. It thus follows that, for each $\theta \in \mathfrak{C}$, $\mathbb{1} \Vdash_{\mathbb{R}_w}$ "κ is θ-supercompact", and thus $\mathbb{1} \Vdash_{\mathbb{R}_w}$ "κ is supercompact", as desired. $\square$

Hereafter assume that κ is a supercompact cardinal and that $w \in \mathcal{U}_\infty$ is the measure sequence given by the previous corollary. In particular, $\mathbb{R}_w$ satisfies (1) of Proposition 3.1.13.

Lemma 3.1.22. *Assume the* GCH *holds. Then,* $\mathbb{1} \Vdash_{\mathbb{R}_w} \mathsf{GCH}$. *In particular, $\mathbb{R}_w$ satisfies (3) of Proposition 3.1.13.*

Proof. Let $G \subseteq \mathbb{R}_u$ generic and θ be an infinite cardinal. We shall distinguish two cases: either $\theta < \kappa_u$ or $\theta \geq \kappa_u$. Assume first that $\theta < \kappa_u$. By Proposition 3.1.18(6), $\mathcal{P}(\theta) \cap V[G] = \mathcal{P}(\theta) \cap V[G_{\alpha_\theta}]$, hence $\mathcal{P}(\theta)$ is determined by the number of $\mathbb{R}_{w_{\alpha_\theta}}$-nice names for subsets of θ. Denote this set by $\mathrm{Nice}(\mathbb{R}_{w_{\alpha_\theta}}; \theta)$. The $\mathsf{GCH}_{<\kappa}$ in V yields the following chain of expressions:

$$|\mathrm{Nice}(\mathbb{R}_{w_{\alpha_\theta}}; \theta)| = |\mathbb{R}_{w_{\alpha_\theta}}|^{\kappa_{w_{\alpha_\theta}} \cdot \theta} = (2^{\kappa_{w_{\alpha_\theta}}})^\theta = (\kappa^+_{w_{\alpha_\theta}})^\theta \leq \theta^+.$$

Thus, GCH_θ holds in $V[G]$. For the other case assume $\theta \geq \kappa_u$ and observe that the $\mathsf{GCH}_{\geq\kappa}$ in V yields

$$|\mathrm{Nice}(\mathbb{R}_u;\theta)| = |\mathbb{R}_u|^{\kappa(u)\cdot\theta} = (2^{\kappa_u})^\theta = (\kappa_u^+)^\theta \leq \theta^+.$$

Again this yields GCH_θ in $V[G]$. Finally, the in particular claim follows from clauses (1) and (6) of Proposition 3.1.18. □

Remark 3.1.23. The same holds if the GCH is replaced by the $\mathsf{GCH}_{>\lambda}$, for some $\lambda < \kappa$.

Lemma 3.1.24. *There is $p \in \mathbb{R}_w$ such that p forces the statement of Proposition 3.1.13(2).*

Proof. Set $A := \{v \in \mathcal{U}_\infty \cap V_{\kappa_w} \mid \kappa_v \text{ is measurable}\}$ and observe that $A \in \mathcal{F}(w)$. Set $\mathbb{R} := \mathbb{R}_w \downarrow \langle(w, A)\rangle$ and let $G \subseteq \mathbb{R}$ generic. In $V[G]$ there is a sequence $\langle w_\alpha \mid \alpha < \kappa\rangle$ of measure sequences which induces a club $C \subseteq \kappa$. By construction, all the members of C are V-measurables. Setting $p := \langle(w, A)\rangle$ the result follows. □

Combining Proposition 3.1.13 with the previous lemmas we arrive at the following corollary:

Corollary 3.1.25. *Assume the* GCH *holds and let κ be a supercompact cardinal. Then there is $w \in \mathcal{U}_\infty$ and $p \in \mathbb{R}_w$ such that $\mathbb{R}_w$ and p witness the hypotheses of Corollary 3.1.9. Besides, $\mathbb{1} \Vdash_{\mathbb{R}_w} \kappa$ is supercompact.*

The proof of Theorem 3.1.1 is now easy:

Proof of Theorem 3.1.1. Let $w \in \mathcal{U}$ and $p \in \mathbb{R}_w$ be as in Corollary 3.1.25. Let $G \subseteq \mathbb{R}_w$ a generic filter with $p \in G$. By our choice, in $V[G]$, κ is a supercompact cardinal which is not $C^{(1)}$-supercompact. Finally, κ is the first ω_1-strong compact in $V[G]$ as a consequence of clause (2) of Corollary 3.1.9 and Theorem 1.4.6(4). □

Before closing the section we would like to make some comments about Proposition 3.1.7 and, in general, about the preservation of (very) large cardinals by Radin forcing. We commence with the following question:

Question 3.1.26. Does the conclusion of Proposition 3.1.7 hold if $j(\kappa)$ is a successor cardinal?

Observe that the fact that $j(\kappa)$ was a limit cardinal was crucially used in Proposition 3.1.7 to guarantee the existence of some cardinal $\lambda \in C \cap E^{j(\kappa)}_\omega$ above κ. A solution of the above question would give us some insights about how elementary embeddings look like in Radin model. For instance, suppose that the answer to Question 3.1.26 was affirmative, then this would give evidence that the elementary embeddings witnessing the existence of (e.g.) a supercompact cardinal in Radin model have non-cardinal target.[4]

[4]Of course, modulo the GCH.

It was already known that there is tension between Prikry-type forcings and very large cardinals. For instance, if κ is an extendible cardinal (cf. Definition 1.2.2) the above argument shows that $\mathbb{R}_w$ destroys the extendibility of κ. In this regard, here we have considerably weakened the strength for a large cardinal to be (almost) surely destroyed by Radin forcing. Indeed, we have shown that even minor strengthenings of measurability, such as ω^*-measurables, are fragile under this forcing.

3.2 More on $C^{(1)}$-supercompactness

In this section we will continue our study of $C^{(1)}$-supercompact cardinals by proving Theorem 3.1.4. Hereafter let us assume that the hypotheses of Theorem 3.1.4 hold. As a preliminary caution we need to prepare our ground model to guarantee that the following hold:

($\aleph$) $\min \mathfrak{M}_\infty = \min \mathfrak{K} = \lambda$;

($\beth$) The strongness and strong compactness of λ are indestructible under $<\lambda$-directed closed forcings which are also λ-strategically closed.

We can ensure this by virtue of a theorem of A. Apter which says that there is a forcing iteration $\mathbb{Q} \subseteq V_\lambda$ which forces ($\aleph$) and ($\beth$) [Apt06, Theorem 2]. Actually $\mathbb{Q}$ forces $(\aleph) + (\beth) + \min \mathfrak{K} < \min \mathfrak{S}$, as the first strong cardinal is below the first supercompact (cf. page 8). Altogether the following are true in $V^{\mathbb{Q}}$:

1. ($\aleph$) and ($\beth$).
2. $\mathsf{GCH}_{>\lambda}$.
3. $\min \mathfrak{M} < \min \mathfrak{M}_\infty = \min \mathfrak{K} = \lambda < \min \mathfrak{S}$.
4. κ and μ are supercompact and $C^{(1)}$-supercompact, respectively.

For the ease of notation let us assume that our ground model is $V^{\mathbb{Q}}$. We show now how to add many square sequences at suitable cofinalities. For this purpose we will use a poset introduced by J. Cummings, M. Foreman and M. Magidor in [CFM01].

For a fixed $\lambda \leq \mathrm{cof}(\theta) < \theta$, there is a $\mathrm{cof}(\theta)$-directed closed and $<\theta$-strategically closed forcing $\mathbb{S}_\theta$ which forces $\square_{\theta,\mathrm{cof}(\theta)}$ [CFM01, Theorem 9.1]. Since θ is singular and $\mathbb{S}_\theta$ is $<\theta$-strategically closed this forcing preserve cofinalities $\leq\theta^+$. Also, under the $\mathsf{GCH}_{>\lambda}$, $|\mathbb{S}_\theta| = \theta^+$, hence $\mathbb{S}_\theta$ becomes cofinality-preserving. It should be clear that $\mathbb{S}_\theta$ also preserves the $\mathsf{GCH}_{>\lambda}$ pattern.

The proof idea for Theorem 3.1.4 is to iterate $\mathbb{S}_\theta$ for a *sparse enough* set of $\theta \in (\lambda, \kappa)$. The standard procedure to build these kind of iterations is to guide them with a function $\ell : \kappa \to V_\kappa$ exhibiting some sort of *fast behavior*. In this context the *sparse enough* set that we are seeking for corresponds with the closure points of the function ℓ: i.e. $\mathrm{cl}(\ell) := \{\alpha < \kappa \mid \ell[\alpha] \subseteq \alpha\}$.

Definition 3.2.1 ($\mathscr{L}$-fast function)**.** Let $\mathscr{L}(x)$ be a large-cardinal property which can be characterized by means of the existence of an extender. If δ is a cardinal such that $\mathsf{ZFC} + \exists x \mathscr{L}(x) \vdash \mathscr{L}(\delta)$, a *$\mathscr{L}$-fast function* $\ell : \delta \to V_\delta$ on δ is a function such that, for each $\lambda > \delta$, there is an extender E with $\mathrm{crit}(j_E) = \delta$ and $j_E(\ell)(\delta) > \lambda$.

Remark 3.2.2. Obviously, any fast function $\ell : \delta \to V_\delta$ can be naturally identified with a function $s : \delta \to \delta$ exhibiting the same behavior.

A paradigmatic example of fast function is given by the the so-called Laver function [Lav78]. For a supercompact cardinal δ, $\ell : \delta \to V_\delta$ is a Laver function if for each set x and each $\lambda \geq |\mathrm{TC}(\{x\})|$ there is a λ-supercompact embedding $j : V \to M$ with $\mathrm{crit}(j) = \delta$ such that $j(\ell)(\delta) = x$. This can be rephrased in terms of extenders as follows: for each $\lambda > \delta$ there is a (δ, Y)-Martin Steel Extender E with $j_E(\ell)(\delta) > \lambda$ [Tsa12, Proposition A.13].

Laver proved in [Lav78] that every supercompact cardinal δ carries a Laver function and used this fact to define a forcing iteration $\mathbb{L}$ which makes the supercompactness of δ indestructible under δ-directed closed forcings. In this section we will use this kind of functions to prove our main theorem.

Lemma 3.2.3. *Let $\delta = \mathrm{cof}(\delta) > \omega$. For each function $\ell : \delta \to V_\delta$ the set $\mathrm{cl}(\ell)$ is a club.*

Proof. It is fairly easy to show that $\mathrm{cl}(\ell)$ is closed. Let $\alpha < \delta$ and set $\beta_0 := \alpha$. For each $1 \leq n < \omega$, define $\beta_{n+1} := \sup \ell[\beta_n]$. Since δ is regular, $\beta_n < \delta$, hence, if $\beta_\omega := \sup_{n<\omega} \beta_n$, $\beta_\omega < \delta$. Observe that $\alpha \leq \beta_\omega \in \mathrm{cl}(\ell)$, as wanted. □

Fix $\ell : \kappa \to V_\kappa$ a Laver function on κ. By restricting ℓ we may assume that $\mathrm{dom}(\ell) = \mathrm{cl}(\ell)$ and that ℓ remains a Laver function.

Definition 3.2.4. Let $\mathbb{P}^\ell_\kappa := \langle \mathbb{P}_\theta; \dot{\mathbb{Q}}_\theta \mid \theta < \kappa \rangle$ be the κ-iteration with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal $\lambda < \theta < \kappa$, if $\theta \in dom(\ell)$ and $\mathbb{1} \Vdash_{\mathbb{P}_\theta}$ "$\check{\theta} \in \dot{E}^\kappa_\lambda \cap \mathrm{Card}$" then $\mathbb{1} \Vdash_{\mathbb{P}_\theta}$ "$\dot{\mathbb{Q}}_\theta = \dot{\mathbb{S}}_\theta$". Otherwise, $\mathbb{1} \Vdash_{\mathbb{P}_\theta}$ "$\dot{\mathbb{Q}}_\theta$ is trivial".

Proposition 3.2.5 (Some properties of $\mathbb{P}^\ell_\kappa$)**.**

1. *$\mathbb{P}^\ell_\kappa$ is λ-directed closed and λ^+-strategically closed, hence $\mathbb{1} \Vdash_{\mathbb{P}_\theta} \check{\theta} \in \dot{E}^\kappa_\lambda$ iff $\theta \in E^\kappa_\lambda$. Also, $\mathbb{1} \Vdash_{\mathbb{P}^\ell_\kappa}$ "$\min \mathfrak{M} < \min \mathfrak{K} = \min \mathfrak{M}_\infty = \check{\lambda} < \min \mathfrak{S}$".*

2. $|\mathbb{P}^\ell_\kappa| = \kappa$, *hence* $\mathbb{1} \Vdash_{\mathbb{P}^\ell_\kappa} \mu \in \mathfrak{S}^{(1)}$. *In particular,* $\mathbb{1} \Vdash_{\mathbb{P}^\ell_\kappa} \mathfrak{S}^{(1)} \neq \emptyset$.

3. $\mathbb{P}^\ell_\kappa$ *has the* κ*-cc and preserves both cofinalities and the* $\mathsf{GCH}_{>\lambda}$.

4. $\mathbb{1} \Vdash_{\mathbb{P}^\ell_\kappa}$ "$\forall \theta \in \dot{E}^\kappa_\lambda \cap \mathrm{dom}(\ell) \cap \mathrm{Card}\,(\square_{\theta,\lambda}$ *holds)". In particular, there is no strongly compact cardinal between* λ *and* κ.

Proof. (1) The first part is certainly true since all the forcings $\mathbb{S}_\theta$ are λ-directed closed and λ^+-strategically closed. By clause ($\sqsupset$) in page 45 it remains to check that λ remains the first strong compact and the first strong cardinal in $V^{\mathbb{P}^\ell_\kappa}$. Let us simply prove that λ is the least strong cardinal in $V^{\mathbb{P}^\ell_\kappa}$ as the analogous claim for strong compactness can be proved similarly.

Let $G \subseteq \mathbb{P}^\ell_\kappa$ generic over V and assume that $V[G] \models \min \mathfrak{M}_\infty < \lambda$. Let $\lambda^* < \lambda$ be witnessing this. Since strongness is a Π_2-expressible property of λ^* and strong cardinals are Σ_2-correct (cf. Proposition 1.2.11), $V[G]_\lambda \models$ "λ^* is strong". On the other hand, $\mathbb{P}^\ell_\kappa$ is λ-distributive, hence $V[G]_\lambda = V_\lambda$, hence $V_\lambda \models$ "$\lambda^\star$ is strong". Finally, since λ was strong in V, hence Σ_2-correct, it follows that λ^* is strong in V, which contradicts the minimality of λ.

(2) This is obvious since $\mathbb{P}^\ell_\kappa \subseteq V_\kappa$.

(3) The first claim follows from (2), hence it is enough with checking that it does not change cofinalities below κ. Let $\theta < \kappa$ be a V-regular cardinal. If $\mathrm{cof}(\theta) \leq \lambda$ the cofinality of θ does not change as a consquence of the λ-closedness of $\mathbb{P}^\ell_\kappa$. Thus, assume that $\mathrm{cof}(\theta) > \lambda$ and split $\mathbb{P}^\ell_\kappa$ as $\mathbb{P}^\ell_\kappa \cong \mathbb{P}_\theta * \dot{\mathbb{Q}}_\theta * \dot{\mathbb{R}}$. By the GCH in the ground model, $|\mathbb{P}_\theta| < \theta$, hence $\mathbb{1} \Vdash_{\mathbb{P}_\theta} \check{\theta} \notin \dot{E}^\kappa_\lambda$, and thus $\mathbb{P}_\theta * \dot{\mathbb{Q}}_\theta$ does not change the cofinality of θ. Finally, $\mathbb{1} \Vdash_{\mathbb{P}_\theta * \dot{\mathbb{Q}}_\theta}$ "$\dot{\mathbb{R}}$ is θ^+-distributive", so $\mathrm{cof}(\theta)^V$ is preserved. The argument for $\mathsf{GCH}_{>\lambda}$ follows by combining the $\mathsf{GCH}_{>\lambda}$ in V, counting nice names arguments and the above factorization.

(4) The first claim follows from (1), (3) and the fact that $\mathbb{P}_{\theta+1}$ forces $\square_{\theta,\lambda}$ and that $\mathbb{R}_\theta$ is θ^+-distributive. The latter claim is an outright consequence of Theorem 1.4.6(3). $\square$

Proposition 3.2.6. *Forcing with* $\mathbb{P}^\ell_\kappa$ *preserves the supercompactness of* κ. *Moreover,* $\mathbb{1} \Vdash_{\mathbb{P}^\ell_\kappa} \min \mathfrak{S} = \kappa$.

Proof. Working in V, let $\lambda > \kappa$, $\theta := (2^{\lambda^{<\kappa}})^+$ and $j : V \to M$ be some θ-supercompact embedding with $\mathrm{crit}(j) = \kappa$ and $j(\ell)(\kappa) > \theta$. Let G a $\mathbb{P}^\ell_\kappa$-generic filter over V. By elementarity, $j(\mathbb{P}^\ell_\kappa) = \langle \mathbb{P}_\theta; \dot{\mathbb{Q}}_\theta \mid \theta < j(\kappa)\rangle$ is the iteration with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal $\lambda < \theta < j(\kappa)$, if $\theta \in \mathrm{dom}(\ell)$, $M \models$ "$\mathbb{1} \Vdash_{\mathbb{P}_\theta} \check{\theta} \in \dot{E}^\kappa_\lambda \cap \mathrm{Card} \to \dot{\mathbb{Q}}_\theta = \dot{\mathbb{S}}_\theta$". Otherwise, $M \models$ "$\mathbb{1} \Vdash_{\mathbb{P}_\theta} \dot{\mathbb{Q}}_\theta$ is trivial". Observe that $j(\ell) \restriction \kappa = \ell$, hence $j(\mathbb{P}^\ell_\kappa) \restriction \kappa = \mathbb{P}^\ell_\kappa$. Besides $\mathrm{cof}(\kappa)^{M[G]} > \lambda$, and thus $j(\mathbb{P}^\ell_\kappa) \cong \mathbb{P}^\ell_\kappa * \mathbb{Q} * \mathbb{P}_{tail}$, where $M \models$ "$\mathbb{1} \Vdash_{\mathbb{P}^\ell_\kappa} \mathbb{Q}$ trivial". Also, since $j(\ell)(\kappa) > \theta$,

$$M \models \text{“}\,\mathbb{1} \Vdash_{\mathbb{P}^\ell_\kappa} \mathbb{Q} * \mathbb{P}_{tail} \text{ is } \theta^+\text{-strategically closed”}.$$

Set $\mathbb{P}^* := \mathbb{Q} * \mathbb{P}_{tail}$. Conditions in $\mathbb{P}^\ell_\kappa$ have bounded support in κ so $j \restriction \mathbb{P}^\ell_\kappa = \mathrm{id}$. Thus, $j[G] \subseteq G * H$, for any $\mathbb{P}^*_G$-generic filter H over $M[G]$. This allows us to lift j to $j^* : V[G] \to M[G * H] \subseteq V[G * H]$ (cf. Section 1.3.3).

Since $\mathbb{P}^\ell_\kappa$ has the κ-cc, ${}^\theta M[G] \cap V[G] \subseteq M[G]$ [Cum10, Proposition 8.4.1]. Similarly, since $\mathbb{P}^*_G$ is θ^+-strategically closed in $M[G]$ and ${}^\theta M[G] \subseteq M[G]$, $\mathbb{P}^*_G$ is also θ-strategically closed in $V[G]$, hence ${}^\theta M[G*H] \cap V[G*H] \subseteq M[G*H]$.

Working in $V[G * H]$, define $\mathcal{U} := \{X \in \mathcal{P}_\kappa(\lambda)^{V[G]} \mid j^*[\lambda] \in j^*(X)\}$. It is routine to check that $\mathcal{U}$ defines a λ-supercompact measure over $\mathcal{P}_\kappa(\lambda)^{V[G]}$.

Let us now check that $\mathcal{U} \in V[G]$. For this observe that the set of $\mathbb{P}^*_G$-nice names for subsets of $\mathcal{P}_\kappa(\lambda)^{V[G]}$ over $V[G]$, $\mathrm{Nice}(\mathcal{P}_\kappa(\lambda)^{V[G]}; \mathbb{P}^*_G)$, has cardinality at most θ. Indeed,

$$|\mathrm{Nice}(\mathcal{P}_\kappa(\lambda)^{V[G]}; \mathbb{P}^*_G)| \leq |\mathbb{P}^*_G|^{(\lambda^{<\kappa})^{V[G]} \cdot |\mathbb{P}^*_G|} < \theta^\theta = \theta^+.$$

For the above inequalities we have used that $|\mathbb{P}^*_G| = |j(\kappa)| \leq (2^{\lambda^{<\kappa}})^V < \theta$ [Kan09, Proposition 22.11] and $\theta^\theta = \theta^+$, as GCH_θ holds in $V[G]$. Since $\mathbb{P}^*_G$ is θ-strategically closed in $V[G]$ it follows that $\mathcal{U} \in V[G]$, so κ is λ-supercompact in $V[G]$. Since the choice of λ was arbitrary, this shows that κ remains supercompact in $V[G]$. Finally, the moreover part follows from Proposition 3.2.5(4) $\square$

We are now in conditions to prove Theorem 3.1.4:

Proof of Theorem 3.1.4. Assume the $\mathsf{GCH}_{>\lambda}$ holds and let $\lambda < \kappa$ be two supercompact cardinals and μ be a $C^{(1)}$-supercompact cardinal above κ. Let $\mathbb{R} := \mathbb{Q} * \mathbb{P}^\ell_\kappa$ where $\mathbb{Q}$ is Apter's forcing from [Apt06, Theorem 2] and $\mathbb{P}^\ell_\kappa$ is as in Definition 3.2.4. Combining propositions 3.2.5 and 3.2.6 it follows that $\mathbb{R}$ forces the following large-cardinal configuration

$$\min \mathfrak{M} < \min \mathfrak{M}_\infty = \min \mathfrak{K} = \lambda < \min \mathfrak{S} = \kappa \leq \min \mathfrak{C}^{(1)}.$$

Now, propositions 3.2.5(4) and 3.1.7 and Remark 3.1.8 imply $\kappa \notin \mathfrak{C}^{(1)}$ and $\mathfrak{C}^{(1)} \neq \emptyset$. Altogether, $\mathbb{R}$ forces the desired large cardinal configuration. $\square$

We close this section leaving the next open question:

Question 3.2.7. Can we separate $\min \mathfrak{K}_{\omega_1}$ from $\min \mathfrak{K}$ in this model?

CHAPTER 4

The identity crisis phenomenon at $C^{(n)}$–supercompact cardinals

We will continue here the discussion initiated in previous chapters on the possible cofigurations of the $C^{(n)}$–hierarchy. Observe that Corollary 2.0.10 does not provide any (non-trivial) information about the relative position between the first $C^{(n)}$–supercompact and the first $C^{(n+1)}$-supercompact. Assuming the consistency of WEEA we proved in Corollary 2.0.14 that $C^{(n)}$–supercompactness forms a hierarchy in the strong sense; that is, $\min \mathfrak{S}^{(n)} < \min \mathfrak{S}^{(n+1)}$, for all $n \geq 1$. For the moment it is still not clear if other configurations are likewise consistent. For instance, can the first $C^{(n+1)}$-supercompact be the first $C^{(n)}$-supercompact? In this section we will address this problematic and prove the following:

Theorem 4.0.1 (Hayut, Magidor, P. [HMP20])**.** *Let $n \geq 1$ and κ be a $C^{(n)}$–supercompact cardinal. Also, assume that κ carries a $\mathfrak{S}^{(n)}$-fast function $\ell\colon \kappa \to \kappa$. Then there is a cardinal-preserving generic extension of the universe where κ is $C^{(n)}$–supercompact and the first ω_1-strongly compact cardinal. In particular, the following configuration holds in the said generic extension:*

$$\min \mathfrak{M} < \min \mathfrak{M}_\infty < \min \mathfrak{K}_{\omega_1} = \min \mathfrak{K} = \min \mathfrak{S} = \min \mathfrak{S}^{(n)} < \min \mathfrak{a}\text{-}\mathfrak{C}^{(1)}.$$

Observe that for obtaining the above configuration the large cardinal hierarchy between $\min \mathfrak{K}_{\omega_1}$ and $\min \mathfrak{S}^{(n)}$ must be collapsed. In particular, WEEA fails there. These sort of situations are well-known by set theorist since Magidor's discovering of the Identity crises phenomenon [Mag76]:

Theorem 4.0.2 (Magidor)**.**

1. *Assume that the existence of a strong compact cardinal is consistent. Then, "$\mathfrak{K} \neq \emptyset + \min \mathfrak{M} = \min \mathfrak{K} < \min \mathfrak{S}$" is also consistent.*
2. *Assume that the existence of a supercompact cardinal is consistent. Then, "$\mathfrak{S} \neq \emptyset + \min \mathfrak{M} < \min \mathfrak{K} = \min \mathfrak{S}$" is also consistent.*

Certainly this means that it is not possible to determine the exact position of the first strong compact. Actually this says more: since $\min \mathfrak{M} \leq \min \mathfrak{K} \leq \min \mathfrak{S}$ is always true, Magidor's theorem shows that $\min \mathfrak{K}$ can coincide with any of the two extreme points of its potential area of location. This is what Magidor called the *Identity crises phenomemon*. Therefore it is clear that Theorem 3.1.1 and Theorem 4.0.1 provide the natural analogous of Magidor's Identity crises theorems at the scale of $C^{(n)}$–supercompact cardinals.

One may also be more ambitious and ask whether the whole hierarchy of $C^{(n)}$–supercompact cardinals can be collapsed to the first supercompact cardinal. Following Magidor's terminology, one may ask if an *Ultimate identity crises* for $C^{(n)}$–supercompactness is possible. The following theorem answers this affirmatively:

Theorem 4.0.3 (Hayut, Magidor, P. [HMP20])**.** *Let $\langle V, \in, \kappa\rangle$ be a model of* $\mathsf{ZFC}^\star$ *plus the scheme $C^{(<\omega)}$-EXT. Then there is a generic extension of the universe exhibiting the following configuration:*

$$\min \mathfrak{M} < \min \mathfrak{M}_\infty < \min \mathfrak{K}_{\omega_1} = \min \mathfrak{K} = \min \mathfrak{S} = \min \mathfrak{S}^{(<\omega)} < \min \mathfrak{a}\text{-}\mathfrak{C}^{(1)}.$$

Here $C^{(<\omega)} - \text{EXT}$ and $\mathsf{ZFC}^\star$ are defined as follows:

Definition 4.0.4. Let $\mathcal{L}$ be the language of Set Theory augmented with an additional constant symbol $\mathbf{k}$.

- $\mathsf{ZFC}^\star$ denotes the version of ZFC where we allow to use the constant symbol $\mathbf{k}$ at any instance of the axioms of replacement and separation.
- We will denote by $C^{(<\omega)} - \text{EXT}$ the scheme of formulas φ_n that for each $n \geq 1$, $\varphi_n \equiv$ "$\mathbf{k}$ is $C^{(n)}$–extendible".
- If is $\mathfrak{M}$ a $\mathcal{L}$-structure, we write $\mathfrak{M} \models C^{(<\omega)} - \text{EXT}$ if for every natural number $n \geq 1$, $\mathfrak{M} \models \mathbf{k}$ is $C^{(n)}$–extendible.

A cardinal κ is $C^{(<\omega)}$-extendible if $\langle V, \in, \kappa\rangle \models C^{(<\omega)} - \text{EXT}$.

Similarly one defines the scheme $C^{(<\omega)}$-SUP and the class of $C^{(<\omega)}$-supercompact cardinals. Notice that theorems 4.0.1 and 4.0.3 solve the following questions of Bagaria:

Question 4.0.5 ([Bag12, §5])**.**

1. Is the first $C^{(1)}$-supercompact a Σ_3-cardinal?
2. Does the class of $C^{(n)}$–supercompact form a hierarchy in the strong sense?

Observe that in both cases the answer is negative. In the first case because is consistent that $\min \mathfrak{S}^{(1)} = \min \mathfrak{S}$, hence $\min \mathfrak{S}^{(1)} \in C^{(2)} \setminus C^{(3)}$. On the other hand the answer to Bagaria's second question is an outright consequence of Theorem 4.0.3.

4.1 Some preliminary comments

Let $n \geq 1$ and κ be a $C^{(n)}$–supercompact cardinal. In Theorem 4.0.1 we have additionally assumed that κ carries a $\mathfrak{S}^{(n)}$-fast function (cf. Definition 3.2.1). This sort of functions are known to exist for $C^{(n)}$–extendible cardinals as shown by Tsaprounis [Tsa18, Theorem 4.2] and thus the consistency of our hypotheses follow from the consistency of a $C^{(n)}$–extendible cardinal.

Tsaprounis' proof mimics Laver's original argument for the existence of $\mathfrak{S}$-fast functions. The key ingredient for the argument to work is that supercompact and $C^{(n)}$–extendible cardinals are as correct as the complexity of the property defining them (cf. Proposition 1.2.11). Specifically, supercompact cardinals are Σ_2-correct and Π_2 definable and $C^{(n)}$–extendible cardinals are Σ_{n+2}-correct and Π_{n+2} definable.

However the above situation is not extensible to $C^{(n)}$–supercompact cardinals: On one hand, the complexity of being a $C^{(n)}$–supercompact cardinal is Π_{n+2}. On the other hand, we have just argued that the first $C^{(n)}$–supercompact cardinal is not necessarily more correct than the first supercompact, hence at most Σ_2-correct. This disagreement poses many difficulties at the time of finding $\mathfrak{S}^{(n)}$-fast functions for $C^{(n)}$–supercompact cardinals. This discussion lead us to ask the following:

Question 4.1.1. Assume that κ is a $C^{(n)}$–supercompact cardinal. Does κ carry a $\mathfrak{S}^{(n)}$-fast function?

Nonetheless, one may obtain $\mathfrak{S}^{(n)}$-fast functions starting with considerably weaker assumptions than $C^{(n)}$–extendibility.

Lemma 4.1.2. *Let κ be a $\mathfrak{a}$-$C^{(n)}$–extendible cardinal. Then κ is a $C^{(n)}$–supercompact cardinal which carries a $\mathfrak{S}^{(n)}$-fast function.*

Proof. Since κ is supercompact we may let a Laver function $\ell : \kappa \to \kappa$ [Lav78]. Fix $\lambda > \kappa$ and let us show that there is a $C^{(n)}$–supercompact embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\ell)(\kappa) > \lambda$. Let $i : V \to N$ be a λ-supercompact embedding derived by some normal measure on $\mathcal{P}_\kappa(\lambda)$ such that $i(\ell)(\kappa) > \lambda$. Since κ is $\mathfrak{a}$-$C^{(n)}$–extendible we may let $\mu \in C^{(n)} \setminus \lambda^+$ with $\mathrm{cof}(\mu) > \lambda$ such that κ is superstrong with target μ. By elementarity, and since $j(\mu) = \mu$,

$$N \models i(\kappa) \text{ is superstrong with target } \mu.$$

Let $E \in N$ be a $(i(\kappa), \mu)$-extender witnessing the superstrongness of κ and $j_E : V \to M_E$ be the corresponding extender embedding. We can now argue as in Theorem 2.0.4 that $j := j_E \circ i$ is a λ-$C^{(n)}$–supercompact embedding with $\mathrm{crit}(j) = \kappa$. Observe that $\mathrm{crit}(j_E) = i(\kappa) > \lambda$ so that $\lambda < j_E(i(\ell)(\kappa)) = j(\ell)(\kappa)$, as desired. $\square$

4.2 Magidor Products of Prikry forcing

Through this section κ will be a $C^{(n)}$–supercompact cardinal which carries a $\mathfrak{S}^{(n)}$-fast function $\ell : \kappa \to \kappa$. By possibly shrinking we may further assume that $\mathrm{dom}(\ell) = \mathrm{cl}(\ell) \cap \mathfrak{M}$ (cf. Lemma 3.2.3). Clearly, $\mathrm{dom}(\ell)$ is stationary in κ. Set

$$\mathrm{supp}(\ell) := \mathrm{dom}(\ell) \cap \{\theta < \kappa \mid \mathrm{dom}(\ell) \cap \theta \text{ is bounded}\}.$$

Let $\langle \kappa_\alpha \mid \alpha < \kappa \rangle$ be an increasing enumeration of $\mathrm{supp}(\ell)$. By construction observe that $\alpha \leq \sup_{\alpha<\beta} \kappa_\alpha < \kappa_\beta$. We will consider the following natural variation of a Magidor iteration:

Definition 4.2.1 (Magidor product)**.** Let χ be an ordinal and for each $\alpha < \chi$, $\mathbb{P}_\alpha := (P_\alpha, \leq_\alpha, \leq^*_\alpha)$ be a Prikry-type forcing [Git10, §6]. Set $\mathscr{P} := \{\mathbb{P}_\alpha \mid \alpha < \chi\}$. The Magidor product of the family $\mathscr{P}$ is the forcing $\mathbb{M}(\mathscr{P}) := (\prod_{\alpha<\chi} P_\alpha, \leq, \leq^*)$, where $\leq$ and $\leq^*$ are defined as follows: $p \leq q$ iff the following hold:

($\aleph$) for each $\alpha < \chi$, $p \upharpoonright \alpha \leq_\alpha q \upharpoonright \alpha$;

($\beth$) there is $a \in [\chi]^{<\aleph_0}$ such that for each $\alpha \in \chi \setminus a$, $p(\alpha) \leq^*_\alpha q(\alpha)$.

Similarly, we define $p \leq^* q$ iff $p \leq q$ and $a = \emptyset$ is a witness for ($\beth$). Denote $\mathbb{1} := \langle \mathbb{1}_{\mathbb{P}_\alpha} \mid \alpha < \chi \rangle$.

Remark 4.2.2. It is worth to remark that in this context is not necessarily true that $\mathbb{1}$ is the $\leq$-greatest element of $\mathbb{M}(\mathscr{P})$. The reason is that some condition p may not be a $\leq^*_\alpha$-extension of $\mathbb{1}_{\mathbb{P}_\alpha}$, for all but finitely many $\alpha < \chi$.

For each $\alpha < \kappa$, let $\mathbb{P}_\alpha$ be Prikry forcing with respect to some normal measure $\mathcal{U}_\alpha$ on κ_α [Git10, §1]. Set $\mathscr{P}_\ell := \{\mathbb{P}_\alpha \mid \alpha < \kappa\}$. The forcing that we will use in the proof of Theorem 4.0.1 is $\mathbb{M}_\ell := \mathbb{M}(\mathscr{P}_\ell) \downarrow \mathbb{1}$. Observe that in this particular case $\mathbb{1} := \langle (\emptyset, \lambda) \mid \lambda \in \mathrm{supp}(\ell) \rangle$ and $p \in \mathbb{M}_\ell$ if and only if for each $\alpha < \kappa$, $p(\alpha) \in \mathbb{P}_\alpha$ and $\{\alpha < \kappa \mid \forall A \in \mathcal{U}_\alpha\, p(\alpha) \neq (\emptyset, A)\} \in [\kappa]^{<\aleph_0}$.

Remark 4.2.3. It is actually not hard to check that $\mathbb{M}_\ell$ is isomorphic to a Magidor iteration of Prikry forcings (see [Git10, §6.1]). This follows from the fact that the places where we force (i.e. $\mathrm{supp}(\ell)$) is a sparse enough set and thus the iteration behaves as a product.

The following series of concepts will be necessary in future arguments:

Definition 4.2.4. A function $s \in \prod_{\alpha<\kappa} {}^{<\omega}(\kappa_\alpha \cup \{\emptyset\})$ is a *stem* if

$$\{\alpha < \kappa \mid s(\alpha) \neq \emptyset\} \in [\kappa]^{<\aleph_0}$$

and for all such α, $s(\alpha)$ is a strictly increasing sequence of cardinals. We will denote by St the set of all stems.

Definition 4.2.5. For $s \in \mathrm{St}$ the *support of* s is the increasing enumeration of the set $\{\alpha < \kappa \mid s(\alpha) \neq \emptyset\}$. We will denote this latter by $\mathrm{supp}(s)$. If $s, t \in \mathrm{St}$ we will write $s \leq_{\mathrm{St}} t$ iff $\mathrm{supp}(t) \subseteq \mathrm{supp}(s)$ and for each $\alpha \in \mathrm{supp}(t)$, $t(\alpha) \sqsubseteq s(\alpha)$.

Definition 4.2.6. The *stem* of a condition $p \in \mathbb{M}_\ell$ is the unique $s \in \mathrm{St}$ such that $p(\alpha) = (s(\alpha), A)$, for some $A \in \mathcal{U}_\alpha$ and $\alpha < \kappa$. We will denote this sequence by $\mathrm{stem}(p)$. The support of p is $\mathrm{supp}(\mathrm{stem}(p))$.

Proposition 4.2.7 (Some properties of $\mathbb{M}_\ell$)**.**

1. *$\mathbb{M}_\ell$ is κ^+-Knaster. In particular, cardinals $\geq\kappa$ are preserved.*

2. *$\mathbb{M}_\ell$ preserves cardinals $<\kappa$, hence $\mathbb{M}_\ell$ is cardinal-preserving.*

3. *For each $\lambda \in \mathrm{supp}(\ell)$, $\mathbb{1} \Vdash_{\mathbb{M}_\ell}$ "$\square_{\lambda,\omega}$ holds". In particular, there are no ω_1-strong compact cardinals $<\kappa$.*

Proof. (1) Let $X \in [\mathbb{M}_\ell]^\kappa$. For each $p \in X$ denote by S_p the support of p. By the Δ-system Lemma one may find $Y \in [X]^\kappa$ and $R \in [\kappa]^{<\aleph_0}$ such that $R = S_p \cap S_q$, for all $p, q \in Y$. Actually, by shrinking Y, we may assume that $p(\alpha) = q(\alpha)$, for all $p, q \in Y$ and $\alpha \in R$.

Let $p, q \in Y$ and for each $\alpha < \kappa$ set $p(\alpha) := (s^p(\alpha), A^p(\alpha))$ and $q(\alpha) := (s^q(\alpha), A^q(\alpha))$. Define

$$r := \begin{cases} p(\alpha), & \text{if } \alpha \in R; \\ (s^p(\alpha), A^p(\alpha) \cap A^q(\alpha)), & \text{if } \alpha \in S_p \setminus R; \\ (s^q(\alpha), A^p(\alpha) \cap A^q(\alpha)), & \text{if } \alpha \in S_q \setminus R; \\ (\emptyset, A^p(\alpha) \cap A^q(\alpha)), & \text{otherwise.} \end{cases}$$

Clearly r is well-defined and $r \leq p, q$.

(2) This easily follows from the fact that Prikry forcing is cardinal preserving.

(3) Let $\lambda \in \mathrm{supp}(\ell)$ and $\alpha < \kappa$ be such that $\lambda = \kappa_\alpha$. Define $\pi_\alpha \colon \mathbb{M}_\ell \to \mathbb{P}_\alpha$ as $p \mapsto p(\alpha)$. Clearly, π_α establishes a projection between $\mathbb{M}_\ell$ and $\mathbb{P}_\alpha$, hence $V^{\mathbb{P}_\alpha} \subseteq V^{\mathbb{M}_\ell}$ (cf. Lemma 1.3.11). Observe that λ is a V-measurable cardinal which has countable cofinality in $V^{\mathbb{M}_\ell}$. Since $\mathbb{M}_\ell$ is cardinal preserving we may appeal to Theorem 3.1.11 and infer the desired result. As customary the in particular claim follows from Theorem 1.4.6(4). $\square$

In the next section we will show that forcing with $\mathbb{M}_\ell$ preserves the $C^{(n)}$–supercompactness of κ. Specifically, for a given $\lambda > \kappa$ and a λ-$C^{(n)}$–supercompact embedding $j : V \to M$ with $j(\ell)(\kappa) > \lambda$, we will show that we can derive from j a λ-$C^{(n)}$–supercompact embedding $j^* : V[G] \to M[H]$ with $\mathrm{crit}(j) = \kappa$ in $V[G]$. Provided we manage to show this it is clear that Proposition 4.2.7 yields the Theorem 4.0.1.

Let us briefly summarize the main ideas to produce these $C^{(n)}$–supercompact embeddings. Let $G \subseteq \mathbb{M}_\ell$ generic over V and $j : V \to M$ with $j(\ell)(\kappa) > \lambda$. It is not hard to show that there is a generic filter $H \subseteq j(\mathbb{M}_\ell)$ over M such that $j[G] \subseteq H$ and $H = G * \tilde{H}$. Thus, j lifts to $j^* : V[G] \to M[G * \tilde{H}] \subseteq V[G * \tilde{H}]$. The problem is thus if we can pick H in such a way that j^* is an inner embedding of $V[G]$ (cf. Remark 1.3.22). More formally, can we pick H in such a way that $\tilde{H} \subseteq j(\mathbb{M}_\ell)/\mathbb{M}_\ell$ is a generic filter definable in $V[G]$? This kind of questions are recurrent in lifting arguments and actually there are many lifting strategies to ensure this. For these strategies to work one needs that the corresponding forcing has good properties in V and that $j(\kappa)$ is a (real) small cardinal. For details see [Cum10, §15].

Nonetheless, non of these techniques will work in our context, as here $j(\kappa)$ is a strong limit cardinal in V. At this point is where we need to appeal to a very special feature of Prikry forcing: namely, Prikry sequences, and thus Prikry generics, can be defined in a very explicit way by using iterated ultrapowers (see [Kan09, Theorem 19.8]).

Our strategy is to show that any generic filter for $j(\mathbb{M}_\ell)/\mathbb{M}_\ell$ over $M[G]$ is determined by a collection of independent Prikry sequences. In particular, we will show that iterated ultrapowers can be used to define generic filters for the iteration $j(\mathbb{M}_\ell)/\mathbb{M}_\ell$. To this aim we will need to secure that $\mathbb{M}_\ell$ satisfies a strong form of Prikry property: the so-called *Strong Prikry property* (cf. Lemma 4.2.13).

Definition 4.2.8. Set $\bigoplus_{\alpha<\kappa} \omega := \{\vec{\gamma} \in {}^{\kappa}\omega \mid \{\alpha < \kappa \mid \vec{\gamma}(\alpha) \neq 0\} \in [\kappa]^{<\aleph_0}\}$.

- For $\vec{\gamma} \in \bigoplus_{\alpha<\kappa} \omega$, the support of $\vec{\gamma}$ is the increasing enumeration of the set $\{\alpha < \kappa \mid \vec{\gamma}(\alpha) \neq 0\}$. We shall denote this latter as $\mathrm{supp}(\vec{\gamma})$;
- Given $s \in \mathrm{St}$, the *length sequence* of s is defined as the unique $\vec{\gamma} \in \bigoplus_{\alpha<\kappa} \omega$ such that $\vec{\gamma}(\alpha) = |s(\alpha)|$, for each $\alpha < \kappa$. We shall denote by $\mathrm{len}(s)$ the length sequence of the stem s.

We denote by $\leq^\star$ the pointwise ordering over $\bigoplus_{\alpha<\kappa} \omega$.

Remark 4.2.9. For $s \in \mathrm{St}$ observe that $\mathrm{supp}(s) = \mathrm{supp}(\mathrm{len}(s))$. Actually $\mathrm{len}(s)$ contains all the relevant information about s: i.e. its support and the length of the corresponding sequences.

Lemma 4.2.10 (Finite Diagonal Intersection)**.** *Fix $\vec{\gamma} \in \bigoplus_{\alpha<\kappa} \omega$ and let $\mathcal{B}$ be a sequence $\langle B^s_\alpha \mid s \in \mathrm{St}, \mathrm{len}(s) = \vec{\gamma}, \alpha < \kappa\rangle$, where $B^s_\alpha \in \mathcal{U}_\alpha$ and $B^s_\alpha \cap (\max(s(\alpha)) + 1) = \emptyset$. Then there is a sequence of sets $\langle C_\alpha \mid \alpha < \kappa\rangle$ fulfilling the following requirements:*

1. *for each $\alpha < \kappa$, $C_\alpha \in \mathcal{U}_\alpha$;*
2. *for every $s \in \prod_{\alpha<\kappa} {}^{<\omega}C_\alpha \cap \mathrm{St}$ with $\mathrm{len}(s) = \vec{\gamma}$, and for each $\alpha < \kappa$, $C_\alpha \setminus \max s(\alpha) + 1 \subseteq B^s_\alpha$.*

Under the above conditions we will say that $\langle C_\alpha \mid \alpha < \kappa\rangle$ is the Diagonal Intersection *of the family $\mathcal{B}$.*

Proof. Let us prove the lemma by induction over $|\operatorname{supp}\vec{\gamma}|$. If $|\operatorname{supp}\vec{\gamma}| = 0$, then there is only one length sequence with this support: i.e. the constant function 0. Thus definining $C_\alpha := B^s_\alpha$, we are done. Now assume by induction that for every $\vec{\gamma}' \in \bigoplus_{\alpha<\kappa}\omega$ with $|\operatorname{supp}\vec{\gamma}'| \leq n$ and for every family of large sets $\langle B^s_\alpha \mid s \in \mathrm{St},\, \mathrm{len}(s) = \vec{\gamma}', \alpha < \delta\rangle$ with $\delta \leq \kappa$, there is $\langle C_\alpha \mid \alpha < \delta\rangle$ witnessing the lemma.

Let $\vec{\gamma}$ be a length sequence with $|\operatorname{supp}\vec{\gamma}| = n+1$ and $\mathcal{B} := \langle B^s_\alpha \mid s \in \mathrm{St},\, \mathrm{len}(s) = \vec{\gamma},\, \alpha < \kappa\rangle$ be a family of large sets. Set $\max(\operatorname{supp}(\vec{\gamma})) := \delta$. Notice that $|\{s \in \mathrm{St} \mid \mathrm{len}(s) = \vec{\gamma}\}| \leq \kappa_\delta$, and for each $\delta < \alpha < \kappa$ set $C_\alpha = \bigcap_{s\in\mathrm{St},\mathrm{len}(s)=\vec{\gamma}} B^s_\alpha$.

Let us now work with the truncated family $\mathcal{B} \restriction \delta := \langle B^s_\alpha \mid s \in \mathrm{St},\, \mathrm{len}(s) = \vec{\gamma},\, \alpha < \delta\rangle$. Observe that all the $s \in \mathrm{St}$ with $\mathrm{len}(s) = \vec{\gamma}$ are of the form $s = t \cup \{\langle \delta, \vec{\eta}\rangle\}$, for some $t \in \mathrm{St}$ with $|\operatorname{supp}(\mathrm{len}(t))| = n$ and some increasing sequence of cardinals $\vec{\eta} \in {}^{\vec{\gamma}(\delta)}\kappa_\delta$. For each possible $\vec{\eta} \in {}^{\vec{\gamma}(\delta)}\kappa_\delta$, set $\mathcal{B}_{\vec{\eta}} := \langle B^s_\alpha \mid t \cup \{\langle \delta, \vec{\eta}\rangle\}, t \in \mathrm{St}, \mathrm{len}(t) = \vec{\gamma}^*,\, \alpha < \delta\rangle$, where $\vec{\gamma}^* := (\vec{\gamma} \setminus \{\langle \delta, \vec{\gamma}\rangle\}) \cup \{\langle \delta, 0\rangle\}$. Since $\sup_{\alpha<\delta} \kappa_\alpha < \kappa_\delta$ one may find $A_\delta \in \mathcal{U}_\delta$ and $\mathcal{B}^*$ such that $\mathcal{B}_{\vec{\eta}} = \mathcal{B}^*$, for each $\vec{\eta} \in {}^{\vec{\gamma}(\delta)}A_\delta$. Now appeal to the induction hypothesis with respect to this particular $\mathcal{B}^*$ and find a sequence of sets $\langle C_\alpha \mid \alpha < \delta\rangle$ witnessing the statement of the lemma. Define $C_\delta := A_\delta \cap \triangle\{B^s_\delta \mid s \in \mathrm{St},\, \mathrm{len}(s) = \vec{\gamma}\}$. Here the diagonal intersection of $\{B^s_\delta \mid s \in \mathrm{St},\, \mathrm{len}(s) = \vec{\gamma}\}$ is defined as

$$\{\beta < \kappa_\delta \mid (s \in \mathrm{St} \wedge \mathrm{len}(s) = \vec{\gamma} \wedge \max(s(\delta)) < \beta) \rightarrow \beta \in B^s_\delta\}.$$

Observe that $\langle C_\alpha \mid \alpha < \kappa\rangle$ satisfies (1) and (2). Indeed, (1) outright follows from the completedness and normality of the measures. Also (2) follows by induction in the case $\alpha < \delta$, by the definition of diagonal intersection in the case $\alpha = \delta$ and just by definition in the case $\delta < \alpha < \kappa$. □

Notation 4.2.11. For each $\alpha < \kappa$ and $s \in \mathrm{St}$, $s \restriction \alpha * \emptyset_\alpha$ is the sequence in St such that

$$t(\beta) := \begin{cases} s(\beta), & \text{if } \beta < \alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Lemma 4.2.12 (Röwbottom Lemma)**.** *Let $f : \mathrm{St} \rightarrow 2$ be a function. There is a sequence of sets $\langle C_\alpha \mid \alpha < \kappa\rangle$, $C_\alpha \in \mathcal{U}_\alpha$, and a function $g \colon \bigoplus_{\alpha<\kappa}\omega \rightarrow 2$ such that for every $s \in \mathrm{St} \cap \prod_{\alpha<\kappa} {}^{<\omega}C_\alpha$, $f(s) = g(\mathrm{len}(s))$.*

Proof. Arguing by induction on $n < \omega$ we will define a sequence of functions $\langle f_n \mid n < \omega\rangle$ and sets $\langle A_{\alpha,n} \mid \alpha < \kappa, n < \omega\rangle$ for which the following hold:

(ℵ) $f_0 : \mathrm{St} \times \{\emptyset\} \rightarrow 2$ is defined by $f_0(s, \emptyset) := f(s)$, and for each $n \geq 1$, $f_n : \mathrm{St} \times {}^n\omega \rightarrow 2$;

($\beth$) for each $\alpha < \kappa$, $A_{\alpha,0} := \kappa_\alpha$. Also, for each $n \in [1,\omega)$ and $\alpha < \kappa$, $A_{\alpha,n} \in \mathcal{U}_\alpha$;

($\gimel$) for each $(s,\vec{\gamma}) \in \mathrm{dom}(f_n)$ with $\max(\mathrm{supp}(s)) = \alpha$ and $s(\alpha) \in {}^{<\omega}A_{\alpha,n+1}$;

$$f_{n+1}(s \restriction \alpha * \emptyset_\alpha, \langle |s(\alpha)| \rangle ^\frown \vec{\gamma}) := f_n(s,\vec{\gamma}).$$

Assume that f_n and $\langle A_{\alpha,n} \mid \alpha < \kappa \rangle$ have already been constructed. Fix $\vec{\gamma} \in {}^n\omega \cup \{\emptyset\}$ and $\alpha < \kappa$. For each $s \in \mathrm{St}$ with $\mathrm{supp}(s) \subseteq \alpha$ define $\Psi^n_{s,\alpha,\vec{\gamma}} : {}^{<\omega}A_{\alpha,n} \to 2$, as $\vec{\eta} \mapsto f_n(s^\alpha_{\vec{\eta}}, \vec{\gamma})$, where $s^\alpha_{\vec{\eta}}$ is the sequence s^* such that $s^*(\beta) = s(\beta)$ iff $\beta \neq \alpha$ and $s^*(\alpha) = \vec{\eta}$. By appealing to Röwbottom theorem [Kan09, Theorem 7.17] we may find a homogeneous set $H^n_{s,\alpha,\vec{\gamma}} \subseteq A_{\alpha,n}$ for the function $\Psi^n_{s,\alpha,\gamma}$. Set $A_{\alpha,n+1} := \bigcap\{H^n_{s,\alpha,\vec{\gamma}} \mid s \in \mathrm{St},\ \mathrm{supp}(s) \subseteq \alpha\}$ and observe that $A_{\alpha,n+1} \in \mathcal{U}_\alpha$, as $\sup_{\beta<\alpha} \kappa_\beta < \kappa_\alpha$. Set

$$\mathrm{St}_{\alpha,n+1} := \{s \in \mathrm{St} \mid \max(\mathrm{supp}(s)) = \alpha,\ s(\alpha) \in {}^{<\omega}A_{\alpha,n+1}\},$$

and define $\Psi^n_{\alpha,\vec{\gamma}} : \mathrm{St}_{\alpha,n+1} \to 2$ as $s \mapsto \Psi^n_{s\restriction\alpha,\alpha,\vec{\gamma}}(s(\alpha))$. Clearly, $\Psi^n_{\alpha,\vec{\gamma}}(s) = f_n(s,\vec{\gamma})$. Notice that by construction of $A_{\alpha,n+1}$, this function only depends on $s \restriction \alpha, |s(\alpha)|$ and $\vec{\gamma}$, so we define $f_{n+1}(s \restriction \alpha * \emptyset_\alpha, \langle |s(\alpha)| \rangle ^\frown \vec{\gamma}) := f_n(s,\vec{\gamma})$. Now repeat this process all again for each $\vec{\gamma}$ and each $\alpha < \kappa$.

For each $\alpha < \kappa$, set $C_\alpha := \bigcap_{n<\omega} A_{\alpha,n}$.

Claim 4.2.12.1. $f(s) = f_{|\mathrm{supp}(s)|}(\emptyset_0, \langle |s(\alpha)| \mid \alpha \in \mathrm{supp}(s) \rangle)$, *for each* $s \in \mathrm{St} \cap \prod_{\alpha<\kappa} {}^{<\omega}C_\alpha$.

Proof of claim. One needs to proceed by induction over the size of the supports. If the size is 0 then $s = \emptyset_0$ and the claim easily follows. In other case, it follows by recursion using ($\gimel$) and the fact that $C_\alpha \subseteq A_{\alpha,n}$, $n < \omega$. □

Define $g : \bigoplus_{\alpha<\kappa} \omega \to 2$ by $g(\vec{\gamma}) := f_{|\mathrm{supp}(\vec{\gamma})|}(\emptyset_0, \langle |\vec{\gamma}(\alpha)| \mid \alpha \in \mathrm{supp}(\vec{\gamma}) \rangle)$. For each $s \in \mathrm{St} \cap \prod_{\alpha<\kappa} {}^{<\omega}C_\alpha$, the above claim and $\mathrm{supp}(s) = \mathrm{supp}(\mathrm{len}(s))$ yield $f(s) = g(\mathrm{len}(s))$. □

Lemma 4.2.13. $\mathbb{M}_\ell$ *has the Strong Prikry property. Namely, for each dense open set* $D \subseteq \mathbb{M}_\ell$ *and* $p \in \mathbb{M}_\ell$, *there is* $p^\star \leq^* p$ *and* $\vec{\gamma} \in \bigoplus_{\alpha<\kappa} \omega$ *such that for all* $q \leq p^\star$ *with* $\vec{\gamma} \leq^\star \mathrm{len}(\mathrm{stem}(q))$, $q \in D$.

Proof. Let $s \in \mathrm{St}$ and $\langle A_\alpha \mid \alpha < \kappa \rangle$ be the stem and the large sets of p. Define a function $f\colon \mathrm{St} \to 2$ as follows:

$$f(t) := \begin{cases} 1 & \text{if } \exists q \leq p\,(q \in D \text{ and } \mathrm{stem}(q) = s^\frown t)^1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\langle C_\alpha \mid \alpha < \kappa \rangle$ and $g\colon \bigoplus_{\alpha<\kappa} \omega \to 2$ be witness for Lemma 4.2.12. Since D is dense and open there is $s^* \in \mathrm{St} \cap \prod_{\alpha<\kappa} {}^{<\omega}C_\alpha$, $s^\star \leq_{\mathrm{St}} s$, such that

$f(s^*) = 1$. Thus, for all $t \in \mathrm{St} \cap \prod_{\alpha<\kappa} {}^{<\omega}C_\alpha$ with $\mathrm{len}(s^*) = \mathrm{len}(t)$, $f(t) = 1$. Set $\vec{\gamma} = \mathrm{len}(s^*)$. We now construct $p^\star$.

For each of such t pick $q_t \leq p$ with $q_t \in D$ and $\mathrm{stem}(q_t) = s^\frown t$. Let $\mathcal{B}^t$ be the collection of large sets $\langle B^t_\alpha \mid \alpha < \kappa\rangle$ appearing in q_t and set

$$\mathcal{B} := \langle \mathcal{B}^t \mid \mathrm{len}(t) = \mathrm{len}(s^*)\rangle.$$

By 4.2.10 there is $\langle D_\alpha \mid \alpha < \kappa\rangle$ a diagonal intersection for the family $\mathcal{B}$. For each $\alpha < \kappa$, set $C^*_\alpha := D_\alpha \cap C_\alpha \cap A_\alpha$. Let $p^\star$ be the condition in $\mathbb{M}_\ell$ with $\mathrm{stem}(p^\star) = s^\star$ and large sets $\langle C^*_\alpha \mid \alpha < \kappa\rangle$. Clearly, $p^\star \leq^* p$. Now observe that each condition $q \leq p^\star$ with $\vec{\gamma} \leq^\star \mathrm{len}(\mathrm{stem}(q))$ is an extension of some q_t with $t \in \prod_{\alpha<\kappa} {}^{<\omega}C^*_\alpha$. Thus, by openess, $q \in D$. □

4.3 Preserving $C^{(n)}$–supercompact cardinals

Let κ be a $C^{(n)}$–supercompact cardinal and $\ell : \kappa \to \kappa$ be a $\mathfrak{S}^{(n)}$-fast function as in Section 4.2. Here we will show that forcing with $\mathbb{M}_\ell$ preserves the $C^{(n)}$–supercompactness of κ. By the comments of the previous section this suffices to obtain the proof of Theorem 4.0.1.

Lemma 4.3.1. $\mathbb{1} \Vdash_{\mathbb{M}_\ell}$ *“κ is $C^{(n)}$–supercompact”.*

Proof. Let $\lambda > \kappa$ and $j : V \to M$ be a λ-$C^{(n)}$–supercompact embedding with $\mathrm{crit}(j) = \kappa$ and $j(\ell)(\kappa) > \lambda$. Also, let $G \subseteq \mathbb{M}_\ell$ generic over V.

By elementarity, $j(\mathbb{M}_\ell)$ is the Magidor product of Prikry forcings defined in M with respect to the set

$$\mathrm{supp}(j(\ell)) = \mathrm{dom}(j(\ell)) \cap \{\theta < j(\kappa) \mid \mathrm{dom}(j(\ell)) \cap \theta \text{ bounded}\}.$$

Recall that all $\delta \in \mathrm{supp}(\ell)$ is a closure point for ℓ and thus the same applies for $\mathrm{supp}(j(\ell))$. Since $j(\ell) \restriction \kappa = \ell$ it is obvious that $j(\mathbb{M}_\ell)$ factorizes as $\mathbb{M}_\ell \times \mathbb{M}^M_{j(\ell)\setminus\ell}$, where $\mathbb{M}^M_{j(\ell)\setminus\ell}$ is the M-version of the Magidor product of Prikry forcings defined with respect to $\mathrm{supp}(j(\ell) \setminus \ell)$. Observe that $j(\ell)(\kappa) > \lambda$ yields $\lambda \notin \mathrm{cl}(j(\ell))$ and thus $\mathrm{supp}(j(\ell) \setminus \ell)$ can be written as an increasing sequence of measurable cardinals $\langle \kappa_\alpha \mid \alpha < j(\kappa)\rangle$ such that $\kappa_0 > \lambda$. Also, by elementarity, $\alpha \leq \sup_{\beta<\alpha} \kappa_\beta < \kappa_\alpha$, for every $\alpha < j(\kappa)$.

For ease of notation set $\mu = j(\kappa)$ and $\mathbb{M}^* = \mathbb{M}^M_{j(\ell)\setminus\ell}$. Working in M we will build an iteration of ultrapowers $\mathfrak{M} = \langle M_\alpha, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \omega\cdot\mu\rangle$ and we will show that $\langle C_\alpha \mid \alpha < \mu\rangle$ generates a $M_{\omega\cdot\mu}$-generic for the Magidor product $j_{\omega\cdot\mu}(\mathbb{M}^*)$, where $C_\alpha = \langle \rho^n_\alpha \mid n < \omega\rangle$ is the αth-critical sequence of the iteration [Kan09, §19]. Set $M_0 := M$, $j_{00} :=$ id and $\mathfrak{U} = \langle \mathcal{U}_\alpha \mid \alpha < \mu\rangle$. For limit $\alpha \leq \omega\cdot\mu$, we will let $M_\alpha := \mathrm{dir}\lim\langle M_\beta, j_{\beta,\gamma} \mid \beta \leq \gamma < \alpha\rangle$. For successor ordinals $\alpha < \omega\cdot\mu$ we will let

$$M_{\omega\cdot\alpha+n+1} = \mathrm{Ult}(M_{\omega\cdot\alpha+n}, j_{\omega\cdot\alpha+n}(\mathfrak{U})_\alpha),\ \ n < \omega$$

Let $j_{\omega\cdot\alpha+n,\omega\cdot\alpha+n+1}$ be the corresponding ultrapower map and define $j_{\beta,\omega\cdot\alpha+n+1}$ in the usual way, for $\beta < \omega\cdot\alpha+n+1$. Set $i := j_{\omega\cdot\mu}$. By standard computations of iterated ultrapowers one can show that $i(\mu) = \mu$. In particular, $j_{\omega\cdot\alpha}(\mathfrak{U})_\alpha = j_{\omega\cdot\alpha}(\mathfrak{U}_\alpha)$. Also, by discreteness of the measurables, $j_{\omega\cdot\alpha}(\kappa_\alpha) = \kappa_\alpha$, which is much larger than α. For each $n < \omega$, set $\rho^n_\alpha = \mathrm{crit}(j_{\omega\cdot\alpha+n,\omega\cdot\alpha+n+1})$. Clearly, $\rho^n_\alpha = j_{\omega\cdot\alpha+n}(\kappa_\alpha)$. Also, notice that $\rho^0_0 > \lambda$ and $\kappa_\alpha = \rho^0_\alpha > \alpha$, for every $\alpha < \mu$. For the ease of notation, on the sequel we will write $M^\star = M_{\omega\cdot\mu}$. For each $\alpha < \mu$, set $H_\alpha := \{\langle s, A\rangle \in \mathbb{P}_\alpha \mid s \sqsubseteq C_\alpha,\ C_\alpha \setminus \max(s) \subseteq A\ \}$. Observe that H_α is just the Prikry generic defined by the critical sequence C_α [Git10, §1]. Define H as the set of all conditions $p \in i(\mathbb{M}^*)$ satisfying the following:

($\aleph$) for all $\alpha < \mu$, there is $q \in H_\alpha$ with $q \leq_{\mathbb{P}_\alpha} p(\alpha)$;

($\beth$) for all but finitely many $\alpha < \mu$, there is $q \in H_\alpha$ such that $q \leq^*_{\mathbb{P}_\alpha} p(\alpha)$,

We will next show that H is a generic filter for the Magidor product $i(\mathbb{M}^\star)$ over $M^\star$.

Claim 4.3.1.1. *H is a filter.*

Proof of claim. It is clear that H is upwards closed. Let $p, p' \in H$. For each $\alpha < \mu$ there are $q_\alpha, q'_\alpha \in H_\alpha$ such that $q_\alpha \leq_{\mathbb{P}_\alpha} p(\alpha)$ and $q'_\alpha \leq_{\mathbb{P}_\alpha} p'(\alpha)$. Actually, by ($\beth$), there is $a \in [\mu]^{<\aleph_0}$ for which this is true for the ordering $\leq^*_{\mathbb{P}_\alpha}$, for all $\alpha \in \mu \setminus a$. For all $\alpha \notin a$, let $r(\alpha)$ be a $\leq^*_{\mathbb{P}_\alpha}$ extension of q_α and q'_α. For the $\alpha \in a$ do the same, noticing that this is possible because $q_\alpha, q'_\alpha \in H_\alpha$. Clearly $r = \langle r(\alpha) \mid \alpha < \mu\rangle$ witnesses compatibility of p and p'. □

Claim 4.3.1.2. *The filter H is generic for $i(\mathbb{M}^*)$ over $M^\star$.*

Proof of claim. Let $D \in M^\star$ be a dense open subset of $i(\mathbb{M}^*)$. Then there is some function $f : \prod_{n<n^*} \kappa^{<\omega}_{\alpha_n} \to \mathcal{P}(\mathbb{M}^*)$ such that for all $\vec{\eta} \in \mathrm{dom}(f)$, $f(\vec{\eta})$ is a dense open subset of $\mathbb{M}^*$ and there are sequences $\vec{\rho}_n \in C^{<\omega}_{\alpha_n}$ for $n < n^\star$ such that $D = i(f)(\vec{\rho}_0, \ldots, \vec{\rho}_{n_\star-1})$. We may and do assume that $\vec{\rho}_n \sqsubseteq C_{\alpha_n}$, for every $n < n^\star$.

Let $M' = \{i(g)(\vec{\rho}_0, \ldots, \vec{\rho}_{n_\star-1}) \mid g \in M \wedge \langle \vec{\rho}_n \mid n < n_{\star-1}\rangle \in i(\mathrm{dom}\, g)\}$. Working in M' we apply Lemma 4.2.13 for D and a condition with stem $\vec{\rho} = \langle \vec{\rho}_n \mid n < n_{\star-1}\rangle * \emptyset_{\alpha_{n^*}}$. Let $p^\star$ and $\vec{\gamma}$ be the obtained direct extension and length sequence. It is sufficient to show that $p^\star$ belongs to the filter H:

Subclaim 4.3.1.2.1. *If $p^\star \in H$ then $H \cap D \neq \emptyset$.*

Proof of subclaim. If $p^\star \in H$, for each $\alpha \in \mathrm{supp}(\vec{\gamma})$, we may take $q(\alpha) \in H_\alpha$ such that $q(\alpha) \leq p^\star(\alpha)$ with stem of length $\geq \vec{\gamma}(\alpha)$. Define $q^\star$ as follows:

$$q^\star(\alpha) := \begin{cases} q(\alpha), & \alpha \in \mathrm{supp}(\vec{\gamma}); \\ p^\star(\alpha) & \text{otherwise.} \end{cases}$$

Observe that $q^\star \in H$, $q^\star \leq p^\star$ and $\vec{\gamma} \leq^\star \mathrm{len}(\mathrm{stem}(q^\star))$. Thus,

$$q^\star \in H \cap \{q \leq p^\star \mid \vec{\gamma} \leq \mathrm{len}(\mathrm{stem}(q))\}.$$

Finally notice that this latter set is contained in D so the result follows. □

Let $g\colon \prod^{<\omega}\kappa_\alpha \to M$ be a function representing the sequence of large sets in $p^\star$. Without loss of generality we may assume that $g(s)$ is of the form $\langle B^s_\alpha \mid \alpha < \mu\rangle$ and s goes over stems with length sequence $\mathrm{len}(p^\star)$. Say $\mathcal{B} = \langle B^s_\alpha \mid \alpha < \mu, \mathrm{len}(s) = \mathrm{len}(p^\star)\rangle$. Using Lemma 4.2.10 we obtain a sequence of large sets $\mathcal{A} = \langle A_\alpha \mid \alpha < \mu\rangle$ such that for every $s \in \prod^{<\omega} A_\alpha$, $A_\alpha \setminus \max(s(\alpha)) + 1 \subseteq B^s_\alpha$, for every $\alpha < \mu$. Clearly, the condition $q^\star$ with stem $\vec{\rho}$ and large sets $i(\mathcal{A})$ is stronger than $p^\star$. Let us verify that $q^\star$ enters the filter H and thus $p^\star$ also.

Subclaim 4.3.1.2.2. *$q^\star \in H$.*

Proof of subclaim. Recall that $\mathrm{supp}(q^\star) = \langle \alpha_n \mid n < n^\star\rangle$. We will show that $q^\star \in \prod_{\alpha<\mu} H_\alpha$ and so $q^\star \in H$. We need to distinguish two cases:

► Assume $\alpha \notin \mathrm{supp}(q^\star)$. Then $q^\star(\alpha) = (\emptyset, i(\mathcal{A})_\alpha)$. We need to check that $C_\alpha \subseteq i(\mathcal{A})_\alpha$. By definition, $i(\mathcal{A})_\alpha \in i(\mathfrak{U})_\alpha$. Since $\mathrm{crit}(j_{\omega\cdot\alpha,\omega\cdot\mu}) > \alpha$ this is equivalent to $j_{\omega\cdot\alpha}(\mathcal{A})_\alpha \in j_{\omega\cdot\alpha}(\mathfrak{U})_\alpha$. Once again $\mathrm{crit}(j_{\omega\cdot\alpha,\omega\cdot\alpha+1}) > \alpha$, so the above is equivalent to $\rho^0_\alpha \in j_{\omega\cdot\alpha+1}(\mathcal{A})_\alpha$ and this equivalent to $\rho^0_\alpha \in i(\mathcal{A})_\alpha$.[2] Arguing similarly one can show that actually $\rho^n_\alpha \in i(\mathcal{A})_\alpha$, for $n < \omega$. Thus, $C_\alpha \subseteq i(\mathcal{A})_\alpha$, as wanted.

► Assume $\alpha \in \mathrm{supp}(q^\star)$. Say $\alpha = \alpha_n$, for some $n < n^\star$. By our latter choice $\vec{\rho}_n \sqsubseteq C_{\alpha_n}$ so we are left with proving that $C_{\alpha_n} \setminus \vec{\rho}_n \subseteq i(\mathcal{A})_{\alpha_n}$.

Observe that $i(\mathcal{A})_{\alpha_n} \setminus \max(\vec{\rho}_n) + 1 \in i(\mathfrak{U})_{\alpha_n}$ On the other hand $\max \vec{\rho}_n = \max(j_{\omega\alpha_n+k+1,\omega\mu}[\vec{\rho}_n])$, where $k = |\vec{\rho}_n|$. Hence, $j_{\omega\alpha_n+k+1}(\mathcal{A})_{\alpha_n} \setminus \max(\vec{\rho}_n) + 1 \in j_{\omega\alpha_n+k+1}(\mathfrak{U})_{\alpha_n}$. By definition, $\rho^{k+1}_{\alpha_n} \in j_{\omega\alpha_n+k+2}(\mathcal{A})_{\alpha_n}$ and so $\rho^{k+1}_{\alpha_n} \in i(\mathcal{A})_{\alpha_n}$ since the critical point of $j_{\omega\alpha_n+k+2,\omega\mu}$ is above both $\rho^{k+1}_{\alpha_n}$ and α_n. Repeating this argument for all $l \in [k+1, \omega)$ we finally have $C_{\alpha_n} \setminus \vec{\rho}_n \subseteq i(\vec{A})_{\alpha_n}$. □

Altogether the above argument shows that $H \cap D \neq \emptyset$, as wanted. □

Set $j^\star = i \circ j$. The proof of next claim leads us to the end of the lemma.

Claim 4.3.1.3. *The embedding $j^\star : V \to M^\star$ lifts to an elementary embedding $j^\star : V[G] \to M^\star[G \times H]$ which is a witness for λ-$C^{(n)}$–supercompactness of κ in $V[G]$.*

Proof of claim. Provide that $j^\star$ lifts, it is clear that this embedding will lie in $V[G]$ since H is definable within M. Let us first show that $j^\star$ lifts. Let $p \in G$ and notice that $j(p) = p^\frown q$, where $q \in \mathbb{M}^*$ has trivial stem.[3] To be

[2]Notice that here we are using that $\mathrm{crit}(j_{\omega\cdot\alpha+1,\omega\cdot\mu}) = \rho^1_\alpha > \rho^0_\alpha$.

[3]$p^\frown q$ here stands for the concatenation (in the right interpretation) of both conditions.

more precise, $q = \langle\langle(s(\alpha), B_\alpha\rangle \mid \alpha < \mu\rangle$, $s(\alpha) = \emptyset$ and $B_\alpha \in \mathcal{U}_\alpha$. Since $\mathrm{crit}(i) > \kappa$, the second elementary embedding does not move p. Similarly, $i(q) = \langle\langle\emptyset, i(\vec{B})_\alpha\rangle \mid \alpha \in \mu\rangle^4$. For each $\alpha < \mu$, one can argue as in Claim 4.3.1.2 that $C_\alpha \subseteq i(\vec{B})_\alpha$, hence $i(q) \in H$, and thus $j^\star(p) \in G \times H$. This shows that j^* lifts. Observe that $j^\star(\kappa) = j(\kappa)$ is a $C^{(n)}$-cardinal in $V[G]$ as $\mathbb{M}$ is mild.

To finish the claim it remains to show that $N = M^\star[G \times H]$ is closed by λ-sequences in $V[G]$. Since N is a model of choice, it is sufficient to show that every λ-sequence of ordinals from $V[G]$ belong to N. Note that the forcing $\mathbb{M}$ introduces new ω-sequences. First, since j is a λ-supercompact embedding, M is closed under λ-sequences from V. Let $\sigma \in V$ be an $\mathbb{M}$-name for a λ-sequence of ordinals. By the κ-cc of $\mathbb{M}$ (cf. Proposition 4.2.7), we may assume that $|\sigma| = \lambda$ and that $\sigma \subseteq \mathbb{M} \times \mathrm{ON}$. Therefore $\sigma \in M$, and in $M[G]$ we can interpret it. Let us finally show that $M[G]$ and $M^\star[G \times H]$ contain the same λ-sequences of ordinals.

Let $\langle\xi_\alpha \mid \alpha < \lambda\rangle$ be a sequence of ordinals in $M^\star[G * H]$. In this model, for every α there is a function $f_\alpha \in M[G]$ such that $i(f_\alpha)(\vec{\rho}^{\,\alpha}_0, \ldots, \vec{\rho}^{\,\alpha}_{n^\alpha-1}) = \xi_\alpha$, where $\vec{\rho}^{\,\alpha}_i$ is a finite sequence of elements of C_{ζ_i}, some $\zeta_i < \mu$. Since the critical point of i is above λ and the sequence of functions $\langle f_\alpha \mid \alpha < \lambda\rangle \in M[G]$ we conclude that $i(\langle f_\alpha \mid \alpha < \lambda\rangle) = \langle i(f_\alpha) \mid \alpha < \lambda\rangle \in M^\star[G]$. Thus, it is sufficient to show that the sequence $\vec{R} = \langle\langle\vec{\rho}^{\,\alpha}_i \mid i < n^\alpha\rangle \mid \alpha < \mu\rangle$ is in $M^\star[G \times H]$.

Let us define by induction on $\gamma < \mu$ a sequence of functions p_γ such that $i(p_\gamma)(H) = \gamma$. Intuitively, p_γ is a procedure for extracting γ, given the information of H. Let us assume that p_β is defined for all $\beta < \gamma$. Since the critical point of $j_{\gamma,\mu}$ is above γ, we know that γ is represented in M_γ by

$$\gamma = j_{0,\gamma}(g)(\rho_0, \ldots, \rho_{n-1}),$$

for some elements of the sequences in H, $\rho_0, \ldots, \rho_{n-1}$. Those elements are all below the γ-th member of H in the increasing enumeration and in particular, do not move under $j_{\gamma,\mu}$. Let $h\colon \mu \to \mu$ be a function such that $i(h)$ is an increasing enumeration of H in such a way that $i(h)(\beta_i) = \rho_i$, where $\beta_0, \ldots, \beta_{n-1}$ are the indices of $\rho_0, \ldots, \rho_{n-1}$. We conclude that

$$\gamma = i(g)(i(h \circ p_{\beta_0})(H), \ldots, i(h \circ p_{\beta_{n-1}})(H)),$$

so we can define p_γ.

Finally, let us show that the sequence $\vec{R}$ is in $M^\star[G \times H]$. Indeed, one can obtain $\vec{R}$ from H by just knowing the indices of each $\vec{\rho}^{\,\alpha}_i$. This sequence of indices is equivalent to a sequence $\vec{\epsilon} = \langle\epsilon_\alpha \mid \alpha < \lambda\rangle$ of ordinals below μ. Letting the condition $\vec{p} = \langle p_{\epsilon_\alpha} \mid \alpha < \lambda\rangle \in M$ and applying the components

[4] Here $\vec{B}$ stands for the sequence of large sets of q, $\langle A_\alpha : \alpha < \mu\rangle$.

of $i(\vec{p})$ to H we obtain $\vec{\epsilon}$. Finally, applying h on the components of $\vec{\epsilon}$, we obtain $\vec{R}$, as wanted. $\square$

$\square$

The above completes the proof of Theorem 4.0.1.

4.4 The Ultimate Identity crisis

For the proof of Theorem 4.0.3 one argues as follows. Assume that $C^{(<\omega)}$-EXT holds and let κ be a $C^{(<\omega)}$-extendible cardinal witnessing it. By Tsaprounis' result [Tsa18, Theorem 4.2], for each $n \geq 1$, there is a $\mathfrak{E}^{(n)}$-fast function $\ell_n : \kappa \to \kappa$ in V. Notice that $V_\kappa \prec V$ and thus one can define those functions uniformly in $V_{\kappa+1}$.[5] In particular, the function $\ell := \sup \ell_n$ can be computed in V and thus belongs to $V_{\kappa+1}$. Observe that ℓ is a $\mathfrak{E}^{(n)}$-fast function for each $n \geq 1$. Arguing as in Theorem 4.0.1 one arrives at the proof of Theorem 4.0.3.

Remark 4.4.1. By virtue of Lemma 4.1.2 and Proposition 2.0.2 the corresponding notion of $\mathfrak{a}$-$C^{(<\omega)}$-extendibility is enough to prove Theorem 4.0.3.

A word has to be said about the consistency strength of the principle $C^{(<\omega)}$-EXT. Notice that if $C^{(<\omega)}$-EXT holds, i.e., if there is a $C^{(<\omega)}$-extendible cardinal, then Vopěnka's principle (VP) is true (cf. Theorem 1.2.9). In particular, the consistency strength of $C^{(<\omega)}$-EXT is bounded by below by VP. On the other hand, observe that these principles are not equivalent. Indeed if there is κ a $C^{(<\omega)}$-extendible cardinal, then $V_\kappa \prec V$, hence for each natural number $n \geq 1$, $V_\kappa \models \mathfrak{E}^{(n)} \neq \emptyset$, and thus $V_\kappa \models \mathsf{ZFC} + \text{VP}$.

To give an upper bound for the consistency strength of $C^{(<\omega)}$-EXT we need to seek for stronger large-cardinal notions.

Definition 4.4.2. Let κ be a cardinal and λ be an ordinal above it. The cardinal κ is said to be λ-*superhuge* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{j(\kappa)}M \subseteq M$. If for each $\lambda > \kappa$ the cardinal κ is λ-*superhuge* then it is said that κ is *superhuge*.

Definition 4.4.3 ([BDT84])**.** Let κ be a superhuge cardinal. A cardinal θ it is said to be *a target of* κ (in symbols $\kappa \to (\theta)$) if there is some ordinal $\lambda > \kappa$ and some λ-superhuge embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ such that $j(\kappa) = \theta$.

Remark 4.4.4. It is not hard to show that if κ is superhuge then the collection of all of its targets forms a proper class [BDT84].

[5] Here we are using that definability over V_κ is definable in $V_{\kappa+1}$.

In [BDT84]], J. Barbanel and C. Di Prisco introduced the following natural strengthening of superhugness:

Definition 4.4.5. A cardinal κ is *stationarily superhuge* if κ is superhuge and $\{\theta \in \mathrm{Card} \mid \kappa \to (\theta)\}$ is a stationary proper class.

Observe, since $C^{(n)}$- is a proper club class (cf. Section 1.2), that any stationarily superhuge cardinal is also $C^{(<\omega)}$-extendible. Therefore, the consistency strength of $C^{(<\omega)}$-extendibility is bounded by above by the existence of a stationarily superhuge cardinal.

Nonetheless, this bound does not seem to be too much satisfactory as stationarily superhugeness is not a canonical large-cardinal notion. In [BDT84, Theorem 6b] the authors show that the consistency strength of stationarily superhugeness is bounded by above by the consistency of a 2-huge cardinal, i.e., a cardinal κ for which there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and ${}^{j^2(\kappa)}M \subseteq M$. Altogether, the consistency strength of $C^{(<\omega)}$-EXT is bounded by above by the existence of a 2-huge cardinal and by below by VP.

4.5 $C^{(n)}$–supercompactness and Forcing: a new world to explore

We would like to close this chapter mentioning some of the main issues to develop a theory of preservation of $C^{(n)}$–supercompact cardinals under forcing. In the light of Lemma 4.3.1 it seems evident that this is a much more delicate issue than in the context of supercompact cardinals.

Since long time ago several preservation results and lifting strategies for supercompact cardinals are familiar to set theorist [Lav78][Ham00][Cum10]. Maybe the most paradigmatic example of this is Laver's indestructibility theorem: if κ is supercompact then there is a forcing κ-iteration $\mathbb{L}$ making the supercompactness of κ indestructible by further κ-directed closed forcing [Lav78]. Informally speaking, $\mathbb{L}$ gives a blackbox which avoid us to be concerned about the survival of κ in further generic extensions.

However these type of results seem do not extend to the realm of $C^{(n)}$–supercompact cardinals. Firstly, it is not evident that the first of these cardinals carry $\mathfrak{S}^{(n)}$-fast function, hence Laver's arguments do not seem to adapt to this new context. Secondly, we have already shown that under WEEA $C^{(n)}$–supercompact cardinals are Σ_{n+2}-correct and thus there is no chance for indestructibility results [Bag+16]. This discussion suggest the following general question:

Question 4.5.1. What kind of forcings preserve $C^{(n)}$–supercompactnes?

Let us state more precisely which are the main concerns to preserve a $C^{(n)}$–supercompact cardinal. Let $\mathbb{P}$ be a forcing notion, $G \subseteq \mathbb{P}$ generic and $\lambda > \kappa$ be an arbitrary cardinal. Assume that we have $j : V \to M$ a ground model elementary embedding witnessing the λ-$C^{(n)}$-supercompactness of κ. There are two natural strategies to ensure that κ remains λ-$C^{(n)}$–supercompact in the generic extension $V[G]$:

($\aleph$) **Lifting strategy:** Lift j to a $V[G]$-definable elementary embedding $j^\star : V[G] \to M^\star$ witnessing the λ-$C^{(n)}$-supercompactness of κ.[6]

($\beth$) **Extender strategy:** Use j to derive an extender $E \in V[G]$ witnessing the λ-$C^{(n)}$-supercompactness of κ (see [Bag12, §5]).

4.5.1 The lifting strategy

Under reasonable assumptions on $\mathbb{P}$ it is possible to lift our embedding to an outer one $j^\star : V[G] \to M[G*H]$ with ${}^\lambda M[G*H] \cap V[G*H] \subseteq M[G*H]$. This is the case, for instance, if $\mathbb{P}$ is a κ-iteration with Easton support guided by a $\mathfrak{S}^{(n)}$-fast function. As commented in page 54, the issue now is to guarantee that H can be defined in $V[G]$. Of course, there are many scenarios where one can arrange this (see e.g. [Cum10, Proposition 8.1]), but all of these lifting strategies rely on the fact that $j(\kappa)$ is a small cardinal in V. Thus, non of them are useful in the current context.

A possible solution would be to mimic Lemma 4.3.1 and cook up a generic filter for $j(\mathbb{P})/\mathbb{P}$. Unfortunately, this option is rarely available and thus does not seem to provide a general enough method to lift these embeddings.

4.5.2 The extender strategy

This strategy is used for instance in [Git10, Lemma 6.4]. As before, assume that we can lift j to an outer elementary embedding $j^\star : V[G] \to M[G*H]$ such that ${}^\lambda M[G*H] \cap V[G*H] \subseteq M[G*H]$. Let τ be a $j(\mathbb{P})$-name for the embedding $j^\star$ and E be the natural $V[G]$-extender derived from $j^\star$; that is, $E := \langle E_a \mid a \in [Y]^{<\omega} \rangle$ where

$$(\star) \quad X \in E_a \iff \exists p \in G\, \exists q \leq j(p) \setminus \kappa,\ p^\frown q \Vdash_{j(\mathbb{P})} \dot{a} \in \tau(\dot{X}),^{7}$$

where $\dot{a}, \dot{X}$ are $\mathbb{P}$-names and Y is a set as in [Bag12, §5]. It seems reasonable to think that one can arrange $E \in V[G]$, E_a is κ-complete normal filter and ${}^\lambda M_E \cap V[G] \subseteq M_E$, where $j_E : V[G] \to M_E$ is the corresponding extender embedding. The skeptic reader may look at [Git10, Lemma 6.4] where a similar result is proved for one measure.

[6] Obsere that if $|\mathbb{P}| < j(\kappa)$ then $j^\star(\kappa)$ is a Σ_n-correct cardinal in $V[G]$.

[7] Here $p^\frown q$ stands for the concatenation of the sequences p and q.

The main issue here does not seem to be related with the definability or the combinatorial properties of E but rather with the value of $j_E(\kappa)$. Observe that we need to make sure that $j_E(\kappa)$ is a Σ_n-correct cardinal in $V[G]$ so it is natural to aim for $j_E(\kappa) = j^\star(\kappa)$. Nonetheless, this is hard to secure as now there is no factoring map between j_E and $j^\star$. This latter fact being a consequence of the *generic definition* of E.

CHAPTER 5

$C^{(n)}$–extendibility, Vopěnka's principle and forcing

In this chapter we aim to contribute to the long-standing program in Set Theory which studies the robustness of Large Cardinals under forcing extensions. In this regard, in a joint project with J. Bagaria [BP18], we have studied the effects of Forcing upon the section of the large-cardinal hierarchy comprised between extendibility and *Vopěnka's Principle* (VP). This is one of most important fragments of the large-cardinal hierarchy, with deep connections with fundamental problems both in Set Theory Set Theory [Woo10][Usu19] and in other areas of Pure Mathematics [Bag+15][BBT13].

Our main contribution to the area is the development of a general theory of preservation of $C^{(n)}$–extendible cardinals under class forcing iterations. From this we will manage to obtain many consistency results regarding $C^{(n)}$–extendible cardinals and VP. This chapter will be precisely devoted to expose the theory and its main applications.

5.1 Introduction

In his seminal work [Lav78], R. Laver proved that after a preparatory forcing the supercompactness of a cardinal can be made indestructible under a wide range of forcing notions. Inspired by this discovering several authors subsequently obtained similar results for other classical large-cardinal notions. For instance, M. Gitik and S. Shelah [GS89] show that a strong cardinal κ can be made indestructible under so-called κ^+-weakly closed forcing satisfying the Prikry condition; J. D. Hamkins [Ham00] uses the *lottery preparation* forcing to make various types of large cardinals indestructible under appropriate forcing notions (e.g., a strong cardinal κ becomes indestructible by $\leq \kappa$-strategically closed forcing, and a strongly compact cardinal κ satisfying $2^\kappa = \kappa^+$ becomes indestructible by, among others, $\mathrm{Add}(\kappa, 1)$). More recently, A. Brooke-Taylor [BT11] showed that VP is indestructible under reverse Easton forcing iterations of increasingly directed-closed forcing notions, without the need for any preparatory forcing.

In the present chapter we are concerned with the preservation of $C^{(n)}$-extendible cardinals under forcing. This family of large cardinals was introduced in [Bag+15] (see also [Bag12]) as a strengthening of the classical notion of extendibility and was shown to provide natural milestones in the road from supercompact cardinals up to VP. Recall that VP(Π_{n+1}), namely VP restricted to classes of structures that are Π_{n+1}-definable, is equivalent to the existence of a $C^{(n)}$–extendible cardinal (cf. Theorem 1.2.10). Hence VP is equivalent to the existence of a $C^{(n)}$–extendible cardinal for each $n \geq 1$. It is in this sense that $C^{(n)}$–extendible cardinals can be conceived as canonical representatives of the large-cardinal hierarchy in the region comprissed between the first supercompact cardinal and VP.

Extendible cardinals have experienced a renewed interest after Woodin's proof of the HOD-Dichotomy [Woo10]. Also, $C^{(n)}$-extendible cardinals have found relevant applications in Category Theory and Algebraic Topology [Bag+15][BBT13]. Thus, the investigation of the preservation of such cardinals under forcing is a worthwhile project, which may lead to further applications.

In general, the preservation of very large cardinals by forcing is a delicate issue since it imposes strong forms of agreement between the ground model and the generic forcing extension. For example, suppose $\kappa \in C^{(n)}$ is inaccessible and $\mathbb{P}$ is a $< \kappa$-distributive forcing notion. If $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\kappa \in \dot{C}^{(n)}$" then $V_\kappa \prec_{\Sigma_n} V^{\mathbb{P}}$. The reason for this is that, since $\mathbb{P}$ is $<\kappa$-distributive and preserves that κ is in $C^{(n)}$, we have $V_\kappa = V_\kappa^{\mathbb{P}} \prec_{\Sigma_n} V^{\mathbb{P}}$. This underlines the fact that the more correct a large cardinal is, the harder is to preserve its correctness under forcing, and therefore the more fragile it becomes. Indeed, one runs into trouble when seeking a result akin to Laver's indestructibility for stronger large cardinals such as extendibles. This phenomenon was first pointed out by K. Tsaprounis in his Ph.D. thesis [Tsa12] and it was afterwards extensively studied in [Bag+16]. One of the consequences of the main theorem of [Bag+16] is that if κ is an extendible cardinal and $\mathbb{P}$ is any non trivial strategically $<\kappa$-closed set forcing notion, then forcing with $\mathbb{P}$ destroys the ($\mathfrak{a}$-$C^{(1)}$-)extendibility of κ. Actually the theorem implies that there is no hope to obtain indestructibility results for Σ_3-correct large cardinals. Thus, if one aims for a general theory of preservation of $C^{(n)}$-extendible cardinals one should concentrate on class forcing notions.

Jointly with J. Bagaria we have developed a general theory of preservation of $C^{(n)}$–extendible cardinals under *Suitable forcing iterations* (cf. Definition 5.5.1). Our main preservation theorem reads as follows:

Theorem 5.1.1 (Bagaria, P.)**.** *Suppose $m, n \geq 1$ and $m \leq n+1$. Suppose $\mathbb{P}$ is a weakly homogeneous Γ_m-definable suitable iteration and there exists* *a proper class of* $\mathbb{P}$-Σ_{n+1}*-reflecting cardinals. If* δ *is a* $\mathbb{P}$-Σ_{n+1}*-supercompact cardinal, then*

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“ } \delta \textit{ is } C^{(n)}\textit{–extendible”}.^1$$

The above result reveals an intriguing aspect of the nature of $C^{(n)}$–extendible cardinals. Recall that the main result of [Bag+15] implies that any set-sized segment of a suitable iteration $\mathbb{P}$ destroys the extendibility of δ. However, Theorem 5.1.1 guarantees that if we force with the whole iteration $\mathbb{P}$ then the $C^{(n)}$–extendibility of δ is preserved. This suggest the following: while for class many $\alpha \in \mathrm{ORD}$ the iteration $\mathbb{P}_\alpha$ forces that δ ceases to be $C^{(n)}$–extendible, there is a class forcing iteration (i.e. the tail forcing $\mathbb{P}/\mathbb{P}_\alpha$) which *resurrects* the $C^{(n)}$–extendibility of δ. In particular, while for class many $\alpha \in \mathrm{ORD}$, $V_\delta^{\mathbb{P}_\alpha} \not\prec_{\Sigma_3} V^{\mathbb{P}_\alpha}$, there is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{R}}$ for a class forcing notion such that $V_\delta^{\mathbb{P}_\alpha * \dot{\mathbb{R}}} \prec_{\Sigma_{n+2}} V^{\mathbb{P}_\alpha * \dot{\mathbb{R}}}$.

For the proof of the above theorem we will need that $C^{(n)}$–extendible cardinals are uniformly characterizable in a Magidor-like way (cf. Theorem 5.2.3): that is, similar to Magidor's characterization of supercompact cardinals [Mag71]. This result will reinforce the fact that $C^{(n)}$-extendible cardinals are a natural model-theoretic strengthening of supercompactness, which was first noticed in [Bag+15]. The same characterization has been independently given by W. Boney in [Bon18], and also in [BGS17] for the virtual forms of higher-level analogs of supercompact cardinals (i.e., n-remarkable cardinals) and virtual $C^{(n)}$-extendible cardinals.

In Section 5.6 we will show that many standard class forcing iterations fulfills the hypotheses of Theorem 5.1.1. As a result, we will derive the consistency of $C^{(n)}$–extendible cardinals (and so, of VP) with many combinatorial principles. For instance, we will show that $C^{(n)}$–extendibles cardinals are consistent with any prescribed behavior of the function $\kappa \mapsto 2^\kappa$ at regular cardinals and also with class many instances of weak square at singular cardinals. Among these applications we would like to stood out one which is connected with Woodin's HOD-conjecture. Namely, in theorems 5.6.18 and 5.6.21 we prove that it is possible to force a complete disagreement, and in many possible forms, between V and HOD with respect to the calculation of successors of regular cardinals, while $C^{(n)}$-extendible cardinals are preserved.

We would like to stress that Theorem 5.1.1 just works for weakly homogeneous iterations. This assumption is certainly crucial to conduct the lifting arguments appearing in the proof.

Nonetheless, there are some relevant principles whose consistency cannot be established by means of a weakly homogeneous forcing (cf. Definition 5.5.3). This is the case, for instance, of V = HOD: Assume that $\mathbb{P} \in \mathrm{HOD}$ is an atomless and weakly homogeneous forcing and that for some generic filter $G \subseteq \mathbb{P}$, $V[G] = \mathrm{HOD}^{V[G]}$. By standard forcing arguments (see e.g. [Jec03])

[1] Here Γ stands for either Σ or Π.

$\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}^V$, hence $G \in V[G] \subseteq \mathrm{HOD}^V \subseteq V$, and thus $G \in V$. This yields the desired contradiction.

In Section 5.7 we will address this issue and prove a preservation theorem for $C^{(n)}$–extendible cardinals under general (non weakly homogeneous, non definable) suitable iterations. For this, we introduce the notions of $C^{(n)}$-extendible and Σ_n-supercompact cardinals relative to a predicate, and then prove that under minor assumptions on the iteration $\mathbb{P}$, every $\mathbb{P}$-$C^{(n)}$-extendible cardinal remains $C^{(n)}$-extendible after forcing with $\mathbb{P}$. As an application we derive the consistency of $C^{(n)}$–extendible cardinals with V = HOD.

5.2 A Magidor-like characterization of $C^{(n)}$–extendibility

We shall prove that $C^{(n)}$–extendible cardinals can be characterized in a way analogous to the following characterization of supercompact cardinals due to M. Magidor.

Theorem 5.2.1 ([Mag71])**.** *For a cardinal δ, the following statements are equivalent:*

1. *δ is a supercompact cardinal.*
2. *For every $\lambda > \delta$ in $C^{(1)}$ and for every $a \in V_\lambda$, there exist ordinals $\bar{\delta} < \bar{\lambda} < \delta$ and there exist some $\bar{a} \in V_{\bar{\lambda}}$ and an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:*
   - $\mathrm{crit}(j) = \bar{\delta}$ *and* $j(\bar{\delta}) = \delta$.
   - $j(\bar{a}) = a$.
   - $\bar{\lambda} \in C^{(1)}$.

The existence of a supercompact cardinal is thus characterized by a form of reflection for Σ_1-correct strata of the universe, for it implies that any Σ_1-truth (i.e., any Σ_1 sentence, with parameters, true in V) is *captured* (up to some change of parameters) by some level below the supercompact cardinal. The following notion generalizes this reflection property to higher levels of complexity.

Definition 5.2.2 (Σ_n-supercompact cardinal)**.** Let $n \geq 1$. If $\lambda > \delta$ is in $C^{(n)}$, then we say that δ is λ-Σ_n-*supercompact* if for every $a \in V_\lambda$, there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{a} \in V_{\bar{\lambda}}$, and there exists elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:

- $\mathrm{crit}(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.

- $j(\bar{a}) = a$.
- $\bar{\lambda} \in C^{(n)}$.

We say that δ is a Σ_n–supercompact cardinal if it is λ-Σ_n-supercompact for every $\lambda > \delta$ in $C^{(n)}$.

The next theorem gives the promised Magidor-like characterization of $C^{(n)}$–extendible cardinals.

Theorem 5.2.3. *For $n \geq 1$, δ is a $C^{(n)}$–extendible cardinal if and only if δ is Σ_{n+1}-supercompact.*

Proof. Suppose that δ is $C^{(n)}$–extendible. Fix any $\lambda > \delta$ in $C^{(n+1)}$ and $a \in V_\lambda$. By $C^{(n)}$–extendibility, let $\mu > \lambda$ in $C^{(n+1)}$, and let $j : V_\mu \longrightarrow V_\theta$ be such that $\mathrm{crit}(j) = \delta$, $j(\delta) > \mu$ and $j(\delta) \in C^{(n)}$, for some ordinal θ. Notice that $j \restriction V_\lambda \in V_\theta$.

Claim 5.2.3.1. *V_θ satisfies the following sentence:*

$$\exists \bar{\lambda} < j(\delta)\ \exists \bar{\delta} < j(\bar{\lambda})\ \exists \bar{a} \in V_{\bar{\lambda}}\ \exists j^* : V_{\bar{\lambda}} \longrightarrow V_{j(\lambda)}$$
$$(j^*(\bar{a}) = j(a)\ \wedge\ j^*(\bar{\delta}) = j(\delta)\ \wedge\ V_{\bar{\lambda}} \prec_{\Sigma_{n+1}} V_{j(\lambda)}).$$

Proof of claim. It is sufficient to show that $V_\lambda \prec_{\Sigma_{n+1}} V_{j(\lambda)}$, for then the claim follows as witnessed by λ, δ, a, and $j \restriction V_\lambda$.

On the one hand, notice that $V_\delta \prec_{\Sigma_{n+1}} V_\mu$, because $C^{(n)}$–extendible cardinals are Σ_{n+2}–correct. By elementarity, this implies $V_{j(\delta)} \prec_{\Sigma_{n+1}} V_\theta$. On the other hand, since $j(\delta) > \mu$ and $j(\delta) \in C^{(n)}$, it is true that $V_\mu \prec_{\Sigma_{n+1}} V_{j(\delta)}$ and thus $V_\mu \prec_{\Sigma_{n+1}} V_\theta$. In addition, since μ and λ were both Σ_{n+1}–correct, it is the case that $V_\lambda \prec_{\Sigma_{n+1}} V_\mu$. Hence, $V_\lambda \prec_{\Sigma_{n+1}} V_\theta$. Also, by elementarity, $V_{j(\lambda)} \prec_{\Sigma_{n+1}} V_\theta$. Combining these two facts, we have that $V_\lambda \prec_{\Sigma_{n+1}} V_{j(\lambda)}$. □

By elementarity, V_μ satisfies the sentence displayed above. Hence, since $\mu \in C^{(n+1)}$, the sentence is true in the universe. Since λ was arbitrarily chosen, this implies that δ is a Σ_{n+1}-supercompact cardinal.

For the converse implication, let λ be greater than δ and let us show that there exists an elementary embedding $j : V_\lambda \longrightarrow V_\theta$, for some ordinal θ, such that $\mathrm{crit}(j) = \delta$, $j(\delta) > \lambda$, and $j(\delta) \in C^{(n)}$. Take $\mu > \lambda$ in $C^{(n+1)}$ and let $\bar{\delta}, \bar{\lambda} < \bar{\mu}$ and $j : V_{\bar{\mu}} \longrightarrow V_\mu$ be such that $\mathrm{crit}(j) = \bar{\delta}$, $j(\bar{\delta}) = \delta$, $j(\bar{\lambda}) = \lambda$, and $\bar{\mu} \in C^{(n+1)}$. Now notice that the sentence

$$\exists \alpha\ \exists j^* : V_{\bar{\lambda}} \longrightarrow V_\alpha\, (\mathrm{crit}(j^*) = \bar{\delta}\ \wedge\ j^*(\bar{\delta}) > \bar{\lambda}\ \wedge\ j^*(\bar{\delta}) \in C^{(n)}) \tag{5.1}$$

is Σ_{n+1}-expressible. Moreover, it is true in V witnessed by λ and j because $j(\bar{\delta}) = \delta > \bar{\lambda}$ and $\delta \in C^{(n)}$. Thus, since $V_{\bar{\mu}}$ is Σ_{n+1}-correct and contains $\bar{\delta}$ and $\bar{\lambda}$, it is also true in $V_{\bar{\mu}}$. By elementarity, V_μ thinks that the sentence

$$\exists \alpha\ \exists j^* : V_\lambda \longrightarrow V_\alpha\, (\mathrm{crit}(j^*) = \delta\ \wedge\ j^*(\delta) > \lambda\ \wedge\ j^*(\delta) \in C^{(n)}).$$

is true. Since $\mu \in C^{(n+1)}$, the above displayed sentence is true in V and so δ is λ-$C^{(n)}$–extendible. As λ was arbitrarily chosen, δ is a $C^{(n)}$–extendible cardinal. $\square$

Remark 5.2.4. Notice that in the proof above, if we had chosen λ to be in $C^{(n)}$, then in the displayed sentence (5.1) we could have also have required $\alpha \in C^{(n)}$. In that case, the proof actually shows that Σ_{n+1}-supercompactness implies $C^{(n)+}$-extendibility, which yields an alternative proof of Tsaprounis' result of the equivalence between $C^{(n)}$–extendibility and $C^{(n)+}$-extendibility [Tsa18].

Corollary 5.2.5. *A cardinal is extendible if and only if it is Σ_2-supercompact.*

Proof. This is a direct consequence of the theorem above, as every extendible cardinal is $C^{(1)}$-extendible. $\square$

Observe that the proof of $C^{(n)}$-extendibility from Σ_{n+1}-supercompactness given above only uses the definition of Σ_{n+1}-supercompactness restricted to those $a \in V_\lambda$ that are ordinals (i.e., the λ in the proof). Also, it is not explicitly required that $\bar{\mu} < \delta$. Moreover, one needs only λ-Σ_{n+1}-supercompactness for class-many λ in $C^{(n+1)}$. Thus, we have the following equivalence.

Corollary 5.2.6. *For $n \geq 1$, a cardinal δ is $C^{(n)}$-extendible if and only if for a proper class of λ in $C^{(n+1)}$, for every $\alpha < \lambda$ there exist $\bar{\delta}, \bar{\alpha} < \bar{\lambda}$ and an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:*

- $\mathrm{crit}(j) = \bar{\delta}$ *and* $j(\bar{\delta}) = \delta$.
- $j(\bar{\alpha}) = \alpha$.
- $\bar{\lambda} \in C^{(n+1)}$.

Since $C^{(n)}$-extendible cardinals are Σ_{n+2}-correct in V, it follows from last theorem that Σ_n-supercompact cardinals are in $C^{(n+1)}$. Moreover, since every $C^{(n+1)}$-extendible cardinal is a limit of $C^{(n)}$-extendible cardinals, every Σ_{n+1}–supercompact cardinal is a limit of Σ_n–supercompact cardinals.

It will become apparent in the following sections that the notion of Σ_{n+1}-supercompactness is a useful reformulation of $C^{(n)}$–extendibility in the context of class forcing.

5.3 Some reflection properties for class forcing iterations

In the sequel we will only work with ORD-length forcing iterations, since extendible cardinals are easily destroyed by set-size ones [Bag+16, Main Theorem 2]. Suppose $\mathbb{P}$ is such an iteration, $G \subseteq \mathbb{P}$ is a generic filter over V, and δ is a $C^{(n)}$-extendible cardinal. We will make use of the Magidor like characterization of $C^{(n)}$-extendibility (Theorem 5.2.3) to show that, under some hypotheses on $\mathbb{P}$, the $C^{(n)}$-extendibility of δ is preserved in $V[G]$. For this, one lifts ground model embeddings $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ witnessing the λ-Σ_{n+1}-supercompactness of δ to embeddings $j : V[G]_{\bar{\lambda}} \longrightarrow V[G]_\lambda$ verifying in $V[G]$ the same property. We refer to [Fri00] (see also [Reipt]) for general facts about class forcing iterations.

For the main preservation results given in the following sections we will need to ensure that there are many cardinals λ that satisfy $V_\lambda[G_\lambda] = V[G]_\lambda$. So, let us give them a name.

Definition 5.3.1. Let $\mathbb{P}$ be a forcing iteration. A cardinal λ is $\mathbb{P}$-*reflecting* if $\mathbb{P}$ forces that $V[\dot{G}]_\lambda \subseteq V_\lambda[\dot{G}_\lambda]$. (Hence, if G is $\mathbb{P}$-generic over V, then $V[G]_\lambda = V_\lambda[G_\lambda]$.)

The following proposition gives some sufficient conditions for a cardinal to be $\mathbb{P}$-reflecting.

Proposition 5.3.2. *Suppose λ is an inaccessible cardinal and $\mathbb{P}$ is an iteration such that $\mathbb{P}_\lambda \subseteq V_\lambda$, $\mathbb{P}_\lambda$ is λ-cc and preserves that λ is inaccessible, and $\mathbb{1} \Vdash_{\mathbb{P}_\lambda}$ "$\dot{\mathbb{Q}}$ is λ-distributive", where $\mathbb{P} \cong \mathbb{P}_\lambda * \mathbb{Q}$. Then λ is $\mathbb{P}$-reflecting.*

Proof. Let $V^{\mathbb{P}_\lambda}$ be the class of $\mathbb{P}_\lambda$-names obtained in the usual way, namely: $V_0^{\mathbb{P}_\lambda} = \emptyset$, $V_{\alpha+1}^{\mathbb{P}_\lambda} = V_\alpha^{\mathbb{P}_\lambda} \cup \mathcal{P}(V_\alpha^{\mathbb{P}_\lambda} \times \mathbb{P}_\lambda)$, and $V_\alpha^{\mathbb{P}_\lambda} = \bigcup_{\beta<\alpha} V_\beta^{\mathbb{P}_\lambda}$, whenever $\alpha \leq \mathrm{ORD}$ is a limit.

On the one hand, since the rank of $i_{G_\lambda}(\tau)$ in $V[G_\lambda]$ is never bigger than the rank of τ in V, for any $\tau \in V^{\mathbb{P}_\lambda}$, we clearly have

$$V_\lambda[G_\lambda] \subseteq V[G_\lambda]_\lambda \subseteq V[G]_\lambda.$$

On the other hand, by induction on the rank and using the fact that $\mathbb{P}_\lambda$ is λ-cc and preserves the inaccessibility of λ, one can easily show that $V[G_\lambda]_\lambda \subseteq V_\lambda[G_\lambda]$.

Since $|V_\lambda| = \lambda$, also $|V_\lambda[G_\lambda]| = \lambda$, and therefore $|V[G_\lambda]_\lambda| = \lambda$. Hence, since $\mathbb{1} \Vdash_{\mathbb{P}_\lambda}$ "$\dot{\mathbb{Q}}$ is λ-distributive", and so $i_{G_\lambda}(\dot{\mathbb{Q}})$ does not add any new subsets of $V[G_\lambda]_\lambda$, we have

$$V[G]_\lambda \subseteq V[G_\lambda]_\lambda.$$

Hence, $V[G]_\lambda \subseteq V_\lambda[G_\lambda]$. □

Let us consider next another key property of iterations that, in our constructions, will need to hold for a proper class of cardinals.

In the sequel, let $\mathcal{L}$ denote the language of set theory augmented with an additional unary predicate $\mathbb{P}$. This choice of language will allow us to work with expressions involving a given iteration $\mathbb{P}$.

Given $k \geq 0$, we need to compute the complexity of the notion

$$\langle V_\kappa, \in, \mathbb{P} \cap V_\kappa \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$$

as a property of κ, when $\mathbb{P}$ is a definable iteration. So, assume $\mathbb{P}$ is Γ_m-definable[2] for some $m \geq 1$, where Γ is either Σ or Π . Let $\Sigma_k^{\mathcal{L}}$ (resp. $\Pi_k^{\mathcal{L}}$) denote the set of Σ_k (resp. Π_k) formulas in the language $\mathcal{L}$.

Proposition 5.3.3. *The truth predicate $\models_{\Sigma_0^{\mathcal{L}}}$ for Σ_0-formulae in $\mathcal{L}$ is Γ_m-definable.*

Proof. Note first that the only atomic formulae in the language $\mathcal{L}$ are of the form "$x \in y$", "$x = y$", or "$x \in \mathbb{P}$", where x and y are variable symbols. Hence, the truth predicate for $\mathcal{L}-$atomic formulae is Γ_m-definable (recall that we assume $\mathbb{P}$ is Γ_m-definable).

Now let $\varphi(\bar{x}, y)$ be a Σ_0-formula. Suppose, by induction on the complexity of the formulae, that $\models_{\Sigma_0^{\mathcal{L}}}$ is Γ_m-definable when restricted to proper subformulae of $\varphi(\bar{x}, y)$. The result is clear for Boolean combinations. So, suppose that $\varphi(\bar{x}, y)$ is of the form $\exists z \in y\, \psi(z, \bar{x})$. Then for any $\bar{a}$ and b,

$$\models_{\Sigma_0^{\mathcal{L}}} \exists z \in b\, \psi(z, \bar{a}) \text{ iff } \exists z \in b \models_{\Sigma_0^{\mathcal{L}}} \psi(z, \bar{a})$$

which shows that $\models_{\Sigma_0^{\mathcal{L}}} \exists z \in b\, \psi(z, \bar{a})$ is Γ_m expressible. □

Proposition 5.3.4. *Let $k \geq 1$. The truth predicate $\models_{\Sigma_k^{\mathcal{L}}}$ for Σ_k-formulae in $\mathcal{L}$ is Σ_{m+k-1}-definable if $\Gamma = \Sigma$, and Σ_{m+k} if $\Gamma = \Pi$; and the truth predicate $\models_{\Pi_k^{\mathcal{L}}}$ for Π_k-formulae in $\mathcal{L}$ is Π_{m+k-1}-definable if $\Gamma = \Pi$, and Π_{m+k} if $\Gamma = \Sigma$.*

Proof. By induction over k. For $k = 1$, take any Σ_1 formula $\varphi(\bar{x}, y) \equiv \exists y\, \psi(\bar{x}, y)$ in $\mathcal{L}$, where $\psi(\bar{x}, y)$ is Σ_0. Given $\bar{a}$ any finite sequence of parameters, notice that

$$\models_{\Sigma_1^{\mathcal{L}}} \exists y\, \psi(\bar{a}, y) \text{ iff } \exists y \models_{\Sigma_0^{\mathcal{L}}} \psi(\bar{a}, y).$$

Therefore, by proposition 5.3.3, $\models_{\Sigma_1^{\mathcal{L}}}$ is Σ_m-definable if $\Gamma = \Sigma$, and Σ_{m+1}-definable if $\Gamma = \Pi$. Similarly, $\models_{\Pi_1^{\mathcal{L}}}$ is Π_m-definable for Π_1 formulae in $\mathcal{L}$ if $\Gamma = \Pi$, and is Π_{m+1}-definable if $\Gamma = \Sigma$.

Suppose now by induction that $\models_{\Pi_k^{\mathcal{L}}}$ is a Π_{m+k-1} definable predicate for Π_k formulae in $\mathcal{L}$ if $\Gamma = \Pi$, and Π_{m+k}-definable if $\Gamma = \Sigma$. Let $\varphi(\bar{x}, y) \equiv \exists y\, \psi(\bar{x}, \bar{y})$

[2]When we say that a forcing notion $\mathbb{P}$ is Γ_m-definable, we mean that the ordering relation $\leq_{\mathbb{P}}$ is Γ_m-definable, hence the set of conditions is also Γ_m-definable.

be a Σ_{k+1} formula in $\mathcal{L}$ with $\psi(\bar{x}, \bar{y})$ being a Π_k formula. Given $\bar{a}$ any finite sequence of parameters,

$$\models_{\Sigma^{\mathcal{L}}_{k+1}} \exists y\, \psi(\bar{a}, \bar{y}) \text{ iff } \exists y \models_{\Pi^{\mathcal{L}}_k} \psi(\bar{a}, \bar{y}).$$

Therefore, $\models_{\Sigma^{\mathcal{L}}_{k+1}}$ is a Σ_{m+k}-definable relation if $\Gamma = \Pi$, and is Σ_{m+k+1}-definable if $\Gamma = \Sigma$. $\square$

For $k \geq 0$, an ordinal α, and an iteration $\mathbb{P}$, we shall denote by $C^{(k)}_{\mathbb{P}}$ the class of all ordinals α such that

$$\langle V_\alpha, \in, \mathbb{P} \cap V_\alpha \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle.$$

It is easily seen that the class $C^{(k)}_{\mathbb{P}}$ is closed and unbounded.

Let us compute next the complexity of $C^{(k)}_{\mathbb{P}}$ when $\mathbb{P}$ is a definable iteration. First, observe that $C^{(0)}_{\mathbb{P}} = \{\alpha \mid \mathbb{P}^{V_\alpha} = \mathbb{P} \cap V_\alpha\}$, for if $\psi(x)$ is a Σ_0 formula in $\mathcal{L}$ and α is an ordinal which correctly interprets the predicate $\mathbb{P}$, then

$$\models_{\Sigma^{\mathcal{L}}_0} \psi(\bar{a}) \text{ iff } \langle V_\alpha, \in, \mathbb{P} \cap V_\alpha \rangle \models \psi(\bar{a})$$

for any $\bar{a}$ in V_α. Thus, if $\mathbb{P}$ is Γ_m-definable, then $C^{(m)} \subseteq C^{(0)}_{\mathbb{P}}$. Note that if $\mathbb{P}$ is Δ_1-definable, i.e., both Σ_1 and Π_1-definable, then the class $C^{(0)}_{\mathbb{P}}$ coincides with ORD and is thus Σ_0-definable. If $\mathbb{P}$ is Σ_1-definable, then $C^{(0)}_{\mathbb{P}}$ is Δ_2-definable (i.e., both Σ_2 and Π_2-definable), for if $\varphi(x)$ is a Σ_1 formula defining $\mathbb{P}$, then:

$$\alpha \in C^{(0)}_{\mathbb{P}} \text{ iff } \exists X (X = V_\alpha \wedge \forall x \in X\, (\varphi(x) \longrightarrow X \models \varphi(x)))$$

and also

$$\alpha \in C^{(0)}_{\mathbb{P}} \text{ iff } \forall X (X = V_\alpha \longrightarrow \forall x \in X\, (\varphi(x) \longrightarrow X \models \varphi(x))).$$

Similarly, if $\mathbb{P}$ is Π_1-definable, then $C^{(0)}_{\mathbb{P}}$ is also Δ_2-definable.

Now suppose $\mathbb{P}$ is Γ_m-definable, where $m \geq 2$. then the class $C^{(0)}_{\mathbb{P}}$ is Δ_{m+1}-definable (i.e., both Σ_{m+1} and Π_{m+1}-definable):

$$\alpha \in C^{(0)}_{\mathbb{P}} \text{ iff } \exists X (X = V_\alpha \wedge \forall x \in X\, (X \models \Psi(x) \longleftrightarrow \Psi(x)))$$

and also

$$\alpha \in C^{(0)}_{\mathbb{P}} \text{ iff } \forall X (X = V_\alpha \rightarrow \forall x \in X\, (X \models \Psi(x) \longleftrightarrow \Psi(x)))$$

where $\Psi(x)$ stands for some Γ_m-formula defining $\mathbb{P}$. Note however that if $\mathbb{P}$ is Δ_m-definable, then $C^{(0)}_{\mathbb{P}}$ is also Δ_m-definable (for $m \geq 2$).

Proposition 5.3.5. *The class $C^{(k)}_{\mathbb{P}}$ is*

Δ_0*-definable,*	*if $k = 0$ and $\mathbb{P}$ is Δ_1-definable.*
Δ_2*-definable,*	*if $k \leq 1$ and $\mathbb{P}$ is Γ_1-definable.*
Δ_m*-definable,*	*if $k \leq 1$ and $\mathbb{P}$ is Δ_m-definable, for $m \geq 2$.*
Δ_{m+1}*-definable,*	*if $k \leq 1$ and $\mathbb{P}$ is Γ_m-definable, for $m \geq 2$.*
Δ_{m+k-1}*-definable,*	*if $k \geq 2$ and $\mathbb{P}$ is Γ_m-definable.*

Proof. We have already computed the complexity of the class $C^{(0)}_{\mathbb{P}}$.

If $k \geq 1$, then we have that $\alpha \in C^{(k)}_{\mathbb{P}}$ if and only if $\alpha \in C^{(k-1)}_{\mathbb{P}}$ and

$$\forall X, Y(X = V_\alpha \wedge Y = \mathbb{P} \cap X \to \forall \bar{a} \in X\, \forall \varphi \in \Sigma^{\mathcal{L}}_k(\models_{\Sigma^{\mathcal{L}}_k} \varphi(\bar{a}) \to$$

$$\langle X, \in, Y\rangle \models \varphi(\bar{a})))$$

or

$$\forall X, Y(X = V_\alpha \wedge Y = \mathbb{P} \cap X \to \forall \bar{a} \in X\, \forall \varphi \in \Pi^{\mathcal{L}}_k(\langle X, \in, Y\rangle \models \varphi(\bar{a}) \to$$

$$\models_{\Pi^{\mathcal{L}}_k} \varphi(\bar{a}))).$$

And also if and only if $\alpha \in C^{(k-1)}_{\mathbb{P}}$ and

$$\exists X, Y(X = V_\alpha \wedge Y = \mathbb{P} \cap X \wedge \forall \bar{a} \in X\, \forall \varphi \in \Sigma^{\mathcal{L}}_k(\models_{\Sigma^{\mathcal{L}}_k} \varphi(\bar{a}) \to$$

$$\langle X, \in, Y\rangle \models \varphi(\bar{a})))$$

or

$$\exists X, Y(X = V_\alpha \wedge Y = \mathbb{P} \cap X \wedge \forall \bar{a} \in X\, \forall \varphi \in \Pi^{\mathcal{L}}_k(\langle X, \in, \mathbb{P} \cap X\rangle \models \varphi(\bar{a}) \to$$

$$\models_{\Pi_k, \mathcal{L}} \varphi(\bar{a}))).$$

Now, by induction, and using proposition 5.3.4, the complexity of the definition of the class $C^{(k)}_{\mathbb{P}}$ is easily computed. □

Notice that if a club proper class of ordinals is Σ_k-definable, then it contains $C^{(k)}$; and if it is Π_k-definable, then it contains $C^{(k+1)}$.

The next proposition will be crucial for further arguments.

Proposition 5.3.6. *Suppose $\mathbb{P}$ is a definable iteration. If κ is a $\mathbb{P}$-reflecting cardinal in* V *such that $\kappa \in C^{(k)}_{\mathbb{P}}$ (and so $\langle V_\kappa, \in, \mathbb{P}_\kappa\rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P}\rangle$), then $\mathbb{P}$ forces $V[\dot{G}]_\kappa \prec_{\Sigma_k} V[\dot{G}]$.*

Proof. This is clear for $k = 0$. So, assume $k \geq 1$. Let $\varphi(x)$ be a Σ_k-formula in the language of set theory and let $\tau \in V_\kappa$ be a $\mathbb{P}_\kappa$-name such that $p \Vdash_{\mathbb{P}} \varphi(\tau)$, for some $p \in \mathbb{P}$. Notice that this is a legitimate choice for τ as κ is $\mathbb{P}$-reflecting.

By taking $\mathbb{P}$ as an additional predicate, the forcing relation $\Vdash_{\mathbb{P}}$ for Σ_k-formulae in the forcing language[3] is Σ_k-definable.

Claim 5.3.6.1. *There exists a condition $q \in \mathbb{P}_\kappa$ such that $q \leq p \restriction \kappa$ and $q \Vdash_{\mathbb{P}} \varphi(\tau)$.*

Proof of claim. Suppose otherwise. Since $\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$, and the sentence "$q \nVdash_{\mathbb{P}} \varphi(\tau)$" is Π_k expressible in the language of $\langle V, \in, \mathbb{P} \rangle$, we have that "$q \nVdash_{\mathbb{P}_\kappa} \varphi(\tau)$" holds in $\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle$, for every $q \leq p \restriction \kappa$ in $\mathbb{P}_\kappa$. Therefore,

$$\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \models \text{“}p \restriction \kappa \Vdash_{\mathbb{P}_\kappa} \neg\varphi(\tau)\text{”}.$$

Again, since $\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$, and the displayed sentence is Π_k,

$$\langle V, \in, \mathbb{P} \rangle \models \text{“}p \restriction \kappa \Vdash_{\mathbb{P}} \neg\varphi(\tau)\text{”}$$

which yields the desired contradiction with the fact that $p \Vdash_{\mathbb{P}} \varphi(\tau)$. □

Since $\mathbb{P}_\kappa \subseteq V_\kappa$, we have that $q \in V_\kappa$. The sentence

$$\exists r \leq q\, (r \Vdash_{\mathbb{P}} \varphi(\tau))$$

is equivalent to a Σ_k sentence in the language of $\langle V, \in, \mathbb{P} \rangle$, with parameters q and τ. So, since $\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$, we have

$$\langle V_\kappa, \in \mathbb{P}_\kappa \rangle \vDash \text{“}\exists r \leq q(r \Vdash_{\mathbb{P}} \varphi(\tau))\text{”}.$$

Altogether, this proves that the set of conditions in $\mathbb{P}_\kappa$ forcing $\varphi(\tau)$ is dense and thus $V_\kappa[G_\kappa] \models \varphi(a)$. Since κ is $\mathbb{P}$-reflecting, $V[G]_\kappa \models \varphi(a)$, as wanted.

Now suppose $V[G]_\kappa \vDash \varphi(a)$. Since $V[G]_\kappa = V_\kappa[G_\kappa]$, there is some condition $p \in G_\kappa$ such that

$$\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \models \text{“}p \Vdash_{\mathbb{P}_\kappa} \varphi(\tau)\text{”}.$$

Hence, since $\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$,

$$\langle V, \in, \mathbb{P} \rangle \models \text{“}p \Vdash_{\mathbb{P}} \varphi(\tau)\text{”}$$

As $p \in G_\kappa \subseteq G$ and $i_G(\tau) = i_{G_\kappa}(\tau) = a$, it follows that $V[G] \models \varphi(a)$, as wanted. □

The last proposition motivates the following strengthening of the notion of $\mathbb{P}$-reflection (cf. Definition 5.3.1).

[3]Namely, in the language of set theory expanded with constant symbols for each $\mathbb{P}$-name.

Definition 5.3.7. If $k \geq 1$ and $\mathbb{P}$ is a definable iteration, then a cardinal κ is $\mathbb{P}$-Σ_k-*reflecting* if it is $\mathbb{P}$-reflecting and belongs to $C^{(k)}_{\mathbb{P}}$.

Proposition 5.3.6 shows that if $\mathbb{P}$ is a definable iteration, then $\mathbb{P}$-Σ_k-reflecting cardinals remain Σ_k-correct in any $\mathbb{P}$-generic extension of V. So, although the main motivation behind proposition 5.3.6 is the lifting of elementary embeddings under suitable iterations, it also sheds some light into the question of the preservation of Σ_n-correct cardinals under forcing (see [Tsa12]), an interesting topic in its own right. Besides proposition 5.3.6, here is a summary of what is known: If $\kappa \in C^{(1)}$ and $\mathbb{P}$ is a forcing notion that preserves V_κ, then $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\kappa \in C^{(1)}$" ([Tsa14]). Also, since V $=$ L is a Π_2 assertion, if in L a cardinal κ belongs to $C^{(2)}$, then any non-trivial forcing notion that preserves L_κ will force that κ is not in $C^{(2)}$, for it will force V $\neq$ L. If $\kappa \in C^{(1)}$, then, as observed by Carmody [Car15], one can easily preserve it being in $C^{(1)}$ while forcing it not being in $C^{(2)}$. Namely, first force the GCH below κ, which preserves $\kappa \in C^{(1)}$, and then force the failure of GCH at κ without changing V_κ.

For the record, let us compute the complexity of the notion of $\mathbb{P}$-Σ_n-reflecting cardinal.

Proposition 5.3.8. *Suppose $m \geq 1$ and $\mathbb{P}$ is a Γ_m-definable iteration. The predicate "κ is a $\mathbb{P}$-Σ_k-reflecting cardinal" is:*

1. Π_{m+1} *if $k \leq 2$ and $\Gamma = \Sigma$*
2. Π_{m+2}, *if $k \leq 3$ and $\Gamma = \Pi$*
3. Δ_{m+k-1} *if either $k \geq 3$ and $\Gamma = \Sigma$, or $k > 3$ and $\Gamma = \Pi$.*

Proof. In general, the assertion "$\mathbb{P}$ forces that $V[\dot{G}]_\kappa \subseteq V_\kappa[\dot{G}_\kappa]$" is Π_{m+1}-expressible if $\Gamma = \Sigma$, and Π_{m+2} if $\Gamma = \Pi$. To see this, first note that the class $V^{\mathbb{P}}$ of $\mathbb{P}$-names is Σ_m-definable if $\Gamma = \Sigma$, and Δ_{m+1}-definable if $\Gamma = \Pi$. Next, note that $\mathbb{P}$ forces $V[\dot{G}]_\kappa \subseteq V_\kappa[\dot{G}_\kappa]$ if and only if

$$\forall \tau, p\, (\tau \in V^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P}} \text{“rank}(\tau) < \kappa\text{”} \rightarrow$$

$$\exists \sigma, q\, (\sigma \in V^{\mathbb{P}} \wedge q \leq p \wedge \text{rank}(\sigma) < \kappa \wedge q \Vdash_{\mathbb{P}} \text{“}\sigma = \tau\text{”})).$$

However, notice that if $\mathbb{P}$ is absolute for V_κ, then the assertion "$\mathbb{P}$ forces that $V[\dot{G}]_\kappa \subseteq V_\kappa[\dot{G}_\kappa]$" is Π_{m+1}-expressible, because the last displayed assertion is equivalent to

$$\forall \tau, p, X\, (\tau \in V^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P}} \text{“rank}(\tau) < \kappa\text{”} \wedge X = V_\kappa \rightarrow$$

$$\exists \sigma, q\, (\sigma \in X \wedge X \models \text{“}\sigma \in V^{\mathbb{P}}\text{”} \wedge q \leq p \wedge \text{rank}(\sigma) < \kappa \wedge q \Vdash_{\mathbb{P}} \text{“}\sigma = \tau\text{”})).$$

Finally, by Proposition 5.3.5, the fact that $\langle V_\kappa, \in, \mathbb{P}_\kappa\rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P}\rangle$ is Δ_{m+1}-expressible[4] if $k = 1$, and Δ_{m+k-1}-expressible if $k \geq 2$. □

5.4 $\mathbb{P}$-Σ_n-supercompactness

The following definition gives a refinement of the notion of Σ_n-supercompact cardinal, relative to definable iterations.

Definition 5.4.1 ($\mathbb{P}$-Σ_n-supercompactness)**.** If $n \geq 1$ and $\mathbb{P}$ is a definable iteration, then we say that a cardinal δ is $\mathbb{P}$-Σ_n-supercompact if there exists a proper class of $\mathbb{P}$-Σ_n-reflecting cardinals, and for every such cardinal $\lambda > \delta$ and every $a \in V_\lambda$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{a} \in V_{\bar{\lambda}}$, and there exists an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:

- $\mathrm{crit}(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.
- $j(\bar{a}) = a$.
- $\bar{\lambda}$ is $\mathbb{P}$-Σ_n-reflecting.

Next proposition unveils the connections between Σ_n-supercompact and $\mathbb{P}$-Σ_n-supercompact cardinals.

Proposition 5.4.2. *Suppose $n \geq 1$ and $\mathbb{P}$ is a Γ_m-definable iteration for some $m \geq 1$. Suppose there is a proper class of $\mathbb{P}$-Σ_n-reflecting cardinals. Then,*

1. *Every $\mathbb{P}$-Σ_n-supercompact cardinal is Σ_n-supercompact.*
2. *If δ is Σ_{m+1}-supercompact, in case $n \leq 2$, or Σ_{m+n-1}-supercompact, in case $n \geq 3$, then δ is $\mathbb{P}$-Σ_n-supercompact.*

In particular, if $\mathbb{P}$ is a Γ_1-definable iteration and there exists a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals, then every Σ_{n+1}-supercompact cardinal is $\mathbb{P}$-Σ_{n+1}-supercompact.

Proof. (1): Assume δ is a $\mathbb{P}$-Σ_n-supercompact. Let $\lambda > \delta$ be a Σ_n-correct cardinal and let $\kappa > \lambda$ be a $\mathbb{P}$-Σ_n-reflecting cardinal. Notice that $V_\kappa \models$ "$V_\lambda \prec_{\Sigma_n} V$" and thus by $\mathbb{P}$-Σ_n-supercompactness, there is some $j : V_{\bar{\kappa}} \longrightarrow V_\kappa$ such that $j(\bar{\lambda}) = \lambda$ for some $\bar{\lambda} < \delta$ and some $\bar{\kappa}$ being $\mathbb{P}$-Σ_n-reflecting. By elementarity, $V_{\bar{\kappa}}$ thinks that $\bar{\lambda}$ is a Σ_n-correct cardinal and thus $V_{\bar{\lambda}} \prec_{\Sigma_n} V$.

(2): Let us prove the case case $n \geq 3$, the case $n \leq 2$ being proved similarly. So, let $\lambda > \delta$ be a $\mathbb{P}$-Σ_n-reflecting cardinal and $\kappa > \lambda$ be a Σ_{m+n-1}-correct cardinal. Since being a $\mathbb{P}$-Σ_n-reflecting cardinal is a Π_{m+n-1} property

[4] I.e., both Σ_{m+1}-expressible and Π_{m+1}-expressible.

(Proposition 5.3.8), V_κ thinks that λ is $\mathbb{P}$-Σ_n-reflecting. Since δ is Σ_{m+n-1}-supercompact, there exist $\bar\delta < \bar\lambda < \bar\kappa$ with $V_{\bar\kappa} \prec_{\Sigma_{m+n-1}} V$, and there exists an elementary embedding $j : V_{\bar\kappa} \longrightarrow V_\kappa$ such that $\mathrm{crit}(j) = \bar\delta$, $j(\bar\delta) = \delta$, and $j(\bar\lambda) = \lambda$. By elementarity, $V_{\bar\kappa}$ thinks that $\bar\lambda$ is $\mathbb{P}$-Σ_n-reflecting, and since $V_{\bar\kappa} \prec_{\Sigma_{m+n-1}} V$, we have that $\bar\lambda$ is $\mathbb{P}$-Σ_n-reflecting in V. Thus, the restricted embedding $j \restriction V_{\bar\lambda}$ witnesses the $\mathbb{P}$-Σ_n-supercompactness of δ. $\square$

The proposition above together with Theorem 5.2.3 yield the following.

Corollary 5.4.3. *Suppose $n \geq 1$ and $\mathbb{P}$ is a Γ_m-definable iteration, some $m \geq 1$. Then, assuming there is a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals,*

1. *Every $\mathbb{P}$-Σ_{n+1}-supercompact cardinal is $C^{(n)}$–extendible.*
2. *Every $C^{(n)}$-extendible cardinal is $\mathbb{P}$-Σ_{n+1}-supercompact, in the case $m = 1$.*
3. *Every $C^{(m+n-1)}$-extendible cardinal is $\mathbb{P}$-Σ_{n+1}-supercompact, in the case $m \geq 2$.*

In particular, if $\mathbb{P}$ is a Γ_1-definable iteration and there exists a proper class of $\mathbb{P}$-Σ_n-reflecting cardinals, a cardinal is $C^{(n)}$-extendible if and only if it is $\mathbb{P}$-Σ_{n+1}-supercompact.

5.5 Suitable iterations

The following is a property enjoyed by many well-known ORD-length forcing iterations, such as Jensen's canonical class forcing for obtaining the global GCH, or the standard class forcing iteration for forcing V=HOD. The property will be needed to prove a general result (Theorem 5.1.1) on the preservation of $C^{(n)}$-extendibility.

Definition 5.5.1 (Suitable iterations)**.** Given κ a cardinal (with possibly $\kappa = \mathrm{ORD}$) a forcing iteration $\mathbb{P}$ of length κ is *suitable* if it is the direct limit of an Easton support iteration[5] $\langle\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa\rangle\rangle$ such that for each $\lambda < \kappa$ there is some $\theta < \kappa$ greater than λ such that

$$\mathbb{1} \Vdash_{\mathbb{P}_\nu} \text{“}\, \dot{\mathbb{Q}}_\nu \text{ is } \lambda\text{-directed closed}\,\text{”}$$

for all $\nu \geq \theta$.

Notice that if an iteration $\mathbb{P}$ has Easton support, then for any inaccessible cardinal λ, if $\mathbb{P}_\lambda \subseteq V_\lambda$ and G is a $\mathbb{P}$-generic filter over V, then $G_\lambda := G \cap \mathbb{P}_\lambda$ is a $\mathbb{P}_\lambda$-generic filter over V_λ.

[5]Recall that an Easton support iteration is a forcing iteration where direct limits are taken at inaccessible stages and inverse limits elsewhere.

It is well-known that suitable class forcing iterations preserve ZFC (see [Fri00]). The condition of eventual λ-directed closedness in the definition above can be strengthened on a club proper class. Namely,

Proposition 5.5.2. *Let $\mathbb{P}$ be a suitable iteration. The class*

$$C = \{\lambda \mid \forall \eta \geq \lambda, \mathbb{1} \Vdash_{\mathbb{P}_\eta} \text{“} \dot{\mathbb{Q}}_\eta \textit{ is } \lambda\textit{-directed closed”}\}$$

is a club class.

Proof. Closedness is obvious. As for unboundedness, fix any λ and build inductively a sequence $\langle \theta_n \mid n < \omega \rangle$ of ordinals greater than λ such that for all $\eta \geq \theta_{n+1}$,

$$\mathbb{1} \Vdash_{\mathbb{P}_\eta} \text{“} \dot{\mathbb{Q}}_\eta \text{ is } \theta_n\text{-directed closed”}.$$

Notice now that $\theta^* := \sup_n \theta_n$ is an element of C. □

Definition 5.5.3. A forcing poset $\mathbb{P}$ is *weakly homogeneous* if for any conditions $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ and q are compatible.

We are now in conditions to present the proof of Theorem 5.1.1:

Proof. Proof of Theorem 5.1.1 Suppose G is $\mathbb{P}$-generic over V. By Corollary 5.2.6 and Proposition 5.3.6, it is sufficient to take an arbitrary $\mathbb{P}$-Σ_{n+1}-reflecting cardinal $\lambda > \delta$, and any $\alpha < \lambda$, and find a $\mathbb{P}$-Σ_{n+1}-reflecting cardinal $\bar{\lambda}$, ordinals $\bar{\delta}, \bar{\alpha} < \bar{\lambda}$, and an elementary embedding $j : V[G]_{\bar{\lambda}} \longrightarrow V[G]_\lambda$ such that $\mathrm{crit}(j) = \bar{\delta}$, $j(\bar{\delta}) = \delta$, and $j(\bar{\alpha}) = \alpha$.

So pick a $\mathbb{P}$-Σ_{n+1}-reflecting cardinal $\lambda > \delta$, and any $\alpha < \lambda$. Since δ is $\mathbb{P}$-Σ_{n+1}-supercompact there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{\alpha} < \bar{\lambda}$, and an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that

- $\mathrm{crit}(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$
- $j(\bar{\alpha}) = \alpha$.
- $\bar{\lambda}$ is $\mathbb{P}$-Σ_{n+1}-reflecting

Notice that since both λ and $\bar{\lambda}$ are $\mathbb{P}$-Σ_{n+1}-reflecting, and $m \leq n+1$, we have that $\mathbb{P}^{V_\lambda} = \mathbb{P}_\lambda = \mathbb{P} \cap V_\lambda$ and $\mathbb{P}^{V_{\bar{\lambda}}} = \mathbb{P}_{\bar{\lambda}} = \mathbb{P} \cap V_{\bar{\lambda}}$.

It will suffice to show that $j \restriction V_{\bar{\lambda}}$ can be lifted to an elementary embedding $j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \longrightarrow V_\lambda[G_\lambda]$, for then, since both λ and $\bar{\lambda}$ are $\mathbb{P}$-reflecting in V, we have that $V_\lambda[G_\lambda] = V[G]_\lambda$ and $V_{\bar{\lambda}}[G_{\bar{\lambda}}] = V[G]_{\bar{\lambda}}$. Thus, j is an elementary embedding from $V[G]_{\bar{\lambda}}$ into $V[G]_\lambda$ with the properties we wanted.

The iterations $\mathbb{P}_{\bar{\lambda}}$ and $\mathbb{P}_\lambda$ factorize as follows:

(i) $\mathbb{P}_{\bar{\lambda}} \cong \mathbb{P}_{\bar{\delta}} * \mathbb{Q}$ with $|\mathbb{Q}| = \bar{\lambda}$.

(ii) $\mathbb{P}_\lambda \cong \mathbb{P}_\delta * \mathbb{Q}^*$ with

$$\mathbb{1} \Vdash_{\mathbb{P}_\delta} \text{“}\mathbb{Q}^* \text{ is weakly homogeneous and } \delta\text{-directed closed”}.$$

Indeed, (i) is clear since $\bar{\lambda}$ is a strong limit. For (ii), since $\mathbb{P}$ is weakly homogeneous, $\mathbb{P}_\delta$ forces that $\mathbb{Q}^*$ is so. Thus, we only need to see that $\mathbb{1} \Vdash_{\mathbb{P}_\delta}$ "$\mathbb{Q}^*$ is δ-directed closed". Recall from proposition 5.5.2 that the class

$$C = \{\mu \mid \forall \eta \geq \mu,\ \mathbb{1} \Vdash_{\mathbb{P}_\eta} \text{“}\dot{\mathbb{Q}}_\eta \text{ is } \mu\text{-directed closed”}\}$$

is a club class. Thus, it will be sufficient to show that δ is a limit point of C, and therefore it belongs to C. So, let $\mu < \delta$ and notice that since $\mathbb{P}$ is a suitable iteration, the sentence $\varphi(\mu)$ asserting:

$$\exists \theta > \mu\, \forall \eta \geq \theta\ (\ \mathbb{1} \Vdash_{\mathbb{P}_\eta} \text{“}\dot{\mathbb{Q}}_\eta \text{ is } \mu\text{-directed closed”})$$

holds in V. Also notice that $\varphi(\mu)$ is equivalent to the Σ_{m+2} sentence:

$$\exists \theta > \mu\, \forall \eta \geq \theta\, \forall \alpha > \eta\, (\alpha \in C^{(m)} \rightarrow$$

$$V_\alpha \models \text{“}\ \mathbb{1} \Vdash_{\mathbb{P}_\eta}\ \dot{\mathbb{Q}}_\eta \text{ is } \mu\text{-directed closed ”}).$$

Since δ is a Σ_{m+2}-correct cardinal (by Proposition 5.4.2 and the following remarks), there must be a witness for $\varphi(\mu)$ below δ. This shows that C is unbounded in δ, as wanted.

Since δ is $C^{(n)}$-extendible (Corollary 5.4.3), and therefore Σ_{n+2}-correct in V, and since j is elementary with $j(\bar{\delta}) = \delta$, we have that $j(\mathbb{P}_{\bar{\delta}}) = \mathbb{P}_\delta$. Also, since $\bar{\delta}$ is the critical point of j, we have that $j[G_{\bar{\delta}}] = G_{\bar{\delta}} \subseteq G_\delta$, and so $j \restriction V_{\bar{\lambda}}$ can be lifted to an elementary embedding

$$j : V_{\bar{\lambda}}[G_{\bar{\delta}}] \longrightarrow V_\lambda[G_\delta].$$

Let us denote by $G_{[\bar{\delta},\bar{\lambda})}$ and $G_{[\delta,\lambda)}$ the filters $G \cap \mathbb{Q}$ and $G \cap \mathbb{Q}^*$, respectively. Notice that these filters are generic for $\mathbb{Q}$ and $\mathbb{Q}^*$ over $V_{\bar{\lambda}}[G_{\bar{\delta}}]$ and $V_\lambda[G_\delta]$, respectively. In order to lift the embedding j to the further generic extension $V_{\bar{\lambda}}[G_{\bar{\lambda}}] = V_{\bar{\lambda}}[G_{\bar{\delta}}][G_{[\bar{\delta},\bar{\lambda})}]$, notice first that $j[G_{[\bar{\delta},\bar{\lambda})}]$ is a directed subset of $\mathbb{Q}^*$ of cardinality $\leq \bar{\lambda}$. Since $\mathbb{Q}^*$ does not add any new subsets of $V_\lambda[G_\delta]$ of size $< \delta$, we have that $j[G_{[\bar{\delta},\bar{\lambda})}]$ belongs to $V_\lambda[G_\delta]$. Therefore, since $\mathbb{Q}^*$ is a δ-directed closed forcing notion in $V_\lambda[G_\delta]$, there is some condition $p \in \mathbb{Q}^*$ such that $p \leq q$, for every $q \in j[G_{[\bar{\delta},\bar{\lambda})}]$. Thus, p is a master condition in $\mathbb{Q}^*$ for the embedding j and the generic filter $G_{[\bar{\delta},\bar{\lambda})}$. So, if $H \subseteq \mathbb{Q}^*$ is a generic filter over $V_\lambda[G_\delta]$ containing p, then j can be lifted to an elementary embedding

$$j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \longrightarrow V_\lambda[G_\delta * H].$$

Claim 5.5.3.1. *In $V[G]$ there exists some generic filter $H \subseteq \mathbb{Q}^*$ over $V_\lambda[G_\delta]$ containing p such that $V_\lambda[G_\delta * H] = V_\lambda[G_\lambda]$.*

Proof of claim. By (ii) above, $\mathbb{Q}^*$ is a weakly homogeneous class forcing in $V_\lambda[G_\delta]$. Thus, the set of conditions $r \in \mathbb{Q}^*$ for which there is an automorphism π of $\mathbb{Q}^*$ such that $\pi(r) \leq p$ is dense. Therefore there is some such r in $G_{[\delta,\lambda)}$. Now, notice that the filter H generated by the set $\pi[G_{[\delta,\lambda)}]$ contains $\pi(r)$ and therefore it contains p. Since H is definable by means of π and $G_{[\delta,\lambda)}$, we conclude that $V_\lambda[G_\delta * H] = V_\lambda[G_\lambda]$. □

By taking $H \subseteq \mathbb{Q}^*$ as in the claim above, we thus obtain a lifting

$$j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \longrightarrow V_\lambda[G_\lambda]$$

as wanted. □

An immediate corollary of our main preservation result is the following:

Corollary 5.5.4. *Suppose $n \geq 1$, $\mathbb{P}$ is a weakly homogeneous Γ_1-definable suitable iteration, δ is a $C^{(n)}$–extendible cardinal, and there is a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals. Then*

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“}\,\delta \textit{ is } C^{(n)}\textit{–extendible''}.$$

Proof. By Theorem 5.2.3, δ is Σ_{n+1}-supercompact, and by Proposition 5.4.2, since $\mathbb{P}$ is a Γ_1-definable suitable iteration, δ is $\mathbb{P}$-Σ_{n+1}-supercompact. Now, Theorem 5.1.1 applies to get the desired conclusion. □

Let us briefly look into the conditions under which there is a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals (this was one of the assumptions in the statement of Theorem 5.1.1). Recall that $C^{(n+1)}_{\mathbb{P}}$ and

$$C = \{\lambda \mid \forall \eta \geq \lambda \ (\mathbb{1} \Vdash_{\mathbb{P}_\eta} \text{“}\,\dot{\mathbb{Q}}_\eta \text{ is } \lambda\text{-directed closed''})\}$$

are club proper classes (cf. Proposition 5.5.2). Also, if $\mathbb{P}$ is Γ_m-definable, then we have seen (cf. Proposition 5.3.5) that $C^{(n+1)}_{\mathbb{P}}$ is Δ_{m+n}-definable, for $n \geq 1$. Moreover, C is easily seen to be Δ_{m+1}-definable. Further, the unbounded class D of all cardinals κ such that $\mathbb{P}$ forces that $V[\dot{G}]_\kappa \subseteq V_\kappa[\dot{G}_\kappa]$ is Π_{m+1}-definable if $\Gamma = \Sigma$, and Π_{m+2}-definable if $\Gamma = \Pi$ (see the proof of Proposition 5.3.8). Note that every inaccessible cardinal κ that is a limit point of the class is $\mathbb{P}$-reflecting. Thus, we have the following.

Proposition 5.5.5. *If* ORD *is Π_{m+n}-Mahlo (i.e., every Π_{m+n}-definable club proper class of ordinals contains an inaccessible cardinal), in case $\Gamma = \Sigma$ or $n > 1$, or is Π_{m+2}-Mahlo in case $\Gamma = \Pi$ and $n = 1$, then the class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals is proper.*

This yields the following corollary.

Corollary 5.5.6. *Suppose that $1 \leq m, n$ with $m \leq n+1$, $\mathbb{P}$ is a weakly homogeneous Γ_m-definable suitable iteration, and δ is a $\mathbb{P}$-Σ_{n+1}-supercompact cardinal. If* ORD *is Π_{m+n}-Mahlo (case $\Gamma = \Sigma$ or $n > 1$), or Π_{m+2}-Mahlo (case $\Gamma = \Pi$ and $n = 1$), then*

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“}\delta \textit{ is } C^{(n)}\text{–}\textit{extendible}\text{”}.$$

5.6 Applications

In this section we show that Theorem 5.1.1 can be used to obtain several consistency results about $C^{(n)}$–extendible cardinals and VP. For the sake of readability we have divided each particular application in a separate subsection.

5.6.1 Vopěnka's principle and suitable iterations

Recall that VP can be characterized in terms of the existence of a $C^{(n)}$–extendible cardinal, for each $n \geq 1$ (cf. Theorem 1.2.9). Similarly, Vopěnka's Principle can be also characterized in terms of the existence of $\mathbb{P}$-Σ_n-supercompact cardinals, for any Γ_m-definable suitable iteration $\mathbb{P}$. Namely,

Theorem 5.6.1. *The following are equivalent:*

1. VP *holds.*
2. *For every $n, m \geq 1$ and every Γ_m-definable suitable iteration $\mathbb{P}$, there exists a $\mathbb{P}$-Σ_n-supercompact cardinal.*

Proof. (1) ⇒ (2): Let $n, m \geq 1$ and let $\mathbb{P}$ be a Γ_m-definable suitable iteration. By Theorem 1.2.10, (1) implies that there is a proper class of $\mathbb{P}$-Σ_n-reflecting cardinals. Then, again by Theorem 1.2.10 and Corollary 5.4.3, there is a $\mathbb{P}$-Σ_n-supercompact cardinal.

(2) ⇒ (1): If (2) holds, then there exists a $C^{(n)}$-extendible cardinal, for every $n \geq 1$ (Theorem 5.2.3), hence by Theorem 1.2.10, VP holds. □

One also obtains a parametrised version of the last theorem, similar to Theorem 1.2.10, using the following lemma.

Lemma 5.6.2. *Let $n \geq 1$. Then* VP(Π_n) *implies that* ORD *is Σ_{n+1}-Mahlo.*

Proof. Let us prove the lemma for $n > 1$. The case $n = 1$ is similar, using the fact that VP(Π_1) is equivalent to the existence of a supercompact, and that every supercompact cardinal belogs to $C^{(2)}$. So, let $n > 1$ and assume that VP(Π_n) holds. Let κ be a $C^{(n-1)}$-extendible cardinal, which exists by

Theorem 1.2.10. Let $\mathcal{C}$ be a Σ_{n+1}-definable club proper class of ordinals and let $\varphi(x)$ be some Σ_{n+1}-formula defining it. We claim that $\mathcal{C}\cap\kappa$ is unbounded. For if $\alpha < \kappa$, then the sentence "$\exists\beta > \alpha(\beta \in \mathcal{C})$" is Σ_{n+1}, hence it is true in V_κ because κ is $C^{(n-1)}$-extendible and so it belongs to $C^{(n+1)}$ (cf. Proposition 1.2.11). Since $\mathcal{C}$ is closed, $\kappa \in \mathcal{C}$. Since κ is inaccessible the result follows. □

Theorem 5.6.3. *Let $m, n \geq 1$ and let $\mathbb{P}$ be a Γ_m-definable suitable iteration. Then*

1. *If there is a $\mathbb{P}$-Σ_{n+1}-supercompact cardinal then* $\mathrm{VP}(\Pi_{n+1})$ *holds.*
2. *If either $\Gamma = \Sigma$ or $n > 1$, then* $\mathrm{VP}(\Pi_{m+n})$ *implies that there exists a $\mathbb{P}$-Σ_{n+1}-supercompact cardinal.*
3. *If $\Gamma = \Pi$ and $n = 1$, then* $\mathrm{VP}(\Pi_{m+1})$ *holds and* ORD *is Π_{m+2}-Mahlo, then there exists a $\mathbb{P}$-Σ_2-supercompact cardinal*

Proof. Item (1) is a direct consequence of Corollary 5.4.3 and Theorem 1.2.10. On the other hand, Theorem 1.2.10 shows that $\mathrm{VP}(\Pi_{m+n})$ is equivalent to the existence of a $C^{(m+n-1)}$-extendible cardinal, which in turn, by Corollary 5.4.3 (1) and (2), and assuming the existence of a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals, implies that there exists a $\mathbb{P}$-Σ_{n+1}-supercompact cardinal. So, since by the lemma above $\mathrm{VP}(\Pi_{\mathrm{m+n}})$ implies that ORD is Π_{m+n}-Mahlo, by Proposition 5.5.5 we have that in the case $\Gamma = \Sigma$ or $n > 1$, there exists a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals. This shows (2). Finally, (3) also follows from Theorem 1.2.10, Corollary 5.4.3, and Proposition 5.5.5. □

Let us close this section by proving Brooke-Taylor's result on the preservation of Vopěnka's Principle under definable suitable iterations, and also by giving a level-by-level version of it.

Theorem 5.6.4 ([BT11])**.** *Let $\mathbb{P}$ be a weakly-homogeneous definable suitable iteration. If* VP *holds in* V*, then* VP *holds in $V^{\mathbb{P}}$.*

Proof. Let $\mathbb{P}$ be any Γ_m-definable weakly-homogeneous suitable iteration. If VP holds in the ground model V then Theorem 5.6.1 shows that for any $n \geq 1$ there is some $\mathbb{P}$-Σ_{n+1}-supercompact cardinal. Also, from Proposition 5.5.5 and Lemma 5.6.2, there is a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals. For each $n \geq 1$, let us denote by δ_n the least $\mathbb{P}$-Σ_{n+1}-supercompact cardinal. Now applying Theorem 5.1.1 we get that $\mathbb{1} \Vdash_{\mathbb{P}}$ "δ_n is $C^{(n)}$-extendible", for every $n \geq 1$ such that $m \leq n+1$. This implies, by Theorem 1.2.10, that $V^{\mathbb{P}} \models \mathrm{VP}$. □

The following is a level-by-level analogue of Brooke-Taylor's theorem.

Theorem 5.6.5. *Let $n, m \geq 1$ be such that $m \leq n+1$, and let $\mathbb{P}$ be a weakly-homogeneous Γ_m-definable suitable iteration. Then,*

1. *If* $\Gamma = \Sigma$ *or* $n > 1$, *and* $\mathrm{VP}(\Pi_{m+n})$ *holds, then* $\mathrm{VP}(\Pi_{n+1})$ *holds in* $V^{\mathbb{P}}$.

2. *If* $\Gamma = \Pi$ *and* $n = 1$, $\mathrm{VP}(\Pi_{m+1})$ *holds, and* ORD *is* Π_{m+2}*-Mahlo, then* $\mathrm{VP}(\Pi_2)$ *holds in* $V^{\mathbb{P}}$.

Proof. For (1), notice that if $\mathrm{VP}(\Pi_{\mathrm{m+n}})$ holds, then by Lemma 5.6.2 ORD is Π_{m+n}-Mahlo, hence by Proposition 5.5.5 there is a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals. Also, by Theorem 5.6.3 (1) and (2) there exists a $\mathbb{P}$-Σ_{n+1}-supercompact cardinal. So, by Theorem 5.1.1 there is in $V^{\mathbb{P}}$ a $C^{(n)}$–extendible cardinal and hence $\mathrm{VP}(\Pi_{n+1})$ holds in the generic extension. The argument for (2) is similar, using Theorem 5.6.3 (2). □

As the reader may have noticed, our statement of Brooke-Taylor's result differs from the original one in that we require $\mathbb{P}$ to be weakly homogeneous. This additional hypothesis seems to be necessary to carry out the lifting arguments of the proof, for without weak homogeneity there is no guarantee that the master condition lies in a segment of the generic G. However, thanks to the weak homogeneity assumption our proof shows more than Brooke-Taylor's, for it shows that *every* relevant elementary embedding from the ground model lifts to an elementary embedding in the forcing extension. Even though the assumption of weak homogeneity is fulfilled by a wide family of forcing notions, it puts some restrictions on the sort of statements that can be forced. One example, as commented in the introduction to this chapter, is the statement $V = \mathrm{HOD}$. In Section 5.7 we will discuss in more detail the situation arising with non weakly homogeneous iterations. We are very grateful to Andrew Brooke-Taylor for his comments on this matter.

5.6.2 Forcing the GCH and related cardinal configurations

Let $\mathbb{P} = \langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{ORD} \rangle$ be the standard Jensen's proper class iteration for forcing the global GCH. Namely, the direct limit of the iteration with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "α is an uncountable cardinal", then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathrm{Add}(\alpha^+, 1)$", and $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise. It is easily seen that the iteration is weakly homogeneous, suitable, and Π_1-definable.

In [Tsa18] Tsaprounis shows that $\mathbb{P}$ preserves $C^{(n)}$-extendible cardinals. We give next a simpler proof of this result.

Theorem 5.6.6 ([Tsa18]). *Forcing with* $\mathbb{P}$ *preserves* $C^{(n)}$*-extendible cardinals.*

Proof. Let us show first that every inaccessible cardinal λ is $\mathbb{P}$-reflecting. So, suppose G is $\mathbb{P}$-generic over V and $a \in V[G]_\lambda$. As $\mathbb{P}$ preserves the innaccessibility of λ, we have $V[G]_\lambda = H_\lambda^{V[G]}$, and so there exists some $\mu < \lambda$

and some binary relation E on μ such that $\langle \mathrm{TC}(a), \in\rangle \cong \langle \mu, E\rangle$. Since the remaining part of the iteration after stage λ is λ-closed, we have that $E \in V[G_\lambda]$, and since λ is inaccessible, and so the direct limit was taken at stage λ of the iteration, $E \in V[G_\alpha]$, for some $\alpha < \lambda$ such that $|\mathbb{P}_\alpha| = \alpha$. So, since $\mathbb{P}_\alpha$ is α^+-cc, we can easily find a nice $\mathbb{P}_\alpha$-name $\tau \in V_\lambda$ such that $i_{G_\lambda}(\tau) = E$. Thus, we have shown that $E \in V_\lambda[G_\lambda]$, hence by taking the transitive collapse of $\langle \mu, E\rangle$, we obtain $a \in V_\lambda[G_\lambda]$.

Since $\mathbb{P}$ is Π_1-definable, the class $C^{(n+1)}_{\mathbb{P}}$ is Δ_{n+1}-definable (Proposition 5.3.5). If λ is a $C^{(n)}$-extendible cardinal, then $\mathrm{VP}(\Pi_{\mathrm{n+1}})$ holds (Theorem 1.2.10), hence by Lemma 5.6.2 ORD is Π_{n+1}-Mahlo. It follows that there is a proper class of regular cardinals in $C^{(n+1)}_{\mathbb{P}}$. Since every such cardinal is inaccessible, there is a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals, hence Corollary 5.5.4 implies that $\mathbb{P}$ preserves λ being $C^{(n)}$-extendible. $\square$

A classical result of Easton shows that for regular cardinals the value of the power-set function can be (almost) arbitrarily chosen.[6] Namely, a class function E from the class REG of infinite regular cardinals to the class of cardinals is called an Easton function if it satisfies König's theorem (i.e., $\mathrm{cof}(E(\kappa)) > \kappa$, for all $\kappa \in \mathrm{REG}$) and is increasingly monotone. Let $\mathbb{P}_E$ be the direct limit of the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "α is a regular cardinal", then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathrm{Add}(\alpha, E(\alpha))$", and $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise. Standard arguments (see [Jec03]) show that if the GCH holds in the ground model, then $\mathbb{P}_E$ preserves all cardinals and cofinalities and forces that $2^\kappa = E(\kappa)$ for each regular cardinal κ. Moreover, for each regular cardinal λ, the remaining part of the iteration after stage λ is λ-closed. Notice that if the Easton function E is Π_m-definable ($m \geq 1$), then $\mathbb{P}_E$ is also Π_m-definable: If $m = 1$, then $p \in \mathbb{P}_E$ if and only if $M \models$ "$p \in \mathbb{P}_E$", for every transitive model of some big-enough finite fragment of ZFC that contains p. And if $m > 1$, then $p \in \mathbb{P}_E$ if and only if $V_\alpha \models$ "$p \in \mathbb{P}_E$", for every $\alpha \in C^{(m-1)}$ such that $p \in V_\alpha$. Moreover, $\mathbb{P}_E$ is suitable and weakly homogeneous. Similarly as in the proof of theorem 5.6.6 we can now show the following.

Theorem 5.6.7. *If E is a Δ_2-definable Easton function, then $\mathbb{P}_E$ preserves $C^{(n)}$-extendible cardinals, all $n \geq 1$. More generally, if E is a Π_m-definable Easton function ($m > 1$) and λ is $C^{(m+n-1)}$-extendible, then $\mathbb{P}_E$ forces that λ is $C^{(n)}$-extendible, all $n \geq 1$ such that $m \leq n+1$.*

Proof. We argue similarly as in the proof of Theorem 5.6.6. First, as observed above, if E is Δ_2-definable, then so is $\mathbb{P}_E$. Also, every inaccessible λ closed

[6] The situation is completely different in the case of singular cardinals, where there are ZFC upper bounds (e.g., Shelah's bound on $2^{\aleph_\omega}$) or eventually constant behaviour assuming the existence of large cardinals (e.g., Solovay's result that SCH holds above the first strongly compact cardinal).

under E is $\mathbb{P}_E$-reflecting because $\mathbb{P}_E$ preserves the inaccessibility of λ. So, since every inaccessible cardinal in $C^{(2)}$ is closed under E, similarly as in the proof of Theorem 5.6.6 we have that $\mathbb{P}_E$ preserves $C^{(n)}$-extendible cardinals.

In general, if E is Π_m-definable ($m > 1$), then $\mathbb{P}_E$ is also Π_m-definable. Also, the class $C^{(n+1)}_{\mathbb{P}_E}$ is Δ_{m+n}-definable (Proposition 5.3.5). If λ is a $C^{(m+n-1)}$-extendible cardinal, then VP($\Pi_{\mathrm{m+n}}$) holds (Theorem 1.2.10), hence by Lemma 5.6.2 ORD is Π_{m+n}-Mahlo. Thus, there exists a proper class of regular cardinals in $C^{(n+1)}_{\mathbb{P}_E}$. Since every such cardinal is inaccessible and closed under E, there is a proper class of $\mathbb{P}_E$-Σ_{n+1}-reflecting cardinals, hence by Corollary 5.4.3 (3) and Theorem 5.1.1, $\mathbb{P}_E$ preserves λ being $C^{(n)}$-extendible. $\square$

Remark 5.6.8. The last theorem is sharp, in the sense that we cannot hope to prove that $\mathbb{P}_E$ preserves $C^{(n)}$-extendible cardinals for every E. For suppose κ is the least $C^{(n)}$-extendible cardinal. Then the Easton function E that sends $\aleph_0$ to κ and every uncountable regular cardinal λ to $\max\{\lambda^+, \kappa\}$ is Π_{n+2}-definable and destroys κ being inaccessible. In the case $n = 1$ this gives, in fact, an example of a Π_2-definable Easton function E such that $\mathbb{P}_E$ destroys the least extendible cardinal. Indeed, in this case $E(\aleph_0) = \kappa$ if and only if

(i) $\forall\lambda > \kappa\exists\mu > \lambda(\mu$ is a limit ordinal and $V_\mu \models$ "κ is λ-extendible").

(ii) $V_\kappa \models \forall\lambda(\lambda$ is not extendible), and

The point for (ii) is that every extendible cardinal κ belongs to $C^{(3)}$, hence V_κ is correct about the non-extendibility of cardinals $\lambda < \kappa$.

Theorem 5.6.7, together with the equivalence given in Theorem 1.2.9, yield the following.

Corollary 5.6.9. *For every definable Easton function E the class forcing $\mathbb{P}_E$ preserves* VP.

Moreover, by combining theorems 5.6.6 and 5.6.7 we also obtain the following.

Corollary 5.6.10. *If* VP *holds in* V*, then in some class forcing extension of* V *that preserves* VP*, for every definable Easton function E there is a further class forcing extension that preserves* VP *and where for every infinite regular cardinal κ, $2^\kappa = E(\kappa)$.*

Proof. First force with the standard Jensen's iteration for forcing the GCH. Then in the forcing extension, given a definable Easton function E, force with $\mathbb{P}_E$, which by theorem 5.6.7 preserves VP and, since the GCH holds, forces $2^\kappa = E(\kappa)$ for every infinite regular cardinal κ. $\square$

5.6.3 On diamonds

Some of the combinatorial principles that fall within our framework are the so-called Diamond Principles.

Definition 5.6.11. Given an infinite regular cardinal κ, recall that a $\diamondsuit_\kappa$-*sequence* is a sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ of sets $A_\alpha \subseteq \alpha$ such that for every $A \subseteq \kappa$ the set $\{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$ is stationary. We say that $\diamondsuit_\kappa$ holds if there exists a $\diamondsuit_\kappa$-sequence.

One straightforward implication of the existence of such sequences over a regular cardinal κ is that $2^{<\kappa} = \kappa$. In particular, if $\kappa = \lambda^+$, then the existence of a $\diamondsuit_\kappa$-sequence implies that the GCH_λ holds. In general, the implication cannot be reversed (see [Rin11] for a full discussion on this matter). The Diamond Principles were introduced by Jensen, who proved that $\diamondsuit_{\kappa^+}$ holds in L, for every infinite cardinal κ. Among its many applications, $\diamondsuit_{\omega_1}$ was firstly used by Jensen to construct a Suslin tree on ω_1, thereby proving the consistency of the negation of Suslin's Hypothesis.

Jensen also considered a natural strengthening of $\diamondsuit_\kappa$ based on stationary subsets of κ.

Definition 5.6.12. Given a stationary set $S \subseteq \kappa$, a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ is a $\diamondsuit_S$-*sequence* if $A_\alpha \subseteq \alpha$ and for every $A \subseteq \kappa$ the set $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary. We say that $\diamondsuit_S$ holds if there is a $\diamondsuit_S$-*sequence*.

There is a natural forcing notion for adding this kind of sequences. Namely, given an infinite cardinal κ, let $\mathbb{D}_{\kappa^+}$ be the forcing notion whose conditions are functions p with $\mathrm{dom}(p) = \alpha + 1$ for some $\alpha < \kappa^+$ and such that $p(\beta) \subseteq \beta$ for each β in the domain, ordered by: $p \leq q$ if and only if $dom(p) \supseteq dom(q)$ and $p \restriction dom(q) = q$. Using density arguments it is not hard to check that given any generic filter $G \subseteq \mathbb{D}_{\kappa^+}$ over V, the sequence $\bigcup G$ is a $\diamondsuit_{\kappa^+}$-sequence in $V[G]$. Furthermore, $\mathbb{D}_{\kappa^+}$ is κ^+-closed and $(2^\kappa)^+$-cc. In particular, if $2^\kappa = \kappa^+$ holds in the ground model, then forcing with $\mathbb{D}_{\kappa^+}$ preserves all cardinals and cofinalities. It is also straightforward to show that $\mathbb{D}_{\kappa^+}$ is isomorphic to a dense subset of $\mathrm{Add}(\kappa^+, 1)$. Therefore, forcing with Jensen's iteration $\mathbb{P}$ for the GCH produces a model where $\diamondsuit_{\kappa^+}$ holds for every κ. Moreover, it is well-known that forcing with $\mathrm{Add}(\kappa^+, 1)$ automatically forces $\diamondsuit_S$, for every stationary $S \subseteq \kappa^+$ in $V^{\mathrm{Add}(\kappa^+,1)}$. Thus, from Theorem 5.6.6, we have the following.

Corollary 5.6.13. *Let $n \geq 1$ and suppose λ is a $C^{(n)}$-extendible cardinal. Then in $V^{\mathbb{P}}$ the cardinal λ is still $C^{(n)}$-extendible and $\diamondsuit_S$ holds, for every κ and every stationary $S \subseteq \kappa^+$. Hence (Theorem 1.2.9), if* VP *holds in* V, *then it also holds in $V^{\mathbb{P}}$, together with $\diamondsuit_S$, for every κ and every stationary $S \subseteq \kappa^+$.*

There is a further generalization of $\diamondsuit_{\kappa^+}$ called $\diamondsuit^+_{\kappa^+}$.

Definition 5.6.14. A sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa^+ \rangle$ is a $\diamondsuit^+_{\kappa^+}$-sequence if $\mathcal{A}_\alpha \in [\mathcal{P}(\alpha)]^{\leq\kappa}$ and for every $A \subseteq \kappa^+$ there is a club $C \subseteq \kappa^+$ such that

$$C \subseteq \{\alpha < \kappa^+ \mid A \cap \alpha \in \mathcal{A}_\alpha \wedge C \cap \alpha \in \mathcal{A}_\alpha\}.$$

We say that $\diamondsuit^+_{\kappa^+}$ holds if there is a $\diamondsuit^+_{\kappa^+}$-sequence.

Theorem 22 of [CFM01] shows that, assuming $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$, there is a κ^+-closed and κ^{++}-cc forcing notion that forces $\diamondsuit^+_{\kappa^+}$. The forcing is essentially an iteration $\mathbb{D}^+_{\kappa^{++}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha \leq \kappa^{++} \rangle$ with supports of size $\leq \kappa$, where $\mathbb{P}_0$ is the natural forcing notion that introduces a sequence $\vec{\mathcal{A}}$ of the right form to be a $\diamondsuit^+_{\kappa^+}$-sequence whereas the rest of the iterates will force the club sets $C \subseteq \kappa^+$ which will witness that $\vec{\mathcal{A}}$ is indeed a $\diamondsuit^+_{\kappa^+}$-sequence. In particular, $\mathbb{D}^+_{\kappa^{++}}$ is a Π_1-definable forcing as its definition can be rendered within any transitive model M of a big-enough fragment of ZFC. The interested reader may find further details in [CFM01].

Let $\mathbb{D}$ be the standard Easton support iteration of the forcings $\mathbb{D}^+_{\kappa^{++}}$, for any cardinal κ. It is not hard to see that $\mathbb{D}$ is a weakly homogeneous, suitable, and Π_1-definable iteration. Indeed, on the one hand, weak homogeneity follows easily from the very definition of the iteration, whilst suitability comes from the κ^+-directed closedness of the iterates $\mathbb{D}^+_{\kappa^{++}}$. On the other hand, the forcing $\mathbb{D}$ is clearly Π_1-definable as it is the direct limit of a family of Π_1-definable forcings: $p \in \mathbb{D}$ if and only if $M \models$ "$\exists\alpha\,(p \in \mathbb{D}_\alpha)$", for every transitive model M of some big-enough finite fragment of ZFC that contains p. The next result now follows from Corollary 5.5.4 and the argument given at the end of the proof of Theorem 5.6.6.

Theorem 5.6.15. *Let $n \geq 1$ and assume that the* GCH *holds. If λ is a $C^{(n)}$-extendible cardinal, then in $V^{\mathbb{D}}$ the cardinal λ is still $C^{(n)}$-extendible and $\diamondsuit^+_{\kappa^+}$ holds for every cardinal κ. Hence (Theorem 1.2.9), if* VP *and the* GCH *hold in* V*, then* VP *also holds in $V^{\mathbb{D}}$, together with $\diamondsuit^+_{\kappa^+}$, for every cardinal κ.*

5.6.4 On weak square sequences

In this section we prove that $C^{(n)}$–extendible cardinals are consistent with class many instances of weak square at singular cardinals. Our result extends a previous theorem Cummings, Foreman and Magidor [CFM01, Theorem 9.1] to the context of $C^{(n)}$–extendible cardinals (cf. Remark 1.4.7). In the said paper the authors took advantage of the indestructibility phenomenon available at supercompact cardinals to obtain the proof of the result. Since this is not longer true in our context, here we will need to appeal instead to a different aspect: namely, the robustness of $C^{(n)}$–extendible cardinals under suitable class forcing iterations.

Theorem 5.6.16. *There is a class forcing iteration that preserves $C^{(n)}$-extendible cardinals, all $n < \omega$, and forces that for every uncountable cardinal λ, if $K(\lambda)$ is the first singular cardinal of cofinality λ^+, then $\square_{K(\lambda),\lambda^+}$ holds.*

Proof. For each singular cardinal θ, denote by $\mathbb{S}_\theta$ the forcing notion from [CFM01, Theorem 9.1]. Recall that this is the forcing we used for the proof of Theorem 3.1.4 and that it is $\mathrm{cof}(\theta)$-directed closed and $<\theta$-strategically closed. Moreover, if the GCH_θ holds then $|\mathbb{S}_\theta| = \theta^+$ and thus is cofinality-preserving. We now define a class forcing iteration of the forcings $\mathbb{S}_\theta$ which yields the desired result.

By Theorem 5.6.6 we may assume that in our ground model there is a $C^{(n)}$–extendible cardinal and that the GCH holds. Let $\mathbb{P} = \langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ be the iteration with Easton support where $\mathbb{P}_0$ is the trivial forcing and for $\alpha \in \mathrm{ORD}$, if $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\alpha \in \mathrm{Card}$ and $\alpha \geq \aleph_1$", then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{S}}_{K(\alpha)}$", where $K(\alpha)$ is the first singular cardinal of cofinality α^+, and $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \{\mathbb{1}\}$", otherwise. Observe that above there is no confusion between cofinalities in V and cofinalities in $V^{\mathbb{P}_\alpha}$ as we preliminary forced the GCH in V.

The iteration $\mathbb{P}$ is easily seen to be Δ_2-definable, since p is a condition if and only if M thinks p is a condition, for every (for some) transitive Σ_1-correct model M of a sufficiently big fragment of ZFC that contains p. Also, for every α, if $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\alpha \in \mathrm{Card}$ and $\alpha \geq \aleph_1$", then the remaining part of the iteration after stage α is α^+-directed closed, hence $\mathbb{P}$ is suitable. Moreover, the $\mathbb{S}_{K(\alpha)}$ are weakly homogeneous, and hence so is $\mathbb{P}$.

Arguing similarly as in the proof of Theorem 5.6.6 (see also the proof of Theorem 5.6.7), we have that $\mathbb{P}$ preserves $C^{(n)}$–extendible cardinals. Suppose now that λ is an uncountable cardinal in $V^{\mathbb{P}}$. Then λ is also an uncountable cardinal in $V^{\mathbb{P}_\lambda}$, and therefore $\mathbb{1} \Vdash_{\mathbb{P}_\lambda}$ "$\dot{\mathbb{Q}}_\lambda = \dot{\mathbb{S}}_{K(\lambda)}$", hence $\square_{K(\lambda),\lambda^+}$ holds in $V^{\mathbb{P}_{\lambda+1}}$. Since the remaining part of the iteration after stage $\lambda + 1$ is $K(\lambda)^+$-strategically closed, it adds no new bounded subsets of $K(\lambda)^+$, hence it preserves $\square_{K(\lambda),\lambda^+}$. □

From Theorem 1.2.9 we obtain the following corollary.

Corollary 5.6.17. *If* VP *holds, then there is a class forcing iteration that preserves* VP *and forces $\square_{\lambda,\mathrm{cof}(\lambda)}$, for a proper class of singular cardinals λ.*

5.6.5 A remark on Woodin's HOD-Conjecture

The remarkable HOD-Dichotomy theorem of Woodin says that if there exists an extendible cardinal, then either V is close to HOD or is far from it. Specifically, if κ is an extendible cardinal, then either (1): for every singular cardinal $\lambda > \delta$, λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$, or (2): every regular cardinal $\lambda > \kappa$ is ω-strongly measurable in HOD (see [Woo10]). Woodin's HOD-Hypothesis asserts that there is a proper class of regular cardinals that are not ω-strongly measurable in HOD, and therefore that the first option of the HOD-Dichotomy is the true one. Woodin's HOD-Conjecture asserts that the HOD-Hypothesis is provable in the theory ZFC + "There exists an extendible cardinal". Our arguments may be used to show that if the HOD-Conjecture holds, and therefore it is provable in ZFC + "There exists an extendible cardinal" that above the first extendible cardinal every singular cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$, there may still be no agreement at all between V and HOD about successors of regular cardinals. Moreover, many singular cardinals in HOD need not be cardinals in V. Let us give some examples.

For α an infinite regular cardinal, and $\beta > \alpha$, let $\mathrm{Coll}(\alpha, \beta)$ be the corresponding Lévy collapse [Jec03, §26]. Let $\mathbb{P}$ be the direct limit of the iteration $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ with Easton support, where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "α is regular" then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{C}}\mathrm{oll}(\alpha, \alpha^+)$", and $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise.

Theorem 5.6.18. *Forcing with $\mathbb{P}$ preserves $C^{(n)}$-extendible cardinals and forces " $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$, for every regular cardinal λ".*

Proof. For the preservation of $C^{(n)}$-extendible cardinals we may argue as in the proof of Theorem 5.6.6, using the fact that $\mathbb{P}$ preserves inaccessible cardinals, and that it is suitable and weakly homogeneous. To prove the claim about successors of regular cardinals, note that if λ is a regular cardinal in $V^{\mathbb{P}}$, then it was also a regular cardinal at stage λ of the iteration, hence its successor was collapsed at stage $\lambda + 1$. Thus, on the one hand,

$$(\lambda^+)^V < (\lambda^+)^{V^{\mathbb{P}}}.$$

On the other hand, since $\mathbb{P}$ is weakly homogeneous and ordinal definable, $\mathrm{HOD}^{\mathbb{V}^{\mathbb{P}}} \subseteq \mathrm{HOD}^{\mathrm{V}}$ (see, e.g., [Jec03] for details). Thus, in $V^{\mathbb{P}}$,

$$(\lambda^+)^{\mathrm{HOD}} < \lambda^+.$$

□

Corollary 5.6.19. *Forcing with $\mathbb{P}$ preserves* VP *and forces $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$ for every regular cardinal λ.*

Theorem 5.6.18 yields the analogous result to the main theorem from [DF08], at the level of $C^{(n)}$-extendible cardinals. Namely,

Corollary 5.6.20. *Let $n \geq 1$. If the theory "ZFC + There is a $C^{(n)}$-extendible cardinal" is consistent, then it is also consistent with* ZFC *that there exists a $C^{(n)}$-extendible cardinal and $\kappa^+ > (\kappa^+)^{\mathrm{HOD}}$, for every regular cardinal κ.*

Suppose now that K is a function on the class of infinite cardinals such that $K(\lambda) > \lambda$, and K is increasingly monotone, for every λ. Let $\mathbb{P}_K$ be the direct limit of an iteration $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ with Easton support, where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "α is regular" then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Coll}}(\alpha, K(\alpha))$", and $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise. Standard arguments show that $\mathbb{P}_K$ preserves all inaccessible cardinals that are closed under K. Moreover, for each α such that $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "α is regular", the remaining part of the iteration after stage α is α-closed, hence it preserves α. Also note that if K is Π_m-definable ($m \geq 1$), then $\mathbb{P}_K$ is also Π_m-definable. Clearly, $\mathbb{P}_K$ is suitable and weakly homogeneous.

Theorem 5.6.21. *If K is Δ_2-definable, then $\mathbb{P}_K$ preserves $C^{(n)}$-extendible cardinals, all $n \geq 1$. More generally, if K is Π_m-definable ($m > 1$) and λ is $C^{(m+n-1)}$-extendible, then $\mathbb{P}_K$ forces that λ is $C^{(n)}$-extendible, all $n \geq 1$ such that $m \leq n+1$. Moreover, $\mathbb{P}_K$ forces*

$$(\lambda^+)^{\mathrm{HOD}} \leq K(\lambda) < \lambda^+$$

for all infinite regular cardinals λ.

Proof. One can argue similarly as in the proof of Theorem 5.6.7 to show that $\mathbb{P}_K$ preserves $C^{(n)}$-extendible cardinals. If G is $\mathbb{P}_K$-generic over V and λ is regular in $V[G]$, then it is also regular at the λ-stage of the iteration. Hence, $\mathbb{Q}_\lambda = \mathrm{Coll}(\lambda, K(\lambda))$, and therefore $K(\lambda) < \lambda^+$ holds in $V[G]$. The inequality $(\lambda^+)^{\mathrm{HOD}} \leq K(\lambda)$ follows from the fact that $\mathbb{P}_K$ is weakly homogeneous, and thus $\mathrm{HOD}^{\mathrm{V[G]}} \subseteq \mathrm{HOD}^{\mathrm{V}}$. □

The theorem above implies that many kinds of disagreement between successors of regulars in HOD and in V may be forced while preserving $C^{(n)}$-extendible cardinals. It also implies that one can destroy many singular cardinals in HOD while preserving $C^{(n)}$-extendible cardinals. For example, let K be such that $K(\lambda)$ is the least singular cardinal in HOD greater than λ, i.e., $K(\lambda) = (\lambda^{+\omega})^{\mathrm{HOD}}$. It is easily seen that K, and therefore also $\mathbb{P}_K$ as defined above, is Δ_2-definable. Then we have the following.

Corollary 5.6.22. *$\mathbb{P}_K$ preserves $C^{(n)}$-extendible cardinals and forces*

$$(\lambda^{+\omega})^{\mathrm{HOD}} < \lambda^+$$

for every regular cardinal λ.

5.7 Non homogeneous suitable iterations and V = HOD

In this section we follow up the discussion at the end of Section 5.6.1 about non weakly homogeneous suitable iterations. One prominent example is K.

McAloon iteration's [McA71] that forces V = HOD by coding the universe throughout the GCH pattern. This iteration is suitable but not weakly homogenous, as argued in page 68. One may also want to consider class forcing iterations $\mathbb{P}$ over some model M such that $\mathbb{P}$ is not definable in M. To deal with such general class forcing notions we shall work within the theory ZFC_P, namely ZFC with the axiom schemata of Separation and Replacement allowing for formulas in the language of set theory an additional predicate symbol P. Let us next consider the relativization to some predicate P of some of the key notions and results from previous sections.

For $n \geq 1$ and P any class, let $C_P^{(n)}$ be the club class of P-Σ_n-correct cardinals, namely the class of all ordinals α such that

$$\langle V_\alpha, \in, P \cap V_\alpha \rangle \prec_{\Sigma_n} \langle V, \in, P \rangle.$$

The next definition is a natural strengthening of the notion of $C^{(n)}$-extendibility relative to a predicate P.

Definition 5.7.1 (P-$C^{(n)}$-extendible cardinal)**.** For $n \geq 1$, we say that a cardinal δ is P-$C^{(n)}$-*extendible* if for every cardinal $\lambda \in C_P^{(n)}$, $\lambda > \kappa$, there is an ordinal θ and an elementary embedding

$$j : \langle V_\lambda, \in, P \cap V_\lambda \rangle \to \langle V_\theta, \in, P \cap V_\theta \rangle$$

with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$. If, moreover, we can pick θ in $C_P^{(n)}$, then we say that δ is P-$C^{(n)+}$-extendible.

Notice that if P is a Δ_{n+1}-definable class, then every $C^{(n)}$-extendible cardinal is P-$C^{(n)}$-extendible.

Similarly, we may also consider the notion of P-Σ_n-supercompactness, for any class P.

Definition 5.7.2 (P-Σ_n-supercompactness)**.** If $n \geq 1$, then we say that a cardinal δ is P-Σ_n-supercompact if for every $\lambda \in C_P^{(n)}$ greater than δ, and every $a \in V_\lambda$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{a} \in V_{\bar{\lambda}}$, and there exists an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:

- $\mathrm{crit}(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.
- $j(\bar{a}) = a$.
- $\bar{\lambda} \in C_P^{(n)}$.

(cf. Definition 5.4.1).

Then, the same arguments as in the proof of Theorem 5.2.3 yield the following equivalence.

Theorem 5.7.3. *For every $n \geq 1$, every class P, and every cardinal κ, the following are equivalent:*

1. *κ is P-$C^{(n)}$-extendible.*

2. *κ is P-Σ_{n+1}-supercompact.*

3. *κ is P-$C^{(n)+}$-extendible.*

Clearly, any P-$C^{(n)}$-extendible cardinal is $C^{(n)}$-extendible, but the converse need not hold.

We are interested here in the case when the predicate P is a suitable iteration $\mathbb{P}$. Then the notion of $\mathbb{P}$-$C^{(n)}$-extendible cardinal is precisely what is needed to prove the following.

Theorem 5.7.4. *Let $\mathbb{P}$ be a (not necessarily definable) suitable iteration. If δ is a $\mathbb{P}$-$C^{(n)}$-extendible cardinal, and there is a proper class of $\mathbb{P}$-reflecting cardinals, then $\mathbb{P}$ forces that δ is $C^{(n)}$-extendible.*

Proof. Let $\lambda > \delta$ be $\mathbb{P}$-reflecting. It will be sufficient to prove that if G_λ is $\mathbb{P}_\lambda$-generic over V, then in the generic extension $V[G_\lambda]$, the set D of conditions $r \in \mathbb{P}_{[\lambda,Ord)}$ that force the existence of an elementary embedding

$$j : V[G_\lambda][\dot{G}_{[\lambda,Ord)}]_\lambda \to V[G_\lambda][\dot{G}_{[\lambda,Ord)}]_\theta$$

some θ, with $\mathrm{crit}(j) = \delta$, $j(\delta) > \lambda$, and $j(\delta) \in C^{(n)}$, is dense in $\mathbb{P}_{[\lambda,Ord)}$.

So, in $V[G_\lambda]$, let r be a condition in $\mathbb{P}_{[\lambda,Ord)}$. Back in V, let $\mu \in C^{(n)}_{\mathbb{P}}$ be greater than λ and such that

$$\mathbb{1} \Vdash_{\mathbb{P}_\mu} \text{“}\mathbb{P}_{[\mu,Ord)} \text{ is } \lambda^+\text{-directed closed”}.$$

Since δ is $\mathbb{P}$-$C^{(n)^+}$-extendible (Theorem 5.7.3), in the ground model V there exists an elementary embedding

$$j : \langle V_\mu, \in, \mathbb{P} \cap V_\mu \rangle \to \langle V_\theta, \in, \mathbb{P} \cap V_\theta \rangle$$

with critical point δ such that $j(\delta) > \mu$, and $\theta, j(\delta) \in C^{(n)}$.

For each $q \in \mathbb{P}_\lambda$ there is an ordinal $\alpha < \delta$ such that $\mathrm{supp}(q) \cap \delta \subseteq \alpha$. Hence, $\mathrm{supp}(j(q)) \cap j(\delta) \subseteq \alpha$, and so $j(q)$ is a $\mathbb{P}_{j(\lambda)}$-condition such that

$$j(q)(\beta) = \begin{cases} q(\beta) & \text{if} \quad \beta < \alpha. \\ \mathbb{1} & \text{if} \quad \beta \in [\alpha, j(\delta)). \end{cases}$$

Since $\mu < j(\delta)$ we have that $\mathrm{supp}(j(q)) \cap [\lambda, \mu) = \emptyset$. So, by our choice of the ordinal μ, in $V[G_\lambda]$ we can take $r^* \in \mathbb{P}_{[\mu,Ord)}$ such that

$$\mathbb{1} \Vdash_{\mathbb{P}_{[\lambda,\mu)}} \text{“}r^* \leq j(q) \restriction [\mu, j(\lambda))\text{”}$$

for all $q \in G_\lambda$. Then, the condition $r \wedge r^*$ such that

$$r \wedge r^*(\beta) = \begin{cases} r(\beta) & \text{if} \quad \beta \in [\lambda, \mu). \\ r^*(\beta) & \text{if} \quad \beta \in [\mu, j(\lambda)). \end{cases}$$

is well-defined and works as a master condition for j and the forcing $\mathbb{P}_{j(\lambda)}/G_\lambda$, because

$$r \wedge r^* \Vdash_{\mathbb{P}_{j(\lambda)}/G_\lambda} j[G_\lambda] \subseteq \dot{G}_{j(\lambda)}.$$

Thus, for any $\mathbb{P}_{j(\lambda)}$-generic filter $G_{j(\lambda)}$ over V extending G_λ and containing $r \wedge r^*$, the elementary embedding

$$j \restriction V_\lambda : \langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle \to \langle V_{j(\lambda)}, \in, \mathbb{P} \cap V_{j(\lambda)} \rangle$$

lifts to an elementary embedding

$$j^* : \langle V_\lambda[G_\lambda], \in, \mathbb{P} \cap V_\lambda[G_\lambda] \rangle \to \langle V_{j(\lambda)}[G_{j(\lambda)}], \in, \mathbb{P} \cap V_{j(\lambda)}[G_{j(\lambda)}] \rangle.$$

Now, since λ is $\mathbb{P}$-reflecting, $\mathbb{P}$ forces that $V_\lambda[\dot{G}_\lambda] = V[\dot{G}]_\lambda$. Hence, by the choice of μ, the same is forced by $\mathbb{P}_\mu$. By the elementarity of j, the structure $\langle V_\theta, \in, \mathbb{P} \cap V_\theta \rangle$ thinks that the forcing $\mathbb{P} \cap V_\theta$ forces $V_{j(\lambda)}[\dot{G}_{j(\lambda)}] = V[\dot{G}]_{j(\lambda)}$. So, since $\theta \in C^{(n)}_{\mathbb{P}}$, $\mathbb{P}$ forces the same. We have thus found a condition below r, namely $r \wedge r^*$, forcing the existence of an elementary embedding

$$j^* : \langle V[\dot{G}]_\lambda, \in, \mathbb{P} \cap V[\dot{G}]_\lambda \rangle \to \langle V[\dot{G}]_{j(\lambda)}, \in, \mathbb{P} \cap V[\dot{G}]_{j(\lambda)} \rangle$$

with $\mathrm{crit}(j^*) = \delta$, $j^*(\delta) > \lambda$, and $j^*(\delta) \in C^{(n)}$, as wanted. □

5.7.1 $C^{(n)}$–extendible cardinals, $\mathrm{V} = \mathrm{HOD}$ and the Ground Axiom

Assuming the existence of a proper class of inaccessible cardinals, let $\mathbb{P}$ be McAloon class forcing iteration that forces $\mathrm{V} = \mathrm{HOD}$, i.e., $\mathbb{P}$ is the iteration of ORD-length, with Easton support, such that at every stage α of the iteration, if α is inaccessible, then $\dot{\mathbb{Q}}$ is the direct sum[7] of all standard forcing notions that code V_α into the GCH pattern along the next α-many cardinals, and $\dot{\mathbb{Q}}$ is trivial otherwise. It is easily seen that $\mathbb{P}$ is both Σ_2-definable and Π_2-definable: $p \in \mathbb{P}$ if and only if there exists $\alpha \in C^{(1)}$ such that $p \in V_\alpha$ and $V_\alpha \models$ "$p \in \mathbb{P}$", if and only if for all $\alpha \in C^{(1)}$, $V_\alpha \models$ "$p \in \mathbb{P}$".

Corollary 5.7.5. *The standard class forcing $\mathbb{P}$ that forces $\mathrm{V} = \mathrm{HOD}$ preserves $C^{(n)}$-extendible cardinals.*

[7]Recall that given Γ a family of forcing notions the direct sum of Γ, $\bigoplus \Gamma$, is the set $\{\langle \mathbb{P}, p \rangle : \mathbb{P} \in \Gamma, p \in \mathbb{P}\}$ endowed with the order $\langle \mathbb{P}, p \rangle \leq_{\bigoplus \Gamma} \langle \mathbb{Q}, q \rangle$ if and only if $\mathbb{P} = \mathbb{Q}$ and $p \leq_{\mathbb{P}} q$.

Proof. As in the proof of Theorem 5.6.6, every inaccessible cardinal is $\mathbb{P}$-reflecting. Hence, since every $C^{(n)}$-extendible cardinal is $\mathbb{P}$-$C^{(n)}$-extendible, Theorem 5.7.4 yields the desired conclusion. □

Remark 5.7.6. A. Brooke-Taylor [BT11] proves that Vopěnka's Principle (equivalently, the existence of a $C^{(n)}$-extendible cardinal, for every n) is preserved by suitable class forcing iterations. However, the proof does not yield a level-by-level preservation, in the sense that it does not show that $C^{(n)}$-extendible cardinals are preserved. Our Theorem 5.7.4 shows that they are preserved for most suitable iterations, assuming a bit more than $C^{(n)}$-extendibility, namely $\mathbb{P}$- $C^{(n)}$-extendibility.

Recall the following notions from [Reipt] and [HRW08]:

Definition 5.7.7 (Hamkins-Reitz)**.**

1. The *Continuum Coding Axiom*, in symbols CCA, is the assertion that for every $\alpha \in \mathrm{ORD}$ and every $a \in \mathcal{P}(\alpha)$, there is $\theta \in \mathrm{ORD}$ such that $\beta \in a$ if and only if $2^{\aleph_{\theta+\beta+1}} = \aleph_{\theta+\beta+2}$.
2. The *Ground Axiom*, in symbols GA, asserts that the universe of sets V is not a forcing extension of an inner model W by a non-trivial set forcing $\mathbb{P} \in W$.

The very same argument used in Corollary 5.7.5 actually proves the consistency of $C^{(n)}$–extendible cardinals with the CCA, and thus also with the GA (see [Reipt, Theorem 9]).

Corollary 5.7.8. *The standard class forcing iteration $\mathbb{P}$ that forces the* CCA *preserves $C^{(n)}$–extendible cardinals. In particular, $C^{(n)}$–extendible cardinals are consistent with the* GA*.*

This generalizes [Reipt, Theorem 16] to $C^{(n)}$–extendible cardinals. Moreover, $C^{(n)}$–extendible cardinals are consistent with "$\mathrm{V} \neq \mathrm{HOD} + \mathsf{GA}$", which extends [HRW08, Corollary 4] to $C^{(n)}$–extendible cardinals.

Theorem 5.7.9. *There is a class forcing $\mathbb{P}$ that forces "$\mathrm{V} \neq \mathrm{HOD} + \mathsf{GA}$" and preserves $C^{(n)}$–extendible cardinals. In particular, $C^{(n)}$–extendible cardinals are consistent with "$\mathrm{V} \neq \mathrm{HOD} + \mathsf{GA}$".*

Proof. Let $\mathbb{Q}$ be the standard class forcing iteration that forces CCA and let $\dot{\mathbb{R}}$ be a $\mathbb{Q}$-name for the Easton-support iteration which forces with $\mathrm{Add}(\kappa, 1)$ at each regular cardinal κ such that $2^{<\kappa} = \kappa$. Set $\mathbb{P} := \mathbb{Q} * \dot{\mathbb{R}}$. By the argument in [HRW08, Theorem 3], $V^{\mathbb{P}} \models$ "$\mathrm{V} \neq \mathrm{HOD} + \mathsf{GA}$". Combining Theorem 5.6.7 and Corollary 5.7.8 it follows that $\mathbb{P}$ preserves $C^{(n)}$–extendible cardinals. □

CHAPTER 6

A successor cardinal can be $C^{(n)}$–extendible in HOD

Digressing from the previous topic we will take advantage of this chapter to present some other results regarding $C^{(n)}$–extendible cardinals. In this occasion, instead of seeking for preservation results, we want to produce generic extensions where the $C^{(n)}$–extendible cardinals are destroyed, but keep their $C^{(n)}$–extendibility in HOD. Thus, our aim here is to produce generic extensions where the $C^{(n)}$–extendible hierarchies of $V[G]$ and $\mathrm{HOD}^{V[G]}$ are quite different. Certainly this will entail the consistency of stronger disagreements between V and HOD than those obtained in theorems 5.6.18 and 5.6.21. These results are aimed to contribute to the area of Set theory which studies the resemblance between V and HOD, which is currently of great interest [GM18a][BNU17][CFG15][Cum+18][Woo10].

Our main result is the following:

Theorem 6.0.1. *Let $n \geq 1$ and assume that δ is a $C^{(n)}$–extendible cardinal. Then there is a forcing extension $V[G]$ by a poset of size δ where the following hold:*

($\aleph$) *$\delta = \kappa_\omega^+$, where κ_ω is a strong limit cardinal with $\mathrm{cof}(\kappa_\omega) = \omega$;*

($\beth$) *$\mathfrak{K}_{\omega_1} \cap \delta^+ = \emptyset$, i.e. there are no ω_1-strong compact cardinals $\leq\delta$.*

($\gimel$) *for each $x \subseteq \kappa_\omega$ in $V[G]$, $\mathrm{HOD}_x^{V[G]} \models \delta$ is $C^{(n)}$–extendible.*

In particular, the $C^{(n)}$–extendible hierarchy of $V[G]_\delta$ and $\mathrm{HOD}_\delta^{V[G]}$ are completely different. Observe that while $\mathrm{HOD}_\delta^{V[G]}$ is a rich model of ZFC in terms of Large Cardinals and Structural Reflection (see Theorem 1.2.10) $V[G]_\delta$ is not. Actually, this latter is not even a model of ZFC.

Remark 6.0.2. In the conclusions of Theorem 6.0.1 one may also include that $\mathrm{HOD}_x^{V[G]}$ is an intermediate generic extension between V and $V[G]$.

The forcing of Theorem 6.0.1 is the Supercompact Extender Based forcing of [Cum+18], also known as the *AIM forcing.* In Definition 10.2.13 we will define this poset and show that it is Σ-Prikry (cf. Definition 10.1.3). For more details the reader is referred to [Cum+18]. We are very grateful to D. Sinapova for useful explanations about the AIM poset.

Proof of Theorem 6.0.1. Let δ be a $C^{(n)}$–extendible cardinal and $\mathbb{P}$ be the McAloon class iteration that forces V = HOD (see page 94). Since we are assuming the existence of a $C^{(n)}$–extendible, hence the existence of class many inaccessibles, $\mathbb{P}$ encodes each set into the GCH pattern class many times. By Corollary 5.7.5 after forcing with $\mathbb{P}$ the cardinal δ remains $C^{(n)}$–extendible. For simplicity, let us denote this resulting extension by V.

Fix $\langle \kappa_n \mid n < \omega \rangle$ an increasing sequence of supercompact cardinals below δ. Let $\mathbb{Q}$ be the AIM forcing defined with respect to $\langle \kappa_n \mid n < \omega \rangle$ and δ (see Definition 10.2.13). Clearly, $|\mathbb{Q}| = \delta$. Let $G \subseteq \mathbb{Q}$ generic over V. In [Cum+18] the following properties of the generic extension $V[G]$ are proved:

1. $\kappa_\omega := \sup_{n<\omega} \kappa_n$ is a strong limit cardinal;
2. all V-cardinals $\theta \in (\kappa, \delta)$ are collapsed, hence $\delta = (\kappa_\omega^+)^{V[G]}$;
3. for each $x \subseteq \kappa_\omega$ there is $\vec{\alpha} \in {}^\omega[\kappa_\omega, \delta)$ and a forcing poset $\mathbb{Q}_{\vec{\alpha}}$ such that
   (a) $\mathbb{Q}$ projects onto $\mathbb{Q}_{\vec{\alpha}}$ and $|\mathbb{Q}_{\vec{\alpha}}| < \delta$;
   (b) $\mathrm{HOD}_x^{V[G]} \subseteq V[H]$, where H is the generic filter generated by G together with the projection between $\mathbb{Q}$ and $\mathbb{Q}_{\vec{\alpha}}$.

For the reader's benefit let us say that (1) and (2) are consequences of Lemma 4, Corollary 1 and Lemma 14, (3)(a) follows from the mere definition of $\mathbb{Q}_{\vec{\alpha}}$ and Lemma 12, and finally (3)(b) follows from Lemma 15. Observe that (1) and (2) above yield $(\aleph)$.

Claim 6.0.2.1. *For each $G \subseteq \mathbb{Q}$ generic, $V \subseteq \mathrm{HOD}^{V[G]}$.*

Proof of claim. Let $x \in V$ and $\mu > (|\mathrm{TC}(\mathbb{Q})|, \mathrm{rank}(x))$ be an inaccessible cardinal. Notice that such μ exists as $\mathbb{Q}$ is a set forcing and there are proper class many inaccessibles. By definition of the iteration $\mathbb{P}$, the set x is coded into the GCH pattern along the μ-next steps of V_μ so that, since $\mu > |\mathrm{TC}(\mathbb{Q})|$, the encoding of x is absolute between V and $V[G]$. Altogether, $x \in \mathrm{HOD}^{V[G]}$. □

Combining the above claim with (3)(b), for each $x \subseteq \kappa_\omega$ in $V[G]$, the following set of inclusions are true:

$$V \subseteq \mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}_x^{V[G]} \subseteq V[H].$$

By the intermediate generic extension theorem [Jec03, Lemma 15.43], $\mathrm{HOD}_x^{V[G]}$ is a generic extension of V by a complete boolean subalgebra of $\mathbb{Q}_{\vec{\alpha}}$, for some $\vec{\alpha} \in {}^{\omega}[\kappa_\omega, \delta)$. Further, by (3)(a), this Boolean algebra has cardinality $<\delta$, hence δ remains $C^{(n)}$–extendible in $\mathrm{HOD}_x^{V[G]}$, for each $x \subseteq \kappa_\omega$ in $V[G]$. This yields ($\gimel$). Thus we are left with establishing ($\beth$). For this we will use the fact that $\square^*_{\kappa_\omega}$ holds in $V[G]$ (cf. Definition 1.4.1).

Claim 6.0.2.2. *For each $G \subseteq \mathbb{Q}$ generic, $V[G] \models \square^*_{\kappa_\omega}$.*

Proof of claim. For the ease of notation, set $\kappa := \kappa_\omega$. Let

$$\mathscr{C} := \langle \mathcal{C}_\alpha \mid \alpha \in \mathrm{Lim},\, \alpha < \delta \rangle$$

be the sequence such that, for each $\alpha < \lambda$, $\mathcal{C}_\alpha$ is the collection of all V-clubs in α with order-type $\leq\kappa$. If $\mathrm{cof}(\alpha) > \kappa$ observe that $\mathcal{C}_\alpha = \emptyset$. Since δ is inaccessible, clearly $|\mathcal{C}_\alpha| < \delta$. Now, observe that in $V[G]$, $\mathscr{C}$ and $\mathcal{C}_\alpha$ have size κ^+ and $\leq\kappa$, respectively. Moreover, for each $\alpha < \delta = (\kappa^+)^{V[G]}$ limit, if $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{acc}(C)$, $C \cap \beta \in \mathcal{C}_\beta$, as $C \cap \beta$ is a V-club with order-type at most κ. Altogether, $\mathscr{C}$ defines a $\square^*_\kappa$-sequence in $V[G]$.

□

Towards a contradiction, assume that there is a ω_1-strong compact cardinal $\theta \leq \delta$ in $V[G]$. By Theorem 2.1 and Theorem 2.3 of [BM14b], $\theta < \kappa_\omega$. Since κ_ω is singular, Theorem 1.4.9 guarantees that κ_ω carries a scale which, by Theorem 1.4.13, is bad.[1] However, by virtue of Theorem 1.4.12, this collides with the fact that $\square^*_{\kappa_\omega}$ holds in $V[G]$, thus there are no ω_1-strong compact $\leq\delta$. □

Remark 6.0.3. Observe that in $V[G]$ there are many measurable cardinals below κ_ω. Indeed, since κ_ω was a limit of supercompact cardinals there were unboundedly many V-measurables $<\kappa_\omega$ and thus, since $\mathbb{Q}$ does not introduce bounded subsets to κ_ω (see Lemma 10.1.10 or [Cum+18]), this also holds in $V[G]$.

Corollary 6.0.4. *Let $n \geq 1$, and assume that δ is a $C^{(n)}$–extendible cardinal. Then there is a forcing extension of the universe where the following hold:*

1. *if $n = 1$, there is $S \in \mathrm{HOD} \cap \mathcal{P}(\delta)$ a* HOD*-stationary set of $\mathfrak{a}$-$C^{(1)}$-extendible cardinals in* HOD *but no ω_1-strong compact cardinals $\leq\delta$.*

2. *if $n \geq 1$, for each $m < n$, there is $S_m \in \mathrm{HOD} \cap \mathcal{P}(\delta)$ a* HOD*-stationary set of $C^{(n)}$–extendible cardinals in* HOD *but no ω_1-strong compact cardinals $\leq\delta$.*

Proof. (1) follows from Theorem 6.0.1 and Theorem 2.0.6. Similarly, (2) follows from Theorem 6.0.1 and Proposition 3.5 of [Bag12]. □

[1] Observe that $\mathrm{cof}(\kappa_\omega) = \omega$.

Corollary 6.0.5. *Let $n \geq 1$, and assume that δ is a $C^{(n)}$–extendible cardinal. Then there is M a transitive model of* ZFC *where the following are true:*

1. *$\delta = \kappa^+$, where κ is strong limit with $\mathrm{cof}(\kappa) = \omega$;*

2. *there are no ω_1-strong compact cardinals $\leq \delta$;*

3. *$M \models \neg \mathrm{VP}(\Pi_\mathrm{n})$.*

Whereas, for each $x \subseteq \kappa$ in M, HOD_x^M satisfies the following:

(4) $\mathrm{HOD}_x^M \models$ *"δ is $C^{(n)}$–extendible", hence* $\mathrm{HOD}_x^M \models \mathrm{VP}(\Pi_{\mathrm{n+1}})$*;*

(5) $(\mathrm{HOD}_x^M)_\delta \models$ *"* ORD *is $C^{(n)}$–extendible", namely, every club proper class contained in $(\mathrm{HOD}_x^M)_\delta$ contains, either a supercompact cardinal, if $n = 1$, or a $C^{(n-1)}$-extendible cardinal, if $n \geq 2$.*

In particular, if $n = 1$, there are no supercompact cardinals in M, while in HOD_x^M there is an extendible cardinal.

Proof. As in Theorem 6.0.1, assume that δ is $C^{(n)}$–extendible and that $\mathrm{V} = \mathrm{HOD}$. By Lemma 5.6.2, $C^{(n)}$–extendibility implies that ORD is Σ_{n+2}-Mahlo, hence we may pick $\mu > \delta$ the least inaccessible cardinal with $V_\mu \prec_{n+1} V$ above δ. Now let $\mathbb{Q}$ be the forcing from Theorem 6.0.1 and $G \subseteq \mathbb{Q}$ generic. Observe that $V_\mu[G] = V[G]_\mu$, as $|\mathbb{Q}| = \delta < \mu$. For each $x \subseteq \kappa$ in $V[G]$, both $\mathrm{HOD}_x^{V[G]}$ and $V[G]$ think that μ is the first inaccessible above δ which is Σ_{n+1}-correct. For this we use that $\mathrm{HOD}_x^{V[G]}$ is a generic extension of V by a forcing of size $<\mu$.

Claim 6.0.5.1. $V[G]_\mu \models \mathsf{ZFC} + \neg\mathrm{VP}(\Pi_\mathrm{n})$.

Proof of claim. That $V[G]_\mu \models \mathsf{ZFC}$ follows from inaccessibility of μ in $V[G]$. For the second, it is enough to show that in $V[G]_\mu$ there are no supercompacts, if $n = 1$, or $C^{(n-1)}$-extendibles, if $n \geq 2$ (cf. Theorem 1.2.10). Let us reproduce the argument just for $n = 1$ as the other case is analogous. Assume that there was a supercompact cardinal in $V[G]_\mu$. Clearly, if exists, it should in (δ, μ) because of Theorem 6.0.1. But, if μ is the first inaccessible such that $V[G]_\mu \prec_2 V[G]$, there cannot be supercompacts in that interval. □

Now set $M := V[G]_\mu$. Observe that since μ is at least Σ_2-correct in $V[G]$ and the class $\mathrm{HOD}_x^{V[G]}$ is Δ_2-definable with parameter $x \in M$, then $\mathrm{HOD}_x^{V[G]} \cap V[G]_\mu = (\mathrm{HOD}_x^{V[G]})_\mu = \mathrm{HOD}_x^{V[G]_\mu} = \mathrm{HOD}_x^M$. Since δ and μ were respectively $C^{(n)}$–extendible and Σ_{n+1}-correct in $\mathrm{HOD}_x^{V[G]}$,

$$\mathrm{HOD}_x^M \models \text{“}\delta \text{ is } C^{(n)}\text{–extendible.}$$

Thus, (4) follows. From this it is clear that (5) also holds as any $C^{(n)}$–extendible cardinal is a strationary limit of supercompact cardinals, if $n = 1$, or of $C^{(n-1)}$-extendibles, if $n \geq 2$. □

Remark 6.0.6. If $n \geq 2$, in M there are proper class many supercompacts, if $n = 2$, or $C^{(n-2)}$-extendibles, if $n > 2$: Since δ was $C^{(n)}$–extendible the set of supercompacts, if $n = 2$, or $C^{(n-2)}$-extendibles, if $n > 2$, is cofinal in δ. For simplicity, let us assume that $n > 2$ as the other case can be covered analogously. Observe that the statement

$$(\star) \ \ \forall\alpha\exists\beta(\beta > \alpha \wedge \beta \text{ is } C^{(n-2)}\text{-extendible})$$

is Π_{n+2} definable and true in V_δ. Thus, since $V_\delta \prec_{n+1} V_\mu$, it already holds in V_μ. Now, apply the same argument as before observing that $V[G]_\mu \models (\star)$ as the forcing $\mathbb{Q}$ is mild.

Part II

Tree property at successors of singular strong limit cardinals

Introduction

Infinite trees play an essential role in Combinatorial Set Theory and in Set-theoretic Topology [Kun14][Jec03][Tod84]. Given an infinite cardinal κ, a tree $\mathcal{T} = \langle T, \preceq \rangle$ is called a κ-tree if its height is κ and all of its levels are of size $<\kappa$ [Kun14]. If moreover κ is a regular cardinal[2], a κ-tree $\mathcal{T}$ is called κ-Aronszajn if it has no cofinal branches, i.e., no $\preceq$-linearly order subsets of size κ. A regular cardinal κ is said to have the ***Tree Property*** (in symbols, $\mathrm{TP}(\kappa)$) if there are no κ-Aronszajn trees. For economy of the language we will tend to omit the distinction between the pair $\mathcal{T}$ and its underlying set T. In this part we are interested in the following classical question about infinite trees.

Question 6.0.7. Let κ be an infinite regular cardinal. Does $\mathrm{TP}(\kappa)$ hold?

It is worth noticing that $\mathrm{TP}(\kappa)$ is a compactness assertion about κ. Indeed, $\mathrm{TP}(\kappa)$ claims that for every κ-tree T, if every subtree $T' \in [T]^{<\kappa}$ has a branch, then T has also a branch. Since our intuition would lead us to expect such behaviour for any regular cardinal κ, a failure of $\mathrm{TP}(\kappa)$ can be morally conceived as the existence of a *pathological* κ-tree.

A classical result due to J. König is that $\mathrm{TP}(\aleph_0)$ holds. One may expect a similar result for $\aleph_1$ but, as shown by N. Aronszajn, the situation in that case is completely different. Specifically, ZFC proves the existence of a $\aleph_1$-Aronszajn tree and thus $\mathrm{TP}(\aleph_1)$ fails. In the light of these discoverings it seems natural to pursue a similar investigation for bigger regular cardinals.

Question 6.0.8. Does ZFC prove the existence of a κ-Aronszajn tree, for some regular cardinal $\kappa \geq \aleph_2$?

The first partial (negative) answer was given by W. Micthell [Mit72] and J. Silver who showed that "$\mathsf{ZFC} + \mathrm{TP}(\aleph_2)$" is equiconsistent with ZFC plus the existence of a weakly compact cardinal. Using Forcing, W. Mitchell first showed that the consistency of ZFC plus the existence of a weakly compact cardinal yields the consistency of ZFC plus $\mathrm{TP}(\aleph_2)$. The converse implication was later obtained by J. Silver, who proved that if $\mathrm{TP}(\aleph_2)$ holds then $\aleph_2$ is weakly compact in L. Both theorems combined emphasize the need of Large Cardinals to understand the tree property configurations above $\aleph_1$, which contrast with the situation of $\aleph_0$ and $\aleph_1$.

In this part we are particularly interested in the forcing notion introduced by Mitchell in his proof of the above result. Following the tradition of the field we will refer to this forcing as *Mitchell Forcing*.

Given a weakly compact cardinal κ, Mitchell forcing $\mathbb{M}(\kappa)$ yields a generic extension where $\kappa = \aleph_2$, $2^{\aleph_0} = \aleph_2$ and $\mathrm{TP}(\aleph_2)$ holds [Mit72]. Morally, $\mathbb{M}(\kappa)$

[2] The regularity hypothesis is crucial as for singular cardinals it is always possible to construct a κ-Aronszajn tree. For details, see [Kun14, §3.5].

can be seen as the amalgam of two components: the first one intended to blow up the continuum to κ (*Cohen component*) and the second one aimed to collapse the interval $(\aleph_1, \kappa)$ (*Collapsing component*). An important feature of Mitchell's model is the failure of the CH. Rather than being contingent, this failure is mandatory on the basis of a theorem of Specker [Spe90]: if $\kappa^{<\kappa} = \kappa$ then there is a (special) κ^+-Aronszajn tree.[3]

One can be more ambitious and ask whether it is consistent to have the tree property at both $\aleph_2$ and $\aleph_3$. The first result in this direction amounts to 1983 and is due to Abraham [Abr83].

Theorem 6.0.9 (Abraham)**.** *Assume the* GCH *holds. Also, assume that there is a supercompact cardinal* κ *joint with a weakly compact cardinal* λ *above it. Then, there is a generic extension of the universe where* $\kappa = \aleph_2$*,* $\lambda = \aleph_3$ *and both* $\mathrm{TP}(\aleph_2)$ *and* $\mathrm{TP}(\aleph_3)$ *hold. Moreover, in this generic extension* $2^{\aleph_0} = \aleph_2$ *and* $2^{\aleph_1} = \aleph_3$ *hold.*

Prima facie it may seem surprising that for obtaining the consistency of $\mathsf{ZFC} + \mathrm{TP}(\aleph_2) + \mathrm{TP}(\aleph_3)$ one may need to require much stronger hypotheses than those assumed by Mitchell. Especially, bearing in mind that the consistency of $\mathsf{ZFC} + \mathrm{TP}(\aleph_2) + \mathrm{TP}(\aleph_4)$ follows from a straightforward application of Mitchell's arguments to two weakly compact cardinals. But, as M. Magidor observed, both questions are radically different. More precisely, Magidor showed that the consistency of the tree property at two consecutive successors of a regular cardinal implies that $0^\sharp$ exists [Abr83, Theorem 1.1]. Of course that is far beyond from any *Mitchell-like assumption*, and thus each of these problems deserve a completely different treatment.

A decade after, and building on Abraham's ideas, J. Cummings and M. Foreman proved the following theorem [CF98].

Theorem 6.0.10 (Cummings & Foreman)**.** *Assume the* GCH *holds. Also, assume that* $\langle \kappa_n \mid n < \omega \rangle$ *is an strictly increasing sequence of supercompact cardinals. Then, there is a generic extension of the universe where* $\mathrm{TP}(\aleph_n)$ *holds, for each* $2 \leq n < \omega$*.*

Subsequent improvements of this result have been obtained by I. Neeman [Nee14] and S. Unger [Ung16].

The above consistency results provide a non-negligible evidence that the answer to Question 6.0.8 is negative: specifically, that using large cardinals one may obtain a model of ZFC where $\mathrm{TP}(\kappa)$ holds, for each regular cardinal $\kappa \geq \aleph_2$. Nonetheless, the construction of a model bearing this thesis remains as one of the main open challenges of Set Theory.

There is a remarkable connection between the configurations of the continuum function $\aleph_\alpha \mapsto 2^{\aleph_\alpha}$ and the tree property. Indeed, an outright consequence of Specker's theorem [Spe90] is that a global failure of the GCH

[3] For the definition of *special* κ^+-Aronszajn tree see [Kun14, §3.5].

is a necessary condition for a negative solution to Question 6.0.8. The first model of ZFC where the GCH fails at any infinite cardinal was constructed in [FW91] and uses a supercompact cardinal and *Supercompact Radin forcing with interleaved collapses*. Thus, the consistency strength for a model of ZFC where $\mathrm{TP}(\kappa)$ holds, for each regular $\kappa \geq \aleph_2$, is at least the existence of a supercompact cardinal.

In this part we aim to contribute to this collective endeavour and analyze the tree property configurations at the first and double successor of a singular strong limit cardinal. This is closely connected with the *Singular Cardinal Problem* and is part of the major area of research in Set Theory of *Singular Cardinal Combinatorics* [She94][Jec95][Git10][Eis10].

The first investigations in this direction were carried out by J. Cummings and M. Foreman [CF98]. As we will see, the inquiries of these authors have had a remarkable influence upon the subsequent developments of the field. In [CF98] the authors prove the following *Mitchell-like* theorem at the scale of strong limit singular cardinals:

Theorem 6.0.11 (Cummings & Foreman)**.** *Assume the* GCH *holds. Also, assume that there is a supercompact cardinal* κ *joint with a weakly compact cardinal* λ *above it. Then there is a (Mitchell-like) forcing* $\mathbb{R}$ *which yields a generic extension where the following hold:*

1. κ *is a strong limit cardinal with* $\mathrm{cof}(\kappa) = \omega$.
2. $2^\kappa = \kappa^{++} = \lambda$*, hence the* SCH_κ *fails.*
3. $\mathrm{TP}(\kappa^{++})$ *holds.*

Later developments due to S. D. Friedman and A. Halilović [FH11] have shown how to obtain the above theorem for $\kappa = \aleph_\omega$ starting from almost optimal hypotheses. Subsequently, M. Gitik [Git14] refined this result and obtain the exact consistency strength for this theory. For this, Gitik needed an increasing sequence of measurable cardinals $\langle \kappa_n \mid n < \omega \rangle$ with $o(\kappa_n) = \kappa_n^{+n+2}$, joint with a weakly compact cardinal λ above $\kappa_\omega := \sup_{n<\omega} \kappa_n$.

The main novelty of Cummings-Foreman (**CF**) approach is that it provides a general template to combine the Prikry-type technology with Mitchell's arguments. This aspect has been subsequently exploited in [Ung13][Sin16][GM18b][FHS18], where several generalizations of the **CF**-Theorem have been obtained. For instance, in [FHS18] it is showed how to get arbitrary failures of the SCH_κ in the **CF**-model. This is useful to test if one can obtain e.g. $\mathrm{TP}(\kappa^{+3})$ in the said model.

A parallel discussion is concerned with the existence of κ^+-Aronszajn trees at singular strong limit cardinals κ. This problem is intimately connected with the Silver-Prikry proof of the consistency of the failure of SCH_κ at a strong limit cardinal [Pri70]. Assume that κ is a supercompact cardinal and that GCH holds.[4] Firstly, Silver proved that one can force $2^\kappa = \kappa^{++}$ while preserving that κ is supercompact [Cum10, §12]. Secondly, Prikry defined a forcing notion (*Prikry forcing*) such that for a given measurable cardinal κ it produces a cardinal-preserving generic extension where κ is a strong limit singular cardinal with $\mathrm{cof}(\kappa) = \omega$. Combining both arguments one arrives at a model of ZFC where the SCH fails at κ, a singular strong limit cardinal with $\mathrm{cof}(\kappa) = \omega$. Besides, since $\kappa^{<\kappa} = \kappa$ and $\mathbb{P}$ preserves κ^+, $\square^*_\kappa$ holds in Prikry's model. In particular, in this later generic extension both SCH_κ and $\mathrm{TP}(\kappa^+)$ fail (cf. Section 1.4). It is worth noticing that here what is crucial is not the use of Prikry forcing but rather that it preserves κ^+. Thus, this situation is extensible to other Prikry-type posets, such as Magidor forcing [Mag78] or Radin forcing [Rad82].

The natural question is if this is in essence the only possible way to produce a model where the SCH_κ fails. More formally,

Question 6.0.12. If κ is a strong limit singular cardinal with $\mathrm{cof}(\kappa) = \omega$, does the failure of SCH_κ imply a failure of $\mathrm{TP}(\kappa^+)$?

This question was originally posed in 1989 by W. H. Woodin [For05] and remained unanswered for long time. The most decided attempt towards settling Woodin's problem was due to M. Gitik and A. Sharon. For a strong limit singular cardinal κ of countable cofinality, in [GS08] the authors produced a generic extension of $\mathsf{ZFC} + \neg\mathsf{SCH}_\kappa + \neg\square^*_\kappa$. For this they started with the consistency of ZFC with the existence of a $\kappa^{+\omega+2}$-supercompact cardinal κ and used a *Supercompact-type* Prikry poset to produce the desired model. This forcing is nowadays known as the *Gitik-Sharon poset*.

Another relevant property of Gitik-Sharon's (**GS**) model is the existence of a very good κ^+-scale (cf. Definition 1.4.10). Shortly after, Cummings and Foreman [CF] observed that the failure of $\square^*_\kappa$ was actually produced by the existence of a bad scale at κ (cf. Theorem 1.4.12). Thus, in the **GS**-model there are additionally a very good and a bad κ^+-scale.

The construction of a model of $\mathsf{ZFC} + \neg\mathsf{SCH}_\kappa + \mathrm{TP}(\kappa^+)$ finally came from I. Neeman [Nee09], who starting with ω-many supercompact cardinals was able to combine the ideas from [GS08] with the analysis of narrow systems of [MS96] to give rise the desired result. Following up on Neeman's ideas, D. Sinapova latter proved [Sin12] that this result can be extended to arbitrary cofinalities. This author also proved in [Sin16] that a failure of the SCH_κ is consistent with $\mathrm{TP}(\kappa^+)$ and $\mathrm{TP}(\kappa^{++})$. To this aim, Sinapova defined a *Mitchell-like forcing* akin to that of [CF98] but replacing Prikry forcing by the **GS**-poset of [GS08]. A subsequent work of D. Sinapova and S. Unger revealed that the same result can be obtained for $\kappa = \aleph_{\omega^2}$ [SU18].

[4]Actually it suffices with κ being a measurable cardinal with $o(\kappa) = \kappa^{++}$.

Here we will be inspired in the approach taken in [FHS18] to obtain arbitrary failures of the SCH_κ in the **CF**-model. As noticed in the introduction of [FHS18] this is somewhat conflictive with Mitchell's original approach. Generally speaking, if one aims to force a generic extension where $2^\kappa \geq \kappa^{+3}$ and $\mathrm{TP}(\kappa^{++})$ holds, then the *Mitchell-like* forcing from [CF98] will exhibit a mismatch between the lengths of its Cohen and Collapsing component. If this happens, there are many troubles at the time of implementing Mitchell analysis of the quotient forcings (see [Mit72] or [Abr83] for details). The above paper is precisely devoted to show how to surround this difficulty.

In this part we will generalize the main results of [FHS18] and [Sin16].

The generalization of the main theorem [FHS18] was obtained in collaboration with M. Golshani [GP20].

Theorem 6.0.13 (Golshani-P.)**.** *Assume the* $\mathsf{GCH}_{\geq\kappa}$ *holds. Let* κ *be a strong cardinal and* $\lambda > \kappa$ *be a weakly compact. Fix* $\delta < \kappa$ *be regular and* $\Theta \geq \lambda$ *be a cardinal with* $\mathrm{cof}(\Theta) > \kappa$*. Then, there is a generic extension of the universe of sets* V *where the following properties hold:*

1. κ *is a strong limit singular cardinal with* $\mathrm{cof}(\kappa) = \delta$*;*
2. *All cardinals and cofinalities outside* $((\kappa^+)^V, \lambda)$ *are preserved. In particular,* $\lambda = (\kappa^{++})^V$*.*
3. $2^\kappa = \Theta$*, hence the* SCH_κ *fails;*
4. $\mathrm{TP}(\kappa^{++})$ *holds.*

Taking $\delta = \omega$ one obtains the main result of [FHS18].

For the proof of Theorem 6.0.13 we have been inspired by the ideas of [FHS18], where in our context the role of Prikry forcing is played by Magidor forcing. This generalization, as we will see, is arguably not immediate nor trivial. Broadly speaking, the reason is that now we also need to deal with the projection of the measures used in the definition of the forcing. This subtlety will make the analysis of the quotients considerably more involved.

Another worth mentioning aspect of the model of Theorem 6.0.13 is the failure of $\mathrm{TP}(\kappa^+)$ (see Proposition 7.3.34). As mentioned before, this is a consequence of the fact that our forcing does not collapse κ^+. To avoid this failure of the tree property one needs to consider *Supercompact-type Prikry forcings*, which will make the arguments to be considerably more involved.

A Supercompact-type forcing that will be important in Chapter 8 is the *Diagonal Supercompact Magidor forcing* or, shortly, the *Sinapova forcing* [Sin08]. In Chapter 8 we will combine the ideas of [Sin12] and [Sin16] with those developed for the proof of Theorem 6.0.13 and prove the following generalization of the main result of [Sin16].

Theorem 6.0.14. *Assume the* $\mathsf{GCH}_{\geq\kappa}$ *holds. Let* $\mathrm{cof}(\mu) = \mu$ *and* κ *be a supercompact cardinal, with* $\mu < \kappa$. *Assume that there is an increasing and continuous sequence of cardinals* $\langle \kappa_\xi \mid \xi < \mu\rangle$ *with* $\kappa_0 := \kappa$ *and* $\kappa_{\xi+1}$ *being supercompact, for each* $\xi < \mu$. *Besides, assume that there is a weakly compact cardinal* λ *with* $\sup_{\xi<\mu}\kappa_\xi < \lambda$, *and let* $\Theta \geq \lambda$ *be a cardinal with* $\mathrm{cof}(\Theta) > \kappa$. *Then, there is a generic extension of the universe where the following holds:*

1. κ *is a strong limit cardinal with* $\mathrm{cof}(\kappa) = \mu$.
2. *All cardinals and cofinalities* $\geq\lambda$ *are preserved,* $(\sup_{\xi<\mu}\kappa_\xi)^{+V} = \kappa^+$ *and* $\lambda = \kappa^{++}$.
3. $2^\kappa = \Theta$, *hence the* SCH_κ *fails.*
4. $\mathrm{TP}(\kappa^+)$ *and* $\mathrm{TP}(\kappa^{++})$ *hold.*
5. *There is a very good scale and a bad scale at* κ.

Letting $\mu = \omega$ the above yields a generalization of the main result of [Sin16], allowing an arbitrary failure of the SCH. A proof of the above theorem also appears in [Pov20].

An additional word has to be said on the gap between the large-cardinal assumptions necessary for Theorem 6.0.13 and Theorem 6.0.14. This is motivated by the following well-known fact: it is inherently harder to obtain $\mathrm{TP}(\kappa^+)$ versus $\mathrm{TP}(\kappa^{++})$, for singular cardinal κ. Indeed, from [Git14] it is known that the latter needs an increasing sequence of measurable cardinals $\langle \kappa_n \mid n < \omega\rangle$ with $o(\kappa_n) = \kappa_n^{+n+2}$ and a weakly compact cardinal λ above them. Instead, the former requires a failure of $\square^*_\kappa$, which requires stronger large cardinals. Specifically, a failure of weak square at a singular cardinals entails $\mathrm{AD}^{L(\mathbb{R})}$, which yields the existence of an inner model with infinitely many Woodin cardinals [Eis10, Theorem 2.3].

Before tackling the proofs of Theorem 6.0.13 and Theorem 6.0.14 we will provide the reader with a brief, though close to be self-contained, exposition of Magidor and Sinapova forcing. Among the results that we will show in the said sections the reader will find some of them which are originally ours. We are speaking about the geometric characterization of Sinapova generics appearing in [Pov20]. From this we will later show how to define Sinapova generics by means of iterated ultrapowers. These results extend classical theorems due to A. Mathias [Mat73] (resp. W. Mitchell [Mit82]) and R. Solovay [Kan09, Theorem 19.18(a)] (resp. G. Fuchs [Fuc14]) in the context of Prikry forcing (resp. Magidor forcing).

Theorem 6.0.15. *Let* V *be an inner model of* W *and* $\mathbb{S} \in V$. *For a sequence* $g^* \in [\prod_{\xi<\mu} B_\xi] \cap W$, g^* *is* $\mathbb{S}$*-generic over* V *if and only if the following properties hold:*

1. *For each sequence $H \in V \cap \prod_{\xi<\mu} U_\xi$, there is $\xi_H < \mu$ such that for all ordinal $\eta \in (\xi_H, \mu)$, $g^*(\eta) \in H(\eta)$.*

2. *For each $\xi < \mu$ limit and each $H \in V \cap \prod_{\theta<\xi} U^\theta_{\xi, g^*(\xi)}$, there is $\xi_H < \xi$ such that for all ordinal $\eta \in (\xi_H, \xi)$, $g^*(\eta) \in H(\eta)$.*

We close this part with a brief chapter analysing to what extend the presence of Large Cardinals force the universe of sets to exhibit certain tree property configurations. In this regard, we will be particularly interested in the classical Magidor-Shelah theorem on the tree property at the successor of $\kappa_\omega := \sup_{n<\omega} \kappa_n$, where $\langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of strong compact cardinals [MS96]. In Chapter 9 we will weaken the large-cardinal hypotheses nedeed for this theorem to be true. Specifially, we will prove the following:

Theorem 6.0.16. *Let $\mathcal{K} := \langle \kappa_n \mid n < \omega \rangle$ and $\mathcal{D} := \langle \delta_n \mid n < \omega \rangle$ be two sequences of cardinals for which the following hold:*

1. *$\omega_1 \leq \delta_0$;*

2. *$\delta_n \leq \kappa_n < \delta_{n+1}$;*

3. *κ_n is the first δ_n-strong compact cardinal.*

Set $\kappa_\omega := \sup_{n<\omega} \kappa_n$ and $\Theta := \kappa_\omega^+$. Then, $\mathrm{TP}(\Theta)$ *holds.*

Through this part we will rely on the following standard convention.

Convention 6.0.17. If $\mathbb{P}$ is a forcing notion and $p \in \mathbb{P}$, we will denote by $\mathbb{P} \downarrow p$ the set of conditions in $\mathbb{P}$ below p.

CHAPTER 7

The Tree Property at double successors of singular cardinals

7.1 An introduction to Magidor forcing

One of the main forcing tools of the present chapter is the so-called *Magidor forcing*. This poset was originally introduced by Magidor in [Mag78] and is devised to change the cofinality of a measurable cardinal κ with $o(\kappa) = \delta$ to some regular cardinal $\delta' \leq \delta$.[1] Thus, in a broad sense, Magidor forcing can be conceived as a more general form of Prikry forcing [Pri70].

Magidor's original approach to this forcing was based on $\lhd$-increasing sequences of measures $\mathcal{U} = \langle U_\alpha \mid \alpha < \delta\rangle$ (see [Mit10, Definition 2.2]). Nonetheless, later developments of Mitchell suggested to use *Coherent Sequences of Measures* instead (c.f. Definition 7.1.1). This is due to the fact that these sequence of measures can be also used to define more sophisticated forcings, such as Radin forcing [Rad82]. In this section we will follow Mitchell's approach to Magidor forcing.

The purpose of this section is just to review the definition and main properties of Magidor forcing. This will become important in Section 7.2 and Section 7.3 where we exchange Prikry by Magidor forcing in Cummings-Foreman poset $\mathbb{R}$ [CF98]. For the reader's benefit we will provide a self-contained approach to the main aspects of this forcing. More details can be found in Magidor's original paper [Mag78] or in Gitik's excellent article [Git10, §5] with the exception of Lemma 7.1.20 and Lemma 7.1.21.

Definition 7.1.1 (Coherent sequence)**.** A coherent sequence of measures $\mathcal{U}$ is a function with domain $\{(\alpha,\beta) \mid \alpha < \ell^{\mathcal{U}} \text{ and } \beta < o^{\mathcal{U}}(\alpha)\}$ such that for $(\alpha,\beta) \in dom(\mathcal{U})$ the following conditions are true:

1. $\mathcal{U}(\alpha,\beta)$ is a normal measure over α;

[1] For a definition of $o(\kappa)$ see [Mit10, §2].

2. If $j^\alpha_\beta : V \longrightarrow \mathrm{Ult}(V, \mathcal{U}(\alpha,\beta))$ stands for the usual ultrapower embedding, then $j^\alpha_\beta(\mathcal{U}) \restriction \alpha+1 = \mathcal{U} \restriction (\alpha,\beta)$, where $\mathcal{U} \restriction \alpha := \mathcal{U} \restriction \{(\alpha',\beta') \mid \alpha' < \alpha \,\&\, \beta' < o^{\mathcal{U}}(\alpha')\}$ and
$$\mathcal{U} \restriction (\alpha,\beta) := \mathcal{U} \restriction \{(\alpha',\beta') \mid (\alpha' < \alpha \,\&\, \beta' < o^{\mathcal{U}}(\alpha')) \text{ or } (\alpha = \alpha' \,\&\, \beta' < \beta)\}.$$

The ordinals $\ell^{\mathcal{U}}$ and $o^{\mathcal{U}}(\alpha)$ are called respectively the length of $\mathcal{U}$ and the Mitchell order at α of $\mathcal{U}$.

Definition 7.1.2. Let $\mathcal{U} = \langle \mathcal{U}(\alpha,\beta) \mid \alpha \leq \kappa, \beta < o^{\mathcal{U}}(\alpha)\rangle$ be a coherent sequence of measures with $\ell^{\mathcal{U}} = \kappa+1$ and $o^{\mathcal{U}}(\kappa) = \delta$. For each $\alpha < \ell^{\mathcal{U}}$, define $\mathcal{F}_{\mathcal{U}}(\alpha) := \bigcap_{\beta < o^{\mathcal{U}}(\alpha)} \mathcal{U}(\alpha,\beta)$, if $o^{\mathcal{U}(\alpha)} > 0$, and otherwise, set $\mathcal{F}_{\mathcal{U}}(\alpha) := \{\emptyset\}$.

Observe that $\mathcal{F}_{\mathcal{U}}(\alpha)$ is the set of all subsets of α which are measure one with respect to all the measures $\mathcal{U}(\alpha,\beta)$, $\beta < o^{\mathcal{U}}(\alpha)$. It is fairly easy to check that in the non-trivial case where $o^{\mathcal{U}}(\alpha) > 0$, $\mathcal{F}_{\mathcal{U}}(\alpha)$ yields an α-complete normal filter over α. In the cases where $\mathcal{U}$ is clear from the context we will tend to omit its mention when referring to $\mathcal{F}_{\mathcal{U}}(\alpha)$.

Definition 7.1.3 (Magidor forcing)**.** Let $\mathcal{U}$ be a coherent sequence of measures with $\ell^{\mathcal{U}} = \kappa+1$ and $o(\kappa) = o^{\mathcal{U}}(\kappa) = \delta$.

(a) Magidor forcing relative to $\mathcal{U}$, denoted by $\mathbb{M}_{\mathcal{U}}$, consists of all finite sequences of the form $p = \langle\langle \alpha^p_0, A^p_0\rangle, \dots, \langle \alpha^p_n, A^p_n\rangle\rangle$ where:

- (א) $\delta < \alpha_0 < \cdots < \alpha_n = \kappa$,
- (ב) $A^p_i \in \mathcal{F}(\alpha^p_i)$,
- (ג) $A^p_i \cap (\alpha^p_{i-1}+1) = \emptyset$ (where, $\alpha^p_{-1} := \delta+1$).

The sequence $\langle \alpha^p_0, \dots, \alpha^p_{n-1}\rangle$ is called the *stem* of p and the integer n^p the *length* of p. Whenever the condition is clear from the context we shall tend to suppress the corresponding superscript.

(b) For $p = \langle\langle \alpha_0, A_0\rangle, \dots, \langle \alpha_n, A_n\rangle\rangle$ and $q = \langle\langle \beta_0, B_0\rangle, \dots, \langle \beta_m, B_m\rangle\rangle$ be two conditions in $\mathbb{M}_{\mathcal{U}}$ we will say that q is stronger than p ($q \leq p$) if the following conditions are fulfilled:

- (א) $m \geq n$,
- (ב) $\forall i \leq n \;\; \exists j \leq m \;\; \alpha_i = \beta_j$ and $B_j \subseteq A_i$,
- (ג) For all j be such that $\beta_j \notin \{\alpha_1, \dots, \alpha_n\}$, $\;B_j \subseteq A_k \cap \beta_j$ and $\beta_j \in A_k$, where $k := \min\{k \leq n \mid \beta_j < \alpha_k\}$.

(c) q is a direct extension or a Prikry extension of p ($q \leq^* p$) if $q \leq p$ and $m = n$.

In the cases where $\mathcal{U}$ is clear from the context, we will tend to write $\mathbb{M}$ rather than $\mathbb{M}_{\mathcal{U}}$. Given a condition $p \in \mathbb{M}$ we will write

$$p := \langle\langle\alpha_0^p, A_0^p\rangle, \dots, \langle\alpha_{n^p-1}^p, A_{n^p-1}^p\rangle, \langle\alpha_{n^p}^p, A_{n^p}^p\rangle\rangle.$$

If $p, q \in \mathbb{M}_{\mathcal{U}}$ are two conditions with the same stem, we define $p \wedge q := \langle\langle\alpha_0^p, A_0^p \cap A_0^q\rangle, \dots, \langle\alpha_{n^p}^p, A_{n^p}^p \cap A_{n^q}^q\rangle\rangle$.

Definition 7.1.4. Let p be a sequence witnessing clauses ($\aleph$) and ($\beth$) of Definition 7.1.3(a). For $i \leq n^p$ and $\alpha \in A_i^p$, define $p^\frown\langle\alpha\rangle$ as the sequence $\langle\langle\alpha_0^p, A_0^p\rangle \dots \langle\alpha_{i-1}^p, A_{i-1}^p\rangle, \langle\alpha, A_i^p \cap \alpha\rangle, \langle\alpha_i^p, A_i^p\rangle, \dots \langle\kappa, A_{n^p}^p\rangle\rangle$. For a sequence $\vec{\alpha} \in [A_i^p]^{<\omega}$, define by recursion $p^\frown\vec{\alpha} := (p^\frown(\vec{\alpha} \restriction |\vec{\alpha}|)^\frown\langle\vec{\alpha}(|\vec{\alpha}|)\rangle$.[2]

Remark 7.1.5. Observe that not for all $\alpha \in A_i^p$, $p^\frown\langle\alpha\rangle \in \mathbb{M}$, since it may be the case that $\alpha \cap A_i^p \notin \mathcal{F}(\alpha)$. Actually, $p^\frown\langle\alpha\rangle \in \mathbb{M}$ if and only if $A_i^p \cap \alpha \in \mathcal{F}(\alpha)$.

Definition 7.1.6. Let $p \in \mathbb{M}$. A finite sequence of ordinals $\vec{x}$ is a *block sequence* for p if $\vec{x} \in [\biguplus_{i \leq n^p} A_i^p]^{<\omega}$. For each $i \leq n^p$, set $\vec{x}_i := \vec{x} \cap A_i^p$. Also, let $i_{\vec{x}}$ be denote an enumeration of the $i \leq n^p$ for which $\vec{x}_i \neq \emptyset$.

Here we use the symbol $\biguplus$ rather than $\bigcup$ just to emphasize the fact that the sets A_i^p and A_j^p are disjoint, for $i \neq j$.

Definition 7.1.7 (Minimal extensions of $\mathbb{M}$)**.** Let $p \in \mathbb{M}$ and $\vec{x}$ be a block sequence for p. Mimicking Definition 7.1.4, we define recursively $p^\frown\vec{\alpha} := (p^\frown\vec{x} \setminus \cup_{j<i}\vec{x}_j)^\frown\vec{x}_i$, where $i = \max i_{\vec{x}}$.

Definition 7.1.8 (Pruned condition)**.** A condition $p \in \mathbb{M}$ is said to be pruned if for every $\vec{x} \in [\biguplus_{i \leq n^p} A_i^p]^{<\omega}$, $p^\frown\vec{x} \in \mathbb{M}$.

Proposition 7.1.9. *Let $p \in \mathbb{M}$. Then, p is a pruned condition if and only if $p^\frown\langle\alpha\rangle \in \mathbb{M}$, for all $\alpha \in \biguplus_{i \leq n^p} A_i^p$.*

Proof. The first implication is obvious and the second follows easily from the recursive definition of $p^\frown\vec{x}$. □

One can use the previous result to show that any condition $p \in \mathbb{M}$ has a $\leq^*$-extension which is pruned.

Proposition 7.1.10. *For each $p \in \mathbb{M}$, there is $p^* \leq^* p$ which is pruned.*

Proof. Fix $p \in \mathbb{M}$. Without loss of generality assume that all large sets in p are non empty, as otherwise the argument is similar. For each $i \leq n^p$, set $A_i^{*,0} := A_i^p$, $A_i^{*,n+1} := \{\alpha \in A_i^{*,n} \mid A_i^{*,n} \cap \alpha \in \mathcal{F}(\alpha)\}$ and $A_i^* := \bigcap_{n<\omega} A_i^{*,n}$.

[2] Here by convention $p^\frown\emptyset := p$.

Claim 7.1.10.1. *For each $i \leq n^p$ and $n < \omega$, $A_i^{*,n} \in \mathcal{F}(\alpha_i^p)$. In particular, for each $i \leq n^p$, $A_i^* \in \mathcal{F}(\alpha_i^p)$.*

Proof of claim. If the first assertion is true then the second follows automatically from the α_i^p-completeness of $\mathcal{F}(\alpha_i^p)$. Thus, we want to argue by induction that $A_i^{*,n} \in \mathcal{F}(\alpha_i^p)$, for $n < \omega$. Clearly this is true for $n = 0$. For the inductive step, let $\beta < o^{\mathcal{U}}(\alpha_i^p)$ and observe that $\alpha_i^p \in j_\beta^{\alpha_i^p}(A_i^{*,n})$ and $j_\beta^{\alpha_i^p}(A_i^{*,n}) \cap \alpha_i^p = A_i^{*,n} \in \mathcal{F}(\alpha_i^p)$, so $A_i^{*,n+1} \in \mathcal{F}(\alpha_i^p)$. □

Let p^* be the $\leq^*$-extension of p with $A_i^{p^*} = A_i^*$, each $i \leq n^p$. We claim that p^* is pruned. For showing this we use Proposition 7.1.9. Let $\alpha \in \biguplus_{i \leq n^p} A_i^*$ and say that $\alpha \in A_i^*$. By construction it is easy to check that $A_i^{*,n} \cap \alpha \in \mathcal{F}(\alpha)$, for all $n < \omega$, hence $A_i^* \cap \alpha \in \mathcal{F}(\alpha)$. Finally, Remark 7.1.5 yields $p^\frown\langle\alpha\rangle \in \mathbb{M}$. □

We will take advantage of the previous result when we will analyze the quotient forcings $\mathbb{R}/\mathbb{R} \upharpoonright \xi$ at Section 7.3. Let us now address the question of cardinals preservation in Magidor extensions.

Proposition 7.1.11. $\mathbb{M}$ *is κ^+-Knaster. That is, any set $S \in [\mathbb{M}]^{\kappa^+}$ contains a set $\mathcal{I} \in [S]^{\kappa^+}$ of compatible conditions.*

Proof. Let $\{p_\alpha \mid \alpha < \kappa^+\}$ be an enumeration of S. For each $\alpha < \kappa^+$, let s_α be the stem of p_α and define $\varphi : \kappa^+ \to [\kappa]^{<\omega}$ as $\varphi(\alpha) := s_\alpha$. By counting arguments, there is $\mathcal{I} \in [S]^{\kappa^+}$ and $s^* \in [\kappa]^{<\omega}$ for which $\varphi[\mathcal{I}] = \{s^*\}$. Let $\alpha, \beta \in \mathcal{I}$ and observe that $p_\alpha \wedge p_\beta \leq_{\mathbb{M}} p_\alpha, p_\beta$, as wanted. □

In particular the above implies that all cardinals $\geq\kappa^+$ are preserved after forcing with $\mathbb{M}$. Let us now describe the combinatorics of the generic extensions by $\mathbb{M}$ below κ^+. Hereafter we will assume that δ is an infinite cardinal and that $G \subseteq \mathbb{M}$ is a generic filter over V. Set

$$C_G := \{\alpha < \kappa \mid \exists p \in G \, \exists n < n^p \, (\alpha = \alpha_n^p)\}.$$

One may argue as in [Git10, Lemma 5.10] that, below a direct extension of $\mathbb{1}_{\mathbb{M}}$, C_G is a closed unbounded subset of κ of order type ω^δ, which will be referred as the Magidor club induced by G.[3] In particular, there is a direct extension of $\mathbb{1}_{\mathbb{M}}$ which forces $\mathrm{cof}(\check{\kappa}) = \mathrm{cof}^V(\check{\delta})$. However, notice that it is still necessary to prove that $\mathrm{cof}^V(\delta)$ and κ have not been collapsed after adding this generic club. As usual, the key property that provides us of the necessary control on the combinatorics of $V_\kappa^{\mathbb{M}}$ is the Prikry property.

Proposition 7.1.12 (Prikry property). $\langle \mathbb{M}, \leq^* \rangle$ *satisfies the Prikry property: namely, for each sentence φ in the language of forcing and a condition $p \in \mathbb{M}$, there is $q \leq^* p$ such that $q \parallel \varphi$.*

[3]Here ω^δ stands for ordinal exponentiation rather than cardinal exponentiation.

There is a remarkable feature concerning the structure of Magidor forcing which we would like to mention. This property is the following: for a condition $p \in \mathbb{M}$, the forcing $\mathbb{M} \downarrow p$ is isomorphic to a product $\mathbb{M}_1 \times \mathbb{M}_2$, where $\mathbb{M}_1$ and $\mathbb{M}_2$ are two Magidor forcing. This *fractal structure* is actually shared with other classical Prikry-type forcing that center around uncountable cofinalities, such as Radin forcing [Rad82] or Sinapova forcing (cf. Proposition 8.1.9). Here $\mathbb{M}_1$ is a Magidor forcing adding a club subset of θ, where θ is a point in the Magidor club $C_{\dot{G}}$ and $\dot{G}$ is a $\mathbb{M} \downarrow p$-name for a generic filter. On the other hand, $\mathbb{M}_2$ is essentially the Magidor forcing $\mathbb{M}$, though this time adding a generic club on κ of points above θ. Let us phrase this in more formal terms.

Definition 7.1.13. Let $p \in \mathbb{M}$ and $m \leq n^p$. We will respectively denote by $p^{\leq m}$ and $p^{>m}$ the sequences

$$p^{\leq m} := \langle\langle \alpha_0^p, A_0^p\rangle, \dots, \langle \alpha_m^p, A_m^p\rangle\rangle,$$
$$p^{>m} := \langle\langle \alpha_{m+1}^p, A_{m+1}^p\rangle, \dots, \langle \alpha_{n^p}^p, A_{n^p}^p\rangle\rangle.$$

If $m < n^p$, is immediate that $p^{\leq m} \in \mathbb{M}_{\mathcal{U}\restriction\alpha_m^p+1}$ and $p^{>m} \in \mathbb{M}_{\mathcal{U}}$.

Lemma 7.1.14. *Let $p \in \mathbb{M}$ and $m < n^p$. There is an isomorphism between $\mathbb{M}_{\mathcal{U}} \downarrow p$ and $\mathbb{M}_{\mathcal{U}\restriction\alpha_m^p+1} \downarrow p^{\leq m} \times \mathbb{M}_{\mathcal{U}} \downarrow p^{>m}$. In particular, $\mathbb{M}_{\mathcal{U}} \downarrow p$ projects onto $\mathbb{M}_{\mathcal{U}\restriction\alpha_m^p+1} \downarrow p^{\leq m}$.*

Proof. For a condition $q \leq p$, let q^- be the initial segment of q for which α_m^p is the last ordinal with $\langle \alpha_m^p, A\rangle \in q^-$, for some $A \in \mathcal{F}(\alpha_m^p)$. Analogously, let q^+ be the sequence such that $q^- {}^\frown q^+ = q$. It is routine to check that $q \mapsto \langle q^-, q^+\rangle$ yields the desired isomorphism. □

Let $p \in \mathbb{M}$ and $\alpha < \kappa$. We will say that α appears in p if for some $m < n^p$ and $A \in \mathcal{F}(\alpha)$, $p(m) = \langle \alpha, A\rangle$. Analogously, define the notion "α appears in p at m". In a mild abuse of notation, let us write $\langle \mathbb{M}_{\mathcal{U}} \downarrow p, \leq^*\rangle$ for the subforcing of $\mathbb{M}_{\mathcal{U}}$ consisting of conditions $\leq^*$-below p.

Lemma 7.1.15. *In the conditions of the above lemma, $\mathbb{M}_{\mathcal{U}\restriction\alpha_m^p+1}$ is $(\alpha_m^p)^+$-Knaster and $\langle \mathbb{M}_{\mathcal{U}} \downarrow p^{>m}, \leq^*\rangle$ is $\beta^{p^{>m}}$-closed, where $\beta^{p^{>m}} := \min\{\alpha \leq \kappa \mid \alpha_m^p < \alpha,\ o^{\mathcal{U}}(\alpha) > 0,\ \alpha$ appears in $p\}$. In particular, for each $\varrho < \alpha_{m+1}^p$, the poset $\langle \mathbb{M}_{\mathcal{U}} \downarrow p^{>m}, \leq^*\rangle$ is $|\varrho \times \mathbb{M}_{\mathcal{U}\restriction\alpha_m^p+1}|^+$-closed.*

Proof. The first claim is consequence of Proposition 7.1.11. For the second, observe that $\beta := \beta^{p^{>m}}$ always exists, as κ is always part of the defining set. Let $\gamma < \beta$ and $\langle p_\alpha \mid \alpha < \gamma\rangle$ be a $\leq^*$-decreasing sequence of conditions $\leq^*$-below $p^{>m}$. Observe that there is a sequence s such that $s(i) = \langle \gamma_i, \emptyset\rangle$, $i < |s|$, and $p_\alpha = s^\frown q_\alpha$, for each $\alpha < \gamma$. Of course, $|s|$ may be 0. In any case, notice that q_α is a sequence of pairs where all the filters involved are β-complete. Thus, $p^* := s^\frown \bigwedge_{\alpha<\gamma} q_\alpha$ provides the desired lower bound. The last claim follows from noticing that β is a measurable $\geq \alpha_{m+1}^p$. □

By combining Proposition 7.1.12, Lemma 7.1.14 and Lemma 7.1.15 one can obtain a complete picture of the combinatorics of $V[G]_\kappa$. For convenience, let $\langle \kappa_\alpha \mid \alpha < \theta \rangle$ be an enumeration of the generic club induced by the generic filter G.

Proposition 7.1.16. *Work in $V[G]$. For each ordinal $\varrho < \kappa$, set $\alpha_\varrho := \min\{\alpha < \theta \mid \kappa_\alpha \leq \varrho < \kappa_{\alpha+1}\}$ and let $p \in G$ where both κ_{α_ϱ} and $\kappa_{\alpha_{\varrho+1}}$ appear at m and $m+1$, respectively. Then,*

$$\mathcal{P}(\varrho)^{V[G]} = \mathcal{P}(\varrho)^{V[G_{\alpha_\varrho}]},$$

where G_{α_ϱ} is the filter generated by G joint with the natural projection between $\mathbb{M} \downarrow p$ and $\mathbb{M}_{\mathcal{U} \restriction \kappa_{\alpha_\varrho}+1} \downarrow p^{\leq m}$.

Proof. Let $\varrho < \kappa$ and $\tau_G \in \mathcal{P}(\varrho)^{V[G]}$. Pick $p \in G$ such that $p \Vdash_{\mathbb{M}_\mathcal{U}} \tau \subseteq \check{\varrho}$. By extending if necessary, we may further assume that p is as in the above statement. Let σ be the $\mathbb{M}_\mathcal{U} \downarrow p^{>m}$-name for a subset of $\varrho \times \mathbb{M}_{\mathcal{U} \restriction \kappa_{\alpha_\varrho}+1}$ defined as follows:

$$\sigma := \{((\check{\theta}, r), s) \mid \theta < \varrho,\ r \in G_{\alpha_\varrho},\ (r, s) \Vdash_{\mathbb{M}_\mathcal{U}} \check{\theta} \in \tau\}.$$

Set $\mu := |\varrho \times \mathbb{M}_{\mathcal{U} \restriction \kappa_{\alpha_\varrho}+1}|$ and let $\langle x_\theta \mid \theta < \mu \rangle$ be an enumeration of the set $\varrho \times \mathbb{M}_{\mathcal{U} \restriction \kappa_{\alpha_\varrho}+1}$. Clearly, $\mu < \beta^{p^{>m}}$ as this latter is a measurable cardinal $\geq \kappa_{\alpha_{\varrho+1}}$. Using Proposition 7.1.12 and Lemma 7.1.15 one may easily define a $\leq^*$-decreasing sequence $\vec{q} = \langle q_\theta \mid \theta < \mu \rangle$ of conditions below $p^{>m}$ such that, for each $\theta < \mu$, $q_\theta \parallel^{V[G_{\alpha_\varrho}]}_{\mathbb{M}_\mathcal{U}}$ "$x_\theta \in \tau$". Indeed this is possible by appealing to the Prikry property (c.f. Proposition 7.1.12) at successor stages, and at limit combine Lemma 7.1.15 with the Prikry property. Once again, using Lemma 7.1.15 we may let q^* be a $\leq^*$-lower bound for $\vec{q}$. Set

$$a := \{x \in \varrho \times \mathbb{M}_{\mathcal{U} \restriction \kappa_{\alpha_\varrho}+1} \mid q^* \Vdash^{V[G_{\alpha_\varrho}]}_{\mathbb{M}_\mathcal{U}} x \in \sigma\}.$$

Clearly, $a \in V[G_{\alpha_\varrho}]$ and $q^* \Vdash^{V[G_{\alpha_\varrho}]}_{\mathbb{M}_\mathcal{U}} \check{a} \subseteq \sigma$. Conversely, let $x \in \varrho \times \mathbb{M}_{\mathcal{U} \restriction \kappa_{\alpha_\varrho}+1}$ and $r \leq_{\mathbb{M}_\mathcal{U}} q^*$ such that $r \Vdash^{V[G_{\alpha_\varrho}]}_{\mathbb{M}_\mathcal{U}} x \in \sigma$. Since $q^* \leq_{\mathbb{M}_\mathcal{U}} q_\theta$ and q_θ decides "$x_\theta \in \sigma$" it follows that $q_\theta \Vdash^{V[G_{\alpha_\varrho}]}_{\mathbb{M}_\mathcal{U}}$ "$\theta \in \sigma$", hence $q^* \Vdash^{V[G_{\alpha_\varrho}]}_{\mathbb{M}_\mathcal{U}} \theta \in \sigma$. Thus, $q^* \Vdash^{V[G_{\alpha_\varrho}]}_{\mathbb{M}_\mathcal{U}} \sigma = \check{a}$. Standard genericity arguments yield $\sigma_H \in V[G_{\alpha_\varrho}]$, where H is the filter generated by G joint with the natural projection between $\mathbb{M} \downarrow p$ and $\mathbb{M}_\mathcal{U} \downarrow p^{>m}$. Now it is not hard to show that $(\theta, r) \in \sigma_H$ if and only if $\theta \in \tau_G$, which yields $\tau_G \in V[G_{\alpha_\rho}]$, as wanted. ☐

The above proposition yields Magidor's theorem:

Theorem 7.1.17 (Magidor)**.** *Let $\delta < \kappa$ be two cardinals with κ being measurable of $o(\kappa) = \delta$. Then there is a cardinal-preserving generic extension of the universe where κ is strong limit and $\mathrm{cof}(\kappa) = \mathrm{cof}(\delta)^V$.*

Proof. Let $\mathbb{M}$ be Magidor forcing with respect to a coherent sequence of measures $\mathcal{U}$ witnessing $o(\kappa) = \delta$. We split the proof of the theorem in a series of claims:

Claim 7.1.17.1. $\mathbb{M}$ *preserves cardinals.*

Proof of claim. The preservation of cardinals $\geq\kappa^+$ is guaranteed by Proposition 7.1.11. Thus, it suffices to analyze the preservation of cardinals $\leq\kappa$. Observe that if all the cardinals $<\kappa$ are preserved then κ is preserved, hence everything amounts to check that this is indeed the case.

Assume otherwise and let $\varrho < \kappa$ be a V-cardinal which is collapsed after forcing with $\mathbb{M}$. Let $G \subseteq \mathbb{M}$ be some generic filter for which there is a surjection $\varphi : \varrho \to \vartheta$, $\varphi \in V[G]$, for some $\vartheta < \varrho$. Since we may encode φ as a subset of ϑ , Proposition 7.1.16 yields $\varphi \in V[G_{\alpha_\vartheta}]$. Thus, φ witnesses that $\mathbb{M}_{\mathcal{U}\restriction\kappa_{\alpha_\vartheta}+1}$ collapses ϱ, which is a cardinal $\geq\kappa^+_{\alpha_\vartheta}$. Observe however that this collides with the $\kappa^+_{\alpha_\vartheta}$-Knasterness of $\mathbb{M}_{\mathcal{U}\restriction\kappa_{\alpha_\vartheta}+1}$, which yields the desired contradiction. □

Claim 7.1.17.2. $\mathbb{M}$ *forces that* κ *is a strong limit cardinal.*

Proof of claim. This is very much in the spirit of the above claim. Let $G \subseteq \mathbb{M}$ generic and $\varrho < \kappa$ be a cardinal.[4] By Proposition 7.1.16, $\mathcal{P}(\varrho)^{V[G]} = \mathcal{P}(\varrho)^{V[G_{\alpha_\rho}]}$, hence $(2^\varrho)^{V[G]} = (2^\varrho)^{V[G_{\alpha_\varrho}]}$. Since $|\mathbb{M}_{\mathcal{U}\restriction\kappa_{\alpha_\varrho}+1}| < \kappa$ it is routine to check that $(2^\varrho)^{V[G_{\alpha_\varrho}]} < \kappa$. Altogether, this shows that κ is strong limit in any generic extension by $\mathbb{M}$. □

Claim 7.1.17.3. $\mathbb{M}$ *forces* "$\mathrm{cof}(\kappa) = \mathrm{cof}(\delta)^V$".

Proof of claim. By Definition 7.1.3 (a)($\aleph$) it is easy to check that $\langle\mathbb{M}, \leq^*\rangle$ is δ^+-closed. Combining this with the Prikry property of $\langle\mathbb{M}, \leq, \leq^*\rangle$ it follows that that $\mathbb{M}$ does not add bounded subsets to δ^+, hence $\mathbb{1}_\mathbb{M} \Vdash_\mathbb{M} \mathrm{cof}(\check{\delta}) = \mathrm{cof}(\check{\delta})^V$. Now the claim follows from the fact that $\mathbb{M}$ adds a club with $\mathrm{otp}(\omega^\delta) = \delta$. □

□

Remark 7.1.18. Despite easily noticeable, it is worth mentioning that $\mathbb{M}$ is not cofinality-preserving: Assume that the hypotheses of Theorem 7.1.17 apply. Set $A := \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \alpha \ \&\ \alpha > \delta\}$. It is routine to check that $A \in \mathcal{F}(\kappa)$, hence $p := \langle\langle\kappa, A\rangle\rangle \in \mathbb{M}$. Let C_G be the Magidor club induced by some generic filter $G \subseteq \mathbb{M} \downarrow p$ over V. Notice that C_G yields a continuous δ-sequence of V-regular cardinals above δ, hence all its limit points become singular in $V[G]$.

[4]By virtue of the former claim observe that there is no confusion here.

There is another remarkable feature of Magidor forcing which is essential for our purposes. This aspect is related with the way Magidor generics are generated. Recall that if $G \subseteq \mathbb{M}$ is generic over V then it generates a Magidor club which we denote by C_G. Conversely, if $C_G \subseteq \kappa$ is the Magidor club generated by G, then the set of conditions $p \in G(C_G)$ defined as

($\aleph$) $\forall m < n^p\, \alpha^p_m \in C_G$,

($\beth$) $\forall \vartheta \in C_G\, \exists q \leq_{\mathbb{M}} p\, \exists m < n^q\, (\vartheta \text{ appears in } q)$,

generates a filter which contains G, hence it is also generic and $G(C_G) = G$. In particular, $V[G] = V[C_G]$.

Let us say that a sequence $\vec{\gamma}$ with $\mathrm{ran}(\vec{\gamma}) \subseteq \kappa$ is a Magidor sequence for $\mathbb{M}_{\mathcal{U}}$ over V if the set $G(\vec{\gamma})$ is a $\mathbb{M}_{\mathcal{U}}$-generic filter over V.[5] By definition, any Magidor sequence for $\mathbb{M}_{\mathcal{U}}$ over V generates a Magidor generic over V. Conversely, any Magidor generic G for $\mathbb{M}_{\mathcal{U}}$ generates a Magidor sequence $\vec{\gamma}_G$ over V, as witnessed by any increasing enumeration of C_G. From this it is clear that any Magidor extension is ultimately determined by a Magidor sequence. It is thus natural to ask whether there is any criterion which allows to establish when a sequence $\vec{\gamma}$ is indeed a Magidor sequence for some $\mathbb{M}_{\mathcal{U}}$. The following result due to Mitchell [Mit82] provides the desired characterization:

Theorem 7.1.19 (Mitchell)**.** *Assume that V is an inner model of W with $\mathbb{M}_{\mathcal{U}} \in V$. A sequence $\vec{\gamma} \in W$ is a Magidor sequence over V if and only if the following hold true:*

1. *for $\alpha < |\vec{\gamma}|$, $\vec{\gamma} \restriction \alpha$ is a Magidor sequence for $\mathbb{M}_{\mathcal{U}\restriction(\vec{\gamma}\restriction\alpha+1)}$ over V;*
2. *for each $X \in \mathcal{P}(\kappa)^V$, $X \in \mathcal{F}_{\mathcal{U}}(\kappa)^V$ if and only if for a tail end of $\alpha < |\vec{\gamma}|$, $\vec{\gamma}(\alpha) \in X$.*

We will be using this result in the next section when we show that $\mathbb{R}$ projects onto $\mathrm{RO}^+(\mathbb{R} \restriction \xi)$ (c.f. Proposition 7.3.2).

Finally, we prove a generalization of the classical Röwbottom's Lemma [Kan09, Theorem 7.17] which will be used in our future analysis of the quotients $\mathbb{R}/\mathbb{R} \restriction \xi$. The following lemma is the key in proving Lemma 7.1.21.

Lemma 7.1.20. *Let $f : [\kappa]^{<\omega} \to \tau, \tau < \kappa$, and let $\langle U_\alpha \mid \alpha < \delta \rangle, \delta < \kappa$, be a sequence of normal measures on κ. Then there are sets $A_\alpha \in U_\alpha$, for $\alpha < \delta$, such that whenever $\langle \alpha_0, \cdots, \alpha_{n-1} \rangle$ is a finite sequence of ordinals less than δ, the function f is constant on the set of increasing sequences $\langle \nu_0, \cdots, \nu_{n-1} \rangle \in \prod_{i \leq n-1} A_{\alpha_i}$.*

[5] Here we are identifying $\vec{\gamma}$ with a club subset of κ.

Proof. We prove, by induction on $n < \omega$, that for each $f : [\kappa]^n \to \tau$, where $\tau < \kappa$, there are sets $A_\alpha \in U_\alpha$, for $\alpha < \delta$ and a function $g : [\delta]^n \to \tau$ such that for each finite sequence $\langle \alpha_0, \cdots, \alpha_{n-1} \rangle$ of ordinals less than δ and all increasing sequences $\langle \nu_0, \cdots, \nu_{n-1} \rangle \in A_{\alpha_0} \times \cdots \times A_{\alpha_{n-1}}$, we have

$$f(\langle \nu_0, \cdots, \nu_{n-1} \rangle) = g(\langle \alpha_0, \cdots, \alpha_{n-1} \rangle).$$

From this the result follows easily by using the σ-completeness of the ultrafilters U_α. Observe that when $n = 1$, this is clear: for each $\alpha < \delta$ let $A_\alpha \in U_\alpha$ be such that $f \restriction A_\alpha$ is constant, and let $g(\alpha)$ be this constant value.

Now suppose that the lemma holds for $n \geq 1$ and we prove it for $n + 1$. Thus let $f : [\kappa]^{n+1} \to 2$. For each $\langle \nu_0, \cdots, \nu_{n-1} \rangle \in [\kappa]^n$, let $f_{\langle \nu_0, \cdots, \nu_{n-1} \rangle}$ be defined by

$$f_{\langle \nu_0, \cdots, \nu_{n-1} \rangle}(\nu) = f(\langle \nu_0, , \cdots, \nu_{n-1}, \nu \rangle).$$

By the induction hypothesis, we can find sets $A_\alpha^{\langle \nu_0, \cdots, \nu_{n-1} \rangle} \in U_\alpha, \alpha < \delta$, and a function $g_{\langle \nu_0, \cdots, \nu_{n-1} \rangle} : \delta \to \tau$ such that whenever $\alpha < \delta$, then for all $\nu \in A_\alpha^{\langle \nu_0, \cdots, \nu_{n-1} \rangle}$,

$$f(\langle \nu_0, \cdots, \nu_{n-1}, \nu \rangle) = g_{\langle \nu_0, \cdots, \nu_{n-1} \rangle}(\langle \alpha \rangle).$$

Define $H : [\kappa]^n \to {}^\delta\tau$ by $H(\langle \nu_0, \cdots, \nu_{n-1} \rangle) := g_{\langle \nu_0, \cdots, \nu_{n-1} \rangle}$. By the induction hypothesis, we can find sets $B_\alpha \in U_\alpha$ and a function $G : [\delta]^n \to {}^\delta\tau$ such that for each finite sequence $\langle \alpha_0, \cdots, \alpha_{n-1} \rangle$ of ordinals less than δ and all increasing sequences $\langle \nu_0, \cdots, \nu_{n-1} \rangle \in \prod_{i \leq n-1} B_{\alpha_i}$, we have

$$g_{\langle \nu_0, \cdots, \nu_{n-1} \rangle} = G(\langle \alpha_0, \cdots, \alpha_{n-1} \rangle).$$

This gives us a definable function $g : [\delta]^{n+1} \to \tau$, defined by

$$g(\langle \alpha_0, \cdots, \alpha_n \rangle) = g_{\langle \nu_0, \cdots, \nu_{n-1} \rangle}(\langle \alpha_n \rangle),$$

for some (and hence any) increasing sequence $\langle \nu_0, \cdots, \nu_{n-1} \rangle \in \prod_{i \leq n-1} B_{\alpha_i}$. Now let $A_\alpha := B_\alpha \cap \triangle_{\langle \nu_0, \cdots, \nu_{n-1} \rangle \in [\kappa]^n} A_\alpha^{\langle \nu_0, \cdots, \nu_{n-1} \rangle} \in U_\alpha$. Let $\langle \alpha_0, \cdots, \alpha_n \rangle$ be a finite set of ordinals less than δ and let $\langle \nu_0, \cdots, \nu_n \rangle \in \prod_{i \leq n} A_{\alpha_i}$ be an increasing sequence. Then $\nu_n \in A_{\alpha_n}^{\langle \nu_0, \cdots, \nu_{n-1} \rangle}$, and we have

$$f(\langle \nu_0, \cdots, \nu_n \rangle) = g_{\langle \nu_0, \cdots, \nu_{n-1} \rangle}(\alpha_n) = g(\langle \alpha_0, \cdots, \alpha_n \rangle),$$

as required. $\square$

Suppose $p = \langle \kappa, A \rangle \in \mathbb{M}$ and $c : [A]^{<\omega} \to \tau$, where $\tau < \kappa$. By shrinking A, we may assume that $A = \biguplus_{\alpha < \delta} A(\alpha)$, where $A(\alpha) \in \mathcal{U}(\kappa, \alpha)$. Then we can apply the above lemma and find sets $B(\alpha) \in \mathcal{U}(\kappa, \alpha), B(\alpha) \subseteq A(\alpha)$, such that whenever $\langle \alpha_0, \cdots, \alpha_{n-1} \rangle$ is a finite sequence of ordinals less than δ, the function c is constant on the set of increasing sequences $\langle \nu_0, \cdots, \nu_{n-1} \rangle \in$

$\prod_{i\leq n-1} B(\alpha_i)$. Note that $B = \biguplus_{\alpha<\delta} B(\alpha) \in \mathcal{F}(\kappa)$ and hence $q = \langle \kappa, B\rangle \in \mathbb{M}$ and is an extension of p.

Lemma 7.1.21 (Generalized Röwbottom's Lemma). *Let $p \in \mathbb{M}$. For each function $c : [\biguplus_{i\leq n^p} A_i^p]^{<\omega} \to \tau$, where $\tau < \alpha_0^p$, there is a sequence of sets $\langle B_i \mid i \leq n^p\rangle$ which is homogeneous for c. Here homogeneity means the following:*[6]

1. *for each $i \leq n^p$, $B_i \subseteq A_i^p$, $B_i \in \mathcal{F}(\alpha_i^p)$ and $B_i = \biguplus_{\alpha<\delta} B_i(\alpha)$, for some $B_i(\alpha) \in \mathcal{U}(\alpha_i^p, \alpha)$;*
2. *for each $m < \omega$ and $\vec{x}, \vec{y} \in [\biguplus_{i\leq n^p} B_i]^m$, if for $k < m$, $\vec{x}(k), \vec{y}(k)$ belong to the same $B_i(\alpha)$, for some $i \leq n^p$ and $\alpha < \delta$, then $c(\vec{x}) = c(\vec{y})$.*

Proof. Let us argue by induction over the length of p and over the coherent sequences of measures. If $n^p = 1$ the argument is covered by Lemma 7.1.20, so let us suppose that $n^p > 1$ and that the result holds for all conditions of length less than n^p.

Set $n := n^p$ and say $p = \langle\langle \alpha_0, A_0\rangle, \cdots, \langle \alpha_n, A_n\rangle\rangle$. Fix $\tau < \alpha_0$ and c: $[\biguplus_{i\leq n} A_i]^{<\omega} \to \tau$ a function. For a sequence $\vec{y} \in [\biguplus_{i\leq n-1} A_i]^{<\omega}$, define $c_{\vec{y}}$: $[A_n]^{<\omega} \to \tau$ by $c_{\vec{y}}(\vec{x}) := c(\vec{y}^\frown\vec{x})$. Arguing as in the base case we can find $A^{\vec{y}} \subseteq A_n$ witnessing clauses (1) and (2) for $c_{\vec{y}}$. In particular, for each such $\vec{y}$, we can find a function $g_{\vec{y}} : [\delta]^{<\omega} \to \tau$ such that for each $\vec{\alpha} \in [\delta]^{<\omega}$ and all increasing sequences $\vec{x} \in \prod A^{\vec{y}}(\vec{\alpha})$, $c(\vec{y}^\frown\vec{x}) = g_{\vec{y}}(\vec{\alpha})$. Set

$$B_n := \bigcap\{A^{\vec{y}} \mid \vec{y} \in [\biguplus_{i\leq n-1} A_i]^{<\omega}\}.$$

Define d on $[\biguplus_{i\leq n-1} A_i]^{<\omega}$ by $d(\vec{y}) := g_{\vec{y}}$. As $\tau^{\delta^{<\omega}} < \alpha_0$, the induction hypothesis give us a sequence $\langle B_i \mid i \leq n-1\rangle$ of sets witnessing clauses (1) and (2) with respect to d.

Claim 7.1.21.1. *$\langle B_i \mid i \leq n\rangle$ witnesses clause (1) and (2) for c.*

Proof of claim. We are left with checking that clause (2) is holds. Suppose that $m < \omega$, $\vec{z}_1, \vec{z}_2 \in [\biguplus_{i\leq n} B_i]^m$ and that for each $k < m$, $\vec{z}_1(k), \vec{z}_2(k)$ belong to the same $B_i(\alpha)$, for some $i \leq n$ and $\alpha < \delta$. Then we can find $\vec{x}_1, \vec{x}_2 \in [B_n]^{\leq m}$ and $\vec{y}_1, \vec{y}_2 \in [\biguplus_{i\leq n-1} B_i]^{\leq m}$ with $|\vec{x}_1| = |\vec{x}_2|$ and $|\vec{y}_1| = |\vec{y}_2|$ such that $\vec{z}_1 = \vec{y}_1^\frown\vec{x}_1$ and $\vec{z}_2 = \vec{y}_2^\frown\vec{x}_2$. By the choice of $\vec{y}_1$ and $\vec{y}_2$, $d(\vec{y}_1) = d(\vec{y}_2)$ hence, by homogeneity, $g_{\vec{y}_1} = g_{\vec{y}_2}$. Similarly our choice of $\vec{x}_1, \vec{x}_2$, yields $c(\vec{z}_1) = c(\vec{y}_1^\frown\vec{x}_1) = g_{\vec{y}_1}(\vec{x}_1) = g_{\vec{y}_2}(\vec{x}_2) = c(\vec{y}_2^\frown\vec{x}_2) = c(\vec{z}_2)$, which gives the desired result. □

The above claim finishes the proof of the induction step and thus yields the lemma. □

[6] For simplicity we will require in the notion of homogeneity that (1) holds, though this is not strictly necessary.

7.2 The main forcing construction

We will devote the current section to introduce the main forcing construction used in the proof of Theorem 6.0.13, when $\Theta = \lambda^+$. Hereafter, δ, κ, λ will be assumed as in the statement of Theorem 6.0.13 and $G \subseteq \mathrm{Add}(\kappa, \lambda^+)$ will be a fixed generic filter over V.

Notation 7.2.1.

- For each $x \subseteq \lambda^+$, $\mathbb{A}_x := (\mathrm{Add}(\kappa, x), \supseteq)$.
- For each $y \subseteq x \subseteq \lambda^+$ and $H \subseteq \mathbb{A}_x$ generic filter over V, $H \restriction y$ will denote the generic filter induced by H and the standard projection between $\mathbb{A}_x$ and $\mathbb{A}_y$.

By a result of Woodin (see e.g. [GS89, §2]) it is feasible to prepare the ground model and make the strongness of κ indestructible under adding arbitrary many Cohen subsets of κ. Thus we may assume that κ is strong in $V[G]$. The following argument is standard but we provide details for the reader's benefit:

Proposition 7.2.2. *For a $(\kappa+2)$-strong cardinal κ, $o(\kappa) = (2^\kappa)^+$. Thus, $(\kappa+2)$-strong cardinals κ have maximal Mitchell-order.*

Proof. The last claim follows from the standard fact that for a measurable cardinal λ, $o(\lambda) \leq (2^\lambda)^+$ [Mit10, §2]. In order to show that $o(\kappa) = (2^\kappa)^+$ we shall need to construct a $\lhd$-increasing sequence $\langle U_\alpha \mid \alpha < (2^\kappa)^+\rangle$ of measures over κ. We will do so proceeding by induction on $\alpha < (2^\kappa)^+$: that is, we will assume that $\mathcal{U} \restriction \alpha = \langle U_\beta \mid \beta < \alpha\rangle$ has been already defined and will show how to define U_α in such a way that $\mathcal{U} \restriction \alpha \in \mathrm{Ult}(V, U_\alpha)$. The next is the key claim:

Claim 7.2.2.1. *Let κ be a $(\kappa+2)$-strong cardinal and $\mathcal{X} \subseteq \mathcal{P}(\kappa)$. Then there is a normal measure U on κ such that $\mathcal{X} \in \mathrm{Ult}(V, U)$.*

Proof of claim. Let $\exists \mathcal{Y}(\mathcal{Y} \subseteq \mathcal{P}(\kappa) \wedge \varphi(\mathcal{Y}, \kappa))$ be the formula where $\varphi(\mathcal{Y}, \kappa) \equiv \forall U\, (U \text{ measure over } \kappa \to \mathcal{Y} \notin \mathrm{Ult}(V, U))$. Assume the above claim was false. Then there is $\mathcal{Y}^*$ witnessing $\varphi(\mathcal{Y}^*, \kappa)$. Let $j : V \to M$ be an elementary embedding witnessing the $(\kappa+2)$-strongness of κ: that is, $\mathrm{crit}(j) = \kappa$, M is transitive and $V_{\kappa+2} \subseteq M$. Notice that $\mathcal{Y}^* \in M$, hence $M \models \varphi(\mathcal{Y}^*, \kappa)$. Now let U and $k : \mathrm{Ult}(V, U) \to M$ be, respectively, the canonical normal measure derived from j and the canonical elementary embedding between $\mathrm{Ult}(V, U)$ and M (see e.g. [Kan09, p. 52]). Since $\mathrm{crit}(k) > \kappa$, by elementarity, $\mathrm{Ult}(V, U) \models \exists \mathcal{Y}(\mathcal{Y} \subseteq \mathcal{P}(\kappa) \wedge \varphi(\mathcal{Y}, \kappa))$. Let $\overline{\mathcal{Y}} \in \mathrm{Ult}(V, U) \cap \mathcal{P}(\mathcal{P}(\kappa))$ be such that $\mathrm{Ult}(V, U) \models \varphi(\overline{\mathcal{Y}}, \kappa)$. Since $k(\overline{\mathcal{Y}}) = \overline{\mathcal{Y}}$ it follows that $M \models \varphi(\overline{\mathcal{Y}}, \kappa)$. But, $\overline{\mathcal{Y}} \in \mathrm{Ult}(V, U)$, $U \in M$ and $\mathrm{Ult}(V, U) = \mathrm{Ult}(V, U)^M$, which yields the desired contradiction. □

Assume that $\mathcal{U} \upharpoonright \alpha$ has been already defined. Since $\alpha < (2^\kappa)^+$, modulo encodings, we may regard this sequence as a member of $\mathcal{P}(\mathcal{P}(\kappa))$. Thus, the previous claim give us a normal measure U_α on κ for which $\mathcal{U} \upharpoonright \alpha \in \mathrm{Ult}(V, U_\alpha)$. It follows that there is a $\lhd$-increasing sequence $\langle U_\alpha \mid \alpha < (2^\kappa)^+\rangle$ witnessing $o(\kappa) = (2^\kappa)^+$.

□

By virtue of the previous proposition κ is a measurable cardinal with $o(\kappa) = \delta$ in $V[G]$. Thus, we may let $\mathcal{U} \in V[G]$ be a coherent sequence of measures $\mathcal{U} = \langle \mathcal{U}(\alpha,\beta) \mid \alpha \leq \kappa, \beta < o^{\mathcal{U}}(\alpha)\rangle$ with $o(\kappa) = o^{\mathcal{U}}(\kappa) = \delta$. For each pair $(\alpha,\beta) \in \mathrm{dom}(\mathcal{U})$, $\dot{\mathcal{U}}(\alpha,\beta)$ will denote a $\mathbb{A}_{\lambda^+}$-name for the measure $\mathcal{U}(\alpha,\beta)$.

Lemma 7.2.3. *There exists an unbounded set of ordinals $\mathcal{A} \subseteq \lambda^+$, closed under taking limits of $\geq\kappa^+$-sequences, such that, for every $\xi \in \mathcal{A}$ and every $\mathbb{A}_{\lambda^+}$-generic filter G, $\mathcal{U}_\xi := \langle \dot{\mathcal{U}}(\alpha,\beta)_G \cap V[G \upharpoonright \xi] \mid \ \alpha \leq \kappa,\ \beta < o^{\dot{\mathcal{U}}}(\alpha)\rangle$ is a coherent sequence of measures in $V[G \upharpoonright \xi]$.*

Proof. Arguing as in [FHS18, Lemma 3.3], for each $(\alpha,\beta) \in \mathrm{dom}(\mathcal{U})$, there is an unbounded set $\mathcal{A}_{(\alpha,\beta)} \subseteq \lambda^+$, closed under taking limits of $\geq\kappa^+$-sequences for which $\dot{\mathcal{U}}(\alpha,\beta)_G \cap V[G \upharpoonright \xi]$ is a normal measure on α in $V[G \upharpoonright \xi]$, for all $\xi \in \mathcal{A}_{(\alpha,\beta)}$. Clearly, the collection of unbounded sets in λ^+ which are closed under taking limits of $\geq\kappa^+$-sequences is closed under taking intersections of κ-many sets. Set $\mathcal{A} := \bigcap_{(\alpha,\beta)\in\mathrm{dom}(\mathcal{U})} \mathcal{A}_{(\alpha,\beta)}$ and observe that $\mathcal{A} \subseteq \lambda^+$ is unbounded and closed in the above mentioned sense. For each $\xi \in \mathcal{A}$, set $\mathcal{U}_\xi := \langle \mathcal{U}_\xi(\alpha,\beta) \mid \alpha \leq \kappa,\ \beta < o^{\mathcal{U}}(\alpha)\rangle$, where $(\alpha,\beta) \in \mathrm{dom}(\mathcal{U})$ and $\mathcal{U}_\xi(\alpha,\beta) := \dot{\mathcal{U}}(\alpha,\beta)_G \cap V[G \upharpoonright \xi]$.

We claim that $\mathcal{A}$ is as desired. Fix $\xi \in \mathcal{A}$ and let $(\alpha,\beta) \in \mathrm{dom}(\mathcal{U})$. By construction $\mathcal{U}_\xi(\alpha,\beta)$ is a normal measure on α, hence (1) of Definition 7.1.1 holds. Let $i^{\alpha,\xi}_\beta$ be the ultrapower by $\mathcal{U}_\xi(\alpha,\beta)$ in $V[G \upharpoonright \xi]$. We are left with showing that $\mathcal{U}_\xi$ satisfies clause (2) of Definition 7.1.1.

Claim 7.2.3.1. *$i^{\alpha,\xi}_\beta(\mathcal{U}_\xi)(\varrho,\nu) = \mathcal{U}_\xi(\varrho,\nu)$, for each $(\varrho,\nu) \in \mathrm{dom}(\mathcal{U}_\xi \upharpoonright (\alpha,\beta))$.*

Proof of claim. Let $(\varrho,\nu) \in \mathrm{dom}(\mathcal{U}_\xi \upharpoonright (\alpha,\beta))$. By definition, (ϱ,ν) is member of $\mathrm{dom}(\mathcal{U} \upharpoonright (\alpha,\beta))$. We now check that the claim holds. Let us distinguish two cases: $\varrho < \alpha$ and $\varrho = \alpha$.

► Assume $\varrho < \alpha$. If $X \in \mathcal{P}(\varrho) \cap V[G \upharpoonright \xi]$, observe that $i^{\alpha,\xi}_\beta(X) = X$, $i^{\alpha,\xi}_\beta(\varrho) = \varrho$ and $i^{\alpha,\xi}_\beta(\nu) = \nu$. Then, it is not hard to check that the desired equality holds.

► Assume $\varrho = \alpha$. Then, $\nu < \beta$. By coherence of $\mathcal{U}$, $\mathcal{U}_\xi(\alpha,\nu) = j^\alpha_\beta(\mathcal{U})(\alpha,\nu) \cap V[G \upharpoonright \xi]$. Let $X \in \mathcal{U}_\xi(\alpha,\nu)$ and $f \in V[G \upharpoonright \xi]$ be such that $X = [f]_{\mathcal{U}_\xi(\alpha,\beta)}$. Since $f \in V[G \upharpoonright \xi]$, it is not hard to check that $[f]_{\mathcal{U}_\xi(\alpha,\beta)} = [f]_{\mathcal{U}(\alpha,\beta)}$.

Notice that $\mathrm{Ult}(V[G], \mathcal{U}(\alpha,\beta)) \models$ "$X \in j^\alpha_\beta(\mathcal{U})(\alpha,\nu)$" and that this is true if and only if $Y := \{\delta < \alpha \mid f(\delta) \in \mathcal{U}(\delta,\nu)\} \in \mathcal{U}(\alpha,\beta)$. Since $f \in V[G \upharpoonright \xi]$,

$f(\delta) \in \mathcal{U}_\xi(\delta,\nu)$ iff $f(\delta) \in \mathcal{U}(\delta,\nu)$, hence $Y = \{\delta < \alpha \mid f(\delta) \in \mathcal{U}_\xi(\delta,\nu)\} \in \mathcal{U}(\alpha,\beta)$. Observe that $Y \in V[G \restriction \xi]$, hence the above is equivalent to $Y \in \mathcal{U}_\xi(\alpha,\beta)$. Thus $[f]_{\mathcal{U}_\xi(\alpha,\beta)} \in i^{\alpha,\xi}_\beta(\mathcal{U}_\xi)(\alpha,\nu)$. Combining all the previous equivalences we arrive at $i^{\alpha,\xi}_\beta(\mathcal{U}_\xi)(\alpha,\nu) = j^\alpha_\beta(\mathcal{U})(\alpha,\nu) \cap V[G \restriction \xi] = \mathcal{U}_\xi(\alpha,\nu)$, as desired. □

Claim 7.2.3.2. $\mathrm{dom}(i^{\alpha,\xi}_\beta(\mathcal{U}_\xi) \restriction \alpha+1) = \mathrm{dom}(\mathcal{U}_\xi \restriction (\alpha,\beta))$.

Proof of claim. The above argument already gives the right to left inclusion. Let $(\varrho,\nu) \in \mathrm{dom}(i^{\alpha,\xi}_\beta(\mathcal{U}_\xi) \restriction \alpha+1)$. It is not hard to check that if $\varrho < \alpha$ then $(\varrho,\nu) \in \mathrm{dom}(\mathcal{U}_\xi \restriction (\alpha,\beta))$, so that we may assume $\varrho = \alpha$. We have to show that $o^{i^{\alpha,\xi}_\beta(\mathcal{U}_\xi)}(\alpha) = \beta$. Let

$$Y = \{\varrho < \alpha \mid o^{\mathcal{U}}(\varrho) = \beta\}.$$

Since $\mathcal{U}$ is a coherent sequence of measures, $o^{j^\alpha_\beta(\mathcal{U})}(\alpha) = \beta$, i.e., $\alpha \in j^\alpha_\beta(Y)$, and hence $Y \in \mathcal{U}(\alpha,\beta)$. Since $\mathcal{U} \restriction \alpha = \mathcal{U}_\xi \restriction \alpha$, we have $Y \in V[G \restriction \xi]$ and hence $Y \in \mathcal{U}_\xi(\alpha,\beta)$. Thus $\alpha \in i^{\alpha,\xi}_\beta(Y)$, which means $o^{i^{\alpha,\xi}_\beta(\mathcal{U}_\xi)}(\alpha) = \beta$, as required. □

The above claims yield $i^{\alpha,\xi}_\beta(\mathcal{U}_\xi) \restriction \alpha+1 = \mathcal{U}_\xi \restriction (\alpha,\beta)$, thus completing the proof of the lemma. □

Remark 7.2.4. For each $\xi \in \mathcal{A}$ and $\alpha < \kappa$, observe that $\mathcal{U}_\xi(\alpha,\beta) = \mathcal{U}(\alpha,\beta)$ and $\mathcal{U}(\alpha,\beta) \in V$, as $\mathbb{A}_\xi$ does not add bounded subsets of κ.

Let $\mathcal{A}$ be a set given by Lemma 7.2.3. Hereafter we will be relying on the following notation.

Notation 7.2.5. For each $\xi \in \mathcal{A}$, let $\mathcal{U}_\xi$ be the coherent sequence of measures resulting of Lemma 7.2.3 and let $\dot{\mathcal{U}}_\xi$ be a $\mathbb{A}_\xi$-name such that $\mathcal{U}_\xi = (\dot{\mathcal{U}}_\xi)_{G\restriction\xi}$. Similarly, $\dot{\mathbb{M}}_\xi$ will be a $\mathbb{A}_\xi$-name such that $\mathbb{M}_{\mathcal{U}_\xi} = (\dot{\mathbb{M}}_\xi)_{G\restriction\xi}$. By convention, $\mathcal{U}_{\lambda^+} := \mathcal{U}$ and $\dot{\mathbb{M}}_{\lambda^+}$ will be a $\mathbb{A}_{\lambda^+}$-name such that $\mathbb{M}_{\mathcal{U}_{\lambda^+}} = (\dot{\mathbb{M}}_{\mathcal{U}_{\lambda^+}})_G$.

Proposition 7.2.6. *Work in V. For each $\xi \in \mathcal{A}$, $\mathbb{A}_{\lambda^+} * \mathbb{M}_{\lambda^+}$ projects onto $\mathbb{A}_\xi * \mathbb{M}_\xi$.*

Proof. Let $\xi \in \mathcal{A}$. It suffices to prove that any $\mathbb{M}_{\lambda^+}$-generic over $V[G]$ induces a $\mathbb{M}_\xi$-generic over $V[G \restriction \xi]$. Notice that by the discussion carried out at the end of §7.1 it suffices with showing that any Magidor sequence $\vec{\gamma}$ for $\mathbb{M}_{\lambda^+}$ over $V[G]$ is also a Magidor sequence for $\mathbb{M}_\xi$ over $V[G \restriction \xi]$. To this aim we will check that $\vec{\gamma}$ witnesses (1) and (2) of Theorem 7.1.19, when $\mathcal{U} = \mathcal{U}_\xi$ and $V = V[G \restriction \xi]$.

Claim 7.2.6.1. *For each $\alpha < |\vec{\gamma}|$, $\vec{\gamma} \restriction \alpha$ is a Magidor sequence for $\mathbb{M}_{\mathcal{U}_\xi\restriction\gamma(\alpha)+1}$ over $V[G \restriction \xi]$*

Proof of claim. Let us argue this by induction on $\alpha < |\vec{\gamma}|$. Assume that for each $\beta < \alpha$, $\vec{\gamma} \restriction \beta$ is a Magidor sequence for $\mathbb{M}_{\mathcal{U}_\xi \restriction (\vec{\gamma} \restriction \beta+1)}$ over $V[G \restriction \xi]$. In order to check that $\vec{\gamma} \restriction \alpha$ witnesses the inductive step we need to verify that (1) and (2) of Theorem 7.1.19 hold. Clearly our induction assumption yields (1), hence we are left with verifying (2). Fix $X \in \mathcal{P}(\kappa)^{V[G \restriction \xi]}$. First let us assume that $X \in \mathcal{F}_{\mathcal{U}_\xi}(\kappa)^{V[G \restriction \xi]}$. Since $\mathcal{F}_{\mathcal{U}_\xi}(\kappa)^{V[G \restriction \xi]} \subseteq \mathcal{F}_{\mathcal{U}_{\lambda^+}}(\kappa)^{V[G]}$ and $\vec{\gamma}$ is a Magidor sequence for $\mathbb{M}_{\lambda^+}$, $\vec{\gamma}(\sigma) \in X$, for a tail end of $\sigma < |\vec{\gamma}|$. Conversely, assume that for a tail end of $\sigma < |\vec{\gamma}|$, $\vec{\gamma}(\sigma) \in X$. As before, $X \in \mathcal{F}_{\mathcal{U}_{\lambda^+}}(\kappa)^{V[G]}$. However, notice that $\mathcal{F}_{\mathcal{U}_\xi}(\kappa)^{V[G \restriction \xi]} = \mathcal{F}_{\mathcal{U}_{\lambda^+}}(\kappa)^{V[G]} \cap \mathcal{P}(\kappa)^{V[G \restriction \xi]}$, hence $X \in \mathcal{F}_{\mathcal{U}_\xi}(\kappa)^{V[G \restriction \xi]}$, as wanted □

The verification of Theorem 7.1.19 (2) for $\vec{\gamma}$ is essentially contained in the proof of the above claim. It thus follows that $\vec{\gamma}$ is as desired.

□

Fix $\xi_0 \in \mathcal{A} \setminus \lambda+1$ and $\pi : \xi_0 \to \mathrm{Even}(\lambda)$ be a bijection.[7] Hereafter, ξ_0 will be fixed. The particular choice of this ordinal is not relevant, we could just have taken any other in $\mathcal{A} \setminus \lambda + 1$. Evidently, π entails an $\in$-isomorphism between the generic extensions $V^{\mathbb{A}_{\xi_0}}$ and $V^{\mathbb{A}_{\mathrm{Even}(\lambda)}}$. Thus, defining $\dot{\mathcal{U}}^\pi_{\xi_0} := \pi(\dot{\mathcal{U}}_{\xi_0})$, $(\dot{\mathcal{U}}^\pi_{\xi_0})_{\pi[G \restriction \xi_0]} = (\dot{\mathcal{U}}_{\xi_0})_{G \restriction \xi_0} = \mathcal{U}_{\xi_0}$. Say that $\mathcal{U}^\pi_{\xi_0}(\alpha, \beta)$ are the measures of this sequence. For enlighten the notation, let H be the $\mathbb{A}_{\mathrm{Even}(\lambda)}$-generic filter generated by $\pi[G \restriction \xi_0]$.

Lemma 7.2.7. *There exists an unbounded set of ordinals $\mathcal{B} \subseteq \lambda$, closed under taking limits of $\geq\kappa^+$-sequences, such that, for every $\gamma \in \mathcal{B}$ and every $\mathbb{A}_{\mathrm{Even}(\lambda)}$-generic filter H, $\mathcal{U}^\pi_\gamma := \langle \dot{\mathcal{U}}^\pi_{\xi_0}(\alpha, \beta)_H \cap V[H \restriction \mathrm{Even}(\gamma)] \mid \alpha \leq \kappa,\ \beta < o^{\dot{\mathcal{U}}}(\alpha) \rangle$ is a coherent sequence of measures in $V[H \restriction \mathrm{Even}(\gamma)]$.*

Proof. Similar to the proof of Lemma 7.2.3. □

Notation 7.2.8. For each $\gamma \in \mathcal{B}$, let $\mathcal{U}^\pi_\gamma$ denote the coherent sequence of measure witnessing Lemma 7.2.7. By convention, $\mathcal{U}^\pi_\lambda := \mathcal{U}_{\xi_0}$. For each $\gamma \in \mathcal{B} \cup \{\lambda\}$, let $\dot{\mathbb{M}}^\pi_\gamma$ be a $\mathbb{A}_{\mathrm{Even}(\gamma)}$-name for the Magidor forcing $\mathbb{M}_{\mathcal{U}^\pi_\gamma}$ in the generic extension $V[H \restriction \mathrm{Even}(\gamma)]$.

Lemma 7.2.9. *Let $\hat{\mathcal{A}} = (\mathcal{A} \cap (\xi_0, \lambda^+)) \cup \{\lambda^+\}$.*

1. *For every $\xi, \tilde{\xi} \in \hat{\mathcal{A}}$ with $\xi < \tilde{\xi}$, there is a projection*

$$\sigma^{\tilde{\xi}}_\xi : \mathbb{A}_{\tilde{\xi}} * \dot{\mathbb{M}}_{\tilde{\xi}} \to \mathrm{RO}^+(\mathbb{A}_\xi * \dot{\mathbb{M}}_\xi).$$

2. *For every $\xi \in \hat{\mathcal{A}}$ and $\gamma \in \mathcal{B}$, there is a projection*

$$\sigma^\xi_\gamma : \mathbb{A}_\xi * \dot{\mathbb{M}}_\xi \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{M}}^\pi_\gamma).$$

[7] For an ordinal α, $\mathrm{Even}(\alpha)$ stands for the set of all even and limit ordinals $\leq\alpha$.

3. *For every* $\xi \in \hat{\mathcal{A}}$ *and* $\gamma \in \mathcal{B}$*, let* $\hat{\sigma}^{\xi}_{\gamma}$ *be the extension of* σ^{ξ}_{γ} *to the Boolean completion of* $\mathbb{A}_{\xi} * \dot{\mathbb{M}}_{\xi}$

$$\hat{\sigma}^{\xi}_{\gamma} : \mathrm{RO}^{+}(\mathbb{A}_{\xi} * \dot{\mathbb{M}}_{\xi}) \to \mathrm{RO}^{+}(\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{M}}^{\pi}_{\gamma}).$$

Then the projections commute with $\sigma^{\lambda^+}_{\gamma}$*:*

$$\sigma^{\lambda^+}_{\gamma} = \hat{\sigma}^{\xi}_{\gamma} \circ \sigma^{\lambda^+}_{\xi}.$$

Proof. The argument for (3) is the same as in [FHS18, Lemma 3.8]. Also, (1) and (2) follow in the same fashion, so let us give details only for (1). Let $G \subseteq \mathbb{A}_{\bar{\xi}}$ generic over V and $\vec{\gamma}$ be a Magidor sequence for $\mathbb{M}_{\bar{\xi}}$ over $V[G]$. Appealing to Proposition 7.2.6 it is clear that $G \restriction \xi * \dot{G}(\vec{\gamma})$ is $\mathbb{A}_{\xi} * \dot{\mathbb{M}}_{\xi}$-generic over V. Then any generic filter for $\mathbb{A}_{\bar{\xi}} * \dot{\mathbb{M}}_{\bar{\xi}}$ induces a generic filter for $\mathbb{A}_{\xi} * \dot{\mathbb{M}}_{\xi}$ and thus a generic filter for the Boolean completion $\mathrm{RO}^{+}(\mathbb{A}_{\xi} * \dot{\mathbb{M}}_{\xi})$. $\square$

Definition 7.2.10 (Main forcing)**.** A condition in $\mathbb{R}$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, \dot{q}) \in \mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}$;
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}]^{<\kappa^+}$;
3. For every $\gamma \in \mathrm{dom}(r)$, $r(\gamma)$ is a $\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{M}}^{\pi}_{\gamma}$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{M}}^{\pi}_{\gamma}} \Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{M}}^{\pi}_{\gamma}} \text{“} r(\gamma) \in \dot{\mathrm{Add}}(\kappa^+, 1) \text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}$ we will write $(p_0, \dot{q}_0, r_0) \leq_{\mathbb{R}} (p_1, \dot{q}_1, r_1)$ iff $(p_0, \dot{q}_0) \leq_{\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\gamma \in \mathrm{dom}(r_1)$, $\sigma^{\lambda^+}_{\gamma}(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{M}}^{\pi}_{\gamma}} \text{“} r_0(\gamma) \leq r_1(\gamma) \text{”}$.

Definition 7.2.11. $\mathbb{U}$ will denote the *termspace forcing.* That is, the pair $(U, \leq)$ where $U := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}\}$ and $\leq$ is the order inherited from $\mathbb{R}$. Set $\bar{\mathbb{R}} := (\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}) \times \mathbb{U}$.

Proposition 7.2.12.

1. $\mathbb{U}$ *is* κ^+*-directed closed.*
2. *The function* $\rho : \bar{\mathbb{R}} \to \mathbb{R}$ *given by* $\langle (p, \dot{q}), (\mathbb{1}, \dot{\mathbb{1}}, r) \rangle \mapsto (p, \dot{q}, r)$ *entails a projection. In particular,* $V^{\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}} \subseteq V^{\mathbb{R}} \subseteq V^{\bar{\mathbb{R}}}$.
3. $V^{\mathbb{A}_{\lambda^+} * \mathbb{M}_{\lambda^+}}$ *and* $V^{\mathbb{R}}$ *have the same* $<\kappa^+$*-sequences.*

Proof. (1) Let $\langle(\mathbb{1},\dot{\mathbb{1}},r_\alpha)\mid\alpha<\kappa\rangle$ be a $\leq_{\mathbb{R}}$-decreasing sequence of conditions in $\mathbb{U}$. Set $\mathrm{dom}(r^*):=\bigcup_{\alpha<\kappa}\mathrm{dom}(r_\alpha)$ and observe that $\mathrm{dom}(r^*)\in[\mathcal{B}]^{<\kappa^+}$. For each $\gamma\in\mathrm{dom}(r^*)$, let $\alpha_\gamma:=\min\{\alpha<\kappa\mid\gamma\in\mathrm{dom}(r_\alpha)\}$. Clearly, $\gamma\in\mathrm{dom}(r_\alpha)$, for each $\alpha\geq\alpha_\gamma$. Since our sequence is $\leq_{\mathbb{R}}$-decreasing this yields

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\gamma)}*\dot{\mathbb{M}}^\pi_\gamma}\Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)}*\dot{\mathbb{M}}^\pi_\gamma}\text{“}\langle r_\alpha(\gamma)\mid\alpha_\gamma\leq\alpha<\kappa\rangle\text{ is }\leq_{\dot{\mathrm{Add}}(\kappa^+,1)}\text{-decreasing”}.$$

Let $r^*(\gamma)$ be a $\mathbb{A}_{\mathrm{Even}(\gamma)}*\dot{\mathbb{M}}^\pi_\gamma$-name for a condition forced by $\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\gamma)}*\dot{\mathbb{M}}^\pi_\gamma}$ to be a lower bound for the above sequence. It is clear that $(\mathbb{1},\dot{\mathbb{1}},r^*)$ provides the desired $\leq_{\mathbb{R}}$-lower bound.

(2) Clearly, ρ is order preserving and $\rho(\mathbb{1}_{\bar{\mathbb{R}}})=\mathbb{1}_{\mathbb{R}}$. Let $(p_1,\dot{q}_1,r_1)\leq_{\mathbb{R}}\rho(\langle(p_2,\dot{q}_2),(\mathbb{1},\dot{\mathbb{1}},r_2)\rangle)$. Define r_3 with $\mathrm{dom}(r_3)=\mathrm{dom}(r_1)$ such that, for each $\gamma\in\mathrm{dom}(r_3)$, $\langle\sigma,b\rangle\in r_3(\gamma)$ iff the following hold: if $b\leq\sigma^{\lambda^+}_\gamma(p_1,\dot{q}_1)$ then $b\Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)}*\dot{\mathbb{M}}^\pi_\gamma}\sigma\in r_1(\gamma)$ and, otherwise, if $b\perp\sigma^{\lambda^+}_\gamma(p_1,\dot{q}_1)$, $b\Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)}*\dot{\mathbb{M}}^\pi_\gamma}\sigma\in r_2(\gamma)$. It is not hard to check that $\langle(p_1,\dot{q}_1),(\mathbb{1},\dot{\mathbb{1}},r_3)\rangle$ is $\leq_{\bar{\mathbb{R}}}$-below the condition $\langle(p_2,\dot{q}_2),(\mathbb{1},\dot{\mathbb{1}},r_2)\rangle$ and $(p_1,\dot{q}_1,r_3)\leq_{\mathbb{R}}(p_1,\dot{q}_1,r_1)$. This shows that ρ defines a projection and thus that $V^{\mathbb{R}}\subseteq V^{\bar{\mathbb{R}}}$. The remaining inclusion follows from the trivial fact that $\mathbb{R}$ projects onto $\mathbb{A}_{\lambda^+}*\dot{\mathbb{M}}_{\lambda^+}$.

(3) Before proving the result we need to begin with an easy observation. Let $(p,\dot{q})\in\mathbb{A}_{\lambda^+}*\dot{\mathbb{M}}_{\lambda^+}$ and say that

$$p\Vdash_{\mathbb{A}_{\lambda^+}}\dot{q}=\langle\langle\tau_0,\dot{A}_0\rangle,\dots,\langle\tau_{n-1},\dot{A}_{n-1}\rangle,\langle\check{\kappa},\dot{A}_n\rangle\rangle,$$

for $\tau_i,\dot{A}_i$ being $\mathbb{A}_{\lambda^+}$-names. Clearly, one may extend p to a condition p^* ensuring that, for each $i<n$, $p^*\Vdash_{\mathbb{A}_{\lambda^+}}\tau_i=\check{\alpha}_i$, for some $\alpha_i<\kappa$. In other words, the set of conditions of the form

$$\langle p,\langle\langle\check{\beta}_0,\dot{A}_0\rangle,\dots,\langle\check{\beta}_{n-1},\dot{A}_{n-1}\rangle,\langle\check{\kappa},\dot{A}_n\rangle\rangle,$$

endowed with the induced order, forms a dense subposet of $\mathbb{A}_{\lambda^+}*\dot{\mathbb{M}}_{\lambda^+}$. Call this forcing $\mathbb{Q}$. Notice that $\mathbb{Q}$ and $\mathbb{A}_{\lambda^+}*\dot{\mathbb{M}}_{\lambda^+}$ are forcing equivalent, hence $V^{\mathbb{Q}}=V^{\mathbb{A}_{\lambda^+}*\dot{\mathbb{M}}_{\lambda^+}}$. By combining Proposition 7.1.11 with the κ^+-Knasterness of $\mathbb{A}_{\lambda^+}$ it is easy to show that $\mathbb{Q}$ is κ^+-Knaster. By Lemma 1.3.18, $\mathbb{1}_{\mathbb{Q}}\Vdash_{\mathbb{Q}}$ “$\mathbb{U}$ is κ^+-distributive”, hence $V^{\mathbb{Q}}$ and $V^{\mathbb{Q}\times\mathbb{U}}$ have the same $<\kappa^+$-sequences, thus $V^{\mathbb{Q}}$ and $V^{\bar{\mathbb{R}}}$ also. The latter assertion yields the desired result. □

Let $\bar{R}\subseteq\bar{\mathbb{R}}$ a generic filter whose projection onto $\mathbb{A}_{\lambda^+}$ generates the generic filter G. Also, let $R\subseteq\mathbb{R}$ be the generic filter generated by $\rho[\bar{R}]$ and $S\subseteq\mathbb{M}_{\lambda^+}$ be the generic filter over $V[G]$ induced by $\bar{R}$. We next prove some important properties of the forcing $\mathbb{R}$.

Proposition 7.2.13 (Some properties of $\mathbb{R}$)**.**

1. *$\mathbb{R}$ is λ-Knaster. In particular, all V-cardinals $\geq\lambda$ are preserved.*

2. *$\mathbb{R}$ preserves all the cardinals outside $((\kappa^+)^V, \lambda)$, while collapses the cardinals there to $(\kappa^+)^V$. In particular,*

$$V[R] \models \text{“}(\kappa^+)^V = \kappa^+ \wedge \lambda = \kappa^{++}\text{”}.$$

3. *$V[R] \models \text{“}2^\kappa = \lambda^+ = \kappa^{+3}\text{”}$.*

4. *$V[R] \models$ “κ is strong limit with $\mathrm{cof}(\kappa) = \delta$”.*

Proof.

1. Let $A \in [\mathbb{R}]^\lambda$. By extending if necessary the conditions of A we may further assume that A is of the form $\{(p_\alpha, \dot{q}_\alpha, r_\alpha) \mid \alpha < \lambda\}$, where

$$p_\alpha \Vdash_{\mathbb{A}_{\lambda^+}} \dot{q}_\alpha = \langle (\check{\beta}^\alpha_0, \dot{A}^\alpha_0), \dots, (\check{\beta}^\alpha_{m_\alpha - 1}, \dot{A}^\alpha_{m_\alpha - 1}), (\check{\kappa}, \dot{A}^\alpha_{m_\alpha}) \rangle.^8$$

Since $\mathbb{A}_{\lambda^+}$ is κ^+-Knaster by passing to a set $\mathcal{I} \in [\lambda]^\lambda$ we may assume that, for all $\alpha, \gamma \in \mathcal{I}$, $p_\alpha \parallel p_\gamma$, $m^* = m_\alpha = m_\gamma$ and $\beta^\alpha_i = \beta^\gamma_i$, for $i < m^*$. Observe that for all $\alpha, \gamma \in \mathcal{I}$, $(p_\alpha \cup p_\gamma, \dot{q}_\alpha \wedge \dot{q}_\gamma)$ witnesses compatibility of $(p_\alpha, \dot{q}_\alpha)$ and $(p_\gamma, \dot{q}_\gamma)$.

On the other hand, appealing to the Δ-system lemma [Kun14, Ch. 3, Lemma 6.15], we may refine $\mathcal{I}$ to $\mathcal{J} \in [\mathcal{I}]^\lambda$ and find $\Delta \in [\mathcal{B}]^{<\kappa^+}$ and r^* in such a way that $\{\mathrm{dom}(r_\alpha) \mid \alpha \in \mathcal{J}\}$ forms a Δ-system with $r_\alpha \upharpoonright \Delta = r^*$, for $\alpha \in \mathcal{J}$. Indeed, this is feasible because the set of $\bigcup_{\gamma \in \Delta} (\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{M}}^\pi_\gamma)$-names has cardinality less than λ. Altogether this shows that $\{p_\alpha \in A \mid \alpha \in \mathcal{J}\}$ is a subset of A of pairwise compatible conditions with cardinality λ.

2. The preservation of cardinals $\geq\lambda$ is a consequence of item (1), so we are left with discussing what occurs with V-cardinals $<\lambda$. Let us begin arguing that cardinals $\leq(\kappa^+)^V$ are preserved. To this aim observe that $\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}$ preserves cardinals $\leq(\kappa^+)^V$, hence Proposition 7.2.12(3) implies that $\mathbb{R}$ preserves cardinals $\leq(\kappa^+)^V$.[9]

Let us now argue that $\mathbb{R}$ collapses all V-cardinals in (κ^+, λ).[10] Assuming this was true it is clear that they have to be collapsed to κ^+. For a V-cardinal $\theta \in (\kappa^+, \lambda)$, let $\eta_\theta := \min \mathcal{B} \cap (\theta, \lambda)$. Notice that $\mathbb{R}$ projects onto $\mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\eta_\theta)} * \dot{\mathbb{M}}^\pi_{\eta_\theta}) * \dot{\mathrm{Add}}(\kappa^+, 1)$ via the map $(p, \dot{q}, r) \mapsto (\sigma^{\lambda^+}_{\eta_\theta}(p, \dot{q}), r(\eta_\theta))$, so if this latter forcing collapses θ then $\mathbb{R}$ also.

Claim 7.2.13.1. $\mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\eta_\theta)} * \dot{\mathbb{M}}^\pi_{\eta_\theta}) * \dot{\mathrm{Add}}(\kappa^+, 1)$ *collapses the interval $(\kappa^+, |\eta_\theta|]$. In particular, θ is collapsed.*

[8] See the discussion carried out in the proof Proposition 7.2.12 (3).

[9] Actually, $\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}$ is cardinal-preserving.

[10] Observe that there is no confusion between κ^+ and $(\kappa^+)^V$.

Proof of claim. Observe that $\mathbb{A}_{\mathrm{Even}(\eta_\theta)} * \dot{\mathbb{M}}^\pi_{\eta_\theta}$ yields a generic extension where $2^\kappa \geq |\eta_\theta|$ and $(\kappa^+)^{\mathbb{A}_{\mathrm{Even}(\eta_\theta)} * \dot{\mathbb{M}}^\pi_{\eta_\theta}} = \kappa^+$. Working there, let $\langle f_\xi \mid \xi < |\eta_\theta| \rangle$ be an enumeration of $|\eta_\theta|$-many different Cohen functions added by this forcing. For each $\xi \in |\eta_\theta|$, set

$$D_\xi := \{r \in \mathrm{Add}(\kappa^+, 1) \mid \exists \zeta < \kappa^+ \, \forall \gamma < \kappa \, f_\xi(\gamma) = p(\zeta + \gamma)\}.$$

It is fairly easy to check that D_ξ is a dense subset of $\mathrm{Add}(\kappa^+, 1)$. Let $A \subseteq \mathrm{Add}(\kappa^+, 1)$ generic over $V^{\mathbb{A}_{\mathrm{Even}(\eta_\theta)} * \dot{\mathbb{M}}^\pi_{\eta_\theta}}$ and define $\Phi : |\eta_\theta| \to \kappa^+$ by $\Phi(\xi) := \min\{\zeta < \kappa^+ \mid \exists r \in A \, (\zeta \text{ witnesses that } r \in D_\xi)\}$. To prove the claim observe that it would suffice with showing that Φ is an injective function. For so, let $\xi \neq \xi'$ and assume that $\Phi(\xi) = \Phi(\xi')$. Denote this common value by σ. By definition, there are $r, s \in A$ such that, for all $\gamma < \kappa$, $f_\xi(\gamma) = r(\sigma + \gamma)$ and $f_{\xi'}(\gamma) = s(\sigma + \gamma)$ but, since A is a filter, this entails $f_\xi = f_{\xi'}$, which yields a contradiction. $\square$

3. By using the $\mathsf{GCH}_{\geq \kappa}$ in the ground model, counting $\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}$-nice names arguments yield "$2^\kappa = \lambda^+$" in the corresponding generic extension. Now use Proposition 7.2.12(3).

4. Once again, this follows by combing Proposition 7.2.12(3) with the fact that $\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}$ forces this property. $\square$

7.3 $\mathrm{TP}(\kappa^{++})$ holds

In the present section we will prove that $V[R] \models \mathrm{TP}(\kappa^{++})$. For a more neat presentation we will simply give details in case $\Theta = \lambda^+$. The main ideas involved in the proof of the general case can be found in Section 7.4.

Let us briefly summarize the structure of the argument. First we beging proving that any counterexample for $\mathrm{TP}(\lambda)$ in $V[R]$ lies in an intermediate extension of $\mathbb{R}$. More formally, any λ-Aronszajn tree in $V[R]$ is a λ-Aronszajn tree in a generic extension given by some truncation of $\mathbb{R}$ (see Proposition 7.3.13). These truncations have the important feature that they are isomorphic to a Mitchell forcing $\mathbb{R}^*$ without mismatches between the Cohen and the collapsing component.

In latter arguments we shall again consider truncations of $\mathbb{R}^*$, $\mathbb{R}^* \restriction \xi$, and use the weak compactness of λ to prove that any λ-Aronszajn tree in $V^{\mathbb{R}^*}$ reflects to a ξ-Aronszajn tree in $V^{\mathbb{R}^* \restriction \xi}$ (see Lemma 7.3.13). Then, we will be in conditions to use Unger's ideas [Ung13] to show that there are no ξ-Aronszajn trees in $V^{\mathbb{R}^* \restriction \xi}$, and thus that $V[R] \models \mathrm{TP}(\lambda)$. For the record of this section, recall that $\xi_0 \in \mathcal{A} \setminus \lambda + 1$ is the ordinal fixed in the previous section.

Definition 7.3.1 (Truncations of $\mathbb{R}$)**.** Let $\xi \in \mathcal{A} \cap (\xi_0, \lambda^+)$. A condition in $\mathbb{R} \restriction \xi$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_\xi * \dot{\mathbb{M}}_\xi$;
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}]^{<\kappa^+}$;
3. For every $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a $\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{M}}^\pi_\alpha$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{M}}^\pi_\alpha} \Vdash_{\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{M}}^\pi_\alpha} \text{“}\dot{r}(\alpha) \in \dot{\mathrm{Add}}(\kappa^+, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R} \restriction \xi$ we will write $(p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \leq_{\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{M}}^\pi_\alpha} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\alpha \in \mathrm{dom}(r_1)$, $\sigma^\alpha_\beta(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{M}}^\pi_\alpha}$ “$\dot{r}_0(\alpha) \leq \dot{r}_1(\alpha)$”.

The following is an immediate consequence of Lemma 7.2.9.

Proposition 7.3.2. *Let $\xi \in \mathcal{A} \cap (\xi_0, \lambda^+)$. Then there is a projection between $\mathbb{R}$ and $\mathrm{RO}^+(\mathbb{R} \restriction \xi)$.*

Proposition 7.3.3. *Let $\dot{T}$ be a $\mathbb{R}$-name for a λ-Aronszajn tree. There is $\xi^* \in \mathcal{A} \cap (\xi_0, \lambda^+)$, such that $V^{\mathbb{R} \restriction \xi^*} \models$ “T is a λ-Aronszajn tree”*

Proof. Let $\dot{T}$ be a $\mathbb{R}$-name for a λ-Aronszajn tree T. Without loss of generality we may assume $\mathbb{1}_\mathbb{R} \Vdash_\mathbb{R} \dot{T} \subseteq \check{\lambda}$. Let $\{A_\alpha\}_{\alpha<\lambda}$ be a family of maximal antichains deciding “$\check{\alpha} \in \dot{T}$”. Set $A^* := \bigcup_{\alpha \in \lambda} A_\alpha$ and observe that $|A^*| \leq \lambda$. In particular, there is some $\xi^* \in \mathcal{A} \cap (\xi_0, \lambda^+)$ be such that $\mathrm{dom}(p) \subseteq \kappa \times \xi^*$, for any condition $(p, \dot{q}, r) \in A^*$. Clearly $\{A_\alpha\}_{\alpha<\lambda}$ is a family of maximal antichains in $\mathbb{R} \restriction \xi^*$ deciding the same assertions, hence $V^{\mathbb{R} \restriction \xi^*} \models$ “T is λ-Aronszajn”. $\square$

Let $\pi^* : \xi^* \to \lambda$ be a bijection extending π. We use π^* to define an $\in$-isomorphism between $V^{\mathbb{A}_{\xi^*}}$ and $V^{\mathbb{A}_\lambda}$.[11] Again, $\mathcal{U}^{\pi^*}_\lambda := (\pi^*(\dot{\mathcal{U}}_{\xi^*}))_{\pi^*[G \restriction \xi^*]}$ is a coherent sequence of measures which (pointwise) extends $\mathcal{U}^\pi_\lambda$. Let $\mathbb{M}^{\pi^*}_\lambda := \mathbb{M}_{\mathcal{U}^{\pi^*}_\lambda}$. Let us denote by $\mathcal{U}^{\pi^*}_\lambda(\alpha, \beta)$ the measures appearing in $\mathcal{U}^{\pi^*}_\lambda$. For the ease of notation, let H^* be the $\mathbb{A}_{\mathrm{Even}(\lambda)}$-generic filter generated by $\pi^*[G \restriction \xi^*]$.

Proposition 7.3.4.

1. *There is an isomorphism $\varphi : \mathbb{A}_{\xi^*} * \dot{\mathbb{M}}_{\xi^*} \to \mathbb{A}_\lambda * \dot{\mathbb{M}}^{\pi^*}_\lambda$.*
2. *For each $\xi \in \mathcal{B}$ the function $\varrho^\lambda_\xi = \sigma^{\xi^*}_\xi \circ \varphi^{-1}$ establishes a projection between $\mathbb{A}_\lambda * \dot{\mathbb{M}}^{\pi^*}_\lambda$ and $\mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi)$.*

[11] This choice will guarantee that our future construction coheres with the previous one.

Proof. For (1), recall that the subposet of $\mathbb{A}_{\xi^*} * \dot{\mathbb{M}}_{\xi^*}$ with conditions of the form $s := (p, \langle\langle\check{\beta}_0, \dot{A}_0\rangle, \dots, \langle\check{\beta}_{n-1}, \dot{A}_{n-1}\rangle, \langle\check{\kappa}, \dot{A}_n\rangle\rangle)$ is dense. Analogously, the same is true for $\mathbb{A}_\lambda * \dot{\mathbb{M}}^{\pi^*}_\lambda$. It is now routine to check that the map $s \mapsto (\pi^*(p), \langle\langle\check{\beta}_0, \dot{A}_0\rangle, \dots, \langle\check{\beta}_{n-1}, \dot{A}_{n-1}\rangle, \langle\check{\kappa}, \dot{A}_n\rangle\rangle)$ defines an isomorphism between these two dense subposets, which is enough to prove the desired result. Observe that now (2) is immediate as $\sigma^{\xi^*}_\xi$ is a projection. □

Definition 7.3.5. A condition in $\mathbb{R}^*$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_\lambda * \dot{\mathbb{M}}^{\pi^*}_\lambda$;
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}]^{<\kappa^+}$;
3. For every $\xi \in \mathrm{dom}(r)$, $r(\xi)$ is an $\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^{\pi}_\xi$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^{\pi}_\xi} \Vdash_{\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^{\pi}_\xi} \text{“}\dot{r}(\xi) \in \dot{\mathrm{Add}}(\kappa^+, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}^*$ we will write $(p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \leq_{\mathbb{A}_{\mathrm{Even}(\lambda)} * \dot{\mathbb{M}}^{\pi^*}_\lambda} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\xi \in \mathrm{dom}(r_1)$, $\varrho^\lambda_\xi(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^{\pi}_\xi} \dot{r}_0(\xi) \leq \dot{r}_1(\xi)$.

Proposition 7.3.6. *$\mathbb{R}^*$ and $\mathbb{R} \restriction \xi^*$ are isomorphic. In particular, $\mathbb{R}^*$ forces that $\dot{T}$ is a λ-Aronszajn tree.*

Proof. It is not hard to check that $(p, \dot{q}, r) \mapsto (\varphi(p, \dot{q}), r)$ defines an isomorphism between both forcings. □

We briefly digress from our previous discussion to introduce the notion of Π^1_1-indescribability, which will be necessary in future arguments.

Definition 7.3.7 (Π^1_1-indescribability)**.** Let θ be a cardinal and $X \in \mathcal{P}(\theta)$. We will say that X is Π^1_1-indescribable in θ if for each $Y \subseteq V_\theta$ and each Π^1_1 sentence Φ, if $\langle V_\theta, \in, Y\rangle \models \Phi$, then there is an ordinal $\eta \in X$ be such that $\langle V_\eta, \in, Y \cap V_\eta\rangle \models \Phi$. The cardinal θ is said to be Π^1_1-indescribable if θ is Π^1_1-indescribable in θ.

A classical result of Hanf and Scott [Kan09, Theorem 6.4] establishes that θ is weakly compact if and only if θ is Π^1_1-indescribable. Thus, assuming that θ is weakly compact, it is not hard to prove that

$$\mathcal{F}_\theta := \{X \in \mathcal{P}(\theta) \mid \text{“}\theta \setminus X \text{ is not } \Pi^1_1\text{-indescribable in } \theta\text{”}\}$$

forms a proper filter on θ. An important property of $\mathcal{F}_\theta$ discovered by Levy is normality [Kan09, Proposition 6.11]. This implies in particular that $\mathcal{F}_\theta$ extends $\mathrm{Club}(\theta)$, *the club filter on* θ, and thus concetrates on the set of Mahlo cardinals below θ. We shall use this to obtain the following refinement of the set $\mathcal{B}$. To this aim, recall that H^* stands for the generic filter $\pi^*[G \restriction \xi^*]$.

Lemma 7.3.8. *There is $\mathcal{B}^* \in (\mathcal{F}_\lambda)^V$, $\mathcal{B}^* \subseteq \mathcal{B}$, with $\kappa^+ < \min \mathcal{B}^*$ such that for every $\xi \in \mathcal{B}^*$, $\langle \dot{\mathcal{U}}^{\pi^*}_\lambda(\alpha,\beta)_{H^*} \cap V[H^* \restriction \xi] \mid \alpha \leq \kappa,\, \beta < o^{\dot{\mathcal{U}}^{\pi^*}_\lambda}(\alpha)\rangle$ is a coherent sequence of measures in $V[H^* \restriction \xi]$.*

Proof. The construction of $\mathcal{B}^*$ is the same as for $\mathcal{B}$ but starting from $\mathcal{B}$ instead of λ (c.f. Lemma 7.2.7). By construction, $\mathcal{B}^*$ is an unbounded set closed by increasing sequences of length $\geq \kappa^+$, hence $\mathcal{B}^* \in (\mathcal{F}_\lambda)^V$. □

Notation 7.3.9. For each $\xi \in \mathcal{B}^*$, let $\mathcal{U}^{\pi^*}_\xi$ denote the sequence witnessing Lemma 7.3.8. Set $\mathbb{M}^{\pi^*}_\xi := \mathbb{M}_{\mathcal{U}^{\pi^*}_\xi}$.

Lemma 7.3.10. *Let $\hat{\mathcal{B}}^* = \mathcal{B}^* \cup \{\lambda\}$ and $\xi \leq \eta \in \hat{\mathcal{B}}^*$. There are projections*

1. $\varrho^\eta_\xi : \mathbb{A}_\eta * \dot{\mathbb{M}}^{\pi^*}_\eta \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi)$,
2. $\tau^\eta_\xi : \mathbb{A}_{\mathrm{Even}(\eta)} * \dot{\mathbb{M}}^\pi_\eta \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi)$,
3. $\hat{\varrho}^\eta_\xi : \mathrm{RO}^+(\mathbb{A}_\eta * \dot{\mathbb{M}}^{\pi^*}_\eta) \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi)$.
4. $\hat{\tau}^\eta_\xi : \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\eta)} * \dot{\mathbb{M}}^\pi_\eta) \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi)$,

such that $\tau^\lambda_\xi = \hat{\tau}^\eta_\xi \circ \tau^\lambda_\eta$ and $\varrho^\eta_\xi = \hat{\tau}^\eta_\xi \circ \varrho^\xi_\xi$. Moreover, $\varrho^\eta_\xi = \sigma^\eta_\xi$.

Proof. The construction of ϱ^η_ξ, τ^η_ξ, $\hat{\varrho}^\eta_\xi$ and $\hat{\tau}^\eta_\xi$ is analogous to Lemma 7.2.9, again using the adequate version of Proposition 7.2.6. A proof for the moreover part can be found in [FHS18, Lemma 3.18]. □

Definition 7.3.11 (Truncations of $\mathbb{R}^*$)**.** Let $\xi \in \mathcal{B}^*$. A condition in $\mathbb{R}^* \restriction \xi$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_\xi * \dot{\mathbb{M}}^{\pi^*}_\xi$,
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}^* \cap \xi]^{<\kappa^+}$;
3. For every $\zeta \in \mathrm{dom}(r)$, $r(\zeta)$ is a $\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^\pi_\zeta$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^\pi_\zeta} \Vdash_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^\pi_\zeta} \text{“}\dot{r}(\zeta) \in \mathrm{Add}(\kappa^+, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}^* \restriction \xi$ we will write $(p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \leq_{\mathbb{A}_\xi * \dot{\mathbb{M}}^{\pi^*}_\xi} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\zeta \in \mathrm{dom}(r_1)$, $\varrho^\xi_\zeta(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^\pi_\zeta} \dot{r}_0(\zeta) \leq \dot{r}_1(\zeta)$.

The proof of the next result is analogous to Proposition 7.3.2.

Proposition 7.3.12. *For each $\xi \in \mathcal{B}^*$, there is a projection between $\mathbb{R}^*$ and $\mathrm{RO}^+(\mathbb{R}^* \restriction \xi)$. In particular, $\mathbb{R}^*$ is isomorphic to the iteration $\mathbb{R}^* \restriction \xi * (\mathbb{R}^* / \mathbb{R}^* \restriction \xi)$.*

Lemma 7.3.13. *Assume there is a λ-Aronszajn tree T in $V^{\mathbb{R}^*}$. Then there is $\xi \in \mathcal{B}^*$ such that $T \cap \xi$ is a ξ-Aronszajn tree in $V^{\mathbb{R}^* \restriction \xi}$.*

Proof. Let $\dot{T}$ be a $\mathbb{R}^*$-name such that $\mathbb{1}_{\mathbb{R}^*} \Vdash_{\mathbb{R}^*}$ "$\dot{T}$ is a λ-Aronszajn tree". Without loss of generality we may assume that $\dot{T}$ is a $\mathbb{R}^*$-name for a subset of λ. It is not hard to check that the above forcing sentence is equivalent to a Π^1_1 sentence Φ in the language $\mathcal{L} = \{\in, \mathbb{R}^*, \dot{T}, \lambda\}$. Since λ is weakly compact, hence Π^1_1-indescribable, there is a set $X \in (\mathcal{F}_\lambda)^V$ such that for each $\xi \in X$, $\langle V_\xi, \in, \mathbb{R}^* \cap V_\xi, \dot{T} \cap \xi, \xi\rangle \models \Phi$. By Lemma 7.3.8 and the former discussion we can assume that all these ξ are Mahlo and that $\xi \in \mathcal{B}^*$. In particular, $\mathbb{R}^* \cap V_\xi = \mathbb{R}^* \restriction \xi$, and thus $\langle V_\xi, \in, \mathbb{R}^* \restriction \xi, \dot{T} \cap \xi, \xi\rangle \models \Phi$. Notice that Φ is absolute between the universe of sets and this structure, hence $\mathbb{1}_{\mathbb{R}^* \restriction \xi} \Vdash_{\mathbb{R}^* \restriction \xi}$ "$\dot{T} \cap \xi$ is a ξ-Aronszajn tree". □

Lemma 7.3.14. *Assume that there is a λ-Aronszajn tree $T \subseteq \lambda$ in $V^{\mathbb{R}^*}$. Let $\xi \in \mathcal{B}^*$ be as in the previous lemma. Then $\mathbb{R}^*/(\mathbb{R}^* \restriction \xi)$ adds b_ξ, a cofinal branch throughout $T \cap \xi$.*

Proof. Observe that in $V^{\mathbb{R}^*}$ there is a cofinal branch b_ξ for $T \cap \xi$, as T is a λ-tree. Nonetheless, $T \cap \xi$ is ξ-Aronszajn in $V^{\mathbb{R}^* \restriction \xi}$ so this branch must be added by the quotient $\mathbb{R}^*/(\mathbb{R}^* \restriction \xi)$. □

By combining Proposition 7.3.3 and 7.3.6 with the above lemma it follows that if the quotients $\mathbb{R}^*/(\mathbb{R}^* \restriction \xi)$ do not add ξ-branches then $\mathrm{TP}(\lambda)$ holds in $V[R]$.

In the next series of lemmas we will prove that for each $\xi \in \mathcal{B}^*$ there are forcings $\mathbb{P}_\xi$ and $\mathbb{Q}^{\mathrm{Even}}_\xi$ fulfilling the following properties:

(α_ξ) $\mathbb{P}_\xi \times \mathbb{Q}^{\mathrm{Even}}_\xi$ projects onto $\mathbb{R}^*/(\mathbb{R}^* \restriction \xi)$ in $V^{\mathbb{R}^* \restriction \xi}$.

(β_ξ) $\mathbb{P}_\xi \times \mathbb{Q}^{\mathrm{Even}}_\xi$ does not add new branches to $T \cap \xi$ over $V^{\mathbb{R}^* \restriction \xi}$.

Combining (α_ξ) and (β_ξ) we would conclude that $\mathbb{R}^*/(\mathbb{R}^* \restriction \xi)$ does not add ξ-branches to $T \cap \xi$. In particular, if this is true for each $\xi \in \mathcal{B}^*$ then $V[R] \models \mathrm{TP}(\lambda)$.

We now introduce a subforcing of $\mathbb{R}^* \restriction \xi$ which we will use to prove properties (α_ξ) and (β_ξ).

Definition 7.3.15. Let $\xi \in \mathcal{B}^*$. A condition in the poset $\mathbb{R}^*_{\mathrm{Even}} \restriction \xi$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi$,
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}^* \cap \xi]^{<\kappa^+}$;
3. For every $\zeta \in \mathrm{dom}(r)$, $r(\zeta)$ is a $\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^\pi_\zeta$-name such that

$$\underset{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^\pi_\zeta}{\mathbb{1}} \Vdash_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^\pi_\zeta} \text{“}\dot{r}(\zeta) \in \mathrm{Add}(\kappa^+, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}^* \restriction \xi$ we will write $(p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \leq_{\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^{\pi}_{\xi}} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\zeta \in \mathrm{dom}(r_1)$, $\tau^{\xi}_{\zeta}(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{M}}^{\pi}_{\zeta}} \dot{r}_0(\zeta) \leq \dot{r}_1(\zeta)$.

Clearly $\mathbb{R}^* \restriction \xi$ projects onto $\mathbb{R}^*_{\mathrm{Even}} \restriction \xi$, for each $\xi \in \mathcal{B}^* \cup \{\lambda\}$. The following is a key lemma:

Lemma 7.3.16. *For each $\xi \in \mathcal{B}^*$, $\psi_\xi : \mathbb{R}^* \restriction \xi \to \mathbb{A}_{\mathrm{Odd}(\xi)} \times \mathbb{R}^*_{\mathrm{Even}} \restriction \xi$ given by $(p, \dot{q}, r) \mapsto \langle p \restriction \mathrm{Odd}(\xi), (\varrho^{\xi}_{\xi}(p, \dot{q}), r)\rangle$ defines a dense embedding. In particular, both posets are forcing equivalent and thus $V^{\mathbb{R}^* \restriction \xi}$ can be seen as a κ^+-cc extension of $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}$.*

Proof. It is routine to check that ψ_ξ is order-preserving and that it preserves incompatibility. Now let $\langle p', (p, \dot{q}, r)\rangle \in \mathbb{A}_{\mathrm{Odd}(\xi)} \times \mathbb{R}^*_{\mathrm{Even}} \restriction \xi$. Letting $\dot{q}^*$ the usual identification of $\dot{q}$ as a $\mathbb{A}_\xi$-name it follows that $\psi_\xi((p' \cup p, \dot{q}^*, r)) \leq \langle p', (p, \dot{q}, r)\rangle$. $\square$

Definition 7.3.17. For each $\xi \in \mathcal{B}^* \cup \{\lambda\}$, define $\mathbb{C}_\xi := \mathbb{A}_\xi * \dot{\mathbb{M}}^{\pi^*}_{\xi}$, $\mathbb{C}^{\mathrm{Even}}_{\xi} := \mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^{\pi}_{\xi}$ and $\mathbb{P}_\xi := \mathbb{C}_\lambda / \mathbb{C}_\xi$ and $\mathbb{U}_\xi := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}^* \restriction \xi\}$. Over $V^{\mathbb{R}^* \restriction \xi}$, define $\mathbb{Q}_\xi := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}^* / \mathbb{R}^* \restriction \xi\}$. Also, over $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}$, define $\mathbb{Q}^{\mathrm{Even}}_{\xi} := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}^* / \mathbb{R}^*_{\mathrm{Even}} \restriction \xi\}$,

Arguing respectively as in Proposition 7.2.12 and Proposition 7.2.13 one obtains the following:

Proposition 7.3.18. *For each $\xi \in \mathcal{B}^*$, the following hold:*

1. *$\mathbb{U}_\xi$ is κ^+-directed closed.*
2. *$\mathbb{C}_\xi \times \mathbb{U}_\xi$ projects onto $\mathbb{R}^* \restriction \xi$ via the map $\langle (p, \dot{q}), (\mathbb{1}, \dot{\mathbb{1}}, r)\rangle \mapsto (p, \dot{q}, r)$.*
3. *$V^{\mathbb{C}_\xi}$ and $V^{\mathbb{R}^* \restriction \xi}$ have the same $<\kappa^+$-sequences. The same is true for $V^{\mathbb{C}^{\mathrm{Even}}_{\xi}}$ and $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}$.*

Proposition 7.3.19. *For each $\xi \in \mathcal{B}^*$, the following hold:*

1. *$\mathbb{R}^* \restriction \xi$ is ξ-Knaster. In particular, all V-cardinals $\geq \xi$ are preserved.*
2. *$\mathbb{R}^* \restriction \xi$ preserves all the cardinals outside the interval $((\kappa^+)^V, \xi)$, while collapses the cardinals there to $(\kappa^+)^V$. In particular,*
$$V^{\mathbb{R}^* \restriction \xi} \models \text{“}(\kappa^+)^V = \kappa^+ \wedge \xi = \kappa^{++}\text{”}.$$
3. *$V^{\mathbb{R}^* \restriction \xi} \models$ “κ is strong limit with $\mathrm{cof}(\kappa) = \delta$”.*
4. *$V^{\mathbb{R}^* \restriction \xi} \models$ “$2^\kappa \geq \xi$”.*

The above results are also true regarded in $V^{\mathbb{R}^*_{\mathrm{Even}}\restriction\xi}$.

Lemma 7.3.20. *For each $\xi \in \mathcal{B}^*$, $\mathbb{Q}^{\mathrm{Even}}_\xi$ is κ^+-directed closed over $V^{\mathbb{R}^*_{\mathrm{Even}}\restriction\xi}$.*

Proof. The argument is the same as in the proof of Proposition 7.2.12(1), using the fact that $V^{\mathbb{R}^*_{\mathrm{Even}}\restriction\xi}$ and $V^{\mathbb{C}^{\mathrm{Even}}_\xi}$ have the same subsets of κ. $\square$

Remark 7.3.21. Despite in [FHS18] is claimed that $\mathbb{Q}_\xi$ is κ^+-closed over $V^{\mathbb{R}^*\restriction\xi}$ this is not the case. Let $x \in {}^\kappa 2$ be in $V^{\mathbb{R}^*\restriction\xi} \setminus V^{\mathbb{A}_{\mathrm{Even}(\xi)}*\dot{\mathbb{M}}^\pi_\xi}$ and fix $\gamma \in \mathcal{B}^* \cap \xi$. For each $\alpha < \kappa$, set $r_\alpha(\gamma) := x \restriction \alpha \in V$. Clearly, $\langle r_\alpha(\gamma) \mid \alpha < \kappa\rangle$ defines a decreasing sequence of conditions of $\mathrm{Add}(\kappa^+, 1)$ in $V^{\mathbb{A}_{\mathrm{Even}(\gamma)}*\dot{\mathbb{M}}^\pi_\gamma}$. Observe however that the only possible value for a lower bound is $r_\kappa(\gamma) = x$, which is not an element of the inner model $V^{\mathbb{A}_{\mathrm{Even}(\xi)}*\dot{\mathbb{M}}^\pi_\xi}$.

In the next series of lemmas we show that $\mathbb{P}_\xi \times \mathbb{Q}^{\mathrm{Even}(\xi)}_\xi$ satisfies (α_ξ) and (β_ξ), which in particular shows that the argument of [FHS18, §3] is repairable.

Lemma 7.3.22. *For each $\xi \in \mathcal{B}^*$, the identity map defines a projection between $\mathbb{Q}^{\mathrm{Even}}_\xi$ and $\mathbb{Q}_\xi$.*

Proof. Clearly $\mathbb{R}^*$ projects onto $\mathbb{R}^* \restriction \xi$ and this latter onto $\mathbb{R}^*_{\mathrm{Even}} \restriction \xi$. Let us denote each of these projections by π_ξ and π^{Even}_ξ, respectively. Observe that it is enough to show that our intended projection is well-defined: Let $(\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{Q}^{\mathrm{Even}}_\xi$. By definition, $(\mathbb{1}, \dot{\mathbb{1}}, \pi^{\mathrm{Even}}_\xi \circ \pi_\xi(r)) \in \dot{G}_{\mathbb{R}^*_{\mathrm{Even}}\restriction\xi}$. Again, by definition, $(\mathbb{1}, \dot{\mathbb{1}}, \pi_\xi(r)) \in \dot{G}_{\mathbb{R}^*\restriction\xi}$, which yields $(\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{Q}_\xi$, as desired. $\square$

Proposition 7.3.23. *For each $\xi \in \mathcal{B}^*$, $\mathbb{P}_\xi \times \mathbb{Q}^{\mathrm{Even}}_\xi$ satisfies (α_ξ).*

Proof. By the above lemma it is enough with checking that $\mathbb{P}_\xi \times \mathbb{Q}_\xi$ satisfies (α_ξ). By definition, a condition in $\mathbb{R}^*/\mathbb{R}^* \restriction \xi$ is a triple $(p, \dot{q}, r)$ such that $(\pi^\lambda_\xi(p, \dot{q}), r \restriction \xi) \in \mathbb{R}^* \restriction \xi$, where π^λ_ξ is the composition of ϱ^λ_ξ with the standard isomorphism between $\mathbb{C}_\xi$ and $\mathrm{RO}^+(\mathbb{C}_\xi)$. In particular, $(p, \dot{q}) \in \mathbb{P}_\xi$. Now, it is immediate to check that $\tau : \mathbb{P}_\xi \times \mathbb{Q}_\xi \to \mathbb{R}^*/\mathbb{R}^* \restriction \xi$ given by $\langle (p, \dot{q}), (\mathbb{1}, \mathbb{1}, r)\rangle \mapsto (p, \dot{q}, r)$ defines a projection. $\square$

It thus remains to prove that $\mathbb{P}_\xi \times \mathbb{Q}^{\mathrm{Even}}_\xi$ satisfies (β_ξ). For this we will need the following preservation lemmas from [Ung12] and [Ung13], respectively.

Lemma 7.3.24.

- *(a) Assume $2^\tau \geq \eta$, $\mathbb{P}$ is τ^+-c.c., and $\mathbb{R}$ is τ^+-closed. Suppose $\dot{T}$ is a $\mathbb{P}$-name for an η-tree. Then in $V[G_\mathbb{P}]$, forcing with $\mathbb{R}$ can not add any new cofinal branches through $\dot{T}_{G_\mathbb{P}}$.*
- *(b) Suppose κ is a regular cardinal, T is a κ-tree and $\mathbb{P}$ is such that $\mathbb{P} \times \mathbb{P}$ is κ-c.c. Then forcing with $\mathbb{P}$ can not add new cofinal branches through T.*

Proposition 7.3.25. *Let $\xi \in \mathcal{B}^*$. If $\mathbb{P}_\xi \times \mathbb{P}_\xi$ is κ^+-cc over $V^{\mathbb{C}_\xi}$ then $\mathbb{P}_\xi \times \mathbb{Q}_\xi^{\mathrm{Even}}$ witnesses (β_ξ).*

Proof. Let us first prove that if $\mathbb{P}_\xi \times \mathbb{P}_\xi$ is κ^+-cc over $V^{\mathbb{R}^* \restriction \xi}$ then $\mathbb{P}_\xi \times \mathbb{Q}_\xi^{\mathrm{Even}}$ witnesses (β_ξ). By Proposition 7.3.16 we can identify $V^{\mathbb{R}^* \restriction \xi}$ as $V^{\mathbb{A}_{\mathrm{Odd}(\xi)} \times \mathbb{R}^*_{\mathrm{Even}} \restriction \xi}$. Clearly, $\mathbb{A}_{\mathrm{Odd}(\xi)} * (\mathbb{P}_\xi \times \mathbb{P}_\xi)$ is κ^+-cc over $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}$.

Let $G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}$ be a $\mathbb{R}^*_{\mathrm{Even}} \restriction \xi$-generic filter over V and let $\tau \in V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}]$ be an $\mathbb{A}_{\mathrm{Odd}(\xi)} (\cong \mathbb{R}^* \restriction \xi / \dot{G}_{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi})$-name for $T \cap \xi$. Then we can consider τ as an $\mathbb{A}_{\mathrm{Odd}(\xi)} * \mathbb{P}_\xi$-name for $T \cap \xi$ as well. It then follows from Lemma 7.3.24(a) that the tree $T \cap \xi$ has the same cofinal branches in the models

$$V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}][G_{\mathbb{A}_{\mathrm{Odd}(\xi)}} * G_{\mathbb{P}_\xi}][G_{\mathbb{Q}_\xi^{\mathrm{Even}}}]$$

and

$$V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}][G_{\mathbb{A}_{\mathrm{Odd}(\xi)}} * G_{\mathbb{P}_\xi}]$$

On the other hand, recall that $T \cap \xi$ has the same cofinal branches in $V^{\mathbb{R}^* \restriction \xi} = V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi * \mathbb{A}_{\mathrm{Odd}(\xi)}}$. By our assumption, $\mathbb{P}_\xi \times \mathbb{P}_\xi$ is κ^+-cc over $V^{\mathbb{R}^* \restriction \xi}$ hence, by Lemma 7.3.24(b), $T \cap \xi$ has no cofinal branches in $V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}][G_{\mathbb{A}_{\mathrm{Odd}(\xi)}} * G_{\mathbb{P}_\xi}]$. The result follows as

$$V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \xi}][G_{\mathbb{A}_{\mathrm{Odd}(\xi)}} * G_{\mathbb{P}_\xi}][G_{\mathbb{Q}_\xi^{\mathrm{Even}}}] = V[G_{\mathbb{R}^* \restriction \xi}][G_{\mathbb{P}_\xi} \times G_{\mathbb{Q}_\xi^{\mathrm{Even}}}].$$

We are now left with showing that if $\mathbb{P}_\xi \times \mathbb{P}_\xi$ is κ^+-cc over $V^{\mathbb{C}_\xi}$ then it is also κ^+-cc over $V^{\mathbb{R}^* \restriction \xi}$. Indeed, observe that then $\mathbb{C}_\xi * (\mathbb{P}_\xi \times \mathbb{P}_\xi)$ is κ^+-cc over V, hence, by Easton's lemma, this is κ^+-cc over $V^{\mathbb{U}_\xi}$. Thus, $\mathbb{P}_\xi \times \mathbb{P}_\xi$ is κ^+-cc over $V^{\mathbb{C}_\xi \times \mathbb{U}_\xi}$. Since $\mathbb{C}_\xi \times \mathbb{U}_\xi$ projects onto $\mathbb{R}^* \restriction \xi$ the desired result follows. $\square$

Therefore, we are left with showing that $\mathbb{P}_\xi \times \mathbb{P}_\xi$ is κ^+-cc over $V^{\mathbb{C}_\xi}$. We will devote the next series of lemmas to this purpose.

Lemma 7.3.26. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions and $\pi : \mathbb{P} \to \mathbb{Q}$ be a projection. For every $p \in \mathbb{P}$ and $q \in \mathbb{Q}$, $q \Vdash_{\mathbb{Q}} p \notin (\mathbb{P}/\dot{\mathbb{Q}})$ if and only if for every generic filter $G \subseteq \mathbb{P}$ with $p \in G$, q is not in H, the generic filter generated by $\pi[G]$. In particular, $\pi(p) \perp q$ iff $q \Vdash_{\mathbb{Q}} p \notin (\mathbb{P}/\dot{\mathbb{Q}})$.*

Proof. The first implication is obvious. Conversely, assume that there is $q' \leq_{\mathbb{Q}} q$ be such that $q' \Vdash_{\mathbb{Q}} p \in (\mathbb{P}/\dot{\mathbb{Q}})$. Let $H \subseteq \mathbb{Q}$ be some generic filter over V containing q. Hence, $p \in \mathbb{P}/H$. Now let $G \subseteq \mathbb{P}/H$ be some generic filter over $V[H]$ containing p. Clearly $\pi[G] = H$ and $q \in H$, which yields the desired contradiction. $\square$

Convention 7.3.27. Let $\mathbb{M}$ be Magidor forcing with respect to a coherent sequence of measures $\mathcal{V}$. Hereafter, we will identify each condition $p \in \mathbb{M}$ with the sequence $\langle \vec{\alpha}^p, \vec{A}^p \rangle$, where $\vec{\alpha}^p := \langle \alpha_0^p, \dots, \alpha_{n^p}^p \rangle$ and $\vec{A}^p := \langle A_0^p, \dots, A_{n^p}^p \rangle$. We will tend to omit the superscript, which will mean that $\langle \vec{\alpha}, \vec{A} \rangle = \langle \vec{\alpha}^p, \vec{A}^p \rangle$, for some p in the corresponding Magidor forcing.

Remark 7.3.28. Let $\xi \in \mathcal{B}^* \cup \{\lambda\}$. Observe that $\tilde{\mathbb{C}}_\xi$, the subposet of $\mathbb{C}_\xi$ with conditions $(p, \dot{q})$ such that $p \Vdash_{\mathbb{A}_\xi} \dot{q} = \langle \check{\vec{\alpha}}^q, \dot{\vec{A}}^q \rangle$, is dense in $\mathbb{C}_\xi$. Thus, for our current purposes it is enough to assume that $\mathbb{C}_\xi = \tilde{\mathbb{C}}_\xi$.

Lemma 7.3.29. *Let $\xi \in \mathcal{B}^*$, $r = (p, \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle) \in \mathbb{C}_\lambda$ and $r' = (q, \langle \check{\vec{\beta}}, \dot{\vec{B}} \rangle) \in \mathbb{C}_\xi$, Then, $r' \Vdash_{\mathbb{C}_\xi}$ "$r \notin \mathbb{P}_\xi$" if and only if one of the following hold:*

1. *$p \restriction \xi \perp_{\mathbb{A}_\xi} q$;*
2. *$p \restriction \xi \parallel_{\mathbb{A}_\xi} q$ and*

$$p \cup q \Vdash_{\mathbb{A}_\lambda} \langle \check{\vec{\beta}}, \dot{\vec{B}} \rangle^\frown (\check{\vec{\alpha}} \setminus \check{\vec{\beta}}) \notin \dot{\mathbb{M}}^{\pi^*}_\xi \ \vee \ \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown (\check{\vec{\beta}} \setminus \check{\vec{\alpha}}) \notin \dot{\mathbb{M}}^{\pi^*}_\lambda.$$ [12]

Proof. First, observe that two conditions $\langle \vec{\alpha}, \vec{A} \rangle, \langle \vec{\beta}, \vec{B} \rangle \in \mathbb{M}^{\pi^*}_\lambda$ are compatible if and only if $\langle \vec{\alpha}, \vec{A} \rangle^\frown (\vec{\beta} \setminus \vec{\alpha}), \langle \vec{\beta}, \vec{B} \rangle^\frown (\vec{\alpha} \setminus \vec{\beta}) \in \mathbb{M}^{\pi^*}_\lambda$. Thereby, if some of the above conditions is true, $\varrho^\lambda_\xi(r) \perp_{\mathbb{C}_\xi} r'$. Thus, Lemma 7.3.26 yields $r' \Vdash_{\mathbb{C}_\xi}$ "$r \notin \mathbb{P}_\xi$". Conversely, assume that (1) and (2) are false and set $\vec{\gamma} := \vec{\alpha} \cup \vec{\beta}$. Since (1) is false, $p \cup q \in \mathbb{A}_\lambda$. Also, since (2) is false, we may let a condition $a \leq_{\mathbb{A}_\lambda} p \cup q$ forcing the opposite. Let $A \subseteq \mathbb{A}_\lambda$ generic (over V) containing a. By the above, in $V[A]$, $\langle \vec{\beta}, \vec{B} \rangle^\frown (\vec{\alpha} \setminus \vec{\beta}) \in \mathbb{M}^{\pi^*}_\xi$ and $\langle \vec{\alpha}, \vec{A} \rangle^\frown (\vec{\beta} \setminus \vec{\alpha}) \in \mathbb{M}^{\pi^*}_\lambda$, hence both Magidor conditions are compatible. Let $\langle \vec{\gamma}, \vec{C} \rangle \in \mathbb{M}^{\pi^*}_\lambda$ be a condition witnessing this compatibility and $S \subseteq \mathbb{M}^{\pi^*}_\lambda$ be generic (over $V[A]$) containing $\langle \vec{\gamma}, \vec{C} \rangle$. Set $r^* := (a, \langle \check{\vec{\gamma}}, \dot{\vec{C}} \rangle)$. Clearly, $r^* \in A * \dot{S}$ and $r^* \leq_{\mathbb{C}_\lambda} r$, so $r \in A * \dot{S}$. On the other hand, $\varrho^\lambda_\xi[A * \dot{S}]$ generates a $\mathbb{C}_\xi$-generic filter containing r', hence Lemma 7.3.26 yields $r' \nVdash_{\mathbb{C}_\xi}$ "$r \notin \mathbb{P}_\xi$", as wanted. □

For each $\xi \in \mathcal{B}^* \cup \{\lambda\}$ we will assume that for each $r = (p, \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle) \in \mathbb{C}_\xi$, $p \Vdash_{\mathbb{A}_\xi}$ "$\langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle$ is pruned". This is of course feasible by virtue of Proposition 7.1.10.

Lemma 7.3.30. *Let $\xi \in \mathcal{B}^*$, $r = (p, \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle) \in \mathbb{C}_\lambda$ and $r' = (q, \langle \check{\vec{\beta}}, \dot{\vec{B}} \rangle) \in \mathbb{C}_\xi$. Assume that $q \leq_{\mathbb{A}_\xi} p \restriction \xi$, $\vec{\alpha} \subseteq \vec{\beta}$ and*

$$(\Upsilon) \ \ p \cup q \Vdash_{\mathbb{A}_\lambda} \text{“}\forall \gamma \in \vec{\beta} \setminus \vec{\alpha} \left(\dot{\vec{A}}(\check{k}_\gamma) \cap \gamma \in \dot{\mathcal{F}}(\gamma) \right)\text{”},$$

where $k_\gamma := \min\{k < |\vec{\alpha}| \mid \gamma < \vec{\alpha}(k)\}$. Then there is a $\mathbb{A}_\xi$-name $\dot{\vec{C}}$ for which all the following hold:

(I) $q \Vdash_{\mathbb{A}_\xi}$ "$q^* := \langle \check{\vec{\beta}}, \dot{\vec{C}} \rangle \leq_{\mathbb{M}^{\pi^*}_\xi} \langle \check{\vec{\beta}}, \dot{\vec{B}} \rangle \wedge q^*$ is pruned".

[12]Here we are identifying the $\mathbb{A}_\xi$-name $\mathbb{M}^{\pi^*}_\xi$ with its standard extension to a $\mathbb{A}_\lambda$-name.

(II) $q \Vdash_{\mathbb{A}_\xi}$ "$\forall \tau \in [\biguplus_i \dot{\vec{C}}(i)]^{<\omega} \left(p \nVdash_{\mathbb{A}_\lambda/\mathbb{A}_\xi} \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown \tau \notin \dot{\mathbb{M}}^{\pi^*}_\lambda \right)$".

Proof. Let us work over $V^{\mathbb{A}_\xi \downarrow q}$. Let $c : [\biguplus_{i \le |\vec{\beta}|} \vec{B}(i)]^{<\omega} \to 2$ be defined as

$$c(\vec{x}) := \begin{cases} 0, & \text{if } p \Vdash_{\mathbb{A}_\lambda/\mathbb{A}_\xi} \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown \vec{x} \notin \dot{\mathbb{M}}^{\pi^*}_\lambda; \\ 1, & \text{if } p \nVdash_{\mathbb{A}_\lambda/\mathbb{A}_\xi} \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown \vec{x} \notin \dot{\mathbb{M}}^{\pi^*}_\lambda. \end{cases}$$

There is a sequence $\vec{C} := \langle C_i \mid i < |\vec{\beta}| \rangle$, such that $C_i \subseteq \vec{B}(i)$ is in $\mathcal{F}(\beta_i)$, $C_i = \biguplus_\alpha C_i(\alpha)$ with $C_i \in \mathcal{U}(\beta_i, \alpha)$, and $\vec{C}$ is homogeneous in the sense of Lemma 7.1.21. In particular, $\langle \vec{\beta}, \vec{C} \rangle \le_{\mathbb{M}^{\pi^*}_\xi} \langle \vec{\beta}, \vec{B} \rangle$. By refining $\vec{C}$ we may further assume that $\langle \vec{\beta}, \vec{C} \rangle$ is pruned (cf. Proposition 7.1.10). Thus, (I) holds. Towards a contradiction, assume that (II) is false. Let $r \le_{\mathbb{A}_\xi} q$ be such that r forces the negation of the above formula. By shrinking r we may assume that there is a block sequence $\vec{x}$ for q^* such that $r \Vdash_{\mathbb{A}_\xi}$ "$\left(p \Vdash_{\mathbb{A}_\lambda/\mathbb{A}_\xi} \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown \check{\vec{x}} \notin \dot{\mathbb{M}}^{\pi^*}_\lambda \right)$". Since $r \le_{\mathbb{A}_\xi} q$, $r \cup p \in \mathbb{A}_\lambda$, hence $r \cup p \Vdash_{\mathbb{A}_\lambda}$ $\langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown \check{\vec{x}} \notin \dot{\mathbb{M}}^{\pi^*}_\lambda$. Now, since r forces $\dot{\vec{C}}$ to be homogeneous for $\dot{c}$ (in the sense of Lemma 7.1.21), it follows that for all block sequence $\vec{y}$ with the same length as $\vec{x}$, if for all $k < |\vec{x}|$, $\vec{x}(k)$ and $\vec{y}(k)$ belong to the same set $C_i(\alpha)$, then $r \cup p \Vdash_{\mathbb{A}_\lambda} \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown \check{\vec{y}} \notin \dot{\mathbb{M}}^{\pi^*}_\lambda$. Let a consists of pairs (i, α) where for some $k < |\vec{x}|$, $\vec{x}(k) \in C_i(\alpha)$. Since p forces $\langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle$ to be pruned the only chance for this property to hold is that $r \cup p \Vdash_{\mathbb{A}_\lambda}$ "$\exists (i, \alpha) \in a,\ \dot{C}_i(\alpha) \cap \dot{\vec{A}}(\check{k}_{\beta_i})(\alpha) \cap \beta_i = \emptyset$", where $k_{\beta_i} := \min\{k < |\vec{\alpha}| \mid \beta_i \le \vec{\alpha}(k)\}$. Now let $A \subseteq \mathbb{A}_\lambda$ be a generic filter with $r \cup p \in A$. Then the above property would hold at $V[A]$, which implies that there is some $(i, \alpha) \in a$ be such that $\dot{C}_i(\alpha)_G \cap \dot{\vec{A}}(k_{\beta_i})(\alpha)_G \cap \beta_i = \emptyset$. By (Υ), $\dot{\vec{A}}(k_{\beta_i})_G \cap \beta_i \in \mathcal{F}(\beta_i)$. It thus follows that $\dot{C}_i(\alpha)_G$ and $\dot{\vec{A}}(k_{\beta_i})(\alpha)_G \cap \beta_i$ are in $\mathcal{U}(\beta_i, \alpha)$, which yields the desired contradiction. $\square$

Lemma 7.3.31. *Let $\xi \in \mathcal{B}^*$, $r = (p, \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle) \in \mathbb{C}_\lambda$ and $r' = (q, \langle \check{\vec{\beta}}, \dot{\vec{B}} \rangle) \in \mathbb{C}_\xi$. Assume that*

(ℵ) $q \le_{\mathbb{A}_\xi} p \restriction \xi$;

(ב) $\vec{\alpha} \subseteq \vec{\beta}$;

(ג) $p \cup q \Vdash_{\mathbb{A}_\lambda}$ "$\langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle^\frown (\check{\vec{\beta}} \setminus \check{\vec{\alpha}}) \in \dot{\mathbb{M}}^{\pi^*}_\lambda$".

Let $\dot{\vec{C}}$ be the sequence obtained from Lemma 7.3.30 with respect to r and r'. Then, $(q, \langle \check{\vec{\beta}}, \dot{\vec{C}} \rangle) \Vdash_{\mathbb{C}_\xi} (p, \langle \check{\vec{\alpha}}, \dot{\vec{A}} \rangle) \in \mathbb{P}_\xi$.

Proof. Otherwise, let $r^* := (r, \langle \check{\vec{\gamma}}, \dot{\vec{D}} \rangle) \le_{\mathbb{C}_\xi} (q, \langle \check{\vec{\beta}}, \dot{\vec{C}} \rangle)$ be forcing the opposite. By using Lemma 7.3.29 with respect to r^* and r it follows that either (1) or

(2) must hold. It is not hard to check that (ℵ)-(ℶ) implies that (2) holds: particularly, that $r \cup p \Vdash_{\mathbb{A}_\lambda}$ "$\langle \check{\vec{\alpha}}, \dot{\vec{A}}\rangle^\frown(\check{\vec{\gamma}} \setminus \check{\vec{\alpha}}) \notin \dot{\mathbb{M}}^{\pi^*}_\lambda$" holds. By (ℶ) and since $r \cup p \leq_{\mathbb{A}_\lambda} p \cup q$, $r \cup p \Vdash_{\mathbb{A}_\lambda}$ "$\langle \check{\vec{\alpha}}, \dot{\vec{A}}\rangle^\frown(\check{\vec{\gamma}} \setminus \check{\vec{\beta}}) \notin \dot{\mathbb{M}}^{\pi^*}_\lambda$". Clearly, $r \leq_{\mathbb{A}_\xi} q$ and $r \Vdash_{\mathbb{A}_\xi} \check{\vec{\gamma}} \setminus \check{\vec{\beta}} \in [\biguplus_i \dot{\vec{C}}(i)]^{<\omega}$. Observe that (ℶ) yields (Υ) of Lemma 7.3.30, and this latter implies $r \cup p \nVdash_{\mathbb{A}_\lambda}$ "$\langle \check{\vec{\alpha}}, \dot{\vec{A}}\rangle^\frown(\check{\vec{\gamma}} \setminus \check{\vec{\beta}}) \notin \dot{\mathbb{M}}^{\pi^*}_\lambda$", which produces the desired contradiction. □

Lemma 7.3.32. *Let $\xi \in \mathcal{B}^*$, $(q, \langle \check{\vec{\beta}}, \dot{\vec{B}}\rangle) \in \mathbb{C}_\xi$ and $\dot{r}_0, \dot{r}_1$ be two $\mathbb{C}_\xi$-names forced by $\mathbb{1}_{\mathbb{C}_\xi}$ to be in $\mathbb{P}_\xi$. Then, there are $(q^*, \langle \check{\vec{\beta}}^*, \dot{\vec{B}}^*\rangle) \in \mathbb{C}_\xi$, $(p_0, \langle \check{\vec{\alpha}}_0, \dot{\vec{A}}_0\rangle)$, $(p_1, \langle \check{\vec{\alpha}}_1, \dot{\vec{A}}_1\rangle) \in \mathbb{P}_\xi$ and $\bar{p}_0, \bar{p}_1 \in \mathbb{A}_\lambda$ be such that the following hold: For $i \in \{0,1\}$,*

- *(a)* $(q^*, \langle \check{\vec{\beta}}^*, \dot{\vec{B}}^*\rangle) \leq_{\mathbb{C}_\xi} (q, \langle \check{\vec{\beta}}, \dot{\vec{B}}\rangle)$,
- *(b_i)* $(q^*, \langle \check{\vec{\beta}}^*, \dot{\vec{B}}^*\rangle) \Vdash_{\mathbb{C}_\xi}$ "$\dot{r}_i = (p_i, \langle \check{\vec{\alpha}}_i, \dot{\vec{A}}_i\rangle) \in \mathbb{P}_\xi$",
- *(c_i)* $\bar{p}_i \leq_{\mathbb{A}_\lambda} p_i$ *and* $(q^*, \langle \check{\vec{\beta}}^*, \dot{\vec{B}}^*\rangle)$ *and* $(\bar{p}_i, \langle \check{\vec{\alpha}}_i, \dot{\vec{A}}_i\rangle)$ *satisfy conditions (1)-(3) of Lemma 8.5.29.*

Proof. Let $(q^*, \langle \check{\vec{\beta}}^*, \dot{\vec{B}}^*\rangle) \leq_{\mathbb{C}_\xi} (q, \langle \check{\vec{\beta}}, \dot{\vec{B}}\rangle)$ and $(p_0, \langle \check{\vec{\alpha}}_0, \dot{\vec{A}}_0\rangle)$, $(p_1, \langle \check{\vec{\alpha}}_1, \dot{\vec{A}}_1\rangle) \in \mathbb{P}_\xi$ be such that (b_0) and (b_1) hold. By extending q^* and $\vec{\beta}^*$ if necessary, we may further assume that $q^* \leq_{\mathbb{A}_\xi} p_0 \restriction \xi \cup p_1 \restriction \xi$ and $\vec{\alpha}_0 \cup \vec{\alpha}_1 \subseteq \vec{\beta}^*$. For each $i \in \{0,1\}$, combining this with Lemma 7.3.29 it follows that condition (3) must fail. Thus, there is $\bar{p}_i \leq_{\mathbb{A}_\lambda} q^* \cup p_i$ with $\bar{p}_i \Vdash_{\mathbb{A}_\lambda} \langle \check{\vec{\alpha}}_i, \dot{\vec{A}}_i\rangle^\frown(\check{\vec{\beta}}^* \setminus \check{\vec{\alpha}}_i) \in \dot{\mathbb{M}}^{\pi^*}_\lambda$. Again, extend p^* to ensure $q^* \leq_{\mathbb{A}_\xi} \bar{p}_0, \bar{p}_1$. It should be clear at this point that, for $i \in \{0,1\}$, $(q^*, \langle \check{\vec{\beta}}^*, \dot{\vec{B}}^*\rangle)$ and $(\bar{p}_i, \langle \check{\vec{\alpha}}_i, \dot{\vec{A}}_i\rangle)$ witness (c_i). □

We are finally in conditions to prove the κ^+-ccness of $\mathbb{P}_\xi \times \mathbb{P}_\xi$.

Lemma 7.3.33. *Let $\xi \in \mathcal{B}^*$. Then, $\mathbb{1}_{\mathbb{C}_\xi} \Vdash_{\mathbb{C}_\xi}$ "$\mathbb{P}_\xi \times \mathbb{P}_\xi$ is κ^+-cc".*

Proof. Let $\{(\dot{r}^0_\theta, \dot{r}^1_\theta)\}_{\theta<\kappa^+}$ be a collection of $\mathbb{C}_\xi$-names that $\mathbb{1}_{\mathbb{C}_\xi}$ forces to be in a maximal antichain of $\mathbb{P}_\xi \times \mathbb{P}_\xi$. Appealing to Lemma 7.3.32 we find families $\{(q^*_\theta, \langle \check{\vec{\beta}}^*_\theta, \dot{\vec{B}}^*_\theta\rangle)\}_{\theta<\kappa^+}$, $\{\langle (p^0_\theta, \langle \check{\vec{\alpha}}^0_\theta, \dot{\vec{A}}^0_\theta\rangle), (p^1_\theta, \langle \check{\vec{\alpha}}^1_\theta, \dot{\vec{A}}^1_\theta\rangle)\rangle\}_{\theta<\kappa^+}$ and $\{\langle \bar{p}^0_\theta, \bar{p}^1_\theta\rangle\}_{\theta<\kappa^+}$ witnessing it.

It is not hard to check that for each $\varrho \in \mathcal{B}^* \cup \{\lambda\}$, $\mathbb{C}_\varrho$ is κ^+-Knaster, hence $\mathbb{C}_\xi \times \mathbb{C}^2_\lambda$ also. In particular, $\mathbb{C}_\xi \times \mathbb{C}^2_\lambda$ is κ^+-cc, and thus we may assume that all the above conditions are compatible. Modulo a further refinement, we may also assume that $\vec{\beta}^*_\theta = \vec{\beta}^*$, $\vec{\alpha}^0_\theta = \vec{\alpha}^0$ and $\vec{\alpha}^1_\theta = \vec{\alpha}^1$, for each $\theta < \kappa^+$. For each $\theta < \eta < \kappa^+$, set $r_{\theta,\eta} := (q^*_\theta \cup q^*_\eta, (\vec{\beta}^*, \dot{\vec{B}}^*_\theta \wedge \dot{\vec{B}}^*_\eta))$ and $r'_{i,\theta,\eta} := (\bar{p}^i_\theta \cup \bar{p}^i_\eta, (\vec{\alpha}^i, \dot{\vec{A}}^i_\theta \wedge \dot{\vec{A}}^i_\eta))$. It is routine to check that, for each $i \in \{0,1\}$, $r_{\theta,\eta}$ and $r'_{i,\theta,\eta}$ witness the hypotheses of Lemma 8.5.29, hence there is $r^*_{\theta,\eta} \leq_{\mathbb{C}_\xi} r_{\theta,\eta}$ forcing that both $r'_{0,\theta,\eta}$ and $r'_{1,\theta,\eta}$ are in $\mathbb{P}_\xi$. In particular, $r^*_{\theta,\eta} \Vdash_{\mathbb{C}_\xi} (\dot{r}^0_\theta, \dot{r}^1_\theta) \parallel_{\mathbb{P}_\xi\times\mathbb{P}_\xi} (\dot{r}^0_\eta, \dot{r}^1_\eta)$, which entails the desired contradiction. $\square$

The combination of the above lemma with Proposition 7.3.23, Proposition 7.3.25 and the preceding discussion yields $V[R] \models \mathrm{TP}(\kappa^{++})$. However, as advanced in the introduction, in $V^{\mathbb{R}}$ the principle $\mathrm{TP}(\kappa^+)$ fails.

Proposition 7.3.34. *Assume that κ is a strong cardinal and that $\mathbb{R}$ is the forcing of Definition 7.2.10. Then, $\mathbb{1}_{\mathbb{R}} \Vdash_{\mathbb{R}}$ "$\mathrm{TP}(\kappa^+)$ fails".*

Proof. Since κ is strong, $\kappa^{<\kappa} = \kappa$, hence $\square^*_\kappa$ holds in V. Since $\mathbb{A}_\Theta * \dot{\mathbb{M}}_\Theta$ preserves κ^+ it is clear that $\mathbb{1}_{\mathbb{A}_\Theta*\dot{\mathbb{M}}_\Theta} \Vdash_{\mathbb{A}_\Theta*\dot{\mathbb{M}}_\Theta}$ "$\square^*_\kappa$ holds", hence $\mathbb{1}_{\mathbb{R}} \Vdash_{\mathbb{R}}$ "$\square^*_\kappa$ holds" (see Proposition 7.2.12(3)). By virtue of Jensen's theorem (cf. page 23) this finally yields $\mathbb{1}_{\mathbb{R}} \Vdash_{\mathbb{R}}$ "$\mathrm{TP}(\kappa^+)$ fails", as wanted. $\square$

In the next chapter we will show how to modify the forcing $\mathbb{R}$ to obtain the tree property at κ^+. At the light of Proposition 7.3.34 this will require to collapse κ^+.

7.4 Forcing arbitrary failures of the SCH_κ

In this last section we address the issue of obtaining arbitrary failures for the SCH_κ in the generic extension of Theorem 6.0.13. Rather than providing full details we will just enumerate the necessary modifications in the arguments. After our exposition we hope to have convinced the reader that the results proved through Section 7.3 still apply in the current context.

1. Assume the $\mathsf{GCH}_{\geq\kappa}$. Let $\kappa < \lambda$ be a strong and a weakly compact cardinal, respectively. Let $\Theta \geq \lambda^{++}$ be a cardinal with $\mathrm{cof}(\Theta) > \kappa$ and $\delta = \mathrm{cof}(\delta) < \kappa$. By preparing the ground model we may further assume that κ is a strong cardinal which is indestructible under adding Cohen subsets of κ. This preparatory forcing does not mess up our initial hypotheses.

2. Set $\mathbb{A}_\Theta := \mathrm{Add}(\kappa, \Theta)$ and, for each $x \in [\Theta]^\lambda$, $\mathbb{A}_x := \mathrm{Add}(\kappa, x)$. Let $G \subseteq \mathbb{A}_\Theta$ a generic filter over V and $\mathcal{U} \in V[G]$ be a coherent sequence of measures with $\ell^{\mathcal{U}} = \kappa + 1$ and $o^{\mathcal{U}}(\kappa) = \delta$. For each pair $(\alpha, \beta) \in \mathrm{dom}(\mathcal{U})$, let $\dot{\mathcal{U}}(\alpha, \beta)$ be a $\mathbb{A}_\Theta$-name such that $\mathcal{U}(\alpha, \beta) = \dot{\mathcal{U}}(\alpha, \beta)_G$. Arguing as in Lemma 7.2.3 we can prove the following:

   Lemma 7.4.1. *There exists an unbounded set $\mathcal{A} \subseteq [\Theta]^\lambda$, closed under taking limits of $\geq\kappa^+$-sequences, such that, for every $x \in \mathcal{A}$ and every $\mathbb{A}_\Theta$-generic filter $\bar{G}$, $\mathcal{U}_x := \langle \dot{\mathcal{U}}(\alpha, \beta)_{\bar{G}} \cap V[\bar{G} \restriction x] \mid \alpha \leq \kappa,\, \beta < o^{\dot{\mathcal{U}}}(\alpha)\rangle$ is a coherent sequence of measures in $V[\bar{G} \restriction x]$.*

Here we are taking advantage of Notation 7.2.1. Let $\dot{\mathcal{U}}_x$ be a $\mathbb{A}_x$-name such that $\mathcal{U}_x = (\dot{\mathcal{U}}_x)_{G\restriction x}$ Similarly, let $\dot{\mathbb{M}}_x$ denote a $\mathbb{A}_x$-name such that $\mathbb{M}_{\mathcal{U}_x} = (\dot{\mathbb{M}}_{\mathcal{U}_\xi})_{G\restriction x}$. By convention, $\mathcal{U}_\Theta := \mathcal{U}$ and $\dot{\mathbb{M}}_\Theta$ will denote a $\mathbb{A}_\Theta$-name such that $\mathbb{M}_{\mathcal{U}_\Theta} = (\dot{\mathbb{M}}_\Theta)_G$. Arguing as in Proposition 7.2.6 one may argue that $\mathbb{M}_\Theta$ projects onto $\mathbb{M}_x$, for each $x \in \mathcal{A}$.

3. Choose $x_0 \in \mathcal{A}$ with $\lambda + 1 \subseteq x_0$ be arbitrary and let $\pi : \mathbb{A}_{x_0} \to \mathbb{A}_{\mathrm{Even}(\lambda)}$ be an isomorphism. Define $\dot{\mathcal{U}}^\pi_{x_0} := \pi(\dot{\mathcal{U}}_{x_0})$. Clearly, $(\dot{\mathcal{U}}^\pi_{x_0})_{\pi[G\restriction x_0]} = (\dot{\mathcal{U}}_{x_0})_{G\restriction x_0} = \mathcal{U}_{x_0}$. Say that $\mathcal{U}^\pi_{x_0}(\alpha,\beta)$ are the measures of $\mathcal{U}^\pi_{x_0}$ and that $\dot{\mathcal{U}}^\pi_{x_0}(\alpha,\beta)$ is a $\mathbb{A}_{x_0}$-name such that $\mathcal{U}^\pi_{x_0}(\alpha,\beta) = \dot{\mathcal{U}}^\pi_{x_0}(\alpha,\beta)_{\pi[G\restriction x_0]}$. Let $\mathcal{B} \subseteq \lambda$ be as given in Lemma 7.2.7. From this point on we will be relying on Notation 7.2.8.

4. Set $\hat{\mathcal{A}} := \{x \in \mathcal{A} \mid x_0 \subseteq x\}$. Arguing as in Lemma 7.2.9 we obtain a system of projections

$$\langle \sigma^\Theta_x : \mathbb{A}_\Theta * \dot{\mathbb{M}}_\Theta \to \mathrm{RO}^+(\mathbb{A}_x * \dot{\mathbb{M}}_x) \mid \ x \in \hat{\mathcal{A}} \rangle,$$

$$\langle \hat{\sigma}^x_\xi : \mathrm{RO}^+(\mathbb{A}_x * \dot{\mathbb{M}}_x) \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi) \mid \ x \in \hat{\mathcal{A}},\, \xi \in \mathcal{B} \rangle,$$

$$\langle \sigma^\Theta_\xi : \mathbb{A}_\Theta * \dot{\mathbb{M}}_\Theta \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\xi)} * \dot{\mathbb{M}}^\pi_\xi) \mid \xi \in \mathcal{B} \rangle.$$

Also, one may guarantee that these projections commute; namely, $\sigma^\Theta_\xi = \hat{\sigma}^x_\xi \circ \sigma^\Theta_x$, for $x \in \hat{\mathcal{A}}$ and $\xi \in \mathcal{B}$.

5. Using these projections define $\mathbb{R}$ as in Definition 7.2.10. It is easy to check that $\mathbb{R}$ forces the statements of propositions 7.2.12 and 7.2.13. For each $x \in \hat{\mathcal{A}}$, let $\mathbb{R} \restriction x$ as in Definition 8.5.1. Now assume that $\mathbb{R}$ forces a failure of $\mathrm{TP}(\lambda)$. Arguing in the same fashion as in Lemma 7.3.3 one obtains a set $x^* \in \hat{\mathcal{A}}$, $x_0 \subsetneq x^*$ for which $\mathbb{R} \restriction x^*$ forces the same failure.

6. Let π^* be a bijection between $\mathbb{A}_{x^*}$ and $\mathbb{A}_\lambda$ extending π. Set $\mathcal{U}^{\pi^*}_\lambda := \pi^*(\dot{\mathcal{U}}_{x^*})_{\pi^*[G\restriction x^*]}$. It is evident that this is a coherent sequence of measures which (pointwise) extends $\mathcal{U}^\pi_\lambda$. Set $\mathbb{M}^{\pi^*}_\lambda := \mathbb{M}_{\mathcal{U}^{\pi^*}_\lambda}$.

7. Argue as in Lemma 7.3.4 to show that π^* extends to an isomorphism between $\mathbb{A}_{x^*} * \dot{\mathbb{M}}_{x^*}$ and $\mathbb{A}_\lambda * \dot{\mathbb{M}}^{\pi^*}_\lambda$, and use it to define $\mathbb{R}^*$ as in Definition 7.3.5. It can be argued that $\mathbb{R}^*$ and $\mathbb{R} \restriction x^*$ are isomorphic, hence $\mathbb{R}^*$ forces the existence of a λ-Aronszajn tree. From this point on it can be checked that all the arguments of Section 7.3 concerning $\mathbb{R}^*$ and its truncations $\mathbb{R}^* \restriction \xi$ still apply in the current context.

CHAPTER 8

The Tree Property at first and double successors of singular cardinals

8.1 Sinapova forcing

In this section we will review a forcing construction due to D. Sinapova. Originally, *Sinapova forcing* (or also *Diagonal Supercompact Magidor forcing*) was conceived to generalize Gitik-Sharon's (**GS**) theorem to uncountable cofinalities [GS08]. Also, inspired by the subsequent inquiries of Cummings and Foremann [CF] on **GS**-model, Sinapova devised this forcing to obtain a generic extension where the following hold:

1. There is a strong limit singular cardinal κ of arbitrary cofinality.

2. The SCH_κ fails.

3. There is a very good and a bad scale at κ (cf. Definition 1.4.10).

Hereafter, μ, κ, $\langle \kappa_\xi \mid \xi < \mu \rangle$, λ and Θ will be as in the statement of Theorem 6.0.14. Besides, we define $\varepsilon := \sup_{\xi<\mu} \kappa_\xi$ and $\delta := \varepsilon^+$. Since we are assuming the $\mathsf{GCH}_{\geq\kappa}$ in the ground model, modulo a suitable preparation, we may assume that $\mathsf{GCH}_{\geq\varepsilon}$ holds, $2^{\kappa_\xi} = \kappa_\xi^+$, for each $\xi < \mu$, and that $\{\kappa\} \cup \langle \kappa_{\xi+1} \mid \xi < \mu \rangle$ are Laver indestructible supercompact cardinals.[1]

Let $\mathbb{A} := \mathrm{Add}(\kappa, \Theta)$, $G \subseteq \mathbb{A}$ a generic filter and $\langle f_\eta \mid \eta \in \Theta \rangle$ be an enumeration of the generic functions added by this filter. During this section our ground model will be $V[G]$. The proof of the next series of result can be found in Sinapova's dissertation [Sin08, §2].

[1]In this section and in the latter sections 8.4 and 8.5 we will simply use that κ is Laver indestructible. The indestructibility of $\langle \kappa_{\xi+1} \mid \xi < \mu \rangle$ will be important in Section 8.6 for the proof of Lemma 8.6.11.

Proposition 8.1.1. *There is a Θ^+-supercompact embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, such that, for each $\eta < \delta$, $j(f_\eta)(\kappa) = \eta$. Also, $\kappa_\xi^{<\kappa} \leq \kappa_\xi^+$, for each $\xi < \mu$ limit.*

Proposition 8.1.2. *For all $\xi < \mu$ and all $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}_\kappa(\kappa_\xi))$, there is a κ_ξ-supercompact measure U_ξ on $\mathcal{P}_\kappa(\kappa_\xi)$ such that $\mathcal{X} \in \mathrm{Ult}(V, U_\xi)$. Also, there are functions $\langle F_\eta^\xi \mid \eta < \delta \rangle$, $F_\eta^\xi \colon \kappa \to \kappa$ such that, for each $\eta < \delta$, $j_{U_\xi}(F_\eta^\xi)(\kappa) = \eta$.*

Proposition 8.1.3. *There is a $\lhd$-sequence of measures $\langle U_\xi \mid \xi < \mu \rangle$ (i.e. $U_\xi \in \mathrm{Ult}(V, U_{\xi'})$, for $\xi < \xi'$) and functions $\langle F_\eta^\xi \mid \xi < \mu, \eta < \delta \rangle$, $F_\eta^\xi \colon \kappa \to \kappa$ such that, U_ξ is a κ_ξ-supercompact measure on $\mathcal{P}_\kappa(\kappa_\xi)$, and for all $\xi < \mu$ and $\eta < \delta$, $j_{U_\xi}(F_\eta^\xi)(\kappa) = \eta$.*

Notation 8.1.4.

- For $\xi < \mu$, $x \in \mathcal{P}_\kappa(\kappa_\xi)$ and $\kappa \leq \tau \leq \kappa_\xi$, $\tau_x := \mathrm{otp}(\tau \cap x)$.
- For $\xi < \mu$ and $x, y \in \mathcal{P}_\kappa(\kappa_\xi)$, $x \prec y$ iff $x \subseteq y$ and $\kappa_{\xi_x} < \kappa_y$.

Let $\mathfrak{U} = \langle U_\xi \mid \xi < \mu \rangle$ and $\mathfrak{F} = \langle F_\eta^\xi \mid \xi < \mu, \eta < \delta \rangle$ be witness for Proposition 8.1.3. Since $\mathfrak{U}$ is a $\lhd$-chain , for each $\zeta < \xi < \mu$, there is a function $x \mapsto \overline{U}_{\xi,x}^\zeta$, over $\mathcal{P}_\kappa(\kappa_\xi)$ representing U_ζ in the ultrapower by U_ξ. Moreover, by restricting this function to a U_ξ-large set, we may assume that each $\overline{U}_{\xi,x}^\zeta$ is a κ_{ζ_x}-supercompact measure on $\mathcal{P}_{\kappa_x}(\kappa_{\zeta_x})$.

Definition 8.1.5. For $\xi < \mu$, let X_ξ be the U_ξ-large set of $x \in \mathcal{P}_\kappa(\kappa_\xi)$ such that

(α) κ_x is a $(\kappa_\xi)_x$-supercompact cardinal above μ.

(β) For each $\zeta \leq \xi$, $\kappa_{\zeta_x}^{<\kappa_x} \leq \kappa_{\zeta_x}^+$. If ξ is limit, $\sup_{\zeta<\xi} \kappa_{\zeta_x} = \kappa_{\xi_x}$.

(γ) $\kappa_x < \kappa_{\xi_x}$.[2]

Similarly to other Prikry-type forcing, Sinapova forcing is articulated by two components: the first one (*stem*) is responsible of adding a generic club on κ, while the second one (*large set*) plays the role of supplying the stem with new extensions. For technical reasons it is standard to require for the stems to be $\prec$-increasing sequences. Roughly, this constraint guarantees that these stems are sound promises for a generic club in κ and also that two different *local versions* of the forcing do not interfere between them.

Let us succinctly describe how Sinapova conditions should look like. Recall that we have started with $\mathfrak{U} = \langle U_\xi \mid \xi < \mu \rangle$ a $\lhd$-increasing sequence of measures in $\mathcal{P}_\kappa(\kappa_\xi)$ and we said that any condition p is composed by a *stem* and a *large set*. Thus, p should be a pair (g, H), where

[2] This means that our choice of the x's is coherent with the fact that $\kappa < \kappa_\xi$.

- g is a finite $\prec$-increasing sequence in $\prod_{\xi<\mu} \mathcal{P}_\kappa(\kappa_\xi)$,
- H is a sequence in $\prod_{\xi<\mu} U_\xi$.

This description is, however, still a bit premature and something else has to be said about the sequence H. To this purpose let us consider the following mental exercise: Let $p = (g, H)$ be a condition in Sinapova forcing and assume that $g = \{\langle \xi, x\rangle\}$. Assume that we want to extend g by adding a point $y \in \mathcal{P}_\kappa(\kappa_\zeta)$, where $\zeta < \xi$. Since we want $g \cup \{\langle \zeta, y\rangle\}$ to be $\prec$-increasing it follows that $y \in \mathcal{P}_{\kappa_x}(\kappa_\zeta \cap x)$. Thus, to extend g, one needs to consider measures over $\mathcal{P}_{\kappa_x}(\kappa_\zeta \cap x)$ and not over $\mathcal{P}_{\kappa_x}(\kappa_{\zeta_x})$. The solution for this is *to lift* our system of measures $\langle \overline{U}^\zeta_{\xi x} : \zeta < \xi, x \in X_\xi\rangle$. More precisely, let $\zeta < \xi$ and $x \in X_\xi$ and denote by $\pi^{\zeta,x} : \mathcal{P}_{\kappa_x}(\kappa_\zeta \cap x) \to \mathcal{P}_{\kappa_x}(\kappa_{\zeta_x})$ the usual projection. Set $U^\zeta_{\xi,x} := \{A \subseteq \mathcal{P}_{\kappa_x}(\kappa_\zeta \cap x) \mid \pi^{\zeta,x}[A] \in \overline{U}^\zeta_{\xi,x}\}$. It is an easy exercise to check that this *lifting* of $\overline{U}^\zeta_{\xi,x}$ yields a supercompact measure over $\mathcal{P}_{\kappa_x}(\kappa_\zeta \cap x)$. In [Sin08, Section 2.2] the following coherence properties for the above measures are proved:

Proposition 8.1.6 (Coherence properties)**.**

(ξ) *For each $\rho < \zeta < \xi < \mu$ and for U_ξ-many x's, $U^\rho_{\xi,x} \lhd U^\zeta_{\xi,x}$.*

(ξ') *For each $\xi < \mu$,*

$$B_\xi = \{x \in X_\xi \mid \forall \zeta, \eta \in \xi\, (\zeta < \eta \to \overline{U}^\zeta_{\xi,x} = [y \mapsto \overline{U}^\zeta_{\eta,y}]_{U^\eta_{\xi,x}})\} \in U_\xi.$$

($\star$) *For $\zeta < \xi$ and $A \in U_\zeta$, $\forall_{U_\xi} x\, (A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa_\zeta)) \in U^\zeta_{\xi,x}$.*

($\diamond$) *For each $\zeta < \eta < \xi$, $z \in B_\xi$ and $A \in U^\zeta_{\xi,z}$,*

$$\forall_{U^\eta_{\xi,z}} x\, (A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa_\eta)) \in U^\zeta_{\eta,x}.$$

Set $\mathfrak{B} = \langle B_\xi \mid \xi < \mu\rangle$.

Definition 8.1.7 (Sinapova forcing)**.** Under the above assumptions, the Sinapova forcing with respect to $(\kappa, \mu, \mathfrak{U}, \mathfrak{B})$ is the partial order $\mathbb{S}_{(\kappa,\mu,\mathfrak{U},\mathfrak{B})}$[3] whose conditions are pairs (g, H) for which the following hold:

1. $\mathrm{dom}(g) \in [\mu]^{<\omega}$ and $\mathrm{dom}(H) = \mu \setminus \mathrm{dom}(g)$.
2. For each $\xi \in \mathrm{dom}(g)$, $g(\xi) \in B_\xi$ and $\kappa_{g(\xi)} > \theta^{+\mu+1}$.[4] Also, g is $\prec$-increasing.

[3]Formally this definition depends also of the functions representing the different measures.

[4] Here θ is an inaccessible cardinal witnessing [Sin08, Lemma 2.7]. This requirement is technical and is necessary for the construction of the bad and the very good scale in the generic extension.

3. For each $\xi \in \mathrm{dom}(H)$,

   (a) If $\xi > \max(\mathrm{dom}(g))$, $H(\xi) \subseteq B_\xi$ and $H(\xi) \in U_\xi$;

   (b) If $\xi < \max(\mathrm{dom}(g))$ then, setting $\xi_g := \min(\mathrm{dom}(g) \setminus \xi + 1)$ and $x := g(\xi_g)$, $H(\xi) \in U^{\xi}_{\xi_g,x}$.

4. For $\xi < \zeta$ with $\xi \in \mathrm{dom}(g)$ and $\zeta \in \mathrm{dom}(H)$, $g(\xi) \prec x$, for all $x \in H(\zeta)$.

For a condition $p = (g, H)$ we say that g is the stem and H the large set of p. For $\eta \in \mathrm{dom}(g^p)$, denote $(g, H)_{\restriction\eta} := (g \restriction \eta, H \restriction \eta)$ and $(g, H)_{\setminus\eta} := (g \setminus \eta, H \setminus \eta)$.

Definition 8.1.8. Let $p, q \in \mathbb{S}$.

(a) $p \leq q$ iff

   1. $g^p \supseteq g^q$,
   2. If $\xi \in \mathrm{dom}(g^p) \setminus \mathrm{dom}(g^q)$ then $g^p(\xi) \in H^q(\xi)$,
   3. If $\xi \notin \mathrm{dom}(g^p)$, $H^p(\xi) \subseteq H^q(\xi)$,

(b) $p \leq^* q$ iff $p \leq q$ and both conditions have the same stem.

Let $p, q \in \mathbb{S}$ with $g^p = g^q = g$. Define $p \wedge q$ as the condition $r := (g, H^p \wedge H^q)$, where $H^p \wedge H^q$ is the function $\xi \mapsto H^p(\xi) \cap H^q(\xi)$, where $\xi \in \mathrm{dom}(H^p)$.

An important feature of $\mathbb{S}$ is that below any condition p the forcing $\mathbb{S} \downarrow p$ can be decomposed as the product of two Sinapova forcings. This feature, as commented in Section 7.1, is also shared with other Prikry-type forcings, such as Magidor (see Lemma 7.1.14) or Radin forcing [Rad82]. Once again, we emphasize that this *fractal structe* of Sinapova forcing is crucial to control the combinatorics of $V^{\mathbb{S}}_\kappa$. Let us phrase this in more formal terms.

Let $(g, G) \in \mathbb{S}$, $\{\langle \xi, x\rangle\} \subseteq g$ and $\xi < \mu$ be limit. For each $\eta < \xi$, set $\mathcal{V}_\eta := U^\eta_{\xi,x}$ and $\mathfrak{V} = \langle \overline{U^\eta_{\zeta,x}} \mid \eta < \zeta < \xi\rangle$. Also, for each $\zeta < \xi$, find a sequence $\mathfrak{C} = \langle C_\eta \mid \eta < \xi\rangle$ of $\mathcal{V}_\eta$-large sets witnessing Proposition 8.1.6 with respect to $\mathfrak{V}$. Now let $S_{\langle \xi,x\rangle} := \{(g, G) \mid \exists (h, H) \in \mathbb{S}\, (g, G) = (h, H)_{\restriction\xi}, \wedge\, h(\xi) = x\}$, and set $\mathbb{S}_{\langle \xi,x\rangle} := (S_{\langle \xi,x\rangle}, \leq_{\langle \xi,x\rangle})$, where $\leq_{\langle \xi,x\rangle}$ is the induced order by $\leq$. One may easily argue that $\mathbb{S}_{\langle \xi,x\rangle}$ is $\mathbb{S}_{(\kappa_x,\xi,\mathfrak{V},\mathfrak{C})}$, the Sinapova forcing with respect to $(\kappa_x, \xi, \mathfrak{V}, \mathfrak{C})$. The following proposition follows essentially in the same abstract way as Lemma 7.1.14.

Proposition 8.1.9 (Factorization)**.** *Let $(g, G) \in \mathbb{S}$, $\{\langle \xi, x\rangle\} \subseteq g$ and $\xi < \mu$ be limit. There is $(g, G') \leq^* (g, G)$ such that the following hold:*

1. *The restriction map π between $\mathbb{S} \downarrow (g, G')$ and $\mathbb{S}_{\langle \xi,x\rangle} \downarrow (\emptyset, G' \restriction \xi)$ defines a projection.*

2. *$\mathbb{S} \downarrow (g, G')$ is isomorphic to $\mathbb{S}_{\langle \xi, x\rangle} \downarrow (\emptyset, G' \restriction \xi) \times \mathbb{S}_{(\kappa,\mu,\mathfrak{U}\setminus\xi+1,\mathfrak{B}\setminus\xi+1)} \downarrow (g \setminus \xi+1, G' \setminus \xi+1)$.*

Let $S \subseteq \mathbb{S}$ be a generic filter for Sinapova forcing. Set $g^* := \bigcup_{p\in S} g^p$, $\kappa^*_\xi := \kappa_{g^*(\xi)}$ and $\vartheta_\xi := \kappa_{\xi_{g^*(\xi)}}$, for each $\xi < \mu$. The following proposition provides a summary of the main properties of $\mathbb{S}$ and $V[S]$. The proofs are, respectively, in the same spirit of the analogous results for Magidor forcing proved at Section 7.1. For more details we refer the reader to [Sin08, §2].

Theorem 8.1.10 (Properties of $\mathbb{S}$)**.**

1. *$\mathbb{S}$ is a δ-Knaster forcing notion.*

2. *$\mathbb{S}$ has the Prikry property: namely, for each $p \in \mathbb{S}$ and each sentence φ in the language of forcing, there is $q \leq^* p$ so that q decides φ.*

3. *Let $\rho < \kappa$ and let ξ be a limit ordinal such that $\vartheta^+_\xi \leq \rho < \kappa^*_{\xi+1}$. Then, $\mathcal{P}(\rho)^{V[S]} = \mathcal{P}(\rho)^{V[S\restriction\xi]}$. Further, if $\rho \leq \kappa^*_0$, $\mathcal{P}(\rho)^{V[S]} = \mathcal{P}(\rho)^V$.*

Proposition 8.1.11. *The following hold in $V[S]$:*

1. *All cardinals and cofinalities $\geq\delta$ are preserved.*

2. *Let $\rho < \kappa$ be a V-cardinal such that for some limit $\xi < \mu$ and some $k < \omega$, $\vartheta^+_\xi \leq \rho < \kappa^*_{\xi+k}$. Then ρ is preserved and $\mathrm{cof}(\rho) = \mathrm{cof}^V(\rho)$. In particular, for each $\xi < \mu$, κ^*_ξ is preserved and thus κ also.*

3. *κ is a strong limit cardinal with $\mathrm{cof}(\kappa) = \mu$ and $2^\kappa = \Theta$. Hence, the SCH_κ fails.*

4. *If $\rho \in (\kappa, \varepsilon]$ is a V-regular cardinal, $\mathrm{cof}(\rho) = \mu$. Thus, all V-cardinals $\rho \in (\kappa, \varepsilon]$ are collapsed to κ.*

Another remarkable property of Sinapova model is the existence of a bad and a very good scale at κ. The concept of scale is the cornerstone of Shelah's PCF theory and has found many applications in Set Theory, Algebra or Topology [She94]. For definitions see Section 1.4. Further information about these objects can be found in [She94][CFM01][AM10]. In [Sin08, §2.5] it is showed how to define in $V[S]$ these scales by using the sequence $\mathfrak{F}$. Combining all of this the next theorem follows:

Theorem 8.1.12 (Sinapova)**.** *In $V[S]$ the following hold true:*

1. *κ is a strong limit cardinal with $\mathrm{cof}(\kappa) = \mu$ and $\delta = \kappa^+$.*

2. *$2^\kappa \geq \Theta$, hence the SCH_κ fails.*

3. *There is a very good and a bad scale at κ.*

8.2 Geometric criterion for genericity

Hereafter $\mathbb{S}$ will be a shorthand for $\mathbb{S}_{(\kappa,\mu,\mathfrak{U},\mathfrak{B})}$. The present section we will devoted to the proof a geometric criterion of genericity for $\mathbb{S}$. This result is the analogous of the respective characterizations due to A. Mathias for Prikry forcing [Mat73] and W. Mitchell for Magidor forcing [Mit82] (cf. Theorem 7.1.19). Our exposition will be inspired in [Fuc14].

Notation 8.2.1.

- $[\prod_{\xi<\mu}\mathcal{P}_\kappa(\kappa_\xi)]$ will denote the set of all $\prec$-increasing sequences in $\prod_{\xi<\mu}\mathcal{P}_\kappa(\kappa_\xi)$ (cf. Notation 8.1.4).
- For $n<\omega$, $[\prod_{\xi<\mu}\mathcal{P}_\kappa(\kappa_\xi)]^n$ denotes the set of $\prec$-sequences of length n in $\prod_{\xi<\mu}\mathcal{P}_\kappa(\kappa_\xi)$. Analogously, $[\prod_{\xi<\mu}\mathcal{P}_\kappa(\kappa_\xi)]^{<\omega}$ denotes the set of finite $\prec$-sequences.
- For $g\in[\prod_{\xi<\mu}\mathcal{P}_\kappa(\kappa_\xi)]^{<\omega}$ we respectively denote by $\max(g)$ and $\min(g)$ the $\prec$-maximum and $\prec$-minimum value of g.

Let S be a $\mathbb{S}$-generic filter over V. The filter S yields a function $g^*\in[\prod_{\xi<\mu}B_\xi]$, which we will call the Sinapova sequence induced by S. Observe that $V[g^*]\subseteq V[S]$. As in Prikry forcing [Git10, §1.1] and Magidor forcing (cf. Section 7.1) there is a way to recover the generic S from its induced Sinapova sequence g^*.

Definition 8.2.2. For each $g^*\in[\prod_{\xi<\mu}B_\xi]$, define

$$\begin{aligned} S(g^*):=\{(h,H)\in\mathbb{S}\mid h\subseteq g^*\wedge\forall\xi\notin\operatorname{dom}(h)\,\exists(f,F)\in\mathbb{S}\\ (f,F)\leq(h,H)\wedge\xi\in\operatorname{dom}(f)\wedge f(\xi)=g^*(\xi)\}.\end{aligned}$$

Proposition 8.2.3. *For each $g^*\in[\prod_{\xi<\mu}B_\xi]$, $S(g^*)$ is a filter on $\mathbb{S}$. Moreover, if $S\subseteq\mathbb{S}$ is a generic filter and g^* is the induced Sinapova sequence, $S(g^*)=S$.*

Proof. The proof is a routine verification. The only point that it is worth mentioning is the following. Suppose that g^* is the sequence induced by S, for some generic filter $S\subseteq\mathbb{S}$. It is easy to check that $S\subseteq S(g^*)$. In particular, by maximality of generic filters, $S=S(g^*)$. □

It follows from the above that if S is $\mathbb{S}$-generic over V and g^* is the corresponding Sinapova sequence then $V[S]=V[g^*]$. The previous proposition suggests the next concept:

Definition 8.2.4. Let V be an inner model of W and suppose that $\mathbb{S}\in V$. A sequence $g^*\in[\prod_{\xi<\mu}B_\xi]\cap W$ is $\mathbb{S}$-generic over V if $S(g^*)$ is a $\mathbb{S}$-generic filter over V.

Proposition 8.2.5. *Let V be an inner model of W and $\mathbb{S} \in V$. If $g^* \in [\prod_{\xi<\mu} B_\xi] \cap W$ is $\mathbb{S}$-generic over V then the following hold:*

1. *For each sequence $H \in V \cap \prod_{\xi<\mu} U_\xi$, there is $\xi_H < \mu$ such that for all ordinal $\eta \in (\xi_H, \mu)$, $g^*(\eta) \in H(\eta)$.*
2. *For each $\xi < \mu$ limit and each $H \in V \cap \prod_{\theta<\xi} U^\theta_{\xi,g^*(\xi)}$, there is $\xi_H < \xi$ such that for all ordinal $\eta \in (\xi_H, \xi)$, $g^*(\eta) \in H(\eta)$.*

Proof. We shall just sketch the proof for property (2) as the proof for (1) is analogous. Let $\xi < \mu$ be a limit ordinal and a function $H \in V \cap \prod_{\theta<\xi} U^\theta_{\xi,x}$. Since g^* is generic, we may let $(g, G) \in S(g^*)$ with $g = \{\langle \xi, g^*(\xi)\rangle\}$. Set $D_H := \{(i, I) \leq (g, G) \mid \exists \theta \in \xi \, \forall \eta \in (\theta, \xi) \, I(\eta) \subseteq H(\eta)\}$. It is not hard to check that D_H is dense below (g, G), hence $D_H \cap S(g^*) \neq \emptyset$. Let (i, I) be a condition in this set and $\theta_i < \xi$ be a witness for $(i, I) \in D_H$. Setting $\xi_H := \theta_i$ it is routine to check that, for all $\eta \in (\xi_H, \xi)$, $g^*(\eta) \in H(\eta)$. □

The goal of this section is precisely to prove that the above properties already characterize those sequences which are $\mathbb{S}$-generic over V. The main result of this section is the following:

Theorem 8.2.6 (Criterion for genericity)**.** *Let V be an inner model of W and $\mathbb{S} \in V$. For a sequence $g^* \in [\prod_{\xi<\mu} B_\xi] \cap W$, g^* is $\mathbb{S}$-generic over V if and only if the following hold:*

1. *For each sequence $H \in V \cap \prod_{\xi<\mu} U_\xi$, there is $\xi_H < \mu$ such that for all ordinal $\eta \in (\xi_H, \mu)$, $g^*(\eta) \in H(\eta)$.*
2. *For each $\xi < \mu$ limit and each $H \in V \cap \prod_{\theta<\xi} U^\theta_{\xi,g^*(\xi)}$, there is $\xi_H < \xi$ such that for all ordinal $\eta \in (\xi_H, \xi)$, $g^*(\eta) \in H(\eta)$.*

We will tackle the proof of Theorem 8.2.6 in the next three subsections.

8.2.1 One step extensions and pruned conditions

Definition 8.2.7. For each $s \in [\mu]^{<\omega}$, define:

- The *left* operator ℓ_s is the map $\ell_s \colon \mu \to \mu \cup \{-1\}$ defined by

$$\ell_s(\xi) := \begin{cases} \max(s \cap \xi), & \text{if } s \cap \xi \neq \emptyset; \\ -1, & \text{otherwise.} \end{cases}$$

- The *right* operator r_s is the map $r_s \colon \mu \to \mu + 1$ defined by $r_s(\xi) := \min((s \cup \{\mu\}) \setminus \xi + 1)$.

Definition 8.2.8 (One-step extension)**.** Let $(g, G) \in \mathbb{S}$, $\xi \in \mathrm{dom}(G)$ and $x \in G(\xi)$. Define $(g, G)^\frown\{\langle \xi, x\rangle\}$ as the pair (f, F), where $f := g \cup \{\langle \xi, x\rangle\}$ and F is the function with $\mathrm{dom}(F) = \mathrm{dom}(G) \setminus \{\xi\}$ defined as

$$F(\eta) := \begin{cases} G(\eta) \cap \mathcal{P}_{\kappa_x}(\kappa_\eta \cap x), & \text{if } r_{\mathrm{dom}(f)}(\eta) = \xi; \\ \{y \in G(\eta) \mid x \prec y\}, & \text{if } \ell_{\mathrm{dom}(f)}(\eta) = \xi; \\ G(\eta), & \text{otherwise.} \end{cases}$$

For $1 \leq n < \omega$ and a function $f \in [\prod_{\xi \in s} B_\xi]$ with $s \in [\mathrm{dom}(G)]^n$, $(g, G)^\frown f$ is defined by recursion as $((g, G)^\frown f \restriction n - 1)^\frown\{\langle s_{n-1}, f(s_{n-1})\rangle\}$.[5]

Remark 8.2.9. Observe that not for all functions $f \in [\prod_{\xi \in s} G(\xi)]$ the pair $(g, G)^\frown f$ yields a condition in $\mathbb{S}$: it may be the case that, for some $\langle \xi, f(\xi)\rangle \in f$, $G(\eta) \cap \mathcal{P}_{\kappa_{f(\xi)}}(\kappa_\eta \cap f(\xi)) \notin U^\eta_{\xi, f(\xi)}$, for $r_{\mathrm{dom}(g \cup f)}(\eta) = \xi$.

Proposition 8.2.10. *Let $(g, G) \in \mathbb{S}$ and $\xi \in \mathrm{dom}(G)$.*

1. *If there is a condition $(f, F) \leq (g, G)$ with $g \cup \{\langle \xi, x\rangle\} = f$, then $(g, G)^\frown\{\langle \xi, x\rangle\} \in \mathbb{S}$. Moreover, this is the $\leq$-greatest condition witnessing this property.*

2. *There is $(g, G^{\xi,+}) \leq^* (g, G)$ such that for all $x \in G^{\xi,+}$,*

$$(g, G)^\frown\{\langle \xi, x\rangle\} \in \mathbb{S}.$$

Proof. For (1), observe that it is enough with guaranteeing that $G(\eta) \cap \mathcal{P}_{\kappa_x}(\kappa_\eta \cap x) \in U^\eta_{\xi,x}$, for $\eta < \xi$. Notice that this outright follows from $(f, F) \leq (g, G)$. For (2) we argue as follows. For $\eta \in \mathrm{dom}(G) \setminus \{\xi\}$, set $G^{\xi,+}(\eta) := G(\eta)$. Now let $\nu := r_{\mathrm{dom}(g)}(\xi)$ and $\sigma := \ell_{\mathrm{dom}(g)}(\xi)$. Without loss of generality assume that $\nu < \mu$, as otherwise the argument is similar. By using $(\diamond)$ of Proposition 8.1.6 it follows that for each $\rho \in (\sigma, \xi)$, there is $A_\rho \in U^\xi_{\nu, g(\nu)}$ such that for each $x \in A_\rho$, $G(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa_\rho \cap x) \in U^\rho_{\xi,x}$. Set $G^{\xi,+}(\xi) := G(\xi) \cap \bigcap_{\rho \in (\sigma,\xi)} A_\rho$. It is routine to check that $(g, G^{\xi,+})$ is as desired. $\square$

One can appeal recursively to Proposition 8.2.10 (1) to obtain the analogous result for functions $f \in [\prod_{\xi \in s} B_\xi]$, $s \in [\mathrm{dom}(G)]^{<\omega}$. The next concept will be useful in future arguments.

Definition 8.2.11. A condition $(g, G) \in \mathbb{S}$ is said to be pruned if for all $s \in [\mathrm{dom}(G)]^{<\omega}$ and all $f \in [\prod_{\xi \in s} G(\xi)]$, $(g, G)^\frown f \in \mathbb{S}$.

Proposition 8.2.12. *A condition (g, G) is pruned iff for each $\langle \xi, x\rangle \in G$, $(g, G)^\frown\{\langle \xi, x\rangle\} \in \mathbb{S}$.*

[5]By convention, $(g, G)^\frown\emptyset := (g, G)$.

Proof. The first implication is obvious. For the converse let us argue, by induction over $n \geq 1$, that for each $s \in [\mathrm{dom}(G)]^n$ and $f \in [\prod_{\xi \in s} G(\xi)]$, $(g, G)^\frown f \in \mathbb{S}$. For $n = 1$ this follows from our hypothesis. Also, the inductive step follows by combining the recursive definition of $(g, G)^\frown f$, the induction hypothesis and our assumption. $\square$

Arguing similarly to Proposition 8.2.10 one can prove the next strengthening of clause (2).

Proposition 8.2.13. *Let $(g, G) \in \mathbb{S}$. There is a condition $(g, G^*) \in \mathbb{S}$ $\leq^*$-below (g, G) which is pruned.*

8.2.2 The Strong Prikry Property

In this section we will prove that the usual strengthening of the Prikry property known as Strong Prikry property (see [Git10, Lemma 1.13]) also holds for $\mathbb{S}$. For the sake of completeness we formulate this principle in the particular context of Sinapova forcing.

Notation 8.2.14. For $(g, G) \in \mathbb{S}$ and $s \in [\mathrm{dom}(G)]^{<\omega}$, set $S^{(g,G)}_s := \{(i, I) \leq (g, G) \mid \mathrm{dom}(i) = \mathrm{dom}(g) \cup s\}$. Let $\mathbb{S}^{(g,G)}_s$ be $S^{(g,G)}_s$ endowed with the induced order. Define $\mathbb{S}^{(g,G)}_{\supseteq s}$ analogously.

Definition 8.2.15 (Strong Prikry Property)**.** We will say that $\mathbb{S}$ has the Strong Prikry Property (SPP, for short) if the following property holds: For each condition $(g, G) \in \mathbb{S}$ and each dense open set $D \subseteq \mathbb{S}$, there is $(g, G^*) \leq^* (g, G)$ and $s \in [\mathrm{dom}(G)]^{<\omega}$ such that $\mathbb{S}^{(g,G^*)}_{\supseteq s} \subseteq D$.

Lemma 8.2.16. *Let $(g, G) \in \mathbb{S}$, $D \subseteq \mathbb{S}$ be dense open and $s \in [\mathrm{dom}(G)]^{<\omega}$. There is a condition $(g, G_s) \leq^* (g, G)$ be such that*

$$(*_s)\ \ \mathbb{S}^{(g,G_s)}_s \cap D \neq \emptyset \implies \mathbb{S}^{(g,G_s)}_{\supseteq s} \subseteq D.$$

Proof. We argue by induction over $n = |s|$. If $n = 0$, then we ask whether there is $(g, \tilde{G}) \leq^* (g, G)$ witnessing $(*_\emptyset)$. If the answer to our query is affirmative then we let $G_\emptyset$ be such $\tilde{G}$. Otherwise, set $G_\emptyset := G$. It is easy to check that $(g, G_\emptyset)$ is as desired.

Now assume that for $(h, H) \in \mathbb{S}$ and each $t \in [\mathrm{dom}(G)]^n$, there is $(h, G_t) \leq^* (h, H)$ witnessing $(*_t)$. Let s be with $|s| = n+1$. Say $\delta := \min(s)$. Set $t := s \setminus \{\delta\}$ and $\xi := r_{\mathrm{dom}(g)}(\delta)$. For each $y \in G(\delta)$, let $(g_y, G_y) := (g, G)^\frown\{\langle \delta, y\rangle\}$ and $(g_y, G_{y,t}) \leq^* (g_y, G_y)$ witnessing $(*_t)$. Now look at the set of $y \in G(\delta)$ for which the property $(*_t)$ is non-trivial. Namely, consider $X := \{y \in G(\delta) \mid \mathbb{S}^{(g_y,G_{y,t})}_t \cap D \neq \emptyset\}$. If $X \in U^\delta_{\xi,g(\xi)}$, set $Y := X$ and, otherwise, let Y to be the complement. Let $(g, G_s) \leq^* (g, G)$ be the diagonalization of $\{(g_y, G_{y,t}) \mid y \in Y\}$ (see [Sin08, Proposition 2.12]).

Claim 8.2.16.1. *$(g, G_s) \leq^* (g, G)$ and witnesses $(*_s)$.*

Proof of claim. The first property is obvious so we are left with verifying that $(*_s)$ holds. Without loss of generality, assume that $\mathbb{S}_s^{(g,G_s)} \cap D \neq \emptyset$. Let $(i, I) \in \mathbb{S}_s^{(g,G_s)} \cap D$. By definition of diagonalization, $(i, I) \leq (g_y, G_{y,t})$, where $y = i(\delta) \in Y$. Hence, $(i, I) \in \mathbb{S}_t^{(g_y,G_{y,t})} \cap D$, and thus $y \in X \cap Y$. This shows that $Y = X$.

Now let $(f, F) \in \mathbb{S}_{\supseteq s}^{(g,G_s)}$. Again, by the definition of diagonalization, $(f, F) \in \mathbb{S}_{\supseteq t}^{(g_y,G_{y,t})}$, for $y = f(\delta) \in Y$. Since $X = Y$, $\mathbb{S}_t^{(g_y,G_{y,t})} \cap D \neq \emptyset$, hence, by $(*_t)$, $\mathbb{S}_{\supseteq t}^{(g_y,G_{y,t})} \subseteq D$, and thus $(f, F) \in D$. Altogether, $\mathbb{S}_{\supseteq s}^{(g,G_s)} \subseteq D$, which yields $(*_s)$. □

This finishes the proof of the lemma. □

Lemma 8.2.17. *Let $(g, G) \in \mathbb{S}$ and $D \subseteq \mathbb{S}$ be dense open. There is a condition $(g, G)^* \leq^* (g, G)$ such that*

$$(*) \ \ \forall s \in [\mathrm{dom}(G)]^{<\omega} \, (\mathbb{S}_s^{(g,G)^*} \cap D \neq \emptyset \implies \mathbb{S}_{\supseteq s}^{(g,G)^*} \subseteq D).$$

In particular, $\mathbb{S}$ has the SPP.

Proof. For each $s \in [\mathrm{dom}(G)]^{<\omega}$, let $(g, G_s) \leq (g, G)$ be given by Lemma 8.2.16. For each $\xi \in \mathrm{dom}(G)$, set $G^*(\xi) := \bigcap\{G_s(\xi) \mid \xi \in s\}$. Observe that $(g, G^*) \in \mathbb{S}$ by Definition 8.1.5(α) and $\mu^{<\aleph_0} = \mu$. Evidently, $(g, G)^* := (g, G^*)$ satisfies $(*)$. For the last clause, since D is dense, there is s with $\mathbb{S}_s^{(g,G)^*} \cap D \neq \emptyset$, so that $S_{\supseteq s}^{(g,G)^*} \subseteq D$. □

One can be a bit more ambitious and require that $(g, G)^*$ and (g, G) would be equal up to some $\xi \in \mathrm{dom}(g)$. More formally, $(g, G)^*_{\restriction \xi+1} = (g, G)_{\restriction \xi+1}$ (c.f. Definition 8.1.7). This more general result follows by combining Lemma 8.2.17 with the following result:

Lemma 8.2.18 (Diagonalization)**.** *Let $(g, G) \in \mathbb{S}$, $\xi \in \mathrm{dom}(G)$ and $\eta \in \mathrm{dom}(g) \cap \xi$. Assume that $A \in U_\xi$ and $\mathcal{A} = \langle (g_x, G_x) \mid x \in A \rangle$ is a family of conditions below (g, G) with $g_x := g \cup \{\langle \xi, x \rangle\}$ and $(g_x, G_x)_{\restriction \eta+1} = (g, G)_{\restriction \eta+1}$. Then, there is $(g, G^*) \leq (g, G)$ such that $(g, G^*)_{\restriction \eta+1} = (g, G)_{\restriction \eta+1}$ which diagonalizes the family $\mathcal{A}$.*

We omit the proof of the above as it is identical to the proof of [Sin08, Proposition 2.12]. Bearing this in mind, one can use Lemma 8.2.17 to prove the following:

Lemma 8.2.19. *Let $(g, G) \in \mathbb{S}$, $D \subseteq \mathbb{S}$ be dense open and $\eta \in \mathrm{dom}(g)$. There is $(g, G)^{*,\eta} \leq (g, G)$ such that if $(i, I) \leq (g, G)^{*,\eta}$ is in D then, for each $(j, J) \leq (i, I)_{\restriction \eta+1} {}^\frown (g, G)^{*,\eta}_{\setminus \eta+1}$, $(j, J) \in D$.*

The proof runs in parallel to [Sin08, Corollary 2.14]. The only relevant difference here is that one needs to invoke Lemma 8.2.17 instead of [Sin08, Proposition 2.13].

8.2.3 The proof of the criterion

We are now in conditions to complete the proof of Theorem 8.2.6. Recall that we are left with showing that if $g^* \in [\prod_{\xi<\mu} B_\xi] \cap W$ witnesses properties (1) and (2) of Proposition 8.2.5 then g^* is $\mathbb{S}$-generic over V.

Proof of Theorem 8.2.6. Towards a contradiction, assume that the implication was false. Let κ be the first cardinal for which we can define a Sinapova forcing $\mathbb{S} := \mathbb{S}_{(\kappa,\mu,\mathfrak{U},\mathfrak{B})}$ and for which there is some $g^* \in [\prod_{\xi<\mu} \mathfrak{B}(\xi)]$ satisfying (1) and (2) but not being generic.

Henceforth $D \subseteq \mathbb{S}$ will be an arbitrary but fixed dense open set. We aim to prove that $D \cap S(g^*) \neq \emptyset$. We will be arguing in a similar fashion to [Git10, Theorem 1.12].

Set $\mathrm{St} := \{g \in [\prod_{\xi<\mu} B_\xi]^{<\omega} \mid \exists G\,(g, G) \in \mathbb{S}\}$. For each $g \in \mathrm{St}$, set $(g, G_g) := \mathbb{1}^\frown g$ and[6]

$$(g, \tilde{G}_g) := \begin{cases} (g, G_g)^*, & \text{if } g = \emptyset; \\ (g, G_g)^{*,\xi} & \text{if } \max(\mathrm{dom}(g)) = \xi, \end{cases}$$

where $(g, G_g)^*$ and $(g, G_g)^{*,\xi}$ are the conditions given by Lemma 8.2.17 and Lemma 8.2.19, respectively.

For each $\xi < \mu$ and $x \in \mathcal{P}_\kappa(\kappa_\xi)$ set $\mathrm{St}_{\xi,x} := \{g \in \mathrm{St} \mid \mathrm{dom}(g) \subseteq \xi, \max(g) \prec x\}$. Observe that $|\mathrm{St}_{\xi,x}| \leq |\mathcal{P}_{\kappa_x}(x)| < \kappa$. Thus, $G^*(\xi) := \Delta_{x\in\mathcal{P}_\kappa(\kappa_\xi)} \left(\bigcap_{g\in\mathrm{St}_{\xi,x}} \tilde{G}_g(\xi)\right) \in U_\xi$. This process yields a function $G^* \in V \cap \prod_{\xi<\mu} U_\xi$. Set $s := (\emptyset, G^*)$. Appealing to property (1) we find $\xi^* < \mu$ limit such that $g^*(\eta) \in G^*(\eta)$, for each $\eta \in (\xi^*, \mu)$. Set $g^*_- := g^* \restriction \xi^*$, $\mathcal{V}_\eta := \overline{U}^\eta_{\xi^*,g^*(\xi^*)}$ and $C_\eta := \overline{B_\eta \cap \mathcal{P}_{\kappa_{g^*(\xi^*)}}(\kappa_\eta \cap g^*(\xi^*))}$, for each $\eta < \xi^*$. Set $\mathfrak{V} := \langle \mathcal{V}_\eta \mid \eta < \xi^* \rangle$, $\mathfrak{C} := \langle C_\eta \mid \eta < \xi^* \rangle$ and $\mathbb{S}_{(\kappa_{g^*(\xi^*)},\xi^*,\mathfrak{V},\mathfrak{C})}$ be the corresponding Sinapova forcing. Clearly, g^*_- witnesses (1) and (2), and $\kappa_{g^*(\xi^*)} < \kappa$, hence $S(g^*_-)$ is a generic filter for $\mathbb{S}_{(\kappa_{g^*(\xi^*)},\xi^*,\mathfrak{V},\mathfrak{C})}$. Let $p^*_- := (\emptyset, \overline{I \restriction \xi^*}) \in S(g^*_-)$. Define $p^* := (\{\langle \xi^*, g^*(\xi^*) \rangle\}, H^*)$, where $\mathrm{dom}(H^*) := \mu \setminus \{\xi^*\}$ and

$$H^*(\eta) := \begin{cases} I(\eta), & \text{if } \eta < \xi^*, \\ \{x \in G^*(\eta) \mid g^*(\xi^*) \prec x\}, & \text{if } \xi^* < \eta, \end{cases}$$

where $I(\eta)$ denotes the lifting of $\overline{I}(\eta)$ to $\mathcal{P}_{\kappa_{g^*(\xi^*)}}(\kappa_\eta \cap g^*(\xi^*))$. Clearly, $p^* \in \mathbb{S}$. Moreover, by appealing to Proposition 8.2.13, we may assume that p^* is pruned. By a very similar argument to Proposition 8.1.9 (1), there is a projection between $\mathbb{S} \downarrow p^*$ and $\mathbb{S}_{(\kappa_{g^*(\xi^*)},\xi^*,\mathfrak{V},\mathfrak{C})} \downarrow p^*_-$. Let π be such projection and set $D_{p^*} := D \cap \mathbb{S} \downarrow p^*$. Clearly, $\pi[D_{p^*}]$ is dense and open in $\mathbb{S}_{(\kappa_{g^*(\xi^*)},\xi^*,\mathfrak{V},\mathfrak{C})} \downarrow$

[6] Since $g \in \mathrm{St}$ observe that Proposition 8.2.10 and the subsequent comments guarantee that $\mathbb{1}^\frown g \in \mathbb{S}$.

p^*_-. Since $p^*_- \in S(g^*_-)$, it follows that $S(g^*_-) \cap \pi[D_{p^*}] \neq \emptyset$. Let $(f, F) \in D_{p^*}$ be such that $\pi(f, F) \in S(g^*_-) \cap \pi[D_{p^*}]$.

Claim 8.2.19.1. *$(f, F) \leq (g, \tilde{G}_g)$, where $g := f \restriction \xi^* + 1$.*

Proof of claim. Clearly, $g \subseteq f$.

► Let $\xi \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)$. Then $\xi^* < \xi$, so that, since $(f, F) \leq p^*$, $F(\xi) \subseteq H^*(\xi)$. By definition of diagonal intersection, and since $\max(g) = g^*(\xi^*)$, $H^*(\xi) \subseteq \tilde{G}_g(\xi)$.

► Let $\xi \in \mathrm{dom}(\tilde{G}_g)$. If $\xi^* < \xi$ then one may argue as before that $F(\xi) \subseteq \tilde{G}_g(\xi)$. Thus, assume $\xi < \xi^*$. Since $(g, \tilde{G}_g)_{\restriction \xi+1} = (g, G_g)_{\restriction \xi+1} = (\mathbb{1}^\frown g)_{\restriction \xi+1}$, we have $\tilde{G}_g(\xi) = G_g(\xi) = \mathcal{P}_{\kappa_{g(\eta)}}(\kappa_\xi \cap g(\eta))$, where $\eta := r_{\mathrm{dom}(g)}(\xi)$. Since $f \restriction \xi^* + 1 = g \restriction \xi^* + 1$, clearly $F(\xi) \in U^\xi_{\eta, g(\eta)}$ and thus $F(\xi) \subseteq \tilde{G}_g(\xi)$. □

Now let (f^*, F^*) be defined as

$$(f, F)_{\restriction \xi^*+1}{}^\frown(p^{*\frown} g^* \restriction (\mathrm{dom}(f) \setminus \xi^* + 1))_{\setminus \xi^*+1}$$

This gives a condition in $\mathbb{S}$, because p^* was pruned and $g^*(\xi) \prec g^*(\eta) \in G^*(\eta)$, for $\eta \in (\xi^*, \mu)$. Observe that (f^*, F^*) is also pruned.

Claim 8.2.19.2. *$(f^*, F^*) \in D \cap S(g^*)$.*

Proof of claim. By combining the definition of $(g, \tilde{G}_g)$, the above claim and the fact that $(f, F) \in D$, it follows that $(f^*, F^*) \in D$. The verification that $(f^*, F^*) \in S(g^*)$ is mere routine. □

From the above arguments we infer that $D \cap S(g^*) \neq \emptyset$ hence, g^* is $\mathbb{S}$-generic over V. This produces a contradiction with our initial assumption on κ and $\mathbb{S}$. □

For future reference we also include the proof of a general version of the classical Röwbottom lemma [Kan09, Theorem 7.17].

Definition 8.2.20. Let $g \in [\prod_{\xi \in \mathrm{dom}(g)} B_\xi]$ and $s \in [\mu \setminus \mathrm{dom}(g)]^{<\omega}$. A sequence $\langle H_\theta \mid \theta \in s \rangle$, is *amenable* to $\langle g, s \rangle$ if for each $\theta \in s$, if $\eta := r_{\mathrm{dom}(g)}(\theta) < \mu$, then $H_\theta \in U^\theta_{\eta, g(\eta)}$ and, otherwise, $H_\theta \in U_\theta$.

A sequence $\langle H_\theta \mid \theta \in \mu \setminus \mathrm{dom}(g) \rangle$ is said to be *amenable* to $\langle g \rangle$ if, for each $s \in [\mu \setminus \mathrm{dom}(g)]^{<\omega}$, $\langle H_\theta \mid \theta \in s \rangle$ is amenable to $\langle g, s \rangle$.

Lemma 8.2.21 (Generalized Röwbottom's lemma)**.** *Let g be a sequence in $[\prod_{\xi \in \mathrm{dom}(g)} B_\xi]$ and $\langle H_\theta \mid \theta \in \mu \setminus \mathrm{dom}(g) \rangle$ be amenable to $\langle g \rangle$.*

For each function $c : [\prod_{\theta \in \mu \setminus \mathrm{dom}(g)} H_\theta]^{<\omega} \to \vartheta$ with $\vartheta \leq \mu$, there is $\langle H^*_\theta \mid \theta \in \mu \setminus \mathrm{dom}(g) \rangle$ amenable to $\langle g \rangle$ such that the following hold:

1. *for each $\theta \in \mu \setminus \mathrm{dom}(g)$, $H^*_\theta \subseteq H_\theta$;*

2. $\langle H^*_\theta \mid \theta \in \mu \setminus \mathrm{dom}(g)\rangle$ *is homogeneous for c: namely, for each* $n < \omega$ *and each* $s \in [\mu \setminus \mathrm{dom}(g)]^n$*, the function* $c \restriction [\prod_{\theta \in s} H^*_\theta]$ *is constant.*

Proof. Arguing by induction over $n < \omega$, we will prove that for each function $\bar{c} : [\prod_{\theta \in \mu \setminus \mathrm{dom}(g)} H_\theta]^n \to \vartheta$ and $s \in [\mu \setminus \mathrm{dom}(g)]^n$, there is a sequence $\mathcal{H}^s = \langle H^s_\theta \mid \theta \in s\rangle$ which is amenable to $\langle g, s\rangle$ and such that $c \restriction [\prod_{\theta \in s} H^s_\theta]$ is constant. If $n = 1$ the claim follows by appealing to the μ^+-completedness of all the measures involved (see Definition 8.1.5(α)). Thus, we shall assume that the result holds for each $1 \leq m \leq n$ and will infer from this that it holds for $n + 1$.

Fix $\bar{c} : [\prod_{\theta \in \mu \setminus \mathrm{dom}(g)} H_\theta]^{n+1} \to \vartheta$ be a function and let $s \in [\mu \setminus \mathrm{dom}(g)]^{n+1}$. Set $\max(s) = \eta_s$. Say, $\xi_s := r_{\mathrm{dom}(g)}(\eta_s)$ and assume, for instance, that $\xi_s < \mu$. Thus, $H_{\eta_s} \in U^{\eta_s}_{\xi_{\eta_s}, g(\xi_{\eta_s})}$. For each $g \in [\prod_{\theta \in s \cap \eta_s} H_\theta]$, let $c_g : H_{\eta_s} \to \vartheta$ be the function defined by $x \mapsto \bar{c}(g \cup \{\langle \eta_s, x\rangle\})$, provided $\max(g) \prec x$, or 0 otherwise. Appealing to the case $n = 1$, for each such g we obtain $\langle H^s_g\rangle$ which is amenable to $\langle g, \{\eta_s\}\rangle$ and homogeneous with respect to c_g. Pick $\vartheta_g \in \vartheta$ be the constant value of c_g witnessing this. Let $H^s_{\eta_s} = \triangle\{H^s_g : g \in [\prod_{\theta \in s \cap \eta_s} H_\theta]\}$, where recall that this diagonal intersection is defined as

$$\{x \in \mathcal{P}_{\kappa_{g(\xi_s)}}(\kappa_{\eta_s} \cap g(\xi_s)) \mid \forall g \in [\prod_{\theta \in s \cap \eta_s} H_\theta]\ (\max(g) \prec x \ \to\ x \in H^s_g)\}.$$

By normality of the measure $U^{\eta_s}_{\xi_s, g(\xi_s)}$, $H^s_{\eta_s} \in U^{\eta_s}_{\xi_{\eta_s}, g(\xi_{\eta_s})}$. On the other hand, let $c^* : [\prod_{\theta \in \mu \setminus \mathrm{dom}(g)} H_\theta]^n \to \vartheta$ be the function sending each g to ϑ_g, in case $g \in [\prod_{\theta \in s \cap \eta_s} H_\theta]$, or 0 otherwise. By the induction hypothesis there is $\mathcal{H}^{s \cap \eta_s} = \langle H^{s \cap \eta_s}_\theta \mid \theta \in s \cap \eta_s\rangle$ which is amenable to $\langle g, s \cap \eta_s\rangle$ and $c^\star \restriction [\prod_{\theta \in s \cap \eta_s} H^{s \cap \eta_s}_\theta]$ has constant value ϑ^*.

We claim that $\mathcal{H}^s = \mathcal{H}^{s \cap \eta_s} \cup \{\langle \eta_s, H^s_{\eta_s}\rangle\}$ witnesses the inductive step relative to the function $\bar{c}$ and the set s. It is easy to check that $\mathcal{H}^s$ is amenable to $\langle g, s\rangle$. For homogeneity, let $f \in [\prod_{\theta \in s} H^s_\theta]$ and say that $f = g \cup \{\langle \eta_s, x\rangle\}$, where $g \in [\prod_{\theta \in s \cap \eta_s} H^s_\theta]$. Since $x \in H^s_{\eta_s}$ and $\max(g) \prec x$, by definition of diagonal intersection, $x \in H^s_g$. Thus, $c_g(x) = \vartheta_g = \vartheta^*$. On the other hand, $\bar{c}(f) = c_g(x)$, so that $\bar{c}(f) = \vartheta^*$. Since the choice of s was arbitrary, the inductive step follows.

For each $n < \omega$ use the previous argument to obtain a sequence $\langle \mathcal{H}^s \mid s \in [\mu \setminus \mathrm{dom}(g)]^n\rangle$, $\mathcal{H}^s = \langle H^s_\theta \mid \theta \in s\rangle$, such that $\mathcal{H}^s$ is amenable to $\langle g, s\rangle$ and $c \restriction [\prod_{\theta \in s} H^s_\theta]$ is constant. Define $\langle H^*_\theta \mid \theta \in \mu \setminus \mathrm{dom}(g)\rangle$ as $H^*_\theta := \bigcap\{H^s_\theta : s \in [\mu \setminus \mathrm{dom}(g)]^{<\omega}, \theta \in s\}$. Since all the measures involved are μ^+-complete this process yields a sequence $\langle H^*_\theta \mid \theta \in \mu \setminus \mathrm{dom}(g)\rangle$ which is amenable to $\langle g\rangle$. Finally, it is routine to check that this sequence is homogeneous for c. $\square$

8.3 Sinapova sequences and iterated ultrapowers

In this section we use Theorem 8.2.6 to show how iterated ultrapowers can be used to define Sinapova sequences. A classical theorem of R. Solovay shows that this is possible for Prikry forcing [Kan09, Theorem 19.18(a)]. Subsequent results due to G. Fuch [Fuc14] and to J. Cummings and W.H. Woodin [CW] revealed that a similar situation can be arranged for Magidor forcing and Radin forcing, respectively. As we have shown in Lemma 4.3.1, this is not just a worth proving result by its own but also may have many potential applications. For further information about iterated ultrapowers the reader is referred to [Kan09, §19].

Let κ be a supercompact cardinal, $\mu < \kappa$ a regular cardinal and an increasing sequence of cardinals $\langle \theta_\xi \mid \xi < \mu \rangle$ above κ. Hereafter we assume that $\mathfrak{U} := \langle U_\eta \mid \eta < \mu \rangle$ is a $\lhd$-increasing sequence of supercompact measures over $\mathcal{P}_\kappa(\theta_\xi)$. Arguing as in Section 8.1 we may find a sequence of large sets $\mathfrak{B} := \langle B_\xi \mid \xi < \mu \rangle$ making the measures of $\mathfrak{U}$ *cohere*. Let $\mathbb{S}$ denote the Sinapova forcing with respect to the tuple $(\kappa, \mu, \mathfrak{U}, \mathfrak{B})$.

Set $M_0 := V$, $\mathfrak{U}^0 := \mathfrak{U}$, $j_{0,0} := \mathrm{id}$ and $\kappa_0 := \kappa$. Fix $\xi < \mu$, and assume that $\mathfrak{M} := \langle \langle \langle M_\zeta, \in, \mathfrak{U}^\zeta \rangle \mid \zeta < \xi \rangle, \langle j_{\zeta,\eta} \mid \zeta \leq \eta < \xi \rangle \rangle$ has already been defined. Let us now show how to define $\langle M_\xi, \in, \mathfrak{U}^\xi \rangle$ and $\langle j_{\zeta,\xi} \mid \zeta \leq \xi \rangle$. If ξ is a limit ordinal, let $\langle M_\xi, \mathfrak{U}^\xi, \langle j_{\eta,\xi} \mid \eta \leq \xi \rangle \rangle := \operatorname{dir lim} \mathfrak{M}$, the direct limit of the ξ-iteration $\mathfrak{M}$. Otherwise, if $\xi = \eta + 1$, we let $j_{\eta,\xi}$ to be the ultrapower embedding induced by $\mathfrak{U}^\eta(\eta)$, $\mathfrak{U}^\xi := j_{\eta,\xi}(\mathfrak{U}^\eta)$, $j_{\zeta,\xi} := j_{\eta,\xi} \circ j_{\zeta,\eta}$ and $M_\xi := \mathrm{Ult}(M_\eta, \mathfrak{U}^\eta(\eta))$. Let $\langle M_\mu, \mathfrak{U}^\mu, \langle j_{\xi,\mu} \mid \xi < \mu \rangle \rangle := \operatorname{dir lim} \mathfrak{M}$, the direct limit of the above μ-iteration of ultrapowers.

Definition 8.3.1. Define $\langle \kappa_\xi \mid \xi < \mu \rangle$, $\langle \lambda_\xi \mid \xi < \mu \rangle$, $\langle \sigma_\xi \mid \xi < \mu \rangle$ and $\langle x^*_\xi \mid \xi < \mu \rangle$ as $\kappa_0 := \kappa$, $\kappa_\xi := \mathrm{crit}(j_{\xi,\xi+1})$, $\lambda_\xi := j_{0,\xi}(\theta_\xi)$, $\sigma_\xi := j_{\xi,\xi+1}[\lambda_\xi]$ and $x^*_\xi := j_{\xi,\mu}[\lambda_\xi]$, respectively.[7]

By fineness of the measures $\langle \mathfrak{U}^\xi(\xi) \mid \xi < \mu \rangle$, for each $\xi < \mu$, $\kappa_\xi < \lambda_\xi < \kappa_{\xi+1}$. Set $\kappa_\mu := j_{0,\mu}(\kappa)$, $\mathfrak{U}^\mu = j_{0,\mu}(\mathfrak{U})$ and $\mathfrak{B}^\mu := j_{0,\mu}(\mathfrak{B})$. For each $\eta < \xi$, say $\mathfrak{U}(\eta) = [f^\eta_\xi]_{\mathfrak{U}(\xi)}$. By shrinking if necessary, we may assume that $f^\eta_\xi(x)$ is a supercompact measure over $\mathcal{P}_{\kappa_x}(\theta_{\eta_x})$. For each such $\eta < \xi$ and $x \in \mathcal{P}_\kappa(\theta_\xi)$, let $g^\eta_\xi(x)$ denote the lifting of $f^\eta_\xi(x)$ to a measure over $\mathcal{P}_{\kappa_x}(\theta_\eta \cap x)$.

Let $\mathbb{S}^\mu$ be Sinapova forcing defined with respect to $(\kappa_\mu, \mu, \mathfrak{U}^\mu, \mathfrak{B}^\mu)$ and the family of functions $\langle j_{0,\mu}(f^\eta_\xi), j_{0,\mu}(g^\eta_\xi) \mid \eta < \xi < \mu \rangle$.

Theorem 8.3.2. *The sequence $\vec{x} = \langle x^*_\xi \mid \xi < \mu \rangle$ yields a Sinapova sequence for $\mathbb{S}^\mu$ over M_μ. Thus, $\vec{x}$ generates a $\mathbb{S}^\mu$-generic filter over M_μ which is definable in V.*

[7] σ_ξ stands for the *seed* of $\mathfrak{U}^\xi(\xi)$.

Proof. By virtue of Theorem 8.2.6 it is sufficient to check that $\vec{x} \in \prod_{\xi<\mu} \mathfrak{B}^\mu(\xi)$ and that (1) and (2) hold in the inner model M_μ. We will divide the proof in a series of claims.

Claim 8.3.2.1. $\vec{x} \in \prod_{\xi<\mu} \mathfrak{B}^\mu(\xi)$.

Proof of claim. First observe that $\mathrm{crit}(j_{0,\mu}) = \kappa > \mu$, hence $\mathfrak{B}^\mu = j_{0,\mu}[\mathfrak{B}]$. Fix $\xi < \mu$. Since $B_\xi \in \mathfrak{U}^0(\xi)$, $j_{0,\xi}(B_\xi) \in \mathfrak{U}^\xi(\xi)$, so $\sigma_\xi \in j_{0,\xi+1}(B_\xi)$. By elementarity, $j_{\xi+1,\mu}(\sigma_\xi) \in j_{0,\mu}(B_\xi)$. Now observe that $|\sigma_\xi| < \mathrm{crit}(j_{\xi+1,\mu})$, hence $j_{\xi+1,\mu}(\sigma_\xi) = j_{\xi,\mu}[\lambda_\xi]$. Observe that this yields $x^*_\xi \in \mathfrak{B}^\mu(\xi)$. □

Claim 8.3.2.2. *For all $H \in M_\mu \cap \prod_{\xi<\mu} \mathfrak{U}^\mu(\xi)$, there is $\xi_H < \mu$ such that, for all $\xi \in (\xi_H, \mu)$, $x^*_\xi \in H(\xi)$.*

Proof of claim. By definition of direct limit there is ξ_H such that for all $\xi \in (\xi_H, \mu)$, $H \in \mathrm{ran}(j_{\xi,\mu})$. Let ξ be some of such ordinals and H^* be a witness for this. By elementarity, $H(\xi) \in \mathfrak{U}^\mu(\xi)$ if and only if $H^*(\xi) \in \mathfrak{U}^\xi(\xi)$, hence $\sigma_\xi \in j_{\xi,\xi+1}(H^*(\xi))$. Arguing as before, $x^*_\xi \in j_{\xi,\mu}(H^*(\xi)) = j_{\xi,\mu}(H^*)(\xi) = H(\xi)$. Altogether, this shows that for all $\xi \in (\xi_H, \mu)$, $x^*(\xi) \in H(\xi)$. □

Claim 8.3.2.3. *For all $H \in M_\mu \cap \prod_{\eta<\xi} j_{0,\mu}(g^\eta_\xi)(x^*_\xi)$, there is $\xi_H < \xi$ such that, for all $\eta \in (\xi_H, \xi)$, $x^*_\eta \in H(\eta)$.*

Proof of claim. We divide the argument in a series of subclaims.

Subclaim 8.3.2.3.1. *For all $\eta < \xi$, $j_{0,\mu}(f^\eta_\xi)(x^*_\xi) = \mathfrak{U}^\xi(\eta)$ and $j_{0,\mu}(g^\eta_\xi)(x^*_\xi)$ is its lifting to a measure in $\mathcal{P}_{\kappa_{\mu_{x^*_\xi}}}(j_{0,\mu}(\theta_\eta) \cap x^*_\xi)$.*

Proof of subclaim. By definition, $\mathfrak{U}(\eta) = [f^\eta_\xi]_{\mathfrak{U}(\xi)}$, hence $\mathfrak{U}^\xi(\eta) = [j_{0,\xi}(f^\eta_\xi)]_{\mathfrak{U}^\xi(\xi)}$. At step ξ we iterate the measure $\mathfrak{U}^\xi(\xi)$, so $\mathfrak{U}^\xi(\eta) = j_{0,\xi+1}(f^\eta_\xi)(\sigma_\xi)$. Observe that $\mathfrak{U}^\xi(\eta)$ is a measure over $\mathcal{P}_{\kappa_\xi}(j_{0,\xi}(\theta_\eta)) = \mathcal{P}_{\kappa_\xi}(j_{0,\xi}(\theta_\eta))^{M_{\xi+1}}$ and $\mathrm{crit}(j_{\xi+1,\eta}) = \kappa_{\xi+1}$ is a supercompact cardinal in $M_{\xi+1}$ above the cardinality of this set, so $j_{\xi+1,\mu}(\mathfrak{U}^\xi(\eta)) = \mathfrak{U}^\xi(\eta)$. Thus, $j_{0,\mu}(f^\eta_\xi)(x^*_\xi) = \mathfrak{U}^\xi(\eta)$, which yields the first result. Similarly, for $\mathfrak{U}(\xi)$-many x's, $g^\eta_\xi(x)$ is the lifting of $f^\eta_\xi(x)$, hence $j_{0,\xi+1}(g^\eta_\xi)(\sigma_\xi)$ is the lifting of $j_{0,\xi+1}(f^\eta_\xi)(\sigma_\xi)$ in $M_{\xi+1}$. From this point is easy to infer the desired result. □

Subclaim 8.3.2.3.2. $j_{0,\mu}(g^\eta_\xi)(x^*_\xi)$ *is a measure over $\mathcal{P}_{\kappa_\xi}(j_{\xi,\mu}[j_{0,\xi}(\theta_\eta)])$.*

Proof of subclaim. By elementarity, it is clear that $j_{0,\mu}(g^\eta_\xi)(x^*_\xi)$ is a measure over $\mathcal{P}_{\kappa_{\mu_{x^*_\xi}}}(j_{0,\mu}(\theta_\eta) \cap x^*_\xi)$. Thus we are left with calculating $\kappa_{\mu_{x^*_\xi}}$ and $j_{0,\mu}(\theta_\eta) \cap x^*_\xi$.

► By definition, $\kappa_{\mu_{x^*_\xi}} = \mathrm{otp}(\kappa_\mu \cap x^*_\xi)$. Now, since $\kappa_\mu = j_{\xi,\mu}(\kappa_\xi)$ and $x^*_\xi = j_{\xi,\mu}[\lambda_\xi]$ it follows that $\kappa_\mu \cap x^*_\xi = j_{\xi,\mu}[\kappa_\xi]$, hence $\kappa_{\mu_{x^*_\xi}} = \kappa_\xi$.

► $j_{0,\mu}(\theta_\eta) \cap x^*_\xi = j_{\xi,\mu}(j_{0,\xi}(\theta_\eta)) \cap j_{\xi,\mu}[\lambda_\xi]$. Since $j_{0,\xi}(\theta_\eta) < \lambda_\xi$, this latter value coincides with $j_{\xi,\mu}[j_{0,\xi}(\theta_\eta)]$, as wanted. □

Since $j_{0,\mu}(g^\eta_\xi)(x^*_\xi)$ is a measure over $\mathcal{P}_{\kappa_\xi}(j_{\xi,\mu}[j_{0,\xi}(\theta_\eta)])$ and $M^{\kappa_\xi}_\xi = M^{\kappa_\xi}_\mu$, it follows that

$$M_\mu \cap \prod_{\eta<\xi} j_{0,\mu}(g^\eta_\xi)(x^*_\xi) = M_\xi \cap \prod_{\eta<\xi} j_{0,\mu}(g^\eta_\xi)(x^*_\xi).$$

Thus, $H \in M_\xi \cap \prod_{\eta<\xi} j_{0,\mu}(g^\eta_\xi)(x^*_\xi)$. Now let $\overline{H}$ be the sequence defined as $\langle \overline{H(\eta)} \mid \eta < \xi\rangle$, where $\overline{H(\eta)}$ is the projection of $H(\eta)$ onto the set $\mathcal{P}_{\kappa_\xi}(j_{0,\xi}(\theta_\eta))$. By Claim 8.3.2.3.1, $\overline{H} \in M_\xi \cap \prod_{\eta<\xi} \mathfrak{U}^\xi(\eta)$. Since M_ξ is a direct limit again there is $\xi_H < \xi$ such that for each $\zeta \in (\xi_H, \xi)$, $\overline{H} \in \operatorname{ran}(j_{\zeta,\xi})$. Let ζ be some of such ordinals and F be a witness for it. Thus, $j_{\zeta,\xi}(F) \in M_\xi \cap \prod_{\eta<\xi} \mathfrak{U}^\xi(\eta)$. By elementarity, $F \in M_\zeta \cap \prod_{\eta<\xi} \mathfrak{U}^\zeta(\eta)$, hence $F(\zeta) \in \mathfrak{U}^\zeta(\zeta)$, and thus $\sigma_\zeta \in j_{\zeta,\zeta+1}(F(\zeta))$. Altogether, $j_{\zeta+1,\xi}(\sigma_\zeta) \in \overline{H}(\zeta)$. Now observe that $|\sigma_\zeta| < \operatorname{crit}(j_{\zeta+1,\xi})$, so $j_{\zeta+1,\xi}(\sigma_\zeta) = j_{\zeta,\xi}[\lambda_\zeta]$.

Subclaim 8.3.2.3.3. *$j_{\zeta,\xi}[\lambda_\zeta] = y^*(\zeta)$, where $y^*(\zeta)$ is the projection of $x^*(\zeta)$ onto $\mathcal{P}_{\kappa_\xi}(j_{0,\xi}(\theta_\zeta))$.*

Proof of subclaim. By definition, $y^*(\zeta) := \{\operatorname{otp}(\alpha \cap x^*(\xi)) \mid \alpha \in x^*(\zeta)\}$. Since $x^*(\zeta) = j_{\zeta,\mu}[\lambda_\zeta]$, it follows that

$$y^*(\zeta) = \{\operatorname{otp}(j_{\zeta,\mu}(\alpha) \cap j_{\xi,\mu}[\lambda_\xi]) \mid \alpha \in \lambda_\zeta\}.$$

Now, for each $\alpha \in \lambda_\zeta$ observe that $j_{\zeta,\mu}(\alpha) \cap j_{\xi,\mu}[\lambda_\xi] = j_{\xi,\mu}(j_{\zeta,\xi}(\alpha)) \cap j_{\xi,\mu}[\lambda_\xi] = j_{\xi,\mu}[j_{\zeta,\xi}(\alpha)]$. Thus, $\operatorname{otp}(j_{\zeta,\mu}(\alpha) \cap j_{\xi,\mu}[\lambda_\xi]) = j_{\zeta,\xi}(\alpha)$. Altogether, this entails $y^*(\zeta) = j_{\zeta,\xi}[\lambda_\zeta]$, as wanted. □

From the above subclaim it follows that for all $\zeta \in (\xi_H, \xi)$, $y^*(\zeta) \in \overline{H}(\zeta)$. By lifting the sequences y^* and H we infer that, for all $\zeta \in (\xi_H, \xi)$, $x^*(\zeta) \in H(\zeta)$. This concludes the proof of the claim. □

The above series of claims yield the desired result. □

8.4 The main forcing construction

The present section will be devoted to introduce the main forcing construction of the chapter. This forcing is a variation of the forcing of Section 7.2 where Magidor forcing is now replaced by Sinapova forcing. This new choice will be the responsible of the scales and the $\mathrm{TP}(\kappa^+)$ in the generic extension. For enlightening the argument we will simply give details for the construction in case $\Theta = \lambda^+$. The general argument runs in parallel to the exposition we made at Section 7.4. Throughout this section we will be relying on the notation established in Notation 7.2.1.

Let $G \subseteq \mathbb{A}_{\lambda^+}$ generic over V. Since κ is Laver indestructible there is in $V[G]$ a $\lhd$-increasing sequence $\mathfrak{U}_{\lambda^+} = \langle U_\xi \mid \xi < \mu\rangle$ of supercompact measures on $\mathcal{P}_\kappa(\kappa_\xi)$, $\xi < \mu$. With $\mathfrak{U}_{\lambda^+}$ we find a sequence $\mathfrak{B}_{\lambda^+} = \langle B_\xi \mid \xi < \mu\rangle$ witnessing Proposition 8.1.6 and later we define the corresponding Sinapova forcing $\mathbb{S}_{\lambda^+} := \mathbb{S}_{(\kappa,\mu,\mathfrak{U},\mathfrak{B})} \in V[G]$. For each such ξ, let $\dot{U}_\xi$ and $\dot{B}_\xi$ be $\mathbb{A}_{\lambda^+}$-nice names for each of such objects. The next result shows that there are many intermediate extensions of $V[G]$ where $(\mathfrak{U}_{\lambda^+}, \mathfrak{B}_{\lambda^+})$ *projects.* For details the reader is referred to Lemma 7.2.3 where a similar result is proved.

Lemma 8.4.1. *There is an unbounded set of ordinals $\mathcal{A} \subseteq \lambda^+$, closed under taking limits of $\geq\delta^+$-sequences, such that, for each $\alpha \in \mathcal{A}$ and each generic filter $G \subseteq \mathbb{A}_{\lambda^+}$, $\langle(\dot{U}_\xi)_G \cap V[G \restriction \alpha] \mid \xi < \mu\rangle$, $\langle(\dot{B}_\xi)_G \cap V[G \restriction \alpha] \mid \xi < \mu\rangle$ are suitable to define Sinapova forcing in $V[G \restriction \alpha]$.*

Notation 8.4.2. For each $\alpha \in \mathcal{A}$, let $\mathfrak{U}_\alpha$ and $\mathfrak{B}_\alpha$ be the sequences witnessing Lemma 8.4.1. Let $\dot{\mathbb{S}}_\alpha$ be a $\mathbb{A}_\alpha$-name representing the Sinapova forcing $\mathbb{S}_{(\kappa,\mu,\mathfrak{U}_\alpha,\mathfrak{B}_\alpha)} \in V[G \restriction \alpha]$.

Proposition 8.4.3. *Work in $V[G]$. For each $\alpha \in \mathcal{A}$, $\mathbb{S}_{\lambda^+}$ projects onto $\mathbb{S}_\alpha$.*

Proof. Let $\alpha \in \mathcal{A}$. Let $g^* \in [\prod_{\xi<\mu} B_\xi]$ a $\mathbb{S}_{\lambda^+}$-generic sequence and set $h^*_\alpha := \langle g^*(\xi) \cap V[G \restriction \alpha] \mid \xi < \mu\rangle$. Clearly, $h^*_\alpha \in [\prod_{\xi<\mu} \mathfrak{B}_\alpha(\xi)]$. By appealing to Theorem 8.2.6 we infer that h^*_α is $\mathbb{S}_\alpha$-generic over V. In particular, each $\mathbb{S}_{\lambda^+}$-generic filter induces a $\mathbb{S}_\alpha$-generic filter, hence $\mathbb{S}_{\lambda^+}$ projects onto $\mathbb{S}_\alpha$. $\square$

Let $\beta_0 \in \mathcal{A} \setminus \lambda + 1$ and $\pi : \beta_0 \to \mathrm{Even}(\lambda)$ be a bijection[8]. Hereafter, β_0 will be fixed. The particular choice of this ordinal is not relevant, we could just have taken any other in $\mathcal{A} \setminus \lambda + 1$. Clearly, π entails an $\in$-isomorphism between $V^{\mathbb{A}_{\beta_0}}$ and $V^{\mathbb{A}_{\mathrm{Even}(\lambda)}}$. Thus, defining $\dot{\mathfrak{U}}^\pi_{\beta_0} := \pi(\dot{\mathfrak{U}}_{\beta_0})$, $(\mathfrak{U}^\pi_{\beta_0})_{\pi[G\restriction\beta_0]} = (\dot{\mathfrak{U}}_{\beta_0})_{G\restriction\beta_0} = \mathfrak{U}_{\beta_0}$. Similarly with $\mathfrak{B}_{\beta_0}$. Say that U^π_ξ and B^π_ξ are the components of these sequences. For the ease of notation, let H be the $\mathbb{A}_{\mathrm{Even}(\lambda)}$-generic filter generated by $\pi[G \restriction \beta_0]$. The proof of the next result is analogous to Lemma 8.4.1.

Lemma 8.4.4. *There is an unbounded set of cardinals $\mathcal{B} \subseteq \lambda$ closed under taking limits of $\geq\delta^+$-sequences, such that for each $\alpha \in \mathcal{B}$ and each generic filter $K \subseteq \mathbb{A}_{\mathrm{Even}(\lambda)}$, the sequences $\langle(\dot{U}^\pi_\xi)_G \cap V[K \restriction \mathrm{Even}(\alpha)] \mid \xi < \mu\rangle$ and $\langle(\dot{B}^\pi_\xi)_G \cap V[K \restriction \mathrm{Even}(\alpha)] \mid \xi < \mu\rangle$ are suitable to define Sinapova forcing in $V[K \restriction \mathrm{Even}(\alpha)]$.*

Notation 8.4.5. For each $\alpha \in \mathcal{B}$, let $\mathfrak{U}^\pi_\alpha$ and $\mathfrak{B}^\pi_\alpha$ denote the sequences witnessing Lemma 8.4.4. By convention, $\mathfrak{U}^\pi_\lambda := \mathfrak{U}_{\beta_0}$ and $\mathfrak{B}^\pi_\lambda := \mathfrak{B}_{\beta_0}$. For each $\alpha \in \mathcal{B} \cup \{\lambda\}$, let $\dot{\mathbb{S}}^\pi_\alpha$ be a $\mathbb{A}_{\mathrm{Even}(\alpha)}$-name such that $\mathbb{S}_{(\kappa,\mu,\mathfrak{U}^\pi_\alpha,\mathfrak{B}^\pi_\alpha)} = (\dot{\mathbb{S}}^\pi_\alpha)_{H\restriction\mathrm{Even}(\alpha)}$.

The next lemma follows from Proposition 8.4.3 in the same abstract manner that Lemma 8.4.6 followed from Proposition 7.2.6.

[8]For an ordinal α, $\mathrm{Even}(\alpha)$ stands for the set of all even and limit ordinals $\leq\alpha$.

Lemma 8.4.6. *Let $\hat{\mathcal{A}} = (\mathcal{A} \cap (\beta_0, \lambda^+)) \cup \{\lambda^+\}$.*

1. *For every $\gamma, \tilde{\gamma} \in \hat{\mathcal{A}}$ with $\gamma < \tilde{\gamma}$, there is a projection*

$$\sigma_\gamma^{\tilde{\gamma}} : \mathbb{A}_{\tilde{\gamma}} * \dot{\mathbb{S}}_{\tilde{\gamma}} \to \mathrm{RO}^+(\mathbb{A}_\gamma * \dot{\mathbb{S}}_\gamma).$$

2. *For every $\gamma \in \hat{\mathcal{A}}$ and $\alpha \in \mathcal{B}$, there is a projection*

$$\sigma_\alpha^\gamma : \mathbb{A}_\gamma * \dot{\mathbb{S}}_\gamma \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi).$$

3. *For every $\gamma \in \hat{\mathcal{A}}$ and $\alpha \in \mathcal{B}$, let $\hat{\sigma}_\alpha^\gamma$ be the extension of σ_α^γ to the Boolean completion of $\mathbb{A}_\gamma * \dot{\mathbb{S}}_\gamma$*

$$\hat{\sigma}_\alpha^\gamma : \mathrm{RO}^+(\mathbb{A}_\gamma * \dot{\mathbb{S}}_\gamma) \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi).$$

Then the projections commute with $\sigma_\alpha^{\lambda^+}$:

$$\sigma_\alpha^{\lambda^+} = \hat{\sigma}_\alpha^\gamma \circ \sigma_\gamma^{\lambda^+}.$$

Definition 8.4.7 (Main forcing)**.** A condition in $\mathbb{R}$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, \dot{q}) \in \mathbb{A}_{\lambda^+} * \dot{\mathbb{S}}_{\lambda^+}$;
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}]^{<\delta}$;
3. For every $\gamma \in \mathrm{dom}(r)$, $r(\gamma)$ is a $\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}_\gamma^\pi$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}_\gamma^\pi} \Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}_\gamma^\pi} \text{“} r(\gamma) \in \dot{\mathrm{Add}}(\delta, 1) \text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}$ we will write $(p_0, \dot{q}_0, r_0) \leq_{\mathbb{R}} (p_1, \dot{q}_1, r_1)$ iff $(p_0, \dot{q}_0) \leq_{\mathbb{A}_{\lambda^+} * \dot{\mathbb{S}}_{\lambda^+}} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\gamma \in \mathrm{dom}(r_1)$, $\sigma_\gamma^{\lambda^+}(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}_\gamma^\pi} \text{“} r_0(\gamma) \leq r_1(\gamma) \text{”}$.

Definition 8.4.8. $\mathbb{U}$ will denote the pair $(U, \leq)$ where $U := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}\}$ and $\leq$ is the order inherited from $\mathbb{R}$. Set $\bar{\mathbb{R}} := (\mathbb{A}_{\lambda^+} * \dot{\mathbb{S}}_{\lambda^+}) \times \mathbb{U}$.

Proposition 8.4.9.

1. *$\mathbb{U}$ is δ-directed closed.*
2. *The function $\rho : \bar{\mathbb{R}} \to \mathbb{R}$ given by $\langle (p, \dot{q}), (\mathbb{1}, \dot{\mathbb{1}}, r) \rangle \mapsto (p, \dot{q}, r)$ entails a projection. In particular, $V^{\mathbb{A}_{\lambda^+} * \dot{\mathbb{S}}_{\lambda^+}} \subseteq V^{\mathbb{R}} \subseteq V^{\bar{\mathbb{R}}}$.*
3. *$V^{\mathbb{A}_{\lambda^+} * \mathbb{S}_{\lambda^+}}$ and $V^{\mathbb{R}}$ have the same $<\delta$-sequences.*

Proof. (1) and (2) follows exactly as in Proposition 7.2.12. For (3), let $\mathbb{C}$ be denote the subforcing of $\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}$ consisting of conditions (p, q) such that $p \in \mathbb{A}_{\lambda^+}$ and $p \Vdash_{\mathbb{A}_{\lambda^+}} \dot{q} = (\check{g}, \dot{H})$, for some $g \in [\prod_{\xi<\mu} \mathcal{P}_\kappa(\kappa_\xi)]^{<\omega} \cap V$ and some $\mathbb{A}_{\lambda^+}$-name for a function $\dot{H}$. Clearly, $\mathbb{C}$ is a dense subposet of $\mathbb{A}_{\lambda^+} * \dot{\mathbb{M}}_{\lambda^+}$. Arguing as in Proposition 7.2.12(3) the result follows. $\square$

Let $\bar{R} \subseteq \bar{\mathbb{R}}$ a generic filter whose projection onto $\mathbb{A}_{\lambda^+}$ generates the generic filter G. Also, let $R \subseteq \mathbb{R}$ be the generic filter generated by $\rho[\bar{R}]$ and $S \subseteq \mathbb{S}_{\lambda^+}$ be the generic filter over $V[G]$ induced by $\bar{R}$.

Proposition 8.4.10 (Some properties of $\mathbb{R}$)**.**

1. *$\mathbb{R}$ is λ-Knaster. In particular, all V-cardinals $\geq\lambda$ are preserved.*
2. *$\mathbb{R}$ preserves κ and δ. Also, it collapses all the V-cardinals of (κ, δ) to κ and all the V-cardinals of (δ, λ) to δ. In particular, $V[R] \models$ "$\delta = \kappa^+ \wedge \lambda = \kappa^{++}$".*
3. *$V[R] \models$ "$2^\kappa = \lambda^+ = \kappa^{+3}$".*
4. *$V[R] \models$ "κ is strong limit with $\mathrm{cof}(\kappa) = \mu$".*
5. *In $V[R]$ there is a bad and a very good scale at κ. In particular, $\square^*_\kappa$ fails and thus there are no special κ^+-Aronszajn trees.*

Proof.

1. The proof is essentially the same as Proposition 7.2.13(1) but we provide details for completeness. Let $K \in [\mathbb{R}]^\lambda$. By extending if necessary the conditions of K we may further assume that K is of the form $\{(p_\alpha, \dot{q}_\alpha, r_\alpha) \mid \alpha < \lambda\}$, where $p_\alpha \Vdash_{\mathbb{A}_{\lambda^+}} \dot{q}_\alpha = (\check{g}_\alpha, \dot{H}_\alpha)$. Here $g_\alpha \in [\prod_{\xi<\mu} \mathcal{P}_\kappa(\kappa_\xi)]^{<\omega} \cap V$ and $\dot{H}$ is a $\mathbb{A}_{\lambda^+}$ for a large set in Since κ^+-Knaster by passing to a set $\mathcal{I} \in [\lambda]^\lambda$ we may assume that, for all $\alpha, \gamma \in \mathcal{I}$, $p_\alpha \parallel p_\gamma$ and $\check{g}_\alpha := g^*$. Observe that for all $\alpha, \gamma \in \mathcal{I}$, $(p_\alpha \cup p_\gamma, \dot{q}_\alpha \wedge \dot{q}_\gamma)$ witnesses compatibility of $(p_\alpha, \dot{q}_\alpha)$ and $(p_\gamma, \dot{q}_\gamma)$.

   On the other hand, appealing to the Δ-system lemma [Kun14, §6], we may refine $\mathcal{I}$ to $\mathcal{J} \in [\mathcal{I}]^\lambda$ and find $\Delta \in [\mathcal{B}]^{<\delta}$ and r^* in such a way that $\{\mathrm{dom}(r_\alpha) \mid \alpha \in \mathcal{J}\}$ forms a Δ-system. Moreover, we may assume that $r_\alpha \restriction \Delta = r^*$, for $\alpha \in \mathcal{J}$. Indeed, this is feasible because the set of $\bigcup_{\gamma\in\Delta}(\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}^\pi_\gamma)$-names has cardinality less than λ. Altogether this shows that $\{p_\alpha \in S \mid \alpha \in \mathcal{J}\}$ is a subset of S of compatible conditions with cardinality λ.

2. Let $\theta \in \{\kappa, \delta\} \cup (\kappa, \delta) \cup (\delta, \lambda)$ and let us discuss what happens in each case. If $\theta = \kappa$ it is enough to prove that $\mathbb{A}_{\lambda^+} * \dot{\mathbb{S}}_{\lambda^+}$ preserves it, and this follows from a standard argument combining the κ-closedness of

$\mathbb{A}_{\lambda^+}$ with the Prikry property and the κ-closedness of $\langle \dot{\mathbb{S}}_{\lambda^+}, \leq^* \rangle$. If $\theta = \delta$ the argument is similar but now appealing to Easton's lemma (cf. Lemma 1.3.18). If $\theta \in (\kappa, \delta)$, it is clear that $\mathbb{R}$ collapses θ because there is a projection between $\mathbb{R}$ and $\mathbb{A}_{\lambda^+} * \dot{\mathbb{S}}_{\lambda^+}$, and this last forcing collapses the interval (κ, δ) (cf Proposition 8.1.11(4)). Finally, assume that $\theta \in (\delta, \lambda)$ and let $\eta \in \mathcal{B} \cap (\delta, \lambda)$ with $\eta > \theta$. It is easy to see that there is a projection between $\mathbb{R}$ and $\mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\eta)} * \dot{\mathbb{S}}^\pi_\eta) * \dot{\mathrm{Add}}(\delta, 1)$. By standard arguments this latter iteration collapses the interval $(\delta, \eta]$ and thus θ.

(3) The first equality follows in the same abstract way that Proposition 7.2.13 (3). For the latter equality use item (2).

(4) By Proposition 7.2.12(3) it suffices to argue that in $V[G * \dot{S}]$ the property holds. Observe that this is already true by Proposition 8.1.11(3).

(5) This follows from the existence of a very good (resp. bad) scale in $V[G * \dot{S}]$ (see Theorem 8.1.12), $(\kappa^+)^{V[G*\dot{S}]} = (\kappa^+)^{V[R]} = \delta$ and the fact that $V[G * \dot{S}]$ and $V[R]$ have the same $<\delta$-sequences. □

8.5 $\mathrm{TP}(\kappa^{++})$ holds

In this section we will prove that $V[R] \models \mathrm{TP}(\kappa^{++})$. For enlightening the presentation, once again, we will simply give details for the proof in case $\Theta = \lambda^+$. We will follow the structure sketched through Section 7.3, and for so we encourage the reader to look there for further details and intuitions. For the record of the section, recall that $\beta_0 \in \mathcal{A} \setminus \lambda + 1$ is the ordinal fixed at the beginning of Section 8.4.

Definition 8.5.1 (Truncations of $\mathbb{R}$)**.** Let $\alpha \in \mathcal{A} \cap (\beta_0, \lambda^+)$. A condition in $\mathbb{R} \upharpoonright \alpha$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_\alpha * \dot{\mathbb{S}}_\alpha$;
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}]^{<\delta}$;
3. For every $\beta \in \mathrm{dom}(r)$, $r(\beta)$ is a $\mathbb{A}_{\mathrm{Even}(\beta)} * \dot{\mathbb{S}}^\pi_\beta$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\beta)} * \dot{\mathbb{S}}^\pi_\beta} \Vdash_{\mathbb{A}_{\mathrm{Even}(\beta)} * \dot{\mathbb{S}}^\pi_\beta} \text{“}\dot{r}(\beta) \in \dot{\mathrm{Add}}(\delta, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R} \upharpoonright \alpha$ we will write $(p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \leq_{\mathbb{A}_{\mathrm{Even}(\beta)} * \dot{\mathbb{S}}^\pi_\beta} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\beta \in \mathrm{dom}(r_1)$, $\sigma^\alpha_\beta(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\beta)} * \dot{\mathbb{S}}^\pi_\beta}$ “$\dot{r}_0(\beta) \leq \dot{r}_1(\beta)$”.

The proof of the next result is exactly the same as Proposition 7.3.2 and Proposition 7.3.3

Proposition 8.5.2. *Let $\alpha \in \mathcal{A} \cap (\beta_0, \lambda^+)$. Then there is a projection between $\mathbb{R}$ and $\mathrm{RO}^+(\mathbb{R} \restriction \alpha)$.*

Proposition 8.5.3. *Let $\dot{T}$ be a $\mathbb{R}$-name for a λ-Aronszajn tree. There is $\beta^* \in \mathcal{A} \cap (\beta_0, \lambda^+)$, such that $V^{\mathbb{R}\restriction\beta^*} \models$ "T is a λ-Aronszajn tree"*

Let $\pi^* : \beta^* \to \lambda$ be a bijection extending π. We use π^* to define an $\in$-isomorphism between $V^{\mathbb{A}_{\beta^*}}$ and $V^{\mathbb{A}_\lambda}$.[9] Again, $\mathfrak{U}^{\pi^*}_\lambda := \pi^*(\dot{\mathfrak{U}}_{\beta^*})_{\pi^*[G\restriction\beta^*]}$ is a $\lhd$-increasing sequence of measures which (pointwise) extends the sequence $\mathfrak{U}^\pi_\lambda$. Similarly, define $\mathfrak{B}^{\pi^*}_\lambda := \pi^*(\mathfrak{B}_{\beta^*})_{\pi^*[G\restriction\beta^*]}$. Let $\mathbb{S}^{\pi^*}_\lambda := \mathbb{S}_{(\kappa,\mu,\mathfrak{U}^{\pi^*}_\lambda,\mathfrak{B}^{\pi^*}_\lambda)}$. For the ease of notation, let H^* be the $\mathbb{A}_{\mathrm{Even}(\lambda)}$-generic filter generated by $\pi^*[G \restriction \beta^*]$.

Proposition 8.5.4.

1. *There is an isomorphism $\varphi : \mathbb{A}_{\beta^*} * \dot{\mathbb{S}}_{\beta^*} \to \mathbb{A}_\lambda * \dot{\mathbb{S}}^{\pi^*}_\lambda$.*

2. *For each $\beta \in \mathcal{B}$ the function $\varrho^\lambda_\beta = \sigma^{\beta^*}_\beta \circ \varphi^{-1}$ establishes a projection between $\mathbb{A}_\lambda * \dot{\mathbb{S}}^{\pi^*}_\lambda$ and $\mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\beta)} * \dot{\mathbb{S}}^\pi_\beta)$.*

Proof. For (1), observe that the subposet of $\mathbb{A}_{\beta^*} * \dot{\mathbb{S}}_{\beta^*}$ consisting of conditions of the form $(p, (\check{g}, \dot{H}))$, is dense. Analogously, for $\mathbb{A}_\lambda * \dot{\mathbb{S}}^{\pi^*}_\lambda$. It is routine to check that $(p, (\check{g}, \dot{H})) \mapsto (\pi^*(p), (\check{g}, \pi^*(\dot{H})))$ defines an isomorphism between these two dense subposets. Observe that now (2) is immediate as $\sigma^{\beta^*}_\beta$ is a projection. □

Definition 8.5.5. A condition in $\mathbb{R}^*$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_\lambda * \dot{\mathbb{S}}^{\pi^*}_\lambda$;

2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}]^{<\delta}$;

3. For every $\beta \in \mathrm{dom}(r)$, $r(\beta)$ is a $\mathbb{A}_{\mathrm{Even}(\beta)} * \dot{\mathbb{S}}^\pi_\beta$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\beta)}*\dot{\mathbb{S}}^\pi_\beta} \Vdash_{\mathbb{A}_{\mathrm{Even}(\beta)}*\dot{\mathbb{S}}^\pi_\beta} \text{“}\dot{r}(\beta) \in \dot{\mathrm{Add}}(\delta, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}^*$ we will write $(p_0, \dot{q}_0, r_0) \le (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \le_{\mathbb{A}_{\mathrm{Even}(\beta)}*\dot{\mathbb{S}}^\pi_\beta} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\beta \in \mathrm{dom}(r_1)$, $\varrho^\lambda_\beta(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\beta)}*\dot{\mathbb{S}}^\pi_\beta} \dot{r}_0(\beta) \le \dot{r}_1(\beta)$.

Proposition 8.5.6. *$\mathbb{R}^*$ and $\mathbb{R} \restriction \beta^*$ are isomorphic. In particular, $\mathbb{R}^*$ forces that $\dot{T}$ is a λ-Aronszajn tree.*

[9] This choice will guarantee that our future construction coheres with the previous one.

Proof. It is not hard to check that $(p, \dot{q}, r) \mapsto (\varphi(p, \dot{q}), r)$ defines an isomorphism between both forcings. $\square$

The next lemma can be proved identically as in Lemma 7.3.8:

Lemma 8.5.7. *There is $\mathcal{B}^* \in (\mathcal{F}_\lambda)^V$, $\mathcal{B}^* \subseteq \mathcal{B}$, with $\delta < \min \mathcal{B}^*$ such that for every $\alpha \in \mathcal{B}^*$, the sequences $\langle (\dot{U}_\xi^{\pi^*})_{H^*} \cap V[H \restriction \alpha] \mid \xi < \mu \rangle$, $\langle (\dot{B}_\xi^{\pi^*})_{H^*} \cap V[H^* \restriction \alpha] \mid \xi < \mu \rangle$ are suitable to define Sinapova forcing $V[H^* \restriction \alpha]$.*

Notation 8.5.8. For each $\alpha \in \mathcal{B}^*$, let $\mathfrak{U}_\alpha^{\pi^*}$ and $\mathfrak{B}_\alpha^{\pi^*}$ denote the sequences witnessing Lemma 8.5.7 and set $\mathbb{S}_\alpha^{\pi^*} := \mathbb{S}_{(\kappa, \mu, \mathfrak{U}_\alpha^{\pi^*}, \mathfrak{B}_\alpha^{\pi^*})}$.

Lemma 8.5.9. *Let $\hat{\mathcal{B}}^* = \mathcal{B}^* \cup \{\lambda\}$ and $\alpha < \gamma \in \hat{\mathcal{B}}^*$. There are projections*

1. $\varrho_\alpha^\gamma : \mathbb{A}_\gamma * \dot{\mathbb{S}}_\gamma^{\pi^*} \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi)$,
2. $\hat{\varrho}_\alpha^\gamma : \mathrm{RO}^+(\mathbb{A}_\gamma * \dot{\mathbb{S}}_\gamma^{\pi^*}) \to \mathrm{RO}^+(\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi)$.

Moreover, for each $\alpha < \gamma \in \mathcal{B}^*$, $\varrho_\alpha^\gamma = \sigma_\alpha^\gamma$.

Proof. The construction of ϱ_α^γ and $\hat{\varrho}_\alpha^\gamma$ is analogous to Lemma 8.4.6, again using a suitable version of Proposition 8.4.3. A proof for the moreover part can be found in [FHS18, Lemma 3.18]. $\square$

The moreover clause of the previous lemma is crucial since it guarantees that there are no disagreements between the projections defining $\mathbb{R}^*$ and the projections intended to define its truncations.

Definition 8.5.10 (Truncations of $\mathbb{R}^*$)**.** Let $\gamma \in \mathcal{B}^*$. A condition in $\mathbb{R}^* \restriction \gamma$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_\gamma * \dot{\mathbb{S}}_\gamma^{\pi^*}$,
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}^* \cap \gamma]^{<\delta}$;
3. For every $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a $\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi} \Vdash_{\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi} \text{“}\dot{r}(\alpha) \in \mathrm{Add}(\delta, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}^* \restriction \gamma$ we will write $(p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\alpha \in \mathrm{dom}(r_1)$, $\varrho_\alpha^\gamma(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\alpha)} * \dot{\mathbb{S}}_\alpha^\pi} \dot{r}_0(\alpha) \leq \dot{r}_1(\alpha)$.

The proof of the next result is analogous to Proposition 8.5.2.

Proposition 8.5.11. *For each $\gamma \in \mathcal{B}^*$, there is a projection between $\mathbb{R}^*$ and $\mathrm{RO}^+(\mathbb{R}^* \restriction \gamma)$. In particular, $\mathbb{R}^*$ is isomorphic to the iteration $\mathbb{R}^* \restriction \gamma * (\mathbb{R}^* / \mathbb{R}^* \restriction \gamma)$.*

The next lemmas can be derived in the same way as Lemma 7.3.13 and Lemma 7.3.14.

Lemma 8.5.12. *Assume there is a λ-Aronszajn tree T in $V^{\mathbb{R}^*}$. Then there is $\gamma \in \mathcal{B}^*$ such that $T \cap \gamma$ is a γ-Aronszajn tree in $V^{\mathbb{R}^* \upharpoonright \gamma}$.*

Lemma 8.5.13. *Assume that there is a λ-Aronszajn tree $T \subseteq \lambda$ in $V^{\mathbb{R}^*}$. Let $\gamma \in \mathcal{B}^*$ be as in the previous lemma. Then $\mathbb{R}^*/(\mathbb{R}^* \upharpoonright \gamma)$ adds b_γ, a cofinal branch throughout $T \cap \gamma$.*

By combining Proposition 8.5.3 and 8.5.6 with the above lemma it follows that if the quotients $\mathbb{R}^*/(\mathbb{R}^* \upharpoonright \gamma)$ do not add γ-branches then $\mathrm{TP}(\lambda)$ holds in $V[R]$. As in Section 7.3 we will show that for each $\gamma \in \mathcal{B}^*$ there are forcings $\mathbb{P}_\gamma$ and $\mathbb{Q}_\gamma^{\mathrm{Even}}$ fulfilling the following properties:

(α_γ) $\mathbb{P}_\gamma \times \mathbb{Q}_\gamma^{\mathrm{Even}}$ projects onto $\mathbb{R}^*/(\mathbb{R}^* \upharpoonright \gamma)$ in $V^{\mathbb{R}^* \upharpoonright \gamma}$.

(β_γ) $\mathbb{P}_\gamma \times \mathbb{Q}_\gamma^{\mathrm{Even}}$ does not add new branches to $T \cap \gamma$ over $V^{\mathbb{R}^* \upharpoonright \gamma}$.

Combining (α_γ) and (β_γ) we would again conclude that $\mathbb{R}^*/(\mathbb{R}^* \upharpoonright \gamma)$ does not add γ-branches to $T \cap \gamma$. In particular, if this is true for each $\gamma \in \mathcal{B}^*$ then $V[R] \models \mathrm{TP}(\lambda)$.

Definition 8.5.14. Let $\gamma \in \mathcal{B}^*$. A condition in the poset $\mathbb{R}^*_{\mathrm{Even}} \upharpoonright \gamma$ is a triple $(p, \dot{q}, r)$ for which all the following hold:

1. $(p, q) \in \mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}^\pi_\gamma$,
2. r is a partial function with $\mathrm{dom}(r) \in [\mathcal{B}^* \cap \xi]^{<\kappa^+}$;
3. For every $\zeta \in \mathrm{dom}(r)$, $r(\zeta)$ is a $\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{S}}^\pi_\zeta$-name such that

$$\mathbb{1}_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{S}}^\pi_\zeta} \Vdash_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{S}}^\pi_\zeta} \text{“}\dot{r}(\zeta) \in \mathrm{Add}(\kappa^+, 1)\text{”}.$$

For conditions $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1)$ in $\mathbb{R}^* \upharpoonright \gamma$ we will write $(p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)$ in case $(p_0, \dot{q}_0) \leq_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}^\pi_\gamma} (p_1, \dot{q}_1)$, $\mathrm{dom}(r_1) \subseteq \mathrm{dom}(r_0)$ and for each $\zeta \in \mathrm{dom}(r_1)$, $\tau^\xi_\zeta(p_0, q_0) \Vdash_{\mathbb{A}_{\mathrm{Even}(\zeta)} * \dot{\mathbb{S}}^\pi_\zeta} \dot{r}_0(\zeta) \leq \dot{r}_1(\zeta)$.

Clearly $\mathbb{R}^* \upharpoonright \gamma$ projects onto $\mathbb{R}^*_{\mathrm{Even}} \upharpoonright \gamma$, for each $\gamma \in \mathcal{B}^* \cup \{\lambda\}$. The analogous of Lemma 7.3.16 is again true.

Lemma 8.5.15. *For each $\gamma \in \mathcal{B}^*$, $\psi_\gamma : \mathbb{R}^* \upharpoonright \gamma \to \mathbb{A}_{\mathrm{Odd}(\gamma)} \times \mathbb{R}^*_{\mathrm{Even}} \upharpoonright \gamma$ given by $(p, \dot{q}, r) \mapsto \langle p \upharpoonright \mathrm{Odd}(\gamma), (\varrho^\gamma_\gamma(p, \dot{q}), r)\rangle$ defines a dense embedding. In particular, both posets are forcing equivalent and thus $V^{\mathbb{R}^* \upharpoonright \gamma}$ can be seen as a κ^+-cc extension of $V^{\mathbb{R}^*_{\mathrm{Even}} \upharpoonright \gamma}$.*

Definition 8.5.16. For each $\gamma \in \mathcal{B}^* \cup \{\lambda\}$, define $\mathbb{C}_\gamma = \mathbb{A}_\gamma * \dot{\mathbb{S}}^{\pi^*}_\gamma$, $\mathbb{C}^{\mathrm{Even}}_\gamma := \mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}^{\pi}_\gamma$, $\mathbb{P}_\gamma := \mathbb{C}_\lambda / \mathbb{C}_\gamma$ and $\mathbb{U}_\gamma := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}^* \restriction \gamma\}$. Now over $V^{\mathbb{R}^* \restriction \gamma}$, define $\mathbb{Q}_\gamma := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}^* / \mathbb{R}^* \restriction \gamma\}$. Now over $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$, define $\mathbb{Q}^{\mathrm{Even}}_\gamma := \{(\mathbb{1}, \dot{\mathbb{1}}, r) \mid (\mathbb{1}, \dot{\mathbb{1}}, r) \in \mathbb{R}^* / \mathbb{R}^*_{\mathrm{Even}} \restriction \gamma\}$,

Arguing respectively as in Proposition 8.4.9 and Proposition 8.4.10 one obtains the following:

Proposition 8.5.17. *For each $\gamma \in \mathcal{B}^*$, the following hold:*

1. *$\mathbb{U}_\gamma$ is δ-directed closed.*
2. *$\mathbb{C}_\gamma \times \mathbb{U}_\gamma$ projects onto $\mathbb{R}^* \restriction \gamma$ via the map $\langle (p, \dot{q}), (\mathbb{1}, \dot{\mathbb{1}}, r) \rangle \mapsto (p, \dot{q}, r)$.*
3. *$V^{\mathbb{C}_\gamma}$ and $V^{\mathbb{R}^* \restriction \gamma}$ have the same $<\delta$-sequences. The same is true for $V^{\mathbb{C}^{\mathrm{Even}}_\gamma}$ and $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$*

Proposition 8.5.18. *For each $\gamma \in \mathcal{B}^*$, the following hold:*

1. *$\mathbb{R}^* \restriction \gamma$ and (resp. $\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma$) is γ-Knaster. In particular, all V-cardinals $\geq \gamma$ are preserved.*
2. *$\mathbb{R}^* \restriction \gamma$ (resp. $\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma$) preserves all the cardinals outside the interval $((\kappa^+)^V, \gamma)$, while collapses the cardinals there to $(\kappa^+)^V$. In particular,*
$$V^{\mathbb{R}^* \restriction \gamma} \models \text{“}(\kappa^+)^V = \kappa^+ \wedge \gamma = \kappa^{++}\text{”}$$
and the same is true in $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$
3. *$V^{\mathbb{R}^* \restriction \gamma} \models$ “κ is strong limit with $\mathrm{cof}(\kappa) = \delta$” and the same is true in $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$.*
4. *$V^{\mathbb{R}^* \restriction \gamma} \models$ “$2^\kappa \geq \gamma$” and the same is true in $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$.*

Lemma 8.5.19. *For each $\gamma \in \mathcal{B}^*$, $\mathbb{Q}^{\mathrm{Even}}_\gamma$ is δ-directed closed over $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$.*

Proof. The argument is the same as in the proof of Proposition 8.4.9(1), using the fact that $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$ and $V^{\mathbb{C}^{\mathrm{Even}}_\gamma}$ have the same $<\delta$-sequences. □

Remark 8.5.20. As in Remark 7.3.21, it is not true that $\mathbb{Q}_\gamma$ is δ-closed over $V^{\mathbb{R}^* \restriction \gamma}$.

Lemma 8.5.21. *For each $\gamma \in \mathcal{B}^*$, the identity map defines a projection between $\mathbb{Q}^{\mathrm{Even}}_\gamma$ and $\mathbb{Q}_\gamma$.*

Proof. The proof is the same as in Lemma 7.3.22. □

Proposition 8.5.22. *For each $\gamma \in \mathcal{B}^*$, $\mathbb{P}_\gamma \times \mathbb{Q}^{\mathrm{Even}}_\gamma$ satisfies (α_γ).*

Proof. The proof is the same as in Propoposition 7.3.23. □

It thus remains to prove that $\mathbb{P}_\gamma \times \mathbb{Q}_\gamma^{\mathrm{Even}}$ satisfies (β_γ). The argument is the same as in Proposition 7.3.25 but we provide details for completeness.

Proposition 8.5.23. *Let $\gamma \in \mathcal{B}^*$. If $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is δ-cc over $V^{\mathbb{C}_\gamma}$ then $\mathbb{P}_\gamma \times \mathbb{Q}_\gamma^{\mathrm{Even}}$ witnesses (β_γ).*

Proof. Let us first prove that if $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is δ-cc over $V^{\mathbb{R}^* \restriction \gamma}$ then $\mathbb{P}_\gamma \times \mathbb{Q}_\gamma^{\mathrm{Even}}$ witnesses (β_γ). By Proposition 8.5.15 we can identify $V^{\mathbb{R}^* \restriction \gamma}$ as $V^{\mathbb{A}_{\mathrm{Odd}(\gamma)} \times \mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$. Clearly, $\mathbb{A}_{\mathrm{Odd}(\gamma)} * (\mathbb{P}_\gamma \times \mathbb{P}_\gamma)$ is δ-cc over $V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$.

Let $G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}$ be a $\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma$-generic filter over V and let $\tau \in V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}]$ be an $\mathbb{A}_{\mathrm{Odd}(\gamma)}(\cong \mathbb{R}^* \restriction \gamma / \dot{G}_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma})$-name for $T \cap \gamma$. Then we can consider τ as an $\mathbb{A}_{\mathrm{Odd}(\gamma)} * \mathbb{P}_\gamma$-name for $T \cap \gamma$ as well. Since in $V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}]$, $2^\delta \geq \gamma$, $\mathbb{A}_{\mathrm{Odd}(\gamma)} * \mathbb{P}_\gamma$ is δ-cc and $\mathbb{Q}_\gamma^{\mathrm{Even}}$ is δ-closed, it follows from Lemma 7.3.24(a) that the tree $T \cap \gamma$ has the same cofinal branches in the models

$$V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}][G_{\mathbb{A}_{\mathrm{Odd}(\gamma)}} * G_{\mathbb{P}_\gamma}][G_{\mathbb{Q}_\gamma^{\mathrm{Even}}}]$$

and

$$V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}][G_{\mathbb{A}_{\mathrm{Odd}(\gamma)}} * G_{\mathbb{P}_\gamma}].$$

Recall that $T \cap \gamma$ had no cofinal branches in $V^{\mathbb{R}^* \restriction \gamma} = V^{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma * \mathbb{A}_{\mathrm{Odd}(\gamma)}}$. By our assumption, $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is δ-cc over $V^{\mathbb{R}^* \restriction \gamma}$ hence, by Lemma 7.3.24(b), $T \cap \gamma$ has the same cofinal branches in $V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}][G_{\mathbb{A}_{\mathrm{Odd}(\gamma)}} * G_{\mathbb{P}_\gamma}]$ and in $V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}][G_{\mathbb{A}_{\mathrm{Odd}(\gamma)}}] = V[G_{\mathbb{R}^* \restriction \gamma}]$. The result follows as

$$V[G_{\mathbb{R}^*_{\mathrm{Even}} \restriction \gamma}][G_{\mathbb{A}_{\mathrm{Odd}(\gamma)}} * G_{\mathbb{P}_\gamma}][G_{\mathbb{Q}_\gamma^{\mathrm{Even}}}] = V[G_{\mathbb{R}^* \restriction \gamma}][G_{\mathbb{P}_\gamma} \times G_{\mathbb{Q}_\gamma^{\mathrm{Even}}}].$$

We are now left with showing that if $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is δ-cc over $V^{\mathbb{C}_\gamma}$ then it is also δ-cc over $V^{\mathbb{R}^* \restriction \gamma}$. Indeed, observe that then $\mathbb{C}_\gamma * (\mathbb{P}_\gamma \times \mathbb{P}_\gamma)$ is δ-cc over V, hence, by Lemma 1.3.18, this is δ-cc over $V^{\mathbb{U}_\gamma}$. Thus, $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is δ-cc over $V^{\mathbb{C}_\gamma \times \mathbb{U}_\gamma}$. Since $\mathbb{C}_\gamma \times \mathbb{U}_\gamma$ projects onto $\mathbb{R}^* \restriction \gamma$ the desired result follows. $\square$

Thus, we are left with checking that $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is δ-cc over $V^{\mathbb{C}_\gamma}$.

Remark 8.5.24. Let $\gamma \in \mathcal{B}^* \cup \{\lambda\}$. As mentioned in the proof of Proposition 8.4.9(3), observe that

$$\tilde{\mathbb{C}}_\gamma := \{(p, (\check{g}, \dot{H})) \mid p \in \mathbb{A}_\gamma,\, g \in V,\, p \Vdash_{\mathbb{A}_\gamma} (\check{g}, \dot{H}) \in \dot{\mathbb{S}}_\gamma^{\pi^*}\}$$

endowed with the induced order yields a dense subposet of $\mathbb{C}_\gamma$. Hence, for our current purposes it is enough to assume that $\mathbb{C}_\gamma = \tilde{\mathbb{C}}_\gamma$.

Notation 8.5.25. For each $\gamma \in \mathcal{B}^* \cup \{\lambda\}$, set $g(\mu) := \varepsilon$ and $\kappa_{g(\mu)} := \kappa$, for every g which is a stem for some $q \in \dot{\mathbb{S}}_\gamma^{\pi^*}$. Observe that $\mathcal{P}_{\kappa_{g(\mu)}}(\kappa_\eta \cap g(\mu)) = \mathcal{P}_\kappa(\kappa_\eta)$, for each $\eta < \mu$.

Convention 8.5.26. For the ease of notation we shall tend to omit the mention to the particular family of measures that we are working with. For instance, instead of writting $(\mathfrak{U}^{\pi^*}_{\gamma})^{\xi}_{\eta,x}$ we shall simply write $U^{\xi}_{\eta,x}$.

Lemma 8.5.27. *Let $\gamma \in \mathcal{B}^*$, $r = (p, (\check{h}, \dot{H})) \in \mathbb{C}_\lambda$ and $r' = (q, (\check{f}, \dot{F})) \in \mathbb{C}_\gamma$. Then, $r' \Vdash_{\mathbb{C}_\gamma}$ "$r \notin \mathbb{P}_\gamma$" if and only if one of the following hold:*

1. *$p \restriction \gamma \perp_{\mathbb{A}_\gamma} q$;*
2. *$p \restriction \gamma \parallel_{\mathbb{A}_\gamma} q$ and $h \cup f$ is not a $\prec$-increasing function;*
3. *$p \restriction \gamma \parallel_{\mathbb{A}_\gamma} q$, $h \cup f$ is a $\prec$-increasing function and*

$$p \cup q \Vdash_{\mathbb{A}_\lambda} (\check{f}, \dot{F})^\frown(\check{h} \setminus \check{f}) \notin \dot{\mathbb{S}}^{\pi^*}_{\gamma} \ \vee\ (\check{h}, \dot{H})^\frown(\check{f} \setminus \check{h}) \notin \dot{\mathbb{S}}^{\pi^*}_{\lambda}.^{10}$$

Proof. First, observe that two conditions $(h, H), (f, F) \in \mathbb{S}^{\pi^*}_{\lambda}$ are compatible if and only if $h \cup f$ is a $\prec$-increasing function and $(h, H)^\frown(f \setminus h), (f, F)^\frown(h \setminus f) \in \mathbb{S}^{\pi^*}_{\lambda}$. Thereby, if some of the above conditions is true, $\varrho^{\lambda}_{\gamma}(r) \perp_{\mathbb{C}_\gamma} r'$. Thus, Lemma 7.3.26 yields $r' \Vdash_{\mathbb{C}_\gamma}$ "$r \notin \mathbb{P}_\gamma$". Conversely, assume that (1)-(3) are false. Since (1) and (2) are false, $p \cup q \in \mathbb{A}_\lambda$ and $i := f \cup h$ is $\prec$-increasing. Also, since (3) is false, we may let a condition $a \leq_{\mathbb{A}_\lambda} p \cup q$ forcing the opposite. Let $A \subseteq \mathbb{A}_\lambda$ generic (over V) containing a. By the above, in $V[A]$, $(f, F)^\frown(h \setminus f) \in \mathbb{S}^{\pi^*}_{\gamma}$ and $(h, H)^\frown(f \setminus h) \in \mathbb{S}^{\pi^*}_{\lambda}$, hence both Sinapova conditions are compatible. Let $(i, I) \in \mathbb{S}^{\pi^*}_{\lambda}$ be a condition witnessing this compatibility and $S \subseteq \mathbb{S}^{\pi^*}_{\lambda}$ generic (over $V[A]$) containing (i, I). Set $r^* := (a, (\check{i}, \dot{I}))$. Clearly, $r^* \in A * \dot{S}$ and $r^* \leq_{\mathbb{C}_\lambda} r$, so $r \in A * \dot{S}$. On the other hand, $\varrho^{\lambda}_{\gamma}[A * \dot{S}]$ generates a $\mathbb{C}_\gamma$-generic filter containing r', hence Lemma 7.3.26 yields $r' \nVdash_{\mathbb{C}_\gamma}$ "$r \notin \mathbb{P}_\gamma$", as wanted. $\square$

For each $\gamma \in \mathcal{B}^* \cup \{\lambda\}$, and unless otherwise stated, we will assume that for each $r = (q, (\check{f}, \dot{F})) \in \mathbb{C}_\gamma$, $q \Vdash_{\mathbb{A}_\gamma}$ "$(\check{f}, \dot{F})$ is pruned". This is of course feasible by virtue of Proposition 8.2.13.

Lemma 8.5.28. *Let $\gamma \in \mathcal{B}^*$, $r = (p, (\check{h}, \dot{H})) \in \mathbb{C}_\lambda$ and $r' = (q, (\check{f}, \dot{F})) \in \mathbb{C}_\gamma$. Assume that $q \leq_{\mathbb{A}_\gamma} p \restriction \gamma$, $h \subseteq f$ and*

$$(\Upsilon)\ \ p \cup q \Vdash_{\mathbb{A}_\lambda} \text{“}\forall \theta \in \mathrm{dom}(\dot{H})\ \left(\dot{H}(\theta) \cap \dot{\mathcal{P}}_{\kappa_{\check{f}(\tau_\theta)}}(\kappa_\theta \cap \check{f}(\tau_\theta)) \in \dot{U}^{\theta}_{\tau_\theta, \check{i}(\tau_\theta)}\right)\text{”},$$

where $q \cup p \Vdash_{\mathbb{A}_\lambda}$ "$\tau_\theta = r_{\mathrm{dom}(\check{f})}(\check{\theta})$". Then there is a $\mathbb{A}_\gamma$-name $\dot{I}$ for which all the following hold:

(I) $q \Vdash_{\mathbb{A}_\gamma}$ "$(\check{f}, \dot{I}) \leq_{\mathbb{S}^{\pi^*}_{\gamma}} (\check{f}, \dot{F}) \wedge (\check{f}, \dot{I})$ is pruned".

(II) $q \Vdash_{\mathbb{A}_\gamma}$ "$\forall \tau \in [\prod_\xi \dot{I}(\xi)]^{<\omega} \left(p \nVdash_{\mathbb{A}_\lambda / \mathbb{A}_\gamma} (\check{h}, \dot{H})^\frown\tau \notin \dot{\mathbb{S}}^{\pi^*}_{\lambda}\right)$".

[10] Here we are identifying the $\mathbb{A}_\gamma$-name $\mathbb{S}^{\pi^*}_{\gamma}$ with its standard extension to a $\mathbb{A}_\lambda$-name.

Proof. Let us work over $V^{\mathbb{A}_\gamma \downarrow q}$. Let $c : [\prod_\xi F(\xi)] \to 2$ be defined as

$$c(i) := \begin{cases} 0, & \text{if } p \Vdash_{\mathbb{A}_\lambda/\mathbb{A}_\gamma} (\check{h}, \dot{H})^\frown i \notin \dot{\mathbb{S}}^{\pi^*}_\lambda; \\ 1, & \text{if } p \nVdash_{\mathbb{A}_\lambda/\mathbb{A}_\gamma} (\check{h}, \dot{H})^\frown i \notin \dot{\mathbb{S}}^{\pi^*}_\lambda. \end{cases}$$

By Lemma 8.2.21 there is $I \subseteq F$ a suitable function for $\langle f \rangle$ and homogeneous for c. In particular, $(f, I) \leq_{\mathbb{S}^{\pi^*}_\gamma} (f, F)$ and (f, I) is pruned, as (f, F) was. Thus, (I) holds. Towards a contradiction, assume that (II) is false. Let $r \leq_{\mathbb{A}_\gamma} q$ be such that r forces the negation of the above formula. By shrinking r we may assume that there is a $\prec$-increasing function i such that $r \Vdash_{\mathbb{A}_\gamma} \check{i} \in [\prod_\xi \dot{I}(\xi)]^{<\omega}$ and $r \Vdash_{\mathbb{A}_\gamma}$ "$\left(p \Vdash_{\mathbb{A}_\lambda/\mathbb{A}_\gamma} (\check{h}, \dot{H})^\frown \check{i} \notin \dot{\mathbb{S}}^{\pi^*}_\lambda\right)$". Since $r \leq_{\mathbb{A}_\gamma} q$, $r \cup p \in \mathbb{A}_\lambda$, hence $r \cup p \Vdash_{\mathbb{A}_\lambda} (\check{h}, \dot{H})^\frown \check{i} \notin \dot{\mathbb{S}}^{\pi^*}_\lambda$. Now, since r forces $\dot{I}$ to be homogenous for $\dot{c}$, it follows that for all j with the same domain as i, $r \cup p \Vdash_{\mathbb{A}_\lambda} (\check{h}, \dot{H})^\frown \check{j} \notin \dot{\mathbb{S}}^{\pi^*}_\lambda$. Since p forces $(\check{h}, \dot{H})$ to be pruned the only chance for this property to hold is that $r \cup p \Vdash_{\mathbb{A}_\lambda} \prod_{\theta \in \mathrm{dom}(i)} \dot{I}(\theta) \cap \prod_{\theta \in \mathrm{dom}(i)} \dot{H}(\theta) = \emptyset$. Let us show that this is impossible.

Let $\theta \in \mathrm{dom}(i)$. If $\theta > \max(\mathrm{dom}(f))$, $\dot{I}(\theta)$ and $\dot{H}(\theta)$ are names for sets in the measure U_θ, and thus they are not forced to be disjoint. Otherwise, if $\theta < \max(\mathrm{dom}(f))$, since $r \cup p \leq_{\mathbb{A}_\lambda} q \cup p$ and (Υ) holds, we may find $s \leq_{\mathbb{A}_\lambda} r \cup p$, such that $s \Vdash_{\mathbb{A}_\lambda} \dot{H}(\theta) \cap \dot{\mathcal{P}}_{\kappa_{\check{f}(\tau_\theta)}}(\kappa_\theta \cap \check{f}(\tau_\theta)) \in \dot{U}^\theta_{\tau_\theta, \check{f}(\tau_\theta)}$. In particular, $s \Vdash_{\mathbb{A}_\lambda} \dot{I}(\theta) \cap \dot{H}(\theta) \cap \dot{\mathcal{P}}_{\kappa_{\check{f}(\tau_\theta)}}(\kappa_\theta \cap \check{f}(\tau_\theta)) \in \dot{U}^\theta_{\tau_\theta, \check{f}(\tau_\theta)}$. Altogether, this produces the desired contradiction. □

Lemma 8.5.29. *Let $\gamma \in \mathcal{B}^*$, $r = (p, (\check{h}, \dot{H})) \in \mathbb{C}_\lambda$ and $r' = (q, (\check{f}, \dot{F})) \in \mathbb{C}_\gamma$. Assume that*

(א) $q \leq_{\mathbb{A}_\gamma} p \restriction \gamma$;

(ב) $h \subseteq f$;

(ג) $p \cup q \Vdash_{\mathbb{A}_\lambda}$ "$(\check{h}, \dot{H})^\frown (\check{f} \setminus \check{h}) \in \dot{\mathbb{S}}^{\pi^*}_\lambda$".

Let $\dot{I}$ be the function obtained from Lemma 8.5.28 with respect to r and r'. Then, $(q, (\check{f}, \dot{I})) \Vdash_{\mathbb{C}_\gamma} (p, (\check{h}, \dot{H})) \in \mathbb{P}_\gamma$.

Proof. Otherwise, let $r^* := (r, (\check{j}, \dot{J})) \leq_{\mathbb{C}_\gamma} (q, (\check{f}, \dot{I}))$ forcing the opposite. By using Lemma 8.5.27 with respect to r^* and r it follows that some of the conditions (1)-(3) must hold. It is not hard to check that (א)-(ג) implies that (3) holds: particularly, that $r \cup p \Vdash_{\mathbb{A}_\lambda}$ "$(\check{h}, \dot{H})^\frown (\check{j} \setminus \check{h}) \notin \dot{\mathbb{S}}^{\pi^*}_\lambda$" holds. By (ג) and since $r \cup p \leq_{\mathbb{A}_\lambda} p \cup q$, $r \cup p \Vdash_{\mathbb{A}_\lambda}$ "$(\check{h}, \dot{H})^\frown (\check{j} \setminus \check{f}) \notin \dot{\mathbb{S}}^{\pi^*}_\lambda$". Clearly, $r \leq_{\mathbb{A}_\gamma} q$ and $r \Vdash_{\mathbb{A}_\gamma} \check{j} \setminus \check{f} \in [\prod_\xi \dot{I}(\xi)]$. Observe that (ג) yields (Υ) of Lemma 8.5.28, and this latter implies $r \cup p \nVdash_{\mathbb{A}_\lambda}$ "$(\check{h}, \dot{H})^\frown (j \setminus f) \notin \dot{\mathbb{S}}^{\pi^*}_\lambda$". This produces the desired contradiction. □

Lemma 8.5.30. *Let $\gamma \in \mathcal{B}^*$, $(q, (\check{f}, \dot{F})) \in \mathbb{C}_\gamma$ and $\dot{r}_0, \dot{r}_1$ be two $\mathbb{C}_\gamma$-names forced by $\mathbb{1}_{\mathbb{C}_\gamma}$ to be in $\mathbb{P}_\gamma$. Then, there are $(q^*, (\check{f}^*, \dot{F}^*)) \in \mathbb{C}_\gamma$, $(p_0, (\check{h}_0, \dot{H}_0))$, $(p_1, (\check{h}_1, \dot{H}_1)) \in \mathbb{P}_\gamma$ and $\bar{p}_0, \bar{p}_1 \in \mathbb{A}_\lambda$ be such that the following hold: For $i \in \{0, 1\}$,*

(a) $(q^*, (\check{f}^*, \dot{F}^*)) \leq_{\mathbb{C}_\gamma} (q, (\check{f}, \dot{F}))$,

(b_i) $(q^*, (\check{f}^*, \dot{F}^*)) \Vdash_{\mathbb{C}_\gamma}$ "$\dot{r}_i = (p_i, (\check{h}_i, \dot{H}_i)) \in \mathbb{P}_\gamma$",

(c_i) $\bar{p}_i \leq_{\mathbb{A}_\lambda} p_i$ *and* $(q^*, (\check{f}^*, \dot{F}^*))$ *and* $(\bar{p}_i, (\check{h}_i, \dot{H}_i))$ *satisfy conditions (1)-(3) of Lemma 8.5.29.*

Proof. Let $(q^*, (\check{f}^*, \dot{F}^*)) \leq_{\mathbb{C}_\gamma} (q, (\check{f}, \dot{F}))$ and $(p_0, (\check{h}_0, \dot{H}_0))$, $(p_1, (\check{h}_1, \dot{H}_1)) \in \mathbb{P}_\gamma$ be such that (b_0) and (b_1) hold. By extending q^* and f^* if necessary, we may further assume that $q^* \leq_{\mathbb{A}_\gamma} p_0 \restriction \gamma \cup p_1 \restriction \gamma$ and $h_0 \cup h_1 \subseteq f^*$. For each $i \in \{0, 1\}$, combining this with Lemma 8.5.27 it follows that condition (3) must fail. Thus, there is $\bar{p}_i \leq_{\mathbb{A}_\lambda} q^* \cup p_i$ with $\bar{p}_i \Vdash_{\mathbb{A}_\lambda} (\check{h}_i, \dot{H}_i)^\frown(\check{f}^* \setminus \check{h}_i) \in \dot{\mathbb{S}}^{\pi^*}_\lambda$. Again, extend p^* to ensure $q^* \leq_{\mathbb{A}_\gamma} \bar{p}_0, \bar{p}_1$. It should be clear at this point that, for $i \in \{0, 1\}$, $(q^*, (\check{f}^*, \dot{F}^*))$ and $(\bar{q}_i, (\check{h}_i, \dot{H}_i))$ witness (c_i). □

Finally, we are in conditions to prove the δ-ccness of $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$.

Lemma 8.5.31. *Let $\gamma \in \mathcal{B}^*$. Then, $\mathbb{1}_{\mathbb{C}_\gamma} \Vdash_{\mathbb{C}_\gamma}$ "$\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is δ-cc".*

Proof. Let $\{(\dot{r}^0_\alpha, \dot{r}^1_\alpha)\}_{\alpha<\delta}$ be a collection of $\mathbb{C}_\gamma$-names that $\mathbb{1}_{\mathbb{C}_\gamma}$ forces to be in a maximal antichain of $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$. Appealing to Lemma 8.5.30 we find families $\{(q^*_\alpha, (\check{f}^*_\alpha, \dot{F}^*_\alpha))\}_{\alpha<\delta}$, $\{\langle(p^0_\alpha, (\check{h}^0_\alpha, \dot{H}^0_\alpha)), (p^1_\alpha, (\check{h}^1_\alpha, \dot{H}^1_\alpha))\rangle\}_{\alpha<\delta}$ and $\{\langle \bar{p}^0_\alpha, \bar{p}^1_\alpha\rangle\}_{\alpha<\delta}$ witnessing it.

It is not hard to check that for each $\varrho \in \mathcal{B}^* \cup \{\lambda\}$, $\mathbb{C}_\varrho$ is δ-Knaster, hence $\mathbb{C}_\gamma \times \mathbb{C}^2_\lambda$ also. In particular, $\mathbb{C}_\gamma \times \mathbb{C}^2_\lambda$ is δ-cc, and thus we may assume that all the above conditions are compatible. Modulo a further refinement, we may also assume that $f^*_\alpha = f^*$, $h^0_\alpha = h^0$ and $h^1_\alpha = h^1$, for each $\alpha < \delta$. For each $\alpha < \beta < \delta$, set $r_{\alpha,\beta} := (q^*_\alpha \cup q^*_\beta, (f^*, \dot{F}^*_\alpha \wedge \dot{F}^*_\beta))$ and $r'_{i,\alpha,\beta} := (\bar{p}^i_\alpha \cup \bar{p}^i_\beta, (h^i, \dot{H}^i_\alpha \wedge \dot{H}^i_\beta))$, for $i \in \{0, 1\}$. It is routine to check that, for each $i \in \{0, 1\}$, $r_{\alpha,\beta}$ and $r'_{i,\alpha,\beta}$ witness the hypotheses of Lemma 8.5.29, hence there is $r^*_{\alpha,\beta} \leq_{\mathbb{C}_\gamma} r_{\alpha,\beta}$ forcing that both $r'_{0,\alpha,\beta}$ and $r'_{1,\alpha,\beta}$ are in $\mathbb{P}_\gamma$. In particular, $r^*_{\alpha,\beta} \Vdash_{\mathbb{C}_\gamma} (\dot{r}^0_\alpha, \dot{r}^1_\alpha) \parallel_{\mathbb{P}_\gamma \times \mathbb{P}_\gamma} (\dot{r}^0_\beta, \dot{r}^1_\beta)$, which entails the desired contradiction.

□

8.6 TP(κ^+) holds

In this section we conclude the proof of Theorem 6.0.14 by showing that TP(κ^+) holds in $V[R]$. Once again, we only give details when $\Theta = \lambda^+$, as the more general case is completely parallel. In essence the arguments exposed here are due to Sinapova [Sin16] and Neeman [Nee09]. The only reason in favour of presenting them is to point out some subtle differences between their argument and ours. Also, by showing explicitly the arguments, we hope to convince the skeptic reader that similar ideas indeed do the job in our context. To avoid repetitions, we sometimes tend to sketch the main ideas and refer the reader to [Sin16], [Sin12] or [Nee09] for more details. The proof of $V[R] \models \mathrm{TP}(\delta)$, at least as conceived in [Sin16], uses a family of intermediate forcings between $\mathbb{R}$ and $\bar{\mathbb{R}}$ (see Section 8.4). These forcings $\mathbb{R}_{\dot{p}}$ have the particularity that its generics $R_{\dot{p}}$ resemble R. For the record of the section let us recall that G, S and R are, respectively, the generic filters for $\mathbb{A}_{\lambda^+}$, $\mathbb{S}_{\lambda^+}$ and $\mathbb{R}$ considered at Section 8.4.

Convention 8.6.1. For each $\mathbb{A}_{\lambda^+}$-name $\dot{q}$ for a condition in $\mathbb{S}_{\lambda^+}$, we shall denote by q its interpretation by G. Also, set $\hat{q} := \langle(\mathbb{1}, \dot{q}), (\mathbb{1}, \dot{\mathbb{1}}, \mathbb{1})\rangle$ and $q^* := (\mathbb{1}, \dot{q}, \mathbb{1})$.

Definition 8.6.2. Let $\dot{q}$ be a $\mathbb{A}_{\lambda^+}$-name for a condition in $\mathbb{S}_{\lambda^+}$. Let $\mathbb{R}_{\dot{q}}$ be the set of $(p, \dot{q}', r) \in \mathbb{R}$ endowed with the order $(p_1, \dot{q}_1, r_1) \leq_{\mathbb{R}_{\dot{p}}} (p_2, \dot{q}_2, r_2)$ if and only if $(p_1, \dot{q}_1) \leq_{\mathbb{A}_{\lambda^+} * \dot{\mathbb{S}}_{\lambda^+}} (p_2, \dot{q}_2)$, $\mathrm{dom}(r_2) \subseteq \mathrm{dom}(r_1)$ and for each $\gamma \in \mathrm{dom}(r_2)$, $\sigma^{\lambda^+}_{\gamma}(p_1, \dot{q}) \Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}^{\pi}_{\gamma}}$ "$\dot{r}_1(\gamma) \leq_{\dot{\mathrm{Add}}(\delta,1)} \dot{r}_2(\gamma)$".

The next proposition shows that there is a system of projections between the forcings $\bar{\mathbb{R}}$, $\mathbb{R}$ and $\mathbb{R}_{\dot{q}}$ (see [Sin16, §2] for details).

Proposition 8.6.3. *Let $\dot{q}$ be a $\mathbb{A}_{\lambda^+}$-name for a condition in $\mathbb{S}_{\lambda^+}$.*

1. *The map $\langle(p, \dot{t}), (\mathbb{1}, \mathbb{1}, r)\rangle \mapsto (p, \dot{t}, r)$ defines a projection between $\bar{\mathbb{R}}$ and $\mathbb{R}_{\dot{q}}$ and also between $\bar{\mathbb{R}} \downarrow \hat{q}$ and $\mathbb{R}_{\dot{q}} \downarrow q^*$.*
2. *The identity entails a projection between $\mathbb{R}_{\dot{q}} \downarrow q^*$ and $\mathbb{R} \downarrow q^*$.*

Let q, t be conditions in $\mathbb{S}_{\lambda^+}$ such that $t \leq_{\mathbb{S}_{\lambda^+}} q$. Then the identity establishes a projection between $\mathbb{R}_q$ and $\mathbb{R}_t$.

Definition 8.6.4. Work in $V[G]$. For each $q \in \mathbb{S}$ define the forcing $\mathbb{U}_q$ whose conditions are all $r \in \mathbb{U}$ such that $r_1 \leq_{\mathbb{U}_p} r_2$ if and only if $\mathrm{dom}(r_2) \subseteq \mathrm{dom}(r_1)$ and there is $p \in G$ such that for each $\gamma \in \mathrm{dom}(r_2)$,

$$\sigma^{\lambda^+}_{\gamma}(p, \dot{q}) \Vdash_{\mathbb{A}_{\mathrm{Even}(\gamma)} * \dot{\mathbb{S}}^{\pi}_{\gamma}} \text{"}r_1(\gamma) \leq r_2(\gamma)\text{"}.$$

The next lemma corresponds with [Sin16, Lemma 2.7].

Lemma 8.6.5. *Let $\dot{q}$ be a $\mathbb{A}_{\lambda^+}$-name for a condition in $\mathbb{S}_{\lambda^+}$. Then $\mathbb{R}_{\dot{q}}$ and $\mathbb{A}_{\lambda^+} * (\dot{\mathbb{S}}_{\lambda^+} \times \dot{\mathbb{U}}_q)$ are isomorphic. In particular, in $V[G]$, there is a projection between $\mathbb{R}_q$ and $\mathbb{U}_q$.*

Proposition 8.6.6. *Work in $V[G]$. For each condition $q \in \mathbb{S}$, the identity yields a projection between $\mathbb{U}$ and $\mathbb{U}_q$. Moreover, for each $t \leq_{\mathbb{S}_{\lambda^+}} q$ the same holds between $\mathbb{U}_q$ and $\mathbb{U}_t$.*

Let $\bar{R} \subseteq \bar{\mathbb{R}}$ a generic filter whose respective projections onto $\mathbb{R}$, $\mathbb{A}_{\lambda^+}$ and $\mathbb{S}_{\lambda^+}$ induce R, G and S.[11] Let $U \subseteq \mathbb{U}$ be the generic filter induced by $\bar{R}$. We also need generics for the family $\langle \mathbb{R}_p, \mathbb{U}_p \mid p \in \mathbb{S}\rangle$. For this, we will use the following standard lemma. For a proof see, for instance, [Ung13, Proposition 4.7].

Lemma 8.6.7. *Let $\mathbb{P}, \mathbb{Q}, \mathbb{C}$ be posets and $\pi : \mathbb{P} \to \mathbb{Q}$ and $\sigma : \mathbb{Q} \to \mathbb{C}$ be projections. For any generic filter $H \subseteq \mathbb{C}$, the restriction $\pi \restriction \mathbb{P}/H$ is a projection between $\mathbb{P}/H$ and $\mathbb{Q}/H$ in $V[H]$.*

For $q \in S$, $q^* \in R$, hence $R \downarrow q^*$ is a generic filter for $\mathbb{R} \downarrow q^*$. Since there are projections π_q between $\bar{\mathbb{R}} \downarrow \hat{q}$ and $\mathbb{R}_{\dot{q}} \downarrow q^*$ and π^q between $\mathbb{R}_{\dot{q}} \downarrow q^*$ and $\mathbb{R} \downarrow q^*$, the previous lemma ensures that $\pi_q \restriction \bar{\mathbb{R}}/R$ is a projection between $\bar{\mathbb{R}}/R$ and $\mathbb{R}_{\dot{q}}/R$. For each $q \in S$, let $R_q \subseteq \mathbb{R}_{\dot{q}} \downarrow q^*$ be the generic filter over $V[R]$ induced by $\bar{R}$ and π_q. Analogously, let $U_q \subseteq \mathbb{U}_q$ be the generic filter over $V[G]$ induced by R_q and the corresponding projection.

Remark 8.6.8.

1. By Proposition 8.6.3, $R_q \subseteq R_{q'} \subseteq R$, for each $q' \leq_{\mathbb{S}_{\lambda^+}} q$ in S. Moreover, for each $s \in \bar{\mathbb{R}}/R$, there is $p \in S$ such that $s \in R_p$ (see [Sin08, Lemma 3.8]).
2. By Proposition 8.6.6, $U \subseteq U_q \subseteq U_{q'}$, each $q' \leq_{\mathbb{S}_{\lambda^+}} q$ in S.

Aiming for a contradiction, assume that $V[R] \models \neg\mathrm{TP}(\delta)$ and let a δ-tree $(T, <_T) \in V[R]$ be witnessing this. For each $\alpha < \delta$, set $T_\alpha := \{u \in T \mid \mathrm{level}(u) = \alpha\}$. Modulo isomorphism, we may assume $T_\alpha = \{\alpha\} \times \kappa$, for each $\alpha < \delta$. Let $\tau \in V[G]$ be a $\mathbb{R}/G$-name for T and assume that $\mathbb{1}_{\mathbb{R}/G} \Vdash_{\mathbb{R}/G}$ "τ is a δ-tree". Analogously, let $\dot{T} \in V[G][U]$ and, for each $q \in \mathbb{S}_{\lambda^+}$, $\dot{T}_q \in V[G][U_q]$ be, respectively, the $\mathbb{S}_{\lambda^+}$-name for the tree T induced by τ. Notice that the interpretation of the names τ, $\dot{T}$ and $\dot{T}_q$ by the corresponding generic filters gives the same set; i.e. T. Thus, the only formal difference between these names is the ground model where they are regarded.

Definition 8.6.9. For a condition $p \in \mathbb{S}_{\lambda^+}$, write $\mathfrak{m}^p := \max(\mathrm{dom}(g^p))$. Denote by $\mathcal{S}$ the set of pairs (g, H^*) for which there is $p \in \mathbb{S}_{\lambda^+}$ with $p_{\restriction \mathfrak{m}^p+1} = (g, H^*)$ (c.f. Definition 8.1.7). We will consider $\mathcal{S}$ endowed with $\leq_{\mathcal{S}}$, the induced order by $\leq_{\mathbb{S}_{\lambda^+}}$: i.e. $(g, H^*) \leq_{\mathcal{S}} (i, I^*)$ iff there are $p, q \in \mathbb{S}_{\lambda^+}$ witnessings that $(g, H^*), (i, I^*) \in \mathcal{S}$ and $p \leq_{\mathbb{S}_{\lambda^+}} q$.

The following property is implicitly considered in [Sin12].

[11] Recall that these are the generic filters of Section 8.4

Definition 8.6.10 (Dagger property)**.** Work in $V[G][U]$. For a pair $(g, H^*) \in \mathcal{S}$, we will say that $\dagger_{(g,H^*)}$ holds if there is $J \subseteq \delta$ unbounded, $\langle p_\alpha \mid \alpha \in J\rangle$ a sequence of conditions in $\mathbb{S}_{\lambda^+}$ and $\xi < \kappa$ such that for each $\alpha \in J$ setting $u_\alpha := \langle \alpha, \xi\rangle$, the following are true:

1. For each $\alpha \in J$, p_α witnesses that $(g, H^*) \in \mathcal{S}$.
2. For each $\alpha < \beta$ in J, $p_\alpha \wedge p_\beta \Vdash^{V[G][U]}_{\mathbb{S}} u_\alpha <_{\dot{T}} u_\beta$.

Since $\mathbb{U}$ is δ-directed closed (in V), $V[U]$ thinks that κ is supercompact and the same holds for the sequence $\langle \kappa_{\xi+1} \mid \xi < \mu\rangle$. By appealing to the arguments of [Sin12, §3] one has the following:

Lemma 8.6.11. *In $V[G][U]$ the set $\{p \in \mathbb{S}_{\lambda^+} \mid \dagger_{p\restriction \mathfrak{m}^p+1}$ holds$\}$ is dense.*

An immediate consequence of the previous lemma is the existence of a cofinal branch of T in $V[\bar{R}]$ (see [Sin12, Proposition 21] and the subsequent discussion).

Proposition 8.6.12. *There is a cofinal branch $b \in V[\bar{R}]$ through T.*

Now we are left to prove that b induces a cofinal branch for T in $V[R]$. Let $\dot{b}$ be a $\bar{\mathbb{R}}/G$-name for b. Moreover, let us assume that $(\mathbb{1}_{\mathbb{S}}, \mathbb{1}_{\mathbb{U}}) \Vdash^{V[G]}_{\mathbb{S}\times\mathbb{U}}$ "$\dot{b}$ cofinal branch in τ". We will need to consider a minor variation of the property $\dagger_h$ of [Sin16, Definition 3.3].

Notation 8.6.13. Work in $V[G]$. For a pair $(g, H^*) \in \mathcal{S}$, denote by $E_{(g,H^*)}$ the set of $u \in T$ for which there are $(q, r) \in \mathbb{S} \times \mathbb{U}$ such that q witnesses $(g, H^*) \in \mathcal{S}$, $r \in U$ and $(q, r) \Vdash^{V[G]}_{\mathbb{S}\times\mathbb{U}} u \in \dot{b}$.

Definition 8.6.14. Work in $V[G]$. For a pair $(g, H^*) \in \mathcal{S}$ and $\alpha < \delta$, we say that there is a (g, H^*)-splitting at $u \in T_\alpha \cap E_{(g,H^*)}$ if, provided that (q, r) witnesses $u \in E_{(g,H^*)}$, there are $\beta \geq \alpha$, $v_1, v_2 \in T_\beta$ and $r_1, r_2 \leq_{\mathbb{U}} r$ in U_q be such that

- $(q, r_k) \Vdash^{V[G]}_{\mathbb{S}\times\mathbb{U}} v_k \in \dot{b}$, $k \in \{0, 1\}$,
- $q \Vdash^{V[G][U]}_{\mathbb{S}} v_1 \perp_{\dot{T}} v_2$.

Remark 8.6.15. If there is a (g, H^*)-splitting at u and $(g, I^*) \in \mathcal{S}$ then there is $(g, F^*) \leq_{\mathcal{S}} (g, I^*), (g, H^*)$ and a (g, F^*)-splitting at u. Indeed, let q, r, v_1, v_2, r_1 and r_2 witnessing the existence of a (g, H^*)-splitting at u. Now set $q^* := (g, F)$, where

$$F(\eta) := \begin{cases} H^*(\eta) \cap I^*(\eta), & \text{if } \eta \in \mathfrak{m}^p \setminus \mathrm{dom}(g^*); \\ H^q(\eta), & \mathfrak{m}^p < \eta. \end{cases}$$

Set $F^* := F \restriction \mathfrak{m}^p + 1$. Clearly $q^* \leq_{\mathbb{S}_{\lambda^+}} q$. By Remark 8.6.8, $r_1, r_2 \in U_{q^*}$. Evidently, q^*, r, v_1, v_2, r_1 and r_2 witness a (g, F^*)-splitting at u and $(g, F^*) \leq_{\mathcal{S}} (g, I^*), (g, H^*)$. The same is true for $(g, F^*) = (g, I^*)$ if $(g, I^*) \leq_{\mathcal{S}} (g, H^*)$.

This remark suggest the following definition:

Definition 8.6.16. Work in $V[G]$. For a stem g, we will say that there is a g-splitting at u if there is some (g, H^*)-splitting at u, for some $(g, H^*) \in \mathcal{S}$.

Definition 8.6.17. Work in $V[G][U]$. For a pair $(g, H^*) \in \mathcal{S}$ we will say that $\dagger^b_{(g,H^*)}$ holds if there is $J \subseteq \delta$ unbounded, $\langle p_\alpha \mid \alpha \in J\rangle$ a sequence of conditions in $\mathbb{S}_{\lambda^+}$ and $\xi < \kappa$ such that for each $\alpha \in J$ setting $u_\alpha := \langle \alpha, \xi\rangle$, the following are true:

1. For each $\alpha \in J$, p_α witnesses that $(g, H^*) \in \mathcal{S}$.
2. For each $\alpha \in J$, $p_\alpha \Vdash^{V[G][U]}_{\mathbb{S}_{\lambda^+}} u_\alpha \in \dot{b}$.
3. For each $\alpha < \beta$ in J, $p_\alpha \wedge p_\beta \Vdash^{V[G][U]}_{\mathbb{S}_{\lambda^+}} u_\alpha <_{\dot{T}} u_\beta$.

A straightforward modification of the arguments involved in the proof of Lemma 8.6.11 yields that $\{p \in \mathbb{S}_{\lambda^+} \mid \dagger^b_{p\restriction\mathfrak{m}^p+1}$ holds$\}$ is dense.

Remark 8.6.18. If $(g, I^*) \in \mathcal{S}$ and $\dagger^b_{(g,H^*)}$ holds then there is $(g, F^*) \leq_{\mathcal{S}} (g, I^*), (g, H^*)$ for which $\dagger^b_{(g,F^*)}$ holds. Indeed, let $J \subseteq \delta$, $\langle p_\alpha \mid \alpha \in J\rangle$ and $\xi < \kappa$ witnessing $\dagger^b_{(g,H^*)}$. For each $\alpha \in J$, define $q_\alpha := (g, F_\alpha)$, where F_α is defined as in Remark 8.6.15 but with respect to $H^{p_\alpha} \setminus \mathfrak{m}^{p_\alpha} + 1$ rather than $H^p \setminus \mathfrak{m}^p + 1$. It is obvious that J, $\langle q_\alpha \mid \alpha \in J\rangle$ and $\xi < \kappa$ are witness for $\dagger^b_{(g,F^*)}$. The same is true for $(g, F^*) = (g, I^*)$ if $(g, I^*) \leq_{\mathcal{S}} (g, H^*)$.

Definition 8.6.19. Work in $V[G][U]$. For a stem g, we will say that $\dagger^b_g$ holds if $\dagger^b_{(g,H^*)}$ holds, for some $(g, H^*) \in \mathcal{S}$. Define

$$\alpha_{(g,H^*)} := \sup\{\alpha < \delta \mid \exists u \in T_\alpha \cap E_{(g,H^*)} \text{ and there is } (g, H^*)\text{-splitting at } u\},$$

and set $\alpha_g := \sup\{\alpha_{(g,H^*)} \mid \exists H^* \, (g, H^*) \in \mathcal{S}\}$.

By a very similar argument to Remark 8.6.15 if $(g, I^*) \leq_{\mathcal{S}} (g, H^*)$, then every (g, H^*)-splitting at some u yields a (g, I^*)-splitting at u, and thus $\alpha_{(g,H^*)} \leq \alpha_{(g,I^*)}$.

Lemma 8.6.20. *If there is a g-splitting at u then there is some stem $i \supseteq g$ for which there is a i-splitting at u and $\dagger^b_i$ holds.*

Proof. Let u be some node where a (g, H^*)-splitting occurs, for some H^*. Say $(q, r) \Vdash^{V[G]}_{\mathbb{S}\times\mathbb{U}} u \in \dot{b}$, $(q, r_k) \Vdash^{V[G]}_{\mathbb{S}\times\mathbb{U}} v_k \in \dot{b}$, $r_k \leq_{\mathbb{U}} r$ and $r_k \in U_q$, for $k \in \{0, 1\}$. By previous comments, find $\tilde{q} \leq_{\mathbb{S}_{\lambda^+}} q$ for which $\dagger^b_{\tilde{q}\restriction\mathfrak{m}^{\tilde{q}}+1}$ holds. Set $(i, I^*) := \tilde{q}_{\restriction\mathfrak{m}^{\tilde{q}}+1}$. Hence, $\dagger^b_i$ holds. By Remark 8.6.8, $r_0, r_1 \in U_{\tilde{q}}$. Clearly, $\tilde{q}, r, v_1, v_2, r_1$ and r_2 witness the existence of a (i, I^*)-splitting at u. $\square$

Now we need to show that if $\dagger_g^b$ holds then $\alpha_g < \delta$. This is essentially what is proved in [Sin16, Proposition 3.4] for Gitik-Sharon forcing. We will give some details just to convince the reader that similar arguments also work for Sinapova forcing.

Lemma 8.6.21. *In $V[G][U]$, for each stem g, if $\dagger_g^b$ holds then $\alpha_g < \delta$.*

Proof. Assume otherwise and let $\bar{r}$ be a condition in U such that $\bar{r} \Vdash_{\mathbb{U}}^{V[G]}$ "$\dagger_g^b$ holds and $\dot{\alpha}_g = \check{\delta}$". Since $\mathbb{1}_{\mathbb{U}} \Vdash_{\mathbb{U}}^{V[G]}$ "δ is regular" and $|\{H^* \mid (g, H^*) \in \mathcal{S}\}|^{V[G]} < \delta$, it follows that

$$\bar{r} \Vdash_{\mathbb{U}}^{V[G]} \text{“}\exists \check{H}^* \, (\dagger^b_{(\check{g}, \check{H}^*)} \text{ holds and } \dot{\alpha}_{(\check{g}, \check{H}^*)} = \check{\delta})\text{”}.$$

By extending $\bar{r}$ if necessary, we may assume that there is $(g, H^*) \in \mathcal{S}$ be such that $\bar{r} \Vdash_{\mathbb{U}}^{V[G]}$ "$\dagger^b_{(\check{g}, \check{H}^*)}$ holds and $\dot{\alpha}_{(\check{g}, \check{H}^*)} = \check{\delta}$".

Claim 8.6.21.1. *Let $r \leq_{\mathbb{Q}} \bar{r}$ and $r \in U_q$, for some $q \in S$ witnessing $(g, I^*) \in \mathcal{S}$ and $(g, I^*) \leq_{\mathcal{S}} (g, H^*)$. Then in $V[G]$ there are $\langle v_i^* \mid i < \varepsilon \rangle$ nodes and $\langle \langle p_i^*, r_i^* \rangle \mid i < \varepsilon \rangle$ conditions in $\mathbb{S}_{\lambda^+} \times \mathbb{U}$ be such that:*

1. *For each $i < \varepsilon$, $p_i^* \leq_{\mathbb{S}_{\lambda^+}} q$, $r_i^* \leq_{\mathbb{U}} r$, $r_i \in U_{p_i^*}$;*
2. *for each $i < \varepsilon$, p_i^* has stem g,*
3. *for each $i < \varepsilon$, $\langle p_i^*, r_i^* \rangle \Vdash_{\mathbb{S}_{\lambda^+} \times \mathbb{U}} v_i^* \in \dot{b}$, and*
4. *for each $i < j < \varepsilon$, $p_i^* \wedge p_j^* \Vdash_{\mathbb{S}_{\lambda^+}} v_i^* \perp_{\dot{T}} v_j^*$.*

Proof of claim. Let U' be a $\mathbb{U}/U_q$ generic over $V[G][U_q]$ and $r \in U'$. Since $r \leq_{\mathbb{Q}} \bar{r}$, $\alpha_{(g,H^*)} = \delta$ and $\dagger^b_{(g,H^*)}$ hold in $V[G][U']$. By the previous remarks we have that $\dagger^b_{(g,I^*)}$ and $\alpha_{(g,I^*)}$ also hold in this model. Denote by $E_{(g,I^*)}$, J, $\langle p_\alpha \mid \alpha \in J \rangle$ and ξ the objects in $V[G][U']$ that witness $\dagger^b_{(g,I^*)}$. Let us now work over $V[G][U']$.

Subclaim 8.6.21.1.1. *For every $u \in E_{(g,I^*)}$, there is $p \in \mathbb{S}_{\lambda^+}$ with $p \leq^*_{\mathbb{S}_{\lambda^+}} q$, $r_1, r_2 \in U_p$ and nodes v_1, v_2 of higher levels, such that $\langle p, r_k \rangle \Vdash_{\mathbb{S}_{\lambda^+} \times \mathbb{U}} v_k \in \dot{b}$ and $p \Vdash_{\mathbb{S}_{\lambda^+}}^{V[G][U']} v_1 \perp_{\dot{T}} v_2$, $u <_{\dot{T}} v_1$, $u <_{\dot{T}} v_2$.*

Proof of subclaim. Let $u \in E_{(g,I^*)}$ and $(p', t') \Vdash_{\mathbb{S}_{\lambda^+} \times \mathbb{U}}^{V[G]} u \in \dot{b}$ with $t' \in U$ and p' witnessing $(g, I^*) \in \mathcal{S}$. Since $\alpha_{(g,I^*)} = \delta$, there is v in a higher level of the tree for which there is a (g, I^*)-splitting. Namely, there are p, r, v_1, v_2, r_1, r_2 as follows:

1. $p \in \mathbb{S}_{\lambda^+}$ witnesses $(g, I^*) \in \mathcal{S}$, $r \in U$, $\langle p, r \rangle \Vdash_{\mathbb{S}_{\lambda^+} * \mathbb{U}}^{V[G]} v \in \dot{b}$,
2. v_k is a node in a higher level than v and $\langle p, r_k \rangle \Vdash_{\mathbb{S}_{\lambda^+} \times \mathbb{U}}^{V[G]} v_k \in \dot{b}$, with $r_k \leq_{\mathbb{U}} r$ and $r_k \in U_p$, for $k \in \{1, 2\}$,

3. $p \Vdash^{V[G][U']}_{\mathbb{S}_{\lambda^+}} v_1 \perp_{\dot{T}} v_2$.

Observe that we may further assume $r_1, r_2 \leq_{\mathbb{U}} t'$. Also, $p^* := p \wedge p'$ is a condition $\leq^*_{\mathbb{S}_{\lambda^+}}$-below p and p'. Remark 8.6.8 yields $r_1, r_2 \in U_{p^*}$. Finally, notice that p^*, r_1, r_2, v_1, v_2 is a witness for our statement. $\square$

By extending r if necessary, we may assume that r forces the conclusion of the above subclaim. Let C be the set of all $\alpha < \delta$ such that for each $\beta < \alpha$ and $u \in T_\gamma$, if there is some $r' \leq_{\mathbb{U}/U_q} r$ with $r' \Vdash^{V[G][U_q]}_{\mathbb{U}/U_q} u \in \dot{E}_{(g,I^*)}$, then there are levels $\beta < \gamma_1 \leq \gamma_2 < \alpha$ and nodes $v_1 \in T_{\gamma_1}$ and $v_2 \in T_{\gamma_2}$ witnessing the above subclaim, for some conditions $p \in \mathbb{S}_{\lambda^+}$ and $r_1, r_2 \in \mathbb{U}$. Clearly, C is closed. Also, since $\alpha_{(g,I^*)} = \delta$, is unbounded, hence C is a club on δ. Observe that $C \in V[G][U_q]$.

Working in $V[G][U']$ define $\langle p^i, \gamma_i, \alpha_i \mid i < \varepsilon\rangle$ as follows: $\gamma_i \in J$, $p^i := p_{\gamma_i}$ and $\alpha_i \in C$ is such that $\gamma_i < \alpha_i \leq \gamma_{i+1}$. For each $i < \varepsilon$, set $u_i := \langle \gamma_i, \xi\rangle$ and let $s_i \in U'$, $s_i \leq_{\mathbb{Q}} r$, be such that $s_i \Vdash^{V[G]}_{\mathbb{U}}$ "$\gamma_i \in \dot{J}$ and $p^i = \dot{p}_{\gamma_i}$". Since $\mathbb{A}$ is κ-cc and $\mathbb{U}$ is δ-directed closed, Easton's lemma implies that $\mathbb{A}$ forces that $\mathbb{U}$ is δ-distributive, hence $\langle p^i, \gamma_i, \alpha_i, s_i \mid i < \varepsilon\rangle \in V[G]$. By construction,

- for each $i < \varepsilon$, p^i witnesses $(g, I^*) \in \mathcal{S}$,
- for each $i < \varepsilon$, $\langle p^i, s_i\rangle \Vdash^{V[G]}_{\mathbb{S}_{\lambda^+} \times \mathbb{U}} u_i \in \dot{b}$,
- $i < j < \varepsilon$, $p^i \wedge p^j \Vdash_{\mathbb{S}_{\lambda^+}} u_i <_{\dot{T}} u_j$.

In particular, $s_i \Vdash^{V[G]}_{\mathbb{U}} u_i \in \dot{E}_{(g,I^*)}$. By definition of C, for each $i < \varepsilon$, there is $q^i \leq^*_{\mathbb{S}_{\lambda^+}} q$, $r^i_1, r^i_2 \in U_{q^i}$ and v^i_1, v^i_2 be such that

1. for each $i < \varepsilon$ and $k \in \{1, 2\}$, $\langle q^i, r^i_k\rangle \Vdash_{\mathbb{S}_{\lambda^+} \times \mathbb{U}} v^i_k \in \dot{b}$ and $r^i_k \in U_{q^i}$,
2. for each $i < \varepsilon$, $q_i \Vdash_{\mathbb{S}_{\lambda^+}} v^i_1 \perp_{\dot{T}} v^i_2$, $u_i <_{\dot{T}} v^i_1$, $u_i <_{\dot{T}} v^i_2$,
3. for each $i < \varepsilon$, $\gamma_i < \mathrm{level}(v^i_1), \mathrm{level}(v^i_2) < \gamma_{i+1}$.

Observe that we may further assume that $q^i \leq_{\mathbb{S}_{\lambda^+}} p^i$, as the stems are the same. Let $\varphi(i, k)$ be "$v^i_k <_{\dot{T}} u_{i+1}$". By (2) and the Prikry property, there is $k^* \in \{1, 2\}$ and $p^*_i \leq^* q^i \wedge p^{i+1}$ be such that $p^*_i \Vdash_{\mathbb{S}_{\lambda^+}} \neg\varphi(i, k^*)$. Set $r^*_i = r^i_{k^*}$ and $v^*_i := v^i_{k^*}$. By using Remark 8.6.8 it is immediate that $\langle p^*_i, r^*_i, v^*_i \mid i < \varepsilon\rangle$ is as desired. This finishes the proof of the claim. $\square$

From this point on the argument is identical to [Sin16], so we decline the chance to provide more details.

$\square$

Lemma 8.6.22. $V[R] \models \mathrm{TP}(\delta)$.

Proof skecth. By Lemma 8.6.21, $\alpha^* := \sup_g\{\alpha_g \mid \text{“}\dagger^b_g \text{ holds”}\} + 1 < \delta$. Let $u \in T_{\alpha^*}$ and $s^* \in R$ be such that $s^* \Vdash^{V[G]}_{\mathbb{R}^*/G} u \in \dot{b}$. Define $b^* := \{v \in T \mid u <_T v, (\exists s \in R)\, s \leq_{\bar{\mathbb{R}}} s^*, s \Vdash^{V[G]}_{\bar{\mathbb{R}}/G} v \in \dot{b}\}$. Clearly, $b^* \in V[R]$ and b^* is a cofinal set in T. By our initial assumption, b^* is not a branch through T, hence there is $\gamma > \alpha^*$ with $|T_\gamma \cap b^*| \geq 2$. By Remark 8.6.8, $\bar{\mathbb{R}}/R = \bigcup_{p \in S} R_p$. We can use this to prove that there is a (g, H^*)-splitting at u, for some (g, H^*). Thus, $\alpha^* \leq \alpha_g$. By Lemma 8.6.20, we may further assume that $\dagger^b_{(g,H^*)}$ holds, so that $\alpha^* \leq \alpha_g < \alpha^*$. This forms the desired contradiction. $\square$

CHAPTER 9

Magidor-Shelah Theorem for partial forms of strong compactness

In previous chapters we have argued that the existence of κ-Aronszajn trees is in essence dependent on Large Cardinals. We have also shown that if (very) Large Cardinals exist then it is viable to force the consistency of many tree property configurations. Nonetheless, there is another interesting question that we have omitted and that we would like to address here:

Question 9.0.1. Do the existence of Large Cardinals imply nice tree property configurations in the universe of sets?

This question is fairly natural as it is well-known that Large Cardinals yield a myriad of forms of compactness (cf. Section 1.4). The following compilation of results reinforces this thesis:

Theorem 9.0.2 (Large Cardinals & Compactness Principles)**.**

1. *If κ is a weakly compact cardinal then* $\mathrm{TP}(\kappa)$ *holds.*
2. *Let κ be a supercompact cardinal. Then the following are true:*
   (a) *The logic $\mathscr{L}_{\kappa,\kappa}$ is compact: namely, every κ-consistent set of $\mathscr{L}_{\kappa,\kappa}$-sentences is consistent.*
   (b) *The principle $\square_{\lambda,\theta}$ fails, for each* $\mathrm{cof}(\theta) < \kappa \leq \lambda$.
3. *If κ is extendible then the logic $\mathscr{L}^n_{\kappa,\kappa}$ is compact, for* $1 \leq n < \omega$.
4. VP *holds if and only if every logic $\mathscr{L}$ has a $\mathscr{L}$-strong compact cardinal.*

For the proofs of (1)-(3) see [Kan09] wile for (4) see [Mak85, §3].

Coming back to our question, in this brief chapter we will be concerned on how the presence of Large Cardinals force the universe of sets to exhibit certain tree property configurations. One of the most relevant results in this direction is due to M. Magidor and S. Shelah [MS96] and reads as follows:

Theorem 9.0.3 (Magidor & Shelah)**.** *Let $\langle \kappa_n \mid n < \omega\rangle$ be a strictly increasing sequence of strong compact cardinals. Then* $\mathrm{TP}(\kappa_\omega^+)$ *holds, where* $\kappa_\omega := \sup_{n<\omega} \kappa_n$.

In this chapter we aim to show that the same result of Theorem 9.0.3 can be obtained from apparently more modest assumptions. We are referring to the partial forms of strong compactness introduced by J. Bagaria and M. Magidor in [BM14a] and [BM14b] (see also Definition 1.1.13).

This family of cardinals were discovered during the authors's investigations of Radicals in Infinite Abelian Group Theory and other topological properties such as κ-Lindelöffness. In [BM14a] it is proved that the first ω_1-strong compact cardinal can be singular, hence consistently smaller than the first strong compact. Yet, many reasonable questions about the nature of these cardinals have not found satisfactory answers. For instance, see Question 1.1.14 and Question 1.1.15.

As announced, we will next show that the necessary hypotheses for Theorem 9.0.3 to work can be weaken to these weak forms of strong compactness. Our argument is similar to that showed at Section 8.6 and will sound familiar to the specialists. Since there is still a chance that this weak forms of strong compactness do not yield a regular nor strong limit cardinals, our theorem would provide evidence that for obtaining Theorem 9.0.3 one just needs the following:

1. The cardinals in $\langle \kappa_n \mid n < \omega\rangle$ enjoy certain compactness behaviour.
2. The critical points associated to their elementary embeddings are cofinal in the cardinal $\kappa_\omega := \sup_{n<\omega} \kappa_n$.

Theorem 9.0.4. *Let $\mathcal{K} := \langle \kappa_n \mid n < \omega\rangle$ and $\mathcal{D} := \langle \delta_n \mid n < \omega\rangle$ be two sequences of cardinals for which the following hold:*

1. $\aleph_1 \leq \delta_0$.
2. $\delta_n \leq \kappa_n < \delta_{n+1}$.
3. *κ_n is the first δ_n-strong compact cardinal.*

Set $\kappa_\omega := \sup_{n<\omega} \kappa_n$ and $\Theta := \kappa_\omega^+$. Then, $\mathrm{TP}(\Theta)$ *holds. In particular, if $\mathcal{K} = \mathcal{D}$, Theorem 9.0.3 follows.*

Proof. Let $\langle T, <_T\rangle$ be a Θ-tree and assume that T_α is exactly $\{\alpha\} \times \kappa_\omega$, $\alpha < \Theta$. Let $j_0 : V \to M_0$ with $\mathrm{crit}(j_0) \geq \delta_0$ and $D_0 \in M$ with $j_0[\Theta] \subseteq D_0$ and $M_0 \models$ "$|D_0| < j_0(\kappa_0)$". Set $\Upsilon_0 := \sup j_0[\Theta]$ and observe that the former implies $\Upsilon_0 < j_0(\Theta)$. Let $\eta_0 \in (\Upsilon_0, j_0(\Theta))$. Since "$j_0(T)$ is a $j_0(\Theta)$-tree"^{M_0}, we may let $u \in j_0(T)_{\eta_0}$. By elementarity, for each $\alpha < \Theta$, there is $\xi < j_0(\kappa_\omega)$ be such that $M_0 \models$ "$\langle j_0(\alpha), \xi\rangle <_{j_0(T)} u$". Now let $\varphi : \Theta \to \omega$ be the function $\alpha \mapsto n_\alpha := \min\{n < \omega \mid$ "$\exists \xi < j_0(\kappa_n)\, \langle j_0(\alpha), \xi\rangle <_{j_0(T)} u$"$^{M_0}\}$. Since Θ is regular there is $J \subseteq \Theta$ unbounded and $n^* < \omega$ with $\varphi[J] = \{n^*\}$.

Claim 9.0.4.1 (Property †)**.** *There is $J \subseteq \Theta$ unbounded and $n^* < \omega$ such that for each $\alpha < \beta \in J$, there are $\xi, \zeta < \kappa_{n^*}$ with $\langle \alpha, \xi\rangle <_T \langle \beta, \zeta\rangle$.*

Proof of claim. Let J and n^* as above and $\alpha < \beta \in J$. By definition of $\varphi \restriction J$, we may find $\xi, \zeta < j(\kappa_{n^*})$ with $M_0 \models$ "$\langle j_0(\alpha), \xi\rangle <_{j_0(T)} u$" and $M_0 \models$ "$\langle j_0(\beta), \zeta\rangle <_{j_0(T)} u$". Since $<_{j_0(T)}$ is tree-like, "$\langle j_0(\alpha), \xi\rangle <_{j_0(T)} \langle j_0(\beta), \zeta\rangle$"^{M_0}. The claim now outright follows by elementarity. □

Now let $j : V \to M$ with $\mathrm{crit}(j) \geq \delta_{n^*+1}$ and $D \in M$ be such that $j[\Theta] \subseteq D$ and $M \models$ "$|D| < j(\kappa_{n^*+1})$". Recall that $\kappa_{n^*} < \delta_{n^*+1}$. Once again, set $\Upsilon := j[\Theta]$, and observe that $\Upsilon < j(\Theta)$. Since † holds, $M \models$ †. Actually, this is the case as witnessed by $j(J)$ and n^*. Let $\eta \in j(J) \setminus \Upsilon$. For each $\alpha \in J$, there are $\xi_\alpha, \delta_\alpha < j(\kappa_{n^*}) = \kappa_{n^*}$ be such that $M \models$ "$\langle j(\alpha), \xi_\alpha\rangle <_{j(T)} \langle \eta, \delta_\alpha\rangle$". Let $\psi : J \to \kappa_{n^*}$ be the function $\alpha \mapsto \delta_\alpha := \min\{\delta < \kappa_{n^*} \mid$ "$\exists \xi < \kappa_{n^*}\, \langle j(\alpha), \xi\rangle <_{j(T)} \langle \eta, \delta\rangle$"$^M\}$. Again by regularity of Θ, there is $\mathcal{I} \in [J]^\Theta$ and $\delta^* < \kappa_{n^*}$, for which $\psi[\mathcal{I}] = \{\delta^*\}$. For each $\alpha \in \mathcal{I}$ let ξ_α be the least witness for $\psi(\alpha) = \delta^*$.

Claim 9.0.4.2. *$b := \{v \in T \mid \exists \alpha \in \mathcal{I}\, \langle \alpha, \xi_\alpha\rangle \leq_T v\}$ is a cofinal branch through the tree T.*

Proof. Since $|\mathcal{I}| = \Theta$, b is cofinal. Now let us check that $\langle b, \leq_T\rangle$ is a chain. Let $v, w \in b$ and $\alpha, \beta \in \mathcal{I}$ witnessing this. Thus, $\langle \alpha, \xi_\alpha\rangle \leq_T v$, $\langle \beta, \xi_\beta\rangle \leq_T w$. Say for instance that $\alpha \leq \beta$. By †, $\langle \alpha, \xi_\alpha\rangle \leq_T \langle \beta, \xi_\beta\rangle$, hence $\langle \alpha, \xi_\alpha\rangle \leq_T v, w$. Since $\leq_T$ is tree-like it is immediate that either $v \leq_T w$ or $w \leq_T v$, which yields the desired property. □

□

Observe that in the above theorem the choice of a ω-sequence of partial strong compact cardinals is not relevant. Actually, what is important is that the length of this sequence is below δ_0, the degree of compactness of κ_0. Thereby, arguing as before one can prove the following generalization of Theorem 9.0.4:

Theorem 9.0.5. *Let $\mathcal{K} := \langle \kappa_\xi \mid \xi < \mu\rangle$ and $\mathcal{D} := \langle \delta_\xi \mid \xi < \mu\rangle$ be two sequences of cardinals for which the following hold:*

1. *$\aleph_0 \leq \mu < \delta_0$;*

2. *$\delta_\xi \leq \kappa_\xi < \delta_{\xi+1}$;*

3. *κ_ξ is the first δ_ξ-strong compact cardinal.*

Set $\kappa_\mu := \sup_{\xi<\mu} \kappa_\xi$ and $\Theta := \kappa_\mu^+$. Then, $\mathrm{TP}(\Theta)$ *holds.*

Part III

Σ-Prikry forcings and their iterations

Introduction

In Part I and Part II of this dissertation we have shown first hand that Prikry-type forcings [Git10] are very useful to tackle a wide spectrum of set-theoretic questions. For instance, in Chapter 3 we used Radin forcing (cf. Definition 3.1.17) to prove the consistency of the first supercompact cardinal to not be $C^{(1)}$-supercompact (cf. Theorem 3.1.1). In contrast, in Chapter 8 we used Prikry-type forcings for a complete different purpose: namely, we used Sinapova forcing (cf. Definition 8.1.7) to obtain the consistency of the tree property at the two first successors of a strong limit singular cardinal κ joint with an arbitrary failure of the SCH_κ (cf. Theorem 6.0.14).

This versatility of Prikry-type forcing is well-known from long time ago. For instance, they have had central applications in Singular Cardinal Combinatorics [Pri70][Mag77a][Mag77b][She83][GS08][Nee09]. Relevant applications have also found their place in other different areas Set Theory [Mag76] [BM14a][GS89][Mit10][GM18a]. Actually, Prikry-type forcings have trespassed the borders of Set Theory and its influence can be traced back in Topology [Dow95] or in Group Theory [MS94]. As a result, the investigation of these forcings has become a central theme of research in Set Theory.

In a series of joint papers with A. Rinot and D. Sinapova [PRS19] [PRS20] we have introduced the class of Σ-Prikry forcings, which aims to provide an abstraction of the classical Prikry-type forcing notions [Git10]. Given a non-decreasing sequence $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ of regular uncountable cardinals and $\kappa := \sup(\Sigma)$, a Σ-Prikry forcing is a triple $(\mathbb{P}, \ell, c)$ satisfying, among others, the following requirements:

(α) $\mathbb{P} = (P, \leq)$ is a notion of forcing.

(β) $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\kappa^+ = \check{\mu}$", for some cardinal μ.

(γ) $\ell : P \to \omega$ is a monotone grading function (cf. Definition 10.1.1).

(ε) $c : P \to \mu$ is a function witnessing the μ^+-*Linked*$_0$-*property* for $\mathbb{P}$ (cf. Definition 10.1.3(3))

(δ) For each $p, q \in P$ with $q \leq p$, there is a $\leq$-greatest condition $w(p, q)$ such that $q \leq w(p, q) \leq p$.

(ζ) $(\mathbb{P}, \ell)$ has the *Complete Prikry Property* (cf. Definition 10.1.3(7)).

The aim of the above requirements is to capture the essence of Prikry-type forcings. More precisely, it aims to abstract some of their prevalent features: namely, there is always a notion of length (γ), minimal extension (δ) and some sort of decision by pure extensions (ζ). Moreover, (ε) provides a strengthening of the usual notion of μ^+-Knasterness, which is actually shared by many Prikry-type forcings (cf. Proposition 7.1.11 and Theorem 8.1.10). We will next provide a more precise formulation of clauses (ε) and (ζ). For more details about Σ-Prikry forcings we referrer the reader to Chapter 10.

In [PRS19] and [PRS20] it is proved that the class of Σ-Prikry forcings includes many Prikry-type posets which center on singular cardinals of countable cofinality. Among these one can find, for instance, the standard Prikry forcing [Pri70][Git10, §1], Gitik-Sharon poset [GS08] or the Extender-Based Prikry forcing [GM94].

Also, in [PRS20], a functor $\mathbb{A}(\cdot,\cdot)$ between the class of Σ-Prikry forcing and $\mathbb{P}$-names is defined. For each Σ-Prikry forcing $\mathbb{P}$ and each $\mathbb{P}$-name $\dot{T}$ for a non-reflecting stationary subset of E^{μ}_{ω}, this functor produces a Σ-Prikry notion of forcing $\mathbb{A}(\mathbb{P},\dot{T})$ which destroys the stationarity of $\dot{T}$ and projects onto $\mathbb{P}$. A key feature of $\mathbb{A}(\cdot,\cdot)$ is that the projection from $\mathbb{A}(\mathbb{P},\dot{T})$ to $\mathbb{P}$ *splits*: that is, in addition to a projection map π from $\mathbb{A}(\mathbb{P},\dot{T})$ onto $\mathbb{P}$, there is a map $\pitchfork$ that goes in the other direction, and the two maps commute in a very strong sense. This is what we call a *forking projection* (cf. Definition 11.0.1). The main result of [PRS20] is Corollary 13.4.1 of this dissertation. An easier formulation of this is the following:

Theorem. *There is a functor $\mathbb{A}(\cdot,\cdot)$ such that, for each Σ-Prikry triple $(\mathbb{P},\ell,c)$ and each $\mathbb{P}$-name $\dot{T}$ for a subset of E^{μ}_{ω}, produces a forcing $\mathbb{A} := \mathbb{A}(\mathbb{P},\dot{T})$ for which there is $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ in such a way that $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}})$ is a Σ-Prikry triple. Besides, the following properties are true:*

1. $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}}$ *"$\dot{T}$ is nonstationary";*
2. $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}})$ *admits a* forking projection *to* $(\mathbb{P},\ell,c)$*.*

Our work is narrowly tied with the long-standing program in Set Theory aimed to find viable iteration schema for relevant families of forcing. These schema are crucial, for instance, to prove consistency results at the level of successors of regular cardinals, such as the *Suslin's Hypothesis* (SH).[1]

The first successful transfinite iteration schema was devised by Solovay and Tennenbaum in [ST71], who proved that the $<\aleph_0$-supported iterations of forcings with the *ccc* have the *ccc*. This is crucial to obtain the consistency of $\mathsf{ZFC} + \neg\mathsf{CH} + \mathrm{FA}_{2^{\aleph_0}}(\mathrm{ccc})$ from the consistency of $\mathsf{ZFC} + \mathsf{CH}$. Notice that $\mathrm{FA}_{2^{\aleph_0}}(\mathrm{ccc})$ is nothing but *Martin's axiom*, which implies the SH.

The Solovay-Tennenbaum technique is very versatile but it admits no generalization which allow to address problems concerning objects of size $>\aleph_1$. In particular, it cannot be used to prove the consistency of a similar forcing axiom at $2^{\aleph_1}$. One crucial reason for this lack of generalizations has to do with the poor behaviour of the higher analogues of the *ccc* at the level

[1] Recall that the SH is equivalent to the assertion "There are no $\aleph_1$-Suslin trees", where an $\aleph_1$-Suslin tree is a $\aleph_1$-tree without cofinal branches and whose antichains are countable.

of cardinals $>\aleph_1$ [Rin14; LHR18; Ros18]. Another reason is that, regardless one requires additional properties upon the iterates, the resulting iterations might be still ill-behaved. For instance, there is a $<\aleph_1$-supported iteration of $\aleph_2$-cc and $\aleph_1$-closed forcings which collapses $\aleph_1$ [Kun14, Example V.4].

Still, various iteration schema for posets having strong forms of the κ^+-cc have been devised when κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$.

For $\kappa = \aleph_1$, J. Baumgartner [Tal94] proved, under the CH, that every $<\aleph_1$-supported iteration of $\aleph_1$-linked, $\aleph_1$-closed and well-met forcings is $\aleph_1$-closed and $\aleph_2$-cc. In particular this can be used to obtain the consistency of $\mathsf{ZFC} + \mathrm{FA}_{2^{\aleph_1}}(\Gamma) + \mathsf{CH} + \neg\mathsf{GCH}_{\aleph_1}$, modulo the consistency of $\mathsf{ZFC} + \mathsf{CH}$. Here Γ denotes the family of $\aleph_1$-closed, $\aleph_1$-linked and well-met forcings.

In 1978, S. Shelah [She78] managed to weaken the iteration hypotheses in Baumgartner's theorem. Specifically, Shelah proved, again under the CH, that every $<\aleph_1$-supported iteration of $\aleph_2$-stationary-cc, $\aleph_1$-closed (with exact upper bounds) and well-met forcings is $\aleph_1$-closed and $\aleph_2$-stationary-cc. Using this the author proved the consistency of $\mathsf{ZFC} + \mathrm{FA}_{2^{\aleph_1}}(\Gamma) + \mathsf{CH} + \neg\mathsf{GCH}_{\aleph_1}$, modulo the consistency of $\mathsf{ZFC} + \mathsf{CH}$. This time Γ denotes the family of $\aleph_2$-stationary-cc, $\aleph_1$-closed (with exact upper bounds) and well-met forcings.

More recently, Cummings et. al. [Cum+17] have proved a similar iteration theorem for $<\kappa$-supported iterations, when κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$. In [Cum+17, Theorem 1.2] the authors prove for an uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$ that any $<\kappa$-supported iteration of countably parallel closed, κ-closed and κ^+-stationary-cc forcing has the κ^+-stationary-cc. For other results in this vein see [She03a; RS01; Eis03; RS11; RS13; RS19].

In contrast, there is a dearth of works involving iterations with support the successor of a singular cardinal. This lack of results entails serious difficulties at the time of proving consistency results at the level of successors of singular cardinals.

A few ad-hoc treatments of these iterations may be found in [She84, §2], [CFM01, §10] and [GR12, §1], and a more general framework is offered by [She03b, §3]. In [DS03], the authors took another approach in which they first pursue a forcing iteration along a successor of a regular cardinal κ, and at the very end they singularize κ by appealing to Prikry forcing. This was latter generalized to Radin forcing in [Cum+17].

This scarcity of results has to do with the fact that some fundamental properties of the forcings are not prevalent enough when κ is a singular cardinal. This is the case, for instance, of κ-closedness: let κ be a singular cardinal of countable cofinality and $S \subseteq E^{\kappa^+}_{\mathrm{cof}(\kappa)}$ be a non-reflecting stationary set. Then the usual forcing to shoot a club through the stationary set $S \cup E^{\kappa^+}_{\neq\mathrm{cof}(\kappa)}$, $\mathrm{CU}(\kappa^+, S \cup E^{\kappa^+}_{\neq\mathrm{cof}(\kappa)})$, is not even $\aleph_1$-closed (see [Cum10, Definition 6.10]). Thus, we might be in the situation where even a single component of our iteration is not closed enough.

A natural strategy to overcome this difficulty comes from the world of Prikry-type forcings. To explain it, it is illustrative to think on the proof of the consistency of the failure of the SCH using Prikry forcing.

Clearly Prikry forcing is not $\aleph_1$-closed, but still one can argue that the cardinal structure below κ has not been damaged. To this aim one needs to mix the *Prikry property* and some sort of "κ-closedness by layers". Roughly speaking, the latter means the following: Prikry forcing $\mathbb{P} := (P, \leq)$ can be written as a union of pair-wise disjoint subforcings $\bigcup_{n<\omega} \mathbb{P}_n$, where each $\mathbb{P}_n$ is κ-closed. The combination of these two properties entails that $\mathbb{P}$ does not add bounded subsets to κ and thus that cardinals $\leq\kappa$ are preserved [Git10, §1]. Among other reasons, this is what motivates the notion of Σ-Prikry forcing.

Unlike in [DS03] and [Cum+17], in [PRS19] we allow to put the Prikry-type forcing at κ as the very first step of the iteration, and then continue iteration up to length κ^{++} without collapsing cardinals. Moreover, we allow κ to be singular from the beginning, which is not the case in the approaches taken in [DS03] and [Cum+17]. Our iteration scheme for the class of Σ-Prikry-forcings is presented in [PRS19].

Viable (and successful) iteration schema for Prikry-type posets already exists: namely, Magidor and Gitik iterations (see [Git10, §6]). In both cases the ordering $\leq^* \setminus \leq$ witnessing the Prikry Property of these iterations can roughly be described as the $<\aleph_0$-supported iteration of the $\leq^*$-orderings of its components. As the expectation from the final $\leq^*$ is to have an eventually-high closure degree, these schemes are typically useful in the context where one carries an iteration $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha \mid \alpha < \rho \rangle$ with each $\dot{\mathbb{Q}}_\alpha$ being a $\mathbb{P}_\alpha$-name for either a trivial forcing or a Prikry-type forcing which concentrates on the combinatorics of an inaccessible cardinal α.

In contrast, we are interested in carrying out an iteration of length κ^{++} where κ is a singular cardinal[2] and all components of the iteration are Prikry-type forcings which concentrate on the combinatorics of κ or its successor. Metaphorically speaking, Gitik and Magidor iteration are more in the spirit of the Easton-style iteration to control the power function below a cardinal ρ, while our iterations are more akin to the standard iteration to force $\mathrm{FA}_{2^{\kappa^+}}(\Gamma)$, when κ is a singular cardinal of countable cofinality.

For this, we will need to allow a support of arbitrarily large size below κ. To be able to lift the Prikry property through an infinite-support iteration, members of the Σ-Prikry class are required to have the following stronger form of the Prikry property:

Complete Prikry Property. There is a partition of the ordering $\leq$ into countably many relations $\langle \leq_n \mid n < \omega \rangle$ such that, if we denote $\mathrm{cone}_n(q) := \{r \mid r \leq_n q\}$, then, for every 0-open $U \subseteq P$ (i.e., $q \in U \implies \mathrm{cone}_0(q) \subseteq U$),

[2]Or more generally, forced by the first step of the iteration to become one.

every $p \in P$ and every $n < \omega$, there exists $q \leq_0 p$ such that $\mathrm{cone}_n(q)$ is either a subset of U or disjoint from U.

The above property is inspired by the *Completely Ramsey Property*, a concept arising from the study of topological Ramsey spaces [Tod10b]. This notion entails a novelty with respect previous approximations to the theory of Prikry-type forcings. The Complete Prikry Property was introduced in [PRS20] aiming to simultaneously capture two paradigmatic features of Prikry-type forcings: namely, the *Prikry property* and the *Strong Prikry property*. In the said paper we prove that this latter yields both the Prikry and the Strong Prikry Property.

Another parameter which requires attention when devising an iteration scheme is the chain condition. Towards solving a problem concerning the combinatorics of κ (or its successor) through an iteration of length κ^{++} there is a need to know that all counterexamples to our problem will show up at some intermediate stage of the iteration. Otherwise, any attempt to eliminate them seem hopeless.

The standard way to secure this is to require that the whole iteration $\mathbb{P}_{\kappa^{++}}$ has the κ^{++}-cc. As κ-supported iterations of κ^{++}-cc posets need not have the κ^{++}-cc (see [Ros18] for an explicit counterexample) the Σ-Prikry forcings are required to satisfy the following strong form of the κ^{++}-cc:

Linked$_0$ *Property.* There exists a map $c : P \to \kappa^+$ satisfying that for all $p, q \in P$, if $c(p) = c(q)$, then p and q are compatible, and, furthermore, $\mathrm{cone}_0(p) \cap \mathrm{cone}_0(q)$ is nonempty.

In particular, our verification of the chain condition of $\mathbb{P}_{\kappa^{++}}$ will not go through the Δ-system lemma; rather, we will take advantage of some ideas arising from the density of box products of topological spaces (see Theorem 10.2.34). Once again, this entails another novelty with respect previous developments of the field.

Now, that we have a way to ensure that all counterexamples show up at intermediate stages, we fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++}\rangle$, and shall want that, for any $\alpha < \kappa^{++}$, $\mathbb{P}_{\alpha+1}$ will amount to force over the model $V^{\mathbb{P}_\alpha}$ aiming to solve the problem suggested by z_α. The standard approach to achieve this is to set $\mathbb{P}_{\alpha+1} := \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset that takes care of z_α. However, the disadvantage of this approach is that if $\mathbb{P}$ is a notion of forcing that blows up 2^κ, then any typical poset $\mathbb{Q}_1$ in $V^{\mathbb{P}_1}$ which is designed to add a subset of κ^+ via bounded approximations will fail to have the κ^{++}-cc.

To work around this, in our scheme, $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{A}_\alpha(\mathbb{P}_\alpha, z_\alpha)$, where $\mathbb{A}_\alpha(\cdot,\cdot)$ is a functor that to each Σ-Prikry poset $\mathbb{P}$ and a problem z, produces a Σ-Prikry poset $\mathbb{A}_\alpha(\mathbb{P}, z)$ that projects onto $\mathbb{P}$ and solves the problem z. Intuitively speaking, the functor $\mathbb{A}_\alpha(\cdot,\cdot)$ gives us a way to embed

$\mathbb{P}$ into a "bigger" Σ-Prikry forcing $\mathbb{A}(\mathbb{P}, z)$ that solves the problem z. At limit stages of our iteration we simply take inverse limits with $<\kappa^+$-support.

At the end of this process we will have defined a poset $\mathbb{P}_{\kappa^{++}}$ which will yield a generic extension having the desired property. A special case of our main result from [PRS19] around iterations of Σ-Prikry forcings may be roughly stated as follows.

Theorem. *Suppose that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of regular uncountable cardinals converging to a cardinal κ. For simplicity, let us say that a notion of forcing $\mathbb{P}$ is* nice *if $\mathbb{P} \subseteq H_{\kappa^{++}}$ and $\mathbb{P}$ does not collapse κ^+. Now, suppose that:*

- *$\mathbb{Q}$ is a nice Σ-Prikry notion of forcing;*
- *$\mathbb{A}(\cdot, \cdot)$ is a functor which produces, for every nice Σ-Prikry notion of forcing $\mathbb{P}$ and every $z \in H_{\kappa^{++}}$, a corresponding nice Σ-Prikry notion of forcing $\mathbb{A}(\mathbb{P}, z)$ admitting a forking projection to $\mathbb{P}$;*[3]
- *$2^{2^\kappa} = \kappa^{++}$, so that we may fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++} \rangle$.*

Then there exists a κ-supported sequence $\langle (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha) \mid \alpha \leq \kappa^{++} \rangle$ of nice Σ-Prikry forcings such that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$, $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{A}(\mathbb{P}_\alpha, z_\alpha)$, and, for every pair $\alpha \leq \beta < \kappa^{++}$, $(\mathbb{P}_\beta, \ell_\beta, c_\beta)$ forking projects onto $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ and $(\mathbb{P}_{\kappa^{++}}, \ell_{\kappa^{++}})$ forking projects onto $(\mathbb{P}_\beta, \ell_\beta)$.

In [PRS19, §5] we also present the very first application of our scheme. There our aim was to obtain the consistency of finite simulatenous reflection of stationary subsets of κ^+ joint with a failure of the SCH_κ. This is similar to a classical result of M. Magidor about reflection of stationary subsets of $\aleph_{\omega+1}$ [Mag82], though in Magidor's model $\mathsf{GCH}_{\aleph_\omega}$ holds.

To prove this result we devise an iteration of length κ^{++} of Σ-Prikry forcings where $\mathbb{Q}$ is the Extender Based Prikry Forcing relative to an increasing sequence of Laver-indestructible supercompact cardinals $\Sigma := \langle \kappa_n \mid n < \omega \rangle$. For the definition of the later steps we invoke just one functor: that given by the theorem in page 180. After this one obtains the following:

Theorem. *Let $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals with $\kappa := \sup_{n<\omega} \kappa_n$. Then there exists a cofinality-preserving generic extension of the universe where the following hold:*

1. *κ is a strong limit singular cardinal;*
2. *$2^\kappa = \kappa^{++}$, hence the SCH_κ fails;*
3. *$\mathrm{Refl}(<\omega, \kappa^+)$ holds (cf. Definition 12.1.1).*

[3]Here we need to require some additional properties (cf. page 246).

Moreover, this result is optimal.[4]

The above theorem was first announced by A. Sharon in his doctoral dissertation [Sha05]. Nonetheless a close inspection of Sharon's proof revealed us a gap in his verification of the κ^{++}-chain-condition of the iteration, which is certainly a crucial point. Broadly speaking, the issue is that the κ^{++}-Knasterness of the iterates is not enough to secure the κ^{++}-cc of the iteration. To fix this we need the stronger notion of the κ^{++}-Linked$_0$-property. This property is weak enough to be fulfilled by many classical Prikry-type forcings and, at the same time, stronger enough to develop a general theory of iterations.

An alternative proof of the above theorem was obtained around the same time by O. Ben-Neria, Y. Hayut and S. Unger [BNHU19]. Their proof avoids iterated forcing and instead it is based on iterated ultrapowers.

In this part of the dissertation we aim to provide the reader with a detailed exposition of the theory of Σ-Prikry forcings developed in [PRS19] and [PRS20]. The following are some of the notational convention upon which we will relying:

Convention 9.0.6.

- For a forcing poset $\mathbb{P} = (P, \leq)$, we will tend to distinguish between the poset $\mathbb{P}$ and its underlying set P.
- We denote $E^\mu_\theta := \{\alpha < \mu \mid \mathrm{cof}(\alpha) = \theta\}$. The sets $E^\mu_{<\theta}$ and $E^\mu_{>\theta}$ are defined in a similar fashion.
- For a stationary subset S of a regular uncountable cardinal μ, we write $\mathrm{Tr}(S) := \{\delta \in E^\mu_{>\omega} \mid S \cap \delta \text{ is stationary in } \delta\}$.
- $H_\nu := \{x \mid |\mathrm{trcl}(x)| < \nu\}$.
- For every $x \subseteq \mathrm{ORD}$, denote $\mathrm{cl}(x) := \{\sup(x \cap \gamma) \mid \gamma \in \mathrm{ORD}, x \cap \gamma \neq \emptyset\}$, and $\mathrm{acc}(x) := \{\gamma \in x \mid \sup(x \cap \gamma) = \gamma > 0\}$.
- For two sets of ordinals x, y, we write $x \sqsubseteq y$ iff there exists an ordinal α such that $x = y \cap \alpha$.

[4] cf. Corollary 12.1.4.

CHAPTER 10

The Σ-Prikry framework

10.1 The axioms

In this section we will introduce the class of Σ-Prikry forcing and prove some of its basic properties. Among these, we show that the Complete Prikry Property (cf. Definition 10.1.3 (7)) yields both the Prikry and the Strong Prikry property for any Σ-Prikry forcing.

Definition 10.1.1. We say that $(\mathbb{P}, \ell)$ is a *graded poset* iff $\mathbb{P} = (P, \leq)$ is a poset, $\ell : P \to \omega$ is a surjection, and, for all $p \in P$:

- For every $q \leq p$, $\ell(q) \geq \ell(p)$;
- There exists $q \leq p$ with $\ell(q) = \ell(p) + 1$.

Convention 10.1.2. For a graded poset as above, we denote $P_n := \{p \in P \mid \ell(p) = n\}$, $P^p_n := \{q \in P \mid q \leq p, \ell(q) = \ell(p)+n\}$, and sometime write $q \leq^n p$ (and say the q is *an n-step extension* of p) rather than writing $q \in P^p_n$.

Definition 10.1.3. Suppose that $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}$, and that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal κ. Suppose that μ is a cardinal such that $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$.[1] For functions $\ell : P \to \omega$ and $c : P \to \mu$, we say that $(\mathbb{P}, \ell, c)$ is *Σ-Prikry* iff all of the following hold:

1. $(\mathbb{P}, \ell)$ is a graded poset;
2. For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{\mathbb{1}\}, \leq)$ is κ_n-directed-closed;[2]
3. For all $p, q \in P$, if $c(p) = c(q)$, then $P^p_0 \cap P^q_0$ is non-empty;

[1]More explicitly, $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = (\check{\kappa})^+$.

[2]That is, for every $D \in [P_n \cup \{\mathbb{1}\}]^{<\kappa_n}$ with the property that for all $p, p' \in D$, there is $q \in D$ with $q \leq p, p'$, there exists $r \in P_n$ such that $r \leq p$ for all $p \in D$.

4. For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{r \leq^n p \mid q \leq^m r\}$ contains a greatest element which we denote by $m(p,q)$.[3] In the special case $m = 0$, we shall write $w(p,q)$ rather than $0(p,q)$;[4]

5. For all $p \in P$, the set $W(p) := \{w(p,q) \mid q \leq p\}$ has size $<\mu$;

6. For all $p' \leq p$ in P, $q \mapsto w(p,q)$ forms an order-preserving map from $W(p')$ to $W(p)$;

7. Suppose that $U \subseteq P$ is a 0-open set, i.e., $r \in U$ iff $P_0^r \subseteq U$. Then, for all $p \in P$ and $n < \omega$, there is $q \leq^0 p$, such that, either $P_n^q \cap U = \emptyset$ or $P_n^q \subseteq U$.

Let us elaborate on the above definition.

- Here, q is a "direct extension" of p in the usual Prikry sense iff $q \leq^0 p$. Note that $q \leq^0 w(p,q) \leq p$. Also, it is clear that if $p \leq^n q$ and $q \leq^m r$, then $p \leq^{n+m} r$.

- The sets P_n^p consist of exactly the n-step extensions of p, and P_n is the set of all conditions of "length" n, i.e., the n-step extensions of $\mathbb{1}$. Note that, typically, $\mathbb{P}_n$ is not a complete suborder of $\mathbb{P}$, and that, for all $p, q \in P_n$, $p \leq q$ iff $p \leq^0 q$. Thereby, $\mathbb{P}_n$ is not necessarily separative.

  Convention. Whenever we talk about forcing with one of the $\mathbb{P}_n$'s, we actually mean that we force with its separative quotient.

- Clause (3) is a very strong form of a chain condition, stronger than that of being μ^+-Knaster, and even stronger than the notion of being μ^+-2-linked. Indeed, a poset $(P, \leq)$ is *μ^+-2-linked* iff there exists a function $c : P \to \mu$ with the property that $c(p) = c(q)$ entails that p and q are compatible, whereas, here, we moreover require that such a compatibility will be witnessed by a 0-step extension of p and q.

  Convention. To avoid encodings, we shall often times define the function c as a map from P to some natural set $\mathfrak{M}$ of size $\leq \mu$, instead of a map to the cardinal μ itself. In the special case that $\mu^{<\mu} = \mu$, we may as well take $\mathfrak{M}$ to be H_μ.

- For every $p \in P$, the set $W(p)$ is called *the p-tree*. For every $n < \omega$, write $W_n(p) := \{w(p,q) \mid q \in P_n^p\}$, and $W_{\geq n}(p) := \bigcup_{m=n}^{\infty} W_m(p)$. By Lemma 10.1.8 below, $(W(p), \geq)$ is a tree of height ω whose n^{th} level is a maximal antichain in $\mathbb{P} \downarrow p$ for every $n < \omega$.

[3] By convention, a greatest element, if exists, is unique.

[4] Note that $w(p,q)$ is the weakest extension of p above q.

- Clause (7) is what we call the *Complete Prikry Property* (CPP), an analogue of the notion of a *completely Ramsey* subset of $[\omega]^\omega$. We shall soon show (Corollary 10.1.7 below) that it is a simultaneous generalization of the usual Prikry Property (PP) and the Strong Prikry Property (SPP).

Definition 10.1.4. Let $d : P \to \theta$ be some coloring, with θ a nonzero cardinal.

1. d is said to be 0-*open* iff $d(p) \in \{0, d(q)\}$ for every pair $q \leq^0 p$ of elements of P;
2. We say that $H \subseteq P$ is *a set of indiscernibles for* d iff, for all $p, q \in H$, $(\ell(p) = \ell(q)) \implies (d(p) = d(q))$.

Remark 10.1.5. The characteristic function $d : P \to 2$ of a subset $D \subseteq P$ is 0-open iff D is a 0-open.

Lemma 10.1.6. *For every $p \in P$, every cardinal θ with* $\log(\theta) < \kappa_{\ell(p)}$ *and every* 0*-open coloring $d : P \to \theta$,*[5] *there exists $q \leq^0 p$ such that $\mathbb{P}\downarrow q$ is a set of indiscernibles for d.*

Proof. Let $p \in P$ and $d : P \to \theta$ as above. Fix an infinite cardinal $\chi < \kappa_{\ell(p)}$ such that $2^\chi \geq \theta$. Fix an injective sequence $\vec{f} = \langle f_\alpha \mid \alpha < \theta\rangle$ consisting of functions from χ to 2 such that, in addition, f_0 is the constant function from χ to $\{0\}$.

Claim 10.1.6.1. *Let $i < \chi$. The set $U_i := \{r \in P \mid f_{d(r)}(i) \neq 0\}$ is* 0*-open.*

Proof. Let $r \in U_i$ and $r' \leq^0 r$. As $r \in U_i$, $f_{d(r)}$ is not the constant function from χ to $\{0\}$, so that $d(r) \neq 0$. Since d is a 0-open coloring, it follows that $d(r') = d(r)$. Consequently, $r' \in U_i$, as well. □

Fix a bijection $e : \chi \leftrightarrow \chi \times \omega$. We construct a $\leq^0$-decreasing sequence of conditions $\langle p_\beta \mid \beta \leq \chi\rangle$ by recursion, as follows.

▶ Let $p_0 := p$.

▶ Suppose that $\beta < \chi$ and that $\langle p_\gamma \mid \gamma \leq \beta\rangle$ has already been defined. Denote $(i, n) := e(\beta)$. Now, appeal to Definition 10.1.3(7) with U_i, p_β and n to obtain $p_{\beta+1} \leq^0 p_\beta$ such that, either $P_n^{p_{\beta+1}} \cap U_i = \emptyset$ or $P_n^{p_{\beta+1}} \subseteq U_i$.

▶ For every limit nonzero $\beta \leq \chi$ such that $\langle p_\gamma \mid \gamma < \beta\rangle$ has already been defined, appeal to Definition 10.1.3(2) to find bound p_β for the sequence.

At the end of the above recursion, let us put $q := p_\chi$, so that $q \leq^0 p$. We claim that $\mathbb{P}\downarrow q$ is a set of indiscernibles for d.

Suppose not, and pick two extensions r, r' of q such that $\ell(r) = \ell(r')$ but $d(r) \neq d(r')$. As $d(r) \neq d(r')$ and $\vec{f}$ is injective, let us fix $i < \chi$ such that

[5] Here, $\log(\theta)$ stands for the least cardinal ν to satisfy $2^\nu \geq \theta$.

$f_{d(r)}(i) \neq f_{d(r')}(i)$. Consequently, $|\{r, r'\} \cap U_i| = 1$. Now, put $n := \ell(r) - \ell(p)$, so that $r, r' \in P^q_n$. Set $\beta := e^{-1}(i, n)$. By the choice of $p_{\beta+1}$, then, either $P^{p_{\beta+1}}_n \cap U_i = \emptyset$ or $P^{p_{\beta+1}}_n \subseteq U_i$. As $q \leq^0 p_{\beta+1}$, we have $\{r, r'\} \subseteq P^{p_{\beta+1}}_n$, contradicting the fact that $|\{r, r'\} \cap U_i| = 1$. □

It follows that the Complete Prikry Property (CPP) implies the Prikry property (PP) as well as the Strong Prikry property (SPP).

Corollary 10.1.7. *Let $p \in P$.*

1. *Suppose φ is a sentence in the forcing language. Then there is $q \leq^0 p$ that decides φ;*
2. *Suppose $D \subseteq P$ is a 0-open set which is dense below p. Then there are $q \leq^0 p$ and $n < \omega$ such that $P^q_n \subseteq D$.*[6]

Proof. (1) Define a 0-open coloring $d : P \to 3$, by letting, for all $r \in P$,

$$d(r) := \begin{cases} 2, & \text{if } r \Vdash \neg\varphi; \\ 1, & \text{if } r \Vdash \varphi; \\ 0, & \text{otherwise.} \end{cases}$$

Appeal to Lemma 10.1.6 with d to get a corresponding $q \leq^0 p$. Towards a contradiction, suppose that q does not decide φ. In other words, there exist $q_1 \leq q$ and $q_2 \leq q$ such that $d(q_1) = 1$ and $d(q_2) = 2$. By possibly iterating Clause (1) of Definition 10.1.3 finitely many times, we may find $r_1 \leq q_1$ and $r_2 \leq q_2$ such that $\ell(r_1) = \ell(r_2)$. By definition of d, we have $d(r_1) = 1$ and $d(r_2) = 2$. Finally, as r_1 and r_2 are two extensions q of the same "length", $1 = d(r_1) = d(r_2) = 2$. This is a contradiction.

(2) Define a coloring $d : P \to 2$ via $d(r) := 1$ iff $r \in D$. By Remark 10.1.5, we may appeal to Lemma 10.1.6 with d to get a corresponding $q \leq^0 p$. As D is dense, let us fix $r \in D$ extending q. Let $n := \ell(r) - \ell(p)$, so that $d \restriction P^q_n$ is constant with value $d(r)$. Recalling that $r \in D$ and the definition of d, we infer that $P^q_n \subseteq D$. □

Lemma 10.1.8 (The p-tree)**.** *Let $p \in P$.*

1. *For every $n < \omega$, $W_n(p)$ is a maximal antichain in $\mathbb{P} \downarrow p$;*
2. *Every two compatible elements of $W(p)$ are comparable;*
3. *For any pair $q' \leq q$ in $W(p)$, $q' \in W(q)$;*
4. *$c \restriction W(p)$ is injective.*

[6]Note that if D is open, then, moreover, $P^q_m \subseteq D$ for all $m \geq n$.

Proof. (1) Clearly, $W_0(p) = \{p\}$ is a maximal antichain below p. Thus, hereafter, assume that $n > 0$.

▶ To see that $W_n(p) = \{w(p,q) \mid q \in P^p_n\}$ is an antichain, suppose that $q_1, q_2 \in P^p_n$ are such that $w(p,q_1)$ and $w(p,q_2)$ are compatible, as witnessed by some q. By Definition 10.1.3(1), $q \in P^p_{n+m}$ for some $m < \omega$. By Definition 10.1.3(4), then, $\{r \in P^p_n \mid q \leq r\}$ contains a greatest element, say, r^*. Let $i < 2$ be arbitrary. As $q \leq w(p,q_i)$, it is not hard to see that $w(p,q_i)$ is the greatest element in $\{r \in P^p_n \mid q \leq r\}$, so that $w(p,q_i) = r^*$. Altogether, $w(p,q_1) = r^* = w(p,q_2)$.

▶ To verify maximality of the antichain $W_n(p)$, let $p' \leq p$ be arbitrary. By Definition 10.1.3(1), let us pick some $q \in P^{p'}_n$, so that $q \in P^p_{n+m}$ for some $m < \omega$. Then, by Definition 10.1.3(4), $\{r \in P^p_n \mid q \leq r\}$ contains a greatest element, say, r^*. As $w(p,r^*) = r^*$, we have $r^* \in W_n(p)$. In addition, r^* and p' are compatible, as witnessed by q.

(2) Suppose that $q_0, q_1 \in W(p)$ are two compatible elements. Fix integers n_0, n_1 such that $q_0 \in W_{n_0}(p)$ and $q_1 \in W_{n_1}(p)$.

If $n_0 = n_1$, then by Clause (1), $q_0 = q_1$. Thus, without loss of generality, assume that $n_0 < n_1$. Let r^* be the greatest element of $\{r \in P^p_{n_0} \mid q_1 \leq r\}$. Then $r^* = w(p,r^*) \in W_{n_0}(p)$ and q_1 witnesses that r^* is compatible with q_0. So r^* and q_0 are compatible elements of $W_{n_0}(p)$, and hence $q_1 \leq r^* = q_0$.

(3) Given $q' \leq q$ as above, let $r' \in P^p$ be such that $q' = w(p,r')$. Now, to prove that $w(p,r') \in W(q)$, it suffices to show that $w(p,r') = w(q,r')$. Here goes:

▶ As $r' \leq w(q,r') \leq q \leq p$, we infer that $w(q,r') \in \{s \mid r' \leq s \leq p\}$, so that $w(q,r') \leq w(p,r')$.

▶ As $r' \leq w(p,r') = q' \leq q$, we infer that $w(p,r') \in \{s \mid r' \leq s \leq q\}$, so that $w(p,r') \leq w(q,r')$.

(4) By Definition 10.1.3(3), for all $q, q' \in W(p)$, if $c(q) = c(q')$, then q and q' are compatible, and they have the same ℓ-value. It now follows from Clause (1) that $c \restriction W(p)$ is injective. □

Lemma 10.1.9. *Suppose that $\bar{p} \leq p' \leq p$ and $q \in W(\bar{p})$. Then $w(p,q) = w(p, w(p',q))$.*[7]

Proof. As $\ell(w(p,q)) = \ell(q) = \ell(w(p',q)) = \ell(w(p,w(p',q))$, we infer the existence of some $n < \omega$ such that both $w(p,q)$ and $w(p,w(p',q))$ belong to $W_n(p)$. By Lemma 10.1.8(1), then, it suffices to verify that the two are compatible. And indeed, we have $q \leq w(p,q)$ and $q \leq w(p',q) \leq w(p,w(p',q))$. □

Lemma 10.1.10. *1. $\mathbb{P}$ does not add bounded subsets of κ;*

[7] For future reference, we point out that this fact relies only on clauses (1) and (4) of Definition 10.1.3.

2. *For every regular cardinal $\nu \geq \kappa$, if there exists $p \in P$ for which $p \Vdash_{\mathbb{P}} \mathrm{cof}(\check{\nu}) < \check{\kappa}$, then there exists $p' \leq p$ with $|W(p')| \geq \nu$;*[8]

3. *Suppose $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular". Then $\mu = \kappa^+$ iff, for all $p \in P$, $|W(p)| \leq \kappa$.*

Proof. (1) Suppose that p forces that σ is a name for a subset of some $\theta < \kappa$. By possibly iterating Clause (1) of Definition 10.1.3 finitely many times, we may find $p' \leq p$ with $\kappa_{\ell(p')} > \theta$. Denote $n := \ell(p')$. Then by Corollary 10.1.7(1) and Definition 10.1.3(2), we may find a $\leq^0$-decreasing sequence of conditions, $\langle p_\alpha \mid \alpha \leq \theta\rangle$, with $p_0 \leq^0 p'$, such that, for each $\alpha < \theta$, p_α $\mathbb{P}$-decides whether α belongs to σ. Then p_θ forces that σ is a ground model set.

(2) Suppose θ, ν are regular cardinals with $\theta < \kappa \leq \nu$, $\dot{f}$ is a $\mathbb{P}$-name for a function from θ to ν, and $p \in P$ is a condition forcing that the image of f is cofinal in ν. Denote $n := \ell(p)$. By Definition 10.1.3(1), we may assume that $\kappa_n > \theta$. For all $\alpha < \theta$, let D_α denote the open set of conditions below p that $\mathbb{P}$-decides a value for $f(\alpha)$. As D_α is dense below p, by Corollary 10.1.7(2) and Definition 10.1.3(2), we may find a $\leq^0$-decreasing sequence of conditions $\langle p_\alpha \mid \alpha < \theta\rangle$, with $p_0 \leq^0 p$, and a sequence $\langle n_\alpha \mid \alpha < \theta\rangle$ of elements of ω, such that, for all $\alpha < \theta$, $P^{p_\alpha}_{n_\alpha} \subseteq D_\alpha$.

By Definition 10.1.3(2), let p' be bound for $\{p_\alpha \mid \alpha < \theta\}$. Evidently, $P^{p'}_{n_\alpha} \subseteq D_\alpha$ for every $\alpha < \theta$. Now, let

$$A_\alpha := \{\beta < \nu \mid \exists p \in P^{p'}_{n_\alpha}[p \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\beta}]\}.$$

By Lemma 10.1.8(1), we have $A_\alpha = \{\beta < \nu \mid \exists p \in W_{n_\alpha}(p')[p \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\beta}]\}$. Let $A := \bigcup_{\alpha<\theta} A_\alpha$. As $|A| \leq \sum_{\alpha<\theta} |W_{n_\alpha}(p')| \leq \theta \cdot |W(p')|$, it follows that if $|W(p')| < \nu$, then $\sup(A) < \nu$, and p' forces that the range of f is bounded below ν, which would form a contradiction. So $|W(p')| \geq \nu$.

(3) The forward implication follows from Definition 10.1.3(5).

Next, suppose that, for all $p \in P$, $|W(p)| \leq \kappa$. Towards a contradiction, suppose that there exist $p \in P$ forcing that κ^+ is collapsed. Denote $\nu := \kappa^+$. As $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular", this means that $p \Vdash_{\mathbb{P}} \mathrm{cof}(\check{\nu}) < \check{\kappa}$, contradicting Clause (2). □

10.2 Some examples

In the present section we will argue that the class of Σ-Prikry forcings is rich enough to include many relevant examples of Prikry-type forcings. For

[8]For future reference, we point out that this fact relies only on clauses (1),(2),(4) and (7) of Definition 10.1.3. Furthermore, we do not need to know that $\mathbb{1}$ decides a value for κ^+.

this, we will prove, with relative detail, that among these forcings one may find Prikry forcing [Pri70], Gitik-Sharon poset [GS08] or the Extender Based Prikry forcing [GM94]. Other classical posets, such as the Tree Prikry forcing [Git10, §1.2] or the Extender Based Prikry forcing with a single extender [Git10, §3], are also very likely Σ-Prikry. The proofs of this section are self-contained and do not assume any previous knowledge of Prikry-type forcings.

10.2.1 Prikry forcing

Hereafter assume that κ is a measurable cardinal and that $\mathcal{U}$ is a normal measure over it, i.e., a normal κ-complete ultrafilter on κ (cf. Definition 1.1.5). In this section we will show that the classical Prikry forcing $\mathbb{P}$ devised to singularize κ to cofinality ω fits into the Σ-Prikry framework. Let us begin with the corresponding definition:

Definition 10.2.1. Prikry forcing is the poset $\mathbb{P} := (P, \leq)$, where

- $P := \{(s, A) \mid s \in [\kappa]^{<\omega} \,\&\, A \in \mathcal{U} \,\&\, \max(s) < \min(A)\}$;
- $(s, A) \leq (t, B)$ iff $t \sqsubseteq s$, $A \subseteq B$ and $s \setminus t \subseteq B$.

For a condition $p := (s, A) \in P$, it is customary to call s the stem of p.

Definition 10.2.2 (Diagonal intersection)**.** Let $X \in [{}^{<\omega}\kappa]^{\kappa}$. The diagonal intersection of a family $\{A_s \mid s \in X\} \subseteq \mathcal{U}$ is defined as

$$\triangle\{A_s \mid s \in X\} := \{\alpha < \kappa \mid \forall s \in X(\max(s) < \alpha \to \alpha \in A_s)\}.$$

Since $\mathcal{U}$ is assumed to be normal, for each $\{A_s \mid s \in X\} \subseteq \mathcal{U}$, the diagonal intersection $\triangle\{A_s \mid s \in X\}$ yields a set in $\mathcal{U}$.

Let Σ be the ω-sequence with constant value κ and set $\mu := \kappa^+$. The notion of length associated to $\mathbb{P}$, $\ell : P \to \omega$, is given by $\ell(s, A) := |s|$. Besides, define $c : P \to {}^{<\omega}\kappa$ via $c(s, A) := s$. In the next proposition we verify that $(\mathbb{P}, \ell, c)$ is Σ-Prikry.

Proposition 10.2.3. *$(\mathbb{P}, \ell, c)$ is Σ-Prikry.*

Proof. We go over the clauses of Definition 10.1.3.

1. For $p = (s, A) \in P$, $(s^\frown\langle\nu\rangle, A \setminus \nu + 1) \in P_1^p$, for all $\nu \in A$. Also, by definition of $\leq$, if $q \leq p$ then $\ell(q) \geq \ell(p)$.
2. Let $D \in [P_n \cup \{\mathbb{1}\}]^{<\kappa_n}$ be a $\leq^0$-directed set. Say, $D = \{p^\alpha \mid \alpha < \theta\}$, for some cardinal $\theta < \kappa_n$. Let s be the common stem of these conditions and set $p^* := (s, A^*)$, where $A^* := \bigcap_{\alpha<\theta} A^{p^\alpha}$. By the κ-completeness of $\mathcal{U}$, $p^* \in P$, and clearly $p^* \leq^0 p^\alpha$, for $\alpha < \theta$.

3. Let $p, q \in P$ and assume that $c(p) = c(q) = s$. Set $p := (s, A)$ and $q := (s, B)$. Clearly $(s, A \cap B)$ is in $P_0^p \cap P_0^q$.

4. Let $p := (s, A) \in P$, $n, m < \omega$ and $q := (t, B) \in P_{n+m}^p$. Set $u := t \restriction (|s| + n)$. Then $r^* := (u, A \setminus \max(u) + 1)$ is the greatest element in $\{r \in P_n^p \mid q \leq^m r\}$.

5. Let $p \in P$ and $n < \omega$. Denoting $p := (s, A)$, we have that $W_n(p) = \{(s^\frown t, A \setminus \max(t) + 1) \mid t \in [A]^n \,\&\, t \text{ is increasing }\}$. Clearly, $|W_n(p)| = \kappa < \mu$, hence $|W(p)| < \mu$.

6. Let $p' \leq p$ and $q, q' \in W(p')$ and assume $q' \leq q$. Set $p := (s, A)$, $q := (t, B)$ and $q' := (u, C)$. By the previous items, $w(p, q) = (t, A \setminus \max(t) + 1)$ and $w(p, q') = (u, A \setminus \max(u) + 1)$ and, since $q' \leq q$, it is clear that $w(p, q') \leq w(p, q)$, as desired.

7. This follows in a similar fashion to the classical proof of the SPP [Git10, Lemma 1.13]. Nonetheless, for the reader's benefit, we give a proof-sketch: Let $p = (s, A)$, U be a 0-open set and $n < \omega$. Define $d : [A]^n \to 2$ as, $d(t) = 1$ iff $(s^\frown t, A \setminus \max(t) + 1) \in U$. By shrinking A one obtains a homogeneous set $C \subseteq A$ for d. Let $C^* := C \cap \triangle_{t \in [C]^n} B_t$, where B_t is a $\mathcal{U}$-large sets such that $B_t \subseteq A$ and $(s^\frown t, B_t) \in U$. By the 0-openess of U is not hard to check that $q := (s, C^*)$ is as desired. □

As a corollary, we infer that the product of two Σ-Prikry notions of forcing need not be Σ-Prikry. For this, let $\mathcal{U}$ and $\mathcal{V}$ be normal measures over the same measurable cardinal κ and let $\mathbb{P}$ and $\mathbb{Q}$ be the corresponding Prikry forcings. We claim that $\mathbb{P} \times \mathbb{Q}$ adds a bounded subset of κ, hence, by Lemma 10.1.10(1), it is not Σ-Prikry.

Let $\vec{s} = \langle s_n \mid n < \omega \rangle$ and $\vec{t} = \langle t_n \mid n < \omega \rangle$ be pairwise generic Prikry-sequences with respect to $\mathbb{P}$ and $\mathbb{Q}$. That is, $\vec{s}$ (resp. $\vec{t}$) generates a generic filter for $\mathbb{P}$ (resp. $\mathbb{Q}$) and furthermore $\vec{s} \notin V[\vec{t}]$ and $\vec{t} \notin V[\vec{s}]$.[9] By mutual genericity, $X := \{n \in \omega \mid s_n < t_n\}$ is infinite and it is also not hard to check that $X \notin V$. In particular, $\mathbb{P} \times \mathbb{Q}$ adds a real.

10.2.2 Supercompact Prikry forcing

Let $\kappa < \lambda$ be two cardinals and assume that is $\mathcal{U}$ a λ-supercompact measure on $\mathcal{P}_\kappa(\lambda)$; namely, $\mathcal{U}$ is a κ-complete, normal and fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$ (cf. Definition 1.1.18). In this section we prove that $\mathbb{P}$, the Supercompact Prikry forcing with respect to $\mathcal{U}$, falls also into the Σ-Prikry framework. Recall that this forcing singularizes κ to cofinality ω and collapses all the cardinals in the interval $[\kappa, \lambda^{<\kappa}]$. For details, see [Git10, §1.4].

[9] Observe that here we are implicitly appealing to Mathias criterion for genericity.

Definition 10.2.4 (Magidor order)**.** For $x, y \in \mathcal{P}_\kappa(\lambda)$, $x \prec y$ iff $x \subseteq y$ and $\mathrm{otp}(x) < \mathrm{otp}(y \cap \kappa)$. Denote by $[\mathcal{P}_\kappa(\lambda)]^{<\omega}$ the set of finite $\prec$-increasing sequences on $\mathcal{P}_\kappa(\lambda)$.

Definition 10.2.5 (Supercompact Prikry forcing)**.** The Supercompact Prikry forcing is the poset $\mathbb{P} := (P, \leq)$, where

- $P := \{(\vec{x}, A) \mid \vec{x} \in [\mathcal{P}_\kappa(\lambda)]^{<\omega} \;\&\; A \in \mathcal{U} \;\&\; \forall y \in A\,(\max_\prec \vec{x} \prec y)\}$
- $(\vec{x}, A) \leq (\vec{y}, B)$ iff $\vec{y} \sqsubseteq \vec{x}$, $\vec{x} \setminus \vec{y} \subseteq B$ and $A \subseteq B$.

For a condition $p = (\vec{x}, A) \in P$, is customary say that $\vec{x}$ and A are, respectively, the stem and the large set of p.

Definition 10.2.6. Given a set of stems X with $|X| = \lambda$, the diagonal intersection of a family $\{A_s \mid s \in X\} \subseteq \mathcal{U}$ is defined as

$$\triangle\{A_s \mid s \in X\} := \{y \in \mathcal{P}_\kappa(\lambda) \mid \forall s \in X(s \prec y \rightarrow y \in A_s)\}.$$

Observe that for each of such families $\mathcal{A} := \{A_s \mid s \in X\}$, normality of the measure $\mathcal{U}$ yields $\triangle \mathcal{A} \in \mathcal{U}$.

Let again Σ be the ω-sequence with constant value κ and set $\mu := (\lambda^{<\kappa})^+$. The notion of length associated to $\mathbb{P}$, $\ell : P \to \omega$, is given by $\ell(\vec{x}, A) := |\vec{x}|$. Finally, define $c : P \to {}^{<\omega}(\lambda^{<\kappa})$ via $c(\vec{x}, A) := \vec{x}$. Mimicking the proof of Proposition 10.2.3 one can easily prove that $(\mathbb{P}, \ell, c)$ is Σ-Prikry.

Proposition 10.2.7. $(\mathbb{P}, \ell, c)$ *is* Σ*-Prikry.* □

10.2.3 Gitik-Sharon forcing

Here we show that the Diagonal Supercompact Prikry Forcing, due to Gitik and Sharon [GS08], can be conceived as a Σ-Prikry forcing. For economy of the discourse, we shall refer to this forcing simply as GS forcing, where the abbreviation GS stands for Gitik-Sharon.

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals, and denote $\kappa := \kappa_0$. Let Σ be the ω-sequence with constant value κ and set $\mu := (\sup_{n<\omega} \kappa_n)^+$. Suppose that κ is a μ^+-supercompact cardinal and let $\mathcal{U}$ be a measure on $\mathcal{P}_\kappa(\mu^+)$ witnessing this. For each $n < \omega$, let $\mathcal{U}_n$ be the projection of $\mathcal{U}$ onto $\mathcal{P}_\kappa(\kappa_n)$. Namely, for each $X \subseteq \mathcal{P}_\kappa(\kappa_n)$, $X \in \mathcal{U}_n$ iff $\pi_n^{-1}[X] \in \mathcal{U}$, where π_n is the standard projection between $\mathcal{P}_\kappa(\mu^+)$ and $\mathcal{P}_\kappa(\kappa_n)$. It is routine to check that, for each $n < \omega$, $\mathcal{U}_n$ is a κ_n-supercompact measure over $\mathcal{P}_\kappa(\kappa_n)$. Through this section $\prec$ will denote the Magidor order defined in Definition 10.2.4.

Definition 10.2.8. The GS poset is the partial order $\mathbb{P} := (P, \leq)$ where,

- P is the set of sequences $p = \langle x_0^p, \ldots, x_{n^p-1}^p, A_{n^p}^p, A_{n^p+1}^p, \ldots \rangle$ such that each $x_i^p \in \mathcal{P}_\kappa(\kappa_i)$, $x_i^p \prec x_{i+1}^p$, $A_k^p \in U_k$ and, for each $y \in A_k^p$, $x_{n^p-1}^p \prec y$;
- let $p, q \in P$ be two conditions and assume that

$$p := \langle x_0^p, \ldots, x_{n^p-1}^p, A_{n^p}^p, A_{n^p+1}^p, \ldots \rangle,$$
$$q := \langle x_0^q, \ldots, x_{m^q-1}^q, A_{m^q}^q, A_{m^q+1}^q, \ldots \rangle.$$

We will write $p \leq q$ iff $m^q \leq n^p$, and the following hold:

1. $\forall i < m^q$, $x_i^q = x_i^p$;
2. $\forall i \, (m^q \leq i < n^p \to x_i^p \in A_i^q)$;
3. $\forall i \, (n^p \leq i \to A_i^p \subseteq A_i^q)$.

It is customary to call $\langle x_0, \ldots, x_{n^p-1} \rangle$ the *stem of* p, and will be denoted by $\mathrm{stem}(p)$.

Definition 10.2.9. Let $\ell : P \to \omega$ be defined as $\ell(p) := n^p$.

Definition 10.2.10. Define $c : P \to {}^{<\omega}(P_\kappa(\kappa^{+\omega}))$ via

$$c(\langle x_0^p, \ldots, x_{\ell(p)-1}^p, A_{\ell(p)}^p, A_{\ell(p)+1}^p, \ldots \rangle) := \langle x_0^p, \ldots, x_{\ell(p)-1}^p \rangle.$$

Definition 10.2.11. Let $p = \langle x_0^p, \ldots, x_{n-1}^p, A_n^p, A_{n+1}^p, \ldots, \rangle$ in P. For each $x \in A_{\ell(p)}^p$, $p^\frown\langle x \rangle$ stands for the unique condition

$$q := \langle x_0^p, \ldots, x_{\ell(p)-1}^p, x, B_{\ell(p)+1}^p, B_{\ell(p)+2}^p, \ldots \rangle,$$

where, for each $i \geq \ell(p)$, $B_i^p := \{y \in A_i^p \mid x \prec y\}$. Similarly, for all $n \geq \ell(p)$, and any $\prec$-increasing $\vec{x} := \langle x_{\ell(p)}, \ldots, x_{n+1} \rangle \in \prod_{i=\ell(p)}^{n+1} A_i^p$, we define $p^\frown\vec{x}$ by recursion over $|\vec{x}|$.[10]

Note that whenever $q \leq p$, for some $\vec{x}$, we have that $q \leq^0 p^\frown\vec{x} \leq p$.

Proposition 10.2.12. *$(\mathbb{P}, \ell, c)$ is Σ-Prikry.*

Proof. We go over the clauses of Definition 10.1.3.

1. For $p \in P$, $p^\frown x \in P_1^p$, for all $x \in A_{\ell(p)}^p$. Also, by the mere definition of $\leq$, $p \leq q$ implies $\ell(p) \geq \ell(q)$.
2. This follows in a similar fashion as in the verification of Clause (2) in Propoposition 10.2.3.
3. Let $p, q \in P$ and assume that $c(p) = c(q)$. Let r be the condition with $\mathrm{stem}(r) = \mathrm{stem}(p)$ and $A_i^r := A_i^p \cap A_i^q$, for $i \geq |\,\mathrm{stem}(p)|$. It is obvious that $r \in P_0^p \cap P_0^q$.

[10] By convention, $p^\frown\emptyset = p$.

4. Let $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$. It is not hard to check that $q \leq^m p^\frown \vec{x} \leq^n p$, for some $\vec{x} \in [\prod_{i=\ell(p)}^{\ell(p)+n-1} A_i^p]$. In fact, if $q \leq^m r \leq^n p$, the mere definition of $\leq$ yields $r \leq^0 p^\frown \vec{x}$, as wanted.

5. By the above clause, for each condition p,
$$W_n(p) = \{p^\frown \vec{x} \mid \vec{x} \in \prod_{i=\ell(p)}^{n-1} A_i^p \,\&\, \vec{x} \text{ is } \prec\text{-increasing}\}.$$
Clearly, $|W_n(p)| = \kappa_n$, hence $|W(p)| < \mu$.

6. Let $p' \leq p$ be in P and let $q_0, q_1 \in W(p')$ with $q_1 \leq q_0$. Let $\vec{x} \in \prod_{n=\ell(p)}^{\ell(p')-1} A_n^p$ be the unique sequence such that $p' \leq_0 p^\frown \vec{x}$. Also, for $i \in \{0,1\}$, let $\vec{x}_i \in \prod_{n=\ell(p')}^{\ell(q_i)-1} A_n^{p'}$ be such that $w(p', q_i) = q_i = p'^\frown \vec{x}_i$. In particular, $\vec{x}_1$ is an extension of $\vec{x}_0$. On the other hand, it is not hard to check that, for each $i \in \{0,1\}$, $w(p, q_i) = (p^\frown \vec{x})^\frown \vec{x}_i$. Altogether this yields $w(p, q_1) \leq w(p, q_0)$, as desired.

7. Let U be a 0-open set, $p \in P$ and $n < \omega$. Say, $p := \langle x_0^p, \dots x_{\ell(p)-1}^p, A_{\ell(p)}^p, A_{\ell(p)+1}^p, \dots \rangle$. For each $\vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A_m^p$, provided there exists $r \in U$ with $r \leq_0 p^\frown \vec{x}$, let $\vec{B}_{\vec{x}} \in \prod_{\ell(p)+n \leq m < \omega} \mathcal{U}_m$ be the sequence of measure one sets of some of such r. Otherwise, let $\vec{B}_{\vec{x}}$ be the sequence $\langle \mathcal{P}_\kappa(\kappa_m) \mid m \in [\ell(p)+n, \omega) \rangle$. For each $\ell(p)+n \leq m < \omega$, set
$$A_m^* := \triangle \{\vec{B}_{\vec{x}}(m) \mid \vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A_m^p\} \cap A_m^p.$$[11]
By normality of $\mathcal{U}_m$, $A_m^* \in \mathcal{U}_m$. Now, define $p' := \langle p'_m \mid m < \omega \rangle$ as the sequence
$$p'_m := \begin{cases} x_m^p, & \text{if } m < \ell(p); \\ A_m^p, & \text{if } \ell(p) \leq m \leq \ell(p)+n-1; \\ A_m^*, & \text{otherwise.} \end{cases}$$
Clearly, $p' \leq_0 p$. Define $d : \prod_{m=\ell(p)}^{\ell(p)+n-1} A_m^p \to 2$ by, $d(\vec{x}) := 1$ iff $p'^\frown \vec{x} \in P$ and there is $r \leq_0 p'^\frown \vec{x}$ be such that $r \in U$. Shrinking the sets A_m^p one may find $\langle A_m^* \mid m \in [\ell(p), \ell(p)+n) \rangle$ be such that $A_m^* \subseteq A_m^p$, $A_m^* \in U_m$ and $d \restriction (\prod_{m=\ell(p)}^{\ell(p)+n-1} A_m^*)$ is constant. Say with d-value i. Let q be the condition defined as p' but with A_m^* as large sets, for $m \in [\ell(p), \ell(p)+n)$. Clearly, $q \leq_0 p'$, hence $q \leq_0 p$.

Claim 10.2.12.1. *q witnesses Clause (7). Namely, either $P_n^q \subseteq U$ or $P_n^q \cap U = \emptyset$.*

[11] c.f Definition 10.2.6.

Proof of claim. For the proof we will distinguish two cases.

▶ Assume $i = 0$. For each $\vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A^*_m$, either $p'^\frown\vec{x} \notin P$ or there is no $r \leq_0 p'^\frown\vec{x}$ such that $r \in U$. Shrinking if necessary, we may assume that the first of these alternatives is false. Thus, for each $\vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A^*_m$, there is no $r \leq_0 p'^\frown\vec{x}$ such that $r \in U$. Now, observe that $p'^\frown\vec{x} = q^\frown\vec{x}$. Then, for each such $\vec{x}$, there is no $r \leq_0 q^\frown\vec{x}$ in U, hence $P^q_n \cap U = \emptyset$.

▶ Assume $i = 1$. Then, for each $\vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A^*_m$, there is $r \leq_0 p'^\frown\vec{x}$ in U. Since $p' \leq_0 p$, it follows that there is some $r \leq_0 p^\frown\vec{x}$ in U. By definition this implies that there is a condition $r' \in U$ with $r' \leq_0 p^\frown\vec{x}$ and such that $\vec{B}_{\vec{x}}$ is its sequence of measure one sets.

Subclaim 10.2.12.1.1. *For each $\vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A^*_m$, $q^\frown\vec{x} \leq_0 r'$.*

Proof of subclaim. Fix $\vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A^*_m$ and set $q' := q^\frown\vec{x}$. Clearly, $q'_m = r'_m$, for each $m \leq \ell(p) + n - 1$. If $\ell(p) + n \leq m < \omega$, $q'_m = B^*_m$, where $B^*_m := \{y \in A^*_m \mid \vec{x}(\ell(p) + n - 1) \prec y\}$. By definition of diagonal intersection, $B^*_m \subseteq B_{\vec{x}}(m)$. Altogether, $q' \leq_0 r'$, as wanted. □

Using the 0-openess of U, the previous claim yields $q^\frown\vec{x} \in U$, for each $\vec{x} \in \prod_{m=\ell(p)}^{\ell(p)+n-1} A^*_m$. Once again by 0-openess of U, $P^q_n \subseteq U$, as desired. □

This finishes the proof of the proposition. □

10.2.4 AIM forcing

Throughout this section assume that $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of λ-supercompact cardinals, for some inaccessible cardinal λ above $\kappa := \sup_{n<\omega} \kappa_n$. For each $n < \omega$, let us fix U_n a λ-supercompact measure over $\mathcal{P}_{\kappa_n}(\lambda)$ and, for each $\kappa \leq \alpha < \lambda$, denote by $U_{n,\alpha}$ its projection by the map $\pi_{n,\alpha} : x \mapsto x \cap \alpha$. It is easy to check that $U_{n,\alpha}$ is an α-supercompact measure over $\mathcal{P}_{\kappa_n}(\alpha)$ and that, $\langle \langle U_{n,\alpha}, \pi_{n,\alpha}, \pi^n_{\alpha,\beta} \rangle \mid \kappa \leq \beta \leq \alpha < \lambda \rangle$ forms a directed system of projections, where $\pi^n_{\alpha,\beta}$ denotes the standard projection between $U_{n,\alpha}$ and $U_{n,\beta}$.[12] In this section we will prove that the American Institute of Mathematics forcing introduced in [Cum+18] is Σ-Prikry. For economy of the language we shall refer to this forcing as the AIM forcing.

Definition 10.2.13. The AIM forcing is the poset $\mathbb{P} = (P, \leq)$, where P consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ such that for some $\ell(p) < \omega$, the following hold true:

[12] Namely, $x \mapsto x \cap \beta$. Provided that no confusion arise, we shall tend to omit the superscript n when referring to $\pi^n_{\alpha,\beta}$.

1. for each $n < \ell(p)$, p_n is a function f_n^p with $\mathrm{dom}(f_n^p) \subseteq [\kappa, \lambda)$, $|\mathrm{dom}(f_n^p)| < \lambda$, and for all $\eta \in \mathrm{dom}(f_n^p)$, $f_n^p(\eta) \in \mathcal{P}_{\kappa_n}(\eta)$;
2. for each $n \geq \ell(p)$, p_n is a triple (a_n^p, A_n^p, f_n^p), where:
   (a) a_n^p is a subset of $[\kappa, \lambda)$ with $|a_n^p| < \lambda$. Moreover, a_n^p admits a maximal element α_n^p;
   (b) $A_n^p \in U_{n,\alpha_n^p}$;
   (c) f_n^p is a function with $\mathrm{dom}(f_n^p) \subseteq [\kappa, \lambda) \setminus a_n^p$, $|\mathrm{dom}(f_n^p)| < \lambda$ such that, for all $\eta \in \mathrm{dom}(f_n^p)$, $f_n^p(\eta) \in P_{\kappa_n}(\eta)$.
3. $\langle a_n^p \mid \ell(p) \leq n < \omega \rangle$ is $\subseteq$-increasing.

We let $p \leq q$ if and only if the following are fulfilled:

1. $\ell(p) \geq \ell(q)$.
2. For all n, $f_n^p \supseteq f_n^q$;
3. For n with $\ell(q) \leq n < \ell(p)$, $a_n^q \subseteq \mathrm{dom}(f_n^p)$, $f_n^p(\alpha_n^q) \in A_n^q$, and $f_n^p(\eta) = f_n^p(\alpha_n^q) \cap \eta$ for all $\eta \in a_n^q$.[13]
4. $(f_n^p(\alpha_n^q))_{\ell(q) \leq n < \ell(p)}$ is $\subseteq$-increasing.
5. For $n \geq \ell(p)$, we have $a_n^q \subseteq a_n^p$, and $A_n^p \subseteq \pi^{-1}_{\alpha_n^p, \alpha_n^q}[A_n^q]$.[14]
6. For $n \geq \ell(p)$, if $\ell(q) < \ell(p)$, then $f^p_{\ell(p)-1}(\alpha^q_{\ell(p)-1}) \subseteq x$ for all $x \in A_n^p$.

The notion of length associated to $\mathbb{P}$, $\ell : P \to \omega$, is $p \mapsto \ell(p)$, where $\ell(p)$ is natural number witnessing $p \in Q$. Also, since $|\mathbb{P}| = \lambda$, we find $c : P \to \lambda$ which is an injection. Finally, by virtue of Lemma 4 and Corollary 1 of [Cum+18], $\mathbb{P}$ collapses all cardinals $\theta \in (\kappa, \lambda)$ and makes λ the successor of κ. Thus, we set $\mu := \lambda$.

Next, we verify that $(\mathbb{P}, \ell, c)$ is Σ-Prikry by going over the clauses of Definition 10.1.3. Before that we need to introduce a couple of concepts.

Definition 10.2.14 (Magidor ordering for & Diagonal intersections)**.** Let $k < l < \omega$ and $(\alpha_i)_{k \leq i < l}$ be a $\leq$-increasing sequence in $[\kappa, \lambda)$ and let $S \subseteq \prod_{i=k}^{l-1} \mathcal{P}_{\kappa_i}(\alpha_i)$.

1. If $s \in S$ is a $\subseteq$-increasing sequence and $x \in \mathcal{P}_{\kappa_l}(\alpha_l)$, we shall write $s \prec x$ if and only if $\max s \subseteq x$.

[13]This is the natural analogous of condition $(2)_n$(b) in Definition 10.2.30.
[14]I.e. $x \cap \alpha_n^q \in A_n^q$, for all $x \in A_n^p$.

2. Let $m \in [l, \omega)$, $\alpha \in [\alpha_{l-1}, \lambda)$, and let $\vec{A} = (A_s)_{s \in S}$ be a family with $A_s \in U_{m,\alpha}$, for each $s \in S$. Then, the diagonal intersection of the S-indexed family $\vec{A}$ is defined as

$$\triangle_{s \in S} A_s := \{x \in \mathcal{P}_{\kappa_m}(\alpha) \mid \forall s \in S\,(s \prec x \rightarrow x \in A_s)\}.$$

In [Cum+18, Lemma 1] it is proved that the above diagonal intersection always yields a set in $U_{m,\alpha}$.

Definition 10.2.15. For conditions $r \leq q$, we let $\mathrm{stem}(r, q)$ denote the finite sequence $(f^r_i(\alpha^q_i))_{\ell(q) \leq i < \ell(r)}$.

Definition 10.2.16. Let q be a condition. Let $l \in (\ell(q), \omega)$ and $s \in \prod_{\ell(q) \leq i < l} A^q_i$ be a $\subseteq$-increasing sequence. Define $q + s$ as the ω-sequence $(r_k)_{k<\omega}$ such that:

- For $k < \ell(q)$, $r_k = f^q_k$.
- For $\ell(q) \leq k < l$, r_k is the function with domain $\mathrm{dom}(f^q_k) \cup a^q_k$ such that $r_k(\eta) = f^q_k(\eta)$ for $\eta \in \mathrm{dom}(f^q_k)$ and $r_k(\eta) = s_k \cap \eta$ for $\eta \in a^q_k$.
- For $k \geq l$, $r_k = (f^q_k, a^q_k, B_k)$ where $B_k = \{x \in A^q_k : s_{l-1} \subseteq x\}$.[15]

By convention we also define $q + \langle\rangle := q$.

Let us now check that $(\mathbb{P}, \ell, c)$ fulfills the clauses of Definition 10.1.3.

Proposition 10.2.17. *Clause* (1) *holds for* $(\mathbb{P}, \ell, c)$.

Proof. Clearly, $q \leq p$ yields $\ell(q) \geq p$. Also $p + \langle x \rangle \in P^p_1$, for $x \in A^p_{\ell(p)}$. □

Lemma 10.2.18. *Let $n < \omega$ and $p, q \in P_n$. The conditions p and q are $\leq_0$-compatible iff the following properties hold:*

1. *for all $k < \omega$, $f^p_k \cup f^q_k$ is a function;*
2. *for all $k \geq n$, $a^p_k \cap \mathrm{dom}(f^q_k) = a^q_k \cap \mathrm{dom}(f^p_k) = \emptyset$.*

Proof. The first implication is easy. For the second, use the above properties to define $r := (r_k \mid k < \omega)$, where

$$r_k := \begin{cases} f^p_k \cup f^q_k, & \text{if } k < n; \\ (a^p_k \cup a^q_k, A^*_k, f^p_k \cup f^q_k), & \text{otherwise,} \end{cases}$$

where $A^*_k := \pi^{-1}_{\alpha,\alpha^p_k}[A^p_k] \cap \pi^{-1}_{\alpha,\alpha^q_k}[A^q_k]$ and $\alpha := \max(\alpha^p_k, \alpha^q_k)$. Clearly, $r \in P_n$ and $r \leq_0 p, q$, as wanted. □

[15] Notice that $f^r_{l-1}(\alpha^q_{l-1}) = s_{l-1}$, as $s_{l-1} \subseteq \alpha^q_{l-1}$.

Proposition 10.2.19. *Clause* (2) *holds for* $(\mathbb{P}, \ell, c)$.

Proof. Let $D \in [P_n \cup \{\mathbb{1}\}]^{<\kappa_n}$ be a $\leq^0$-directed set. Say, $D = \{p^\alpha \mid \alpha < \theta\}$, for some cardinal $\theta < \kappa_n$. By Lemma 10.2.18, for each $k \geq n$, and all $\alpha, \beta < \theta$, $\mathrm{dom}(a_k^{p^\alpha}) \cap \mathrm{dom}(f_k^{p^\beta}) = \emptyset$. Define by recursion $\langle (b_k, B_k) \mid k \geq n \rangle$, where $b_k \subseteq [\kappa, \lambda)$, $|b_k| < \lambda$, and $B_k \in U_{k \max(b_k)}$, as follows:

1. Let $k \geq n$ and assume that $\langle b_i \mid n \leq i < k \rangle$ has been defined. Set $b_k^* := (\bigcup_{n \leq i < k} b_i) \cup \bigcup_{\alpha < \theta} a_k^{p^\alpha}$. Since $\theta < \lambda$, $b_k^* \subseteq [\kappa, \lambda)$ and $|b_k^*| < \lambda$. Now let $\delta_k \in \lambda \setminus \bigcup_{j < \omega, \alpha < \theta} \mathrm{dom}(f_j^{p^\alpha})$ such that $\delta_k \geq \sup b_k^*$. Define $b_k := b_k^* \cup \{\delta_k\}$.

2. Define $B_k := \bigcap_{\alpha < \theta} \pi^{-1}_{\delta_k, \alpha_m^{p^\alpha}}[A_m^{p^\alpha}]$. Clearly, $B_k \in U_{k, \delta_k}$.

At the end of this recursive procedure, set $r := \langle r_k \mid k < \omega \rangle$, where

$$r_k := \begin{cases} \bigcup_{\alpha < \theta} f_k^{p^\alpha}, & \text{if } k < n, \\ (b_k, B_k, \bigcup_{\alpha < \theta} f_k^{p^\alpha}), & \text{otherwise,} \end{cases}$$

It is not hard to check $r \in P_n$ and clearly, for each $\alpha < \theta$, $r \leq_0 p^\alpha$. □

Proposition 10.2.20. *Clause* (3) *holds for* $(\mathbb{P}, \ell, c)$.

Proof. It follows from injectivity of c and the fact that $P_0^p \neq \emptyset$, for $p \in P$. □

Proposition 10.2.21. *Clause* (4) *holds for* $(\mathbb{P}, \ell, c)$.

Proof. Let $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$. Set $s := \mathrm{stem}(p, q) \restriction n$. Observe that $q \leq^m p + s$ and $p + s \leq^n p$. By the definition of $\leq$ it is routine to check that $p + s$ is the $\leq$-greatest element of $\{r \leq^n p \mid q \leq^m r\}$. □

For the record, observe that we have implicitly proved, for $q \leq p$, that $w(p, q) = p + \mathrm{stem}(p, q)$. Let us continue with the verification of the clauses.

Proposition 10.2.22. *Clause* (5) *holds for* $(\mathbb{P}, \ell, c)$.

Proof. By the above proof it is clear that $W_n(p) := \{p + s \mid s \in \prod_{k=\ell(p)}^{\ell(p)+n-1} A_k^p\}$, hence $|W_n(p)| < \mu$, and thus $|W(p)| < \mu$. □

Proposition 10.2.23. *Clause* (6) *holds for* $(\mathbb{P}, \ell, c)$.

Proof. Let $p' \leq p$ and $q_1 \leq q_0$ be in $W(p')$. By the above proposition, for each $i \in \{0, 1\}$, there is a sequence s_i such that $q_i = p' + s_i$. It is automatic that $s_i = \mathrm{stem}(p', q_i)$. Now $w(p, q_i) = w(p, p' + \mathrm{stem}(p', q_i))$, which is easily seen to be the same as $p + (\mathrm{stem}(p, p')^\frown \mathrm{stem}(p, q_i))$. Now, since $q_1 \leq q_0$, observe that $\mathrm{stem}(p, q_0)$ is a subsequence of $\mathrm{stem}(p', q_1)$, so that it is routine to check that $p + (\mathrm{stem}(p, p')^\frown \mathrm{stem}(p, q_1)) \leq p + (\mathrm{stem}(p, p')^\frown \mathrm{stem}(p0, q_0))$, as desired. □

We are thus left with showing that Clause (7) of Definition 10.1.3 holds for the triple $(\mathbb{P}, \ell, c)$. To this aim we shall need to prove some auxiliary lemmas and introduce a few more concepts.

Definition 10.2.24. Let $n < \omega$. For $p, q \in P$, we will write $p \sqsubseteq_n q$ if and only if the following conditions hold:

1. $\ell(p) = \ell(q)$.
2. $(a^p_k, A^p_k) = (a^q_k, A^q_k)$, for each $\ell(p) \leq k < \ell(p) + n$.

The proof of the following lemma can be found in [Cum+18, Lemma 6].

Lemma 10.2.25. *Let $\langle p_n \mid n < \omega\rangle$ be a sequence of conditions in P such that, for each $n < \omega$, $p_{n+1} \sqsubseteq_n p_n$. Then there is $q \in P$ such that, for each $n < \omega$, $q \sqsubseteq_n p_n$.*

Under the above conditions we will say that $\langle p_n \mid n < \omega\rangle$ is a *fusion sequence* and that q is its *fusion condition*. The next result, which we call *Diagonalization*, is crucial for the proof of the CPP for the AIM forcing.

Lemma 10.2.26 (Diagonalization)**.** *Let U be a 0-open set and $p \in P$ be a condition. There is $q \in P^p_0$ such that, for each $r \in P^q \cap U$, $w(q, r) \in U$.*

Proof. We follow [Cum+18, Lemma 10] checking that essentially the same arguments work in the current setting. As in [Cum+18] we will begin defining a sequence $\langle p_n \mid n < \omega\rangle$ of conditions witnessing the following requirements:

($\aleph$) $p_0 := p$,

($\beth$) $p_{n+1} \sqsubseteq_n p_n$,

($\gimel$) for all $r \in P^{p_{n+1}}_n \cap U$, $w(p_{n+1}, r) \in U$.

Now assume for a moment that we manage to obtain such a sequence. By construction $\langle p_n \mid n < \omega\rangle$ is a fusion sequence so that Lemma 10.2.25 guarantees the existence of a fusion condition q.

Claim 10.2.26.1. *The fusion condition q witnesses the statement of the lemma.*

Proof of claim. Obviously $q \in P^p_0$, so it suffices to check the other property. Let $r \in P^q \cap U$ and set $n := \ell(r) - \ell(q)$. Notice that $r \in P^{p_{n+1}}_n \cap U$ so that, by ($\gimel$), $w(p_{n+1}, r) \in U$. Clearly, $w(q, r) \leq_0 w(p_{n+1}, r)$ and thus, by 0-openess of U, $w(q, r) \in U$. □

Set $p_0 := p$. For the construction of p_1 we first ask whether there is some $r \in P^{p_0}_0 \cap U$ and, if so, we set $p_1 := r$. Otherwise, $p_1 := p_0$.

Claim 10.2.26.2. *p_1 witnesses ($\beth$) and ($\gimel$).*

Proof of claim. Clearly, $p_1 \sqsubseteq_0 p_0$. On the other hand, if $s \in P_0^{p_1} \cap U$, $w(p_1, s) = p_1$, hence $w(p_1, s) \in U$. This proves ($\beth$) and ($\gimel$) for p_1. □

Now assume that, for some $1 \leq n < \omega$, p_n has been defined. Let S be the set of $\subseteq$-increasing sequences of $\prod_{k=\ell(p)}^{\ell(p)+n-1} A_k^{p_n}$. In other words, S is the set of all sequences s such that $p_n + s \in P$. Since λ is inaccessible, $|S| < \lambda$, hence we may let an enumeration $\langle s_\alpha^n \mid \alpha < \theta\rangle$ of S, for some $\theta < \lambda$.

The proof idea is the following: we need to define, by induction on θ, a $\sqsubseteq_n$-decreasing sequence of conditions $\langle p^{\alpha,n} \mid \alpha < \lambda\rangle$ *catching up* many of the potentials conditions r lying in $P_n^{p^\alpha} \cap U$. At the end of this recursive process we will define p_{n+1} as the *diagonal limit* of $\langle p^{\alpha,n} \mid \alpha < \lambda\rangle$ and check that it fulfills ($\beth$) and ($\gimel$). In order to avoid the cumbersome notation $p^{\alpha,n}$ we shall waive the dependence on n and just write p^α.

Define $p^0 := (p_k^0 \mid k < \omega)$, where

$$p_k^0 := \begin{cases} f_k^{p_n}, & \text{if } k < \ell(p); \\ (a_k^{p_n}, A_k^{p_n}, f_k^{p_n}), & \text{if } \ell(p) \leq k < \ell(p) + n; \\ (a_k^{p_n}, \mathcal{P}_{\kappa_k}(a_k^{p_n}), f_k^{p_n}), & \text{if } k \geq \ell(p) + n. \end{cases}$$

Now assume that p^α has been defined and set $q^\alpha := p^\alpha + s_\alpha^n$. If $P_0^{q_\alpha} \cap U \neq \emptyset$ let r^α be some condition there. Otherwise, set $r^\alpha := q^\alpha$. We need to keep track of the following information:

- ▶ $(a_k^{\alpha+1}, A_k^{\alpha+1}) := (a_k^{r^\alpha}, A_k^{r^\alpha})$, for each $\ell(p) + n \leq k < \omega$,
- ▶ $\gamma_k^{\alpha+1} := \max a_k^{\alpha+1}$, for each $\ell(p) + n \leq k < \omega$,
- ▶ $f_k^{\alpha+1} := f_k^{r^\alpha}$, for each $k < \omega$.

Define $p^{\alpha+1} := (p_k^{\alpha+1} \mid k < \omega)$, where

$$p_k^{\alpha+1} := \begin{cases} f_k^{\alpha+1}, & \text{if } k < \ell(p); \\ (a_k^{p_n}, A_k^{p_n}, f_k^{\alpha+1}), & \text{if } \ell(p) \leq k < \ell(p) + n; \\ (a_k^{\alpha+1}, \mathcal{P}_{\kappa_k}(\gamma_k^{\alpha+1}), f_k^{\alpha+1}), & \text{if } k \geq \ell(p) + n. \end{cases}$$

In case $\alpha < \theta$ is a limit ordinal and $\langle p^\beta \mid \beta < \alpha\rangle$ has been already defined, let $p^\alpha := (p_k^\alpha \mid k < \omega)$ be given by

$$p_k^\alpha := \begin{cases} \bigcup_{\beta<\alpha} f_k^\beta, & \text{if } k < \ell(p); \\ (a_k^{p_n}, A_k^{p_n}, \bigcup_{\beta<\alpha} f_k^\beta), & \text{if } \ell(p) \leq k < \ell(p) + n; \\ (a_k^\alpha, \mathcal{P}_{\kappa_k}(\gamma_k^\alpha), \bigcup_{\beta<\alpha} f_k^\beta), & \text{if } k \geq \ell(p) + n. \end{cases}$$

Here a_k^α and γ_k^α are defined by recursion as follows: For each $\ell(p)+n \leq k < \omega$, set $a_k^* := \bigcup_{m<k} a_m^* \cup \bigcup_{\beta<\alpha} a_k^\beta$. Since $|\bigcup_k \bigcup_{\beta<\alpha} f_k^\beta| < \lambda$, we may find $\gamma_k^\alpha < \lambda$ be such that

(a) $\gamma_k^\alpha \notin \bigcup_k \bigcup_{\beta<\alpha} \mathrm{dom}(f_k^\beta)$,

(b) $\sup a_k^* \leq \gamma_k^\alpha$.

Set $a_k^\alpha := a_k^* \cup \{\gamma_k^\alpha\}$. Since $a_k^\alpha \subseteq a_{k+1}^\alpha$, $p^\alpha \in P$.

Now define p_{n+1} in a similar fashion: namely, $p_{n+1} := (t_k \mid k < \omega)$ where

$$t_k := \begin{cases} \bigcup_{\alpha<\theta} f_k^\alpha, & \text{if } k < \ell(p); \\ (a_k^{p_n}, A_k^{p_n}, \bigcup_{\alpha<\theta} f_k^\alpha), & \text{if } \ell(p) \leq k < \ell(p)+n; \\ (a_k^{p_{n+1}}, A_k^{p_{n+1}}, \bigcup_{\alpha<\theta} f_k^\alpha), & \text{if } k \geq \ell(p)+n; \end{cases}$$

Here $a_k^{p_{n+1}}$, $\alpha_k^{p_{n+1}}$ and $A_k^{p_{n+1}}$ are defined as follows:

(α) $a_k^{p_{n+1}}$ and $\alpha_k^{p_{n+1}}$ are defined according to the previous procedure;

(β) $A_k^{p_{n+1}} := \pi^{-1}_{\alpha_k^{p_{n+1}}, \alpha_k^{p_n}}[A_k^{p_n}] \cap \triangle\{\pi^{-1}_{\alpha_k^{p_{n+1}}, \gamma_k^{\alpha+1}}[A_k^{\alpha+1}] \mid \alpha < \theta\}$.

Observe that $A_k^{p_{n+1}} \in U_{k, \alpha_k^{p_{n+1}}}$ and thus $p_{n+1} \in Q$.

Claim 10.2.26.3. *p_{n+1} witnesses ($\sqsupset$) and ($\beth$).*

Proof of claim. For ($\sqsupset$) let us go over the clauses (1)-(6) of Definition 10.2.13. Notice that $\ell = \ell(p_{n+1}) = \ell(p_n)$ so that clauses (1), (3), (4) and (6) are trivially true. Also (2) is easily seen to be true. For (5) let $k \geq \ell$ and notice that, by construction, $a_k^{p_n} \subseteq a_k^{p_{n+1}}$. On the other hand, either $A_k^{p_{n+1}} = A_k^{p_n}$ or $A_k^{p_{n+1}} \subseteq \pi^{-1}_{\alpha_k^{p_{n+1}}, \alpha_k^{p_n}}[A_k^{p_n}]$ so that (5) holds. Altogether, $p_{n+1} \leq_0 p_n$ and thus $p_{n+1} \sqsubseteq_n p_n$.

For ($\beth$) let $r \in P_n^{p_{n+1}} \cap U$ and set $s := \mathrm{stem}(r, p_{n+1})$. Since $s \in S$ there is some $\alpha < \theta$ such that $s = s_\alpha^n$. A moment of reflection will convince us that, for each $\beta < \theta$, $p_{n+1} \sqsubseteq_n p^\beta$, in particular $p_{n+1} \sqsubseteq_n p^\alpha$, and thus $s = \mathrm{stem}(r, p^\alpha)$. As in the construction, set $q^\alpha := p^\alpha + s$. Notice that $r \in P_0^{q^\alpha} \cap U$, so in the recursion we have necessarily chosen some $r^\alpha \in P_0^{q^\alpha} \cap U$. If we manage to prove $p_{n+1} + s \leq r^\alpha$, namely $w(p_{n+1}, r) \leq_0 r^\alpha$, the 0-openess of U wild yield the desired conclusion. To this aim we shall need to go over the clauses (1), (2), (5) of Definition 10.2.13, but notice that the verification of (1) and (2) are straightforward. For (5) set $t := p_{n+1} + s$ and let $k \geq \ell(t)$. Notice that $a_k^t = a_k^{p_{n+1}} \supseteq a_k^{\alpha+1} = a_k^{r^\alpha}$. On the other hand, for each $x \in A_k^t$, by Definition 10.2.13(6), $s \prec x$, so that $x \in \pi^{-1}_{\alpha_k^{p_{n+1}}, \gamma_k^{\alpha+1}}[A_k^{\alpha+1}]$ and thus $x \cap \gamma_k^{\alpha+1} \in A_k^{\alpha+1}$. Since $A_k^{\alpha+1}$ is by definition $A_k^{r^\alpha}$ it follows that, for each $x \in A_k^t$, $x \cap \gamma_k^{\alpha+1} \in A_k^{r^\alpha}$, as wanted. □

□

Proposition 10.2.27. *Clause (7) holds for $(\mathbb{P}, \ell, c)$.*

Proof. Fix $p \in P$, U a 0-open set and $n < \omega$. Let q be the condition given by Lemma 10.2.26 regarded with respect p and U. Set $\ell := \ell(q)$. Denote by S the set of all sequences of length n for which $q+s \in P$ and define $F : S \longrightarrow 2$

$$F(s) := \begin{cases} 1, & \text{if } q+s \in U; \\ 0, & \text{otherwise.} \end{cases}$$

By shrinking $\{A^q_k\}_{\ell \leq k \leq \ell+n-1}$ we may find a family $\{B_k\}_{\ell \leq k \leq \ell+n-1}$ of large sets such that $B_k \subseteq A^q_k$ and such that $F \upharpoonright \prod_k B_k$ is constant, for each such k. Now define q^* as q but replacing A^q_k by B_k, for each $k \in [\ell, \ell+n-1]$. Clearly $q^* \leq_0 q$, hence $q^* \in P^p_0$. Thus, it remains to check that the dichotomy indicated at Clause (7) occurs.

Assume that $P^{q^*}_n \cap U \neq \emptyset$ and let r be a condition witnessing this. By lemma 10.2.26, $w(q,r) \in U$ so that, by 0-openess, $w(q^*,r) \in U$. Let $s \in \prod_k B_k$ be such that $w(q^*,r) = q^* + s$. Since $q^* + s \in U$, $F(s) = 1$. Now the homogeneity of $\prod_k B_k$ yields $F(t) = 1$, for each $t \in \prod_k B_k$, hence $W_n(q^*) \subseteq U$. Finally observe that each element of $P^{q^*}_n$ is a 0-extension of an condition in $W_n(q^*)$ so that, again by 0-openess, $P^{q^*}_n \subseteq U$. □

Altogether, the above discussion implies that $(\mathbb{P}, \ell, c)$ is Σ-Prikry.

10.2.5 Extender-based Prikry Forcing

In this section, we recall the definition of the Extender Based Prikry Forcing (EBPF) due to Gitik and Magidor [GM94, §3] (see also [Git96] and [Git10, §2]), and verify that it fits into the Σ-Prikry framework. Unlike other expositions of this forcing, we shall not assume the GCH, as in future chapters we want to be able to conduct various forcing preparations (such as Laver's [Lav78]) that messes up the GCH. Our setup along the current section will be the following:

- $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of cardinals;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := 2^\mu$;
- $\mu^{<\mu} = \mu$ and $\lambda^{<\lambda} = \lambda$;
- for each $n < \omega$, κ_n carries a $(\kappa_n, \lambda+1)$-extender E_n.[16]

In particular, we are assuming that, for each $n < \omega$, there is an elementary embedding $j_n : V \to M_n$ such that M_n is a transitive class, ${}^{\kappa_n}M_n \subseteq M_n$, $V_{\lambda+1} \subseteq M_n$ and $j_n(\kappa_n) > \lambda$. For each $n < \omega$, and each $\alpha < \lambda$, define

$$E_{n,\alpha} := \{X \subseteq \kappa_n \mid \alpha \in j_n(X)\}.$$

[16] See Definition 1.1.26.

Note that $E_{n,\alpha}$ is a non-principal κ_n-complete ultrafilter over κ_n, provided that $\alpha \geq \kappa_n$. Moreover, in the particular case of $\alpha = \kappa_n$, E_{n,κ_n} is also normal. For ordinals $\alpha < \kappa_n$ the measures $E_{n,\alpha}$ are principal so the only reason to consider them is for a more neat presentation. For each $n < \omega$, we shall consider an ordering $\leq_{E_n}$ over λ, as follows:

Definition 10.2.28. For each $n < \omega$, set

$$\leq_{E_n} := \{(\beta, \alpha) \in \lambda \times \lambda \mid \beta \leq \alpha, \wedge \exists f \in {}^{\kappa_n}\kappa_n \; j_n(f)(\alpha) = \beta\}.$$

It is routine to check that $\leq_{E_n}$ is reflexive, transitive and antisymmetric, hence $(\lambda, \leq_{E_n})$ is a partial order. The intuition behind the ordering $\leq_{E_n}$ is, provided $\beta \leq_{E_n} \alpha$, that one can represent the seed of $E_{n,\beta}$ by means of the seed of $E_{n,\alpha}$, and so the ultrapower $\mathrm{Ult}(V, E_{n,\beta})$ can be encoded within $\mathrm{Ult}(V, E_{n,\alpha})$. Formally speaking, and it is straightforward to check it, if $\beta \leq_{E_n} \alpha$ then $E_{n,\beta} \leq_{\mathrm{RK}} E_{n,\alpha}$ as witnessed by any function $f : \kappa_n \to \kappa_n$ such that $j_n(f)(\alpha) = \beta$.[17] In case $\beta \leq_{E_n} \alpha$, we shall fix in advance a witnessing map $\pi_{\alpha,\beta} : \kappa_n \to \kappa_n$. In the special case where $\alpha = \beta$, by convention $\pi_{\alpha,\alpha} =: \mathrm{id}$. Observe that $\leq_{E_n} \restriction (\kappa_n \times \kappa_n)$ is exactly the $\in$-order over κ_n so that when we refer to $\leq_{E_n}$ we will really be speaking about the restriction of this order to $\lambda \setminus \kappa_n$.

The following lemma lists some key features of the poset $(\lambda, \leq_{E_n})$:

Lemma 10.2.29. *Let $n < \omega$.*

1. *For every $a \in [\lambda]^{<\kappa_n}$, there are λ-many $\alpha < \lambda$ above $\sup(a)$ such that for every $\gamma, \beta \in x$:*
   - *$\gamma, \beta \leq_{E_n} \alpha$;*
   - *if $\gamma \leq_{E_n} \beta$, then $\{\nu \in \kappa_n \mid \pi_{\alpha,\gamma}(\nu) = \pi_{\beta,\gamma}(\pi_{\alpha,\beta}(\nu))\} \in E_{n,\alpha}$.*
2. *For all $\gamma < \beta$, $\gamma \leq_{E_n} \alpha$, and $\beta \leq_{E_n} \alpha$,*
$$\{\nu \in \kappa_n \mid \pi_{\alpha,\gamma}(\nu) < \pi_{\alpha,\beta}(\nu)\} \in E_{n,\alpha}.$$
3. *For all $\alpha, \beta < \lambda$ with $\beta \leq_{E_n} \alpha$, $\pi_{\alpha,\beta} : \kappa_n \to \kappa_n$ is a projection map, such that for each $A \in E_{n,\alpha}$, $\pi_{\alpha,\beta}[A] \in E_{n,\beta}$.*

Proof. All of this is proved in [Git10, §2], under the unnecessary hypothesis of GCH. Instead, let us define Δ to be the set of all infinite cardinals $\delta \leq \kappa_n$ satisfying $\delta^{<\mathrm{cof}(\delta)} = \delta$. Clearly, Δ is a closed set, and as κ_n is a measurable cardinal, $\max(\Delta) = \kappa_n$. It thus follows that we may recursively construct an enumeration $\langle a_\alpha \mid \alpha < \kappa_n \rangle$ of $[\kappa_n]^{<\kappa_n}$ such that, for every $\delta \in \Delta$:

[17] The notation $\leq_{\mathrm{RK}}$ stands for the usual Rudin-Keisler ordering (cf. [Git10, p. 1366]).

- $\{a_\alpha \mid \alpha < \delta\} \subseteq [\delta]^{<\delta}$;
- for each $a \in [\delta]^{<\mathrm{cof}(\delta)}$, $\{\alpha < \delta \mid a_\alpha = a\}$ has size δ.

Write $\langle a_\alpha \mid \alpha < j_n(\kappa_n)\rangle := j_n(\langle a_\alpha \mid \alpha < \kappa_n\rangle)$.

Claim 10.2.29.1. $\{a_\alpha \mid \alpha < \lambda\} = [\lambda]^{<\lambda}$ *and each element is enumerated cofinally often.*

Proof. As $V_{\lambda+1} \subseteq M_n$ and $j_n(\kappa_n) > \lambda = \lambda^{<\lambda}$, we get that $\lambda \in j(\Delta)$ and:

- $\{a_\alpha \mid \alpha < \lambda\} \subseteq [\lambda]^{<\lambda}$;
- for each $a \in [\lambda]^{<\lambda}$, $\{\alpha < \lambda \mid a_\alpha = a\}$ has size λ. □

The rest of the proof is now identical to that in [Git10, §2]. Specifically:

1. By Lemmas 2.1, 2.2 and 2.4 of [Git10].
2. This is Lemma 2.3 of [Git10].
3. This is obvious. □

Let us now revisit the EBPF and show that it can be interpreted as a Σ-Prikry triple $(\mathbb{P}, \ell, c)$. We shall first need the following building blocks:

Definition 10.2.30. Let $n < \omega$. Define $\mathbb{Q}_{n0}$, $\mathbb{Q}_{n1}$, and $\mathbb{Q}_n$ as follows:

$(0)_n$ $\mathbb{Q}_{n0} := (Q_{n0}, \leq_{n0})$, where elements of Q_{n0} are triples $p = (a^p, A^p, f^p)$ meeting the following requirements:

(a) f^p is a function from some $x \in [\lambda]^{\leq\kappa}$ to κ_n;

(b) $a^p \in [\lambda]^{<\kappa_n}$, and a^p contains a $\leq_{E_n}$-maximal element, which hereafter is denoted by $\mathrm{mc}(a^p)$;

(c) $\mathrm{dom}(f^p) \cap a^p = \emptyset$;

(d) $A^p \in E_{n,\mathrm{mc}(a^p)}$;

(e) if $\beta < \alpha$ is a pair in a, for all $\nu \in A^p$, $\pi_{\mathrm{mc}(a^p)\beta}(\nu) < \pi_{\mathrm{mc}(a^p)\alpha}(\nu)$;

(f) if $\alpha, \beta, \gamma \in a$ with $\gamma \leq_{E_n} \beta \leq_{E_n} \alpha$, then, for all $\nu \in \pi_{\mathrm{mc}(a^p)\alpha}[A]$, $\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu))$.

The ordering $\leq_{n0}$ is defined as follows: $(a^p, A^p, f^p) \leq_{n0} (b^q, B^q, g^q)$ iff the following are satisfied:

(i) $f^p \supseteq g^q$,

(ii) $a^p \supseteq b^q$,

(iii) $\pi_{\mathrm{mc}(a^p)\,\mathrm{mc}(b^q)}[A^p] \subseteq B^q$.

$(1)_n$ $\mathbb{Q}_{n1} := (Q_{n1}, \leq_{n1})$, where $Q_{n1} := \bigcup\{{}^x\kappa_n \mid x \in [\lambda]^{\leq\kappa}\}$ and $\leq_{n1} := \supseteq$.

$(2)_n$ $\mathbb{Q}_n := (Q_{n0} \cup Q_{n1}, \leq_n)$, where the ordering $\leq_n$ is defined as follows: for each $p, q \in Q_n$, $p \leq_n q$ iff

(a) either $p, q \in Q_{ni}$ for some $i \in 2$ and $p \leq_{ni} q$, or

(b) $p \in Q_{n1}$, $q \in Q_{n0}$ and, for some $\nu \in A$, $p \leq_{n1} q^\frown\langle\nu\rangle$, where

$$q^\frown\langle\nu\rangle := f^q \cup \{(\beta, \pi_{\mathrm{mc}(a^q),\beta}(\nu)) \mid \beta \in a^q\}.$$

Remark 10.2.31. By Lemma 10.2.29, Clauses (b)–(f) may indeed hold simultaneously. Also, observe that necessarily $\nu = \mathrm{mc}(a^q)$ in Clause $(2)_n$(b).

Definition 10.2.32 (EBPF)**.** The Extender Based Prikry Forcing is the poset $\mathbb{P} := (P, \leq)$ defined by the following clauses:

- Conditions in P are sequences $p = \langle p_n \mid n < \omega\rangle \in \prod_{n<\omega} Q_n$.
- For all $p, q \in P$, $p \leq q$ iff $p_n \leq_n q_n$ for every $n < \omega$.
- For all $p \in P$:
  - There is $n < \omega$ such that $p_n \in Q_{n0}$;
  - For every $n < \omega$, if $p_n \in Q_{n0}$, then $p_{n+1} \in Q_{n0}$ and $a^{p_n} \subseteq a^{p_{n+1}}$.

Definition 10.2.33. $\ell : P \to \omega$ is defined by letting for all $p = \langle p_n \mid n < \omega\rangle$:

$$\ell(p) := \min\{n < \omega \mid p_n \in Q_{n0}\}.$$

We already have $\mathbb{P}$ and ℓ; we shall soon see that $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$, so that we now need to introduce a map $c : P \to \mu$. As $\mu^{<\mu} = \mu$, we shall instead be defining a map $c : P \to H_\mu$. To define the function c we shall use the following theorem due to R. Engelking and M. Karlowicz:

Theorem 10.2.34 ([EK65])**.** *Let $\kappa \leq \mu \leq \lambda \leq 2^\mu$. Then the following conditions are equivalent:*

1. *$\mu^{<\kappa} = \mu$;*
2. *there exists a sequence $\langle e^i \mid i < \mu\rangle$ of functions from λ to μ with the property that for every $x \in [\lambda]^{<\kappa}$ and every function $e\colon x \to \mu$, there is some $i < \mu$ such that $e \subseteq e^i$.*

Remark 10.2.35. The above result is useful to prove, among other things, that the product of $2^{\aleph_0}$-many separable topological spaces endowed with the product topology is also separable.

As $\mu^\kappa = \mu$ and $2^\mu = \lambda$ we may appeal to Theorem 10.2.34 to fix a sequence $\langle e^i \mid i < \mu\rangle$ of functions from λ to μ with the property that, for every function $e : x \to \mu$ with $x \in [\lambda]^{\leq\kappa}$, there exists $i < \mu$ such that $e \subseteq e^i$.

Definition 10.2.36. For every function $f \in \bigcup_{n<\omega} Q_{n1}$, let

$$i(f) := \min\{i < \mu \mid f \subseteq e^i\}.$$

For every $p = (a, A, f) \in \bigcup_{n<\omega} Q_{n0}$, let $i(p)$ be the least $i < \mu$ such that:

- for all $\alpha \in a$, $e^i(\alpha) = 0$;
- for all $\alpha \in \mathrm{dom}(f)$, $e^i(\alpha) = f(\alpha) + 1$.

Finally, for every condition $p = \langle p_n \mid n < \omega\rangle$ in P, let

$$c(p) := \ell(p)^\frown\langle i(p_n) \mid n < \omega\rangle.$$

Before we turn to the analysis of $(\mathbb{P}, \ell, c)$, let us recall the following motivating theorem.

Theorem 10.2.37 (Gitik-Magidor, [GM94])**.** *$\mathbb{P}$ is cofinality-preserving, adds no new bounded subsets of κ, and forces 2^κ to be λ.*

We now begin verifying that $(\mathbb{P}, \ell, c)$ is indeed Σ-Prikry. The following fact can be proved as Lemma 10.2.18. For more details see [Git10, Lemma 2.15].

Fact 10.2.38. *Let $p, q \in P$ with $\ell(p) = \ell(q)$. Then p and q are $\leq^0$-compatible iff the two holds:*

- *for every $n < \omega$, $f^p_n \cup f^q_n$ is a function;*
- *for every $n \geq \ell(p)$, $\mathrm{dom}(f^p_n) \cup \mathrm{dom}(f^q_n)$ is disjoint from $a^p_n \cup a^q_n$.*

Clause (1) can be verified in the same way as in Proposition 10.2.17.

Lemma 10.2.39. *$(\mathbb{P}, \ell)$ is a graded poset.*

Now, we move forward to verify Clause (2).

Lemma 10.2.40. *Let $n < \omega$. $\mathbb{P}_n := (P_n \cup \{\mathbb{1}\}, \leq)$ is κ_n-directed-closed.*

Proof. Let $D \in [P_n \cup \{\mathbb{1}\}]^{<\kappa_n}$ be a $\leq^0$-directed set. Say, $D = \{p^\alpha \mid \alpha < \theta\}$, for some cardinal $\theta < \kappa_n$. By Fact 10.2.38, for each $m \geq n$, and all $\alpha, \beta < \theta$, $\mathrm{dom}(a^{p^\alpha}_m) \cap \mathrm{dom}(f^{p^\beta}_m) = \emptyset$. Define by recursion $\langle (b_m, B_m) \mid m \geq n\rangle$, where $b_m \in [\lambda]^{<\kappa_m}$ and $B_m \in E_{m\,\mathrm{mc}(b_m)}$, as follows:

1. Let $m \geq n$ and assume that $\langle b_i \mid n \leq i < m\rangle$ has been defined. Set $b^*_m := (\bigcup_{n\leq i<m} b_i) \cup \bigcup_{\alpha<\theta} a^{p^\alpha}_m$. Since $n \leq m$ and $\theta < \kappa_n$, $b^*_m \in [\lambda]^{<\kappa_m}$. By Lemma 10.2.29(1) we may find $\delta_m \in \lambda \setminus \bigcup_{j<\omega,\alpha<\theta} \mathrm{dom}(f^{p^\alpha}_j)$ large enough such that for every $\gamma, \beta \in b^*_m$,
   - $\gamma, \beta \leq_{E_n} \delta_m$;
   - if $\gamma \leq_{E_n} \beta$, then $\{\nu \in \kappa_n \mid \pi_{\delta_m,\gamma}(\nu) = \pi_{\beta,\gamma}(\pi_{\delta_m,\beta}(\nu))\} \in E_{n,\delta_m}$.

   Define $b_m := b^*_m \cup \{\delta_m\}$.
2. Again, appeal to Lemma 10.2.29 to find $B_m \in E_{m\,\mathrm{mc}(b_m)}$ with
$$B_m \subseteq \bigcap_{\alpha<\theta} \pi^{-1}_{\mathrm{mc}(b_m),\mathrm{mc}(a^{p^\alpha}_m)}[A^{p^\alpha}_m],$$
   and B_m witnessing Clauses $(0)_m$(e) and $(0)_m$(f) of Definition 10.2.30.

At the end of this recursive procedure, define $r := \langle r_m \mid m < \omega\rangle$, where

$$r_m := \begin{cases} \bigcup_{\alpha<\theta} f^{p^\alpha}_m, & \text{if } m < n,\\ (b_m, B_m, \bigcup_{\alpha<\theta} f^{p^\alpha}_m), & \text{otherwise,}\end{cases}$$

Now, it is not hard to check that, for each $\alpha < \theta$, $r \leq_0 p^\alpha$. □

Next, we verify Clause (3) of Definition 10.1.3.

Lemma 10.2.41. *Suppose that $p = \langle p_n \mid n < \omega\rangle$ and $q = \langle q_n \mid n < \omega\rangle$ are two conditions, and $c(p) = c(q)$. Then $P^p_0 \cap P^q_0$ is nonempty.*

Proof. Let $\ell^\frown\langle i_n \mid n < \omega\rangle := c(p)$.

▶ For all $n < \ell$, it follows from $c(p) = c(q)$ that $n < \ell(p) = \ell(q)$ and $p_n \cup q_n \subseteq e^{i_n}$, so that $p_n \cup q_n$ is a function.

▶ For all $n \geq \ell$, it follows from $i(p_n) = i_n = i(q_n)$ that $e^{i_n}[a^p_n \cup a^q_n] = \{0\}$, $e^{i_n}[\mathrm{dom}(f^p_n) \cup \mathrm{dom}(f^q_n)] \cap \{0\} = \emptyset$ and $\mathrm{dom}(f^p_n \cap f^q_n) = \mathrm{dom}(f^p_n) \cap \mathrm{dom}(f^q_n)$. So $f^p_n \cup f^q_n$ is a function and $\mathrm{dom}(f^p_n) \cap a^q_n = \mathrm{dom}(f^q_n) \cap a^p_n = \emptyset$.

It thus follows from Fact 10.2.38 that $P^p_0 \cap P^q_0 \neq \emptyset$. □

The following convention will be applied hereafter:

Convention 10.2.42. For every sequence $\{A_k\}_{i\leq k\leq j}$ such that each A_k is a subset of κ_k, we shall identify $\prod^j_{k=i} A_k$ with its subset consisting only of the sequences that are moreover increasing. In addition, for each $p \in P$, we shall refer to $\langle f^p_n \mid n < \ell(p)\rangle$, $\langle f^p_n \mid \ell(p) \leq n < \omega\rangle$ and $\langle a^p_n \mid \ell(p) \leq n < \omega\rangle$, as, respectively, the *stem*, the *f-part* and the *a-part* of p.

Definition 10.2.43. Let $p = \langle f^p_n \mid n < \ell(p)\rangle^\frown\langle(a^p_n, A^p_n, f^p_n) \mid \ell(p) \leq n < \omega\rangle$ in P. Define:

- $p^\frown\emptyset := p$;
- For every $\nu \in A^p_{\ell(p)}$, $p^\frown\langle\nu\rangle := q$ where $q = \langle q_n \mid n < \omega\rangle$ is the unique sequence defined as follows:

$$q_n := \begin{cases} p_n{}^\frown\langle\nu\rangle, & \text{if } n = \ell(p); \\ p_n, & \text{otherwise.} \end{cases}$$

- By recursion, for all $m \geq \ell(p)$ and $\vec\nu = \langle \nu_{\ell(p)}, \ldots, \nu_m, \nu_{m+1}\rangle \in \prod_{n=\ell(p)}^{m+1} A^p_n$, we define $p^\frown\vec\nu := (p^\frown\vec\nu \restriction (m+1))^\frown\langle\nu_{m+1}\rangle$.

Using the definition of the ordering one can prove the following easy fact:

Fact 10.2.44. If $p = \langle f^p_n \mid n < \ell(p)\rangle^\frown\langle(a^p_n, A^p_n, f^p_n) \mid \ell(p) \leq n < \omega\rangle$ in P and $q \leq^m p$, then there exists a unique $\vec\nu \in \prod_{n=\ell(p)}^{\ell(p)+m-1} A^p_n$ such that $q \leq^0 p^\frown\vec\nu$. In fact, $\vec\nu = \langle f^q_i(\mathrm{mc}(a^p_i)) \mid \ell(p) \leq i < \ell(q)\rangle$.

By the above fact, given $n, m < \omega$ and $q \leq^{n+m} p$, let $\vec\nu$ be such that $q \leq^0 p^\frown\vec\nu$, and set $m(p,q) := p^\frown(\vec\nu \restriction n)$. We will soon argue that $m(p,q)$ indeed coincides with the greatest element of $\{r \in P^p_n \mid q \leq^m r\}$. For every $k < \omega$, set $W_k(p) := \{p^\frown\vec\nu \mid \vec\nu \in \prod_{n=\ell(p)}^{\ell(p)+k-1} A^p_n\}$. Next, we address Clause (4).

Lemma 10.2.45. *Let $p \in P$, $n, m < \omega$ and $q \in P^p_{n+m}$. The set $R := \{r \in P^p_n \mid q \leq^m r\}$ contains a greatest element.*

Proof. By Fact 10.2.44, we may let $\vec\nu \in \prod_{k=\ell(p)}^{\ell(p)+n+m-1} A^p_k$ be such that $q \leq^0 p^\frown\vec\nu$. It is routine to check that $p^\frown(\vec\nu \restriction n)$ is the greatest element of R. □

Now, to Clause (5).

Lemma 10.2.46. *For all $p \in P$, $W(p) := \{w(p,q) \mid q \leq p\}$ has size κ.*

Proof. Let $p \in P$, $n < \omega$ and $q \in P^p_n$. By Fact 10.2.44, we have that $|W_n(p)| < \kappa_{n+\ell(p)}$, hence $|W(p)| = \sup_{n<\omega} |W_n(p)| = \kappa < \mu$. □

Let us now proceed with the verification of Clause (6).

Lemma 10.2.47. *Let $p' \leq p$ in P. Then $q \mapsto w(p,q)$ forms an order-preserving map from $W(p')$ to $W(p)$*

Proof. By Fact 10.2.44, let $\vec\sigma \in \prod_{\ell(p)\leq k<\ell(p')} A^p_k$ be the unique sequence such that $p' \leq^0 p^\frown\vec\sigma$. Let $q, r \in W(p')$ and assume that $q \leq r$. By the proof of Lemma 10.2.45, there are $\vec\nu, \vec\mu$ be such that $q = p'^\frown\vec\nu$ and $r = p'^\frown\vec\mu$. Observe that $\vec\nu$ must end-extend $\vec\mu$, and so $w(p,q) = p^\frown\vec\sigma^\frown\vec\nu \leq p^\frown\vec\sigma^\frown\vec\mu = w(p,r)$. □

Our next task is proving that $(\mathbb{P}, \ell, c)$ satisfies the Complete Prikry Property, that is, Clause (7) of Definition 10.1.3. To this end, we shall need to consider the following auxiliary concept:

Definition 10.2.48. Given $m < \omega$ and two conditions $p, q \in P$, say

- $p = \langle f^p_n \mid n < \ell(p)\rangle^\frown\langle (a^p_n, A^p_n, f^p_n) \mid \ell(p) \leq n < \omega\rangle$;
- $q = \langle f^q_n \mid n < \ell(q)\rangle^\frown\langle (a^q_n, A^q_n, f^q_n) \mid \ell(q) \leq n < \omega\rangle$,

we shall write $q \sqsubseteq^m p$ iff $q \leq^0 p$ and, for all $n < \omega$,

$$\ell(p) \leq n \leq m \implies (a^p_n = a^q_n \text{ and } A^p_n = A^q_n).$$

Definition 10.2.49. For an ordinal $\delta \leq \kappa$, a sequence of conditions $\langle p^\alpha \mid \alpha < \delta\rangle$ is said to be a *fusion sequence* iff, for every pair $\beta < \alpha < \delta$, $p^\alpha \sqsubseteq^{m(\beta)+1} p^\beta$, where $m(\beta) := \sup\{m < \omega \mid \kappa_m \leq \beta\}$.[18]

Lemma 10.2.50 (Fusion Lemma)**.** *For every ordinal $\delta \leq \kappa$ and every fusion sequence $\langle p^\alpha \mid \alpha < \delta\rangle$, there exists a condition p' such that, for all $\beta < \delta$, $p' \sqsubseteq^{m(\beta)+1} p^\beta$.*

Proof. This is somehow a standard fact, so we just briefly go over the main points of the proof. Let $\langle p^\alpha \mid \alpha < \delta\rangle$ be an arbitrary fusion sequence and set ℓ for the common length of its conditions. Assume $0 < \delta \leq \kappa$.

▶ If δ is a successor ordinal, say $\delta := \beta+1$, then, for all $\gamma \leq \beta$, $p^\beta \sqsubseteq^{m(\gamma)+1} p^\gamma$. Setting $p' := p^\beta$ we get the desired condition.

▶ If δ is a limit ordinal, define $p' := \langle p'_n \mid n < \omega\rangle$ as follows:

$$p'_n := \begin{cases} \bigcup_{\alpha<\delta} f^{p^\alpha}_n, & \text{if } n < \ell; \\ (a^\beta_n, A^\beta_n, \bigcup_{\alpha<\delta} f^{p^\alpha}_n), & \text{if } n \geq \ell \text{ and } \exists \beta < \delta (n \leq m(\beta)+1); \\ (b_n, B_n, \bigcup_{\alpha<\delta} f^{p^\alpha}_n), & \text{if } n \geq \ell \text{ and } \forall \beta < \delta (m(\beta)+1 < n), \end{cases}$$

where (b_n, B_n) are constructed as in Lemma 10.2.40. It is routine to check that p' is as desired. □

The upcoming argument follows the proof of [Git10, Lemma 2.18], simply verifying that it works for merely 0-open sets, instead of open and dense sets. To clarify the key ideas involved in the proof, we shall split it into two, as follows.

Lemma 10.2.51 (Diagonalization)**.** *Let $p \in P$ and U be a 0-open subset of P. Then there is $q \in P^p_0$ such that, for every $r \in P^q \cap U$, $w(q, r) \in U$.*

Proof. Fix a bijection $h : \kappa \to {}^{<\omega}\kappa$ such that, for every $n < \omega$, $h[\kappa_n] = {}^{<\omega}\kappa_n$. We shall first define by recursion a fusion sequence $\langle p^\alpha \mid \alpha < \kappa\rangle$.

Set $\ell := \ell(p)$ and $p_0 := p$. Next, assume that for some $\alpha < \kappa$, $\langle p^\beta \mid \beta < \alpha\rangle$ has already been defined and let us show how to construct p^α. By Lemma 10.2.50, fix a condition $\tilde{p}^\alpha$ such that, for all $\beta < \alpha$, $\tilde{p}^\alpha \sqsubseteq^{m(\beta)+1} p^\beta$. Let $\vec{\nu} := h(\alpha)$. If $\tilde{p}^{\alpha\frown}\vec{\nu}$ is not well-defined, that is, $\vec{\nu} \notin \prod_{k=\ell}^{\ell+|\vec{\nu}|-1} A^{\tilde{p}^\alpha}_k$, then set $p^\alpha := \tilde{p}^\alpha$. Otherwise, set $q^\alpha := \tilde{p}^{\alpha\frown}\vec{\nu}$. There are two cases to consider:

[18]By convention, $\sup(\emptyset) := 0$.

(a) If $U \cap P_0^{q_\alpha}$ is empty or $\ell + |\vec{\nu}| - 1 < m(\alpha) + 1$, then again set $p^\alpha := \tilde{p}^\alpha$.

(b) Otherwise, pick $r^\alpha \in U \cap P_0^{q_\alpha}$, and define $p^\alpha := \langle p_n^\alpha \mid n < \omega \rangle$ by letting, for all $n < \omega$,

$$p_n^\alpha := \begin{cases} (a_n^{\tilde{p}^\alpha}, A_n^{\tilde{p}^\alpha}, f_n^{r^\alpha} \restriction (\mathrm{dom}(f_n^{r^\alpha}) \setminus a_n^{\tilde{p}^\alpha})), & \text{if } \ell \leq n \leq \ell + |\vec{\nu}| - 1; \\ r_n^\alpha, & \text{otherwise.} \end{cases}$$

Since $m(\alpha) + 1 \leq \ell + |\vec{\nu}| - 1$, $p^\alpha \sqsubseteq^{m(\alpha)+1} \tilde{p}^\alpha$, hence $p^\alpha \sqsubseteq^{m(\beta)+1} p^\beta$ for all $\beta < \alpha$.

Note that if p^α was defined according to case (b), then $p^\alpha {}^\frown h(\alpha) = r^\alpha \in U$. Observe that $\langle p^\alpha \mid \alpha < \kappa \rangle$ is a fusion sequence and thus, by appealing to Lemma 10.2.50, we may pick a condition q which is $\leq^0$-below all of them. By shrinking further, we may assume that, for all $n \geq \ell$, $A_n^q \cap \kappa_{n-1} = \emptyset$. Here, by convention, $\kappa_{-1} := 0$.

Claim 10.2.51.1. *q witnesses the conclusion of the lemma.*

Proof of claim. Let $r \in P^q \cap U$ and α be such that $r \leq^0 q {}^\frown h(\alpha)$. Aiming for a contradiction, assume that p^α has been defined according to case (a). Observe that by our refinement of A_n^q, $h(\alpha) \in \kappa_{\ell+|h(\alpha)|-1}^{<\omega}$. Since $h[\kappa_n] \subseteq \kappa_n^{<\omega}$, $\alpha < \kappa_{\ell+|h(\alpha)|-1}$, which yields $m(\alpha)+1 \leq \ell+|h(\alpha)|-1$ and thus a contradiction with our initial assumption.

Now, it is clear that $p^\alpha {}^\frown h(\alpha)$ is in the 0-open set U and so $w(q, r) = q {}^\frown h(\alpha) \in U$, as well. □

□

We are now ready to complete the verification of Clause (7) for the EBPF.

Lemma 10.2.52. *Let $p \in P$ and U be a 0-open subset of P. For every $n < \omega$, there is $q^* \leq^0 p$, such that either $P_n^{q^*} \cap U = \emptyset$ or $P_n^{q^*} \subseteq U$.*

Proof. Let $q \leq^0 p$ be given by Lemma 10.2.51 with respect to p and U. Set $\ell := \ell(q)$. We want to recursively define a $\leq^0$-decreasing sequence of conditions $\langle q^n \mid n < \omega \rangle$ such that

1. $q^0 \leq^0 q$,
2. for each $n < \omega$, $q^n := \langle q_k^n \mid k < \omega \rangle$, where

$$q_k^n := \begin{cases} (a_k^q, B_k^n, f_k^q), & \text{if } k \in [\ell, \ell + n); \\ q_k, & \text{otherwise;} \end{cases}$$

3. for each $n < \omega$,

$$W_n(q^n) \cap U \neq \emptyset \implies W_n(q^n) \subseteq U.$$

Namely, all the q^n's have the same stem, *a-parts* and *f-parts*, and we only shrink the measure one sets so that for each n, either all weak n-step extensions of q^n are in U, or none of them are.

By convention, $q^{-1} := q$. Now assume that q^{n-1} has already been defined according to (1)-(3). We shall now exhibit a recursive procedure that allows to define q^n. We need to fix some notation beforehand: For each $i \in [1,n]$, set $j_n(i) := n + \ell - i$, $S_n(i) := \prod_{k=\ell}^{j_n(i)-1} A_k^{q^{n-1}}$ and $T_n(i) := \prod_{k=j_n(i)+1}^{j_n(1)} B_k^n$, where $\langle B_k^n \mid k \in [\ell, \ell+n) \rangle$ is a sequence of large sets which we will define recursively. For each $i \in [1,n]$ the recursion goes as follows:

Case (a): Assume $S_n(i) \neq \emptyset$ and distinguish the next two subcases:

1. Suppose $T_n(i) \neq \emptyset$: For each $\vec{\nu} \in S_n(i)$, set
$$X^n_{j_n(i),\vec{\nu}} := \{\vartheta \in A^{q^{n-1}}_{j_n(i)} \mid \forall \vec{\eta} \in T_n(i)\, (q^{n-1} {}^\frown (\vec{\nu} {}^\frown \langle \vartheta \rangle {}^\frown \vec{\eta}) \in U)\}.$$
If this set lies in the corresponding measure, set $B^n_{j_n(i),\vec{\nu}} := X^n_{j_n(i),\vec{\nu}}$. Otherwise, $B^n_{j_n(i),\vec{\nu}} := A^{q^{n-1}}_{j_n(i)} \setminus B^n_{j_n(i),\vec{\nu}}$. Define $B^n_{j_n(i)} := \bigcap_{\vec{\nu} \in S_n(i)} B^n_{j_n(i),\vec{\nu}}$. Clearly, $B^n_{j_n(i)} \subseteq A^{q^{n-1}}_{j_n(i)}$ and $B^n_{j_n(i)} \in U_{j_n(i),\mathrm{mc}(a^q_{j_n(i)})}$, by the $\kappa_{j_n(i)}$-completeness of this measure.[19]

2. Suppose $T_n(i) = \emptyset$: For each $\vec{\nu} \in S_n(i)$, set
$$X^n_{j_n(i),\vec{\nu}} := \{\vartheta \in A^{q^{n-1}}_{j_n(i)} \mid q^{n-1} {}^\frown (\vec{\nu} {}^\frown \langle \vartheta \rangle) \in U\}.$$
Now define $B^n_{j_n(i),\vec{\nu}}$ and $B^n_{j_n(i)}$ as done in case (a)(1).

Case (b): Assume $S_n(i) = \emptyset$ and distinguish the next two subcases:

1. Suppose $T_n(i) \neq \emptyset$: Set
$$X^n_{j_n(i)} := \{\vartheta \in A^{q^{n-1}}_{j_n(i)} \mid \forall \vec{\eta} \in T_n(i)\, (q^{n-1} {}^\frown (\langle \vartheta \rangle {}^\frown \vec{\eta}) \in U)\}.$$
If this latter lies in the corresponding measure, set $B^n_{j_n(i)} := X^m_{j_n(i)}$. Otherwise, $B^n_{j_n(i)} := A^{q^{n-1}}_{j_n(i)} \setminus X^n_{j_n(i)}$.

2. Suppose $T_n(i) = \emptyset$: Set
$$X^m_{j_n(i)} := \{\vartheta \in A^{q^{n-1}}_{j_n(i)} \mid q^{n-1} {}^\frown \langle \vartheta \rangle \in U\}.$$
Now define $B^n_{j_n(i)}$ as done in case (b)(1).

[19] Here we are implicitly using that $|S_n(i)| < \kappa_{j_n(i)}$.

This recursion produces a family of sets $\langle B^n_k \mid k \in [\ell, \ell+n)\rangle$ such that $B^n_k \subseteq A^{q^{n-1}}_k$ and $B^n_k \in U_{k,\mathrm{mc}(a^q_k)}$. Now define $q^n := \langle q^n_k \mid k < \omega\rangle$ as

$$q^n_k := \begin{cases} (a^q_k, B^n_k, f^q_k), & \text{if } k \in [\ell, \ell+n);\\ q_k, & \text{otherwise.}\end{cases}$$

Clearly q^n satisfies (1) and (2). Thus, we are left with verifying (3). For this, let us bear in mind that $B^n_k =: A^{q^n}_k$, for $k \in [\ell, \ell+n)$. Also, it is key to notice that $q^{n\frown}\vec\nu = q^{n-1\frown}\vec\nu$, for each $\vec\nu \in \prod_{k=\ell}^{\ell+n-1} A^{q^n}_k$.

Claim 10.2.52.1. *q^n satisfies (3) of the above.*

Proof of claim. Observe that if $n = 0$, $W_0(q^0) = \{q^0\}$ and the result follows. Thus, we are left with verifying that (3) holds when $n \geq 1$. We will split the proof into two cases: $n = 1$ and $n \geq 2$.

Subclaim 10.2.52.1.1. *q^1 satisfies (3) of the above.*

Proof of subclaim. Assume $W_1(q^1) \cap U \neq \emptyset$ and let $\nu \in A^{q^1}_\ell$ be such that $q^{1\frown}\langle\nu\rangle$ witnesses this. It is not hard to check that $A^{q^1}_\ell$ is defined according to Case (b)(2). Observe that ν witnesses $A^{q^1}_\ell \cap X^1_\ell \neq \emptyset$, hence $A^{q^1}_\ell = X^1_\ell$. Thus, for all $\eta \in A^{q^1}_\ell$, $q^{1\frown}\langle\eta\rangle \in U$, which yields $W_1(q^1) \subseteq U$. □

Subclaim 10.2.52.1.2. *q^n satisfies (3) of the above, provided $n \geq 2$.*

Proof of subclaim. Assume $W_n(q^n) \cap U \neq \emptyset$ and let $\vec\nu := \langle \nu_\ell, \dots, \nu_{\ell+n-1}\rangle \in \prod_{k=\ell}^{\ell+n-1} A^{q^n}_k$ be a witness for it. For each $i \in [1, n)$, set $\vec\nu_i := \langle \nu_\ell, \dots, \nu_{j_n(i+1)}\rangle$. and $\vec\nu_n := \emptyset$. If $i = 1$, it is not hard to check that $A^{q^n}_{\ell+n-1}$ has been defined according to case (a)(2). Moreover, observe that $A^{q^n}_{\ell+n-1} \cap X^n_{\ell+n-1,\vec\nu_1} \neq \emptyset$, as witnessed by $\nu_{\ell+n-1}$. Thereby, $q^{n\frown}(\vec\nu_1{}^\frown\langle\eta\rangle) \in U$, for all $\eta \in A^{q^n}_{\ell+n-1}$.

Now assume recursively that for some $i \in [1, n)$, $q^{n\frown}(\vec\nu_i{}^\frown\vec\eta) \in U$, for each $\vec\eta \in \prod_{k=j_n(i)}^{\ell+n-1} A^{q^n}_k$. We want to derive from this that $q^{n\frown}(\vec\nu_{i+1}{}^\frown\vec\eta) \in U$, for each $\vec\eta \in \prod_{k=j_n(i+1)}^{\ell+n-1} A^{q^n}_k$. Here we need to distinguish two more subcases:

► Assume $i+1 \in [2, n)$. Then, it is the case that $j_n(i+1) - 1 \geq \ell$ and $j_n(i+1) + 1 \leq j_n(1)$, hence we fall into case (a)(1). Observe that $\vec\nu(j_n(i))$ witnesses $X^n_{j_n(i),\vec\nu_i} \cap A^{q^n}_{j_n(i)} \neq \emptyset$. This latter fact being a consequence of the recursion hypothesis, $q^{n\frown}\vec\nu \in U$ and $T_n(i) := \prod_{k=j_n(i)}^{\ell+n-1} A^{q^n}_k$. Combining this with the recursion hypothesis, $q^{n\frown}(\vec\nu_{i+1}{}^\frown\vec\eta) \in U$, for each $\vec\eta \in \prod_{k=j_n(i+1)}^{\ell+n-1} A^{q^n}_k$.

► Assume $i+1 = n$. Then, it is the case that $j_n(n)-1 < \ell$ and $j_n(n)+1 \leq j_n(1)$, hence we fall into case (b)(1). Arguing as before, $\vec\nu(j_n(n))$ witnesses $X^n_{j_n(n)} \cap A^{q^n}_{j_n(n)} \neq \emptyset$. Again, this yields $q^{n\frown}\vec\eta \in U$, for all $\vec\eta \in \prod_{k=j_n(n)}^{n+\ell-1} A^{q_n}_k$. □

□

Appealing to Lemma 10.2.50 we may let q^* be a $\leq^0$-extension of the sequence $\langle q^n \mid n < \omega\rangle$. We claim that q^* is as desired: Let $n < \omega$ and $r \in P_n^{q^*} \cap U$. By Lemma 10.2.51 and 0-openess, $w(q^n, r) \in U$. Since q^n witnesses (3), $W_n(q^n) \subseteq U$. Again, by the 0-openess of U, $P_n^{q^n} \subseteq U$, hence $P_n^{q^*} \subseteq U$, which yields the desired result. $\square$

Corollary 10.2.53. $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$.

Proof. Recall that $\mu = \kappa^+$ and κ is singular. So, if $\mathbb{1}_{\mathbb{P}} \not\Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$, then there exists a condition p in $\mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \mathrm{cof}(\mu) < \kappa$. Now, by Lemmas 10.2.39, 10.2.40, 10.2.45 and 10.2.52, we may appeal to Fact 10.1.10(2), and infer the existence of $p' \leq p$ with $|W(p')| \geq \mu$, contradicting Lemma 10.2.46. $\square$

Altogether, we have established the following:

Corollary 10.2.54. *$(\mathbb{P}, \ell, c)$ is Σ-Prikry.* $\square$

10.2.6 Lottery sum of Σ-Prikry forcings

Suppose that $\Sigma = \langle \kappa_n \mid n < \omega\rangle$ is a non-decreasing sequence of regular uncountable cardinals converging to some cardinal κ. Let μ be a cardinal and $\langle(\mathbb{Q}_i, \ell_i, c_i) \mid i < \nu\rangle$ be a sequence of Σ-Prikry notions of forcing such that $\nu < \mu$. Furthermore, assume that for all $i < \nu$, $\mathbb{1}_{\mathbb{Q}_i} \Vdash_{\mathbb{Q}_i} \check{\mu} = \kappa^+$.

Define $P := \{(i,p) \mid i < \nu, p \in Q_i\} \cup \{\emptyset\}$ and an ordering $\leq$, letting $(i,p) \leq (j,q)$ iff $i = j$ and $p \leq_{\mathbb{Q}_i} q$, as well as setting $\emptyset \leq x$ for any $x \in P$. Set $\mathbb{P} := (P, \leq)$ and note that $\mathbb{1}_{\mathbb{P}} = \emptyset$ and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{Q}_i} \check{\mu} = \kappa^+$. Now, define $\ell : P \to \omega$ by letting $\ell(\emptyset) := 0$ and $\ell(i,p) := \ell_i(p)$. Finally, define $c : P \to \mu \times \mu$ by letting $c(\emptyset) := (0,0)$ and $c(i,p) := (i, c_i(p))$.

Proposition 10.2.55. *$(\mathbb{P}, \ell, c)$ is Σ-Prikry.*

Proof. We go over the clauses of Definition 10.1.3.

1. Let $(i,q) \leq (j,p)$. By definition, $i = j$ and $q \leq_{\mathbb{Q}_i} p$. Since $(\mathbb{Q}_i, \ell_i, c_i)$ is Σ-Prikry it follows that $\ell(i,p) = \ell(p) \leq \ell(q) = \ell(i,q)$, as wanted. Similarly one can prove $P_1^p \neq \emptyset$, for each $p \in P$.

2. Let $D \in [P_n \cup \{\emptyset\}]^{<\kappa_n}$ be directed. Find $i < \nu$ such that $D \setminus \{\emptyset\} \subseteq \{i\} \times (Q_i)_n$. Now, as $(\mathbb{Q}_i, \ell_i, c_i)$ is Σ-Prikry, there exists a lower bound p for $\{q \in (Q_i)_n \mid (i,q) \in D\}$. Evidently, (i,p) is a lower bound for D.

3. Follows from the fact that, for all $i < \nu$, $(\mathbb{Q}_i, \ell_i, c_i)$ being Σ-Prikry.

(4)-(5) Let $x \in P$ and $(i,q) \in P^x$. If $x = \emptyset$ it is not hard to check that $w(\emptyset, \emptyset) = \emptyset$ and that, more generally, $m(\emptyset, (i,q)) = (i, m(\mathbb{1}_{\mathbb{Q}_i}, q))$. Hence, $W(\emptyset) \subseteq \{\emptyset\} \cup \bigcup_{i<\nu} W(\mathbb{1}_{\mathbb{Q}_i})$. Analogously, if $x \neq \emptyset$, say $x = (i,p)$, $m((i,p),(i,q)) = (i, m(p,q))$. In particular, $W_n(i,p) = \{i\} \times W_n(p)$. Since $\nu < \mu$, this yields clauses (4) and (5).

(6) This is obvious.

(7) Let $U \subseteq P$ be a 0-open set and fix $x \in P$ and $n < \omega$. If $x \neq \emptyset$, denote $(i, p) := x$. Otherwise, let $(i, p) := (0, \mathbb{1}_{\mathbb{P}_0})$. In both cases, $(i, p) \leq^0 x$. Now, it is not hard to check that $U_i := \{q \in Q_i \mid (i, q) \in U\}$ is also 0-open. Since $(\mathbb{Q}_i, \ell_i, c_i)$ is Σ-Prikry we may find $q \in (Q_i)_0^p$ such that either $(Q_i)_n^q \subseteq U_i$ or $(Q_i)_n^q \cap U_i = \emptyset$. Set $y := (i, q)$. Clearly $y \leq^0 x$. If $P_n^q \cap U \neq \emptyset$ then clearly $(Q_i)_n^q \cap U_i \neq \emptyset$, hence $(Q_i)_n^q \subseteq U_i$, and thus $P_n^q \subseteq U$. □

CHAPTER 11

Forking projections

In this chapter, we introduce the notion of *forking projection* which will play a key role in Chapter 13.

Definition 11.0.1. Suppose that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is a Σ-Prikry triple, $\mathbb{A} = (A, \trianglelefteq)$ is a notion of forcing, and $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ are functions with $\mathrm{dom}(\ell_{\mathbb{A}}) = \mathrm{dom}(c_{\mathbb{A}}) = A$.

A pair of functions $(\pitchfork, \pi)$ is said to be a *forking projection* from $(\mathbb{A}, \ell_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$ iff all of the following hold:

1. π is a projection from $\mathbb{A}$ onto $\mathbb{P}$, and $\ell_{\mathbb{A}} = \ell_{\mathbb{P}} \circ \pi$;
2. for all $a \in A$, $\pitchfork(a)$ is an order-preserving function from $(\mathbb{P} \downarrow \pi(a), \leq)$ to $(\mathbb{A} \downarrow a, \trianglelefteq)$;
3. for all $p \in P$, $\{a \in A \mid \pi(a) = p\}$ admits a greatest element, which we denote by $\lceil p \rceil^{\mathbb{A}}$;
4. for all $n, m < \omega$ and $b \trianglelefteq^{n+m} a$, $m(a, b)$ exists and satisfies:
$$m(a, b) = \pitchfork(a)(m(\pi(a), \pi(b)));$$
5. for all $a \in A$ and $r \leq \pi(a)$, $\pi(\pitchfork(a)(r)) = r$;
6. for all $a \in A$ and $r \leq \pi(a)$, $a = \lceil \pi(a) \rceil^{\mathbb{A}}$ iff $\pitchfork(a)(r) = \lceil r \rceil^{\mathbb{A}}$;
7. for all $a \in A$, $a' \trianglelefteq^0 a$ and $r \leq^0 \pi(a')$, $\pitchfork(a')(r) \trianglelefteq \pitchfork(a)(r)$.

The pair $(\pitchfork, \pi)$ is said to be a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ iff, in addition to all of the above, the following holds:

8. for all $a, a' \in A$, if $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$, then $c_{\mathbb{P}}(\pi(a)) = c_{\mathbb{P}}(\pi(a'))$ and, for all $r \in P_0^{\pi(a)} \cap P_0^{\pi(a')}$, $\pitchfork(a)(r) = \pitchfork(a')(r)$.

Example 11.0.2. Suppose that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is any Σ-Prikry triple and that $\mathbb{Q}$ is any notion of forcing with a greatest element $\mathbb{1}_{\mathbb{Q}}$. Let $\mathbb{A} = (A, \trianglelefteq)$ be the product forcing $\mathbb{P} \times \mathbb{Q}$. Define $\pi : A \to P$ via $\pi(p, q) := p$, and, for each $a = (p, q)$ in A, define $\pitchfork(a) : \mathbb{P} \downarrow p \to \mathbb{A} \downarrow a$ via $\pitchfork(a)(r) := (r, q)$. Set $\ell_{\mathbb{A}} := \ell_{\mathbb{P}} \circ \pi$. Define $c_{\mathbb{A}} : A \to \operatorname{ran}(c_{\mathbb{P}}) \times Q$ via $c_{\mathbb{A}}(p, q) := (c_{\mathbb{P}}(p), q)$. Then $\lceil p \rceil^{\mathbb{A}} = (p, \mathbb{1}_{\mathbb{Q}})$, $w((p, q), (p', q')) = (w(p, p'), q)$, and the pair $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$.

Lemma 11.0.3. *Suppose that $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$. Let $a \in A$.*

1. *$\pitchfork(a) \restriction W(\pi(a))$ forms a bijection from $W(\pi(a))$ to $W(a)$;*
2. *for all $n < \omega$ and $r \leq^n \pi(a)$, $\pitchfork(a)(r) \in A^a_n$.*

Proof. (1) By Clauses (4) and (5) of Definition 11.0.1.
(2) By Clauses (1), (2) and (5) of Definition 11.0.1. □

Lemma 11.0.4. *Suppose that $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$. Let $U \subseteq A$ and $a \in A$. Denote $U_a := U \cap (\mathbb{A} \downarrow a)$.*

1. *If U_a is* 0*-open, then so is $\pi[U_a]$;*
2. *If U_a is dense below a, then $\pi[U_a]$ is dense below $\pi(a)$.*

Proof. (1) Suppose U_a is 0-open. To see that $\pi[U_a]$ is 0-open, let $p \in \pi[U_a]$ and $p' \leq^0 p$ be arbitrary. Find $b \in U_a$ such that $\pi(b) = p$ and set $b' := \pitchfork(b)(p')$. Clearly, b' is well-defined and by Definition 11.0.1(5), $b' \trianglelefteq^0 b$, so that, by 0-openness of U_a, $b' \in U_a$. Again, Definition 11.0.1(5) yields $\pi(b') = \pi(\pitchfork(b)(p')) = p'$, thus $p' \in \pi[U_a]$, as desired.

(2) Suppose that U_a is dense below a. To see that $\pi[U_a]$ is dense below $\pi(a)$, let $p \leq \pi(a)$ be arbitrary. Since, by Definition 11.0.1(1), π is a projection from $\mathbb{A}$ to $\mathbb{P}$, we may find $a^* \trianglelefteq a$ such that $\pi(a^*) \leq p$. As U_a is dense below a, we may then find $a^\star \trianglelefteq a^*$ in U_a. Clearly, $\pi(a^\star) \leq p$. □

Throughout the rest of this chapter, suppose that:

- $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}_{\mathbb{P}}$;
- $\mathbb{A} = (A, \trianglelefteq)$ is a notion of forcing with a greatest element $\mathbb{1}_{\mathbb{A}}$;
- $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal κ, and μ is a cardinal such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$;
- $\ell_{\mathbb{P}}$ and $c_{\mathbb{P}}$ are functions witnessing that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is a Σ-Prikry;
- $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ are functions with $\operatorname{dom}(\ell_{\mathbb{A}}) = \operatorname{dom}(c_{\mathbb{A}}) = A$;

- $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$.

We shall now go over each of the clauses of Definition 10.1.3 and collect sufficient conditions for the triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to be Σ-Prikry, as well.

Lemma 11.0.5. *$(\mathbb{A}, \ell_{\mathbb{A}})$ is a graded poset.*

Proof. For all $a, b \in A$, $b \trianglelefteq a \implies \pi(b) \leq \pi(a) \implies \ell_{\mathbb{A}}(b) = \ell_{\mathbb{P}}(\pi(b)) \geq \ell_{\mathbb{P}}(\pi(a)) = \ell_{\mathbb{A}}(a)$. In addition, as $(\mathbb{P}, \ell_{\mathbb{P}})$ is a graded poset, for any given $a \in A$, we may pick $r \in P_1^{\pi(a)}$. By Lemma 11.0.3(2), then, $\pitchfork(a)(r)$ witnesses that A_1^a is non-empty. $\square$

Lemma 11.0.6. *Let $n < \omega$. Suppose that for every directed family D of conditions in $\mathbb{A}_n$ with $|D| < \kappa_n$, if the map $d \mapsto \pi(d)$ is constant over D, then D admits bound in $\mathbb{A}_n$.*

Then $\mathbb{A}_n$ is κ_n-directed-closed.

Proof. Suppose that E is a given directed family in $\mathbb{A}_n$ of size less than κ_n. In particular, $\{\pi(e) \mid e \in E\}$ is a directed family in $\mathbb{P}_n$ of size less than κ_n; hence, by Definition 10.1.3(2), we may find bound for it (in $\mathbb{P}_n$), say, r. Put $D := \{\pitchfork(e)(r) \mid e \in E\}$. By Lemma 11.0.3(2), D is a family of conditions in $\mathbb{A}_n$ with $|D| < \kappa_n$. By Definition 11.0.1(5), the map $d \mapsto \pi(d)$ is constant (indeed, with value r) over D.

Claim 11.0.6.1. *D is directed.*

Proof. Given $d_0, d_1 \in D$, fix $e_0, e_1 \in E$ such that $d_i = \pitchfork(e_i)(r)$ for all $i < 2$. As E is directed, let us pick $e^* \in E$ such that $e^* \trianglelefteq e_0, e_1$. Put $d^* := \pitchfork(e^*)(r)$, so that $d^* \in D$. Then, by Definition 11.0.1(7), $d^* \trianglelefteq d_0, d_1$. $\square$

Now, by the hypothesis of the lemma, we may pick bound for D (in $\mathbb{A}_n$), say, b. By Definition 11.0.1(2), for all $a \in E$, $b \trianglelefteq \pitchfork(a)(r) \trianglelefteq a$, and hence b is a bound for E. $\square$

Lemma 11.0.7. *For all $a, a' \in A$, if $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$, then $A_0^a \cap A_0^{a'}$ is non-empty.*

In particular, if $|ran(c_{\mathbb{A}})| \leq \mu$, then $\mathbb{A}$ is μ^+-2-linked.

Proof. By Definition 11.0.1(8), $c(\pi(a)) = c(\pi(a'))$. Since $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is Σ-Prikry, Definition 10.1.3(3) guarantees the existence of some $r \in P_0^{\pi(a)} \cap P_0^{\pi(a')}$ and thus, again by Definition 11.0.1(8), $\pitchfork(a)(r) = \pitchfork(a')(r)$. Finally, Lemma 11.0.3(2) yields that this common value is in $A_0^a \cap A_0^{a'}$, as desired. $\square$

Lemma 11.0.8. *For all $a \in A$, $n, m < \omega$ and $b \trianglelefteq^{n+m} a$, $m(a, b)$ exists.*

Proof. This is covered by Definition 11.0.1(4). $\square$

Lemma 11.0.9. *For all $a \in A$, $|W(a)| < \mu$.*

Proof. This follows from Lemma 11.0.3(1) and Definition 10.1.3(5) for $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$. □

Lemma 11.0.10. *For all $a' \leq a$ in A, $b \mapsto w(a,b)$ forms an order-preserving map from $W(a')$ to $W(a)$.*

Proof. Fix an arbitrary pair $b' \trianglelefteq b$ in $W(a')$, and let us show that $w(a,b') \trianglelefteq w(a,b)$. By Definition 11.0.1(4) with $m = 0$, $w(a,b') = ⋔(a)(w(\pi(a),\pi(b')))$ and $w(a,b) = ⋔(a)(w(\pi(a),\pi(b)))$. On the other hand, π is a projection, in particular order-preserving, hence $\pi(b') \leq \pi(b)$, and also both such conditions extend $\pi(a)$. By Definition 10.1.3(6) for $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$, $w(\pi(a),\pi(b')) \leq w(\pi(a),\pi(b))$, and thus, appealing to Definition 11.0.1(7), it follows that

$$⋔(a)(w(\pi(a),\pi(b'))) \trianglelefteq ⋔(a)(w(\pi(a),\pi(b))),$$

which yields the desired result. □

Definition 11.0.11. The forking projection $(⋔, \pi)$ is said to have the *mixing property* iff for all $a \in A$, $n < \omega$, $q \leq^0 \pi(a)$, and a function $g : W_n(q) \to \mathbb{A} \downarrow a$ such that $\pi \circ g$ is the identity map,[1] there exists $b \trianglelefteq^0 a$ with $\pi(b) = q$ such that $⋔(b)(r) \trianglelefteq^0 g(r)$ for every $r \in W_n(q)$.

Lemma 11.0.12. *Suppose that $(⋔, \pi)$ has the mixing property. Let $U \subseteq A$ be a 0-open set. Then, for all $a \in A$ and $n < \omega$, there is $b \trianglelefteq^0 a$ such that, either $A^b_n \cap U = \emptyset$ or $A^b_n \subseteq U$.*

Proof. Let $a \in A$ and $n < \omega$. Set $U_a := U \cap (\mathbb{A} \downarrow a)$, $\bar{U} := \pi[U_a]$, and $p := \pi(a)$. By Lemma 11.0.4(1), $\bar{U}$ is 0-open. Since $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is Σ-Prikry, we now appeal to Definition 10.1.3(7) and find $q \leq^0 p$ such that, either $P^q_n \cap \bar{U} = \emptyset$ or $P^q_n \subseteq \bar{U}$.

Claim 11.0.12.1. *If $P^q_n \cap \bar{U} = \emptyset$, then there exists $b \trianglelefteq^0 a$ with $\pi(b) = q$ such that $A^b_n \cap U = \emptyset$.*

Proof. Suppose that $P^q_n \cap \bar{U} = \emptyset$. Set $b := ⋔(a)(q)$, so that $b \trianglelefteq a$ and $\pi(b) = q$. As $\ell_{\mathbb{A}}(b) = \ell_{\mathbb{P}}(q) = \ell_{\mathbb{A}}(a)$, we moreover have $b \trianglelefteq^0 a$. Finally, since $d \in A^b_n \cap U \implies \pi(d) \in P^q_n \cap \bar{U}$, we infer that $A^b_n \cap U = \emptyset$. □

Claim 11.0.12.2. *If $P^q_n \subseteq \bar{U}$, then there exists $b \trianglelefteq^0 a$ with $\pi(b) = q$ such that $A^b_n \subseteq U$.*

Proof. Suppose that $P^q_n \subseteq \bar{U}$. So, for every $r \in P^q_n$, we may pick $a_r \in U_a$ such that $\pi(a_r) = r$. Define a function $g : W_n(q) \to U_a$ via $g(r) := a_r$. By the mixing property, we now obtain a condition $b \trianglelefteq^0 a$ such that $⋔(b)(r) \trianglelefteq^0 g(r)$ for every $r \in W_n(q)$. As U is 0-open, it follows that $⋔(b)``W_n(q) \subseteq U$. By Lemma 11.0.3(1), $W_n(b) = ⋔(b)``W_n(q) \subseteq U$; hence, again by 0-openess of U, $A^b_n \subseteq U$, as desired. □

[1] Equivalently, a function $g : W_n(q) \to A$ such that $g(r) \trianglelefteq a$ and $\pi(g(r)) = r$ for every $r \in W_n(q)$.

This completes the proof. $\square$

Corollary 11.0.13. *Suppose that Clauses* (2) *and* (7) *of Definition 10.1.3 are valid for* $(\mathbb{A}, \ell_{\mathbb{A}})$*. If* $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ *is singular*", *then* $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$.

Proof. Suppose that $\mathbb{1}_{\mathbb{A}} \not\Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$. As $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$ and $\mathbb{A}$ projects to $\mathbb{P}$, this means that there exists $a \in A$ such that $a \Vdash_{\mathbb{A}} |\check{\mu}| \leq |\check{\kappa}|$. Towards a contradiction, suppose that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular". As $\mathbb{A}$ projects to $\mathbb{P}$, it altogether follows that $a \Vdash_{\mathbb{A}} \mathrm{cof}(\check{\mu}) < \check{\kappa}$. By Lemma 10.1.10(2), then, there exists $a' \trianglelefteq a$ with $|W(a')| \geq \mu$, contradicting Lemma 11.0.8(2). $\square$

Remark 11.0.14. The message behind this chapter is that if $(\pi, \pitchfork)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to a Σ-Prikry triple $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ then, modulo some few additional requirements (see lemmas 11.0.6, 11.0.7 and 11.0.12 and Corollary 11.0.13), $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ is automatically Σ-Prikry. This explains the key role of forking projections in our iteration scheme.

We close end up the chapter with a useful lemma about Σ-Prikry forcing and forking projections.

Lemma 11.0.15 (Canonical form)**.** *Suppose that* $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ *and* $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ *are both Σ-Prikry notions of forcing. Denote* $\mathbb{P} = (P, \leq)$ *and* $\mathbb{A} = (A, \trianglelefteq)$.

If $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ admits a forking projection *to* $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ *as witnessed by* $\pitchfork$ *and* π*, then we may assume that all of the following hold true:*

1. *each element of A is a pair (x, y) with $\pi(x, y) = x$;*
2. *for all $a \in A$, $\lceil \pi(a) \rceil^{\mathbb{A}} = (\pi(a), \emptyset)$;*
3. *for all $p, q \in P$, if $c_{\mathbb{P}}(p) = c_{\mathbb{P}}(q)$, then $c_{\mathbb{A}}(\lceil p \rceil^{\mathbb{A}}) = c_{\mathbb{A}}(\lceil q \rceil^{\mathbb{A}})$.*

Proof. By applying a bijection, we may assume that $A = |A|$ with $\mathbb{1}_{\mathbb{A}} = \emptyset$. To clarify what we are about to do, we agree to say that "a is a lift" iff $a = \lceil \pi(a) \rceil^{\mathbb{A}}$. Now, define $f : A \to P \times A$ via:

$$f(a) := \begin{cases} (\pi(a), \emptyset), & \text{if } a \text{ is a lift;} \\ (\pi(a), a), & \text{otherwise.} \end{cases}$$

Claim 11.0.15.1. *f is injective.*

Proof. Suppose $a, a' \in A$ with $f(a) = f(a')$.

► If a is not a lift and a' is not a lift, then from $f(a) = f(a')$ we immediately get that $a = a'$.

► If a is a lift and a' is a lift, then from $f(a) = f(a')$, we infer that $\pi(a) = \pi(a')$, so that $a = \lceil \pi(a) \rceil^{\mathbb{A}} = \lceil \pi(a') \rceil^{\mathbb{A}} = a'$.

► If a is not a lift, but a' is a lift, then from $f(a) = f(a')$, we infer that $a = \emptyset = \mathbb{1}_{\mathbb{A}}$, contradicting the fact that $\mathbb{1}_{\mathbb{A}} = \lceil \mathbb{1}_{\mathbb{P}} \rceil^{\mathbb{A}} = \lceil \pi(\mathbb{1}_{\mathbb{A}}) \rceil^{\mathbb{A}}$ is a lift. So this case is void. $\square$

Let $B := \mathrm{ran}(f)$ and $\trianglelefteq_B := \{(f(a), f(b)) \mid a \trianglelefteq b\}$, so that $\mathbb{B} := (B, \trianglelefteq_B)$ is isomorphic to $\mathbb{A}$. Define $\ell_{\mathbb{B}} := \ell_{\mathbb{A}} \circ f^{-1}$ and $\pi_{\mathbb{B}} := \pi \circ f^{-1}$. Also, define $\pitchfork_{\mathbb{B}}$ via $\pitchfork_{\mathbb{B}}(b)(p) := f(\pitchfork(f^{-1}(b))(p))$. It is clear that $b \in B$ is a lift iff $f^{-1}(a)$ is a lift iff $b = (\pi_{\mathbb{B}}(b), \emptyset)$. Next, define $c_{\mathbb{B}} : B \to \mu \times 2$ by letting for all $b \in B$:

$$c_{\mathbb{B}}(b) := \begin{cases} (c_{\mathbb{P}}(\pi_{\mathbb{B}}(b)), 0), & \text{if } b \text{ is a lift;} \\ (c_{\mathbb{A}}(f^{-1}(b)), 1), & \text{otherwise.} \end{cases}$$

Claim 11.0.15.2. *Suppose $b_0, b_1 \in B$ with $c_{\mathbb{B}}(b_0) = c_{\mathbb{B}}(b_1)$. Then $c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_0)) = c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_1))$ and, for all $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$, $\pitchfork_{\mathbb{B}}(b_0)(r) = \pitchfork_{\mathbb{B}}(b_1)(r)$.*

Proof. We focus on verifying that for all $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$, $\pitchfork_{\mathbb{B}}(b_0)(r) = \pitchfork_{\mathbb{B}}(b_1)(r)$. For each $i < 2$, denote $a_i := f^{-1}(b_i)$ and $p_i := \pi_{\mathbb{B}}(b_i)$, so that $\pi(a_i) = p_i$. Suppose $r \in P_0^{p_0} \cap P_0^{p_1}$.

► If b_0 is a lift, then so are b_1, a_0, a_1. Therefore, for each $i < 2$, Definition 11.0.1(6) implies that $\pitchfork_{\mathbb{B}}(b_i)(r) = f(\pitchfork(a_i)(r)) = f(\lceil r \rceil^{\mathbb{A}}) = \lceil r \rceil^{\mathbb{B}}$. In effect, $\pitchfork_{\mathbb{B}}(b_0)(r) = \pitchfork_{\mathbb{B}}(b_1)(r)$, as desired.

► Otherwise, $c_{\mathbb{A}}(a_0) = c_{\mathbb{A}}(a_1)$. As $r \in P_0^{\pi(a_0)} \cap P_0^{\pi(a_1)}$, $\pitchfork_{\mathbb{B}}(b_0)(p) = f(\pitchfork(a_0)(p)) = f(\pitchfork(a_1)(p)) = \pitchfork_{\mathbb{B}}(b_1)(p)$. □

This completes the proof. □

CHAPTER 12

Σ-Prikry forcings and finite simultaneous stationary reflection

In this brief chapter we will discuss the interplay between Σ-Prikry forcings and simultaneous stationary reflection. More precisely, for a given Σ-Prikry forcing, we are seeking for sufficient conditions to force a model of finite simultaneous stationary reflection at the successor of a strong limit singular cardinal.

We will begin the chapter introducing the principle $\mathrm{Refl}({<}\theta, S, T)$ and commenting the restrictions that it imposes upon the behavior of the continuum function. Specifically, for a strong limit singular cardinal κ, we will prove that $\mathrm{Refl}(\mathrm{cof}(\kappa), \kappa^+)$ entails GCH_κ (cf. Corollary 12.1.4). Thus, in the particular case where $\mathrm{cof}(\kappa) = \omega$, and if consistent, $\neg\mathsf{SCH}_\kappa + \mathrm{Refl}({<}\omega, \kappa^+)$ is optimal. In the future Chapter 15 we will show that this combinatorial configuration is indeed consistent, modulo ω-many supercompact cardinals.

In the rest of the chapter we analyse the finite simultaneous reflection in generic extensions by Σ-Prikry forcings. As a result we obtain sufficient conditions for a Σ-Prikry poset to force the principle $\mathrm{Refl}({<}\omega, \Gamma)$.[1] For more details, see Corollary 12.2.7. Finally, the chapter is closed giving sufficient conditions to get a model of $\mathrm{Refl}({<}\omega, \kappa^+)$.

12.1 Stationary reflection and the SCH

Definition 12.1.1. For cardinals $\theta < \mu = \mathrm{cof}(\mu)$, and stationary subsets S, T of μ, the principle $\mathrm{Refl}({<}\theta, S, T)$ asserts that for every collection $\mathcal{S}$ of stationary subsets of S, with $|\mathcal{S}| < \theta$ and $\sup(\{\mathrm{cof}(\alpha) \mid \alpha \in \bigcup \mathcal{S}\}) < \sup(S)$, the set $T \cap \bigcap_{S \in \mathcal{S}} \cap \mathrm{Tr}(S)$ is non-empty. We write $\mathrm{Refl}({<}\theta, S)$ for $\mathrm{Refl}({<}\theta, S, \mu)$ and $\mathrm{Refl}(\theta, S)$ for $\mathrm{Refl}({<}\theta^+, S)$.[2]

[1] For definitions see the paragraph before Lemma 12.2.1.

[2] Where, for θ finite, θ^+ stands for $\theta + 1$.

Definition 12.1.2 (Shelah, [She94, Definition 5.1, p. 85])**.** For infinite cardinals $\mu \geq \nu \geq \theta$, define

$$\mathrm{cov}(\mu,\nu,\theta,2) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq [\mu]^{<\nu}\, \forall X \in [\mu]^{<\theta}\, \exists A \in \mathcal{A}(X \subseteq A)\}.$$

The following proposition is implicit in the work of Solovay on the Singular Cardinal Hypothesis (SCH).

Proposition 12.1.3. *Suppose* $\mathrm{Refl}(<\theta, S, E^{\mu}_{<\nu})$ *holds for a stationary* $S \subseteq \mu$ *and some cardinal* $\nu < \mu$. *Then* $\mathrm{cov}(\mu,\nu,\theta,2) = \mu$.

Proof. Let $\langle S_i \mid i < \mu\rangle$ be a partition of S into mutually disjoint stationary sets. Put $T := \{\alpha < \mu \mid \omega < \mathrm{cof}(\alpha) < \nu\}$. Set $\mathcal{A} := \{A_\alpha \mid \alpha \in T\}$, where for each $\alpha \in T$, $A_\alpha := \{i < \mu \mid S_i \cap \alpha \text{ is stationary}\}$. Since each $\alpha \in T$ admits a club C_α of order-type $< \nu$, and $C_\alpha \cap S_i \neq \emptyset$ for all $i \in A_\alpha$, while $S_i \cap S_j = \emptyset$ for all $i < j < \mu$, we get that $\mathcal{A} \subseteq [\mu]^{<\nu}$. By $\mathrm{Refl}(<\theta, S, E^{\mu}_{<\nu})$, for every $X \in [\mu]^{<\theta}$, there must exist some $A \in \mathcal{A}$ such that $X \subseteq A$. Altogether, $\mathcal{A}$ witnesses that $\mathrm{cov}(\mu,\nu,\theta,2) = \mu$. □

Note that for every singular strong limit κ, $\mathrm{cov}(\kappa^+, \kappa, (\mathrm{cof}(\kappa))^+, 2) = 2^\kappa$. In particular, this yields the first of the announced results:

Corollary 12.1.4. *If* κ *is a singular strong limit cardinal admitting a stationary subset* $S \subseteq \kappa^+$ *for which* $\mathrm{Refl}(\mathrm{cof}(\kappa), S)$ *holds, then* $2^\kappa = \kappa^+$. □

12.2 Simultaneous stationary reflection and Σ-Prikry forcings

Throughout this section, suppose that $(\mathbb{P}, \ell, c)$ is a given Σ-Prikry notion of forcing. Denote $\mathbb{P} = (P, \leq)$ and $\Sigma = \langle \kappa_n \mid n < \omega\rangle$. Also, define κ and μ as in Definition 10.1.3. Our universe of sets is denoted by V, and we write $\Gamma := \{\alpha < \mu \mid \omega < \mathrm{cof}^V(\alpha) < \kappa\}$.[3]

Lemma 12.2.1. *Suppose that* $r^* \in P$ *and that* τ *is a* $\mathbb{P}$*-name. For all* $n < \omega$, *write* $\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in \mu \times P_n \;\&\; p \Vdash_{\mathbb{P}} \check{\alpha} \in \tau\}$. *Then one of the following holds:*

1. $D := \{p \in P \mid (\forall q \leq p)\; q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ *is stationary*"$\}$ *is open and dense below* r^**;*[4]
2. *There exist* $r^\star \leq r^*$ *and* $I \in [\omega]^\omega$ *such that, for all* $q \leq r^\star$ *with* $\ell(q) \in I$,

$$q \Vdash_{\mathbb{P}_{\ell(q)}} \text{“}\dot{T}_{\ell(q)} \textit{ is nonstationary”}.$$

[3]All findings of the analysis in this section goes through if we replace μ by a regular cardinal $\nu \geq \mu$ and replace Γ by $\{\alpha < \nu \mid \omega < \mathrm{cof}^V(\alpha) < \kappa\}$.

[4]Recall that we identify each of the $\mathbb{P}_n$'s with its separative quotient.

Proof. D is clearly open. Suppose that D is not dense below r^*. Then, we may pick some condition $p^* \leq r^*$ such that for all $p \leq p^*$, there is $q \leq p$, such that $q \not\Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is stationary", i.e., there exists $q' \leq q$ in $\mathbb{P}_{\ell(q)}$ such that $q' \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary". Hence, for all $p \leq p^*$, there is $q' \leq p$, such that $q' \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary". In other words, the 0-open set $E := \{q \in \mathbb{P} \mid q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary"$\}$ is dense below p^*.

Now, define a 0-open coloring $d : P \to 2$ via $d(q) := 1$ iff $q \in E$. By virtue of Lemma 10.1.6, find $r^\star \leq^0 p^*$ such that $\mathbb{P} \downarrow r^\star$ is a set of indiscernibles for d. Note that as E is dense below $r^\star$, Clause (1) of Definition 10.1.3 entails that the set $I := \{\ell(q') \mid q' \leq r^\star \ \&\ q' \in E\}$ must be infinite. Finally, as $\mathbb{P} \downarrow r^\star$ is a set of indiscernibles for d, for all $q \leq r^\star$ with $\ell(q) \in I$, we indeed have $q \in E$. □

Lemma 12.2.2. *Suppose that $r^\star \in P$, $I \in [\omega]^\omega$, and $\langle \dot{C}_n \mid n \in I\rangle$ is a sequence such that, for all $q \leq r^\star$ with $\ell(q) \in I$, we have:*

$$q \Vdash_{\mathbb{P}_{\ell(q)}} \dot{C}_{\ell(q)} \text{ is a club in } \check{\mu}.$$

Consider the $\mathbb{P}$-name $\dot{Y} := \{(\check{\alpha}, q) \mid (\alpha, q) \in R\}$, where

$$R := \{(\alpha, q) \in \mu \times P \mid q \leq r^\star \ \&\ \forall r \leq q[\ell(r) \in I \to r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)}]\}.$$

Suppose G is $\mathbb{P}$-generic over V, with $r^\star \in G$. Let Y be the interpretation of R in by G. Then:

1. *$V[G] \models Y$ is unbounded in μ;*
2. *$V[G] \models \mathrm{acc}^+(Y) \cap \Gamma \subseteq Y$.*

Proof. We commence with a claim.

Claim 12.2.2.1. *For every $p \leq r^\star$ and $\gamma < \mu$, there exist $\bar{p} \leq^0 p$ and $\bar{\gamma} \in (\gamma, \mu)$ such that, for every $q \leq \bar{p}$ with $\ell(q) \in I$, $q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{C}_{\ell(q)} \cap (\gamma, \bar{\gamma})$ is non-empty".*

Proof. Given p and γ as above, write:

$$D_{p,\gamma} := \{q \in \mathbb{P} \mid q \leq p \ \&\ \ell(q) \in I \ \&\ \exists \gamma' > \gamma (q \Vdash_{\mathbb{P}_{\ell(q)}} \check{\gamma}' \in \dot{C}_{\ell(q)})\}.$$

Note that $I_{p,\gamma} := \{\ell(q) \mid q \in D_{p,\gamma}\}$ is equal to $I \setminus \ell(p)$.[5] Let $d : P \to 2$ be defined via $d(r) := 1$ iff $r \in D_{p,\gamma}$. As $D_{p,\gamma}$ is 0-open we get from Lemma 10.1.6 a condition $\bar{p} \leq^0 p$ such that $\mathbb{P} \downarrow \bar{p}$ is a set of indiscernibles for d. Thereby, for all $n < \omega$, if $P_n^{\bar{p}} \cap D_{p,\gamma} \neq \emptyset$, then $P_n^{\bar{p}} \subseteq D_{p,\gamma}$. As $\bar{p} \leq p$, $I_{p,\gamma} = I \setminus \ell(p)$,

[5] By standard facts about forcing, if $\mathbb{Q}$ is a notion of forcing, and $q \in \mathbb{Q}$ is a condition that forces that $\dot{C}$ is some cofinal subset of a cardinal μ, then for every ordinal $\gamma < \mu$, there exists an extension q' of q and some ordinal γ' above γ such that $q' \Vdash_{\mathbb{Q}} \check{\gamma}' \in \dot{C}$.

and $W_n(\bar{p}) \subseteq P^{\bar{p}}_n$ for all $n < \omega$, we get in particular that $A_n := W_{n-\ell(\bar{p})}(\bar{p})$ is a subset of $D_{p,\gamma}$ for all $n \in I \setminus \ell(p)$.

For all $n \in I \setminus \ell(p)$ and $r \in A_n$, fix $\gamma_r \in (\gamma, \mu)$ such that

$$r \Vdash_{\mathbb{P}_{\ell(r)}} \gamma_r \in \dot{C}_{\ell(r)}.$$

By Definition 10.1.3(5), $|\bigcup_{n\in I\setminus\ell(p)} A_n| < \mu$, so that $\bar{\gamma} := \sup\{\gamma_r \mid r \in \bigcup_{n\in I\setminus\ell(p)} A_n\} + 1$ is $< \mu$.

Now, let $q \leq \bar{p}$ with length in I be arbitrary. As $I_{p,\gamma} = I \setminus \ell(p)$, we have $\ell(q) \in I_{p,\gamma}$. In particular, $P^{\bar{p}}_{\ell(q)-\ell(\bar{p})} \cap D_{p,\gamma} \neq \emptyset$, and thus $A_{\ell(q)} \subseteq D_{p,\gamma}$. Pick $r \in A_{\ell(q)}$ with $q \leq r$. Then $r \Vdash_{\mathbb{P}_{\ell(r)}} \gamma_r \in \dot{C}_{\ell(r)}$. In particular, $q \Vdash_{\ell(q)}$ "$\dot{C}_{\ell(q)} \cap (\gamma, \bar{\gamma})$ is non-empty". □

Now, let G be a $\mathbb{P}$-generic with $r^\star \in G$. Of course, the interpretation of $\dot{Y}$ in $V[G]$ is

$$Y := \{\alpha < \mu \mid (\exists q \in G)(\forall r \leq q)[\ell(r) \in I \to r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)}]\}.$$

Claim 12.2.2.2. *1. Y is unbounded in $V[G]$;*

2. $\mathrm{acc}^+(Y) \cap \Gamma \subseteq Y$.

Proof. (1) We run a density argument in V. Let $p \leq r^\star$ and $\gamma < \mu$ be arbitrary. By an iterative application of Claim 12.2.2.1, we find a $\leq_0$-decreasing sequence of conditions in $\mathbb{P}$, $\langle p_n \mid n < \omega\rangle$, and an increasing sequence of ordinals below μ, $\langle \gamma_n \mid n < \omega\rangle$, such that $p_0 \leq^0 p$, $\gamma_0 = \gamma$, and such that for every n and every $q \leq p_n$ with $\ell(q) \in I$, we have that $q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{C}_{\ell(q)} \cap (\gamma_n, \gamma_{n+1})$ is non-empty".

By Clause (2) of Definition 10.1.3, $\mathbb{P}_{\ell(p)}$ is σ-closed, so let q^* be a lower for $\langle q_n \mid n < \omega\rangle$. Put $\gamma^* := \sup_n \gamma_n$. Then for every $r \leq q^*$ with length in I, we have $r \Vdash_{\mathbb{P}_{\ell(r)}} \gamma^* \in \dot{C}_{\ell(r)}$. That is, q^* witnesses that $\gamma^* \in Y \setminus \gamma$.

(2) Suppose that $\alpha \in \mathrm{acc}^+(Y) \cap \Gamma$. Set $\eta := \mathrm{cof}^V(\alpha)$, and pick a large enough $k < \omega$ such that $\eta < \kappa_k$. Fix $p \in G$ such that $p \leq r^\star$, $p \Vdash \check{\alpha} \in \mathrm{acc}^+(\dot{Y})$, and $\ell(p) \geq k$.

Work in V. Let $\langle \alpha_j \mid j < \eta\rangle$ be an increasing cofinal sequence in α. For each $j < \eta$, consider the set $D_j := \{q \in P \mid \exists \gamma \in (\alpha_j, \alpha)\ q \Vdash_{\mathbb{P}} \check{\gamma} \in \dot{Y}\}$. Clearly, D_j is open and dense below p. We claim that the intersection $\bigcap_{j<\eta} D_j$ is dense below p, as well. To this end, let $p' \leq p$ be arbitrary. For each $j < \eta$, D_j is 0-open and dense below p', so since $\eta < \kappa_k \leq \kappa_{\ell(p')}$, we obtain from Corollary 10.1.7(2) and Definition 10.1.3(2), a $\leq^0$-decreasing sequence $\langle q_j \mid j \leq \eta\rangle$ along with a sequence of natural numbers $\langle n_j \mid j < \eta\rangle$ such that $q_0 \leq^0 p'$ and $P^{q_j}_{n_j} \subseteq D_j$ for all $j < \eta$. Let $p'' := q_\eta$. As $\eta = \mathrm{cof}^V(\alpha) > \omega$, we may pick a cofinal $J \subseteq \eta$ for which $\{n_j \mid j \in J\}$ is a singleton, say, $\{n\}$. Then $P^{p''}_n \subseteq \bigcap_{j\in J} P^{q_j}_{n_j} \subseteq \bigcap_{j\in J} D_j = \bigcap_{j<\eta} D_j$. Thus, the latter contains an element extending p'', which extends p'.

Fix $q \in G \cap \bigcap_{j<\eta} D_j$ extending p and let us show that q witnesses that α is in Y. That is, we shall verify that, for all $r \leq q$ with $\ell(r) \in I$, $r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)}$. First, notice that for all $j < \eta$, there exists some $\gamma_j \in (\alpha_j, \alpha)$ such that $q \Vdash_{\mathbb{P}} \check{\gamma}_j \in \dot{Y}$. Now let $r \leq q$ with $\ell(r) \in I$ be arbitrary and notice that $r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\gamma}_j \in \dot{C}_{\ell(r)}$ for all $j < \eta$, hence $r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)}$. □

This completes the proof of Lemma 12.2.2. □

Lemma 12.2.3. *Suppose that $r^* \in P$ forces that τ is a $\mathbb{P}$-name for a stationary subset T of Γ. For all $n < \omega$, write $\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in \mu \times P_n \ \&\ p \Vdash_{\mathbb{P}} \check{\alpha} \in \tau\}$. Then $D := \{p \in P \mid (\forall q \leq p)\ q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is stationary"$\}$ is open and dense below r^*.*

Proof. Suppose not. Then, by Lemma 12.2.1, let us pick $r^\star \leq r^*$ and $I \in [\omega]^\omega$ such that, for all $q \leq r^\star$ with $\ell(q) \in I$,

$$q \Vdash_{\mathbb{P}_{\ell(q)}} \text{“}\dot{T}_{\ell(q)} \text{ is nonstationary”}.$$

Now, for each $n \in I$, we appeal to the maximal principle [Kun14, Lemma IV.7.2] to find a $\mathbb{P}_n$-name $\dot{C}_n$ for a club subset of μ, such that, for all $q \leq r^\star$ with $\ell(q) \in I$, we have $q \Vdash_{\mathbb{P}_{\ell(q)}} \dot{C}_{\ell(q)} \cap \dot{T}_{\ell(q)} = \emptyset$. Consider the $\mathbb{P}$-name:

$$\dot{Y} := \{(\check{\alpha}, q) \in \mu \times P \mid q \leq r^\star \ \&\ \forall r \leq q[\ell(r) \in I \rightarrow r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)}]\}.$$

Let G be $\mathbb{P}$-generic over V, with $r^\star \in G$, and Y be the interpretation of $\dot{Y}$ in $V[G]$. By Lemma 12.2.2:

1. $V[G] \models Y$ is unbounded in μ;
2. $V[G] \models \mathrm{acc}^+(Y) \cap \Gamma \subseteq Y$.

As $r^\star \leq r^*$, our hypothesis entails:

(3) $V[G] \models T$ is stationary in μ.

So $V[G] \models Y \cap T \neq \emptyset$. Pick $\alpha < \mu$ and $r \in G$ such that $r \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{Y} \cap \tau$. Of course, we may find such r that in addition satisfies $r \leq r^\star$ and $\ell(r) \in I$. By definition of $\dot{T}_{\ell(r)}$, the ordered-pair $(\check{\alpha}, r)$ is an element of the name $\dot{T}_{\ell(r)}$. In particular, $r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{T}_{\ell(r)}$. From $r \leq r^\star$, $\ell(r) \in I$, and $r \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{Y}$, we have $r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)}$. Altogether $r \Vdash_{\mathbb{P}_{\ell(r)}} \dot{C}_{\ell(r)} \cap \dot{T}_{\ell(r)} \neq \emptyset$, contradicting the choice of $\dot{C}_{\ell(r)}$. □

Recall that a supercompact cardinal χ is said to be *Laver-indestructible* iff for every χ-directed-closed notion of forcing $\mathbb{Q}$, $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "χ is supercompact". Also recall that for every supercompact cardinal χ and every regular cardinal $\nu \geq \chi$, $\mathrm{Refl}({<}\chi, E^\nu_{<\chi}, E^\nu_{<\chi})$ holds. We refer the reader to [CFM01] for further details. For our purposes, we would just need the following:

Lemma 12.2.4. *For all $n<\omega$, if κ_n is a Laver-indestructible supercompact cardinal, then $V^{\mathbb{P}_n} \models \mathrm{Refl}({<}\omega, E^{\mu}_{<\kappa_n}, E^{\mu}_{<\kappa_n})$.*[6]

Proof. By Clause (2) of Definition 10.1.3, $\mathbb{P}_n$ is κ_n-directed-closed, and so $V^{\mathbb{P}_n} \models$ "κ_n is supercompact". In particular, $V^{\mathbb{P}_n} \models \mathrm{Refl}({<}\omega, E^{\mu}_{<\kappa_n}, E^{\mu}_{<\kappa_n})$. □

Lemma 12.2.5. *Suppose:*

- *For all $n<\omega$, $V^{\mathbb{P}_n} \models \mathrm{Refl}({<}\omega, E^{\mu}_{<\kappa_n}, E^{\mu}_{<\kappa_n})$;*
- *$r^* \in P$ forces that $\langle \tau^i \mid i<k\rangle$ is a finite sequence of $\mathbb{P}$-names for stationary subsets of $(E^{\mu}_{<\kappa})^V$;*

Write $\dot{T}^i_n := \{(\check{\alpha}, p) \mid (\alpha,p) \in \mu \times P_n \ \&\ p \Vdash_{\mathbb{P}} \check{\alpha} \in \tau^i\}$ for all $i<k$ and $n<\omega$.

Suppose $D^i := \{p \in P \mid (\forall q \le p) q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}^i_{\ell(q)}$ is stationary"$\}$ is open and dense below r^* for each $i<k$. Then for every $\mathbb{P}$-generic G over V with $r^* \in G$, $\langle T^i \mid i<k\rangle$ reflects simultaneously in $V[G]$.[7]

Proof. As before, we run a density argument below the condition r^*. Given an arbitrary $p_0 \le r^*$, pick $p \in \bigcap_{i<k} D^i$ below p_0 and a large enough $m<\omega$ such that $p \Vdash_{\mathbb{P}}$ "$\forall i<k(\tau^i \cap E^{\mu}_{<\kappa_m})$ is stationary". By possibly extending p using Definition 10.1.3(1), we may assume that $n := \ell(p)$ is $\ge m$. Let G_n be $\mathbb{P}_n$-generic with $p \in G_n$. As $V[G_n] \models \mathrm{Refl}({<}\omega, E^{\mu}_{<\kappa_n}, E^{\mu}_{<\kappa_n})$, let us fix some $q \le^0 p$ in G_n, and some $\delta \in E^{\mu}_{<\kappa_n}$ such that $q \Vdash_{\mathbb{P}_n}$ "$\forall i<k(\dot{T}^i_n \cap \delta$ is stationary)".

In V, pick a club $C \subseteq \delta$ of order type $\mathrm{cof}(\delta)$. Note that $|C|<\kappa_n$. Then for each $i<k$, $q \Vdash_{\mathbb{P}_n}$ "$\dot{T}^i_n \cap C$ is stationary in δ". Working for a moment in $V[G_n]$, write $A^i := C \cap (\dot{T}^i_n)_{G_n}$. Since $\mathbb{P}_n$ is κ_n-closed, we may find $r \in P_n$ extending q that, for all $i<k$, decides A^i to be some ground model stationary subset B^i of δ. Then, for every $i<k$,

$$r \Vdash_{\mathbb{P}_n} \text{“}\dot{T}^i_n \cap \delta \text{ contains the stationary set } \check{B}^i\text{”}.$$

By definition of the name $\dot{T}^i_n$, we have that $r \Vdash_{\mathbb{P}} \check{B}^i \subseteq \tau^i \cap \delta$. Finally, since $\mathrm{otp}(B^i) \le \delta < \kappa$, Lemma 10.1.10(1), B^i remains stationary in $V^{\mathbb{P}}$ for each i. So, $r \le p_0$, and $r \Vdash_{\mathbb{P}} \tau^i \cap \delta$ is stationary for each $i<k$. □

Corollary 12.2.6. *Suppose $V^{\mathbb{P}_n} \models \mathrm{Refl}({<}\omega, E^{\mu}_{<\kappa_n}, E^{\mu}_{<\kappa_n})$ for all $n<\omega$. Then $V^{\mathbb{P}} \models \mathrm{Refl}({<}\omega, \Gamma)$.*

Proof. Let r^* be a condition in G forcing that $\langle \tau^i \mid i<k\rangle$ is a finite sequence of $\mathbb{P}$-names for stationary subsets $\langle T^i \mid i<k\rangle$ of Γ. For each $i<k$ and each $n<\omega$, write $\dot{T}^i_n := \{(\check{\alpha}, p) \mid (\alpha,p) \in (\mu \times P_n) \ \&\ p \Vdash_{\mathbb{P}}$

[6] Note that, as $\mathbb{P}_n$ is κ_n-closed, $(E^{\mu}_{<\kappa_n})^{V^{\mathbb{P}_n}} = (E^{\mu}_{<\kappa_n})^V$.

[7] $\langle T^i \mid i<k\rangle$ stands for the G-interpretation of the sequence of $\mathbb{P}$-names $\langle \tau^i \mid i<k\rangle$.

$\check{\alpha} \in \tau^i\}$. By Lemma 12.2.3, for each $i < k$, $D^i := \{p \in P \mid (\forall q \leq p) q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}^i_{\ell(q)}$ is stationary"$\}$ is open and dense below r^*. Finally by virtue of Lemma 12.2.5, $\langle T^i \mid i < k\rangle$ reflects simultaneously in $V[G]$. □

Putting Lemma 12.2.4 together with Corollary 12.2.6, we arrive at the main result of the chapter.

Corollary 12.2.7. *Suppose that each cardinal in Σ is a Laver-indestructible supercompact cardinal. Then $\mathbb{1} \Vdash_{\mathbb{P}} \mathrm{Refl}({<}\omega, \Gamma)$.* □

12.3 Towards a model of $\mathrm{Refl}({<}\omega, \kappa^+)$

Towards a model $V[G]$ satisfying $\mathrm{Refl}({<}\omega, \kappa^+)$ it will also be necessary to address the reflection of stationary subsets of $\mu \setminus \Gamma$. Observe that in the special case where κ is singular and $\mu = \kappa^+$ the set $\mu \setminus \Gamma$ will be nothing but $(E^\mu_\omega)^V$. The next result implies, in this particular case, that $\mathrm{Refl}({<}\omega, \kappa^+)$ is equivalent to $\mathrm{Refl}({<}\omega, \Gamma) + \mathrm{Refl}(1, (E^\mu_\omega)^V, \Gamma)$.

Proposition 12.3.1. *Suppose that μ is non-Mahlo cardinal, and $\theta \leq \mathrm{cof}(\mu)$. For stationary subsets T, Γ, R of μ, $\mathrm{Refl}({<}2, T, \Gamma) + \mathrm{Refl}({<}\theta, \Gamma, R)$ entails $\mathrm{Refl}({<}\theta, T \cup \Gamma, R)$.*

Proof. Given a collection $\mathcal{S}$ of stationary subsets of $T \cup \Gamma$, with $|\mathcal{S}| < \theta$ and $\sup(\{\mathrm{cof}(\alpha) \mid \alpha \in \bigcup \mathcal{S}\}) < \mu$, we shall first attach to any set $S \in \mathcal{S}$, a stationary subset S' of Γ, as follows.

▶ If $S \cap \Gamma$ is stationary, then let $S' := S \cap \Gamma$.

▶ If $S \cap \Gamma$ is nonstationary, then for every club $C \subseteq \mu$, $S \cap C$ is a stationary subset of T, and so by $\mathrm{Refl}({<}2, T, \Gamma)$, there exists $\alpha \in \Gamma \cap E^\mu_{>\omega}$ such that $(S \cap C) \cap \alpha$ is stationary in α, and in particular, $\alpha \in C$. So, $\{\alpha \in \Gamma \mid S \cap \alpha \text{ is stationary}\}$ is stationary. Since μ is non-Mahlo, we may pick S' which is a stationary subset of it and all of its points consists of the same cofinality. Next, as $|\mathcal{S}| < \mathrm{cof}(\mu)$, we have $\sup(\{\mathrm{cof}(\alpha) \mid \alpha \in S', S \in \mathcal{S}\}) < \mu$, and so, from $\mathrm{Refl}({<}\theta, \Gamma, R)$, we find some $\alpha \in R$ such that $S' \cap \alpha$ is stationary for all $S \in \mathcal{S}$. The next claim completes the proof of the proposition.

Claim 12.3.1.1. *Let $S \in \mathcal{S}$. Then $S \cap \alpha$ is stationary in α.*

Proof. If $S' = S$, then $S \cap \alpha = S' \cap \alpha$ is stationary in α, and we are done. Next, assume $S' \neq S$, and let c be an arbitrary club in α. As $S' \cap \alpha$ is stationary in α, we may pick $\delta \in \mathrm{acc}(c) \cap S'$. As $\delta \in S' \subseteq E^\mu_{>\omega}$, $c \cap \delta$ is a club in δ, and as $\delta \in S'$, $S \cap \delta$ is stationary, so $S \cap c \cap \delta \neq \emptyset$. In particular, $S \cap c \neq \emptyset$. □

□

Thus, for a generic extension $V[G]$, it is equivalent to satisfy $\mathrm{Refl}({<}\omega, \kappa^+)$ or $\mathrm{Refl}({<}\omega, \Gamma) + \mathrm{Refl}(1, (E^{\mu}_{\omega})^V, \Gamma)$. Since we already know about sufficient conditions for $V[G] \models \mathrm{Refl}({<}\omega, \Gamma)$, we are thus left with discussing how to force this latter principle. For this, in the next chapter, we will devise a notion of forcing for killing a given single counterexample to $\mathrm{Refl}(1, (E^{\mu}_{\omega})^V, \Gamma)$. This discussion will be completed in Chapter 15, where we will find a mean to iterate this process. As a result, we will get a generic extension where $\mathrm{Refl}(1, E^{\mu}_{\omega}, \Gamma)$ holds.

CHAPTER 13

Killing one non-reflecting stationary set

Throughout this chapter, suppose that $(\mathbb{P}, \ell, c)$ is a given Σ-Prikry notion of forcing. Denote $\mathbb{P} = (P, \leq)$ and $\Sigma = \langle \kappa_n \mid n < \omega \rangle$. Also, define κ and μ as in Definition 10.1.3, and assume that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular" and that $\mu^{<\mu} = \mu$. Our universe of sets is denoted by V, and we assume that, for all $n < \omega$, $V^{\mathbb{P}_n} \models \mathrm{Refl}(1, E^\mu_\omega, E^\mu_{<\kappa_n})$.[1] Write $\Gamma := \{\alpha < \mu \mid \omega < \mathrm{cof}^V(\alpha) < \kappa\}$.

13.1 The poset $\mathbb{A}(\mathbb{P}, \dot{T})$

Lemma 13.1.1. *Suppose $r^\star \in P$ forces that $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset T of $(E^\mu_\omega)^V$ that does not reflect in Γ. For each $n < \omega$, write $\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in E^\mu_\omega \times P_n \ \&\ p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}$. Then, for every $q \leq r^\star$, we have $q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary".*

Proof. Towards a contradiction, suppose that there exists $q \leq r^\star$ such that $q \nVdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary". Consequently, we may pick $p \leq^0 q$ such that $p \Vdash_{\mathbb{P}_n}$ "$\dot{T}_n$ is stationary", for $n := \ell(q)$. Let G_n be $\mathbb{P}_n$-generic with $p \in G_n$. As $V[G_n] \models \mathrm{Refl}(1, E^\mu_\omega, E^\mu_{<\kappa_n})$, let us fix $p' \leq^0 p$ in G_n, and some $\delta \in E^\mu_{<\kappa_n}$ of uncountable cofinality such that $p' \Vdash_{\mathbb{P}_n}$ "$\dot{T}_n \cap \delta$ is stationary". As $\mathbb{P}_n$ is κ_n-closed, $\delta \in \Gamma$. In V, pick a club $C \subseteq \delta$ of order type $\mathrm{cof}(\delta)$. Note that $|C| < \kappa_n$. Then, $p' \Vdash_{\mathbb{P}_n}$ "$\dot{T}_n \cap C$ is stationary in δ". Working for a moment in $V[G_n]$, write $A := C \cap (\dot{T}_n)_{G_n}$. Since $\mathbb{P}_n$ is κ_n-closed, we may find $r \in P_n$ extending p' that decides A to be some ground model stationary subset B of δ. Namely,

$$r \Vdash_{\mathbb{P}_n} \text{“}\dot{T}_n \cap \delta \text{ contains the stationary set } \check{B}\text{”}.$$

By definition of the name $\dot{T}_n$, we have that $r \Vdash_{\mathbb{P}} \check{B} \subseteq \dot{T} \cap \delta$. Finally, as $\mathrm{otp}(B) < \kappa$, we infer from Lemma 10.1.10(1) that B remains stationary

[1] In particular, $\kappa_n > \aleph_1$ in $V^{\mathbb{P}_n}$.

in any forcing extension by $\mathbb{P}$. So, $r \le p' \le p \le q \le r^\star$, and $r \Vdash_{\mathbb{P}}$ "$\dot{T} \cap \delta$ is stationary", contradicting the fact that $r^\star$ forces $\dot{T}$ to not reflect in Γ. $\square$

Suppose $r^\star \in P$ forces that $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset T of $(E^\mu_\omega)^V$ that does not reflect in Γ. We shall devise a Σ-Prikry notion of forcing $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ such that $\mathbb{A} = \mathbb{A}(\mathbb{P}, \dot{T})$ projects to $\mathbb{P}$ and kills the stationarity of T. Moreover, $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ will admit a forking projection to $(\mathbb{P}, \ell, c)$ with the mixing property.

Here goes. For all $n < \omega$, write $\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in E^\mu_\omega \times P_n \ \& \ p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}$. Let $I := \omega \setminus \ell(r^\star)$. By Lemma 13.1.1, for all $q \le r^\star$ with $\ell(q) \in I$, $q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary". Thus, for each $n \in I$, we may pick a $\mathbb{P}_n$-name $\dot{C}_n$ for a club subset of μ such that, for all $q \le r^\star$ with $\ell(q) = n$,

$$q \Vdash_{\mathbb{P}_n} \dot{T}_n \cap \dot{C}_n = \emptyset.$$

Consider the binary relation R as defined in Lemma 12.2.2 (page 225) with respect to $\langle \dot{C}_n \mid n \in I \rangle$. A moment reflection makes it clear that, for all $(\alpha, q) \in R$, $q \Vdash_{\mathbb{P}} \check{\alpha} \notin \dot{T}$.

Definition 13.1.2. Suppose $p \in P$. A *labeled p-tree* is a function $S : W(p) \to [\mu]^{<\mu}$ such that for all $q \in W(p)$:

1. $S(q)$ is a closed bounded subset of μ;
2. $S(q') \supseteq S(q)$ whenever $q' \le q$;
3. $q \Vdash_{\mathbb{P}} S(q) \cap \dot{T} = \emptyset$;
4. for all $q' \le q$ in $W(p)$, either $S(q') = \emptyset$ or $(\max(S(q')), q) \in R$.

Definition 13.1.3. For $p \in P$, we say that $\vec{S} = \langle S_i \mid i \le \alpha \rangle$ is a *p-strategy* iff all of the following hold:

1. $\alpha < \mu$;
2. S_i is a labeled p-tree for all $i \le \alpha$;
3. for every $i < \alpha$ and $q \in W(p)$, $S_i(q) \sqsubseteq S_{i+1}(q)$;
4. for every $i < \alpha$ and a pair $q' \le q$ in $W(p)$, $(S_{i+1}(q) \setminus S_i(q)) \sqsubseteq (S_{i+1}(q') \setminus S_i(q'))$;
5. for every limit $i \le \alpha$ and $q \in W(p)$, $S_i(q)$ is the ordinal closure of $\bigcup_{j<i} S_j(q)$. In particular, $S_0(q) = \emptyset$ for all $q \in W(p)$.

This section centers around the following notion of forcing, which is –in essence– a *Prikryrize version* of the standard forcing to shoot a club through the complement of a stationary set.

Definition 13.1.4. Let $\mathbb{A}(\mathbb{P}, \dot{T})$ be the notion of forcing $\mathbb{A} := (A, \trianglelefteq)$, where:

1. $(p, \vec{S}) \in A$ iff $p \in P$, and $\vec{S}$ is either the empty sequence, or a p-strategy;
2. $(p', \vec{S'}) \trianglelefteq (p, \vec{S})$ iff:
   (a) $p' \leq p$;
   (b) $\mathrm{dom}(\vec{S'}) \geq \mathrm{dom}(\vec{S})$;
   (c) $S'_i(q) = S_i(w(p, q))$ for all $i \in \mathrm{dom}(\vec{S})$ and $q \in W(p')$.

For all $p \in P$, denote $\lceil p \rceil^{\mathbb{A}} := (p, \emptyset)$.

Remark 13.1.5. The relation $\trianglelefteq$ is well-defined as $w(p, q) \in W(p)$, the domain of the p-labeled trees S_i.

It is easy to see that $\mathbb{1}_{\mathbb{A}} = \lceil \mathbb{1}_{\mathbb{P}} \rceil^{\mathbb{A}}$.

Lemma 13.1.6. *For every $\nu \geq \mu$, if $\mathbb{P}$ is a subset of H_ν, then so is $\mathbb{A}$.*

Proof. Suppose $\mathbb{P} \subseteq H_\nu$ for a given $\nu \geq \mu$. To prove that $\mathbb{A} \subseteq H_\nu$, it suffices to show that $A \subseteq H_\nu$. Now, each element of A is a pair $(p, \vec{S})$, with $p \in P \subseteq H_\nu$ and $\vec{S} \in {}^{<\mu}({}^{W(p)}[\mu]^{<\mu})$, so, as $\nu \geq \mu$, it suffices to show that ${}^{W(p)}[\mu]^{<\mu} \subseteq H_\nu$. Any element of ${}^{W(p)}[\mu]^{<\mu}$ is a subset of $W(p) \times [\mu]^{<\mu}$ of size $|W(p)|$ and, in particular, a subset of $H_\nu \times H_\mu$ of size $<\mu$ because of Definition 10.1.3(5), so that it is indeed an element of H_ν. □

Lemma 13.1.7. *Suppose $(p, \vec{S}) \in A$, where p is compatible with $r^\star$. For every $\epsilon < \mu$, there exist $\alpha > \epsilon$ and $(q, \vec{T}) \trianglelefteq (p, \vec{S})$ such that, for all $r \in W(q)$, $\mathrm{dom}(\vec{T}) = \alpha + 1$ and $\max(T_\alpha(r)) = \alpha$.*

Proof. Fix $p' \leq p, r^\star$. Define a p'-strategy $\vec{S'}$ with $\mathrm{dom}(\vec{S}) = \mathrm{dom}(\vec{S'})$ using Clause (2c) of Definition 13.1.4, $(p', \vec{S'}) \trianglelefteq (p, \vec{S})$. Next, let $\epsilon < \mu$ be arbitrary. Since $(\mathbb{P}, \ell, c)$ is Σ-Prikry, we infer from Definition 10.1.3(5) that $|W(p')| < \mu$. Thus, by possibly extending ϵ, we may assume that $S'_i(q) \subseteq \epsilon$, for all $q \in W(p')$ and $i \in \mathrm{dom}(\vec{S'})$.

Assume for a moment that $\vec{S'} \neq \emptyset$ and write $\delta + 1 := \mathrm{dom}(\vec{S'})$. As $p' \leq r^\star$, by the very same proof of Claim 12.2.2.2(1), we may fix $(\alpha, q) \in R$ with $\alpha > \delta + \epsilon$ and $q \leq p'$. Define $\vec{T} = \langle T_i : W(q) \to [\mu]^{<\mu} \mid i \leq \alpha \rangle$ by letting for all $r \in W(q)$ and $i \in \mathrm{dom}(\vec{T})$:

$$T_i(r) := \begin{cases} S'_i(w(p', r)), & \text{if } i \leq \delta; \\ S'_\delta(w(p', r)) \cup \{\alpha\}, & \text{otherwise.} \end{cases}$$

It is easy to see that T_i is a labeled q-tree for each $i \leq \alpha$. By Definitions 13.1.3 and 13.1.4, we also have that $(q, \vec{T})$ is a condition in $\mathbb{A}$ and $(q, \vec{T}) \trianglelefteq (p', \vec{S'}) \trianglelefteq (p, \vec{S})$. Altogether, α and $(q, \vec{T})$ are as desired.

In case $\vec{S} = \emptyset$, arguing as before we may find $(\alpha, q) \in R$ with $\alpha > \epsilon$ and $q \leq p'$. Define $\vec{T} = \langle T_i : W(q) \to [\mu]^{<\mu} \mid i \leq \alpha \rangle$ by letting for all $r \in W(q)$ and $i \in \mathrm{dom}(\vec{T})$:

$$T_i(r) := \begin{cases} \emptyset, & \text{if } i = 0; \\ \{\alpha\}, & \text{otherwise.} \end{cases}$$

It is clear that $\vec{T}$ is a q-strategy and that $(q, \vec{T})$ is as desired. $\square$

Theorem 13.1.8. $(r^\star, \emptyset) \Vdash_{\mathbb{A}}$ "*$\dot{T}$ is nonstationary*".

Proof. Let G be $\mathbb{A}$-generic over V, with $(r^\star, \emptyset) \in G$. Work in $V[G]$. Let $\bar{G}$ be the induced generic for $\mathbb{P}$ via π, so that $r^\star \in \bar{G}$.

For all $a = (p, \vec{S})$ in G and $i \in \mathrm{dom}(\vec{S})$, write $d_a^i := \bigcup\{S_i(q) \mid q \in \bar{G} \cap W(p)\}$. Then, let

$$d_a := \begin{cases} d_a^{\max(\mathrm{dom}(\vec{S}))}, & \text{if } \vec{S} \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Claim 13.1.8.1. *Suppose that $a = (p, \vec{S})$ is an element of G.*

In $V[\bar{G}]$, for all $i \in \mathrm{dom}(\vec{S})$, the ordinal closure $\mathrm{cl}(d_a^i)$ of d_a^i is disjoint from T.

Proof. Work in $V[\bar{G}]$. By Lemma 10.1.8(1), for all $n < \omega$, there exists a unique element in $\bar{G} \cap W_n(p)$, which we shall denote by p_n. By Lemma 10.1.8(2), it follows that $\langle p_n \mid n < \omega \rangle$ is $\leq$-decreasing and then, by Definition 13.1.2, for each $i \in \mathrm{dom}(\vec{S})$, $\langle S_i(p_n) \mid n < \omega \rangle$ is a weakly $\subseteq$-increasing (though, not $\sqsubseteq$-increasing) sequence of closed sets that converges to d_a^i.

We now argue by induction on $i \in \mathrm{dom}(\vec{S})$. The base case is trivial, since $d_a^0 = \emptyset$.

Next, suppose that the claim holds for a given $i < \max(\mathrm{dom}(\vec{S}))$, and let us prove it for $i+1$. Let $\delta \in \mathrm{cl}(d_a^{i+1}) \setminus \mathrm{cl}(d_a^i)$ be arbitrary. We have to verify that $\delta \notin T$. By Clauses (3) and (4) of Definition 13.1.2, we may assume that $\delta \in \mathrm{cl}(d_a^{i+1}) \setminus d_a^{i+1}$. In particular, as d_a^{i+1} is the countable union of closed sets, we have $\mathrm{cof}(\delta) = \omega$.

Subclaim 13.1.8.1.1. *There exists a sequence $\langle \delta_n \mid n \in N \rangle$ of ordinals in δ such that:*

- *$N \in [\omega]^\omega$;*
- *$\sup_{n \in N} \delta_n = \delta$;*
- *for every $n \in N$, $n = \min\{\bar{n} < \omega \mid \delta_n \in S_{i+1}(p_{\bar{n}}) \setminus S_i(p_{\bar{n}})\}$.*

Proof. Since $\delta \in \mathrm{cl}(d_a^{i+1}) \setminus (\mathrm{cl}(d_a^i) \cup d_a^{i+1})$ and $\mathrm{cof}(\delta) = \omega$, we may find a strictly increasing sequence $\langle \delta^m \mid m < \omega \rangle$ of ordinals in $d_a^{i+1} \setminus d_a^i$ such that $\sup_{m<\omega} \delta^m = \delta$. For each $m < \omega$, let $n_m < \omega$ be the least such that $\delta^m \in S_{i+1}(p_{n_m}) \setminus S_i(p_{n_m})$. Since $S_{i+1}(p_n)$ is closed for every $n < \omega$, we get that $m \mapsto n_m$ is finite-to-one, so that $N := \{n_m \mid m < \omega\}$ is infinite. For each $n \in N$, set $m(n) := \min\{m < \omega \mid n = n_m\}$ and $\delta_n := \delta^{m(n)}$. Evidently,

$$\min\{\bar{n} < \omega \mid \delta_n \in S_{i+1}(p_{\bar{n}}) \setminus S_i(p_{\bar{n}})\} =$$
$$\min\{\bar{n} < \omega \mid \delta^{m(n)} \in S_{i+1}(p_{\bar{n}}) \setminus S_i(p_{\bar{n}})\} =$$
$$n_{m(n)} = n.$$

In particular, $\langle m(n) \mid n \in N \rangle$ is injective, and $\sup_{n\in N} \delta_n = \delta$. □

Let $\langle \delta_n \mid n \in N \rangle$ be given by the subclaim. By Definition 13.1.3(3), for all $n < m < \omega$, we have $(S_{i+1}(p_n) \setminus S_i(p_n)) \sqsubseteq (S_{i+1}(p_m) \setminus S_i(p_m))$, and hence $\delta = \sup_{n\in N} \sup(S_{i+1}(p_n) \setminus S_i(p_n))$. Recalling that $S_i(p_n) \sqsubseteq S_{i+1}(p_n)$ for all $n < \omega$, we conclude that

$$\delta = \sup_{n\in N} \max(S_{i+1}(p_n)).$$

By Definition 13.1.2(4), we have $(\max(S_{i+1}(p_m)), p_n) \in R$ for all $n \in N$ and $m \geq n$. So, since, for each $m \in I$, $\dot{C}_m$ is a $\mathbb{P}_m$-name for a club, we infer that $(\delta, p_n) \in R$ for all $n \in N$. Recalling the definition of R and the fact that $I = \omega \setminus \ell(r^\star)$, we infer that, for every $n \geq \min(N)$, $p_n \leq r^\star$, and

$$p_n \Vdash_{\mathbb{P}_n} \check{\delta} \in \dot{C}_n.$$

Now, for every $n \geq \min(N)$, by the very choice of $\dot{C}_n$ and since $p_n \leq r^\star$, $p_n \Vdash_{\mathbb{P}_n} \dot{T}_n \cap \dot{C}_n = \emptyset$. Altogether, for a tail of $n < \omega$,

$$p_n \Vdash_{\mathbb{P}_n} \check{\delta} \notin \dot{T}_n.$$

It thus follows from the definition of $\langle \dot{T}_n \mid n < \omega \rangle$ and the fact that $\{p_n \mid n < \omega\} \subseteq \bar{G}$, that $\delta \notin T$.

Finally, suppose $i \in \mathrm{acc}^+(\mathrm{dom}(\vec{S}))$, and that the claim holds below i. Let $\delta \in \mathrm{cl}(d_a^i) \setminus d_a^i$ be arbitrary. By the previous analysis, it is clear that we may pick $N \in [\omega]^\omega$ and an increasing sequence of ordinals $\langle \delta_n \mid n \in N \rangle$ that converges to δ, such that $\delta_n \in S_i(p_n)$ for all $n \in N$. By the last clause of Definition 13.1.3, for each $n \in N$, we may let $j_n < i$ be the least for which there exists $\delta'_n \in S_{j_n+1}(p_n)$ with $\delta_n \geq \delta'_n > \sup\{\delta_m \mid m \in N \cap n\}$.

If $\sup_{n\in N} j_n < i$, then by the induction hypothesis, $\delta \notin T$, and we are done. Suppose that $\sup_{n\in N} j_n = i$. By thinning N out, we may assume that $n \mapsto j_n$ is strictly increasing over N. In particular, for all $m < n$ both from N, we have $\delta'_m \in S_{j_m+1}(p_m) \subseteq S_{j_n}(p_m) \subseteq S_{j_n}(p_n) \sqsubseteq S_{j_n+1}(p_n)$, so that $\delta'_m \le \max(S_{j_n}(p_n)) \le \delta'_n$. Altogether, $\delta = \sup_{n\in N}\max(S_{j_n}(p_n))$. By Definition 13.1.2(4), we have $(\max(S_{j_n}(p_m)), p_n) \in R$ whenever $n \in N$ and $m \in \omega \setminus n$. Thus, as in the successor case, we have $(\delta, p_n) \in R$ for all $n \in N$, and hence $\delta \notin T$. □

By appealing to Lemma 13.1.7, we now fix a sequence $\langle a_\alpha \mid \alpha < \mu\rangle$ of conditions in G such that, for all $\alpha < \mu$, letting $(p, \vec{S}) := a_\alpha$, we have $\mathrm{dom}(\vec{S}) = \alpha + 1$. Denote $D_\alpha := \mathrm{cl}(d_{a_\alpha})$. By the preceding claim and regularity of μ we infer:[2]

Claim 13.1.8.2. *For every $\alpha < \mu$, D_α is a closed bounded subset of μ, disjoint from T.* □

Claim 13.1.8.3. *For every $\alpha < \mu$ and $a' = (p', \vec{S'})$ in G with $\mathrm{dom}(\vec{S'}) = \alpha + 1$, $d_{a'} = d_{a_\alpha}$.*

Proof. Denote $a_\alpha = (p, \vec{S})$. As a_α and a' are in G, we may pick $(r, \vec{T})$ that extends both. In particular, $r \le p, p'$, and, for all $q \in W(r)$, $S_\alpha(w(p,q)) = T_\alpha(q) = S'_\alpha(w(p', q))$. Let $m := \ell(r) - \ell(p)$. Then, for all $k < \omega$, $q \in W_k(r) \cap G$ iff $w(p,q) \in W_{m+k}(p) \cap G$. Note that these sets are singletons. Then

$$d_{a_\alpha} = \bigcup\{S_\alpha(q) \mid q \in \bar{G} \cap W_{\ge m}(p)\} = \bigcup\{T_\alpha(q) \mid q \in \bar{G} \cap W(r)\}.$$

Similarly, we have that $d_{a'} = \bigcup\{T_\alpha(q) \mid q \in \bar{G} \cap W(r)\}$, and so $d_{a_\alpha} = d_{a'}$. □

Claim 13.1.8.4. *For every $\alpha < \beta < \mu$, $D_\alpha \sqsubseteq D_\beta$.*

Proof. Let $\alpha < \beta < \mu$. It suffices to show that $d_{a_\alpha} \sqsubseteq d_{a_\beta}$. Let $(p, \vec{S}) := a_\beta$ and set $a := (p, \vec{S} \restriction (\alpha+1))$. As $a_\beta \trianglelefteq a$, we infer that $a \in G$. Thus, the preceding claim yields $d_a = d_{a_\alpha}$. Let $\langle p_n \mid n < \omega\rangle$ be the decreasing sequence of conditions such that p_n is unique element of $\bar{G} \cap W_n(p)$. Then:

- $d_{a_\alpha} = \bigcup\{S_\alpha(p_n) \mid n < \omega\}$, and
- $d_{a_\beta} = \bigcup\{S_\beta(p_n) \mid n < \omega\}$.

Note that by Clauses (3) and (5) of Definition 13.1.3, for all $n < \omega$, $S_\alpha(p_n) \sqsubseteq S_\beta(p_n)$. Now, let $\gamma < \mu$ be arbitrary. We consider two cases:

▶ If $\gamma \in d_{a_\alpha}$, then we may find $n < \omega$ such that $\gamma \in S_\alpha(p_n)$, and as $S_\alpha(p_n) \sqsubseteq S_\beta(p_n)$, we infer that $\gamma \in d_{a_\beta}$.

▶ If $\gamma \in d_{a_\beta} \setminus d_{a_\alpha}$, then we first find $n < \omega$ such that $\gamma \in S_\beta(p_n)$. In particular, $\gamma \in S_\beta(p_n) \setminus S_\alpha(p_n)$, and as $S_\alpha(p_n) \sqsubseteq S_\beta(p_n)$, this means that $\gamma \ge \sup(S_\alpha(p_n))$. By Definition 13.1.2(2), for all $m \ge n$, $S_\beta(p_n) \subseteq S_\beta(p_m)$, and so it likewise follows that, for all $m \ge n$, $\gamma \ge \sup(S_\alpha(p_m))$. By Definition 13.1.2(2), for all $m < n$, $S_\alpha(p_m) \subseteq S_\alpha(p_n)$, and so $\gamma \ge \sup(S_\alpha(p_n)) \ge \sup(S_\alpha(p_m))$. Altogether, $\gamma \ge \sup(d_{a_\alpha})$. □

[2] See Corollary 11.0.13.

Claim 13.1.8.5. *For every $\epsilon < \mu$, there exists $\alpha < \mu$ such that* $\max(D_\alpha) > \epsilon$.

Proof. By Lemma 13.1.7, we may find $(q, \vec{T})$ in G and $\alpha > \epsilon$ such that, for all $r \in W(q)$, $\mathrm{dom}(\vec{T}) = \alpha + 1$ and $\max(T_\alpha(r)) = \alpha$. By Claim 13.1.8.3, then, $\max(D_\alpha) = \alpha > \epsilon$. □

Put $D := \bigcup\{D_\alpha \mid \alpha < \mu\}$. By Claims 13.1.8.2 and 13.1.8.4, D is closed subset of μ, disjoint from T. By Claim 13.1.8.5, D is unbounded. So T is nonstationary in $V[G]$. □

13.2 $\mathbb{A}(\mathbb{P}, \dot{T})$ and forking projections

The present section will be devoted to prove that $\mathbb{A}(\mathbb{P}, \dot{T})$ admits a forking projection onto $(\mathbb{P}, \ell, c)$ as witnessed by the maps π and $\pitchfork$ of Definition 13.2.3.

Definition 13.2.1. Let $\ell_{\mathbb{A}} := \ell \circ \pi$. Denote $A_n := \{a \in A \mid \ell_{\mathbb{A}}(a) = n\}$, $A^a_n := \{a' \in A \mid a' \trianglelefteq a, \ell_{\mathbb{A}}(a') = \ell_{\mathbb{A}}(a) + n\}$, and $\mathbb{A}_n := (A_n \cup \{\mathbb{1}_{\mathbb{A}}\}, \trianglelefteq)$.

Definition 13.2.2. Define $c_{\mathbb{A}} : A \to H_\mu$ by letting, for all $(p, \vec{S}) \in A$,

$$c_{\mathbb{A}}(p, \vec{S}) := (c(p), \{(i, c(q), S_i(q)) \mid i \in \mathrm{dom}(\vec{S}), q \in W(p)\}).$$

The rest of this section is devoted to verifying that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ is a Σ-Prikry triple that admits a forking projection to $(\mathbb{P}, \ell, c)$.

Definition 13.2.3 (Projection and forking)**.**

- Define $\pi : A \to P$ by stipulating $\pi(p, \vec{S}) := p$.
- Given $a = (p, \vec{S})$ in A, define $\pitchfork(a) : \mathbb{P} \downarrow p \to A$ by letting for each $p' \leq p$, $\pitchfork(a)(p') := (p', \vec{S'})$, where $\vec{S'}$ is the sequence $\langle S'_i : W(p') \to [\mu]^{<\mu} \mid i < \mathrm{dom}(\vec{S})\rangle$ to satisfy:

$$S'_i(q) := S_i(w(p, q)) \text{ for all } i \in \mathrm{dom}(\vec{S'}) \text{ and } q \in W(p'). \qquad (*)$$

Lemma 13.2.4. *Let $a \in A$ and $p' \leq \pi(a)$. Then $\pitchfork(a)(p') \in A$ and $\pitchfork(a)(p') \trianglelefteq a$, so that $\pitchfork(a)$ is a well-defined function from $\mathbb{P} \downarrow \pi(a)$ to $\mathbb{A} \downarrow a$.*

Proof. Set $a := (p, \vec{S})$. If $\vec{S} = \emptyset$, then $\pitchfork(a)(p') = \lceil p' \rceil^{\mathbb{A}}$, and we are done.

Next, suppose that $\mathrm{dom}(\vec{S}) = \alpha + 1$. Let $(p', \vec{S'}) := \pitchfork(a)(p')$. Let $i \leq \alpha$ and we shall verify that S'_i is a p'-labeled tree. To this end, let $q' \leq q$ be arbitrary pair of elements of $W(p')$.

- By Definition 10.1.3(6), we have $w(p, q') \leq w(p, q)$, so that $S'_i(q') = S_i(w(p, q')) \supseteq S_i(w(p, q)) = S'_i(q)$.

- As $q \leq w(p,q)$, $w(p,q) \Vdash_{\mathbb{P}} S_i(w(p,q)) \cap \dot{T} = \emptyset$, so that, since $S'_i(q) = S_i(w(p,q))$, we clearly have $q \Vdash_{\mathbb{P}} S'_i(q) \cap \dot{T} = \emptyset$.

- To avoid trivialities, suppose that $S'_i(q') \neq \emptyset$. Write $\gamma := \max(S_i(w(p,q))$. As $(\gamma, w(p,q)) \in R$ and $q \leq w(p,q)$, we clearly have $(\gamma, q) \in R$. Recalling that $\max(S'_i(q)) = \gamma$, we are done.

To prove that $(p', \vec{S}')$ is a condition in A it remains to argue that $\vec{S}'$ fulfills the requirements described in Clauses (3) and (5) of Definition 13.1.3 but this already follows from the definition of $\vec{S}'$ and the fact that $\vec{S}$ is a p-strategy. Finally $\pitchfork(a)(p') = (p', \vec{S}') \trianglelefteq (p, \vec{S}) = a$ by the very choice of p' and by Definition 13.2.3. □

Let us now check that the pair of functions $(\pitchfork, \pi)$ of Definition 13.2.3 is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell, c)$. We prove this by going over the clauses of Definition 11.0.1.

Lemma 13.2.5.

1. *π is a projection from $\mathbb{A}$ onto $\mathbb{P}$, and $\ell_{\mathbb{A}} = \ell \circ \pi$;*

2. *for all $a \in A$, $\pitchfork(a)$ is an order-preserving function from $(\mathbb{P} \downarrow \pi(a), \leq)$ to $(\mathbb{A} \downarrow a, \trianglelefteq)$;*

3. *for all $p \in P$, $(p, \emptyset)$ is the greatest element of $\{a \in A \mid \pi(a) = p\}$;*

4. *for all $n, m < \omega$ and $b \trianglelefteq^{n+m} a$, $m(a,b)$ exists and satisfies:*
$$m(a,b) = \pitchfork(a)(m(\pi(a), \pi(b)));$$

5. *for all $a \in A$ and $p' \leq \pi(a)$, $\pi(\pitchfork(a)(p')) = p'$;*

6. *for all $a \in A$ and $p' \leq \pi(a)$, $a = (\pi(a), \emptyset)$ iff $\pitchfork(a)(p') = (p', \emptyset)$;*

7. *for all $a \in A$, $a' \trianglelefteq a$ and $r \leq \pi(a')$, $\pitchfork(a')(r) \trianglelefteq \pitchfork(a)(r)$;*

8. *for all $a, a' \in A$, if $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$, then $c(\pi(a)) = c(\pi(a'))$ and $\pitchfork(a)(r) = \pitchfork(a')(r)$ for every $r \leq \pi(a), \pi(a')$.*

Proof. 1. The equality between the lengths comes from Definition 13.2.1 so let us concentrate on proving that π forms a projection. Clearly, $\pi(\mathbb{1}_{\mathbb{A}}) = \mathbb{1}_{\mathbb{P}}$. By Definition 13.1.4, for all $a' \trianglelefteq a$ in A, we have $\pi(a') \leq \pi(a)$. Finally, suppose that $a \in A$ and $p' \leq \pi(a)$, and let us find $a' \trianglelefteq a$ such that $\pi(a') \leq p'$. Put $a' := \pitchfork(a)(p')$. Then it is not hard to check that $a' \trianglelefteq a$ and $\pi(\pitchfork(a)(p')) = p'$, so we are done.

2. Let $a = (p, \vec{S})$ be an arbitrary element of A. By Lemma 13.2.4, $\pitchfork(a)$ is a function from $\mathbb{P} \downarrow \pi(a)$ to $\mathbb{A} \downarrow a$. To see that it is order-preserving, fix $r \leq q$ below $\pi(a)$. By Definition 13.2.3, $\pitchfork(a)(r) = (r, \vec{R})$ and $\pitchfork(a)(q) = (q, \vec{Q})$, where $\vec{R}$ and $\vec{Q}$ are as described in Definition 13.2.3(*). In particular, $\mathrm{dom}(\vec{R}) = \mathrm{dom}(\vec{S}) = \mathrm{dom}(\vec{Q})$. So, to establish that $\pitchfork(a)(r) \trianglelefteq \pitchfork(a)(q)$, it suffices to verify Clause (2c) of Definition 13.1.4. Let $i \in \mathrm{dom}(\vec{R})$ and $r' \in W(r)$ be arbitrary and notice that (*) implies $R_i(r') = S_i(w(p, r'))$. Since $r \leq q$, hence $w(q, r') \in W(q)$, again by (*), $Q_i(w(q, r')) = S_i(w(p, w(q, r')))$. Using Lemma 10.1.9, it is the case that $Q_i(w(q, r')) = S_i(w(p, r'))$, hence $R_i(r') = Q_i(w(q, r'))$.

3. This is easy to see.

4. Write $a = (p, \vec{S})$ and $b = (\bar{p}, \vec{T})$. Appealing to Definition 10.1.3(4), set $p' := m(p, \bar{p})$, so that $\bar{p} \leq^m p' \leq^n p$. Now, let $a' := \pitchfork(a)(p')$. By Definition 13.2.3, a' takes the form $(p', \vec{S'})$, where $\mathrm{dom}(\vec{S'}) = \mathrm{dom}(\vec{S})$, and $S'_i(q) := S_i(w(p, q))$, for all $i \in \mathrm{dom}(\vec{S'})$ and $q \in W(p')$. Observe that if we prove $a' = m(a, b)$, i.e., that a' is the greatest element of $\{c \in A^a_n \mid c \in A^b_m\}$, we will be done with both assertions.

   Claim 13.2.5.1. *a' belongs to $\{c \in A^a_n \mid c \in A^b_m\}$.*

   Proof. By Clauses (1) and (2) together with Clause (5) below, a' is an element of A^a_n, so it suffices to show that $b \trianglelefteq a'$.

   We already know that $\bar{p} \leq^m p'$ and $\mathrm{dom}(\vec{T}) \geq \mathrm{dom}(\vec{S}) = \mathrm{dom}(\vec{S'})$, thus, by virtue of Definition 13.1.4, we are left with verifying that $T_i(q) = S'_i(w(p', q))$ for all $i \in \mathrm{dom}(\vec{S'})$ and $q \in W(\bar{p})$.

   Let i and q be as above. As $b \trianglelefteq a$, we infer that $T_i(q) = S_i(w(p, q))$. By definition of S'_i and Lemma 10.1.9, $S'_i(w(p', q)) = S_i(w(p, w(p', q)) = S_i(w(p, q))$, so that, altogether, $T_i(q) = S'_i(w(p', q))$, as desired. □

   Claim 13.2.5.2. *a' is the greatest element of $\{c \in A^a_n \mid b \in A^b_m\}$.*

   Proof. Let $c = (r, \vec{R})$ be a condition with $(\bar{p}, \vec{T}) \trianglelefteq^m (r, \vec{R}) \trianglelefteq^n (p, \vec{S})$. In particular, $\bar{p} \leq^m r \leq^n p$, so that, since $p' = m(p, \bar{p})$, $r \leq^0 p'$.

   We already know that $r \leq p'$ and $\mathrm{dom}(\vec{R}) \geq \mathrm{dom}(\vec{S}) = \mathrm{dom}(\vec{S'})$. Now, let $i \in \mathrm{dom}(\vec{S'})$ and $q \in W(r)$ be arbitrary. By definition of S'_i and Lemma 10.1.9, $S'_i(w(p', q)) = S_i(w(p, w(p', q)) = S_i(w(p, q))$. As $c \trianglelefteq a$, the latter is equal to $R_i(q)$, hence $c \trianglelefteq a'$, as desired. □

5. This follows immediately from Definition 13.2.3.

6. Suppose that $a \in A$ with $a = (\pi(a), \emptyset)$. By Definition 13.2.3(*), for all $p' \leq \pi(a)$, $\pitchfork(a)(p') = (p', \emptyset)$. Conversely, let $a := (\pi(a), \vec{S})$ and suppose that $\pitchfork(a)(q) = (q, \emptyset)$. Again, by Definition 13.2.3, $\mathrm{dom}(\vec{S}) = \emptyset$, and thus $a = (\pi(a), \emptyset)$, as desired.

7. Let $a \in A$, $a' \trianglelefteq a$ and $r \leq \pi(a')$ be arbitrary, say $a' = (p', \vec{S'})$ and $a = (p, \vec{S})$. By Definition 13.1.4, the following three hold:

   - $p' \leq p$;
   - $\mathrm{dom}(\vec{S}) \leq \mathrm{dom}(\vec{S'})$,
   - $S'_i(q) = S_i(w(p, q))$, for all $i \in \mathrm{dom}(\vec{S})$ and $q \in W(p')$.

   By Definition 13.2.3, $\pitchfork(a)(r) := (r, \vec{S^a})$, where $\mathrm{dom}(\vec{S^a}) = \mathrm{dom}(\vec{S})$ and, for all $i < \mathrm{dom}(\vec{S})$ and $q \in W(r)$, $S^a_i(q) = S_i(w(p, q))$. A similar statement is valid for $\pitchfork(a')(r) = (r, \vec{S^{a'}})$. Notice that $\mathrm{dom}(\vec{S^{a'}}) \geq \mathrm{dom}(\vec{S^a})$ and that, for all $i < \mathrm{dom}(\vec{S^a})$ and $q \in W(r)$, Lemma 10.1.9 yields the following chain of equalities:

   $$S^{a'}_i(q) = S'_i(w(p', q)) = S_i(w(p, w(p', q))) = S_i(w(p, q)) = S^a_i(q).$$

   Altogether we have proved $\pitchfork(a')(r) \trianglelefteq \pitchfork(a)(r)$.

8. Let $a = (p, \vec{S})$ and $a' = (p', \vec{S'})$ be elements of A with $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$. By Definition 13.2.2, then, $c(\pi(a)) = c(\pi(a'))$ and $\mathrm{dom}(\vec{S}) = \mathrm{dom}(\vec{S'})$. Now, let $r \leq \pi(a), \pi(a')$ be arbitrary; we shall show that $\pitchfork(a)(r) = \pitchfork(a')(r)$. Recall that $\pitchfork(a)(r) = (r, \vec{T})$ and $\pitchfork(a')(r) = (r, \vec{T'})$, where $\vec{T}$ and $\vec{T'}$ are the r-strategy of length $\mathrm{dom}(\vec{S})$ given by Definition 13.2.3(*) with respect to a and a', respectively. Therefore, it suffices to show that, for all $i \in \mathrm{dom}(\vec{S})$ and $q \in W(r)$, $S_i(w(p, q)) = S'_i(w(p', q))$. Let $i \in \mathrm{dom}(\vec{S})$ and $q \in W(r)$ be arbitrary. By Lemma 10.1.8(4), $c \restriction W(p)$ is injective. Since $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$, Definition 13.2.2 yields $c``W(p) = c``W(p')$. Consequently, $c(w(p, q)) = c(t)$, where t is the unique element of $W(p')$ that is compatible with $w(p, q)$ and has the same length. Thus, it is not hard to check that $t = w(p', q)$, hence $c(w(p, q)) = c(w(p', q))$. Finally, as $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$ and $c(w(p, q)) = c(w(p', q))$, it is the case that $S_i(w(p, q)) = S'_i(w(p', q))$. $\square$

Remark 13.2.6. Note that the above proof only uses the fact that the triple $(\mathbb{P}, \ell, c)$ is Σ-Prikry together with the defining properties of $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ (that is, Definitions 13.1.4, 13.2.1, 13.2.2 and 13.2.3). In particular, we have not relied on any clause of Definition 10.1.3 for $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$, which have not yet been verified.

13.3 $\mathbb{A}(\mathbb{P}, \dot{T})$ is a Σ-Prikry forcing

Out next task is to verify that $\mathbb{A} := \mathbb{A}(\mathbb{P}, \dot{T})$ forms a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ joint with the functions $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ given in Definition 13.2.1 and Definition 13.2.2. To this aim we will show that this latter triple joint with the pair of functions $(\pi, \pitchfork)$ from Definition 13.2.3 witness the hypotheses of Lemma 11.0.6 and Lemma 11.0.12.

Lemma 13.3.1. *Let $n < \omega$. Suppose that D is a directed family of conditions in $\mathbb{A}_n$, $|D| < \kappa_n$, and for some $\bar{p}$, we have $\pi(a) = \bar{p}$ for all $a \in D$. Then D admits bound in $\mathbb{A}_n$.*

Proof. Since D is directed, given any $a, a' \in D$, we may pick $b \in D$ extending a and a'; now, as $\pi[D] = \{\bar{p}\}$, find $\vec{S}, \vec{S}', \vec{T}$ such that $a = (\bar{p}, \vec{S})$, $a' = (\bar{p}, \vec{S}')$ and $b = (\bar{p}, \vec{T})$, and note that, by Definition 13.1.4, for all $q \in W(\bar{p})$ and $i \in \mathrm{dom}(\vec{S}) \cap \mathrm{dom}(\vec{S}')$, $S_i(q) = \vec{T}_i(q) = S'_i(q)$. It thus follows that D is linearly ordered by $\trianglelefteq$, and, for all $(\bar{p}, \vec{S}), (\bar{p}, \vec{S}') \in D$, $(\bar{p}, \vec{S}) \trianglelefteq (\bar{p}, \vec{S}')$ iff $\mathrm{dom}(\vec{S}) \geq \mathrm{dom}(\vec{S}')$. So $(D, \trianglelefteq)$ is order-isomorphic to $(\theta, \ni)$ for some ordinal $\theta < \kappa_n$. In particular, if θ is a successor ordinal, then D admits bound. So let us assume that θ is a limit ordinal.

For every $\tau < \theta$, let $(\bar{p}, \vec{S}^{\tau})$ denote the τ^{th}-element of D. Set $\alpha := \sup_{\tau<\theta} \mathrm{dom}(\vec{S}^{\tau})$. We define a $\bar{p}$-strategy $\vec{S} = \langle S_i \mid i \leq \alpha \rangle$ as follows. Fix $q \in W(\bar{p})$.

► For $i < \alpha$, $S_i(q)$ is defined as the unique element of $\{S^{\tau}_i(q) \mid \tau < \theta, i \in \mathrm{dom}(\vec{S}^{\tau})\}$.

► For $i = \alpha$, we distinguish two cases:

►► If $S_i(q) = \emptyset$ for all $i < \alpha$, then we continue and let $S_\alpha(q) := \emptyset$;

►► Otherwise, let $S_\alpha(q) := \bigcup_{i<\alpha} S_i(q) \cup \{\beta_q\}$, where

$$\beta_q := \sup\{\max(S_i(q)) \mid i < \alpha, S_i(q) \neq \emptyset\}.$$

Claim 13.3.1.1. $(\bar{p}, \vec{S}) \in A_n$*. In particular,* $(\bar{p}, \vec{S})$ *is bound for* D*.*

Proof. Since, for each $\tau < \theta$, $\vec{S}^{\tau}$ is a $\bar{p}$-strategy, a moment of reflection makes it clear that we only need to verify that S_α is a labeled $\bar{p}$-tree. Let $q \in W(\bar{p})$ be arbitrary. As $\langle S_i(q) \mid i < \alpha \rangle$ is weakly $\sqsubseteq$-increasing sequence of closed sets we only need to verify Clauses (3) and (4) of Definition 13.1.2. First we show that $q \Vdash_{\mathbb{P}} S_\alpha(q) \cap \dot{T} = \emptyset$. For this aim observe that Definition 13.1.2(4) yields $(q, \max(S_i(q)) \in R$, for each $i < \alpha$. Now, for each $r \leq q$ with $\ell(r) \in I$ and $i < \alpha$, $r \Vdash_{\mathbb{P}_{\ell(r)}} \max(S_i(q)) \in \dot{C}_{\ell(r)}$, hence $r \Vdash_{\mathbb{P}_{\ell(r)}} \beta_q \in \dot{C}_{\ell(r)}$, and thus, again by definition of R, $(\beta_q, q) \in R$ (cf. Lemma 12.2.2). Combining Definition 13.1.2(3) with $(\beta_q, q) \in R$, it altogether follows that $q \Vdash_{\mathbb{P}} S_\alpha(q) \cap \dot{T} = \emptyset$.

Finally let $q' \leq q$ and let us check that the last bullet holds. For all $i < \alpha$, since S_i is a $\bar{p}$-strategy, either $S_i(q') = \emptyset$ or $(\max(S_i(q')), q) \in R$. If $S_\alpha(q') \neq \emptyset$, then $\max(S_\alpha(q'))$ is the limit of $\langle \max(S_i(q')) \mid i < \alpha, S_i(q') \neq \emptyset \rangle$, so that, arguing as before, $(\max(S_\alpha(q')), q) \in R$.

Thus we have shown that $(\bar{p}, \vec{S}) \in A_n$ and clearly $(\bar{p}, \vec{S})$ gives bound for D. $\square$

This completes the proof. $\square$

Lemma 13.3.2 (Mixing property)**.** *Let $(p, \vec{S}) = a \in A$, $p' \leq^0 p$, and $m < \omega$. Suppose that $g : W_m(p') \to \mathbb{A} \downarrow a$ is a function such that $\pi \circ g$ is the identity map. Then there exists $b \trianglelefteq^0 a$ with $\pi(b) = p'$ such that $\pitchfork(b)(r) \trianglelefteq^0 g(r)$ for every $r \in W_m(p')$.*

Proof. Using Definition 10.1.3(5), we may find some cardinal $\theta < \mu$ and an injective enumeration $\{r^\tau \mid \tau < \theta\}$ of $W_m(p')$. For each $\tau < \theta$, let $\vec{S}^\tau$ be such that $g(r^\tau) = (r^\tau, \vec{S}^\tau)$. As we are seeking $b \trianglelefteq^0 a$ such that, in particular, for every $\tau < \theta$, $\pitchfork(b)(r) \trianglelefteq^0 g(r^\tau)$, we may make our life harder and assume that $\mathrm{dom}(\vec{S}^\tau)$ is nonzero, say $\mathrm{dom}(\vec{S}^\tau) = \alpha_\tau + 1$.

Set $\alpha := \sup(\mathrm{dom}(\vec{S}))$, so that, if $\mathrm{dom}(\vec{S}) > 0$, then $\mathrm{dom}(\vec{S}) = \alpha + 1$. Set $\alpha' := \sup_{\tau<\theta} \alpha_\tau$, and note that, by regularity of μ, $\alpha \leq \alpha' < \mu$. Our goal is to define a sequence $\vec{T} = \langle T_i : W(p') \to [\mu]^{<\mu} \mid i \leq \alpha' \rangle$ for which $b := (p', \vec{T})$ satisfies the conclusion of the lemma.

As $\{r^\tau \mid \tau < \theta\}$ is an enumeration of the m^{th}-level of the p-tree $W(p')$, Lemma 10.1.8 entails that, for each $q \in W(p')$, there is a unique ordinal $\tau_q < \theta$, such that q is comparable with r^{τ_q}. It thus follows from Lemma 10.1.8(3) that, for all $q \in W(p')$, $\ell(q) - \ell(p') \geq m$ iff $q \in W(r^{\tau_q})$.

Now, for all $i \leq \alpha'$ and $q \in W(p')$, let:

$$T_i(q) := \begin{cases} S^{\tau_q}_{\min\{i,\alpha_{\tau_q}\}}(q), & \text{if } q \in W(r^{\tau_q}); \\ S_{\min\{i,\alpha\}}(w(p,q)), & \text{if } q \notin W(r^{\tau_q}) \text{ and } \alpha > 0; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Claim 13.3.2.1. *Let $i \leq \alpha'$. Then T_i is a labeled p'-tree.*

Proof. Fix $q \in W(p')$ and let us go over the Clauses of Definition 13.1.2.

1. It is clear that in any of the three cases, $T_i(q)$ is a closed bounded subset of μ.

2. Let $q' \leq q$. We focus on the non-trivial case in which $\ell(q') - \ell(p') \geq m$, while $\ell(q) - \ell(p') < m$ and $\alpha > 0$.

   ▶ If $i \leq \alpha$, then $T_i(q) = S_i(w(p,q))$ and $T_i(q') = S_i^{\tau_q}(q')$. In this case, since $w(r^{\tau_q}, q) \leq w(p,q)$ and $\vec{S}$ is a p-strategy, $S_i(w(p,q)) \subseteq$

$S_i(w(r^{\tau_q}, q))$. In addition, since $(r^{\tau_q}, \vec{S}^{\tau_q}) \trianglelefteq (p, \vec{S})$, $S_i(w(r^{\tau_q}, q)) = S_i^{\tau_q}(q)$, so that $T_i(q) \subseteq S_i^{\tau_q}(q)$. But $S_i^{\tau_q}(q) \subseteq S_i^{\tau_q}(q')$, so that altogether $T_i(q) \subseteq T_i(q')$, as desired.

► If $i > \alpha$, then $T_i(q) = S_\alpha(w(p, q))$ and $T_i(q') = S_j^{\tau_q}(q')$ for $j := \min\{i, \alpha_{\tau_q}\}$. In this case, as $\vec{S}$ is a p'-strategy and $\vec{S}^{\tau_q}$ is an r^{τ_q}-strategy, we infer from $(r^{\tau_q}, \vec{S}^{\tau_q}) \trianglelefteq (p, \vec{S})$ that:

$$S_\alpha(w(p, q)) \subseteq S_\alpha(w(r^{\tau_q}, q)) = S_\alpha^{\tau_q}(q) \sqsubseteq S_j^{\tau_q}(q) \subseteq S_j^{\tau_q}(q').$$

Altogether, $T_i(q) \subseteq T_i(q')$, as desired.

3. If $q \in W(r^{\tau_q})$, then this follows from the fact that $S^{\tau_q}_{\min\{i,\alpha_{\tau_q}\}}$ is a labeled r^{τ_q}-tree. If $q \notin W(r^{\tau_q})$ and $\alpha > 0$, then this follows from the fact that $S_{\min\{i,\alpha\}}$ is a labeled p-tree and $q \leq w(p, q)$.

4. Let $q' \leq q$ in $W(p')$ and assume that $T_i(q') \neq \emptyset$. We focus on the case $T_i(q') = S_j(w(p, q'))$, for $j := \min\{i, \alpha\}$. In particular, $\beta := \max(S_j(w(p, q')))$ is well-defined. Clearly $w(p, q') \leq w(p, q)$ so, since S_j is a labeled p-tree, $(\beta, w(p, q')) \in R$. But $q' \leq w(p, q')$, so by the nature of R, we have that $(\beta, q') \in R$, as well. □

Claim 13.3.2.2. *The sequence $\vec{T} = \langle T_i : W(p') \to [\mu]^{<\mu} \mid i \leq \alpha' \rangle$ is a p'-strategy.*

Proof. We need to go over the clauses of Definition 13.1.3. However, Clause (1) is trivial, Clause (2) is established in the preceding claim, and Clauses (3) and (5) follow from the corresponding features of $\vec{S}$ and the $\vec{S}^\tau$'s. Thus, we are left with verifying Clause (4).

To this end, fix $i < \alpha$ and a pair $q' \leq q$ in $W(p')$. We have to show that $(T_{i+1}(q) \setminus T_i(q)) \sqsubseteq (T_{i+1}(q') \setminus T_i(q'))$. As before, the only non-trivial case is when $\ell(q') - \ell(p') \geq m$, while $\ell(q) - \ell(p') < m$ and $\alpha > 0$. To avoid arguing about the empty set, we may also assume that $\alpha > i$. In particular, $\alpha_\tau > i$. So

- $T_{i+1}(q) \setminus T_i(q) = S_{i+1}(w(p, q)) \setminus S_i(w(p, q))$, and
- $T_{i+1}(q') \setminus T_i(q') = S_{i+1}^{\tau_q}(q') \setminus S_i^{\tau_q}(q')$.

Now, as $\vec{S}$ is a p-strategy, we infer that $S_{i+1}(w(p, q)) \setminus S_i(w(p, q)) \sqsubseteq S_{i+1}(w(p, q')) \setminus S_i(w(p, q'))$. But $(r^{\tau_{q'}}, \vec{S}^{\tau_{q'}}) \trianglelefteq (p, \vec{S})$, and hence, for each $j \in \{i, i+1\}$, $S_j^{\tau_{q'}}(q') = S_j(w(p, q'))$. The desired equation now follows immediately. □

Thus, we have established that $b := (p', \vec{T})$ is a legitimate condition.

Claim 13.3.2.3. *$\pi(b) = p'$ and $b \trianglelefteq^0 a$.*

Proof. The first assertion is trivial, and it also implies that $b \trianglelefteq^0 a$ iff $b \trianglelefteq a$, hence, we focus on establishing the latter. As $p' \leq p$ and $\alpha' \geq \alpha$, we are left with verifying Clause (2c) of Definition 13.1.4. To avoid trivialities, suppose also that $\alpha > 0$. Now, let $i \leq \alpha$ and $q \in W(p')$ be arbitrary.

▶ If $\ell(q) < \ell(p') + m$, then we have $T_i(q) = S_i(w(p, q))$, and we are done.

▶ If $\ell(q) \geq \ell(p') + m$, then $T_i(q) = S_i^{\tau_q}(q)$ and, since $(r^{\tau_q}, \vec{S}^{\tau_q}) \trianglelefteq (p, \vec{S})$, $T_i(q) = S_i(w(p, q))$, as desired. □

Claim 13.3.2.4. *Let $\tau < \theta$. For each $q \in W(r^\tau)$, $w(p', q) = w(r^\tau, q)$.*

Proof. As $r^\tau \leq p'$, we have $\{s \mid q \leq s \leq r^\tau\} \subseteq \{s \mid q \leq s \leq p'\}$, so that $w(r^\tau, q) \leq w(p', q)$. In addition, as $w(p', q)$ and r^τ are compatible elements of $W(p')$ (as witnessed by q), we infer from Lemma 10.1.8(2), $\ell(w(p', q)) = \ell(q) \geq \ell(r^\tau)$ and Definition 10.1.3(1), that $w(p', q) \leq r^\tau$, so that $w(p', q) \in \{s \mid q \leq s \leq r^\tau\}$, and hence $w(p', q) \leq w(r^\tau, q)$. □

Recalling Claim 13.3.2.3, to complete our proof, we fix an arbitrary $\tau < \theta$, and turn to show that $\pitchfork(b)(r^\tau) \trianglelefteq^0 g(r^\tau)$. By Lemma 13.2.5(5), $\pi(\pitchfork(b)(r^\tau)) = r^\tau = \pi(g(r^\tau))$, so that we may focus on verifying that $\pitchfork(b)(r^\tau) \trianglelefteq g(r^\tau)$.

To this end, let $\vec{T}^\tau$ denote the r^τ-strategy such that $\pitchfork(b)(r^\tau) = (r^\tau, \vec{T}^\tau)$. By Definition 13.2.3(*), $\mathrm{dom}(\vec{T}^\tau) = \mathrm{dom}(\vec{T}) = \alpha' + 1$, hence $\mathrm{dom}(\vec{S}^\tau) = \alpha_\tau + 1 \leq \alpha' + 1 \leq \mathrm{dom}(\vec{T}^\tau)$. Now, let $i \leq \alpha_\tau$ and $q \in W(r^\tau)$. By Definition 13.2.3(*), $T_i^\tau(q) = T_i(w(p', q))$. By the preceding claim $w(p', q) = w(r^\tau, q)$, so that $q' := w(p', q)$ is in $W(r^\tau)$ and $\tau_{q'} = \tau$. In effect, by definition of $T_i(q')$ (just before Claim 13.3.2.1), we get that $T_i(q') = S_i^\tau(q')$. Altogether, $T_i^\tau(q) = S_i^\tau(q') = S_i^\tau(w(r^\tau, q'))$, as required by Clause (2c) of Definition 13.1.4. □

Corollary 13.3.3. $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ *is a Σ-Prikry triple, and $\mathbb{1}_\mathbb{A} \Vdash_\mathbb{A} \check{\mu} = \check{\kappa}^+$.*

Proof. We first go over the clauses of Definition 10.1.3:

1. By Lemma 11.0.5.
2. By Lemma 11.0.6 together with Lemma 13.3.1.
3. By Lemma 11.0.7 and the fact that $|H_\mu| = \mu$.
4. By Lemma 11.0.8.
5. By Lemma 11.0.9.
6. By Lemma 11.0.10.
7. By Lemma 11.0.12 together with Lemma 13.3.2.

Finally, by Corollary 11.0.13 and the fact that $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P}$ "$\check{\kappa}$ is singular", $\mathbb{1}_\mathbb{A} \Vdash_\mathbb{A} \check{\mu} = \check{\kappa}^+$. □

For the record we make explicit one more feature of the poset $\mathbb{A}$.

Lemma 13.3.4 (Transitivity)**.** *Let $a \in A$. For all $q \leq^0 \pi(a)$ and $r \in W(q)$,*

$$\pitchfork(a)(r) = \pitchfork(\pitchfork(a)(q))(r).$$

Proof. Set $(p, \vec{S}) := a$. Fix an arbitrary $q \leq^0 \pi(a)$, and let $b = \pitchfork(a)(q)$. Fix an arbitrary $r \in W(q)$, and set $(t, \vec{T}) := \pitchfork(a)(r)$ and $(u, \vec{U}) := \pitchfork(b)(r)$. By Definition 13.2.3, it follows that $u = r = t$ and $\mathrm{dom}(\vec{T}) = \mathrm{dom}(\vec{S}) = \mathrm{dom}(\vec{U})$. Once again Definition 13.2.3 yields, for each $i \in \mathrm{dom}(\vec{S})$ and $s \in W(t)$, $T_i(s) = S_i(w(p, s))$. Analogously, for each $i \in \mathrm{dom}(\vec{S})$ and $s \in W(u)$, $Q_i(s) = S_i(w(p, s))$. Altogether, $W(t) = W(u)$, and for each $i \in \mathrm{dom}(\vec{S})$ and $s \in W(u)$, $T_i(s) = Q_i(s)$, as desired. $\square$

13.4 The last word about $\mathbb{A}(\mathbb{P}, \dot{T})$

By putting together all the results of Chapter 12 and Chapter 13 we arrive at the following corollary:

Corollary 13.4.1. *Suppose $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of Laver-indestructible supercompact cardinals, and let $\kappa := \sup(\Sigma)$. Suppose:*

(i) $(\mathbb{P}, \ell, c)$ is a Σ-Prikry notion of forcing, and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular";

(ii) $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$, for some cardinal $\mu = \mu^{<\mu}$;

(iii) $\mathbb{P} \subseteq H_{\mu^+}$;

(iv) $r^\star \in P$ forces that z is a $\mathbb{P}$-name for a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in $\{\alpha < \mu \mid \omega < \mathrm{cof}^V(\alpha) < \kappa\}$.

Then, there exists a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ such that:

1. $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ admits a forking projection to $(\mathbb{P}, \ell, c)$ that has the mixing property;

2. $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$;

3. $\mathbb{A} \subseteq H_{\mu^+}$;

4. $\lceil r^\star \rceil^{\mathbb{A}}$ forces that z is nonstationary.

Proof. By Lemma 12.2.4, for all $n < \omega$, $V^{\mathbb{P}_n} \models \mathrm{Refl}({<}\omega, E^\mu_{<\kappa_n}, E^\mu_{<\kappa_n})$. So, all the blanket assumptions of Section 13 are satisfied, and we obtain a notion of forcing $\mathbb{A} := \mathbb{A}(\mathbb{P}, z)$ together with maps $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ such that, by Corollary 13.3.3, $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ is Σ-Prikry.

Now, Clauses (1) and (2) follow from Lemma 13.2.5 and Corollary 13.3.3, Clause (3) follows from Lemma 13.1.6, and Clause (4) follows from Theorem 13.1.8. $\square$

CHAPTER 14

ITERATIONS OF Σ-PRIKRY FORCINGS

In this chapter we present a viable iteration scheme for Σ-Prikry posets. Hereafter let us assume that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, and denote $\kappa := \sup_{n<\omega} \kappa_n$. Also, assume that μ is some cardinal satisfying $\mu^{<\mu} = \mu$, so that $|H_\mu| = \mu$. The following convention will be applied in advance:

Convention 14.0.1. For all ordinals $\gamma \leq \alpha \leq \mu^+$:

1. $\emptyset_\alpha := \alpha \times \{\emptyset\}$ denotes the α-sequence with constant value $\emptyset$;
2. For a γ-sequence p and an α-sequence q, $p * q$ denotes the unique α-sequence satisfying that for all $\beta < \alpha$:

$$(p * q)(\beta) = \begin{cases} q(\beta), & \text{if } \gamma \leq \beta < \alpha; \\ p(\beta), & \text{otherwise.} \end{cases}$$

Our iteration scheme requires three building blocks:

Building Block I. We are given a Σ-Prikry triple $(\mathbb{Q}, \ell, c)$ such that $\mathbb{Q} = (Q, \leq_Q)$ is a subset of H_{μ^+}, $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q} \check{\mu} = \kappa^+$ and $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q}$ "κ is singular". To streamline the matter, we also require that $\mathbb{1}_\mathbb{Q}$ be equal to $\emptyset$.

Building Block II. For every Σ-Prikry triple $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ such that $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+}, $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} \check{\mu} = \kappa^+$ and $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P}$ "κ is singular", every $r^\star \in P$, and every $\mathbb{P}$-name $z \in H_{\mu^+}$, we are given a corresponding Σ-Prikry triple $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ such that:

(a) $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ admits a forking projection $(\pitchfork, \pi)$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ that has the mixing property;

(b) $\mathbb{1}_\mathbb{A} \Vdash_\mathbb{A} \check{\mu} = \kappa^+$;

(c) $\mathbb{A} = (A, \trianglelefteq)$ is a subset of H_{μ^+}.

By Lemma 11.0.15, we may streamline the matter, and also require that:

(d) each element of A is a pair (x, y) with $\pi(x, y) = x$;

(e) for every $a \in A$, $\lceil \pi(a) \rceil^{\mathbb{A}} = (\pi(a), \emptyset)$;

(f) for every $p, q \in P$, if $c_{\mathbb{P}}(p) = c_{\mathbb{P}}(q)$, then $c_{\mathbb{A}}(\lceil p \rceil^{\mathbb{A}}) = c_{\mathbb{A}}(\lceil q \rceil^{\mathbb{A}})$.

Building Block III. We are given a function $\psi : \mu^+ \to H_{\mu^+}$.

Goal 14.0.2. Our goal is to define a system $\langle (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \langle ⋔_{\alpha,\gamma} \mid \gamma \leq \alpha \rangle) \mid \alpha \leq \mu^+ \rangle$ in such a way that for all $\gamma \leq \alpha \leq \mu^+$:

(i) $\mathbb{P}_\alpha$ is a poset $(P_\alpha, \leq_\alpha)$, $P_\alpha \subseteq {}^\alpha H_{\mu^+}$, and, for all $p \in P_\alpha$, $|B_p| < \mu$, where $B_p := \{\beta + 1 \mid \beta \in \mathrm{dom}(p)\ \&\ p(\beta) \neq \emptyset\}$;

(ii) The map $\pi_{\alpha,\gamma} : P_\alpha \to P_\gamma$ defined by $\pi_{\alpha,\gamma}(p) := p \restriction \gamma$ forms a projection from $\mathbb{P}_\alpha$ to $\mathbb{P}_\gamma$ and $\ell_\alpha = \ell_\gamma \circ \pi_{\alpha,\gamma}$;

(iii) $\mathbb{P}_0$ is a trivial forcing, $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I, and $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{A}$ given by Building Block II when invoked with $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ and a pair $(r^\star, z)$ which is decoded from $\psi(\alpha)$;

(iv) If $\alpha > 0$, then $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ is a Σ-Prikry triple whose greatest element is $\emptyset_\alpha$, $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1}$, and $\emptyset_\alpha \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \kappa^+$;

(v) If $0 < \gamma < \alpha \leq \mu^+$, then the pair of maps $(⋔_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha)$ admits a forking projection to $(\mathbb{P}_\gamma, \ell_\gamma)$; in case $\alpha < \mu^+$, this pair furthermore witnesses that $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ admits a forking projection to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$;

(vi) If $0 < \gamma \leq \beta \leq \alpha$, then, for all $p \in P_\alpha$ and $r \leq_\gamma p \restriction \gamma$, $⋔_{\beta,\gamma}(p \restriction \beta)(r) = (⋔_{\alpha,\gamma}(p)(r)) \restriction \beta$.

Remark 14.0.3. Note the asymmetry between the case $\alpha < \mu$ and the case $\alpha = \mu^+$:

1. By Clause (i), we will have that $\mathbb{P}_\alpha \subseteq H_{\mu^+}$ for all $\alpha < \mu$, but $\mathbb{P}_{\mu^+} \not\subseteq H_{\mu^+}$. Still, $\mathbb{P}_{\mu^+}$ will nevertheless be isomorphic to a subset of H_{μ^+}, as we may identify P_{μ^+} with $\{p \restriction (\sup(B_p) + 1) \mid p \in P_{\mu^+}\}$.

2. Clause (v) puts a weaker assertion for $\alpha = \mu^+$. To see this is necessary, note that by the pigeonhole principle, there must exist two conditions $p, q \in P_{\mu^+}$ and an ordinal $\gamma < \mu^+$ for which $c_{\mu^+}(p) = c_{\mu^+}(q)$, $B_p \subseteq \gamma$, but $B_q \not\subseteq \gamma$. Now, towards a contradiction, assume there is a map $⋔$ witnessing together with $\pi_{\mu^+,\gamma}$ that $(\mathbb{P}_{\mu^+}, \ell_{\mu^+}, c_{\mu^+})$ admits a forking projection to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$. By Definition 11.0.1(8), then, $c_\gamma(p \restriction \gamma) = c_\gamma(q \restriction \gamma)$, so that by Definition 10.1.3(3), we should be

able to pick $r \in (P_\gamma)_0^{p\restriction\gamma} \cap (P_\gamma)_0^{q\restriction\gamma}$, and then by Definition 11.0.1(8), $⋔(p)(r) = ⋔(q)(r)$. Finally, as $B_p \subseteq \gamma$, $p = \lceil p \restriction \gamma \rceil^{\mathbb{P}_{\mu^+}}$,[1] so that, by Definition 11.0.1(6), $⋔(p)(r) = \lceil r \rceil^{\mathbb{P}_{\mu^+}}$. But then $⋔(q)(r) = \lceil r \rceil^{\mathbb{P}_{\mu^+}}$, so that, by Definition 11.0.1(6), $q = \lceil q \restriction \gamma \rceil^{\mathbb{P}_{\mu^+}}$, contradicting the fact that $B_q \nsubseteq \gamma$.

14.0.1 Defining the iteration

For every $\alpha < \mu^+$, fix an injection $\phi_\alpha : \alpha \to \mu$. As $|H_\mu| = \mu$, we may appeal to Theorem 10.2.34 and fix a sequence $\langle e^i \mid i < \mu \rangle$ of functions from μ^+ to H_μ such that for every function $e : C \to H_\mu$ with $C \in [\mu^+]^{<\mu}$, there is $i < \mu$ such that $e \subseteq e^i$.

Remark 14.0.4. Instead of appealing to the Δ-system lemma we will use the sequence of functions $\langle e^i \mid i < \mu \rangle$ to prove that the μ^+-Linked$_0$-property of the iterates is preserved along a μ^+-length iteration. In particular, this will guarantee that these iterations have the μ^+-chain-condition.

The upcoming definition is by recursion on $\alpha \leq \mu^+$, and we continue as long as we are successful. We shall later verify that the described process is indeed successful.

► Let $\mathbb{P}_0 := (\{\emptyset\}, \leq_0)$ be the trivial forcing and $⋔_{0,0}(\emptyset)$ be the identity function.

► Let $\mathbb{P}_1 := (P_1, \leq_1)$, where $P_1 := {}^1Q$ and $p \leq_1 p'$ iff $p(0) \leq_Q p'(0)$. Define ℓ_1 and c_1 by stipulating $\ell_1(p) := \ell(p(0))$ and $c_1(p) = c(p(0))$. For all $p \in P_1$, let $⋔_{1,0}(p) : \{\emptyset\} \to \{p\}$ be the constant function, and let $⋔_{1,1}(p)$ be the identity function.

► Suppose $\alpha < \mu^+$ and that $\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta, \langle ⋔_{\beta,\gamma} | \ \gamma \leq \beta \rangle) \mid \beta \leq \alpha \rangle$ has already been defined. We now define $(\mathbb{P}_{\alpha+1}, \ell_{\alpha+1}, c_{\alpha+1})$ and $\langle ⋔_{\alpha+1,\gamma} | \ \gamma \leq \alpha + 1 \rangle$.

►► If $\psi(\alpha)$ happens to be a triple (β, r, σ), where $\beta < \alpha$, $r \in P_\beta$ and σ is a $\mathbb{P}_\beta$-name, then we appeal to Building Block II with $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$, $r^\star := r * \emptyset_\alpha$ and $z := \{(\xi, p * \emptyset_\alpha) \mid (\xi, p) \in \sigma\}$ to get a corresponding Σ-Prikry poset $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$.

►► Otherwise, we obtain $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ by appealing to Building Block II with $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$, $r^\star := \emptyset_\alpha$ and $z := \emptyset$.

In both cases, we also obtain a projection π from $\mathbb{A} = (A, \trianglelefteq)$ to $\mathbb{P}_\alpha$, and a corresponding forking $⋔$. Furthermore, each element of A is a pair (x, y) with $\pi(x, y) = x$, and, for every $p \in P_\alpha$, $\lceil p \rceil^{\mathbb{A}} = (p, \emptyset)$. Now, define $\mathbb{P}_{\alpha+1} := (P_{\alpha+1}, \leq_{\alpha+1})$ by letting $P_{\alpha+1} := \{x^\frown\langle y \rangle \mid (x, y) \in A\}$, and then let $p \leq_{\alpha+1} p'$ iff $(p \restriction \alpha, p(\alpha)) \trianglelefteq (p' \restriction \alpha, p'(\alpha))$. Put $\ell_{\alpha+1} := \ell_1 \circ \pi_{\alpha+1,1}$ and define $c_{\alpha+1} : P_{\alpha+1} \to H_\mu$ via $c_{\alpha+1}(p) := c_{\mathbb{A}}(p \restriction \alpha, p(\alpha))$.

[1] This is consequence of the fact that $p = (p \restriction \gamma) * \emptyset_{\mu^+} = \lceil p \restriction \gamma \rceil^{\mathbb{P}_{\mu^+}}$. See the discussion at the beginning of Lemma 14.0.8.

Next, let $p \in P_{\alpha+1}$, $\gamma \leq \alpha + 1$ and $r \leq_\gamma p \restriction \gamma$ be arbitrary; we need to define $\pitchfork_{\alpha+1,\gamma}(p)(r)$. For $\gamma = \alpha + 1$, let $\pitchfork_{\alpha+1,\gamma}(p)(r) := r$, and for $\gamma \leq \alpha$, let

$$\pitchfork_{\alpha+1,\gamma}(p)(r) := x^\frown\langle y\rangle \text{ iff } \pitchfork(p \restriction \alpha, p(\alpha))(\pitchfork_{\alpha,\gamma}(p \restriction \alpha)(r)) = (x, y). \qquad (*)$$

► Suppose $\alpha \leq \mu^+$ is a nonzero limit ordinal, and that $\langle(\mathbb{P}_\beta, \ell_\beta, c_\beta, \langle\pitchfork_{\beta,\gamma} \mid \gamma \leq \beta\rangle) \mid \beta < \alpha\rangle$ has already been defined. Define $\mathbb{P}_\alpha := (P_\alpha, \leq_\alpha)$ by letting P_α be all α-sequences p such that $|B_p| < \mu$ and $\forall\beta < \alpha(p \restriction \beta \in P_\beta)$. Let $p \leq_\alpha q$ iff $\forall\beta < \alpha(p \restriction \beta \leq_\beta q \restriction \beta)$. Let $\ell_\alpha := \ell_1 \circ \pi_{\alpha,1}$. Next, we define $c_\alpha : P_\alpha \to H_\mu$, as follows.

►► If $\alpha < \mu^+$, then, for every $p \in P_\alpha$, let

$$c_\alpha(p) := \{(\phi_\alpha(\gamma), c_\gamma(p \restriction \gamma)) \mid \gamma \in B_p\}.$$

►► If $\alpha = \mu^+$, then, given $p \in P_\alpha$, first let $C := \mathrm{cl}(B_p)$, then define a function $e : C \to H_\mu$ by stipulating:

$$e(\gamma) := (\phi_\gamma[C \cap \gamma], c_\gamma(p \restriction \gamma)),$$

and then let $c_\alpha(p) := i$ for the least $i < \mu$ such that $e \subseteq e^i$.

Finally, let $p \in P_\alpha$, $\gamma \leq \alpha$ and $r \leq_\gamma p \restriction \gamma$ be arbitrary; we need to define $\pitchfork_{\alpha,\gamma}(p)(r)$. For $\gamma = \alpha$, let $\pitchfork_{\alpha,\gamma}(p)(r) := r$, and for $\gamma < \alpha$, let $\pitchfork_{\alpha,\gamma}(p)(r) := \bigcup\{\pitchfork_{\beta,\gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \alpha\}$.

14.0.2 Verification

We now verify that for all $\alpha \leq \mu^+$, $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \langle\pitchfork_{\alpha,\gamma} \mid \gamma \leq \alpha\rangle)$ fulfills requirements (i)–(vi) of Goal 14.0.2. By the recursive definition given so far, it is obvious that Clauses (i) and (iii) hold, so we focus on the rest. We commence with Clause (ii)

Lemma 14.0.5. *For all $\gamma \leq \alpha \leq \mu^+$, $\pi_{\alpha,\gamma}$ forms a projection from $\mathbb{P}_\alpha$ to $\mathbb{P}_\gamma$, and $\ell_\alpha = \ell_\gamma \circ \pi_{\alpha,\gamma}$.*

Proof. The case $\gamma = \alpha$ is trivial, so assume $\gamma < \alpha \leq \mu^+$. Clearly, $\pi_{\alpha,\gamma}$ is order-preserving and also $\pi_{\alpha,\gamma}(\emptyset_\alpha) = \emptyset_\gamma$. Let $q \in P_\alpha$ and $q' \in P_\gamma$ be such that $q' \leq_\gamma \pi_{\alpha,\gamma}(q)$. Set $q^* := q' * \emptyset_\alpha$ and notice that $\pi_{\alpha,\gamma}(q^*) = q'$. Altogether, $\pi_{\alpha,\gamma}$ is indeed a projection. For the second part, recall that, for all $\beta \leq \mu^+$, $\ell_\beta := \ell_1 \circ \pi_{\beta,1}$, hence $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1} = \ell_1 \circ (\pi_{\gamma,1} \circ \pi_{\alpha,\gamma}) = (\ell_1 \circ \pi_{\gamma,1}) \circ \pi_{\alpha,\gamma} = \ell_\gamma \circ \pi_{\alpha,\gamma}$. □

Next, we deal with an expanded version of Clause (vi).

Lemma 14.0.6. *For all $0 < \gamma \leq \alpha \leq \mu^+$, $p \in P_\alpha$ and $r \in P_\gamma$ with $r \leq_\gamma p \restriction \gamma$, if we let $q := \pitchfork_{\alpha,\gamma}(p)(r)$, then:*

1. *$q \restriction \beta = \pitchfork_{\beta,\gamma}(p \restriction \beta)(r)$ for all $\beta \in [\gamma, \alpha]$;*

2. *$B_q = B_p \cup B_r$;*

3. *$q \restriction \gamma = r$;*

4. *$p = (p \restriction \gamma) * \emptyset_\alpha$ iff $q = r * \emptyset_\alpha$;*

5. *for all $p' \leq^0_\alpha p$, if $r \leq^0_\gamma p' \restriction \gamma$, then $⫛_{\alpha,\gamma}(p')(r) \leq_\alpha ⫛_{\alpha,\gamma}(p)(r)$.*

Proof. Clause (3) follows from Clause (1) and the fact that $⫛_{\gamma,\gamma}(p \restriction \gamma)$ is the identity function. Clause (4) follows from Clauses (2) and (3).

We now prove Clauses (1), (2) and (5) by induction on $\alpha \leq \mu^+$:

- ▶ The case $\alpha = 1$ is trivial, since, in this case, $\gamma = \beta = \alpha$.

- ▶ Suppose $\alpha = \alpha' + 1$ is a successor ordinal and that the claim holds for α'. Fix arbitrary $0 < \gamma \leq \alpha$, $p \in P_\alpha$ and $r \in P_\gamma$ with $r \leq_\gamma p \restriction \gamma$. Denote $q := ⫛_{\alpha,\gamma}(p)(r)$. Recall that $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1}$ was defined by feeding $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ along with maps π and $⫛$, such that each condition in the poset $\mathbb{A} = (A, \trianglelefteq)$ is a pair (x, y) with $\pi(x, y) = x$. Furthermore, by definition of $⫛_{\alpha,\gamma}$, $q = ⫛_{\alpha,\gamma}(p)(r)$ is equal to $x^\frown\langle y\rangle$, where

  $$(x, y) := ⫛(p \restriction \alpha', p(\alpha'))(⫛_{\alpha',\gamma}(p \restriction \alpha')(r)).$$

  In particular, $q \restriction \alpha' = x = \pi(⫛(p \restriction \alpha', p(\alpha'))(⫛_{\alpha',\gamma}(p \restriction \alpha')(r)))$, which, by Definition 11.0.1(5), is equal to $⫛_{\alpha',\gamma}(p \restriction \alpha')(r)$.

  (1) It follows that for all $\beta \in [\gamma, \alpha)$:

  $$q \restriction \beta = (q \restriction \alpha') \restriction \beta = ⫛_{\alpha',\gamma}(p \restriction \alpha')(r) \restriction \beta = ⫛_{\beta,\gamma}(p \restriction \beta)(r),$$

  where the rightmost equality follows from the induction hypothesis. In addition, the case $\beta = \alpha$ is trivial.

  (2) To avoid trivialities, assume $\gamma < \alpha$. By the previous clause, $q \restriction \alpha' = ⫛_{\alpha,\gamma}(p \restriction \alpha')(r)$. So, by the induction hypothesis, $B_{q\restriction\alpha'} = B_{p\restriction\alpha'} \cup B_r$, and we are left with showing that $\alpha \in B_q$ iff $\alpha \in B_p$. As $q \leq_\alpha p$, we have $B_q \supseteq B_p$, so the forward implication is clear. Finally, if $\alpha \notin B_p$, then $p(\alpha') = \emptyset$, and hence

  $$(x, y) = ⫛(p \restriction \alpha', \emptyset)(⫛_{\alpha',\gamma}(p \restriction \alpha')(r)).$$

  It thus follows from Clause (e) of Building Block II together with the fact that $⫛$ satisfies Clause (6) of Definition 11.0.1 that $(x, y) = (⫛_{\alpha',\gamma}(p \restriction \alpha')(r), \emptyset)$. Recalling that $q = x^\frown\langle y\rangle$, we conclude that $\alpha \notin B_q$, as desired.

  (5) To avoid trivialities, assume $\gamma < \alpha$. Fix $p' \leq^0_\alpha p$ with $r \leq^0_\gamma p' \restriction \gamma$. By definition of $\leq_{\alpha'+1}$, proving $⫛_{\alpha,\gamma}(p')(r) \leq_\alpha ⫛_{\alpha,\gamma}(p)(r)$ amounts to

verifying that $(x', y') \trianglelefteq (x, y)$, where

$$(x', y') := \pitchfork(p' \restriction \alpha', p'(\alpha'))(\pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r)).$$

Now, by the induction hypothesis, $\pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r) \leq_{\alpha'} \pitchfork_{\alpha',\gamma}(p \restriction \alpha')(r)$. So, since $\pitchfork(p \restriction \alpha', p(\alpha'))$ is order-preserving, it suffices to prove that

$$(x', y') \trianglelefteq \pitchfork(p \restriction \alpha', p(\alpha'))(\pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r)).$$

Denote $a := (p \restriction \alpha', p(\alpha'))$ and $a' := (p' \restriction \alpha', p'(\alpha'))$. Then, by Clause (7) of Definition 11.0.1, indeed

$$\pitchfork(a')(\pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r)) \trianglelefteq \pitchfork(a)(\pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r)).$$

► Suppose $\alpha \in \mathrm{acc}(\mu^+ + 1)$ is an ordinal such that, for all $\gamma \leq \beta \leq \alpha' < \alpha$, $p \in P_{\alpha'}$ and $r \in P_\gamma$,

$$\pitchfork_{\beta,\gamma}(p \restriction \beta)(r) = (\pitchfork_{\alpha',\gamma}(p \restriction \alpha')(r)) \restriction \beta.$$

Fix arbitrary $0 < \gamma \leq \alpha$, $p \in P_\alpha$ and $r \in P_\gamma$ with $r \leq_\gamma p \restriction \gamma$. Denote $q := \pitchfork_{\alpha,\gamma}(p)(r)$. By our definition of $\pitchfork_{\alpha,\gamma}$ at the limit stage, we have:

$$q = \bigcup\{\pitchfork_{\beta,\gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \alpha\}.$$

By the induction hypothesis, $\langle \pitchfork_{\beta,\gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \alpha \rangle$ is a $\subseteq$-increasing sequence, and $B_{\pitchfork_{\beta,\gamma}(p \restriction \beta)(r)} = B_{p \restriction \beta} \cup B_r$ whenever $\gamma \leq \beta < \alpha$. It thus follows that q is a legitimate condition, and Clauses (1), (2) and (5) are satisfied. □

Actually we can proof the following strengthening of Lemma 14.0.6(2).

Lemma 14.0.7. *For all $\alpha \leq \mu^+$ and $p \in P_\alpha$, $\pitchfork_{\alpha,0}(p)$ is the constant function $\{\emptyset\} \mapsto \{p\}$. In particular, $B_{\pitchfork_{\alpha,0}(p)(\emptyset)} = B_p$.*

Proof. We argue by induction over $\alpha \leq \mu^+$.

► The case $\alpha = 0$ is trivial.

► Suppose $\alpha = \alpha' + 1$ is a successor ordinal and that the lemma holds for α'. Recall that $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1}$ was defined by feeding $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ along with maps π and $\pitchfork$. By definition, $q = \pitchfork_{\alpha,0}(p)(\emptyset)$ is equal to $x^\frown\langle y \rangle$, where

$$(x, y) := \pitchfork(p \restriction \alpha', p(\alpha'))(\pitchfork_{\alpha',0}(p \restriction \alpha')(\emptyset)).$$

Thus, by the induction hypothesis, $(x, y) = \pitchfork(p \restriction \alpha', p(\alpha'))(p \restriction \alpha')$. We now check that the right-side of the previous equality is actually the same as $(p \restriction \alpha', p(\alpha'))$.

Claim 14.0.7.1. *Let $(\mathbb{P}, \ell_{\mathbb{P}})$ and $(\mathbb{A}, \ell_{\mathbb{A}})$ be as in Definition 11.0.1. Suppose that $(⫛, \pi)$ is a pair of functions witnessing clauses* (2) *and* (4) *of Definition 11.0.1 with respect to $(\mathbb{A}, \ell_{\mathbb{A}})$ and $(\mathbb{P}, \ell_{\mathbb{P}})$. Then, for each $a \in A$, $⫛(a)(\pi(a)) = a$.*

Proof of claim. By Definition 11.0.1(2) it is enough to check that, for all $b \in \mathbb{A} \downarrow a$, $b \trianglelefteq ⫛(a)(\pi(a))$. Nonetheless observe that for each $b \in \mathbb{A} \downarrow a$, clauses (2) and (4) of Definition 11.0.1 imply

$$b \trianglelefteq w(a, b) = ⫛(a)(w(\pi(a), \pi(b))) \trianglelefteq ⫛(a)(\pi(a)),$$

which yields the desired result. □

Applying the above claim with respect to $(\mathbb{P}_{\alpha'}, \ell_{\alpha'})$ and $(\mathbb{A}, \ell_{\mathbb{A}})$ it follows that $⫛(p \restriction \alpha', p(\alpha'))(p \restriction \alpha') = (p \restriction \alpha', p(\alpha'))$. Thus, $⫛_{\alpha,0}(p)(\emptyset) = p$, as wanted.

► If α is a non-zero limit ordinal the result follows by combining the induction hypothesis with the fact that

$$⫛_{\alpha,0}(p)(\emptyset) := \bigcup_{0 \leq \beta < \alpha} ⫛_{\beta,0}(p \restriction \beta)(\emptyset).$$

□

Our next task is to verify Clause (v) of Goal 14.0.2.

Lemma 14.0.8. *Suppose that $\alpha \leq \mu^+$ is such that for all nonzero $\gamma < \alpha$, $(\mathbb{P}_\gamma, c_\gamma, \ell_\gamma)$ is Σ-Prikry. Then, for all nonzero $\gamma \leq \alpha$, the pair of maps $(⫛_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha)$ admits a forking projection to $(\mathbb{P}_\gamma, \ell_\gamma)$. If $\alpha < \mu^+$, then this pair furthermore witnesses that $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ admits a forking projection to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$.*

Proof. Let us go over the clauses of Definition 11.0.1.

Clause (1) is covered by Lemma 14.0.5, Clause (5) is covered by Lemma 14.0.6(3), and Clause (7) is covered by Lemma 14.0.6(5). Clause (3) is obvious, since for all nonzero $\gamma < \alpha$ and $p \in P_\gamma$, a straight-forward verification makes clear that $p * \emptyset_\alpha$ is the greatest element of $\{q \in P_\alpha \mid \pi_{\alpha,\gamma}(q) = p\}$. In effect, Clause (6) follows from Lemma 14.0.6(4).

Thus, we are left with verifying Clauses (2), (4), and (8). The next claim takes care of the first two.

Claim 14.0.8.1. *For all nonzero $\gamma \leq \alpha$ and $p \in P_\alpha$:*

1. *$⫛_{\alpha,\gamma}(p)$ defines an order-preserving function from $(\mathbb{P}_\gamma \downarrow (p \restriction \gamma), \leq_\gamma)$ to $(\mathbb{P}_\alpha \downarrow p, \leq_\alpha)$;*

2. *for all $n, m < \omega$ and $q \leq_\alpha^{n+m} p$, $m(p,q)$ exists and, furthermore,*

$$m(p,q) = \pitchfork_{\alpha,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma)).$$

Proof. We prove the two clauses by induction on $\alpha \leq \mu^+$:

- ▶ The case $\alpha = 1$ is trivial, since, in this case, $\gamma = \alpha$.

- ▶ Suppose $\alpha = \alpha' + 1$ is a successor ordinal and that the claim holds for α'. Let $\gamma \leq \alpha$ and $p \in P_\alpha$ be arbitrary. To avoid trivialities, assume $\gamma < \alpha$. By the induction hypothesis, $\pitchfork_{\alpha',\gamma}(p \restriction \alpha')$ is an order-preserving function from $\mathbb{P}_\gamma \downarrow (p \restriction \gamma)$ to $\mathbb{P}_{\alpha'} \downarrow (p \restriction \alpha')$.

  Recall that $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1}$ was defined by feeding $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ along with a pair of maps $(\pitchfork, \pi)$. Now, as $\pitchfork(p \restriction \alpha', p(\alpha'))$ and $\pitchfork_{\alpha',\gamma}(p \restriction \alpha')$ are both order-preserving, the very definition of $\pitchfork_{\alpha,\gamma}(p \restriction \gamma)$ and $\leq_{\alpha'+1}$ implies that $\pitchfork_{\alpha,\gamma}(p \restriction \gamma)$ is order-preserving. In addition, as (x, y) is a condition in $\mathbb{A}$ iff $x^\frown\langle y\rangle \in P_\alpha$ and as $\pitchfork(p \restriction \alpha', p(\alpha'))$ is an order-preserving function from $\mathbb{P}_{\alpha'} \downarrow (p \restriction \alpha')$ to $\mathbb{A} \downarrow (p \restriction \alpha', p(\alpha'))$, we infer that, for all $r \leq_\gamma p \restriction \gamma$, $\pitchfork_{\alpha,\gamma}(p \restriction \gamma)(r)$ is in $\mathbb{P}_\alpha \downarrow p$.

  Let $q \leq_\alpha^{n+m} p$ for some $n, m < \omega$. Let

  $$(x, y) := m((p \restriction \alpha', p(\alpha')), (q \restriction \alpha', q(\alpha'))).$$

  Trivially, $m(p, q)$ exists and is equal to $x^\frown\langle y\rangle$. We need to show that $m(p,q) = \pitchfork_{\alpha,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma))$. By Definition 11.0.1(4),

  $$(x, y) = \pitchfork(p \restriction \alpha', p(\alpha'))(m(p \restriction \alpha', q \restriction \alpha')).$$

  By the induction hypothesis,

  $$m(p \restriction \alpha', q \restriction \alpha') = \pitchfork_{\alpha',\gamma}(p \restriction \alpha')(m(p \restriction \gamma, q \restriction \gamma)),$$

  and so it follows that

  $$(x, y) = \pitchfork(p \restriction \alpha', p(\alpha'))(\pitchfork_{\alpha',\gamma}(p \restriction \alpha')(m(p \restriction \gamma, q \restriction \gamma))).$$

  Thus, by definition of $\pitchfork_{\alpha,\gamma}$ and the above equation, $\pitchfork_{\alpha,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma))$ is indeed equal to $x^\frown\langle y\rangle$.

- ▶ Suppose $\alpha \in \mathrm{acc}(\mu^+ + 1)$ is an ordinal for which the claim holds below α. Let $\gamma \leq \alpha$ and $p \in P_\alpha$ be arbitrary. To avoid trivialities, assume $\gamma < \alpha$. By Lemma 14.0.6(1), for every $r \in \mathbb{P}_\gamma \downarrow (p \restriction \gamma)$:

  $$\pitchfork_{\alpha,\gamma}(p)(r) = \bigcup_{\gamma \leq \alpha' < \alpha} \pitchfork_{\alpha',\gamma}(p \restriction \alpha')(r).$$

As for all $q, q' \in P_\alpha$, $q \leq_\alpha q'$ iff $\forall \alpha' < \alpha (q \restriction \alpha' \leq_{\alpha'} q' \restriction \alpha')$, the induction hypothesis implies that $\pitchfork_{\alpha,\gamma}(p)$ is an order-preserving function from $\mathbb{P}_\gamma \downarrow (p \restriction \gamma)$ to $\mathbb{P}_\alpha \downarrow p$;

Finally, let $q \leq_\alpha p$; we shall show that $m(p, q)$ exists and is, in fact, equal to $\pitchfork_{\alpha,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma))$. By Lemma 14.0.6(1) and the induction hypothesis,

$$\pitchfork_{\alpha,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma)) = \bigcup_{\gamma \leq \alpha' < \alpha} m(p \restriction \alpha', q \restriction \alpha'),$$

call it r. We shall show that r plays the role of $m(p, q)$.

By definition of $\leq_\alpha$, it is clear that $q \leq^m_\alpha r \leq^n_\alpha p$, so it remains to show that it is the greatest condition in $(P^p_\alpha)_n$ to satisfy this. Fix an arbitrary $s \in (P^p_\alpha)_n$ with $q \leq^m_\alpha s$. For each $\alpha' < \alpha$, $q \restriction \alpha' \leq^m_{\alpha'} s \restriction \alpha' \leq^n_{\alpha'} p \restriction \alpha'$, so that $s \restriction \alpha' \leq_{\alpha'} m(p \restriction \alpha', q \restriction \alpha')$, and thus $s \leq_\alpha r$. Altogether this shows that $r = m(p, q)$.

This completes the proof of the claim. □

We are left with verifying Clause (8) of Definition 11.0.1.

Claim 14.0.8.2. *Suppose $\alpha \neq \mu^+$. For all $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$ and all nonzero $\gamma \leq \alpha$, $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$ and $\pitchfork_{\alpha,\gamma}(p)(r) = \pitchfork_{\alpha,\gamma}(p')(r)$ for every $r \in (P_\gamma)^{p\restriction\gamma}_0 \cap (P_\gamma)^{p'\restriction\gamma}_0$.*

Proof. By induction on $\alpha < \mu^+$:

- ▶ The case $\alpha = 1$ is trivial, since, in this case, $\gamma = \alpha$.

- ▶ Suppose $\alpha = \alpha' + 1$ is a successor ordinal and that the claim holds for α'. Fix an arbitrary pair $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$.

  Recall that $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1}$ was defined by feeding $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ along with a pair of maps $(\pitchfork, \pi)$. By definition of $c_{\alpha'+1}$, we have

$$c_{\mathbb{A}}(p \restriction \alpha', p(\alpha')) = c_\alpha(p) = c_\alpha(p') = c_{\mathbb{A}}(p' \restriction \alpha', p'(\alpha')).$$

  So, as $(\pitchfork, \pi)$ witnesses that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ admits a forking projection to $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$, we have $c_{\alpha'}(p \restriction \alpha') = c_{\alpha'}(p' \restriction \alpha')$, and, for all $r \in (P_{\alpha'})^{p\restriction\alpha'}_0 \cap (P_{\alpha'})^{p'\restriction\alpha'}_0$, $\pitchfork(p \restriction \alpha', p(\alpha'))(r) = \pitchfork(p' \restriction \alpha', p'(\alpha'))(r)$.

  Now, as $c_{\alpha'}(p \restriction \alpha') = c_{\alpha'}(p' \restriction \alpha')$, the induction hypothesis implies that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$ for all nonzero $\gamma \leq \alpha'$. In addition, the case $\gamma = \alpha$ is trivial.

  Finally, fix a nonzero $\gamma \leq \alpha$ and $r \in (P_\gamma)^{p\restriction\gamma}_0 \cap (P_\gamma)^{p'\restriction\gamma}_0$, and let us prove that $\pitchfork_{\alpha,\gamma}(p)(r) = \pitchfork_{\alpha,\gamma}(p')(r)$. To avoid trivialities, assume $\gamma <$

α. It follows from the definition of $\pitchfork_{\alpha,\gamma}$ that $\pitchfork_{\alpha,\gamma}(p)(r) = x^\frown\langle y\rangle$ and $\pitchfork_{\alpha,\gamma}(p')(r) = x'^\frown\langle y'\rangle$, where:

- $(x, y) := \pitchfork(p \restriction \alpha', p(\alpha'))(\pitchfork_{\alpha',\gamma}(p \restriction \alpha')(r))$, and
- $(x', y') := \pitchfork(p' \restriction \alpha', p'(\alpha'))(\pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r))$.

But we have already pointed out that the induction hypothesis implies that $\pitchfork_{\alpha',\gamma}(p \restriction \alpha')(r) = \pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r)$, call it, r'. So, we just need to prove that $\pitchfork(p \restriction \alpha', p(\alpha'))(r') = \pitchfork(p' \restriction \alpha', p'(\alpha'))(r')$. But we also have $c_{\mathbb{A}}(p \restriction \alpha, p(\alpha')) = c_\alpha(p) = c_\alpha(p') = c_{\mathbb{A}}(p' \restriction \alpha, p'(\alpha'))$, so, as $(\pitchfork, \pi)$ witnesses that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ admits a forking projection to $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$, Clause (8) of Definition 11.0.1 implies that $\pitchfork(p \restriction \alpha', p(\alpha'))(r') = \pitchfork(p' \restriction \alpha', p'(\alpha'))(r')$, as desired.

► Suppose $\alpha \in \operatorname{acc}(\mu^+)$ is an ordinal for which the claim holds below α. For any condition $q \in \bigcup_{\alpha' \leq \alpha} P_{\alpha'}$, define a function $f_q : B_q \to H_\mu$ via $f_q(\alpha') := c_{\alpha'}(q \restriction \alpha')$. Now, fix an arbitrary pair $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$. By definition of c_α this means that

$$\{(\phi_\alpha(\gamma), c_\gamma(p \restriction \gamma)) \mid \gamma \in B_p\} = \{(\phi_\alpha(\gamma), c_\gamma(p' \restriction \gamma)) \mid \gamma \in B_{p'}\}.$$

As ϕ_α is injective, $f_p = f_{p'}$. Next, let $\gamma \leq \alpha$ be nonzero; we need to show that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$. The case $\gamma = \alpha$ is trivial, so assume $\gamma < \alpha$.

Now, if $\operatorname{dom}(f_p) \setminus \gamma$ is nonempty, then for $\alpha' := \min(\operatorname{dom}(f_p) \setminus \gamma)$, we have $c_{\alpha'}(p \restriction \alpha') = f_p(\alpha') = f_{p'}(\alpha') = c_{\alpha'}(p' \restriction \alpha')$, and then the induction hypothesis entails $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$. In particular, if $\operatorname{dom}(f_p)$ is unbounded in α, then $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$ for all $\gamma \leq \alpha$.

Next, suppose that $\operatorname{dom}(f_p)$ is bounded in α and let $\delta < \alpha$ be the least ordinal to satisfy $\operatorname{dom}(f_p) \subseteq \delta$. We need to prove by induction on $\gamma \in [\delta, \alpha)$ that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$. The successor step follows from Clauses (e) and (f) of Building Block II, and the limit step follows the fact that for any limit ordinal $\gamma \in [\delta, \alpha)$, the injectivity of ϕ_γ and the equality $f_{p\restriction\gamma} = f_p = f_{p'} = f_{p'\restriction\gamma}$ implies that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$.

Finally, fix a nonzero $\gamma \leq \alpha$ and $r \in (P_\gamma)_0^{p\restriction\gamma} \cap (P_\gamma)_0^{p'\restriction\gamma}$, and let us prove that $\pitchfork_{\alpha,\gamma}(p)(r) = \pitchfork_{\alpha,\gamma}(p')(r)$. To avoid trivialities, assume $\gamma < \alpha$. We already know that, for all $\alpha' \in [\gamma, \alpha)$, $c_{\alpha'}(p \restriction \alpha') = c_{\alpha'}(p' \restriction \alpha')$, and so the induction hypothesis implies that $\pitchfork_{\alpha',\gamma}(p \restriction \alpha')(r) = \pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r)$, and then by Lemma 14.0.6(1):

$$\begin{aligned}\pitchfork_{\alpha,\gamma}(p)(r) &= \bigcup_{\gamma \leq \alpha' < \alpha} \pitchfork_{\alpha',\gamma}(p \restriction \alpha')(r) = \\ &= \bigcup_{\gamma \leq \alpha' < \alpha} \pitchfork_{\alpha',\gamma}(p' \restriction \alpha')(r) = \pitchfork_{\alpha,\gamma}(p')(r),\end{aligned}$$

as desired. $\square$

This completes the proof of Lemma 14.0.6. $\square$

By now, we have verified all clauses of Goal 14.0.2 with the exception of Clause (iv). Before we are in conditions to do that, let us verify that $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ has mixing property for every $\alpha \geq 1$.

Lemma 14.0.9. *Let $1 \leq \alpha \leq \mu^+$, and suppose that, for all nonzero $\gamma < \alpha$, $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$ is a Σ-Prirky triple admitting a forking projection to $(\mathbb{P}_1, \ell_1, c_1)$, as witnessed by the pair of maps $(\pitchfork_{\gamma,1}, \pi_{\gamma,1})$.*

Then $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ has the mixing property. That is, for all $p \in P_\alpha$, $p' \leq_1^0 \pi_{\alpha 1}(p)$ and $m < \omega$, for every $g : W_m(p') \to P_\alpha$ such that, for every $r \in W_m(p')$, $g(r) \leq_\alpha p$ and $\pi_{\alpha,1}(g(r)) = r$, there exists $q \in (P_\alpha)_0^p$ with $\pi_{\alpha,1}(q) = p'$ such that, for every $r \in W_m(p')$, $\pitchfork_{\alpha,1}(q)(r) \leq_\alpha g(r)$.

Proof. Notice that, by Lemma 14.0.8, if, for all nonzero $\gamma < \alpha$, $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$ is a Σ-Prirky triple, then $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha)$ admits a forking projection to $(\mathbb{P}_1, \ell_1)$. We shall prove that $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ has the mixing property. The proof is by induction on $\alpha \in [1, \mu^+]$.

▶ The base case $\alpha = 1$ follows by taking $g := \mathrm{id}$ and $q := p'$, since $\pi_{1,1}$ and $\pitchfork_{1,1}(q)$ are the identity maps.

▶ Suppose that $\alpha = \alpha' + 1$ for a nonzero ordinal $\alpha' < \mu^+$ such that $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ is a Σ-Prirky triple admitting a forking projection to $(\mathbb{P}_1, \ell_1, c_1)$ with the mixing property, as witnessed by the pair $(\pitchfork_{\alpha',1}, \pi_{\alpha',1})$. Suppose that we are given p, p', m and $g : W_m(p') \to P_\alpha$ as in the statement of the lemma.

Derive a function $g' : W_m(p') \to P_{\alpha'}$ via $g'(r) := g(r) \restriction \alpha'$. Since $p' \leq_1^0 \pi_{\alpha',1}(p \restriction \alpha')$, the hypothesis on α' provides us a condition $p_{\alpha'} \in (P_{\alpha'})_0^{p\restriction\alpha'}$ with $\pi_{\alpha',1}(p_{\alpha'}) = p'$ such that, for every $r \in W_m(p')$,

$$\pitchfork_{\alpha',1}(p_{\alpha'})(r) \leq_{\alpha'} g'(r) = g(r) \restriction \alpha'. \tag{14.1}$$

Claim 14.0.9.1. *There exists $q \leq_\alpha p$ with $\pi_{\alpha,\alpha'}(q) = p_{\alpha'}$ such that, for every $r \in W_m(p')$, $\pitchfork_{\alpha,1}(q)(r) \leq_\alpha g(r)$.*

Proof. By Fact 11.0.3(1), for each $s \in W_m(p_{\alpha'})$, we may let r_s denote the unique element of $W_m(p')$ to satisfy $s = \pitchfork_{\alpha',1}(p_{\alpha'})(r_s)$. Now, recall that, by definition of $\mathbb{P}_\alpha = (P_\alpha, \leq_\alpha)$, we have that $P_\alpha := \{x^\frown\langle y\rangle \mid (x, y) \in A\}$ for some poset $(A, \trianglelefteq)$ given by Building Block II together with maps $\pi : A \to P_{\alpha'}$ and $\pitchfork$. Furthermore, each element of A is pair (x, y) for which $\pi(x, y) = x$, and, for all $q \in P_\alpha$ and $r \leq_1 \pi_{\alpha,1}(q)$, $\pitchfork_{\alpha,1}(q)(r) := x^\frown\langle y\rangle$ is defined according to (*) on page 249. Thus, define a function $g_{\alpha'} : W_m(p_{\alpha'}) \to A$ by letting, for each $s \in W_m(p_{\alpha'})$,

$$g_{\alpha'}(s) := \pitchfork(g(r_s) \restriction \alpha', g(r_s)(\alpha'))(\pitchfork_{\alpha',1}(p_{\alpha'})(r_s)).$$

By Equation (14.1) above, $g_{\alpha'}$ is indeed well-defined. Let $a := (p \restriction \alpha', p(\alpha'))$ so that $a \in A$ and $p_{\alpha'} \leq_{\alpha'} \pi(a)$. For every $s \in W_m(p_{\alpha'})$, as $g(r_s) \leq_\alpha p$, we have

$$g_{\alpha'}(s) \trianglelefteq (g(r_s) \restriction \alpha', g(r_s)(\alpha')) \trianglelefteq (p \restriction \alpha', p(\alpha')) = a.$$

Observe that here we are also using Definition 11.0.1(2) with respect to $⫛(g(r_s) \restriction \alpha', g(r_s)(\alpha'))$. In addition, by Definition 11.0.1(5), for every condition $s \in W_m(p_{\alpha'})$, $\pi(g_{\alpha'}(s)) = ⫛_{\alpha',1}(p_{\alpha'})(r_s) = s$, as a consequence of the choice of r_s. Thus, we are in conditions to utilize the mixing property of $(⫛, \pi)$ from Building Block II, and find $b \trianglelefteq^0 a$ with $\pi(b) = p_{\alpha'}$ such that, for every $s \in W_m(p_{\alpha'})$, $⫛(b)(s) \trianglelefteq g_{\alpha'}(s)$.

Let $q := p_{\alpha'}{}^\frown\langle y^* \rangle$ for the unique y^* such that $b = (p_{\alpha'}, y^*)$. To see that q is as desired, let $r \in W_m(p')$ be arbitrary.

Let $s \in W_m(p_{\alpha'})$ be such that $r_s = r$, and write $(x_s, y_s) := ⫛(b)(s)$. Since $⫛_{\alpha,1}(q)(r)$ is defined according to equation (*), $⫛_{\alpha,1}(q)(r) = x_s{}^\frown\langle y_s \rangle$. As

$$(x_s, y_s) = ⫛(b)(s) \trianglelefteq g_{\alpha'}(s) \trianglelefteq (g(r) \restriction \alpha', g(r)(\alpha')),$$

this means that $⫛_{\alpha,1}(q)(r) \leq_\alpha g(r)$, as desired. □

Let q be given by the previous claim. As $\pi_{\alpha,1}(q) = \pi_{\alpha',1}(p_{\alpha'}) = p'$, we are done.

► Suppose that $\alpha \in \mathrm{acc}(\mu^+ + 1)$, and, for every nonzero $\alpha' < \alpha$, $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ is a Σ-Prikry triple admitting a forking projection to $(\mathbb{P}_1, \ell_1, c_1)$ with the mixing property, as witnessed by the pair $(⫛_{\alpha',1}, \pi_{\alpha',1})$.

Suppose that we are given p, p', m and $g : W_m(p') \to P_\alpha$ as in the statement of the lemma. Set $C := \mathrm{cl}(\bigcup_{r \in W_m(p')} B_{g(r)}) \cup \{1, \alpha\}$. Since $|W_m(p')| < \mu$ and, for each r, $|B_r| < \mu$, we have $|C| < \mu$.

We now turn to define a $\subseteq$-increasing sequence $\langle p_\gamma \mid \gamma \in C \rangle \in \prod_{\gamma \in C} (P_\gamma)_0^{p \restriction \gamma}$ such that $p_1 = p'$ and, for all $\gamma \in C$ and $r \in W_m(p')$,

$$⫛_{\gamma,1}(p_\gamma)(r) \leq_\gamma g(r) \restriction \gamma. \tag{14.2}$$

The definition is by recursion on $\gamma \in C$:

- For $\gamma = 1$, we clearly let $p_1 := p'$.
- Suppose $\gamma > 1$ is a non-accumulation point of $C \cap \alpha$. By definition of $C \cap \alpha$, this means that there exists β with $\gamma = \beta + 1$. Let $\bar{\beta} := \sup(C \cap \gamma)$, so that $\bar{\beta} \leq \beta$, and then let $p_\beta := p_{\bar{\beta}} * \emptyset_\beta$. We know that, for every $r \in W_m(p')$, $⫛_{\bar{\beta},1}(p_{\bar{\beta}})(r) \leq_{\bar{\beta}} g(r) \restriction \bar{\beta}$. As the interval $(\bar{\beta}, \beta]$ is disjoint from $\bigcup_{r \in W_m(p')} B_{g(r)}$, furthermore, by Lemma 14.0.6(1) and (2), for every $r \in W_m(p)$,

$$⫛_{\beta,1}(p_\beta)(r) = ⫛_{\bar{\beta},1}(p_{\bar{\beta}})(r) * \emptyset_\beta \leq_\beta (g(r) \restriction \bar{\beta}) * \emptyset_\beta = g(r) \restriction \beta.$$

Next, by Claim 14.0.9.1, we obtain $q \leq_\gamma p \restriction \gamma$ with $\pi_{\gamma,\beta}(q) = p_\beta$ such that for all $r \in W_m(p')$, $\pitchfork_{\gamma,1}(q)(r) \leq_\gamma g(r) \restriction \gamma$. Thus, $p_\gamma := q$ is as desired.

- Suppose $\gamma \in \mathrm{acc}(C)$. Define $p_\gamma := \bigcup_{\delta \in (C \cap \gamma)} p_\delta$. By regularity of μ, we have $|B_{p_\gamma}| < \mu$, so that $p_\gamma \in P_\gamma$. As, for all $\beta \in C \cap \gamma$, $p_\beta \leq_\beta p \restriction \beta$, we also have $p_\gamma \leq_\gamma p \restriction \gamma$. Combining the definition of $\pitchfork_{\gamma,1}(p_\gamma)$, Lemma 14.0.6(1), and the fact that $\sup(C \cap \gamma) = \gamma$, it follows that, for each $r \in W_m(p')$, $\pitchfork_{\gamma,1}(p_\gamma)(r) = \bigcup_{\delta \in (C \cap \gamma)} \pitchfork_{\delta,1}(p_\delta)(r)$. By Equation (14.2), which was provided by the induction hypothesis, $\pitchfork_{\gamma,1}(p_\gamma)(r) \leq_\gamma g(r) \restriction \gamma$.

- Suppose $\gamma = \alpha$, but $\gamma \notin \mathrm{acc}(C)$. In this case, let $\bar{\alpha} := \sup(C \cap \alpha)$, and then set $p_\alpha := p_{\bar{\alpha}} * \emptyset_\alpha$. As the interval $(\bar{\alpha}, \alpha]$ is disjoint from $\bigcup_{r \in W_m(p')} B_{g(r)}$, by Lemma 14.0.6, Clauses (1) and (2), for every $r \in W_m(p)$,

$$\pitchfork_{\alpha,1}(p_\alpha)(r) = \pitchfork_{\bar{\alpha},1}(p_{\bar{\alpha}})(r) * \emptyset_\alpha \leq_\alpha (g(r) \restriction \bar{\alpha}) * \emptyset_\alpha = g(r).$$

Clearly, $q := p_\alpha$ is as desired. □

We are now ready to address Clause (iv) of Goal 14.0.2.

Lemma 14.0.10. *For all nonzero $\alpha \leq \mu^+$, $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ is Σ-Prikry with greatest element $\emptyset_\alpha$, $\ell_\alpha := \ell_1 \circ \pi_{\alpha,1}$, and $\emptyset_\alpha \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \kappa^+$.*

Proof. We argue by induction on $\alpha \leq \mu^+$. The base case $\alpha = 1$ follows from the fact that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I. The successor step $\alpha = \beta + 1$ follows from the fact that $\mathbb{P}_{\beta+1}$ was obtained by invoking Building Block II.

Next, suppose that $\alpha \in \mathrm{acc}(\mu^+ + 1)$ is such that the conclusion of the lemma holds below α. In particular, the hypothesis of Lemma 14.0.8 is satisfied, so that, for all nonzero $\beta \leq \gamma \leq \alpha$, $\pitchfork_{\gamma,\beta}$ and $\pi_{\gamma,\beta}$ witness together that $(\mathbb{P}_\gamma, \ell_\gamma)$ admits a forking projection to $(\mathbb{P}_\beta, \ell_\beta)$. We now go over the clauses of Definition 10.1.3:

(1) The first bullet of Definition 10.1.1 follows from the fact that $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1}$. Next, let $p \in P_\alpha$ be arbitrary. Denote $\bar{p} := \pi_{\alpha,1}(p)$. Since $(\mathbb{P}_1, \ell_1, c_1)$ is Σ-Prikry, we may pick $p' \leq_1 \bar{p}$ with $\ell_1(p') = \ell_1(\bar{p}) + 1$. As the pair $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha)$ admits a forking projection to $(\mathbb{P}_1, \ell_1)$, Fact 11.0.3(2) implies that $\pitchfork_{\alpha,1}(p)(p')$ is an element of $(P_\alpha)_1^p$.

(2) Let $n < \omega$. To see that $(\mathbb{P}_\alpha)_n$ is κ_n-directed-closed, fix an arbitrary directed family $D \subseteq (P_\alpha)_n$ of size $<\kappa_n$. Let $C := \mathrm{cl}(\bigcup_{p \in D} B_p) \cup \{1, \alpha\}$. We shall define a $\subseteq$-increasing sequence $\langle p_\gamma \mid \gamma \in C \rangle \in \prod_{\gamma \in C} (P_\gamma)_n$ such, for all $\gamma \in C$, p_γ is bound for $\{p \restriction \gamma \mid p \in D\}$. The definition is by recursion on $\gamma \in C$:

- For $\gamma = 1$, as $\{p \restriction 1 \mid p \in D\}$ is directed. By the induction hypothesis, $(\mathbb{P}_1, \ell_1, c_1)$ is Σ-Prikry, and hence we may find bound $p_1 \in (P_1)_n$ for the set under consideration.

- Suppose $\gamma > 1$ is a non-accumulation point of $C \cap \alpha$. Let $\beta := \sup(C \cap \gamma)$, and consider the set $A_\gamma := \{⫛_{\gamma,\beta}(p \restriction \gamma)(p_\beta) \mid p \in D\}$. By the induction hypothesis, $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$ is Σ-Prikry. By Clause (7) of Definition 11.0.1, A_γ is directed, and hence we may find bound $p_\gamma \in (P_\gamma)_n$ for the set under consideration.

- Suppose $\gamma \in \mathrm{acc}(C)$. Define $p_\gamma := \bigcup_{\beta \in (C \cap \gamma)} p_\beta$. By regularity of μ, we have $|B_{p_\gamma}| < \mu$, so that $p_\gamma \in P_\gamma$. Now, for all $p \in D$ and all $\beta \in C \cap \gamma$, we have $p_\gamma \restriction \beta = p_\beta \leq_\beta p \restriction \beta$. So, p_γ is indeed a bound for $\{p \restriction \gamma \mid p \in D\}$.

- Suppose $\gamma = \alpha$, but $\gamma \notin \mathrm{acc}(C)$. In this case, let $\bar{\alpha} := \sup(C \cap \alpha)$, and then set $p_\alpha := p_{\bar{\alpha}} * \emptyset_\alpha$. As the interval $(\bar{\alpha}, \alpha]$ is disjoint from $\bigcup_{p \in D} B_p$, for every $p \in D$,
$$p_\alpha = (p_{\bar{\alpha}} \restriction \bar{\alpha}) * \emptyset_\alpha \leq_\alpha (p \restriction \bar{\alpha}) * \emptyset_\alpha = p.$$

Clearly, p_α is bound for D, as desired.

The next claim takes care of Clause (3)

Claim 14.0.10.1. *Suppose $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$. Then, $(P_\alpha)_0^p \cap (P_\alpha)_0^{p'}$ is nonempty.*

Proof. If $\alpha < \mu^+$, then since $(⫛_{\alpha,1}, \pi_{\alpha,1})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ admits a forking projection to $(\mathbb{P}_1, \ell_1, c_1)$, we get from Clause (8) of Definition 11.0.1 that $c_1(p \restriction 1) = c_1(p' \restriction 1)$, and then by Clause (3) of Definition 10.1.3, we may pick $r \in (P_1)_0^{p \restriction 1} \cap (P_1)_0^{p' \restriction 1}$. In effect, Clause (8) of Definition 11.0.1 entails $⫛_{\alpha,1}(p)(r) = ⫛_{\alpha,1}(p')(r)$. Finally, Fact 11.0.3(2) implies that $⫛_{\alpha,1}(p)(r)$ is in $(P_\alpha)_0^p$ and that $⫛_{\alpha,1}(p')(r)$ is in $(P_\alpha)_0^{p'}$. In particular, $(P_\alpha)_0^p \cap (P_\alpha)_0^{p'}$ is nonempty.

From now on, assume $\alpha = \mu^+$. In particular, for all nonzero $\beta < \gamma < \mu^+$, $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$ is a Σ-Prikry triple admitting a forking projection to $(\mathbb{P}_\beta, \ell_\beta, c_\beta)$ as witnessed by the pair $(⫛_{\gamma,\beta}, \pi_{\gamma,\beta})$. To avoid trivialities, assume also that $|\{\mathbb{1}_{\mu^+}, p, p'\}| = 3$. In particular, $C_p := \mathrm{cl}(B_p)$ and $C_{p'} := \mathrm{cl}(B_{p'})$ are nonempty and distinct. Consider the functions $e_p : C_p \to H_\mu$ and $e_{p'} : C_{p'} \to H_\mu$ satisfying:

- for all $\gamma \in C_p$, $e_p(\gamma) := (\phi_\gamma[C_p \cap \gamma], c_\gamma(p \restriction \gamma))$,

- for all $\gamma \in C_{p'}$, $e_{p'}(\gamma) := (\phi_\gamma[C_{p'} \cap \gamma], c_\gamma(p' \restriction \gamma))$.

Write i for the common value of $c_{\mu^+}(p)$ and $c_{\mu^+}(p')$. It follows that, for every $\gamma \in C \cap C'$, $e_p(\gamma) = e^i(\gamma) = e_{p'}(\gamma)$, so that $\phi_\gamma[C_p \cap \gamma] = \phi_\gamma[C_{p'} \cap \gamma]$ and hence $C_p \cap \gamma = C_{p'} \cap \gamma$. Consequently, $R := C_p \cap C_{p'}$ is an initial segment of C_p and an initial segment of $C_{p'}$.

Let $\zeta := \max(C_p \cup C_{p'})$, so that $p = (p \restriction \zeta) * \emptyset_{\mu^+}$ and $p' = (p' \restriction \zeta) * \emptyset_{\mu^+}$. Set $\gamma_0 := \max(R \cup \{0\})$. By the above analysis, $C_p \cap (\gamma_0, \zeta]$ and $C_{p'} \cap (\gamma_0, \zeta]$ are two disjoint closed sets. Consequently, there exists a finite increasing sequence $\langle \gamma_{j+1} \mid j \leq k \rangle$ of ordinals from $C_p \cup C_{p'}$ such that $\gamma_{k+1} = \zeta$ and, for all $j \leq \kappa$:

(i) if $\gamma_{j+1} \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_p$;

(ii) if $\gamma_{j+1} \notin C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_{p'}$.

We now define a sequence $\langle r_j \mid j \leq k+1 \rangle$ in $\prod_{j=0}^{k+1} \left((P_{\gamma_j})_0^{p \restriction \gamma_j} \cap (P_{\gamma_j})_0^{p' \restriction \gamma_j} \right)$, as follows.

(1) Assume that $R \neq \emptyset$.

(*a*) For $j = 0$, since $\gamma_0 \in C_p \cap C_{p'}$, we have $e_p(\gamma_0) = e_{p'}(\gamma_0)$. In particular, $c_{\gamma_0}(p \restriction \gamma_0) = c_{\gamma_0}(p' \restriction \gamma_0)$, and we may indeed pick $r_0 \in (P_{\gamma_0})_0^{p \restriction \gamma_0} \cap (P_{\gamma_0})_0^{p' \restriction \gamma_0}$.

(*b*) Suppose that $j < k+1$, where r_j has already been defined. Let $q := \pitchfork_{\gamma_{j+1}, \gamma_j}(p \restriction \gamma_{j+1})(r_j)$ and $q' := \pitchfork_{\gamma_{j+1}, \gamma_j}(p' \restriction \gamma_{j+1})(r_j)$. By Lemma 14.0.6(2), $B_q = (B_p \cap \gamma_{j+1}) \cup B_{r_j}$ and $B_{q'} = (B_{p'} \cap \gamma_{j+1}) \cup B_{r_j}$. In particular, if $\gamma_{j+1} \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (B_q \cup B_{q'}) \subseteq B_q$, so that $q' = r_j * \emptyset_{\gamma_{j+1}}$ and $q \leq_{\gamma_{j+1}} q'$ by Clauses (4) and (5) of Lemma 14.0.6, respectively. Likewise, if $\gamma_{j+1} \notin C_p$, then $q = r_j * \emptyset_{\gamma_{j+1}}$, so that $q' \leq_{\gamma_{j+1}} q$. Thus, $\{q, q'\} \cap (P_{\gamma_j})_0^{p \restriction \gamma_j} \cap (P_{\gamma_j})_0^{p' \restriction \gamma_j}$ is nonempty, and we may let r_{j+1} be an element of that set.

(2) Assume that $R = \emptyset$.

(*a*) For $j = 0$, $\gamma_0 = 0$ and thus $r_0 := \emptyset$ is the desired condition.

(*b*) For $j = 1$, let $q := \pitchfork_{\gamma_1, 0}(p \restriction \gamma_1)(\emptyset)$ and $q' := \pitchfork_{\gamma_1, 0}(p' \restriction \gamma_1)(\emptyset)$. By Lemma 14.0.7, it follows that $q = p \restriction \gamma_1$ and $q' = p' \restriction \gamma_1$. In particular, if $\gamma_1 \in C_p$, then $(\gamma_0, \gamma_1] \cap (B_q \cup B_{q'}) \subseteq B_q$, so that $q' = \emptyset_{\gamma_1}$ and $q \leq_{\gamma_1} q'$. Likewise, if $\gamma_1 \notin C_p$, then $q = \emptyset_{\gamma_1}$, so that $q' \leq_{\gamma_1} q$. Thus, $\{q, q'\} \cap (P_{\gamma_1})_0^{p \restriction \gamma_1} \cap (P_{\gamma_1})_0^{p' \restriction \gamma_1}$ is nonempty, and we may let r_1 be an element of that set.

(*c*) For any $2 \leq j \leq k+1$, one proceeds by recursion as in case (1)(*b*) with respect to r_{j-1}.

After this process one defines $r := r_{k+1} * \emptyset_{\mu^+}$, which by construction is an element of $(P_{\mu^+})_0^p \cap (P_{\mu^+})_0^{p'}$. □

4. Let $p \in P_\alpha$, $n, m < \omega$ and $q \in (P^p_\alpha)_{n+m}$ be arbitrary. Recalling that $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha)$ admits a forking projection to $(\mathbb{P}_1, \ell_1)$, we infer from Clause (4) of Definition 11.0.1 that $\pitchfork_{\alpha,1}(p)(m(p \restriction 1, q \restriction 1))$ is the greatest element of $\{r \leq^n_\alpha p \mid q \leq^m_\alpha r\}$.

5. Recalling that $(\mathbb{P}_1, \ell_1, c_1)$ is Σ-Prikry, and that $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha)$ admits a forking projection to $(\mathbb{P}_1, \ell_1)$, we infer from Fact 11.0.3(1) that, for every $p \in P_\alpha$, $|W(p)| = |W(p \restriction 1)| < \mu$.

6. Let $p', p \in P_\alpha$ with $p' \leq_\alpha p$. Let $q \in W(p')$ be arbitrary. For all $\gamma < \alpha$, the pair $(\pitchfork_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ witnesses that $(\mathbb{P}_\alpha, \ell_\alpha)$ admits a forking projection to $(\mathbb{P}_\gamma, \ell_\gamma)$, so that by the special case $m = 0$ of Clause (4) of Definition 11.0.1,
$$w(p, q) = \pitchfork_{\alpha,\gamma}(p)(w(p \restriction \gamma, q \restriction \gamma)).$$
Now, for all $q' \leq_\alpha q$, the induction hypothesis implies that, for all $\gamma < \alpha$, $w(p \restriction \gamma, q' \restriction \gamma) \leq_\gamma w(p \restriction \gamma, q \restriction \gamma)$. Together with Clause (5) of Definition 11.0.1, it follows that, for all $\gamma < \alpha$,
$$w(p, q') \restriction \gamma = w(p \restriction \gamma, q' \restriction \gamma) \leq_\gamma w(p \restriction \gamma, q \restriction \gamma) = w(p, q) \restriction \gamma.$$
So, by definition of $\leq_\alpha$, $w(p, q') \leq_\alpha w(p, q)$, as desired.

7. This follows from Lemma 11.0.12, using Lemma 14.0.9.

To complete our proof we shall need the following claim.

Claim 14.0.10.2. *For each* $1 \leq \alpha \leq \mu^+$*,* $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \kappa^+$*.*

Proof. The case $\alpha = 1$ is given by Building Block I. Towards a contradiction, suppose that $1 < \alpha \leq \mu^+$ and that $\mathbb{1}_{\mathbb{P}_\alpha} \nVdash_{\mathbb{P}_\alpha} \check{\mu} = \kappa^+$. As $\mathbb{1}_{\mathbb{P}_1} \Vdash_{\mathbb{P}_1} \check{\mu} = \kappa^+$ and $\mathbb{P}_\alpha$ projects to $\mathbb{P}_1$, this means that there exists $p \in P_\alpha$ such that $p \Vdash_{\mathbb{P}_\alpha} |\mu| \leq |\kappa|$. Since $\mathbb{P}_1$ is isomorphic to the poset $\mathbb{Q}$ of Building Block I, and since $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "κ is singular", $\mathbb{1}_{\mathbb{P}_1} \Vdash_{\mathbb{P}_1}$ "κ is singular". As $\mathbb{P}_\alpha$ projects to $\mathbb{P}_1$, in fact $p \Vdash_{\mathbb{P}_\alpha} \mathrm{cof}(\mu) < \kappa$. Thus, Lemma 10.1.10(2) yields a condition $p' \leq_\alpha p$ with $|W(p')| \geq \mu$, contradicting Clause (5) above. □

This completes the proof of Lemma 14.0.10. □

CHAPTER 15

A model for simultaneous stationary reflection and a failure of the SCH

In this chapter we present the first application of our iteration scheme. Starting with a model with ω-many supercompact cardinals we will construct a generic extension where κ is a singular strong limit cardinal with $\mathrm{cof}(\kappa) = \omega$, $2^\kappa = \kappa^{++}$ and $\mathrm{Refl}(<\omega, \kappa^+)$ holds (cf. Theorem 15.0.3). A weaker form of this theorem was first announced by A. Sharon in his Ph.D. dissertation[1] [Sha05] but a close inspection on his arguments revealed us a gap in the verification of the chain condition of the iteration. Here we will take advantage of the iteration scheme for Σ-Prikry forcings developed in the previous chapter to prove the theorem. An alternative proof of this result which does not use iterate forcing is credited to O. Ben-Neria, Y. Hayut and S. Unger [BNHU19].

Through this section we make the following assumptions:

(א) $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals;

(ב) $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;

(ג) $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$;

(ד) $\Gamma := \{\alpha < \mu \mid \omega < \mathrm{cof}^V(\alpha) < \kappa\}$.

We now want to appeal to the iteration scheme of the previous section. For this, we need to introduce our three building blocks of choice.

Building Block I. We let $(\mathbb{Q}, \ell, c)$ be the Σ-Prikry triple of EBPF for blowing up 2^κ to κ^{++}. By the results of Subsection 10.2.5, $\mathbb{Q}$ is a subset

[1] Sharon just announced the result for standard reflection and not for simultaneous one.

of H_{μ^+} and $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \check{\mu} = \kappa^+$. In addition, κ is singular, so that $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "κ is singular".

Building Block II. For every Σ-Prikry triple $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ such that $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+} and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$, every $r^\star \in P$, and every $\mathbb{P}$-name $z \in H_{\mu^+}$, we are given a corresponding Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ such that:

(a) $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ that has the mixing property;

(b) $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \kappa^+$;

(c) $\mathbb{A} = (A, \trianglelefteq)$ is a subset of H_{μ^+};

(d) each element of A is a pair (x, y) with $\pi(x, y) = x$;

(e) if $r^\star \in P$ forces that z is a $\mathbb{P}$-name for a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ, then

$$\lceil r^\star \rceil^{\mathbb{A}} \Vdash_{\mathbb{A}} \text{“}z \text{ is nonstationary”}.$$

Remark 15.0.1. The above block is obtained as follows.

▶ If $r^\star \in P$ forces that z is a $\mathbb{P}$-name for a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ, then we invoke Corollary 13.4.1.

▶ Otherwise, let $\mathbb{A} := (A, \trianglelefteq)$, where $A := P \times \{\emptyset\}$ and $(p, q) \trianglelefteq (p', q')$ iff $p \leq p'$. Define $\pi : A \to P$ via $\pi(x, y) := x$. Define $\pitchfork$ via $\pitchfork(a)(p) := (p, \emptyset)$ and let $\ell_{\mathbb{A}} := \ell_{\mathbb{P}} \circ \pi$ and $c_{\mathbb{A}} := c_{\mathbb{P}} \circ \pi$. It is straight-forward to verify that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ and $(\pitchfork, \pi)$ satisfy all the requirements.

Building Block III. As $2^\mu = \mu^+$, we fix a surjection $\psi : \mu^+ \to H_{\mu^+}$ such that the preimage of any singleton is cofinal in μ^+.

Now, we appeal to the iteration scheme of Chapter 14 with these building blocks, and obtain, in return, a Σ-Prikry triple $(\mathbb{P}_{\mu^+}, \ell_{\mu^+}, c_{\mu^+})$.

Theorem 15.0.2. *In $V^{\mathbb{P}_{\mu^+}}$ all of the following hold true:*

1. *Any cardinal in V remains a cardinal and retains its cofinality;*
2. *κ is a singular strong limit of countable cofinality;*
3. *$2^\kappa = \kappa^{++}$;*
4. Refl$(<\omega, \kappa^+)$.

Proof. (1) By Lemma 10.1.10(1), no cardinal $\leq \kappa$ changes its cofinality; by Lemma 10.1.10(3), κ^+ is not collapsed, and by Definition 10.1.3(3), no cardinal $> \kappa^+$ changes its cofinality.

(2) In V, κ is a singular strong limit of countable cofinality, and so by Lemma 10.1.10(1), this remains valid in $V^{\mathbb{P}_{\mu^+}}$.

(3) In V, we have that $2^\kappa = \kappa^+$. In addition, by Remark 14.0.3(1), $\mathbb{P}_{\mu^+}$ is isomorphic to a subset of H_{μ^+}, so that, from $|H_{\mu^+}| = \kappa^{++}$, we infer that $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa \leq \kappa^{++}$. Finally, as $\mathbb{P}_{\mu^+}$ projects to $\mathbb{P}_1$ which is isomorphic to $\mathbb{Q}$, we get that $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa \geq \kappa^{++}$. Altogether, $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa = \kappa^{++}$.

(4) As $\kappa^+ = \mu$ and κ is singular, $\mathrm{Refl}({<}\omega, \kappa^+)$ is equivalent to $\mathrm{Refl}({<}\omega, E^\mu_{<\kappa})$. By Corollary 12.2.6, we already know that $V^{\mathbb{P}_{\mu^+}} \models \mathrm{Refl}({<}\omega, \Gamma)$. So, by Proposition 12.3.1, it suffices to verify that $\mathrm{Refl}({<}2, (E^\mu_\omega)^V, \Gamma)$ holds in $V^{\mathbb{P}_{\mu^+}}$.

Let G be $\mathbb{P}_{\mu^+}$-generic over V and hereafter work within $V[G]$. Towards a contradiction, suppose that there exists a subset T of $(E^\mu_\omega)^V$ that does not reflect in Γ. Fix $r^* \in G$ and a $\mathbb{P}_{\mu^+}$-name τ such that τ_G is equal to such a T and such that r^* forces τ to be a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ. Furthermore, we may require that τ be a *nice name*, i.e., each element of τ is a pair $(\check{\xi}, p)$ where $(\xi, p) \in (E^\mu_\omega)^V \times P_{\mu^+}$, and, for all $\xi \in (E^\mu_\omega)^V$, the set $\{p \mid (\check{\xi}, p) \in \tau\}$ is an antichain.

As $\mathbb{P}_{\mu^+}$ satisfies Clause (3) of Definition 10.1.3, $\mathbb{P}_{\mu^+}$ has the μ^+-cc. Consequently, there exists a large enough $\beta < \mu^+$ such that

$$B_{r^*} \cup \bigcup \{B_p \mid (\xi, p) \in \tau\} \subseteq \beta.$$

Let $r := r^* \restriction \beta$ and set

$$\sigma := \{(\xi, p \restriction \beta) \mid (\xi, p) \in \tau\}.$$

From the choice of Building Block III, we may find a large enough $\alpha < \mu^+$ with $\alpha > \beta$ such that $\psi(\alpha) = (\beta, r, \sigma)$. As $\beta < \alpha$, $r \in P_\beta$ and σ is a $\mathbb{P}_\beta$-name, the definition of our iteration at step $\alpha + 1$ involves appealing to Building Block II with $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$, $r^\star := r * \emptyset_\alpha$ and $z := \{(\xi, p * \emptyset_\alpha) \mid (\xi, p) \in \sigma\}$. For any ordinal $\eta < \mu^+$, denote $G_\eta := \pi_{\mu^+,\eta}[G]$. By the choice of β, and as $\alpha > \beta$, we have

$$\tau = \{(\xi, p * \emptyset_{\mu^+}) \mid (\xi, p) \in \sigma\} = \{(\xi, p * \emptyset_{\mu^+}) \mid (\xi, p) \in z\},$$

so that, in $V[G]$,

$$T = \tau_G = \sigma_{G_\beta} = z_{G_\alpha}.$$

In addition, $r^* = r^\star * \emptyset_{\mu^+}$.

Finally, as r^* forces τ is a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ, $r^\star$ forces that z is a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ. So, since $\pi_{\mu^+,\alpha+1}(r^*) = r^\star * \emptyset_{\alpha+1} = \lceil r^\star \rceil^{\mathbb{P}_{\alpha+1}}$ is in $G_{\alpha+1}$, Clause (e) of Building Block II entails that, in $V[G_{\alpha+1}]$, there exists a club in μ which is disjoint from T. In particular, T is nonstationary in $V[G]$, contradicting its very choice. $\square$

Thus, we arrive at the following strengthening of the theorem announced by Sharon in [Sha05].

Theorem 15.0.3. *Suppose that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals, converging to a cardinal κ. Then there exists a cofinality-preserving forcing extension where the following properties hold:*

1. *κ is a singular strong limit cardinal of countable cofinality;*
2. *$2^\kappa = \kappa^{++}$, hence SCH_κ fails;*
3. *$\mathrm{Refl}({<}\omega, \kappa^+)$ holds.*

Proof. Let $\mathbb{L}$ be the direct limit of the iteration $\langle \mathbb{L}_n; \dot{\mathbb{Q}}_n \mid n < \omega \rangle$, where $\mathbb{L}_0$ is the trivial forcing and for each $1 \leq n < \omega$, if $\mathbb{1}_{\mathbb{L}_n} \Vdash_{\mathbb{L}_n}$ "κ_{n-1} is supercompact", then $\mathbb{1}_{\mathbb{L}_n} \Vdash_{\mathbb{L}_n}$ "$\dot{\mathbb{Q}}_n$ is a Laver preparation for κ_n". After forcing with $\mathbb{L}$, each κ_n remains supercompact and, moreover, becomes indestructible under κ_n-directed-closed forcing. Also, cardinals and cofinalities are preserved.

Working in $V^{\mathbb{L}}$, set $\mu := \kappa^+$, $\lambda := \kappa^{++}$ and $\mathbb{C} := \mathrm{Add}(\lambda, 1)$. Finally, work in $W := V^{\mathbb{L} * \dot{\mathbb{C}}}$. Since κ is singular strong limit of cofinality $\omega < \kappa_0$ and κ_0 is supercompact, $2^\kappa = \kappa^+$. Also, thanks to the forcing $\mathbb{C}$, $2^\mu = \mu^+$. Altogether, in W, all the following hold:

- $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-Indestructible supercompact cardinals;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
- $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$;

Now, appeal to Theorem 15.0.2. □

We close the chapter with the following open question referring to the first singular cardinal $\aleph_\omega$.

Question 15.0.4. Is it consistent that $\aleph_\omega$ is a strong limit cardinal for which $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $\mathrm{Refl}({<}\omega, \aleph_{\omega+1})$?

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