



Universidad de Valladolid

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TESIS DOCTORAL:

Equivariant algebraic kk -theory and isomorphism conjectures

Presentada por *Dña Maria Eugenia Ellis Raggio* para optar al grado de
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Dirigida por:
Dr. D. Guillermo Cortiñas

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Contents

Agradecimientos	i
Contenido	v
Resumen	vii
Introduction	1
Chapter 1. Equivariant algebraic kk -theory	7
1. \mathcal{H} -algebras and $\hat{\mathcal{H}}$ -algebras	9
2. Homotopy invariance	14
3. Matrix invariance	20
4. Extensions and classifying maps	21
5. Excision	25
6. Algebraic kk -theory	27
7. Equivariant matrix invariance	30
8. Equivariant algebraic kk -theory	38
Chapter 2. Adjointness theorems in kk -theory.	41
1. Crossed product and trivial action	42
2. Induction and Restriction	46
3. A discrete variant of Green's imprimitivity	53
4. Duality	56
5. Green-Julg theorem for $kk^{\mathcal{H}}$	58
Chapter 3. Isomorphism conjectures with proper coefficients	63
1. G -simplicial sets and model category structures	64
2. Equivariant homology	68
3. K -theory	75
4. K -theory and the standing assumptions	79
5. Homotopy K -theory	86
6. Proper G -rings	87
7. Induction and equivariant homology	91
8. Assembly as a connecting map	93
9. Isomorphism conjectures with proper coefficients	97
Conclusiones	99
Bibliography	105

Contenido

- CAPÍTULO 1: KK-TEORÍA ALGEBRAICA EQUIVARIANTE
 - (1) \mathcal{H} -álgebras y $\hat{\mathcal{H}}$ -álgebras
 - (2) Invarianza homotópica
 - (3) Invarianza matricial
 - (4) Extensiones y morfismos clasificantes
 - (5) Escisión
 - (6) kk -teoría algebraica
 - (7) Invarianza matricial equivariante
 - (8) kk -teoría algebraica equivariante
- CAPÍTULO 2: TEOREMAS DE AJUNCIÓN EN LA KK-TEORÍA
 - (1) Producto cruzado y la acción trivial
 - (2) Inducción y Restricción.
 - (3) Una variante discreta del teorema de imprimitividad de Green.
 - (4) Dualidad
 - (5) Teorema de Green-Julg para $kk^{\mathcal{H}}$
- CAPÍTULO 3: CONJETURAS DE ISOMORFISMO CON COEFICIENTES PROPIOS
 - (1) G -conjuntos simpliciales y estructura de categorías de modelos
 - (2) Homología equivariante
 - (3) K -teoría
 - (4) La K -teoría y las hipótesis globales
 - (5) K -teoría homotópica
 - (6) G -anillos propios
 - (7) Inducción y homología equivariante
 - (8) El morfismo de ensamblaje como un morfismo de conexión
 - (9) Conjetura de isomorfismos con coeficientes propios

Resumen

La kk -teoría algebraica fue introducida por G. Cortiñas y A. Thom en [5]. Esta teoría es una K -teoría bivalente en la categoría de ℓ -álgebras siendo ℓ un anillo conmutativo con unidad. Para cada par de álgebras (A, B) se define un grupo $kk(A, B)$. Se obtiene una categoría $\mathfrak{K}\mathfrak{K}$ cuyos objetos son las álgebras y cuyos morfismos son los elementos de $kk(A, B)$. La categoría $\mathfrak{K}\mathfrak{K}$ es una categoría triangulada y existe un funtor canónico $j : \text{Alg}_\ell \rightarrow \mathfrak{K}\mathfrak{K}$ con ciertas propiedades universales. Estas propiedades son la de invarianza homotópica polinomial, invarianza por matrices y la propiedad de escisión. La definición de la kk -teoría algebraica fue motivada por los trabajos de J. Cuntz [6] y N. Higson [12] sobre las propiedades universales de la KK -teoría de C^* -álgebras definida por Kasparov en [19].

En esta tesis se continua la línea de trabajo de [5] verificando que la kk -teoría algebraica admite una versión equivariante. Sean ℓ un anillo conmutativo con unidad, G un grupo numerable y \mathcal{H} un álgebra de Hopf sobre un cuerpo. En el capítulo 1 definimos una k -teoría bivalente para la categoría de G -álgebras, álgebras G -graduadas, \mathcal{H} -módulo álgebras y \mathcal{H} -comódulo álgebras. Denotamos por $\mathcal{X}\text{-Alg}$ a cualquiera de estas categorías. Se introduce el concepto de \mathcal{X} -estabilidad que consiste en una noción equivariante de invarianza matricial. Posteriormente se establecen las propiedades universales que verifica el funtor canónico $j : \mathcal{X}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{X}}$: invarianza por homotopías polinómicas, \mathcal{X} -estabilidad y la propiedad de escisión.

En el capítulo 2 se estudian los teoremas de adjunción en la kk -teoría algebraica equivariante. Este estudio nos permite seguir completando el diccionario iniciado en [5] entre la kk -teoría algebraica y la KK -teoría de Kasparov. Si G es un grupo numerable, se definen funtores que extienden el producto cruzado por G y la acción trivial

$$\rtimes G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K} \quad \tau : \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{K}\mathfrak{K}^G.$$

El primer resultado de adjunción es una versión algebraica del TEOREMA DE GREEN-JULG y nos permite relacionar los kk^G -grupos con los KH -grupos de la K -teoría homotópica de C. Weibel definida en [32]. Si G es un grupo finito, $1/|G| \in \ell$, B es un álgebra y A es una G -álgebra entonces existe un isomorfismo

$$\psi_{GJ} : kk^G(B^\tau, A) \rightarrow kk(B, A \rtimes G)$$

Aquí B^τ indica la G -álgebra con la acción trivial de G . En particular tomando $B = \ell$,

$$kk^G(\ell, A) \simeq KH(A \rtimes G) \quad \text{para toda } A \in G\text{-Alg}.$$

Para cada subgrupo H de G se definen funtores de inducción y restricción

$$\text{Ind}_H^G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^H \quad \text{Res}_G^H : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^H$$

los cuales son funtores adjuntos. En otras palabras,

$$\mathrm{kk}^G(\mathrm{Ind}_H^G(B), A) \simeq \mathrm{kk}^H(B, \mathrm{Res}_G^H(A)) \quad \text{para } A \in G\text{-Alg y } B \in H\text{-Alg}.$$

En particular si A es una G -álgebra,

$$\mathrm{kk}^G(\ell^{(G)}, A) \simeq \mathrm{KH}(A).$$

Aquí $\ell^{(G)} = \bigoplus_{g \in G} \ell$ con la acción regular de G . Una versión algebraica del TEOREMA DE IMPRIMITIVIDAD DE GREEN nos permite identificar en $\mathcal{R}\mathcal{R}$ al álgebra $A \rtimes H$ con el álgebra $\mathrm{Ind}_H^G A \rtimes G$. También se prueba que $\mathcal{R}\mathcal{R}^G$ la categoría de G -álgebras y $\mathcal{R}\mathcal{R}^{\hat{G}}$ la categoría de álgebras G -graduadas son equivalentes. Esta equivalencia está dada por los funtores

$$\rtimes G : \mathcal{R}\mathcal{R}^G \rightarrow \mathcal{R}\mathcal{R}^{\hat{G}} \quad \hat{G} \rtimes : \mathcal{R}\mathcal{R}^{\hat{G}} \rightarrow \mathcal{R}\mathcal{R}^G.$$

inducidos por los productos cruzados. Es el análogo algebraico a la DUALIDAD DE BAAJ-SKANDALIS, [1]. Por último estudiamos el caso de $\mathcal{R}\mathcal{R}^{\mathcal{H}}$ cuando \mathcal{H} es un álgebra de Hopf de dimensión finita sobre un cuerpo. Existen funtores

$$\tau : \mathcal{R}\mathcal{R} \rightarrow \mathcal{R}\mathcal{R}^{\mathcal{H}} \quad \# : \mathcal{R}\mathcal{R}^{\mathcal{H}} \rightarrow \mathcal{R}\mathcal{R}$$

inducidos por la acción trivial y el producto *smash*. Cuando \mathcal{H} es un álgebra semisimple obtenemos una versión del teorema de Green-Julg en este contexto. En particular si A es un \mathcal{H} -módulo álgebra entonces

$$\mathrm{kk}^{\mathcal{H}}(\ell, A) \simeq \mathrm{KH}(A \# \mathcal{H}).$$

En el último capítulo estudiamos conjeturas de isomorfismo siguiendo la línea de trabajo de [8]. Consideramos estructuras de modelo en la categoría de G -conjuntos simpliciales y en la categoría de G -espacios topológicos. Dichas estructuras están definidas en función de una familia \mathcal{F} de subconjuntos de G y son tales que las equivalencias débiles y las fibraciones son punto a punto. Los objetos cofibrantes son aquellos X tales que el estabilizador G_x pertenece a la familia \mathcal{F} para todo $x \in X$. Al probar que el siguiente par de funtores es una equivalencia de Quillen

$$\mathbf{Top}^G \begin{array}{c} \xrightarrow{\mathrm{Sing}_*} \\ \parallel_* \\ \xleftarrow{\quad} \end{array} \mathbb{S}^G.$$

obtenemos que es equivalente trabajar con un modelo simplicial de espacio o con un modelo topológico.

Decimos que un funtor $H : \mathbb{S}^G \rightarrow \mathbf{Spt}$ de la categoría de G -conjuntos simpliciales en la categoría de espectros satisface la (G, \mathcal{F}) -conjetura de isomorfismo, si para el reemplazo cofibrante $\pi : \mathcal{E}(G, \mathcal{F}) \rightarrow *$ en la categoría de modelos mencionada anteriormente el morfismo

$$H(\pi) : H(\mathcal{E}(G, \mathcal{F})) \rightarrow H(*)$$

es una equivalencia. Si $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ es un funtor y R es un G -anillo con unidad, podemos construir, siguiendo la línea Davis-Luck [8], un funtor

$$H^G(-, E(R)) : \mathbb{S}^G \rightarrow \mathbf{Spt}$$

tal que $H^G(*, E(R)) = E(R \rtimes G)$. La (G, \mathcal{F}, E, R) -conjetura de isomorfismo es la (G, \mathcal{F}) -conjetura para el funtor $H^G(-, E(R))$. Probamos que considerando unas condiciones

mínimas sobre E , las **STANDING ASSUMPTIONS** 3.2.5 (condiciones que satisfacen K y KH , ver proposiciones 3.4.18 y 3.5.3), $H^G(-, E(A))$ está definido no sólo para los G -anillos con unidad, si no que está definido también para todo G -anillo A que satisface E -escisión. Más aún, probamos que si

$$(1) \quad 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

es una sucesión exacta de anillos E -escisivos y X es un G -conjunto simplicial, entonces

$$H^G(X, E(A')) \rightarrow H^G(X, E(A)) \rightarrow H^G(X, E(A''))$$

es una fibración homotópica. Esta es una propiedad necesaria para establecer un análogo algebraico al método dual-Dirac que es un método usado en la prueba de la conjetura de Baum-Connes para algunos grupos. Otra propiedad básica que nos proporciona sucesiones (1) en las cuales al menos uno de los anillos satisface la conjetura de isomorfismos, es el teorema 3.7.3. En éste se muestra que si E satisface las *standing assumptions* y A es un anillo E -escisivo de la forma

$$(2) \quad A = \bigoplus_i \text{Ind}_{K_i}^G B_i$$

en donde B_i es un K_i -anillo y $K_i \in \mathcal{F}$ para todo i , entonces el funtor $H^G(-, E(A))$ lleva (G, \mathcal{F}) -equivalencias en equivalencias. En particular la (G, \mathcal{F}, E, A) -conjetura de isomorfismo se satisface. Usamos este hecho en la sección 8 para mostrar que bajo otros supuestos adicionales 3.8.1 (los cuales siguen siendo verificados por $E = K$ y KH), para cada G -anillo E -escisivo A existe de manera funtorial una sucesión de G -anillos E -escisivos

$$\mathfrak{F}^0 B \rightarrow \mathfrak{F}^\infty B \rightarrow \mathfrak{F}^\infty B / \mathfrak{F}^0 B$$

y un morfismo natural $A \rightarrow \mathfrak{F}^0 B$ tal que

- i) $H^G(X, E(A)) \rightarrow H^G(X, E(\mathfrak{F}^0 B))$ es una equivalencia para todo G -conjunto simplicial X .
- ii) $H^G(X, E(\mathfrak{F}^\infty B)) \rightarrow *$ es una equivalencia si X es (G, \mathcal{F}) -cofibrante.
- iii) $H^G(-, E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B))$ lleva (G, \mathcal{F}) -equivalencias en equivalencias.

A partir de esta sucesión obtenemos que

$$H^G(\mathcal{E}(G, \mathcal{F}), E(A)) \rightarrow E(A \rtimes G)$$

es una equivalencia si y sólo si el morfismo de conexión

$$\Omega(E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B \rtimes G)) \rightarrow E(A \rtimes G)$$

es una equivalencia. En particular lo anterior se aplica cuando $E = K, KH$. En el teorema 3.9.2 probamos que bajo hipótesis más fuertes en E , la cual la más importante es que E satisface escisión (e.g. KH satisface pero K no), entonces la (G, \mathcal{F}, E, A) -conjetura de isomorfismo es verdad para cuando A es un anillo (G, \mathcal{F}) -propio. Si X es un conjunto simplicial localmente finito con una acción de G entonces el G -anillo A es propio sobre X si este es un álgebra sobre el anillo $\mathbb{Z}^{(X)}$ de funciones polinomiales finitamente soportadas en X , la acción es compatible con la acción de G en A y en X y $\mathbb{Z}^{(X)} \cdot A = A$. Decimos que A es (G, \mathcal{F}) -propio si es propio sobre algún conjunto simplicial X localmente finito en el cual G actúa de manera que los estabilizadores pertenecen a la familia \mathcal{F} . Por ejemplo un álgebra es de la forma (2) si y solamente si es propia sobre algún G -conjunto simplicial

cero dimensional de la forma $X = \coprod G/K_i$. La noción de anillo (G, \mathcal{F}) -propio usada aquí es la noción algebraica de la G - C^* -álgebra propia, y el teorema 3.9.2 es una versión algebraica del conocido hecho de que la conjetura de Baum-Connes es cierta para las G - C^* -álgebras [11].

Introduction

Some concepts in operator algebras and in non-commutative geometry can fit into algebraic contexts; this observation can lead to interesting results in algebraic settings. In this thesis we analyse some items in a dictionary between operator algebras and algebras without topological structure.

Algebraic kk -theory has been introduced by G. Cortiñas and A. Thom in [5]. This is a bivariant K -theory on the category of ℓ -algebras where ℓ is a commutative ring with unit. For each pair (A, B) of ℓ -algebras a group $kk(A, B)$ is defined. A category $\mathfrak{K}\mathfrak{K}$ is obtained whose objects are the ℓ -algebras and whose morphisms are the elements of the group $kk(A, B)$. The category $\mathfrak{K}\mathfrak{K}$ is triangulated and there is a canonical functor $j : \text{Alg}_\ell \rightarrow \mathfrak{K}\mathfrak{K}$ with universal properties. These properties are algebraic homotopy invariance, matrix invariance and excision.

The definition of algebraic kk -theory was inspired by the work of J. Cuntz [6] and N. Higson [12] on the universal properties of Kasparov KK -theory [19]. The KK -theory of separable C^* -algebras is a common generalization both of topological K -homology and topological K -theory as an additive bivariant functor. Let A, B be separable C^* -algebras then

$$(3) \quad KK_*(\mathbb{C}, B) \simeq K_*^{top}(B) \quad KK^*(A, \mathbb{C}) = K_{hom}^*(A)$$

here $K_*^{top}(B)$ denotes the K -theory of B and $K_{hom}^*(A)$ the K -homology of A . J. Cuntz in [6] gave another equivalent definition of the original one given in [19]. This new approach allowed to put bivariant K -theory in algebraic context. Higson in [12] stated the universal property of KK whose algebraic analogue is studied in [5], where also an analogue of (3) is proved. On the algebraic side, if A is an ℓ -algebra then

$$kk(\ell, A) \simeq KH(A)$$

here KH is Weibel's homotopy K -theory defined in [32]. We can start to build a dictionary between Kasparov's KK -theory and algebraic kk -theory in the following way

Kasparov's KK-theory	algebraic kk -theory
bivariant K -theory on separable C^* -algebras $C^*\text{-Alg}$	bivariant K -theory on ℓ -algebras Alg_ℓ
$k : C^*\text{-Alg} \rightarrow KK$	$j : \text{Alg}_\ell \rightarrow \mathfrak{K}\mathfrak{K}$
k is stable with respect to compact operators \mathcal{K}	j is stable with respect to $M_\infty = \bigcup_{n \in \mathbb{N}} M_n$
k is continuous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above
$KK_*(\mathbb{C}, A) \simeq K_*(A)$	$kk_*(\ell, A) \simeq KH_*(A)$

In this work we obtain an equivariant version of this dictionary. Let G be a countable group and \mathcal{H} be a Hopf algebra over a field. We introduce an algebraic kk -theory for the categories of G -algebras, G -graded algebras, \mathcal{H} -module algebras and \mathcal{H} -comodule algebras. We define an equivariant algebraic notion of matrix invariance. We study the different cases separately. In the category of G -algebras, every object A is stably isomorphic to the equivariant matrix algebra $M_G(A)$. In the category of G -graded algebras, which we call \hat{G} -algebras, stabilization is with respect to the graded matrix algebra $M_{\hat{G}}$. The definition of G -stability was inspired by the definition of equivariant stability in G - C^* -algebras (see [24]). In the case of \mathcal{H} -algebras, we fix a basis of \mathcal{H} as a ℓ -space and we define an \mathcal{H} -algebra called $\text{End}_\ell^F(\mathcal{H})$. The \mathcal{H} -stability identifies A with $\text{End}_\ell^F(\mathcal{H}) \otimes A$. This identification depends on a chosen basis of \mathcal{H} . We put a finiteness condition in $\text{End}_\ell^F(\mathcal{H})$ and M_G but these conditions are different if we take $\mathcal{H} = \ell G$. The equivariant matrix invariance in the case of \mathcal{H} -comodule algebras is similar to that of \mathcal{H} -algebras. After that we introduce the appropriate brand of algebraic kk -theory and we establish its universal properties in each case. We consider several properties which are valid for G -algebras, \hat{G} -algebras, \mathcal{H} -algebras and \mathcal{H} -comodule algebras and we write \mathcal{X} -algebra to refer either of them. We can resume Chapter 1 in the following table

Equivariant Kasparov's KK -theory	Equivariant algebraic kk -theory
bivariant K -theory on separable G - C^* -algebras G - C^* -Alg	bivariant K -theory on \mathcal{X} -algebras \mathcal{X} -Alg
$k : G\text{-}C^*\text{-Alg} \rightarrow KK^G$	$j : \mathcal{X}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{X}}$
k is stable with respect to $\mathcal{K}(\ell^2(G \times \mathbb{N}))$	j is \mathcal{X} -stable
k is continuous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above

In Chapter 2 we study adjointness theorems in equivariant kk -theory. We put in an algebraic context some of the adjointness theorems which appear in Kasparov KK -theory. Let G be a countable group and ℓ a commutative ring with unit. We define the functors of trivial action and crossed product between $\mathfrak{K}\mathfrak{K}$ and $\mathfrak{K}\mathfrak{K}^G$. The first adjointness theorem is Theorem 2.1.4 which is an algebraic version of the GREEN-JULG THEOREM. This result gives us the first computation related with homotopy K -theory. If G is a finite group, A is a G -algebra, B is an algebra and $\frac{1}{|G|} \in \ell$ then there is an isomorphism

$$\psi_{GJ} : kk^G(B^r, A) \rightarrow kk(B, A \rtimes G).$$

In particular, if $B = \ell$ then

$$kk^G(\ell, A) \simeq KH(A \rtimes G).$$

We consider a subgroup H of G , define induction and restriction functors between $\mathfrak{K}\mathfrak{K}^G$ and $\mathfrak{K}\mathfrak{K}^H$ and study the adjointness between them. If B is an H -algebra and A is a G -algebra then there is an isomorphism

$$\psi_{IR} : kk^G(\text{Ind}_H^G B, A) \rightarrow kk^H(B, \text{Res}_G^H A).$$

This result gives us another computation. Taking H the trivial group and $B = \ell$ we obtain that

$$kk^G(\ell^{(G)}, A) \simeq KH(A) \quad \forall A \in G\text{-Alg}.$$

Here $\ell^{(G)} = \bigoplus_{g \in G} \ell$ with the regular action of G . More general, if H is a finite subgroup of G and $1/|H| \in \ell$ we combine ψ_{GJ} and ψ_{IR} and obtain

$$kk^G(\ell^{(G/H)}, A) \simeq KH(A \rtimes H) \quad \forall A \in G\text{-Alg}.$$

We also prove an algebraic version of GREEN IMPRIMITIVITY THEOREM and obtain that

$$\mathrm{KH}(A \rtimes H) \simeq \mathrm{KH}(\mathrm{Ind}_H^G A \rtimes G).$$

We also obtain an algebraic version of the Baaj-Skandalis theorem. We show that the functors

$$\rtimes G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{G}} \quad \hat{G} \rtimes : \mathfrak{K}\mathfrak{K}^{\hat{G}} \rightarrow \mathfrak{K}\mathfrak{K}^G$$

are inverse category equivalences. Let \mathcal{H} be a Hopf algebra with finite dimension. We define functors between $\mathfrak{K}\mathfrak{K}$ and $\mathfrak{K}\mathfrak{K}^{\mathcal{H}}$, the smash product and the trivial action. We study the adjointness between them in theorem 2.5.4. We obtain that if \mathcal{H} is semisimple, B is an algebra and A is an \mathcal{H} -algebra then there is an isomorphism

$$\psi : \mathrm{kk}^{\mathcal{H}}(B^\tau, A) \rightarrow \mathrm{kk}(B, A \# \mathcal{H}).$$

In particular,

$$\mathrm{kk}^{\mathcal{H}}(\ell, A) \simeq \mathrm{KH}(A \# \mathcal{H}).$$

In the last chapter we study isomorphism conjectures in the sense of [8]. If \mathcal{F} is a family of subgroups of G we consider model category structures on \mathbb{S}^G and \mathbf{Top}^G . With this structure weak equivalences and fibration are object-wise. Cofibrant object are those X such that the stabilizer subgroup G_x is a subgroup of \mathcal{F} for all $x \in X$. We prove that the following is a Quillen equivalence

$$\mathbf{Top}^G \begin{array}{c} \xrightarrow{\mathrm{Sing}_*} \\ \parallel_* \\ \xleftarrow{\quad} \end{array} \mathbb{S}^G.$$

We say that a functor $H : \mathbb{S}^G \rightarrow \mathbf{Spt}$ from the category of G -simplicial set to the category of spectra satisfies the (G, \mathcal{F}) -isomorphism conjecture if for the cofibrant replacement $\pi : \mathcal{E}(G, \mathcal{F}) \rightarrow *$ in the \mathcal{F} -model category mentioned aboved, the map

$$H(\pi) : H(\mathcal{E}(G, \mathcal{F})) \rightarrow H(*)$$

is an equivalence. If $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ is a functor and R is a unital G -ring, one constructs, following Davis-Luck [8], a functor

$$H^G(-, E(R)) : \mathbb{S}^G \rightarrow \mathbf{Spt}$$

such that $H^G(*, E(R)) = E(R \rtimes G)$. The (G, \mathcal{F}, E, R) -isomorphism conjecture is the (G, \mathcal{F}) -conjecture for the functor $H^G(-, E(R))$. We show that under very mild assumptions on E , the STANDING ASSUMPTIONS 3.2.5 (which are satisfied for example when E is either K or KH , see propositions 3.4.18 and 3.5.3), $H^G(-, E(A))$ is defined not only for unital G -rings, but also for all E -excisive G -rings A , that is all G -rings on which E -satisfies excision. Moreover we show that if

$$(4) \quad 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

is an exact sequence of E -excisive rings and X is a G -simplicial set, then

$$H^G(X, E(A')) \rightarrow H^G(X, E(A)) \rightarrow H^G(X, E(A''))$$

is a homotopy fibration. This is a basic property needed to establish an algebraic analogue of the Dirac-dual method which is used to prove the Baum-Connes conjecture for some groups. Another basic property, which provide us with enough sequences (4) in which at

least one of the rings satisfies the isomorphism conjecture, is Theorem 3.7.3 which shows that if E satisfies the standing assumptions and A is an E -excisive G -ring of the form

$$(5) \quad A = \bigoplus_i \text{Ind}_{K_i}^G B_i$$

with B_i a K_i -ring and $K_i \in \mathcal{F}$ for all i , then the functor $H^G(-, E(A))$ maps (G, \mathcal{F}) -equivalences to equivalences. In particular the (G, \mathcal{F}, E, A) -isomorphism conjecture holds. We use this in Section 8 to show that under some additional assumptions (which are still satisfied if $E = K$, KH, see 3.8.1), for each E -excisive G -ring A there is a functorial exact sequence of E -excisive G -rings

$$\mathfrak{F}^0 B \rightarrow \mathfrak{F}^\infty B \rightarrow \mathfrak{F}^\infty B / \mathfrak{F}^0 B$$

and a natural map $A \rightarrow \mathfrak{F}^0 B$ such that

- i) $H^G(X, E(A)) \rightarrow H^G(X, E(\mathfrak{F}^0 B))$ is an equivalence for all G -simplicial set X .
- ii) $H^G(X, E(\mathfrak{F}^\infty B)) \rightarrow *$ is an equivalence if X is (G, \mathcal{F}) -cofibrant.
- iii) $H^G(-, E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B))$ maps (G, \mathcal{F}) -equivalences to equivalences.

It follows that the assembly map

$$H^G(\mathcal{E}(G, \mathcal{F}), E(A)) \rightarrow E(A \rtimes G)$$

is an equivalence iff the connecting map

$$\Omega(E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B \rtimes G)) \rightarrow E(A \rtimes G)$$

is an equivalence. In particular all this applies when $E = K$, KH. We also show in Theorem 3.9.2 that under stronger hypothesis on E , of which the main one is that E satisfies excision (e.g. KH satisfies this but K does not), then the (G, \mathcal{F}, E, A) -isomorphism conjecture is true whenever A is (G, \mathcal{F}) -proper. If X is a locally finite simplicial set with a G -action then a G -ring A is proper over X if it is an algebra over the ring $\mathbb{Z}^{(X)}$ of finitely supported polynomial maps on X , the algebra action is compatible with the actions of G on A and on X , and $\mathbb{Z}^{(X)} \cdot A = A$. We say that A is (G, \mathcal{F}) -proper if it is proper over a locally finite simplicial set X on which G acts with all stabilizers in \mathcal{F} . For example an algebra is of the form (5) if and only if it is proper over the zero-dimensional G -simplicial set $X = \coprod G / K_i$.

We remark that the notion of (G, \mathcal{F}) -proper ring used here is the algebraic analogue of the notion of proper G - C^* -algebra, and that Theorem 3.9.2 is an algebraic version of the known fact that Baum-Connes conjecture holds for proper G - C^* -algebras [11].

CHAPTER 1

Equivariant algebraic kk -theory

Algebraic kk -theory has been introduced by G. Cortiñas and A. Thom in [5]. This is a bivariant K -theory on the category of ℓ -algebras where ℓ is a commutative ring with unit. For each pair (A, B) of ℓ -algebras a group $kk(A, B)$ is defined. A category $\mathfrak{K}\mathfrak{K}$ is obtained whose objects are the ℓ -algebras and whose morphisms are the elements of the group $kk(A, B)$. The category $\mathfrak{K}\mathfrak{K}$ is triangulated and there is a canonical functor $j : \text{Alg}_\ell \rightarrow \mathfrak{K}\mathfrak{K}$ with universal properties. These properties are algebraic homotopy invariance, matrix invariance and excision.

The definition of algebraic kk -theory was inspired by the work of J. Cuntz [6] and N. Higson [12] on the universal properties of Kasparov KK -theory [19]. The KK -theory of separable C^* -algebras is a common generalization both of topological K -homology and topological K -theory as an additive bivariant functor. Let A, B be separable C^* -algebras then

$$(6) \quad KK_*(\mathbb{C}, B) \simeq K_*^{top}(B) \quad KK^*(A, \mathbb{C}) = K_{hom}^*(A)$$

here $K_*^{top}(B)$ denotes the K -theory of B and $K_{hom}^*(A)$ the K -homology of A . J. Cuntz in [6] gave another equivalent definition of the original one given in [19]. This new approach allowed to put bivariant K -theory in algebraic context. Higson in [12] stated the universal property of KK whose algebraic analogue is studied in [5], where also an analogue of (6) is proved. On the algebraic side, if A is an ℓ -algebra then

$$kk(\ell, A) \simeq KH(A)$$

here KH is Weibel's homotopy K -theory defined in [32]. We can start to build a dictionary between Kasparov's KK -theory and algebraic kk -theory in the following way

Kasparov's KK -theory	algebraic kk -theory
bivariant K -theory on separable C^* -algebras $C^*\text{-Alg}$	bivariant K -theory on ℓ -algebras Alg_ℓ
$k : C^*\text{-Alg} \rightarrow KK$	$j : \text{Alg}_\ell \rightarrow \mathfrak{K}\mathfrak{K}$
k is stable with respect to compact operators \mathcal{K}	j is stable with respect to $M_\infty = \bigcup_{n \in \mathbb{N}} M_n$
k is continuous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above
$KK_*(\mathbb{C}, A) \simeq K_*(A)$	$kk_*(\ell, A) \simeq KH_*(A)$

In this chapter we obtain an equivariant version of this dictionary. Let G be a countable group and \mathcal{H} be a Hopf algebra over a field. We introduce an algebraic kk -theory for the categories of G -algebras, G -graded algebras, \mathcal{H} -module algebras and \mathcal{H} -comodule algebras. We define an equivariant algebraic notion of matrix invariance. We study the different cases separately. In the category of G -algebras, every object A is stably isomorphic to the equivariant matrix algebra $M_G(A)$. In the category of G -graded algebras, which we call \hat{G} -algebras, stabilization is with respect to the graded matrix algebra $M_{\hat{G}}$. The definition of G -stability was inspired by the definition of equivariant stability in G - C^* -algebras (see [24]). In the case of \mathcal{H} -algebras, we fix a basis of \mathcal{H} as a ℓ -space and we define an \mathcal{H} -algebra called $\text{End}_\ell^F(\mathcal{H})$. The \mathcal{H} -stability identifies A with $\text{End}_\ell^F(\mathcal{H}) \otimes A$. This identification depends on a chosen basis of \mathcal{H} . We put a finiteness condition in $\text{End}_\ell^F(\mathcal{H})$ and M_G but these conditions are different if we take $\mathcal{H} = \ell G$. The equivariant matrix invariance in the case of \mathcal{H} -comodule algebras is similar to that of \mathcal{H} -algebras. After that we introduce the appropriate brand of algebraic kk -theory and we establish its universal properties in each case. We consider several properties which are valid for G -algebras, \hat{G} -algebras, \mathcal{H} -algebras and \mathcal{H} -comodule algebras and we write \mathcal{X} -algebra to refer either of them. We can resume this chapter in the following table

Equivariant Kasparov's KK -theory	Equivariant algebraic kk -theory
bivariant K -theory on separable G - C^* -algebras G - C^* -Alg	bivariant K -theory on \mathcal{X} -algebras \mathcal{X} -Alg
$k : G$ - C^* -Alg $\rightarrow KK^G$	$j : \mathcal{X}$ -Alg $\rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{X}}$
k is stable with respect to $\mathcal{K}(\ell^2(G \times \mathbb{N}))$	j is \mathcal{X} -stable
k is continuous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above

1. \mathcal{H} -algebras and $\hat{\mathcal{H}}$ -algebras

In this section we set up the notation and terminology related to Hopf algebras, module algebras and comodule algebras. We follow [25] for this part.

1.1. Hopf algebras. Let ℓ be a commutative ring with unit. In this section all tensor products are over ℓ ; we write $\otimes = \otimes_{\ell}$. An ℓ -algebra is an ℓ -bimodule A together with an ℓ -linear map called **multiplication** $\mu : A \otimes A \rightarrow A$. It is **associative** if the following diagram is commutative:

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\
 \text{id} \otimes \mu \downarrow & & \downarrow \mu \\
 A \otimes A & \xrightarrow{\mu} & A
 \end{array} \quad (\text{associativity})$$

An ℓ -algebra A with **unit** is an associative ℓ -algebra A together with an ℓ -linear map called **unit** $u : \ell \rightarrow A$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 & & A \otimes A & & \\
 & \nearrow u \otimes \text{id} & \downarrow \mu & \nwarrow \text{id} \otimes u & \\
 \ell \otimes A & & & & A \otimes \ell \\
 & \searrow & \downarrow & \swarrow & \\
 & & A & &
 \end{array} \quad (\text{unit})$$

The two lower maps are given by scalar multiplication. We will denote by $1_A := u(1_{\ell})$ and we will write ab for $\mu(a, b)$. Let V, W be ℓ -modules. Let us denote by $\tau : V \otimes W \rightarrow W \otimes V$ the **twisting map**, $\tau(v \otimes w) = w \otimes v$. An ℓ -algebra A is **commutative** if $\mu \circ \tau = \mu$.

An ℓ -coalgebra is an ℓ -module C together with an ℓ -linear map called **comultiplication** $\Delta : C \rightarrow C \otimes C$ such that the following diagram is commutative

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \Delta \otimes \text{id} \\ C \otimes C & \xrightarrow{\text{id} \otimes \Delta} & C \otimes C \otimes C \end{array} \quad (\text{coassociativity})$$

An ℓ -coalgebra with **counit** is a coalgebra C together with an ℓ -linear map called **counit** $\epsilon : C \rightarrow \ell$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & C & & \\ & \swarrow 1 \otimes & \downarrow \Delta & \searrow \otimes 1 & \\ \ell \otimes C & & & & C \otimes \ell \\ & \nwarrow \epsilon \otimes \text{id} & & \nearrow \text{id} \otimes \epsilon & \\ & & C \otimes C & & \end{array} \quad (\text{counit})$$

We say that C is **cocommutative** if $\tau \circ \Delta = \Delta$. An ℓ -module $(B, \mu, u, \Delta, \epsilon)$ is a **bialgebra** if (B, μ, u) is an algebra with unit and (B, Δ, ϵ) is a coalgebra with counit such that Δ and ϵ are algebra morphisms. We will use the **Sweedler notation** or **sigma notation** for Δ . If $c \in C$ we write

$$\Delta c = \sum c_1 \otimes c_2$$

The subscripts 1 and 2 are symbolic, and do not indicate particular elements of C . The power of the notation becomes apparent when Δ must be applied more than once. If we apply Δ $(n-1)$ -times to c we can write

$$\Delta_{n-1}(c) = \sum c_1 \otimes c_2 \otimes \dots \otimes c_n$$

because of the coassociativity.

An ℓ -module $(\mathcal{H}, \mu, u, \Delta, \epsilon, S)$ is a **Hopf algebra** if $(\mathcal{H}, \mu, u, \Delta, \epsilon)$ is a bialgebra and $S : \mathcal{H} \rightarrow \mathcal{H}$ is a bijective ℓ -linear map such that

$$\sum (Sh_1)h_2 = \epsilon(h)1_{\mathcal{H}} = \sum h_1(S h_2) \quad \forall h \in \mathcal{H}$$

The map S is called an **antipode** of \mathcal{H} . We will write \overline{S} for the inverse of S . A map $f : \mathcal{H} \rightarrow \mathcal{K}$ is a **Hopf morphism** if it is a bialgebra morphism and $f(S_{\mathcal{H}}h) = S_{\mathcal{K}}f(h)$.

EXAMPLE 1.1.1. Let G be a group and ℓ a commutative ring with unit. The **group algebra** ℓG of G is a Hopf algebra with the following structure:

$$\ell G := \{ \sum_{g \in G} a_g \delta_g : a_g \in \ell \text{ and } \{g \in G : a_g \neq 0\} \text{ is a finite set} \}$$

$$\mu : \ell G \otimes \ell G \rightarrow \ell G \quad \delta_g \otimes \delta_h \mapsto \delta_{gh} \quad u : \ell \rightarrow \ell G \quad u(1) = \delta_e$$

$$\Delta : \ell G \rightarrow \ell G \otimes \ell G \quad \delta_g \mapsto \delta_g \otimes \delta_g \quad \epsilon : \ell G \rightarrow \ell \quad \epsilon(\delta_g) = 1$$

$$S : \ell G \rightarrow \ell G \quad \delta_g \mapsto \delta_{g^{-1}}$$

1.2. \mathcal{H} -algebras. Let \mathcal{H} be a Hopf algebra. A left \mathcal{H} -module A is an ℓ -module with an ℓ -linear map $\gamma : \mathcal{H} \otimes A \rightarrow A$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{H} \otimes \mathcal{H} \otimes A & \xrightarrow{\mu \otimes \text{id}} & \mathcal{H} \otimes A \\ \text{id} \otimes \gamma \downarrow & & \downarrow \gamma \\ \mathcal{H} \otimes A & \xrightarrow{\gamma} & A \end{array} \quad \begin{array}{ccc} \ell \otimes A & \xrightarrow{u \otimes \text{id}} & \mathcal{H} \otimes A \\ & \searrow \text{scalar mult.} & \downarrow \gamma \\ & & A \end{array}$$

We will write $h \cdot a$ for $\gamma(h \otimes a)$. A left \mathcal{H} -module algebra is an ℓ -algebra A and a left \mathcal{H} -module structure on A such that

- $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$, for all $h \in \mathcal{H}$, $a, b \in A$.
- $h \cdot 1_A = \epsilon(h)1_A$, for all $h \in \mathcal{H}$.

We call A an \mathcal{H} -algebra for short. A map $f : A \rightarrow B$ is an \mathcal{H} -algebra morphism if

- $f(ab) = f(a)f(b)$ for all $a, b \in A$.
- $f(h \cdot a) = h \cdot f(a)$ for all $a \in A$, $h \in \mathcal{H}$.

EXAMPLE 1.1.2. For any algebra A we can consider the trivial action $h \cdot a = \epsilon(h)a$. In this case we say A is a trivial \mathcal{H} -algebra.

EXAMPLE 1.1.3. Let M be a left \mathcal{H} -module. Put

$$\text{End}_\ell(M) = \{\varphi : M \rightarrow M : \varphi \text{ is } \ell\text{-linear}\},$$

$\text{End}_\ell(M)$ is an ℓ -algebra with the composition and is an \mathcal{H} -algebra with the following action

$$(h \cdot \varphi)(m) = \sum h_1 \cdot \varphi(S(h_2) \cdot m) \quad \varphi \in \text{End}_\ell(M), h \in \mathcal{H}$$

More generally we have the following example.

EXAMPLE 1.1.4. Let M be an \mathcal{H} -module and let A be an \mathcal{H} -algebra. If $\varphi \in \text{End}_\ell(M)$ and $h, k \in \mathcal{H}$ we put

$$(7) \quad \varphi_{h,k} \in \text{End}_\ell(M) \quad \varphi_{h,k}(m) = h \cdot \varphi(S(k) \cdot m)$$

It is easy to check that $(\varphi_{h,k})_{\tilde{h},\tilde{k}} = \varphi_{\tilde{h}h,\tilde{k}k}$. Define the following action in $\text{End}_\ell(M) \otimes A$

$$h \cdot (\varphi \otimes a) = \sum \varphi_{h_1,h_3} \otimes h_2 \cdot a$$

Let us see that $\text{End}_\ell(M) \otimes A$ is an \mathcal{H} -algebra,

$$\begin{aligned} k \cdot (h \cdot (\varphi \otimes a)) &= k \cdot (\sum \varphi_{h_1,h_3} \otimes h_2 \cdot a) \\ &= \sum (\varphi_{h_1,h_3})_{k_1,k_3} \otimes k_2 \cdot (h_2 \cdot a) \\ &= \sum \varphi_{k_1 h_1, k_3 h_3} \otimes (k_2 h_2) \cdot a \\ &= (kh) \cdot (\varphi \otimes a) \end{aligned}$$

$$\begin{aligned}
h \cdot ((\varphi \otimes a)(\tilde{\varphi} \otimes \tilde{a})) &= \sum (\varphi \tilde{\varphi})_{h_1, h_3} \otimes h_2 \cdot (a \tilde{a}) \\
&= \sum (\varphi \tilde{\varphi})_{h_1, h_4} \otimes (h_2 \cdot a)(h_3 \cdot \tilde{a}) \\
&= \sum h_1 \cdot (\varphi(\tilde{\varphi}(S(h_5) \cdot \quad)) \otimes (h_2 \cdot a)(\epsilon(h_3)h_4 \cdot \tilde{a})) \\
&= \sum h_1 \cdot (\varphi(S(h_3)h_4 \cdot \tilde{\varphi}(S(h_6) \cdot \quad)) \otimes (h_2 \cdot a)(h_5 \cdot \tilde{a})) \\
&= \sum \varphi_{h_1, h_3} \tilde{\varphi}_{h_4, h_6} \otimes (h_2 \cdot a)(h_5 \cdot \tilde{a}) \\
&= \sum h_1 \cdot (\varphi \otimes a)h_2 \cdot (\tilde{\varphi} \otimes \tilde{a}) \\
1 \cdot (\varphi \otimes a) &= \varphi_{1,1} \otimes 1 \cdot a = \varphi \otimes a
\end{aligned}$$

$$h \cdot (\text{id} \otimes 1_A) = \sum \text{id}_{h_1, h_3} \otimes h_2 \cdot 1_A = \sum \text{id}_{h_1, h_3} \otimes \epsilon(h_2)1_A = \sum \text{id}_{h_1, h_2} \otimes 1_A = \text{id} \otimes 1_A$$

1.3. $\hat{\mathcal{H}}$ -algebras. A left \mathcal{H} -comodule A is an ℓ -module with an ℓ -linear map $\rho : A \rightarrow \mathcal{H} \otimes A$ such that the following diagrams commute:

$$\begin{array}{ccc}
A & \xrightarrow{\rho} & \mathcal{H} \otimes A \\
\rho \downarrow & & \downarrow \Delta \otimes \text{id} \\
\mathcal{H} \otimes A & \xrightarrow{\text{id} \otimes \rho} & \mathcal{H} \otimes \mathcal{H} \otimes A
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\rho} & \mathcal{H} \otimes A \\
1 \otimes \text{id} \searrow & & \downarrow \epsilon \otimes \text{id} \\
& & \ell \otimes A
\end{array}$$

Following the Sweedler notation we write $\rho(a) = \sum a_{-1} \otimes a_0$. A left \mathcal{H} -comodule algebra A is an ℓ -algebra and a left \mathcal{H} -comodule such that

- μ_A is a morphism of left \mathcal{H} -comodules: $\rho(ab) = \sum a_{-1}b_{-1} \otimes a_0b_0 \quad \forall a, b \in A$
- u_A is a morphism of left \mathcal{H} -comodules: $\rho(1_A) = 1 \otimes 1_A$

We call A an $\hat{\mathcal{H}}$ -algebra for short; note that $\hat{\mathcal{H}}$ does not denote any object. A map $f : A \rightarrow B$ is an $\hat{\mathcal{H}}$ -algebra morphism if it is an algebra morphism and a comodule morphism, that is:

- $f(ab) = f(a)f(b)$ for all $a, b \in A$.
- $\rho_B \circ f = (\text{id} \otimes f) \circ \rho_A$.

EXAMPLE 1.1.5. Suppose ℓ is a field. Let A be an $\hat{\mathcal{H}}$ -algebra and M an \mathcal{H} -comodule with finite dimension over ℓ . Then $\text{End}_\ell(M) \otimes A$ is an $\hat{\mathcal{H}}$ -algebra with the following structure

$$\rho : \text{End}_\ell(M) \otimes A \rightarrow \mathcal{H} \otimes \text{End}_\ell(M) \otimes A \quad \rho(\varphi \otimes a)(m) = \sum (\varphi(m_0))_{-1} a_{-1} \bar{S}(m_{-1}) \otimes (\varphi(m_0))_0 \otimes a_0$$

1.4. Dual of Hopf algebras. Let $(\mathcal{H}, \mu, u, \Delta, \epsilon, S)$ be a Hopf algebra. The dual of \mathcal{H} is

$$\mathcal{H}^* = \{\varphi : \mathcal{H} \rightarrow \ell : \varphi \text{ is } \ell\text{-linear}\}.$$

We consider the dual of the multiplication Δ^* , the dual unit ϵ^* , the dual counit u^* and the dual antipode S^* . We also want to consider the dual comultiplication but some difficulties

arise. The image of $\mu^* : \mathcal{H}^* \rightarrow (\mathcal{H} \otimes \mathcal{H})^*$ may not lie in $\mathcal{H}^* \otimes \mathcal{H}^*$. The finite dual of \mathcal{H} is defined as

$$\mathcal{H}^o = \{\varphi \in \mathcal{H}^* : \mu^*(\varphi) \in \mathcal{H}^* \otimes \mathcal{H}^*\}$$

There are many other equivalent conditions which define \mathcal{H}^o (see for instance [25], Chapter 9). One of them is

$$\mu^*(\varphi) \in \mathcal{H}^* \otimes \mathcal{H}^* \iff \dim(\varphi \leftarrow \mathcal{H}) < \infty \quad \varphi \leftarrow \mathcal{H} := \{\varphi \leftarrow h \in \mathcal{H}^* : (\varphi \leftarrow h)(k) = \varphi(hk)\}$$

If \mathcal{H} is finite dimensional with basis $B = \{g_1, \dots, g_n\}$, the set $B^* = \{g_1^*, \dots, g_n^*\}$ is a basis of $\mathcal{H}^* = \mathcal{H}^o$. In this case we can write the dual comultiplication in the following way

$$\delta = \mu^* : \mathcal{H}^* \rightarrow \mathcal{H}^* \otimes \mathcal{H}^* \quad \delta(\varphi) = \sum_i g_i^* \otimes (\varphi \leftarrow g_i)$$

PROPOSITION 1.1.6. Let $(\mathcal{H}, \mu, u, \Delta, \epsilon, S)$ be a Hopf algebra. Then $(\mathcal{H}^o, \mu^*, u^*, \Delta^*, \epsilon^*, S^*)$ is also a Hopf algebra.

PROOF: See [25] Theorem 9.1.3. □

1.5. ind-categories. In Chapter 1 we are going to work with directed diagrams of \mathcal{H} -algebras and $\hat{\mathcal{H}}$ -algebras.

Let \mathcal{C} be a category. The **category of ind-objects** of \mathcal{C} is the category $\text{ind-}\mathcal{C}$ of directed diagrams in \mathcal{C} . An object in $\text{ind-}\mathcal{C}$ is described by a filtering partially ordered set (I, \leq) and a functor $A : I \rightarrow \mathcal{C}$. The set of homomorphisms in $\text{ind-}\mathcal{C}$ is defined by

$$\text{hom}_{\text{ind-}\mathcal{C}}((A, I), (B, J)) := \lim_{i \in I} \text{colim}_{j \in J} \text{hom}_{\mathcal{C}}(A_i, B_j).$$

Note each homomorphism in $\text{ind-}\mathcal{C}$ is represented by a natural transformation $\sigma : I \rightarrow J$ that is cofinal (i.e. $\sigma(I)$ is cofinal in J) and a family of homomorphisms $\{f_i\}_{i \in I}$ in $\text{hom}_{\mathcal{C}}(A_i, B_{\sigma(i)})$ such that if $i \leq j$ the following diagram is commutative

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_{\sigma(i)} \\ \downarrow & & \downarrow \\ A_j & \xrightarrow{f_j} & B_{\sigma(j)} \end{array}$$

We write $f(i)$ for $\sigma(i)$. Two families of homomorphisms in \mathcal{C} , $\{f_i\}_{i \in I}$ and $\{\tilde{f}_i\}_{i \in I}$ represent the same homomorphism in $\text{ind-}\mathcal{C}$ if for all $i \in I$ there exists $j(i) \in J$ such that $f(i), \tilde{f}(i) \leq j(i)$ and the following diagram is commutative

$$\begin{array}{ccccc} & & B_{f(i)} & & \\ & \nearrow f_i & & \searrow & \\ A_i & & & & B_{j(i)} \\ & \searrow \tilde{f}_i & & \nearrow & \\ & & B_{\tilde{f}(i)} & & \end{array}$$

Note that there is a natural functor $\text{ind}-(\text{ind-}\mathcal{C}) \rightarrow \text{ind-}\mathcal{C}$ mapping

$$((A_i, J_i), I) \mapsto (A_{ij}, \bigcup_{i \in I} J_i \times \{i\})$$

We shall use this functor to collapse any ind-ind-object to an ind-object. Any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ extends to $\text{ind-}F : \text{ind-}\mathcal{C} \rightarrow \text{ind-}\mathcal{D}$ by

$$(8) \quad F(A_i, i \in I) = (F(A_i), i \in I)$$

We shall identify objects of \mathcal{C} with the full subcategory of $\text{ind-}\mathcal{C}$ consisting of the constant ind-objects.

2. Homotopy invariance

In this section we recall the notion of algebraic homotopy and give a brief exposition of some of its properties. In particular we discuss some properties of homotopy invariant functors. All algebras are over ℓ . \mathcal{H} is a fixed Hopf algebra. We consider several properties which are valid for \mathcal{H} -algebras and $\hat{\mathcal{H}}$ -algebras, we write \mathcal{X} -algebra to refer either of them. In this section tensor products are over \mathbb{Z} .

2.1. Algebraic homotopies. Let A be an \mathcal{X} -algebra. Put

$$A^{\Delta^1} := A[t] = A \otimes_{\mathbb{Z}} \mathbb{Z}[t].$$

Note that $A[t]$ is an \mathcal{X} -algebra. Let write by $c_A : A \rightarrow A[t]$ for the inclusion of A as constant polynomials in $A[t]$ and $\text{ev}_i : A[t] \rightarrow A$ for the evaluation of t at i ($i = 0, 1$). Note these morphisms are morphisms of \mathcal{X} -algebras and that c_A is a section of ev_i .

Let $f_0, f_1 : A \rightarrow B$ be morphisms in $\mathcal{X}\text{-Alg}$. We call f_0 and f_1 **elementarily homotopic** if there exists a morphism $H : A \rightarrow B[t]$ such that $\text{ev}_i H = f_i$, $i = 0, 1$. We denote it by $f_0 \sim_e f_1$. A morphism $f : A \rightarrow B$ is an **elementary homotopy equivalence** if there exists a morphism $g : B \rightarrow A$ such that $f \circ g \sim_e \text{id}_B$ and $g \circ f \sim_e \text{id}_A$.

An \mathcal{X} -algebra A is **elementarily contractible** if the null morphism and the identity morphism are elementarily homotopic. In other words, A is elementarily contractible if there exists a morphism $f : A \rightarrow A[t]$ such that $\text{ev}_0 \circ f = 0$ and $\text{ev}_1 \circ f = \text{id}_A$.

EXAMPLE 1.2.1. Let A be an \mathcal{X} -algebra. The **path algebra** $PA := \{p \in A[u] : p(0) = 0\}$ is elementarily contractible and the homotopy is given by

$$PA \rightarrow PA[t] \quad \sum_{i \geq 1} a_i u^i \mapsto \sum_{i \geq 1} a_i (tu)^i.$$

EXAMPLE 1.2.2. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an $\mathbb{N}_{\geq 0}$ -graded \mathcal{X} -algebra. The inclusion $\iota : A_0 \rightarrow A$ is an elementary homotopy equivalence. The projection $p : A \rightarrow A_0$ is the homotopy inverse of ι because $p \circ \iota = \text{id}_{A_0}$ and $\iota \circ p \sim_e \text{id}_A$ where the homotopy is given by

$$H : A \rightarrow A[t] \quad H\left(\sum_{i \in \mathbb{N}} a_i\right) = a_0 + \sum_{i \geq 1} a_i t^i.$$

Note PA is an $\mathbb{N}_{\geq 0}$ -graded \mathcal{X} -algebra.

It is easy to check that elementary homotopy is a reflexive and symmetric relation. In general, it is not transitive.

DEFINITION 1.2.3 (algebraic homotopy). Let $f, g : A \rightarrow B$ be morphisms of \mathcal{X} -algebras. We call f and g **homotopic**, and write $f \sim g$, if they can be connected by a chain of elementary homotopies,

$$f \sim_e h_0 \sim_e \dots \sim_e h_n \sim_e g.$$

We denote the set of homotopy classes by $[A, B]_{\mathcal{X}}$. A morphism $f : A \rightarrow B$ is an **homotopy equivalence** if there exists a morphism $g : B \rightarrow A$ such that $f \circ g \sim \text{id}_B$ and $g \circ f \sim \text{id}_A$.

DEFINITION 1.2.4. An \mathcal{X} -algebra A is **contractible** if the identity morphism and the null morphism are homotopic.

REMARK 1.2.5. Let A, B be \mathcal{X} -algebras. We suppose $f, h, g : A \rightarrow B$ are morphisms of \mathcal{X} -algebras such that $f \sim_e h \sim_e g$. We can consider morphisms $H, G : A \rightarrow B[t]$ such that

$$\text{ev}_0 \circ H = f \quad \text{ev}_1 \circ H = h \quad \text{ev}_1 \circ G = g \quad \text{ev}_0 \circ G = h$$

Then we have a morphism $R : A \rightarrow B[t] \times_B B[t]$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{G} & B[t] \\ \text{dashed } R \searrow & \nearrow \text{pr}_2 & \downarrow \text{ev}_0 \\ & B[t] \times_B B[t] & \\ \text{curved } H \searrow & \downarrow \text{pr}_1 & \downarrow \text{ev}_1 \\ & B[t] & \rightarrow B \end{array} \quad \begin{array}{l} \text{ev}_0 \circ \text{pr}_1 \circ R = f \\ \text{ev}_1 \circ \text{pr}_2 \circ R = g \end{array}$$

Suppose there exists a chain of elementary homotopies

$$f \sim_e h_1 \sim_e \dots \sim_e h_{n-1} \sim_e g.$$

We denote this chain in the following way

$$(9) \quad \bullet \xrightarrow{H_1} \bullet \xrightarrow{H_2} \bullet \xrightarrow{H_3} \dots \xrightarrow{H_{n-1}} \bullet \xrightarrow{H_n} \bullet$$

$f \quad h_1 \quad h_2 \quad \dots \quad h_{n-1} \quad g$

where the beginning of the arrow denotes the value at 0 and the end of the arrow denotes the value at 1. There exists a morphism $R : A \rightarrow B[t] \times_B (B[t] \times_B \dots \times_B (B[t] \times_B B[t]))$ such that the following pull back diagrams commute

$$\begin{array}{ccc} A & \xrightarrow{H_n} & B[t] \\ \text{dashed } R_1 \searrow & \nearrow \text{pr}_2 & \downarrow \text{ev}_0 \\ & B[t] \times_B B[t] & \\ \text{curved } H_{n-1} \searrow & \downarrow \text{pr}_1 & \downarrow \text{ev}_1 \\ & B[t] & \rightarrow B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{R_1} & B[t] \times_B B[t] \\ \text{dashed } R_2 \searrow & \nearrow \text{pr}_2 & \downarrow \text{ev}_0 \circ \text{pr}_1 \\ & B[t] \times_B (B[t] \times_B B[t]) & \\ \text{curved } H_{n-2} \searrow & \downarrow \text{pr}_1 & \downarrow \text{ev}_1 \\ & B[t] & \rightarrow B \end{array}$$

\dots

$$\begin{array}{ccc}
A & \xrightarrow{R_{n-2}} & B[t] \times_B \dots \times_B (B[t] \times_B B[t]) \\
\downarrow R & \searrow pr_2 & \downarrow \text{ev}_0 \circ pr_1 \\
B[t] \times_B (B[t] \times_B \dots \times_B (B[t] \times_B B[t])) & \xrightarrow{pr_2} & B[t] \times_B \dots \times_B (B[t] \times_B B[t]) \\
\downarrow H_1 & \downarrow pr_1 & \downarrow \text{ev}_0 \circ pr_1 \\
B[t] & \xrightarrow{ev_1} & B
\end{array}$$

Note that

(10)

$$B[t] \times_B (B[t] \times_B \dots \times_B (B[t] \times_B B[t])) = \{(p_1(t), \dots, p_n(t)) : p_i(t) \in B[t] \text{ } p_i(1) = p_{i+1}(0)\}$$

We have $\text{ev}_0 \circ pr_1 \circ R = f$ and $\text{ev}_1 \circ pr_2 \circ R = g$ where pr_2 is the projection to the last coordinate. It is equivalent to have the chain of homotopies (9) and to have the morphism R .

2.2. Homotopies of ind- \mathcal{X} -algebras. Let $A = (A, I)$ and $B = (B, J)$ be ind- \mathcal{X} -algebras, we define

$$[A, B]_{\mathcal{X}} = \lim_{i \in I} \text{colim}_{j \in J} [A_i, B_j]_{\mathcal{X}}$$

Note that for each $i \in I$ and $j \in J$ there is a natural map

$$\text{hom}_{\mathcal{X}\text{-Alg}}(A_i, B_j) \mapsto [A_i, B_j]_{\mathcal{X}}$$

which sends each f to its homotopy class $[f]$. We also have a map

$$(11) \quad \text{hom}_{\text{ind-}\mathcal{X}\text{-Alg}}(A, B) \mapsto [A, B]_{\mathcal{X}}.$$

We say two morphisms in $\text{ind-}\mathcal{X}\text{-Alg}$ are **homotopic** if their images by (11) are equal. An ind- \mathcal{X} -algebra A is **contractible** if the null morphism and the identity morphism are homotopic.

2.3. Homotopy invariant functors. A functor $F : \mathcal{X}\text{-Alg} \rightarrow \mathcal{C}$ is **homotopy invariant** if it maps the inclusion $c_A : A \rightarrow A[t]$ to an isomorphism. The following Lemma shows an equivalent definition.

LEMMA 1.2.6. Let $F : \mathcal{X}\text{-Alg} \rightarrow \mathcal{C}$ be a functor. F is a homotopy invariant functor if and only if $F(f) = F(g)$ when $f \sim g$ (or equivalently when $f \sim_e g$).

□

2.4. The simplicial algebra A^{Δ} . Define the following simplicial ring

$$(12) \quad \mathbb{Z}^{\Delta} : [n] \mapsto \mathbb{Z}^{\Delta^n} \quad \mathbb{Z}^{\Delta^n} := \mathbb{Z}[t_0, \dots, t_n] / \langle 1 - \sum_i t_i \rangle$$

$$\begin{aligned}
\Theta : [n] \rightarrow [m] &\mapsto \Theta^* : \mathbb{Z}^{\Delta^m} \rightarrow \mathbb{Z}^{\Delta^n} \\
\Theta^*(t_i) &= \begin{cases} 0 & \text{si } \Theta^{-1}(i) = \emptyset \\ \sum_{j \in \Theta^{-1}(i)} t_j & \text{si } \Theta^{-1}(i) \neq \emptyset \end{cases}
\end{aligned}$$

Let A be an \mathcal{X} -algebra. Define

$$A^{\Delta} : [n] \mapsto A^{\Delta^n} \quad A^{\Delta^n} := A \otimes_{\mathbb{Z}} \mathbb{Z}^{\Delta^n}$$

Note A^Δ is a simplicial \mathcal{X} -algebra because A^{Δ^n} is an \mathcal{X} -algebra with $\gamma \otimes \text{id}$ or $\rho \otimes \text{id}$. The face maps and degeneracy maps are given by

$$\begin{aligned} d_i^n : A^{\Delta^n} &\rightarrow A^{\Delta^{n-1}} & d_i^n(p)(t_0, \dots, t_{n-1}) &= p(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) & 0 \leq i \leq n & \quad n \neq 0 \\ s_j^n : A^{\Delta^n} &\rightarrow A^{\Delta^{n+1}} & s_j^n(p)(t_0, \dots, t_{n+1}) &= p(t_0, \dots, t_j + t_{j+1}, t_{j+2}, \dots, t_{n+1}) & 0 \leq j \leq n \end{aligned}$$

If A has unit, it is easy to check that

$$A^{\Delta^1} = A[t_0, t_1] / \langle 1 - t_0 - t_1 \rangle \cong A[t] \quad d_0^* = \text{ev}_0 \quad d_1^* = \text{ev}_1$$

We can enrich the category $\mathcal{X}\text{-Alg}$ over simplicial sets, as follows. We have a mapping space functor

$$\text{hom}_{\mathcal{X}\text{-Alg}}^\bullet : \mathcal{X}\text{-Alg}^{\text{op}} \times \mathcal{X}\text{-Alg} \rightarrow \mathbb{S} \quad (A, B) \mapsto ([n] \mapsto \text{hom}_{\mathcal{X}\text{-Alg}}(A, B^{\Delta^n}))$$

Let $\mu : \mathbb{Z}^\Delta \otimes \mathbb{Z}^\Delta \rightarrow \mathbb{Z}^\Delta$ be the multiplication map. For $A, B, C \in \mathcal{X}\text{-Alg}$ we define

$$\odot : \text{hom}_{\mathcal{X}\text{-Alg}}^\bullet(B, C) \times \text{hom}_{\mathcal{X}\text{-Alg}}^\bullet(A, B) \rightarrow \text{hom}_{\mathcal{X}\text{-Alg}}^\bullet(A, C) \quad (g \odot f)_n := (\text{id}_C \otimes \mu)(g^{\Delta^n} \otimes f)$$

2.5. Subdivision of simplicial sets. We will recall some concepts from [10], Chapter 3, Section 4. Recall that the nondegenerate simplices of the standard n -simplex

$$\Delta^n = \text{hom}_\Delta(\cdot, [n])$$

are the monic ordinal number maps $[m] \rightarrow [n]$. There is exactly one such monomorphism for each subset of $[n]$ of cardinality $m + 1$. It follows that the nondegenerate simplices of Δ^n form a poset $P\Delta^n$, ordered by the face relation, and this poset is isomorphic to that of the nonempty subsets of the ordinal number $[n]$, ordered by inclusion. The **subdivision** of Δ^n , is the nerve of the poset $P\Delta^n$. We write it as $\text{sd } \Delta^n$. If X is a simplicial set, the **subdivision** of X , is

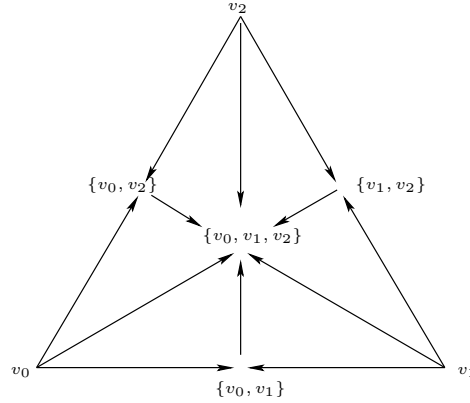
$$\text{sd } X = \text{colim}_{\sigma: \Delta^n \rightarrow X} \text{sd } \Delta^n.$$

The map of posets $P\Delta^n \rightarrow [n]$ given by $[v_0, v_1, \dots, v_k] \mapsto v_k$, induces a natural map

$$(13) \quad h : \text{sd } \Delta^n \rightarrow \Delta^n$$

which it is called **last vertex map**.

EXAMPLE 1.2.7. Consider Δ^2 and its vertices v_0, v_1, v_2 . The vertices of $\text{sd } \Delta^2$ are $v_0, v_1, v_2, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1, v_2\}$, the 1-simplices are $[\{v_0\}, \{v_0, v_1\}], [\{v_1\}, \{v_0, v_1\}], [\{v_1\}, \{v_1, v_2\}], [\{v_2\}, \{v_1, v_2\}], [\{v_0\}, \{v_0, v_2\}], [\{v_2\}, \{v_0, v_2\}], [\{v_0, v_1\}, \{v_0, v_1, v_2\}], [\{v_1, v_2\}, \{v_0, v_1, v_2\}], [\{v_0, v_2\}, \{v_0, v_1, v_2\}], [\{v_2\}, \{v_0, v_1, v_2\}], [\{v_0\}, \{v_0, v_1, v_2\}], [\{v_2\}, \{v_0, v_1, v_2\}]$, etc.



2.6. The algebra of polynomial functions on a simplicial set. Let A be an ℓ -algebra and X be a simplicial set. We define

$$A^X := \lim_{\Delta^n \rightarrow X} A^{\Delta^n} = \int_n \prod_{x \in X_n} A^{\Delta^n} = \text{map}_{\mathbb{S}}(X, A^{\Delta})$$

If (K, \star) is a pointed simplicial set, put

$$A^{(K, \star)} := \text{map}_{\mathbb{S}_*}((K, \star), A^{\Delta}) = \ker(\text{map}_{\mathbb{S}}(K, A^{\Delta}) \rightarrow \text{map}_{\mathbb{S}}(\star, A^{\Delta})) = \ker(A^K \rightarrow A)$$

From [5] we have the following lemmas.

LEMMA 1.2.8. Let $j : K \rightarrow L$ in \mathbb{S} , $\star \in K$ and $A \in \text{Alg}$. If j is a cofibration, then

- The map $A^L \rightarrow A^K$ is surjective.
- The sequence $0 \rightarrow A^{(L/K, \star)} \rightarrow A^{(L, \star)} \rightarrow A^{(K, \star)} \rightarrow 0$ is exact.

PROOF: See Lemma 3.1.2 in [5] □

LEMMA 1.2.9. Let K be a finite simplicial set, \star a vertex of K , and A an algebra. Then \mathbb{Z}^K and $\mathbb{Z}^{(K, \star)}$ are free abelian groups and there are natural isomorphisms

$$A \otimes \mathbb{Z}^K \xrightarrow{\sim} A^K \quad A \otimes \mathbb{Z}^{(K, \star)} \xrightarrow{\sim} A^{(K, \star)}$$

PROOF: See Lemma 3.1.3 in [5] □

Let A be an \mathcal{X} -algebra and K a finite simplicial set, by Lemma 1.2.9 A^K and $A^{(K, \star)}$ are \mathcal{X} -algebras with $\gamma \otimes \text{id}$ or $\rho \otimes \text{id}$. We will denote by $\text{sd}^{\bullet} X$ the following pro-simplicial set

$$\text{sd}^{\bullet} X : \dots \xrightarrow{h} \text{sd}^n X \xrightarrow{h} \text{sd}^{n-1} X \xrightarrow{h} \dots \xrightarrow{h} \text{sd} X \xrightarrow{h} X$$

where h is the morphism defined in (13). If A is an \mathcal{X} -algebra, we consider the following ind- \mathcal{X} -algebra

$$\text{sd}^{\bullet} X^* : A^X \rightarrow A^{\text{sd} X} \rightarrow \dots \rightarrow A^{\text{sd}^{n-1} X} \rightarrow A^{\text{sd}^n X} \rightarrow \dots$$

The next picture shows $\text{sd}^{\bullet} \Delta^1$. The map h contracts the dashed lines to a point.

$$B = \{(p_1, \dots, p_{2^k}) : p_i \in A[u], p_1(0) = 0, p_i(1) = p_{i+1}(0)\} \simeq A^{(\text{sd}^k \Delta^1, \star)}$$

Let $H : B \rightarrow B[t] \times_B \dots \times_B B[t]$ such that

$$H(p_1, \dots, p_{2^k}) = (Q_1(p_1, \dots, p_{2^k}), \dots, Q_{2^k}(p_1, \dots, p_{2^k}))$$

$$Q_i(p_1, \dots, p_{2^k}) = (p_1(u), \dots, p_{i-1}(u), p_i(ut), p_i(t), \dots, p_i(t))$$

Note H is well-defined

$$\begin{aligned} \text{ev}_1 \circ Q_i(p_1, \dots, p_{2^k}) &= (p_1(u), \dots, p_{i-1}(u), p_i(u), p_i(1), \dots, p_i(1)) \\ &= (p_1(u), \dots, p_{i-1}(u), p_i(u), p_{i+1}(0), \dots, p_{i+1}(0)) \\ &= \text{ev}_0 \circ Q_{i+1}(p_1, \dots, p_{2^k}) \end{aligned}$$

and

$$\begin{aligned} \text{ev}_0 \circ pr_1 \circ H(p_1, \dots, p_{2^k}) &= \text{ev}_0(p_1(ut), p_1(t), \dots, p_1(t)) = (p_1(0), p_1(0), \dots, p_1(0)) = 0 \\ \text{ev}_1 \circ pr_2 \circ H(p_1, \dots, p_{2^k}) &= \text{ev}_1(p_1(u), \dots, p_{2^k-1}(u), p_{2^k}(ut)) = (p_1(u), \dots, p_{2^k-1}(u), p_{2^k}(u)) \end{aligned}$$

□

COROLLARY 1.2.12. The ind- \mathcal{X} -algebra $A^{(\text{sd}^\bullet \Delta^1, \star)}$ is contractible. □

3. Matrix invariance

3.1. Matrix algebra. Let Λ be an infinite set and let $\mathcal{P}_F(\Lambda)$ be the set of its finite sets. We consider the inclusion relation in $\mathcal{P}_F(\Lambda)$ and we obtain a poset. Let $X \in \mathcal{P}_F(\Lambda)$ and define

$$M_X := \{\varphi : X \times X \rightarrow \mathbb{Z}\}.$$

For each $(x, y) \in X \times X$ we consider

$$e_{x,y} : X \times X \rightarrow \mathbb{Z} \quad e_{x,y}(s, t) = \begin{cases} 1 & (s, t) = (x, y) \\ 0 & (s, t) \neq (x, y) \end{cases}$$

Every matrix in M_X is a \mathbb{Z} -linear combination of elements $e_{x,y}$, $x, y \in X$. If $X \subset Y$ we define $\iota : M_X \rightarrow M_Y$ in the obvious way

$$\iota : M_X \rightarrow M_Y \quad \iota(\varphi)(x, y) = \begin{cases} \varphi(x, y) & \text{if } (x, y) \in X \times X \\ 0 & \text{otherwise} \end{cases}$$

Put

$$M_\bullet := \{M_X\}_{X \in \mathcal{P}_F(\Lambda)} \quad M_{|\Lambda|} := \text{colim}_X M_X \quad \mathcal{M}_{|\Lambda|} = M_\bullet \otimes M_{|\Lambda|}$$

Note M_\bullet and $\mathcal{M}_{|\Lambda|}$ are ind-algebras and $M_{|\Lambda|}$ is an algebra with the multiplication given by

$$e_{x,y} \cdot e_{z,w} = \begin{cases} e_{x,w} & \text{if } y = z \\ 0 & \text{if } y \neq z \end{cases}$$

The set

$$\Gamma_\Lambda^l := \{m \in \mathbb{Z}^{\Lambda \times \Lambda} : m \cdot M_{|\Lambda|} \subseteq M_{|\Lambda|} \subseteq M_{|\Lambda|} \cdot m\}.$$

consists of those matrices in $\mathbb{Z}^{\Lambda \times \Lambda}$ having finitely many nonzero elements in each row and column. Every element T in Γ_Λ^l can be written as a formal sum

$$T = \sum_{i,j \in \Lambda} a_{i,j} e_{i,j} \quad \text{with } a_{i,j} \in \mathbb{Z} \text{ such that } \forall i \in \Lambda \text{ the sets } \{j : a_{i,j} \neq 0\} \text{ and } \{j : a_{j,i} \neq 0\} \text{ are finite sets}$$

LEMMA 1.3.1. Let $V, W \in \Gamma_\Lambda^l$ such that $VW = 1$. Define

$$\psi : M_{|\Lambda|} \rightarrow M_{|\Lambda|} \quad \psi(A) = WAV \quad \hat{\psi} : M_\bullet \rightarrow M_\bullet \quad \hat{\psi}(A) = WAV \quad 1 \otimes \psi : \mathcal{M}_{|\Lambda|} \rightarrow \mathcal{M}_{|\Lambda|}$$

ψ is an morphism of algebras, $\hat{\psi}$ and $1 \otimes \psi$ are morphisms of ind-algebras. If ι denotes the inclusion in the first entry of M_2 , then $\iota\psi$ is homotopic to ι . The morphisms $\hat{\psi}$ and $1 \otimes \psi$ are homotopic to the identity.

PROOF: See Lemma 4.1.1 in [5]. □

As Λ is a infinite set, we can choose a subset of Λ bijectable to \mathbb{N} . Suppose $\Lambda = \mathbb{N} \times \lambda$. In order to define a direct sum of matrices in $M_{|\Lambda|}$ consider, $n, m \in \mathbb{N}$, $g, h \in \lambda$ and the elements in Γ_Λ^l

$$V(n, g, m, h) = \begin{cases} 1 & \text{if } m = 2n \quad g = h \\ 0 & \text{otherwise} \end{cases} \quad V^*(n, g, m, h) = \begin{cases} 1 & \text{if } 2m = n \quad g = h \\ 0 & \text{otherwise} \end{cases}$$

$$W(n, g, m, h) = \begin{cases} 1 & \text{if } m = 2n + 1 \quad g = h \\ 0 & \text{otherwise} \end{cases} \quad W(n, g, m, h) = \begin{cases} 1 & \text{if } 2m + 1 = n \quad g = h \\ 0 & \text{otherwise} \end{cases}$$

Define $\oplus : M_{|\Lambda|} \times M_{|\Lambda|} \rightarrow M_{|\Lambda|}$ as follows

$$\varphi \oplus \tilde{\varphi} = V\varphi V^* + W\tilde{\varphi} W^*.$$

There is also a tensor product defined in $M_{|\Lambda|}$. Let $\eta : (\mathbb{N} \times \lambda) \rightarrow (\mathbb{N} \times \lambda) \times (\mathbb{N} \times \lambda)$ a bijection. Define $\star : M_{|\Lambda|} \times M_{|\Lambda|} \rightarrow M_{|\Lambda|}$ as

$$(\varphi \star \tilde{\varphi})(x, y) := \varphi(\eta(x))\tilde{\varphi}(\eta(y))$$

We obtain that $(\mathcal{M}_{|\Lambda|}, \oplus, \star)$ is a homotopy semiring, see Section 4.1 of [5]. If $\Lambda = \mathbb{N}$ we will denote $M_{|\mathbb{N}|} = M_\infty$ and $\mathcal{M}_{|\mathbb{N}|} = \mathcal{M}_\infty$.

3.2. Matrix invariant functors. Let A be an \mathcal{X} -algebra. We define

$$M_n A = M_n \otimes_{\mathbb{Z}} A \quad M_\infty A = M_\infty \otimes_{\mathbb{Z}} A$$

which are \mathcal{X} -algebras in the obvious way. Denote by $\iota_n : \mathbb{Z} \rightarrow M_n$ and $\iota_\infty : \mathbb{Z} \rightarrow M_\infty$ the inclusions at the upper left corner. A functor $F : \mathcal{X}\text{-Alg} \rightarrow \mathcal{C}$ is M_n -stable (M_∞ -stable) if $F(\iota_n \otimes \text{id}_A)$ ($F(\iota_\infty \otimes \text{id}_A)$) is an isomorphism for all $A \in \mathcal{X}\text{-Alg}$.

4. Extensions and classifying maps

4.1. Extension. Following [5], a sequence of morphisms in $\text{ind-}\mathcal{X}\text{-Alg}$

$$(14) \quad A \xrightarrow{f} B \xrightarrow{g} C$$

is called an extension if f is a kernel of g and g is a cokernel of f .

Let $\mathcal{X}\text{-mod}$ be the category of \mathcal{X} -modules with linear and equivariant maps. Let $F : \mathcal{X}\text{-Alg} \rightarrow \mathcal{X}\text{-mod}$ be the forgetful functor. This functor can be extended to $F : \text{ind-}\mathcal{X}\text{-Alg} \rightarrow \text{ind-}\mathcal{X}\text{-mod}$ as is shown in (8). We will call an extension (14) weakly split if $F(g)$ has a section in $\mathcal{X}\text{-mod}$.

4.2. Path extension. Let S^1 be the simplicial circle $\Delta^1/\partial\Delta^1$, we define

$$\Omega := \mathbb{Z}^{(S^1, \star)}$$

The path extension of an \mathcal{X} -algebra A is the extension iduced by the cofibration $\partial\Delta^1 \subset \Delta^1$, see 1.2.8.

$$(15) \quad \Omega A \longrightarrow A^{\Delta^1} \xrightarrow{(\text{ev}_0, \text{ev}_1)} A \oplus A$$

The extension (15) is weakly split because we have a linear and \mathcal{X} -equivariant section of $(\text{ev}_0, \text{ev}_1)$

$$(16) \quad (a, b) \mapsto (1 - t)a + tb.$$

4.3. Universal extension. Let M be an \mathcal{H} -module. Consider in

$$\tilde{T}(M) = \bigoplus_{n \geq 1} M^{\otimes n} \quad M^{\otimes n} = \underbrace{M \otimes \dots \otimes M}_{n\text{-times}}$$

the usual structure of \mathcal{H} -module. The action in $M^{\otimes n}$ is given by

$$h \cdot (m_1 \otimes m_2 \otimes \dots \otimes m_n) = \sum h_1 \cdot m_1 \otimes h_2 \cdot m_2 \otimes \dots \otimes h_n \cdot m_n$$

It is easy to check that $\tilde{T}(M)$ is an \mathcal{H} -module algebra with this action. Similarly, let R be an \mathcal{H} -comodule. Consider in

$$\tilde{T}(R) = \bigoplus_{n \geq 1} R^{\otimes n}$$

the usual structure of \mathcal{H} -comodule. The coaction in $R^{\otimes n}$ is given by

$$\rho(r^1 \otimes r^2 \otimes \dots \otimes r^n) = \sum r_{-1}^1 r_{-1}^2 \dots r_{-1}^n \otimes r_0^1 \otimes \dots \otimes r_0^n$$

It is easy to check that $\tilde{T}(R)$ is an \mathcal{H} -comodule algebra with this coaction. Both constructions are functorial hence we consider a functor $\tilde{T} : \mathcal{X}\text{-mod} \rightarrow \mathcal{X}\text{-Alg}$. Put

$$T := \tilde{T} \circ F : \mathcal{X}\text{-Alg} \rightarrow \mathcal{X}\text{-Alg}$$

If A is an \mathcal{X} -algebra there exists an \mathcal{X} -algebra morphism

$$\eta_A : T(A) \rightarrow A \quad \eta_A(a_1 \otimes \dots \otimes a_n) = a_1 \dots a_n$$

and an \mathcal{X} -module morphism $\mu_A : A \rightarrow T(A)$ which is the inclusion at the first summand of $T(A)$.

REMARK 1.4.1. Let A, B be \mathcal{X} -algebras, it is easy to check that

$$\text{hom}_{\mathcal{X}\text{-Alg}}(T(A), B) \simeq \text{hom}_{\mathcal{X}\text{-mod}}(F(A), F(B)).$$

Hence if we have a morphism $A \rightarrow B$ in $\mathcal{X}\text{-mod}$, we can extend it to a morphism $T(A) \rightarrow B$ of \mathcal{X} -algebras. It shows that T is the left adjoint of F .

The counit of the adjunction is $\eta_A : T(A) \rightarrow A$. It is surjective (see [22], IV.3 Thm 1). We define

$$J(A) := \ker \eta_A.$$

The universal extension of A is

$$J(A) \xrightarrow{\iota_A} T(A) \xrightarrow{\eta_A} A.$$

PROPOSITION 1.4.2. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an weakly split extension. There exists a commutative diagram of extensions as follows

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \uparrow \xi & & \uparrow \hat{\xi} & & \uparrow \text{id}_C \\ J(C) & \xrightarrow{\iota_C} & T(C) & \xrightarrow{\eta_C} & C \end{array}$$

where the map ξ is unique up to elementary homotopy.

PROOF: Let s be a section of $F(g)$ and define $\hat{\xi} = \eta_B \circ \tilde{T}(s)$, then

$$\eta_C = \eta_C \circ T(g) \circ \tilde{T}(s) = g \circ \eta_B \circ \tilde{T}(s) = g \circ \hat{\xi}.$$

Define $\xi : J(C) \rightarrow A$ as the restriction of $\hat{\xi}$ to $J(C)$. This construction of ξ depends on which s is chosen. If $\xi_1, \xi_2 : J(C) \rightarrow A$ are morphisms constructed taking different sections of $F(g)$ we will show ξ_1, ξ_2 are homotopic. Define a linear equivariant map

$$H : C \rightarrow A[t] \quad H(c) = (1-t)\xi_1(c) + t\xi_2(c).$$

By 1.4.1 it extends to a homomorphism and there exists a morphism $H : T(C) \rightarrow A[t]$ in $\mathcal{X}\text{-Alg}$ such that

$$\text{ev}_0 \circ H|_{J(C)} = \xi_1 \quad \text{ev}_1 \circ H|_{J(C)} = \xi_2.$$

□

We call ξ the **classifying map** of the extension $A \xrightarrow{f} B \xrightarrow{g} C$. We abuse notation because we shall work with maps up to homotopy.

PROPOSITION 1.4.3. ([5] 4.4.2) Let

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ f \downarrow & & \downarrow h & & \downarrow g \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

be a commutative diagram of weakly split extensions. Then there is a diagram

$$\begin{array}{ccc} J(C) & \longrightarrow & A \\ J(g) \downarrow & & \downarrow f \\ J(C') & \longrightarrow & A' \end{array}$$

of classifying maps, which is commutative up to elementary homotopy.

PROPOSITION 1.4.4. ([5] 4.4.3) Let L be a ring and A an \mathcal{X} -algebra. Then the extension

$$(17) \quad J(A) \otimes_{\mathbb{Z}} L \rightarrow T(A) \otimes_{\mathbb{Z}} L \rightarrow A \otimes_{\mathbb{Z}} L$$

is weakly split, and there is a choice for classifying map $\phi_{A,L} : J(A \otimes_{\mathbb{Z}} L) \rightarrow J(A) \otimes_{\mathbb{Z}} L$ of this extension, which is natural in both variables. \square

COROLLARY 1.4.5. ([5] 4.4.4) Let K be a finite pointed simplicial set. There is a homotopy class of maps

$$(18) \quad \phi_{A,K} : J(A^K) \rightarrow J(A)^K$$

natural with respect to K , which is represented by a classifying map of following the extension

$$J(A)^K \xrightarrow{\iota_A^*} T(A)^K \xrightarrow{\eta_A^*} A^K$$

Naturality means that for any map of finite pointed simplicial sets $f : K \rightarrow L$, the following diagram commutes

$$\begin{array}{ccc} J(A^K) & \longrightarrow & J(A)^K \\ J(f^*) \downarrow & & \downarrow f^* \\ J(A^L) & \longrightarrow & J(A)^L \end{array}$$

\square

4.4. Loop extension. Let A be an \mathcal{X} -algebra. The path algebra of A is $PA := P \otimes_{\mathbb{Z}} A$ with $P := \mathbb{Z}^{(\Delta^1, \star)}$. The loop extension of A is

$$(19) \quad \Omega A \rightarrow PA \xrightarrow{\text{ev}_1} A$$

Note it is naturally weakly split because $a \mapsto at$ is a natural section of $F(\text{ev}_1)$. Thus we can pick a natural choice for the classifying map of (19). We call it

$$\rho_A : J(A) \rightarrow \Omega A$$

Let

$$\mathcal{P} := \mathbb{Z}^{(\text{sd}^\bullet \Delta^1, \star)}$$

we have an extension of ind- \mathcal{X} -algebras

$$A^{\text{S}^1} := A^{(\text{sd}^\bullet \text{S}^1, \star)} \quad A^{\text{S}^1} \rightarrow \mathcal{P}A \xrightarrow{\text{ev}_1} A$$

which is naturally weakly split. The classifying map $J(A) \rightarrow A^{\text{S}^1}$ is the following composition

$$(20) \quad J(A) \xrightarrow{\rho_A} \Omega A \xrightarrow{h} A^{\text{S}^1}$$

where h is induced by the last vertex map. We will sometimes abuse notation and write ρ_A for the map (20).

4.5. Mapping path extension. Let $f : A \rightarrow B$ be a morphism of \mathcal{X} -algebras. The mapping path extension of f is the extension obtained from the loop extension of B by pulling it back to A

$$\begin{array}{ccccc} P_f := PB \times_B A & \Omega B & \xrightarrow{\iota} & PB \times_B A & \xrightarrow{\pi_f} & A & (E') \\ & \downarrow & & \downarrow & & \downarrow f & \\ & \Omega B & \xrightarrow{\iota} & PB & \xrightarrow{\text{ev}_1} & B & (E) \end{array}$$

We call P_f the **path algebra** of f . Note the extension (E') is naturally weakly split because $\tilde{s}(a) = (s \circ f(a), a)$ is a natural section of π_f where s is the natural section of (E) given in the Section 4.4. We define

$$(21) \quad \mathcal{P}_f := \mathcal{P}B \times_B A \quad B^{\mathbb{S}^1} \rightarrow \mathcal{P}B \times_B A \xrightarrow{\pi_f} A.$$

Note $\rho_f := \rho_B J(f)$ is the classifying map of the extension (21).

5. Excision

5.1. Triangulated categories. We recall the definition of a triangulated category [21], [5]. We will express the definition in terms of a loop functor Ω . A **triangulated category** $(\mathcal{T}, \Omega, \mathcal{Q})$ is an additive category \mathcal{T} with an equivalence $\Omega : \mathcal{T} \rightarrow \mathcal{T}$ and a class \mathcal{Q} of sequences in \mathcal{T} called **distinguished triangles**

$$(T) \quad \Omega C \rightarrow A \rightarrow B \rightarrow C$$

satisfying the following axioms:

TR0: Any sequence isomorphic to a distinguished triangle is a distinguished triangle. The sequence

$$\Omega A \rightarrow 0 \rightarrow A \xrightarrow{\text{id}_A} A$$

is a distinguished triangle for every object A in \mathcal{T} .

TR1: For any morphism $\alpha : A \rightarrow B$ in \mathcal{T} , there exists a distinguished triangle of the form

$$\Omega B \rightarrow C \rightarrow A \xrightarrow{\alpha} B$$

TR2: Consider the two triangles

$$\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \quad \Omega B \xrightarrow{-\Omega(h)} \Omega C \xrightarrow{-f} A \xrightarrow{-g} B$$

If one is distinguished, then so is the other.

TR3: For any solid arrow commutative diagram

$$\begin{array}{ccccccc} \Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ \downarrow \Omega k & & \downarrow j & & \downarrow l & & \downarrow k \\ \Omega C' & \xrightarrow{f'} & A' & \xrightarrow{g'} & B' & \xrightarrow{h'} & C' \end{array}$$

there exists a filler $j : A \rightarrow A'$ which makes the whole diagram commutative.

TR4: Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be morphism in \mathcal{T} . We set $\gamma = \beta\alpha$. There exists a commutative diagram

$$\begin{array}{ccccccc}
\Omega^2 C & \longrightarrow & \Omega D' & \longrightarrow & \Omega B & \xrightarrow{\Omega\beta} & \Omega C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & D''' & \xrightarrow{1} & D''' & \longrightarrow & 0 \\
\downarrow & & \downarrow h & & \downarrow & & \downarrow \\
\Omega C & \xrightarrow{j} & D'' & \longrightarrow & A & \xrightarrow{\gamma} & C \\
\downarrow 1 & & \downarrow & & \downarrow \alpha & & \downarrow 1 \\
\Omega C & \longrightarrow & D' & \longrightarrow & B & \xrightarrow{\beta} & C
\end{array}$$

in which each row and column is a distinguished triangle. Furthermore, the square

$$\begin{array}{ccc}
\Omega B & \xrightarrow{\Omega\beta} & \Omega C \\
\downarrow & & \downarrow j \\
D''' & \xrightarrow{h} & D''
\end{array}$$

commutes in \mathcal{T} .

A **triangle functor** from $(\mathcal{T}_1, \Omega_1, \mathcal{Q}_1)$ to $(\mathcal{T}_2, \Omega_2, \mathcal{Q}_2)$ is a pair consisting of an additive functor $F : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ and a natural transformation $\alpha : \Omega_2 F \rightarrow F \Omega_1$ such that

$$\Omega_2 F(C) \xrightarrow{F(f) \circ \alpha_C} F(A) \xrightarrow{F(g)} F(B) \xrightarrow{F(h)} F(C)$$

is a distinguished triangle in \mathcal{T}_2 for each triangle $\Omega_1 C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$.

5.2. Excisive homology theories. Let \mathcal{E} be a class of extensions

$$(22) \quad (E) \quad A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{X}\text{-Alg}$, and let $(\mathcal{T}, \Omega, \mathcal{Q})$ be a triangulated category. An \mathcal{E} -excisive homology theory for \mathcal{X} -algebras with values in \mathcal{T} consists of a functor $H : \mathcal{X}\text{-Alg} \rightarrow \mathcal{T}$, together with a collection $\{\partial_E : E \in \mathcal{E}\}$ of maps

$$\partial_E^H = \partial_E \in \text{hom}_{\mathcal{T}}(\Omega H(C), H(A)).$$

The maps ∂_E are to satisfy the following requirements:

- For all E as in (22)

$$\Omega H(C) \xrightarrow{\partial_E} H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C)$$

is a distinguished triangle in \mathcal{T} .

- If

$$\begin{array}{ccc}
(E) : & A & \xrightarrow{f} B \xrightarrow{g} C \\
& \alpha \downarrow & \beta \downarrow \quad \gamma \downarrow \\
(E') : & A' & \xrightarrow{f'} B' \xrightarrow{g'} C'
\end{array}$$

is a map of extensions, then the following diagram commutes:

$$\begin{array}{ccc} \Omega H(C) & \xrightarrow{\partial_E} & H(A) \\ \Omega H(\gamma) \downarrow & & \downarrow H(\alpha) \\ \Omega H(C') & \xrightarrow{\partial_{E'}} & H(A) \end{array}$$

6. Algebraic kk -theory

6.1. Universal functors. Let P be a property for functors defined on $\mathcal{X}\text{-Alg}$ to some triangulated category. A functor $u : \mathcal{X}\text{-Alg} \rightarrow \mathbb{U}$ is a **universal functor** with P if it has the property P and if $F : \mathcal{X}\text{-Alg} \rightarrow \mathcal{T}$ is another functor with P there exists a unique triangle functor $G : \mathbb{U} \rightarrow \mathcal{T}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{X}\text{-Alg} & \xrightarrow{u} & \mathbb{U} \\ & \searrow F & \downarrow G \\ & & \mathcal{T} \end{array}$$

In the next section we shall see the definition of algebraic kk -theory and a construction of a universal functor with $P = \{P_1, P_2, P_3\}$ where the properties are: excision (see 5.2), homotopy invariance (see 2.3), and matrix invariance (see 3.2). In Section 8 we define equivariant kk -theory and we construct a functor which is universal for the properties: excision, homotopy invariance and equivariant matrix invariance (defined in Section 7).

6.2. kk -groups. Let A, B be \mathcal{X} -algebras. Define

$$E_n(A, B) := [J^n(A), \mathcal{M}_\infty B^{\mathbb{S}^n}]_{\mathcal{X}}.$$

Consider the following morphism $\iota_n : E_n(A, B) \rightarrow E_{n+1}(A, B)$ such that

$$J^n(A) \xrightarrow{f} \mathcal{M}_\infty B^{\mathbb{S}^n} \mapsto J^{n+1}(A) \xrightarrow{J(f)} J(\mathcal{M}_\infty B^{\mathbb{S}^n}) \xrightarrow{\phi_{\mathcal{M}_\infty, B^{\mathbb{S}^n}}} \mathcal{M}_\infty J(B^{\mathbb{S}^n}) \xrightarrow{\rho_{B^{\mathbb{S}^n}}} \mathcal{M}_\infty B^{\mathbb{S}^{n+1}}$$

Define

$$kk^{[\mathcal{X}]}(A, B) = \text{colim}_{n \in \mathbb{N}} E_n(A, B)$$

6.3. Composition product. A morphism of \mathcal{X} -algebras $f : A \rightarrow B$ induces a morphism of weakly split extensions

$$\begin{array}{ccccc} A^{\mathbb{S}^1} & \longrightarrow & \mathcal{P}A & \longrightarrow & A \\ f^{\mathbb{S}^1} \downarrow & & \downarrow & & \downarrow f \\ B^{\mathbb{S}^1} & \longrightarrow & \mathcal{P}B & \longrightarrow & B \end{array}$$

By Proposition 1.4.3 the following diagram is commutative

$$\begin{array}{ccc} J(A) & \xrightarrow{\rho_A} & A^{\mathbb{S}^1} \\ J(f) \downarrow & & \downarrow f^{\mathbb{S}^1} \\ J(B) & \xrightarrow{\rho_B} & B^{\mathbb{S}^1} \end{array}$$

Then $f^{s^1} \rho_A$ and $\rho_f = \rho_B J(f)$ represent the same element in $kk^{|\mathcal{X}|}(A, B)$. Denote by $\gamma_A : J(A^{s^1}) \rightarrow J(A)^{s^1}$ the morphism (18) and note

$$\rho_{J(A)} \simeq \gamma_A J(\rho_A) : J^2(A) \rightarrow J(A)^{s^1}$$

THEOREM 1.6.1. ([5]) Let A , B and C be \mathcal{X} -algebras. There exists an associative product

$$\circ : kk^{|\mathcal{X}|}(B, C) \times kk^{|\mathcal{X}|}(A, B) \rightarrow kk^{|\mathcal{X}|}(A, C)$$

which extends the composition of algebra homomorphisms.

If $[\alpha] \in kk^{|\mathcal{X}|}(B, C)$ is an element represented by $\alpha : J^n(B) \rightarrow C^{s^n}$ and $[\beta] \in kk^{|\mathcal{X}|}(A, B)$ is an element represented by $\beta : J^m(A) \rightarrow B^{s^m}$ then $[\alpha] \circ [\beta]$ is an element $kk^{|\mathcal{X}|}(A, C)$ represented by

$$J^{n+m}(A) \xrightarrow{J^n(\beta)} J^n(B^{s^m}) \rightarrow J^n(B)^{s^m} \xrightarrow{\alpha^{s^m}} C^{s^{n+m}}$$

6.4. The category $\mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}$. The product given in theorem 1.6.1 allows us to define a composition in the following category $\mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}$:

- The objects are the \mathcal{X} -algebras: $\text{ob}(\mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}) = \text{ob}(\mathcal{X}\text{-Alg})$.
- The morphisms are the $kk^{|\mathcal{X}|}$ -groups: $\text{hom}_{\mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}}(A, B) = kk^{|\mathcal{X}|}(A, B)$.

Denote

$$j : \mathcal{X}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}$$

to the functor which at the level of objects is the identity and at level of morphism sends $f : A \rightarrow B$ to $[f] \in kk^{|\mathcal{X}|}(A, B)$.

REMARK 1.6.2. Let C be an \mathcal{X} -algebra. A morphism of \mathcal{X} -algebras $f : C \rightarrow \mathcal{M}_\infty C$ represents an element $j(f)$ in $kk^{|\mathcal{X}|}(C, \mathcal{M}_\infty C)$. But also represents an element in $kk^{|\mathcal{X}|}(C, C)$ because

$$kk^{|\mathcal{X}|}(C, C) = \text{colim } E_n(C, C) \quad \text{and} \quad E_0(C, C) = [C, \mathcal{M}_\infty C].$$

A priori, kk -theory is only defined for \mathcal{X} -algebras. However, if (A, J) is an ind- \mathcal{X} -algebra for which all structure maps are $kk^{|\mathcal{X}|}$ -equivalences, we can equally well speak about its $kk^{|\mathcal{X}|}$ -groups. The $kk^{|\mathcal{X}|}$ -groups with an ind- \mathcal{X} -algebra in the right argument are defined via the colimit of the induces diagram of $kk^{|\mathcal{X}|}$ -groups, i.e.

$$kk^{|\mathcal{X}|}(A, (B, J)) := \text{colim}_{j \in J} kk^{|\mathcal{X}|}(A, B_j)$$

The statement that two ind- \mathcal{X} -algebras B and C are $kk^{|\mathcal{X}|}$ -equivalent, has the rather strict meaning, that all structure maps of B and C are $kk^{|\mathcal{X}|}$ -equivalences and all morphisms that constitute the morphism of ind- \mathcal{X} -algebras are $kk^{|\mathcal{X}|}$ -equivalences as well.

6.5. The triangulated structure of $\mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}$. In this section we describe the triangulated structure (see 5.1) in $\mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}$. We shall define the endofunctor $\Omega : \mathfrak{K}\mathfrak{K}^{|\mathcal{X}|} \rightarrow \mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}$ and the class \mathcal{Q} of distinguished triangles.

6.5.1. *The endofunctor Ω .* The functor $\Omega : \mathfrak{K}\mathfrak{K}^{[\mathcal{X}]} \rightarrow \mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$ sends any \mathcal{X} -algebra A to the path \mathcal{X} -algebra ΩA . Let $[\alpha]$ be an element of $\text{kk}^{[\mathcal{X}]}(A, B)$ represented by $\alpha : J^n(A) \rightarrow B^{S^n}$. The class of $\Omega[\alpha]$ is represented by

$$(23) \quad J^n(A^{S^1}) \rightarrow J^n(A)^{S^1} \xrightarrow{\alpha^{S^1}} B^{S^{n+1}}$$

Note (23) represents an element of $\text{kk}^{[\mathcal{X}]}(A^{S^1}, B^{S^1})$ (see Lemma 6.3.8 [5] to check it is well defined). As $\iota : \Omega A \rightarrow A^{S^1}$ is a $\text{kk}^{[\mathcal{X}]}$ -equivalence (see corollary of Lemma 6.3.2 [5]), we have the natural morphism

$$\text{kk}^{[\mathcal{X}]}(A^{S^1}, B^{S^1}) \xrightarrow{\sim} \text{kk}^{[\mathcal{X}]}(\Omega A, \Omega B)$$

6.5.2. *Distinguished Triangles.* A diagram

$$\Omega C \rightarrow A \rightarrow B \rightarrow C$$

of morphisms in $\mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$ is called a **distinguished triangle** if it is isomorphic in $\mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$ to the path sequence

$$\Omega B' \xrightarrow{j(\iota)} P_f \xrightarrow{j(\pi_f)} A' \xrightarrow{j(f)} B'$$

associated with a homomorphism $f : A' \rightarrow B'$ of \mathcal{X} -algebras. Denote the class of distinguished triangles by \mathcal{Q} .

THEOREM 1.6.3 ([5]). The category $\mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$ is triangulated with respect to the endofunctor $\Omega : \mathfrak{K}\mathfrak{K}^{[\mathcal{X}]} \rightarrow \mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$ and the class \mathcal{Q} of distinguished triangles.

6.6. Universal properties of $\mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$. Let $E : A \xrightarrow{f} B \xrightarrow{g} C$ be an weakly split extension and let $c_E \in \text{kk}^{[\mathcal{X}]}(J(C), A)$ be the classifying map of E . As the natural map $\rho_A : J(A) \rightarrow \Omega A$ induces a kk -equivalence (see Lemma 6.3.10, [5]) we can consider the following morphisms in $\text{kk}^{[\mathcal{X}]}(\Omega C, A)$

$$(24) \quad \partial_E := c_E \circ \rho_C^{-1}.$$

The functor $j : \mathcal{X}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$ with the morphisms $\{\partial_E : E \in \mathcal{E}\}$ is an excisive homology theory, homotopy invariant and M_∞ -stable. In fact, let E be an extension as in (22). Take the path sequence asociated to g and the following diagram in $\mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$

$$\begin{array}{ccccccc} \Omega C & \xrightarrow{j(\iota)} & P_g & \xrightarrow{j(\pi_g)} & B & \xrightarrow{j(g)} & C \\ \text{id} \uparrow & & \iota_f \uparrow & & \uparrow \text{id} & & \uparrow \text{id} \\ \Omega C & \xrightarrow{\partial_E} & A & \xrightarrow{j(f)} & B & \xrightarrow{j(g)} & C \end{array} \quad \begin{array}{l} (T) \\ (T') \end{array}$$

The first square commutes beacuse $\iota_f \circ c_E$ is elementarily homotopic to $\iota \circ \rho_C$. By Lemma 6.3.2 [5], the morphism ι_f is a $\text{kk}^{[\mathcal{X}]}$ -equivalence. Finally (T) and (T') are isomorphic in $\mathfrak{K}\mathfrak{K}^{[\mathcal{X}]}$ and (T') is a distinguished triangle.

THEOREM 1.6.4. ([5]) The functor $j : \mathcal{X}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{|\mathcal{X}|}$ is universal with the properties defined above. In other words, if \mathcal{C} is a triangulated category and $G : \mathcal{X}\text{-Alg} \rightarrow \mathcal{C}$ together a class of morphisms $\{\bar{\partial}_E : E \in \mathcal{E}\}$ is an excisive, homotopy invariant and M_∞ -stable functor, then there exists a unique triangle functor $\bar{G} : \mathfrak{K}\mathfrak{K}^{|\mathcal{X}|} \rightarrow \mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}\text{-Alg} & \xrightarrow{j} & \mathfrak{K}\mathfrak{K}^{|\mathcal{X}|} \\ & \searrow G & \downarrow \bar{G} \\ & & \mathcal{C} \end{array}$$

THEOREM 1.6.5 ([5]). Consider $\mathcal{X} = \ell$, then

$$\mathrm{kk}_*^{|\ell|}(\ell, A) = \mathrm{kk}_*(\ell, A) \simeq \mathrm{KH}_*(A)$$

Here KH_* is the homotopy K-theory of Wiebel, see [32]. We recall this definition in a general context in Section 5 of Chapter 3.

7. Equivariant matrix invariance

In the equivariant setting we will replace the property of M_∞ -stability by an \mathcal{X} -stability condition. Let consider the different cases of \mathcal{X} separately.

7.1. G-stability. Let G be a group and ℓ a commutative ring with unit. In this section we define a G -algebra called M_G . Let $M_{|G|}$ be the matrix algebra defined in 3.1. The translation action in G defines an action in $M_{|G|}$

$$g \cdot e_{s,t} = e_{gs,gt}.$$

Recall a functor is homotopy invariant if it sends the inclusion $A \rightarrow A[t]$ to an isomorphism. A $M_{|G|}$ -stable functor on $G\text{-Alg}$ sends the morphism $a \rightarrow a \otimes e_{1_G, 1_G}$ to an isomorphism. We want a definition of G -stability such that a G -stable functor on $G\text{-Alg}$, identifies an G -algebra A with the algebra $M_G \otimes A$. In this case we may not have an equivariant map between A and $M_G \otimes A$. Note that the map $a \rightarrow a \otimes e_{1_G, 1_G}$ is not equivariant. For this reason we will define G -stability in terms of G -modules.

DEFINITION 1.7.1 (G -module with basis).

- A pair (\mathcal{W}, B) is a G -module with basis if \mathcal{W} is a G -module, free as an ℓ -module and B is a basis of \mathcal{W} .
- A pair (\mathcal{W}', B') is a submodule with basis of (\mathcal{W}, B) if \mathcal{W}' is a submodule of \mathcal{W} and $B' \subset B$.

Note that if (\mathcal{W}_1, B_1) and (\mathcal{W}_2, B_2) are G -modules with basis then $(\mathcal{W}_1 \oplus \mathcal{W}_2, B_1 \sqcup B_2)$ is a G -module with basis.

DEFINITION 1.7.2. Let (\mathcal{W}, B) be a G -module with basis $B = \{v_i : i \in \Lambda\}$. We define

$$\mathcal{L}(\mathcal{W}, B) := \{\psi : \Lambda \times \Lambda \rightarrow \ell : \{i : \psi(i, j) \neq 0\} \text{ is finite for all } j\}$$

Let $p_i : \mathcal{W} \rightarrow \ell$ be the projection to the submodule of \mathcal{W} generated by v_i ,

$$p_i\left(\sum_{j \in \Lambda} a_j v_j\right) = a_i.$$

Note that $\mathcal{L}(\mathcal{W}, B)$ and $\text{End}_\ell(\mathcal{W}) = \{f : \mathcal{W} \rightarrow \mathcal{W} : f \text{ is } \ell\text{-linear}\}$ are isomorphic; indeed we have inverse isomorphisms

$$(25) \quad \text{End}_\ell(\mathcal{W}) \rightarrow \mathcal{L}(\mathcal{W}, B) \quad f \mapsto \psi_f \quad \psi_f(i, j) = p_i(f(v_j))$$

$$(26) \quad \mathcal{L}(\mathcal{W}, B) \rightarrow \text{End}_\ell(\mathcal{W}) \quad \psi \mapsto f_\psi \quad f_\psi(v_k) = \sum_{i \in \Lambda} \psi(i, k) v_i$$

Define

$$\mathcal{C}(\mathcal{W}, B) := \{\psi \in \mathcal{L}(\mathcal{W}, B) : \{j : \psi(i, j) \neq 0\} \text{ is finite for all } i\}$$

$$\mathcal{F}(\mathcal{W}, B) := \{\psi \in \mathcal{L}(\mathcal{W}, B) : \{(i, j) : \psi(i, j) \neq 0\} \text{ is finite}\}$$

and

$$\text{End}_\ell^F(\mathcal{W}, B) := \{f \in \text{End}_\ell(\mathcal{W}) : \psi_f \in \mathcal{F}(\mathcal{W}, B)\} \quad \text{End}_\ell^C(\mathcal{W}, B) := \{f \in \text{End}_\ell(\mathcal{W}) : \psi_f \in \mathcal{C}(\mathcal{W}, B)\}$$

Note that $\mathcal{C}(\mathcal{W}, B)$ is a ring with the matrix product and $\text{End}_\ell^C(\mathcal{W}, B)$ is a ring with the composition. By (25) and (26), $\text{End}_\ell^C(\mathcal{W}, B)$ and $\mathcal{C}(\mathcal{W}, B)$ are isomorphic rings.

DEFINITION 1.7.3 (finiteness conditions for G-modules). Let (\mathcal{W}, B) be a G-module with basis. Consider the representation

$$\rho : G \rightarrow \text{End}_\ell(\mathcal{W}) \quad g \mapsto \rho_g \quad \rho_g(w) = g \cdot w$$

We say that (\mathcal{W}, B) is a G-module by finite automorphisms if $\rho(G) \subset \text{End}_\ell^F(\mathcal{W}, B)$. We say that (\mathcal{W}, B) is a G-module by almost finite automorphisms if $\rho(G) \subset \text{End}_\ell^C(\mathcal{W}, B)$. If (\mathcal{W}, B) is a G-module by almost finite automorphisms, $\text{End}_\ell^F(\mathcal{W}, B)$ is a G-algebra with the following action

$$g \cdot f = \rho(g)f(\rho(g))^{-1}$$

Note that $\text{End}_\ell^F(\mathcal{W}, B)$ is an ideal of $\text{End}_\ell^C(\mathcal{W}, B)$.

EXAMPLE 1.7.4. Let $\mathcal{W} = \ell G$ be the group algebra considered as a G-module via the regular representation with basis $B = \{\delta_g : g \in G\}$,

$$(27) \quad g \cdot \left(\sum_{h \in G} a_h \delta_h \right) = \sum_{h \in G} a_h \delta_{gh}.$$

Note

$$\rho : G \rightarrow \text{End}_\ell(\ell G) \simeq \mathcal{L}(\ell G, B) \quad g \mapsto M_g = \sum_{t \in G} e_{gt, t} \quad (M_g)(s, t) = \begin{cases} 1 & s = gt \\ 0 & s \neq gt \end{cases}$$

As $M_g \in \mathcal{C}(\ell G, B)$ for all $g \in G$, $(\ell G, B)$ is a G-module by almost finite automorphisms. Moreover we have

$$(M_g)^{-1} = M_{g^{-1}} = (M_g)^t.$$

Observe that

$$M_G = \mathcal{F}(\ell G, B).$$

DEFINITION 1.7.5. Let A be a G -algebra. Consider the tensor product

$$M_G A = M_G \otimes A$$

with the diagonal action of G

$$(M \otimes a)(N \otimes b) = MN \otimes ab \quad g \cdot (e_{s,t} \otimes a) = e_{gs,gt} \otimes g \cdot a.$$

PROPOSITION 1.7.6. Let \mathcal{W} be a G -module. Let \mathcal{W}^τ be \mathcal{W} considered as a G -module with trivial action, then

$$\ell G \otimes \mathcal{W} \simeq \ell G \otimes \mathcal{W}^\tau$$

PROOF: The isomorphisms are given by

$$T : \ell G \otimes \mathcal{W}^\tau \rightarrow \ell G \otimes \mathcal{W} \quad S : \ell G \otimes \mathcal{W} \rightarrow \ell G \otimes \mathcal{W}^\tau$$

$$T(\delta_g \otimes h) = \delta_g \otimes g(h) \quad S(\delta_g \otimes h) = \delta_g \otimes g^{-1}(h)$$

It is clear that each is the inverse of the other and that they are equivariant morphisms. \square

If (\mathcal{W}, B) is a G -module with basis we will write $\text{End}_\ell^G(\mathcal{W})$ and $\text{End}_\ell^F(\mathcal{W})$ ommiting the basis when there is no confusion.

PROPOSITION 1.7.7. Let (\mathcal{W}, B) be a G -module with basis. Then

$$\text{End}_\ell^F(\ell G \otimes \mathcal{W}) \simeq \text{End}_\ell^F(\ell G) \otimes \text{End}_\ell^F(\mathcal{W})$$

PROOF: Define $T : \text{End}_\ell^F(\ell G) \otimes \text{End}_\ell^F(\mathcal{W}) \rightarrow \text{End}_\ell^F(\ell G \otimes \mathcal{W})$ by

$$T(e_{g,h} \otimes e_{v,w}) = e_{g,v,h,w} \quad v, w \in B \quad g, h \in G$$

As T is a bijection between the basis, T is an isomorphism. \square

DEFINITION 1.7.8 (G -stability). Let (\mathcal{W}_1, B_1) and (\mathcal{W}_2, B_2) be G -modules by almost finite automorphisms such that $\text{card}(B_i) \leq \text{card}(G) \times \text{card } \mathbb{N}$, $i = 1, 2$. The inclusion $\iota : \mathcal{W}_1 \rightarrow \mathcal{W}_1 \oplus \mathcal{W}_2$ induces a morphism of G -algebras

$$(28) \quad \tilde{\iota} : \text{End}_\ell^F(\mathcal{W}_1) \rightarrow \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \quad f \mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$$

Let A be a G -algebra and consider

$$\tilde{\iota} \otimes 1 : \text{End}_\ell^F(\mathcal{W}_1) \otimes A \rightarrow \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A.$$

A functor $F : G\text{-Alg} \rightarrow \mathcal{C}$ is G -stable if for (\mathcal{W}_1, B_1) , (\mathcal{W}_2, B_2) and A as above $F(\tilde{\iota} \otimes 1)$ is an isomorphism in \mathcal{C} .

PROPOSITION 1.7.9. If $F : G\text{-Alg} \rightarrow \mathcal{C}$ is a G -stable functor then F is M_∞ -stable.

PROOF: Consider $(\mathcal{W}_1, B_1) = (\ell, \{1\})$ and $(\mathcal{W}_2, B_2) = (\ell^{(\mathbb{N})}, \{e_i : i \in \mathbb{N}\})$ with $\ell^{(\mathbb{N})} = \bigoplus_{i=1}^\infty \ell$, $\{e_i : i \in \mathbb{N}\}$ is the canonical basis and both modules have the trivial action of G . Note

$$\text{End}_\ell^F(\ell) = \text{End}_\ell(\ell) = \ell \quad \text{End}_\ell^F(\ell \oplus \ell^{(\mathbb{N})}) = \text{End}_\ell^F(\ell^{(\mathbb{N})}) = M_\infty$$

and $\tilde{\iota} : \ell \rightarrow M_\infty$ is the inclusion at the upper left corner. Then $\tilde{\iota} \otimes 1 = \iota : A \rightarrow M_\infty(A)$ and $F(\iota)$ is an isomorphism. \square

COROLLARY 1.7.10. If $G = \{e\}$, F is G -stable if and only if F is M_∞ -stable.

DEFINITION 1.7.11 (zig-zag). Let A, B be G -algebras and $F : G\text{-Alg} \rightarrow \mathcal{C}$ be a functor. A zig-zag between A and B by F is a diagram in $G\text{-Alg}$

$$A \xrightarrow{f_1} C_1 \xleftarrow{g_1} \dots \xrightarrow{f_n} C_n \xleftarrow{g_n} B$$

such that $F(g_i)$, $i = 1, \dots, n$, are isomorphisms in \mathcal{C} .

REMARK 1.7.12. Let A be a G -algebra and F be a G -stable functor. There exists a zig-zag between A and $M_G A$ by F . Consider $\mathcal{W}_1 = (\ell G, B)$ as in the example 1.7.4 and consider $\mathcal{W}_2 = (\ell, \{1\})$ with the trivial action of G . Put $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $C = \text{End}_\ell^F(\mathcal{W})$ with the induced action, then

$$\iota : A = A \otimes \ell = A \otimes \text{End}_\ell^F(\ell) \rightarrow A \otimes C \leftarrow A \otimes M_G : \iota'$$

is a zig-zag between A and $M_G A$.

PROPOSITION 1.7.13. Suppose G is countable. Let $F : G\text{-Alg} \rightarrow \mathcal{C}$ be an M_∞ -stable functor. Define

$$\hat{F} : G\text{-Alg} \rightarrow \mathcal{C} \quad A \mapsto F(M_G A)$$

where $M_G A$ was defined in 1.7.5. Then \hat{F} is G -stable.

PROOF: Let $(\mathcal{W}_1, B_1), (\mathcal{W}_2, B_2)$ be G -modules by almost finite automorphisms and A be a G -algebra. Consider

$$\tilde{\iota} \otimes 1 : \text{End}_\ell^F(\mathcal{W}_1) \otimes A \rightarrow \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A.$$

Let us check that $\hat{F}(\tilde{\iota} \otimes 1)$ is an isomorphism:

$$\begin{aligned} F(M_G \otimes \text{End}_\ell^F(\mathcal{W}_1) \otimes A) &\simeq F(\text{End}_\ell^F(\ell G \otimes \mathcal{W}_1) \otimes A) && \text{by 1.7.7} \\ &\simeq F(\text{End}_\ell^F(\ell G \otimes \mathcal{W}_1^\tau) \otimes A) && \text{by 1.7.6} \\ &\simeq F(M_G \otimes \text{End}_\ell^F(\mathcal{W}_1^\tau) \otimes A) && \text{by 1.7.7} \\ F(M_G \otimes \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A) &\simeq F(\text{End}_\ell^F((\mathcal{W}_1 \oplus \mathcal{W}_2)^\tau) \otimes M_G \otimes A) \end{aligned}$$

$\hat{F}(\tilde{\iota} \otimes 1) = F(M_G \otimes \tilde{\iota} \otimes 1)$ is an isomorphism because F is M_∞ -stable. \square

EXAMPLE 1.7.14. Suppose G is a finite group of order n such that $1/n \in \ell$. Define $\xi = (1/n) \sum_{g \in G} \delta_g$, then ξ is idempotent. The map $s : \ell G \rightarrow \ell G$, $s(1) = \xi$, is a G -equivariant section of the canonical argumentation $\varphi : \ell G \rightarrow \ell$. Thus the sequence of G -modules

$$(29) \quad 0 \longrightarrow I \longrightarrow \ell G \xrightarrow{\varphi} \ell \longrightarrow 0$$

splits. Then

$$\ell G = \ell \xi \oplus I$$

I is a G -module with basis $\{\delta_e - \delta_g : g \neq e\}$. Define

$$\lambda_g = \begin{cases} \xi & g = e \\ \delta_e - \delta_g & g \neq e \end{cases}$$

The set $\Lambda = \{\lambda_g : g \in G\}$ is a basis of ℓG and the relations with the elements of $B = \{\delta_g : g \in G\}$ are the following

$$\begin{aligned} \lambda_e &= \frac{1}{n} \sum_{g \in G} \delta_g & \lambda_h &= \delta_e - \delta_h \\ \delta_e &= \lambda_e + \frac{1}{n} \sum_{g \neq e} \lambda_g & \delta_h &= \lambda_e - \lambda_h + \frac{1}{n} \sum_{g \neq e} \lambda_g \end{aligned} \quad h \neq e$$

Consider $\mathcal{W}_1 = \ell = (\ell\xi, \{\xi\})$ and $\mathcal{W}_2 = (I, \{\lambda_g\}_{g \neq e})$, in this case the morphism (28) is

$$\iota : \ell \rightarrow M_G \simeq \text{End}(\ell G, \Lambda) \quad 1 \mapsto \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

If we write it in the canonical basis we have

$$\bar{\iota} : \ell \rightarrow M_G = \text{End}(\ell G, B) \quad 1 \mapsto \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$$

If $F : G\text{-Alg} \rightarrow \mathcal{C}$ is a G -stable functor then $F(\bar{\iota})$ is an isomorphism in \mathcal{C} .

REMARK 1.7.15. Let I be a set with a distinguished element ι_0 . Let A be an ℓ -algebra and $\{A_i : i \in I\}$ subalgebras such that $A = \bigoplus_{i \in I} A_i$. For each $i \in I$ we consider the projection $\pi_{A_i} : A \rightarrow A_i$. The following matrices with coefficients in $\text{End}_\ell^F(A)$ and with indices in I are conjugated

$$M(i, j) = \begin{cases} \text{id} & i = j = \iota_0 \\ 0 & \text{otherwise} \end{cases} \quad N(i, j) = \begin{cases} \pi_{A_i} & i = j \\ 0 & i \neq j \end{cases}$$

In fact, if we choose T such that

$$T(i, j) = \begin{cases} \pi_{A_j} & i = \iota_0 \\ \pi_{A_i} & j = \iota_0 \\ 1 - \pi_{A_i} & i = j \neq \iota_0 \\ 0 & \text{otherwise} \end{cases}$$

it is easy to see that $T^2 = Id$ and $TMT = N$.

REMARK 1.7.16. Let H be a subgroup of G . Define

$$\ell(G/H) = \bigoplus_{gH \in G/H} \ell$$

Consider $\mathcal{W}_1 = (\ell(G/H), \{gH : g \in G\})$ with the regular action and $\mathcal{W}_2 = (\ell(G/H), \{gH : g \in G\})$ with the trivial action. Put

$$M_{G/H} = \mathcal{F}(\mathcal{W}_1) \quad M_{|G/H|} = \mathcal{F}(\mathcal{W}_2)$$

There is an isomorphism

$$\begin{aligned} M_{G/H} \otimes M_G &\xrightarrow{\cong} M_{|G/H|} \otimes M_G \\ e_{sH, gH} \otimes e_{t, r} &\mapsto e_{t^{-1}sH, r^{-1}gH} \otimes e_{t, r} \end{aligned}$$

7.2. \hat{G} -stability. In this section we consider the dual of the notion of G -stability. We want to identify a G -graded algebra A with the G -graded matrix algebra $M_{\hat{G}}A$. The definition of \hat{G} -stability is easier than that of G -stability because the morphism $A \rightarrow M_{\hat{G}}A$ is homogeneous.

DEFINITION 1.7.17 (\hat{G} -algebra). A \hat{G} -algebra A is an associative algebra with a family of submodules $\{A_s\}_{s \in G}$ such that

$$A = \bigoplus_{s \in G} A_s \quad A_s A_t \subseteq A_{st} \quad s, t \in G$$

We write $|a| = s$ if $a \in A_s$. A morphism $f : A \rightarrow B$ of \hat{G} -algebras is an algebra morphism such that $f(A_s) \subseteq B_s$, $s \in G$. Write $\hat{G}\text{-Alg}$ for the category of \hat{G} -algebras.

EXAMPLE 1.7.18. An associative algebra A is a \hat{G} -algebra with the trivial grading:

$$A = \bigoplus_{s \in G} A_s \quad A_s = \begin{cases} A & s = e \\ 0 & s \neq e \end{cases}$$

In particular ℓ is a \hat{G} -algebra.

EXAMPLE 1.7.19. Let A be a \hat{G} -algebra. Define the following grading in $M_G A$

$$(30) \quad (M_G A)_g := \langle e_{s,t} \otimes a : g = s|a|t^{-1} \rangle$$

As

$$(e_{s,t} \otimes a)(e_{\tilde{s},\tilde{t}} \otimes b) = \begin{cases} e_{s,\tilde{t}} \otimes ab & t = \tilde{s} \\ 0 & t \neq \tilde{s} \end{cases}$$

we have

$$\begin{aligned} |(e_{s,t} \otimes a)(e_{\tilde{s},\tilde{t}} \otimes b)| &= s|a||b|\tilde{t}^{-1} && \text{if } t = \tilde{s} \\ &= s|a|t^{-1}\tilde{s}|b|\tilde{t}^{-1} \\ &= |e_{s,t} \otimes a||e_{\tilde{s},\tilde{t}} \otimes b| \end{aligned}$$

and $M_G A$ is a \hat{G} -algebra. We will write $M_{\hat{G}}A$ for $M_G A$ with this structure.

DEFINITION 1.7.20 (\hat{G} -stability). Let A be a \hat{G} -algebra. Consider $M_{\hat{G}}A$ as in the example 1.7.19. Observe the map

$$\iota_A : A \rightarrow M_{\hat{G}}A \quad a \mapsto e_{1_G,1_G} \otimes a$$

is homogeneous. A functor $F : \hat{G}\text{-Alg} \rightarrow \mathcal{C}$ is a \hat{G} -stable if $F(\iota_A)$ is an isomorphism in \mathcal{C} for all $A \in \hat{G}\text{-Alg}$.

7.3. \mathcal{H} -stability. In this section we suppose \mathcal{H} is a Hopf algebra over a field ℓ . Recall that a left Hopf module over \mathcal{H} is an ℓ -module M such that

- M is a left \mathcal{H} -module.
- M is a left \mathcal{H} -comodule, via $\rho : M \rightarrow \mathcal{H} \otimes M$.
- ρ is a left \mathcal{H} -module map, where $\mathcal{H} \otimes M$ is a left module via

$$h \cdot (k \otimes m) = \sum h_1 k \otimes h_2 \cdot m.$$

We may also write

$$\rho(h \cdot m) = h \cdot \rho(m) \quad \sum (h \cdot m)_{-1} \otimes (h \cdot m)_0 = \sum h_1 \cdot m_{-1} \otimes h_2 \cdot m_0 \quad \forall m \in M \quad \forall h \in \mathcal{H}$$

PROPOSITION 1.7.21. Let M be a left \mathcal{H} -module, $\mathcal{H} \otimes M$ is a Hopf module over \mathcal{H} via

$$h \cdot (k \otimes m) = \sum h_1 k \otimes h_2 \cdot m \quad \rho(k \otimes m) = \Delta(k) \otimes m$$

Write M^τ for the trivial \mathcal{H} -module, then

$$\mathcal{H} \otimes M \simeq \mathcal{H} \otimes M^\tau$$

PROOF: The isomorphisms are given by

$$\alpha : \mathcal{H} \otimes M^\tau \rightarrow \mathcal{H} \otimes M \quad \alpha(h \otimes m) = \sum h_1 \otimes h_2 \cdot m$$

$$\beta : \mathcal{H} \otimes M \rightarrow \mathcal{H} \otimes M^\tau \quad \beta(h \otimes m) = \sum h_1 \otimes S(h_2) \cdot m$$

One checks that one is the inverse of the each other and that they are Hopf modules morphisms. \square

Let M be an \mathcal{H} -module. In the Example 1.1.3 we have seen an \mathcal{H} -algebra structure on $\text{End}_\ell(M)$. Define

$$(31) \quad \text{End}_\ell^F(M) := \{\varphi \in \text{End}_\ell(M) : \varphi(M) \text{ is a finite dimensional subspace of } M\}.$$

Let $\varphi \in \text{End}_\ell^F(M)$ and $h \in \mathcal{H}$. Consider C_h the subcoalgebra of \mathcal{H} generated by h . By Fundamental theorem of coalgebras we know C_h has finite dimension over ℓ . Because $(h \cdot \varphi)(M) \subset C_h \cdot \varphi(M)$ we conclude $(h \cdot \varphi)(M)$ is finite dimensional and $h \cdot \varphi \in \text{End}_\ell^F(M)$. Hence $\text{End}_\ell^F(M)$ is an \mathcal{H} -subalgebra of $\text{End}_\ell(M)$. It is easy to check that $\text{End}_\ell^F(M)$ is an ideal of $\text{End}_\ell(M)$.

REMARK 1.7.22. Let M be an \mathcal{H} -module, $\varphi \in \text{End}_\ell^F(M)$ and $h, k \in \mathcal{H}$. Recall from (7) the definition of $\varphi_{h,k}$, then $\varphi_{h,k}(M) \subset h \cdot \varphi(M)$. The space $\varphi(M)$ has finite dimension and h is fixed. Thus $h \cdot \varphi(M)$ and $\varphi_{h,k}(M)$ has also finite dimension. We conclude that $\varphi_{h,k} \in \text{End}_\ell^F(M)$.

DEFINITION 1.7.23. Let M_1 and M_2 be \mathcal{H} -modules and let A be an \mathcal{H} -algebra. Let us consider the following homomorphism of \mathcal{H} -algebras

$$\begin{aligned} \iota : \text{End}_\ell^F(M_1) \otimes A &\rightarrow \text{End}_\ell^F(M_1 \oplus M_2) \otimes A \\ f \otimes a &\mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \otimes a \end{aligned}$$

A functor $F : \mathcal{H}\text{-Alg} \rightarrow \mathcal{C}$ is \mathcal{H} -stable if it is M_∞ -stable and $F(\iota)$ is an isomorphism for every M_1, M_2 and A as above and with

$$\dim_\ell(M_i) \leq \dim_\ell(\mathcal{H}) \quad i = 1, 2.$$

REMARK 1.7.24. Let A be an \mathcal{H} -algebra and $F : \mathcal{H}\text{-Alg} \rightarrow \mathcal{C}$ an \mathcal{H} -stable functor. There exists a zig-zag between A and $\text{End}_\ell^F(\mathcal{H}) \otimes A$ by F . Consider \mathcal{H} as a \mathcal{H} -module with the regular action and ℓ with the trivial action. Then

$$A \cong \ell \otimes A \cong \text{End}_\ell^F(\ell) \otimes A \xrightarrow{\iota} \text{End}_\ell^F(\ell \oplus \mathcal{H}) \otimes A \xleftarrow{\iota'} \text{End}_\ell^F(\mathcal{H}) \otimes A$$

is a zig-zag by F .

PROPOSITION 1.7.25. Let \mathcal{H} be a finite dimensional Hopf algebra and M an \mathcal{H} -module with finite dimension. Then

$$\mathrm{End}_\ell(\mathcal{H} \otimes M) \simeq \mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M)$$

as \mathcal{H} -algebras. The structures of $\mathrm{End}_\ell(\mathcal{H} \otimes M)$ and $\mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M)$ are those given in the examples 1.1.3 and 1.1.4 respectively.

PROOF: Define

$$\Gamma : \mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M) \rightarrow \mathrm{End}_\ell(\mathcal{H} \otimes M) \quad \Gamma(\varphi \otimes \psi)(h \otimes m) = \varphi(h) \otimes \psi(m)$$

One checks Γ is an \mathcal{H} -algebra monomorphism. By a dimension argument we conclude that it is an isomorphism. \square

PROPOSITION 1.7.26. Suppose $\dim_\ell \mathcal{H}$ is finite. Let $F : \mathcal{H}\text{-Alg} \rightarrow \mathcal{C}$ be a homotopy invariant and M_∞ -stable functor. The following functor

$$\hat{F} : \mathcal{H}\text{-Alg} \rightarrow \mathcal{C} \quad A \mapsto F(\mathrm{End}_\ell(\mathcal{H}) \otimes A)$$

is homotopy invariant and \mathcal{H} -stable.

PROOF: Let M_1 and M_2 be \mathcal{H} -modules with finite dimension and A be an \mathcal{H} -algebra. Put

$$\tilde{\iota} \otimes 1 : \mathrm{End}_\ell(M_1) \otimes A \rightarrow \mathrm{End}_\ell(M_1 \oplus M_2) \otimes A.$$

Let $\hat{F} : \mathcal{H}\text{-Alg} \rightarrow \mathcal{C}$, $\hat{F}(A) := F(\mathrm{End}_\ell(\mathcal{H}) \otimes A)$. We have to prove $\hat{F}(\tilde{\iota} \otimes 1)$ is an isomorphism. As

$$\begin{aligned} F(\mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M_1) \otimes A) &\simeq F(\mathrm{End}_\ell(\mathcal{H} \otimes M_1) \otimes A) && \text{see 1.7.25} \\ &\simeq F(\mathrm{End}_\ell(\mathcal{H} \otimes M_1^\tau) \otimes A) && \text{by 1.7.21} \\ &\simeq F(\mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M_1^\tau) \otimes A) && \text{see 1.7.25} \\ &\simeq F(\mathrm{End}_\ell(M_1^\tau) \otimes \mathrm{End}_\ell(\mathcal{H}) \otimes A) \\ F(\mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M_1 \oplus M_2) \otimes A) &\simeq F(\mathrm{End}_\ell((M_1 \oplus M_2)^\tau) \otimes \mathrm{End}_\ell(\mathcal{H}) \otimes A) \end{aligned}$$

$\hat{F}(\tilde{\iota} \otimes 1)$ is an isomorphism by the M_∞ -stability of F . \square

In order to obtain an \mathcal{H} -stable functor when \mathcal{H} is not finite dimensional, we have tried to apply the argument used in Proposition 1.7.26. We could not prove an equivalent assumption of Proposition 1.7.25. We are not sure if the right finiteness condition on $\mathrm{End}_\ell(M)$ is that written on (31). If it is the correct one, we have to redefine the $\mathrm{kk}^{[\mathcal{H}]}$ -groups stabilizing by the $\mathbb{N} \times \mathbb{N}$ -matrices with finite rank instead.

7.4. $\hat{\mathcal{H}}$ -stability. Suppose \mathcal{H} is a Hopf algebra over a field ℓ with finite dimension. Let A be an $\hat{\mathcal{H}}$ -algebra.

PROPOSITION 1.7.27. Let N be a \mathcal{H} -comodule and N^τ the trivial comodule. There is a isomorphism

$$\mathcal{H} \otimes N \simeq \mathcal{H} \otimes N^\tau$$

\square

Let N be a \mathcal{H} -comodule with finite dimension. Recall from Example 1.1.5 the structure of \mathcal{H} -algebra on $\mathrm{End}_\ell(N) \otimes A$.

DEFINITION 1.7.28 ($\hat{\mathcal{H}}$ -stability). Let N_1 and N_2 be \mathcal{H} -comodules with finite dimension and let A be an $\hat{\mathcal{H}}$ -algebra. Let us consider the following homomorphism of $\hat{\mathcal{H}}$ -algebras

$$\begin{aligned} \iota : \text{End}_\ell(N_1) \otimes A &\rightarrow \text{End}_\ell(N_1 \oplus N_2) \otimes A \\ f \otimes a &\mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \otimes a \end{aligned}$$

A functor $F : \hat{\mathcal{H}}\text{-Alg} \rightarrow \mathcal{C}$ is $\hat{\mathcal{H}}$ -stable if it is M_∞ -stable and $F(\iota)$ is an isomorphism for every N_1, N_2 and A as above.

PROPOSITION 1.7.29. Let $F : \hat{\mathcal{H}}\text{-Alg} \rightarrow \mathcal{C}$ be a homotopy invariant and M_∞ -stable functor. The following functor

$$\hat{F} : \hat{\mathcal{H}}\text{-Alg} \rightarrow \mathcal{C} \quad A \mapsto F(\text{End}_\ell(\mathcal{H}) \otimes A)$$

is homotopy invariant and $\hat{\mathcal{H}}$ -stable. □

8. Equivariant algebraic kk -theory

Suppose G is a countable group and ℓ is a commutative ring with unit. Let A, B be G -algebras, we define

$$kk^G(A, B) := kk^{|\mathcal{G}|}(M_G \otimes A, M_G \otimes B).$$

Consider the category $\mathfrak{K}\mathfrak{K}^G$ whose objects are the G -algebras and the morphisms between A and B are the elements of $kk^G(A, B)$. In other words,

$$\text{hom}_{\mathfrak{K}\mathfrak{K}^G}(A, B) = \text{hom}_{\mathfrak{K}\mathfrak{K}^{|\mathcal{G}|}}(M_G \otimes A, M_G \otimes B)$$

Let $j^G : G\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^G$ be the functor such that is the identity on objects and sends each morphism of G -algebras $f : A \rightarrow B$ to its class $[f] \in kk^G(A, B)$.

THEOREM 1.8.1. The functor $j^G : G\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^G$ is an excisive, equivariantly homotopy invariant, and G -stable functor. Moreover, it is the universal functor for these properties.

PROOF: Let E be an extension as in (22), define

$$\partial_E^G \in \text{hom}_{\mathfrak{K}\mathfrak{K}^G}(\Omega C, A) = \text{hom}_{\mathfrak{K}\mathfrak{K}^{|\mathcal{G}|}}(M_G \otimes \Omega C, M_G \otimes A) = \text{hom}_{\mathfrak{K}\mathfrak{K}^{|\mathcal{G}|}}(\Omega M_G \otimes C, M_G \otimes A)$$

as the morphism ∂_E^G defined in (24) asociated to the following weakly split extension

$$M_G \otimes A \rightarrow M_G \otimes B \rightarrow M_G \otimes C \quad (E')$$

By theorem 1.6.4 and proposition 1.7.13 the functor $j : G\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^G$ with the family $\{\partial_E^G : E \in \mathcal{E}\}$ is an excisive, homotopy invariant and G -stable functor. Let us check it is universal for these properties. Let $X : G\text{-Alg} \rightarrow \mathcal{C}$ be a functor which has the mentioned

properties with a family $\{\bar{\partial}_E : E \in \mathcal{E}\}$. By theorem 1.6.4 there exists a unique triangle functor $\bar{X} : \mathfrak{K}\mathfrak{K}^{|\mathbf{G}|} \rightarrow \mathcal{C}$ such that the following diagram commutes

$$(32) \quad \begin{array}{ccc} \mathbf{G}\text{-Alg} & \xrightarrow{j^{\mathbf{G}}} & \mathfrak{K}\mathfrak{K}^{\mathbf{G}} \\ & \searrow j \quad \nearrow X' & \\ & \mathfrak{K}\mathfrak{K}^{|\mathbf{G}|} & \\ & \downarrow \bar{X} & \\ & \mathcal{C} & \end{array}$$

We will define $X' : \mathfrak{K}\mathfrak{K}^{\mathbf{G}} \rightarrow \mathcal{C}$. We know that $X' = X$ on objects. As X is \mathbf{G} -stable the following morphisms are a zig-zag between A and $M_{\mathbf{G}} \otimes A$ by X , (see Remark 1.7.12)

$$(33) \quad A \xrightarrow{\iota_A} M_{\mathbf{G} \sqcup \{*\}} \otimes A \xleftarrow{\iota'_A} M_{\mathbf{G}} \otimes A$$

Let $\alpha \in \mathbf{kk}^{\mathbf{G}}(A, B)$ and define

$$X'(\alpha) := X(\iota_B)^{-1} X(\iota'_B) \bar{X}(\alpha) X(\iota'_A)^{-1} X(\iota_A).$$

Note this definition is the unique possibility to make the diagram (32) commutative. \square

Let A, B be $\hat{\mathbf{G}}$ -algebras. We define

$$\mathbf{kk}^{\hat{\mathbf{G}}}(A, B) := \mathbf{kk}^{|\hat{\mathbf{G}}|}(A, B).$$

Consider the category $\mathfrak{K}\mathfrak{K}^{\hat{\mathbf{G}}}$ whose objects are the $\hat{\mathbf{G}}$ -algebras and the morphisms between A and B are the elements of $\mathbf{kk}^{\hat{\mathbf{G}}}(A, B)$. Let $j^{\hat{\mathbf{G}}} : \hat{\mathbf{G}}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{\mathbf{G}}}$ be the functor that is the identity on objects and which sends each morphism of \mathbf{G} -algebras $f : A \rightarrow B$ to its class $[f] \in \mathbf{kk}^{\hat{\mathbf{G}}}(A, B)$.

THEOREM 1.8.2. The functor $j^{\hat{\mathbf{G}}} : \hat{\mathbf{G}}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{\mathbf{G}}}$ is an excisive, graded homotopy invariant, and $\hat{\mathbf{G}}$ -stable functor. Moreover, it is the universal functor for these properties.

PROOF: : It follows from Theorem 1.6.4 and the fact that $\hat{\mathbf{G}}$ -stability property holds by the M_{∞} -stability property. \square

Let \mathcal{H} be a Hopf algebra over a field ℓ with finite dimension. Similary, if A and B are \mathcal{H} -algebras, we define

$$\mathbf{kk}^{\mathcal{H}}(A, B) := \mathbf{kk}^{|\mathcal{H}|}(\text{End}_{\ell}(\mathcal{H}) \otimes A, \text{End}_{\ell}(\mathcal{H}) \otimes B).$$

Consider the category $\mathfrak{K}\mathfrak{K}^{\mathcal{H}}$ whose objects are the \mathcal{H} -algebras and the morphisms between A and B are the elements of $\mathbf{kk}^{\mathcal{H}}(A, B)$. In other words,

$$\text{hom}_{\mathfrak{K}\mathfrak{K}^{\mathcal{H}}}(A, B) = \text{hom}_{\mathfrak{K}\mathfrak{K}^{|\mathcal{H}|}}(\text{End}_{\ell}(\mathcal{H}) \otimes A, \text{End}_{\ell}(\mathcal{H}) \otimes B)$$

Let $j^{\mathcal{H}} : \mathcal{H}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{H}}$ be the functor which is the identity on objects and sends each morphism of \mathcal{H} -algebras $f : A \rightarrow B$ to its class $[f] \in \mathbf{kk}^{\mathcal{H}}(A, B)$.

THEOREM 1.8.3. The functor $j^{\mathcal{H}} : \mathcal{H}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{H}}$ is universal for the properties: excisive, equivariantly homotopy invariant, and \mathcal{H} -stable.

PROOF: The proof is similar to the proof of Theorem 1.8.1. Just we have to replace M_G by $\text{End}_\ell^F(\mathcal{H})$. \square

Similary, if A and B are $\hat{\mathcal{H}}$ -algebras, we define

$$kk^{\hat{\mathcal{H}}}(A, B) := kk^{|\hat{\mathcal{H}}|}(\text{End}_\ell(\mathcal{H}) \otimes A, \text{End}_\ell(\mathcal{H}) \otimes B).$$

Consider the category $\mathfrak{K}\mathfrak{K}^{\hat{\mathcal{H}}}$ whose objects are the $\hat{\mathcal{H}}$ -algebras and the morphisms between A and B are the elements of $kk^{\hat{\mathcal{H}}}(A, B)$. Let $j^{\hat{\mathcal{H}}} : \hat{\mathcal{H}}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{\mathcal{H}}}$ be the functor which is the identity on objects and sends each morphism of $\hat{\mathcal{H}}$ -algebras $f : A \rightarrow B$ to its class $[f] \in kk^{\hat{\mathcal{H}}}(A, B)$.

THEOREM 1.8.4. The functor $j^{\hat{\mathcal{H}}} : \hat{\mathcal{H}}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{\mathcal{H}}}$ is universal for the properties: excisive, equivariantly homotopy invariant, and $\hat{\mathcal{H}}$ -stable.

\square

CHAPTER 2

Adjointness theorems in kk -theory.

In this chapter we study adjointness theorems in equivariant kk -theory. We put in an algebraic context some of the adjointness theorems which appear in Kasparov KK -theory. Let G be a countable group and ℓ a commutative ring with unit. We define the functors of trivial action and crossed product between $\mathfrak{K}\mathfrak{K}$ and $\mathfrak{K}\mathfrak{K}^G$. The first adjointness theorem is Theorem 2.1.4 which is an algebraic version of the GREEN-JULG THEOREM. This result gives us the first computation related with homotopy K -theory. If G is a finite group, A is a G -algebra, B is an algebra and $\frac{1}{|G|} \in \ell$ then there is an isomorphism

$$\psi_{GJ} : \mathfrak{K}\mathfrak{K}^G(B^\tau, A) \rightarrow \mathfrak{K}\mathfrak{K}(B, A \rtimes G).$$

In particular, if $B = \ell$ then

$$\mathfrak{K}\mathfrak{K}^G(\ell, A) \simeq \mathfrak{K}\mathfrak{H}(A \rtimes G).$$

We consider a subgroup H of G , define induction and restriction functors between $\mathfrak{K}\mathfrak{K}^G$ and $\mathfrak{K}\mathfrak{K}^H$ and study the adjointness between them. If B is an H -algebra and A is a G -algebra then there is an isomorphism

$$\psi_{IR} : \mathfrak{K}\mathfrak{K}^G(\text{Ind}_H^G B, A) \rightarrow \mathfrak{K}\mathfrak{K}^H(B, \text{Res}_G^H A).$$

This result gives us another computation. Taking H the trivial group and $B = \ell$ we obtain that

$$\mathfrak{K}\mathfrak{K}^G(\ell^{(G)}, A) \simeq \mathfrak{K}\mathfrak{H}(A) \quad \forall A \in G\text{-Alg}.$$

Here $\ell^{(G)} = \bigoplus_{g \in G} \ell$ with the regular action of G . More general, if H is a finite subgroup of G and $1/|H| \in \ell$ we combine ψ_{GJ} and ψ_{IR} and obtain

$$\mathfrak{K}\mathfrak{K}^G(\ell^{(G/H)}, A) \simeq \mathfrak{K}\mathfrak{H}(A \rtimes H) \quad \forall A \in G\text{-Alg}.$$

We also prove an algebraic version of GREEN IMPRIMITIVITY THEOREM and obtain that

$$\mathfrak{K}\mathfrak{H}(A \rtimes H) \simeq \mathfrak{K}\mathfrak{H}(\text{Ind}_H^G A \rtimes G).$$

We also obtain an algebraic version of the Baaj-Skandalis theorem. We show that the functors

$$\rtimes G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{G}} \quad \hat{G} \rtimes : \mathfrak{K}\mathfrak{K}^{\hat{G}} \rightarrow \mathfrak{K}\mathfrak{K}^G$$

are inverse category equivalences. Let \mathcal{H} be a Hopf algebra with finite dimension. We define functors between $\mathfrak{K}\mathfrak{K}$ and $\mathfrak{K}\mathfrak{K}^{\mathcal{H}}$, the smash product and the trivial action. We study the adjointness between them in theorem 2.5.4. We obtain that if \mathcal{H} is semisimple, B is an algebra and A is an \mathcal{H} -algebra then there is an isomorphism

$$\psi : \mathfrak{K}\mathfrak{K}^{\mathcal{H}}(B^\tau, A) \rightarrow \mathfrak{K}\mathfrak{K}(B, A \# \mathcal{H}).$$

In particular,

$$\mathfrak{K}\mathfrak{K}^{\mathcal{H}}(\ell, A) \simeq \mathfrak{K}\mathfrak{H}(A \# \mathcal{H}).$$

1. Crossed product and trivial action

1.1. Trivial action. Let A be an ℓ -algebra. Recall A^τ is A with the trivial action of G . This gives us a functor $\tau : \text{Alg} \rightarrow G\text{-Alg}$. It is easy to check that $j^G \circ \tau$ satisfies excision and is homotopy invariant and M_∞ -stable. By theorem 1.6.4 there exists a unique functor $\hat{\tau} : \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{K}\mathfrak{K}^G$ such that the following diagram is commutative

$$\begin{array}{ccc} \text{Alg} & \xrightarrow{\tau} & G\text{-Alg} \\ j \downarrow & & \downarrow j^G \\ \mathfrak{K}\mathfrak{K} & \xrightarrow{\hat{\tau}} & \mathfrak{K}\mathfrak{K}^G \end{array}$$

We will write τ for $\hat{\tau}$.

1.2. Crossed product. Let A be a G -algebra. The crossed product algebra $A \rtimes G$ is the ℓ -module $A \otimes \ell G$ with the following multiplication

$$(a \rtimes g)(b \rtimes h) = a(g \cdot b) \rtimes gh \quad a, b \in A, g, h \in G.$$

PROPOSITION 2.1.1. Let A be a G -algebra and \mathcal{W} be a G -module by almost finite automorphisms. The following algebras are naturally isomorphic

$$(A \rtimes G) \otimes \text{End}_\ell^F(\mathcal{W}) \simeq (A \otimes \text{End}_\ell^F(\mathcal{W})) \rtimes G.$$

PROOF: Let $\rho : G \rightarrow (\text{End}_\ell^K(\mathcal{W}))^\times$ be the structure map. Define

$$\phi : (A \rtimes G) \otimes \text{End}_\ell^F(\mathcal{W}) \rightarrow (A \otimes \text{End}_\ell^F(\mathcal{W})) \rtimes G \quad \phi(a \rtimes g \otimes \varphi) = a \otimes \varphi \rho(g^{-1}) \rtimes g$$

It is an algebra morphism:

$$\begin{aligned} \phi((a \rtimes g \otimes \varphi)(\tilde{a} \rtimes \tilde{g} \otimes \tilde{\varphi})) &= \phi(ag(\tilde{a}) \rtimes g\tilde{g} \otimes \varphi\tilde{\varphi}) \\ &= ag(\tilde{a}) \otimes \varphi\tilde{\varphi}\rho(\tilde{g}^{-1}g^{-1}) \rtimes g\tilde{g} \\ &= (a \otimes \varphi\rho(g^{-1}))(g(\tilde{a}) \otimes \rho(g)\tilde{\varphi}\rho(\tilde{g}^{-1})\rho(g^{-1})) \rtimes g\tilde{g} \\ &= (a \otimes \varphi\rho(g^{-1}) \rtimes g)(\tilde{a} \otimes \tilde{\varphi}\rho(\tilde{g}^{-1}) \rtimes \tilde{g}) \\ &= \phi(a \rtimes g \otimes \varphi)\phi(\tilde{a} \rtimes \tilde{g} \otimes \tilde{\varphi}) \end{aligned}$$

On the other hand, define

$$\psi : (A \otimes \text{End}_\ell^F(\mathcal{W})) \rtimes G \rightarrow (A \rtimes G) \otimes \text{End}_\ell^F(\mathcal{W}) \quad \psi(a \otimes \varphi \rtimes g) = a \rtimes g \otimes \varphi\rho(g).$$

It is also an algebra morphism:

$$\begin{aligned}
\psi(a \otimes \varphi \rtimes g)(\tilde{a} \otimes \tilde{\varphi} \rtimes \tilde{g}) &= \psi([a \otimes \varphi][g(\tilde{a}) \otimes \rho(g)\tilde{\varphi}\rho(g^{-1})] \rtimes g\tilde{g}) \\
&= \psi(ag(\tilde{a}) \otimes \varphi\rho(g)\tilde{\varphi}\rho(g^{-1}) \rtimes g\tilde{g}) \\
&= ag(\tilde{a}) \rtimes g\tilde{g} \otimes \varphi\rho(g)\tilde{\varphi}\rho(\tilde{g}) \\
&= (a \rtimes g \otimes \varphi\rho(g))(\tilde{a} \rtimes \tilde{g} \otimes \tilde{\varphi}\rho(\tilde{g})) \\
&= \psi(a \otimes \varphi \rtimes g)\psi(\tilde{a} \otimes \tilde{\varphi} \rtimes \tilde{g})
\end{aligned}$$

It is clear ϕ and ψ are inverse of each other. \square

PROPOSITION 2.1.2. There exists a unique functor $\rtimes G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathbf{G}\text{-Alg} & \xrightarrow{\rtimes G} & \mathbf{Alg} \\
j^G \downarrow & & \downarrow j \\
\mathfrak{K}\mathfrak{K}^G & \xrightarrow{\rtimes G} & \mathfrak{K}\mathfrak{K}
\end{array}$$

PROOF: We shall show $j(- \rtimes G)$ is excisive, homotopy invariant and G -stable. Because $\rtimes G$ maps split sequences to split sequences and j is excisive, then $j(- \rtimes G)$ is excisive. That $j(- \rtimes G)$ is homotopy invariant follows from the fact that

$$A[t] \rtimes G = (A \rtimes G)[t].$$

Let $(\mathcal{W}_1, B_1), (\mathcal{W}_2, B_2)$ be G -modules by almost finite automorphisms and A a G -algebra. Consider the isomorphism ψ defined in Proposition 2.1.1. Note that the following diagram is commutative

$$\begin{array}{ccc}
(A \otimes \text{End}_\ell^F(\mathcal{W}_1)) \rtimes G & \xrightarrow{(1 \otimes \tilde{i}) \rtimes G} & (A \otimes \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2)) \rtimes G \\
\psi \downarrow & & \downarrow \psi \\
(A \rtimes G) \otimes \text{End}_\ell^F(\mathcal{W}_1) & \xrightarrow{(1 \rtimes G) \otimes \tilde{i}} & (A \rtimes G) \otimes \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2)
\end{array}$$

Because j is M_∞ -stable, $j((1 \rtimes G) \otimes \tilde{i})$ is an isomorphism. Hence $j(- \rtimes G)(1 \otimes \tilde{i})$ is an isomorphism by the diagram above. \square

REMARK 2.1.3. Let $[\alpha] \in \text{kk}^G(A, B)$ be an element represented by $\alpha : J^n(M_G A) \rightarrow (M_G B)^{\text{sd}^p \mathcal{S}^n}$ which is a morphism in $[J^n(M_G A), \mathcal{M}_\infty(M_G B)^{\text{sd}^p \mathcal{S}^n}]$. Let us see who $[\alpha] \rtimes G$ is. Consider the classifying map

$$J^n(M_G A \rtimes G) \rightarrow J^n(M_G A) \rtimes G$$

The element $[\alpha] \rtimes G$ is represented by the following composition

$$J^n(M_G A \rtimes G) \rightarrow J^n(M_G A) \rtimes G \xrightarrow{\alpha \rtimes G} (M_G B)^{\text{sd}^p \mathcal{S}^n} \rtimes G.$$

1.3. Adjointness between τ and $\rtimes G$. In this section we shall see an algebraic version of the GREEN-JULG THEOREM. In [24] there is a version of this result in the context of Kasparov KK -theory. In [11] there is a version of this theorem in the context of E -theory.

THEOREM 2.1.4. Let G be a finite group of n elements and $1/n \in \ell$. The functors $\tau : \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{K}\mathfrak{K}^G$ and $\rtimes G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}$ are adjoint functors. Hence

$$kk^G(A^\tau, B) \simeq kk(A, B \rtimes G) \quad A \in \text{Alg} \quad B \in G\text{-Alg}.$$

PROOF: By [[22] Theorem 2, pag 81], it is enough to prove that there exist natural transformations

$$\overline{\alpha}_A \in kk(A, A^\tau \rtimes G) \quad \text{and} \quad \overline{\beta}_B \in kk^G((B \rtimes G)^\tau, B)$$

such that

$$A^\tau \xrightarrow{\tau(\overline{\alpha}_A)} (A^\tau \rtimes G)^\tau \xrightarrow{\overline{\beta}_{\tau(A)}} A^\tau \quad B \rtimes G \xrightarrow{\overline{\alpha}_{B \rtimes G}} (B \rtimes G)^\tau \rtimes G \xrightarrow{\overline{\beta}_{B \rtimes G}} B \rtimes G$$

are the identities in $kk^G(A^\tau, A^\tau)$ and $kk(B \rtimes G, B \rtimes G)$ respectively.

Put $\epsilon = 1/n \sum_{g \in G} g$ in ℓG and define

$$(34) \quad \alpha_A : A \rightarrow A^\tau \rtimes G \quad \alpha(a) = a \otimes \epsilon$$

Note $A^\tau \rtimes G = A \otimes \ell G$ and α_A is an algebra morphism since ϵ is idempotent. Consider the element $\overline{\alpha}_A \in kk(A, A^\tau \rtimes G)$ represented by α_A . Let

$$\beta_B : (B \rtimes G)^\tau \rightarrow M_G B \quad \beta_B(b \rtimes g) = \sum_{s \in G} s(b) e_{s,sg}$$

Let us check that β_B is an algebra morphism:

$$\begin{aligned} \beta_B((b \rtimes g)(a \rtimes h)) &= \beta_B(bg(a) \rtimes gh) \\ &= \sum_{s \in G} s(bg(a)) e_{s,sg}h \\ &= \sum_{s,t \in G} s(b)t(a) e_{s,sg} e_{t,th} \\ &= (\sum_{s \in G} s(b) e_{s,sg}) (\sum_{t \in G} t(a) e_{t,th}) \\ &= \beta_B(b \rtimes g) \beta_B(a \rtimes h) \end{aligned}$$

Let us check that β_B is equivariant:

$$\begin{aligned} \beta_B(h(b \rtimes g)) &= \beta_B(b \rtimes g) \\ &= \sum_{s \in G} s(b) e_{s,sg} \\ &= \sum_{s \in G} h s(b) e_{hs,hs}g \\ &= h(\sum_{s \in G} s(b) e_{s,sg}) \\ &= h(\beta_B(b \rtimes g)) \end{aligned}$$

Let $\bar{\beta}_B \in \text{kk}^G((B \rtimes G)^\tau, B)$ be represented by β_B . The composite $\beta_{\tau(A)}\tau(\alpha_A)$ is

$$(35) \quad A^\tau \xrightarrow{\tau(\alpha_A)} (A^\tau \rtimes G)^\tau \xrightarrow{\beta_{\tau(A)}} M_G A^\tau \quad a \mapsto \frac{1}{n} \sum_{s,t \in G} a e_{s,t}.$$

Thus $\beta_{\tau(A)}\tau(\alpha_A) = \text{id}_{A^\tau} \otimes \bar{\iota}$ where $\bar{\iota}$ is the map defined in 1.7.14. As j^G is G -stable, $j^G(\text{id}_{A^\tau} \otimes \bar{\iota})$ is the identity in $\text{kk}^G(A^\tau, A^\tau)$.

On the other hand take the morphism ψ defined in the proof of the Proposition 2.1.1. By 2.1.3, $\bar{\beta}_B \rtimes G$ is represented by $\psi \circ (\beta_B \rtimes G)$. We want to prove that the identity in $\text{kk}(B \rtimes G, B \rtimes G)$ is represented by the following composition

$$B \rtimes G \xrightarrow{\alpha_{B \rtimes G}} (B \rtimes G)^\tau \rtimes G \xrightarrow{\beta_B \rtimes G} (B \otimes M_G) \rtimes G \xrightarrow{\psi} (B \rtimes G) \otimes M_G$$

$$b \rtimes g \mapsto \frac{1}{n} \sum_{h,s \in G} (s(b) \rtimes h) e_{s,h^{-1}sg}$$

Put $t = h^{-1}sg$ and note

$$(36) \quad \frac{1}{n} \sum_{h,s \in G} (s(b) \rtimes h) e_{s,h^{-1}sg} = \frac{1}{n} \sum_{t,s \in G} (s(b) \rtimes sgt^{-1}) e_{s,t}$$

and

$$s(b) \rtimes sgt^{-1} = (\mathbf{1} \rtimes s)(b \rtimes g)(\mathbf{1} \rtimes t^{-1}) \quad \text{in } \tilde{B} \rtimes G$$

We can write (36) as $TA_{b \rtimes g}T^{-1}$, where

$$A_{b \rtimes g} = \frac{1}{n} \sum_{t,s \in G} (b \rtimes g) e_{s,t} \quad T = \sum_{t \in G} (\mathbf{1} \rtimes t) e_{t,t}$$

Because $b \rtimes g \mapsto A_{b \rtimes g}$ represents the identity, the same is true of $b \rtimes g \mapsto TA_{b \rtimes g}T^{-1}$. \square

EXAMPLE 2.1.5. We give an example to show that the adjointness between of τ and $\rtimes G$ of Theorem 2.1.4 fails to hold at the algebra level. Let $G = \mathbb{Z}_2 = \{1, \sigma\}$, $A = \ell$ and $B = (\ell G)^*$ the dual algebra of ℓG with the regular action. Note $\text{hom}_{G\text{-Alg}}(A^\tau, B)$ has two elements only:

$$\varphi_i : \ell \rightarrow (\ell G)^* \quad \varphi_0(1) = 0 \quad \varphi_1(1) = \chi_1 + \chi_\sigma$$

One the other hand $\text{hom}_{\text{Alg}}(A, B \rtimes G) = \text{hom}_{\text{Alg}}(\ell, (\ell G)^* \rtimes G)$ has at least as many elements as ℓ . For each $\lambda \in \ell$ we can define

$$\varphi_\lambda : \ell \rightarrow (\ell G)^* \rtimes G \quad \varphi_\lambda(1) = \chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma) \quad \lambda \in \ell$$

Note φ_λ is an algebra morphism because $\chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma)$ is an idempotent element:

$$(\chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma))^2 = \chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma) + \lambda(\chi_1 \chi_\sigma \rtimes \sigma) + \lambda^2(\chi_1 \chi_\sigma \rtimes 1) = \chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma)$$

Write ψ_{GJ} for the isomorphism of the Theorem 2.1.4

$$(37) \quad \psi_{GJ} : \text{kk}^G(B^\tau, A) \rightarrow \text{kk}(B, A \rtimes G) \quad \psi_{GJ} = \alpha^* \circ \rtimes G$$

where α is the morphism defined in (34).

COROLLARY 2.1.6. Let G be a finite group such that $1/|G| \in \ell$. Let A be a G -algebra, then

$$\text{kk}^G(\ell, A) \simeq \text{kk}(\ell, A \rtimes G) \simeq \text{KH}(A \rtimes G)$$

□

2. Induction and Restriction

Let H be a subgroup of G . In this section we study the functors of induction and restriction between the categories $\mathfrak{K}\mathfrak{K}^H$ and $\mathfrak{K}\mathfrak{K}^G$

2.1. Finitely supported polynomial maps. Let A be an ℓ -algebra and X a simplicial set. We write $\sigma \in X$ when σ is an n -simplex for some n , in other words when $\sigma \in X_n$. Denote by $\langle \sigma \rangle \subset X$ the subsimplicial set of X generated by σ :

$$\langle \sigma \rangle_i = \{ \tau \in X_i : \exists \alpha : [i] \rightarrow [n] \text{ such that } \alpha^*(\sigma) = \tau \}$$

Let $\sigma \in X$ be a simplex. The **star** of σ in X is the following set of simplices of X

$$\text{St}(\sigma) := \text{St}_X(\sigma) = \{ \tau \in X : \langle \tau \rangle \cap \langle \sigma \rangle \neq \emptyset \}.$$

The **closed star** of σ is the sub simplicial set $\overline{\text{St}}(\sigma)$ generated by $\text{St}(\sigma)$,

$$\overline{\text{St}}(\sigma) = \langle \text{St}(\sigma) \rangle.$$

Let M be the set of simplices of X . We define

$$\text{St}_X(M) := \bigcup_{\sigma \in M} \text{St}_X(\sigma) \quad \overline{\text{St}}_X(M) = \langle \text{St}_X(M) \rangle = \bigcup_{\sigma \in M} \overline{\text{St}}(\sigma).$$

We define the **link** of M as

$$\text{Link}(M) := \overline{\text{St}}_X(M) \setminus \text{St}_X(M).$$

LEMMA 2.2.1. Let X be a simplicial set; write NX for the set of nondegenerate simplices. The following are equivalent.

- i) $(\forall \sigma \in X) \{ \tau \in NX : \langle \tau \rangle \supset \langle \sigma \rangle \}$ is a finite set.
- ii) For every $\sigma \in X$, $\overline{\text{St}}_X(\sigma)$ is a finite simplicial set.

PROOF. If $\sigma \in X$, then $\langle \sigma \rangle$ has finitely many nondegenerate simplices, and thus the set $\{ \langle \tau \rangle \cap \langle \sigma \rangle : \tau \in X \}$ is finite. Hence if i) holds, there are finitely many $\tau \in NX$ such that $\langle \tau \rangle \cap \langle \sigma \rangle \neq \emptyset$; in other words, $NX \cap \text{St}_X(\sigma)$ is a finite set, and therefore $\overline{\text{St}}_X(\sigma)$ is a finite simplicial set. Thus i) \Rightarrow ii). Next note that $\langle \tau \rangle \supset \langle \sigma \rangle$ implies $\tau \in \text{St}_X(\sigma)$, whence ii) \Rightarrow i). □

DEFINITION 2.2.2. A simplicial set X is **locally finite** if for all $\sigma \in X$, $\overline{\text{St}}_X(\sigma)$ is a finite simplicial set.

DEFINITION 2.2.3. The **support** of an element $\phi \in A^X$ is generated by the simplices σ such that $\phi(\sigma) \neq 0$,

$$\text{supp}(\phi) := \langle \sigma \in X : \phi(\sigma) \neq 0 \rangle$$

Let $\phi, \psi \in A^X$ and let $f : Y \rightarrow X$ be a simplicial map then

$$(38) \quad \text{supp}(\phi \cdot \psi) \subset \text{supp}(\phi) \cap \text{supp}(\psi) \quad \text{supp}(f^*(\phi)) \subset f^{-1}(\text{supp}(\phi)).$$

We say ϕ has **finite support** if $\text{supp}(\phi) \subset X$ is a finite simplicial set. We define the **algebra of polynomial maps with finite support in X** as

$$A^{(X)} := \{ \phi \in A^X : \text{supp}(\phi) \text{ is finite} \}$$

If X is finite then $A^X = A^{(X)}$. In general, by (38) we have that $A^{(X)} \triangleleft A^X$ is an ideal. Note that if $f : X \rightarrow Y$ is a morphism of simplicial sets, the image of $A^{(Y)}$ by $f^* : A^Y \rightarrow A^X$ is not necessarily in $A^{(X)}$. By this reason $X \mapsto A^{(X)}$ can not be extended to a functor in \mathbb{S} . But if f is **proper** (i.e. $f^{-1}(K)$ is finite for all finite $K \subset Y$) then $f^*(A^{(Y)}) \subset A^{(X)}$ by (38). We conclude that $A^{(-)}$ is a functor from the category of simplicial sets with proper morphisms. Next we consider the behaviour of this functor with respect to colimits. First of all, if $\{X_i\}$ is a family of simplicial sets, then we have

$$(39) \quad A(\coprod X_i) = \bigoplus_i A^{(X_i)}$$

Here \bigoplus indicates the direct sum of abelian groups, equipped with coordinatewise multiplication. Second, $A^{(-)}$ maps coequalizers of proper maps to equalizers; if $\{f_j : X \rightarrow Y\}$ is a family of proper maps, then

$$(40) \quad A^{\text{coeq}_j\{f_j : X \rightarrow Y\}} = \text{eq}_j\{f_j^* : A^{(Y)} \rightarrow A^{(X)}\}$$

Next recall that if I is a small category and $X : I \rightarrow \mathbb{S}$ is a functor, then the colimit of X can be computed as a coequalizer:

$$\text{colim}_i X_i = \text{coeq}\left(\coprod_{\alpha \in \text{Ar}(I)} X_{s(\alpha)} \xrightleftharpoons[\partial_1]{\partial_0} \coprod_{i \in \text{Ob}(I)} X_i\right)$$

Here Ob and Ar are respectively the sets of objects and of arrows of I , and if $\alpha \in \text{Ar}(I)$ then $s(\alpha) \in \text{Ob}(I)$ is its source; we also write $r(\alpha)$ for the range of α . The maps ∂_0 and ∂_1 are defined as follows. The restriction of ∂_i to the copy of $X_{s(\alpha)}$ indexed by α is the inclusion $X_{s(\alpha)} \subset \coprod_j X_j$ if $i = 0$ and the composite of $X(\alpha)$ followed by the inclusion $X_{r(\alpha)} \subset \coprod_j X_j$ if $i = 1$. The conditions that ∂_0 and ∂_1 be proper are equivalent to the following

- ∂_0) Each object of I is the source of finitely many arrows.
- ∂_1) Each object of I is the range of finitely many arrows, and X sends each map of I to a proper map.

EXAMPLE 2.2.4. For example the functor $\sigma \mapsto \langle \sigma \rangle$ from the set of nondegenerate simplices of X , ordered by $\sigma \leq \tau$ if $\langle \sigma \rangle \subset \langle \tau \rangle$, always satisfies ∂_1 ; condition ∂_0 is precisely condition i) of Lemma 2.2.1. Hence ∂_0 is satisfied if and only if X is locally finite, and in that case we have

$$A^{(X)} = \text{eq}\left(\bigoplus_{\sigma \in NX} A^{\langle \sigma \rangle} \xrightleftharpoons[\partial_1^*]{\partial_0^*} \bigoplus_{\substack{\langle \tau \rangle \subset \langle \sigma \rangle, \\ \sigma, \tau \in NX}} A^{\langle \tau \rangle}\right)$$

2.2. Restriction. Let A be a G -algebra and $H \subset G$ a subgroup. If we restrict the action to H we obtain an H -algebra $\text{Res}_G^H(A)$. It is clear this construction defines a functor $\text{Res}_G^H : G\text{-Alg} \rightarrow H\text{-Alg}$. It is easily seen that we can extend $\text{Res}_G^H : G\text{-Alg} \rightarrow H\text{-Alg}$ to a

triangle functor $\text{Res}_G^H : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^H$ so that the following diagram commutes

$$\begin{array}{ccc} G\text{-Alg} & \xrightarrow{\text{Res}_G^H} & H\text{-Alg} \\ j^G \downarrow & & \downarrow j^H \\ \mathfrak{K}\mathfrak{K}^G & \xrightarrow{\text{Res}_G^H} & \mathfrak{K}\mathfrak{K}^H \end{array}$$

Note that for $H = \{1\}$ we get $j^{\{1\}} = j : \text{Alg} \rightarrow \mathfrak{K}\mathfrak{K}$ and $\text{Res}_G^{\{1\}} : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}$ is the induced by the forgetful functor $G\text{-Alg} \rightarrow \text{Alg}$.

2.3. Induction. We shall define an induction functor $\text{Ind}_H^G : H\text{-Alg} \rightarrow G\text{-Alg}$ and extend it to a functor $\text{Ind}_H^G : \mathfrak{K}\mathfrak{K}^H \rightarrow \mathfrak{K}\mathfrak{K}^G$ such the following diagram is commutative

$$\begin{array}{ccc} H\text{-Alg} & \xrightarrow{\text{Ind}_H^G} & G\text{-Alg} \\ j^H \downarrow & & \downarrow j^G \\ \mathfrak{K}\mathfrak{K}^H & \xrightarrow{\text{Ind}_H^G} & \mathfrak{K}\mathfrak{K}^G \end{array}$$

DEFINITION 2.2.5 ($\text{Ind}_H^G(A)$). Let $H \subset G$ be a subgroup, $\pi : G \rightarrow G/H$ the projection and A an H -algebra. Consider

$$A^{(G,H)} := \{f : G \rightarrow A : \#\pi(\text{supp}(f)) < \infty\}$$

Define

$$\text{Ind}_H^G(A) = \{f \in A^{(G,H)} : f(s) = h(f(sh)) \quad \forall s \in G, \quad h \in H\}.$$

One checks that $\text{Ind}_H^G(A)$ is a G -algebra with the pointwise multiplication and the following action

$$(41) \quad (g \cdot f)(s) = f(g^{-1}s) \quad f \in \text{Ind}_H^G(A) \quad g, s \in G.$$

DEFINITION 2.2.6 ($\text{Ind}_H^G(\varphi)$). Let $\varphi : A \rightarrow B$ be a morphism of H -algebras. Define

$$\text{Ind}_H^G(\varphi) : \text{Ind}_H^G(A) \rightarrow \text{Ind}_H^G(B) \quad \text{Ind}_H^G(\varphi)(f) = \varphi \circ f$$

As φ is equivariant $\varphi \circ f$ lies in $\text{Ind}_H^G(B)$.

Let A be a G -algebra and $H \subset G$ a subgroup. The following identities are easy to check:

$$\text{Ind}_G^G(A) \simeq A \quad \text{Ind}_{\{e\}}^G(A) = A^{(G)} \quad \text{Ind}_H^G(\ell) = \ell^{(G/H)}$$

If H is finite, then

$$A^{(G,H)} = A^{(G)}.$$

Let A be an H -algebra. Consider

$$\text{Big Ind}_H^G(A) := \{f : G \rightarrow A : f(s) = h(f(sh)), \quad \forall s \in G, \quad h \in H\}$$

Note $\text{Big Ind}_H^G(A)$ is a G -algebra with operations defined pointwise, and where G acts like in (41). If $f \in \text{Big Ind}_H^G(A)$ and $x = sH \in G/H$, then the value of f at any $g \in x$ determines f on the whole x ; in particular,

$$\text{supp}(f) \cap sH \neq \emptyset \Rightarrow sH \subset \text{supp}(f) \quad (sH \in G/H)$$

Hence

$$\text{supp}(f) = \coprod_{sH \cap \text{supp}(f) \neq \emptyset} sH$$

Consider the projection $\pi : G \rightarrow G/H$. We obtain

$$\text{Ind}_H^G(A) = \{f \in \text{Big Ind}_H^G(A) : \#\pi(\text{supp}(f)) < \infty\}$$

One checks that $\text{Ind}_H^G(A) \subset \text{Big Ind}_H^G(A)$ is a subalgebra; we shall presently introduce some of its typical elements. If $g \in G$, write $\chi_g : G \rightarrow \mathbb{Z}$ for the characteristic function. If $a \in A$ and $g \in G$, then

$$\xi_H(g, a) = \sum_{h \in H} \chi_{gh} h^{-1}(a) \quad \xi_H(g, a)(s) = \begin{cases} h^{-1}(a) & s = gh \\ 0 & s \notin gH \end{cases}$$

These elements are in $\text{Ind}_H^G(A)$ because

$$\text{supp}(\xi_H(g, a)) = gH \quad \xi_H(g, a)(s) = t \cdot \xi_H(g, a)(st) \quad \forall s \in G, t \in H$$

Let $r : G/H \rightarrow G$ be a pointed section and $\mathcal{R} = r(G/H)$. Every element $\phi \in \text{Big Ind}_H^G(A)$ can be written as a formal sum

$$(42) \quad \phi = \sum_{g \in \mathcal{R}} \xi_H(g, \phi(g))$$

Note that $\phi \in \text{Ind}_H^G(A)$ if and only if the sum above is finite. In particular

$$\text{Ind}_H^G(A) = \sum_{g \in G, a \in A} \ell \xi_H(g, a) \subset \text{Big Ind}_H^G(A)$$

It is easy to see that if $\varphi : A \rightarrow B$ is an H -equivariant map then

$$(43) \quad \text{Ind}_H^G(\varphi)(\xi_H(g, a)) = \xi_H(g, \varphi(a))$$

Observe that for each $s \in G$, the map

$$\xi_H(s, -) : A \rightarrow \text{Big Ind}_H^G(A)$$

is an algebra homomorphism. Moreover, we have the following relations

$$(44) \quad s \cdot \xi_H(g, a) = \xi_H(sg, a)$$

$$(45) \quad \xi_H(g, a) \xi_H(\tilde{g}, \tilde{a}) = \begin{cases} \xi_H(\tilde{g}, \tilde{g}^{-1}g(a)\tilde{a}) & \tilde{g}^{-1}g \in H \\ 0 & \tilde{g}^{-1}g \notin H \end{cases}$$

$$(46) \quad \xi_H(g, a) = \xi_H(gh, h^{-1} \cdot a) \quad h \in H$$

It follows that $(g, a) \mapsto \xi_H(g, a)$ gives a G -equivariant map

$$G \times_H A \rightarrow \text{Ind}_H^G(A).$$

Here $G \times_H A = G \times A / \sim$, where

$$(g_1, a_1) \sim (g_2, a_2) \iff h = g_1^{-1}g_2 \in H \quad \text{and} \quad a_1 = h \cdot a_2.$$

Extending by linearity we obtain an isomorphism of left G -modules

$$\ell[G] \otimes_{\ell[H]} A \rightarrow \text{Ind}_H^G(A)$$

Thus we may think of $\text{Ind}_H^G(A)$ as the G -module induced from the H -module A , equipped with an algebra structure compatible with that of A . In fact (45) implies that if $r : G/H \rightarrow G$ is a section, then

$$(47) \quad \ell^{(G/H)} \otimes A \rightarrow \text{Ind}_H^G(A) \quad \chi_x \otimes a \mapsto \xi_H(r(x), a)$$

is a *nonequivariant* algebra isomorphism.

LEMMA 2.2.7. Let X be an H -simplicial set; put

$$\text{Ind}_H^G(X) = G \times_H X$$

There is a natural, G -equivariant isomorphism $\ell^{(\text{Ind}_H^G(X))} \simeq \text{Ind}_H^G(\ell^{(X)})$.

PROOF: Let $\pi : G \times X \rightarrow \text{Ind}_H^G(X)$ be the projection. We have a G -algebra isomorphism

$$\Theta : \text{Big Ind}_H^G(\ell^X) \rightarrow \ell^{\text{Ind}_H^G(X)}, \quad \Theta(f)(\pi(g, x)) = f(g)(x)$$

For $s \in G$ and $\phi \in \ell^X$,

$$\Theta(\xi_H(s, \phi))(\pi(g, x)) = \begin{cases} \phi(s^{-1}g \cdot x) & \text{if } g \in sH \\ 0 & \text{if } g \notin sH \end{cases}$$

In particular, for $\Theta(\xi_H(s, \phi))$ not to vanish on $\pi(g, x)$, we must have $g = sh$ and $x \in h^{-1} \cdot \{\phi \neq 0\}$ for some $h \in H$. Hence $\text{supp}(\Theta(\xi_H(s, \phi))) \subset \pi(\{s\} \times \text{supp}(\phi))$ which is a finite simplicial set if $\phi \in \ell^{(X)}$. Therefore Θ maps $\text{Ind}_H^G(\ell^{(X)})$ inside $\ell^{(\text{Ind}_H^G(X))}$. It remains to show that $\Theta^{-1}(\ell^{(\text{Ind}_H^G(X))}) \subset \text{Ind}_H^G(\ell^{(X)})$. Let $\{g_i\} \subset G$ be a full set of representatives of G/H . Every element of $G \times_H X$ can be written uniquely as $\pi(g_i, x)$ for some i and some $x \in X$. Hence as a simplicial set, $\text{Ind}_H^G(X)$ is the disjoint union of the $Y_i = \pi(\{g_i\} \times X)$. In particular if $\phi \in \ell^{(\text{Ind}_H^G(X))}$, then its support meets finitely many of the Y_i , and $\text{supp}(\phi) \cap Y_i$ is a finite simplicial set. Thus there is a finite number of i such that $\psi = \Theta^{-1}(\phi)$ is nonzero on $g_i H$, and its restriction to each of these subsets takes values in $\ell^{(X)}$. By (42), this implies that $\psi \in \text{Ind}_H^G(\ell^{(X)})$, as we had to prove. \square

PROPOSITION 2.2.8. The functor $\text{Ind}_H^G : H\text{-Alg} \rightarrow G\text{-Alg}$ is exact.

PROOF: The exactness only depends on the group structure of the objects involved. It follows from the fact that (47) is an isomorphism. \square

PROPOSITION 2.2.9. Let A be a G -algebra and B be an H -algebra, then

$$\text{Ind}_H^G(B \otimes \text{Res}_G^H A) \simeq \text{Ind}_H^G(B) \otimes A$$

PROOF: The isomorphisms are given by

$$\begin{array}{lll} S : \text{Ind}_H^G(B) \otimes A & \rightarrow & \text{Ind}_H^G(B \otimes \text{Res}_G^H A) \\ \xi_H(g, b) \otimes a & \mapsto & \xi_H(g, b \otimes g^{-1} \cdot a) \end{array} \quad \begin{array}{lll} T : \text{Ind}_H^G(B \otimes \text{Res}_G^H A) & \rightarrow & \text{Ind}_H^G(B) \otimes A \\ \xi_H(g, b \otimes a) & \mapsto & \xi_H(g, b) \otimes g \cdot a \end{array}$$

It is easy to check that they are mutually inverse equivariant maps. \square

COROLLARY 2.2.10. Let A be a G -algebra. Then

$$\text{Ind}_H^G \text{Res}_G^H A \rightarrow \ell^{(G/H)} \otimes A \quad \xi_H(s, b) \mapsto \chi_{sH} \otimes s \cdot b$$

is an isomorphism of G -algebras.

□

Let $x \subset G$ and define

$$\text{Ind}_H^G(A)[x] := \{f \in \text{Ind}_H^G(A) : \text{supp}(f) \subset x\}.$$

Note that if $G = \bigsqcup_i x_i$ is a disjoint union with $x_i = g_i H$ then

$$(48) \quad \text{Ind}_H^G(A) = \bigoplus_i \text{Ind}_H^G(A)[x_i]$$

Suppose K is another subgroup of G . Let $x = K\theta H$ for some $\theta \in G$. Consider the subgroup of K

$$(49) \quad K_\theta = K \cap \theta H \theta^{-1}$$

Put $H_{\theta^{-1}} = \theta^{-1} K \theta \cap H$. Conjugation by θ^{-1} defines an isomorphism

$$(50) \quad c_{\theta^{-1}} : K_\theta \rightarrow H_{\theta^{-1}} \quad c_{\theta^{-1}}(k) = \theta^{-1} k \theta$$

Hence we may view an $H_{\theta^{-1}}$ -algebra A as a K_θ -algebra via $c_{\theta^{-1}}$. We denote it by $c_{\theta^{-1}}^*(A)$.

PROPOSITION 2.2.11. The map

$$\alpha : \text{Res}_G^K \text{Ind}_H^G(A)[K\theta H] \rightarrow \text{Ind}_{K_\theta}^K(c_{\theta^{-1}}^*(\text{Res}_H^{H_{\theta^{-1}}}(A))) \quad \alpha(f)(k) = f(k\theta)$$

is an isomorphism of K -algebras.

PROOF: We can check $\alpha(\xi_H(s, a)) = \xi_{K_\theta}(s\theta^{-1}, a)$. The map α is equivariant:

$$\alpha(k \cdot f)(x) = (k \cdot f)(x\theta) = f(k^{-1}x\theta) = \alpha(f)(k^{-1}x) = (k \cdot \alpha(f))(x).$$

It is an isomorphism because $\alpha(\xi_H(k\theta, a)) = \xi_{K_\theta}(k, a)$. □

PROPOSITION 2.2.12. Let T be a simplicial set regarded as an H -simplicial set with the trivial action, then

$$(51) \quad \text{Ind}_H^G(T) = \coprod_{H\theta K} \text{Ind}_{K_{\theta^{-1}}}^K(T)$$

PROOF: It is enough to prove that $G/H \cong \coprod_{H\theta K} K/K_{\theta^{-1}}$, as K -sets with the right regular action. Consider the decomposition

$$G = \coprod_{\theta \in \mathcal{R}} H\theta K.$$

Here θ runs among a full set of representatives \mathcal{R} of the double cosets $H \backslash G / K$. We have

$$G/H = \coprod_{H\theta K} H\theta K / H\theta K \cap H$$

Observe the function $K \rightarrow H\theta K$, $k \mapsto \theta k$ is equivariant and if $k, \tilde{k} \in K$ then

$$\theta k \equiv \theta \tilde{k}_{(H)} \iff \theta k \tilde{k}^{-1} \theta^{-1} \in H \iff k \tilde{k}^{-1} \in \theta^{-1} H \theta \iff k \tilde{k}^{-1} \in K_{\theta^{-1}} \iff k \equiv \tilde{k}_{(K_{\theta^{-1}})}$$

Then

$$\frac{H\theta K}{H\theta K \cap H} \cong \frac{K}{K_{\theta^{-1}}}$$

□

PROPOSITION 2.2.13. Let $\text{Ind} : \mathbf{H}\text{-Alg} \rightarrow \mathbf{G}\text{-Alg}$ be the functor such that $\text{Ind}(A) = \text{Ind}_H^G(M_H \otimes A)$. There exists a functor $\text{Ind}_H^G : \mathfrak{K}\mathfrak{K}^H \rightarrow \mathfrak{K}\mathfrak{K}^G$ such that the following diagram is commutative

$$(52) \quad \begin{array}{ccc} \mathbf{H}\text{-Alg} & \xrightarrow{\text{Ind}} & \mathbf{G}\text{-Alg} \\ j^H \downarrow & & \downarrow j^G \\ \mathfrak{K}\mathfrak{K}^H & \xrightarrow{\text{Ind}_H^G} & \mathfrak{K}\mathfrak{K}^G \end{array}$$

PROOF: The functor $j^G \circ \text{Ind}$ is excisive because Ind maps split extensions to split extensions and j^G is excisive. Let us see the functor $j^G \circ \text{Ind}_H^G$ is homotopy invariant and M_∞ -stable. Let $f, g : A \rightarrow B$ be elementary homotopic morphisms of \mathbf{H} -algebras. Then there exists $H : A \rightarrow B[t]$ such that $f = \text{ev}_0 \circ H$ $g = \text{ev}_1 \circ H$. Hence $\text{Ind}_H^G(f)$ and $\text{Ind}_H^G(g)$ are elementary homotopic morphisms of \mathbf{G} -algebras and the homotopy is given by

$$\text{Ind}_H^G H : \text{Ind}_H^G(A) \rightarrow \text{Ind}_H^G(B[t]) \simeq \text{Ind}_H^G(B)[t] \quad \text{by 2.2.9.}$$

As j^G is homotopy invariant then $j^G \circ \text{Ind}_H^G(f) = j^G \circ \text{Ind}_H^G(g)$. Let $\iota : A \rightarrow M_\infty A$, then $j^G \circ \text{Ind}_H^G(\iota)$ is an isomorphism because by Proposition 2.2.9 we have

$$\text{Ind}_H^G(M_\infty A) \simeq M_\infty \text{Ind}_H^G(A)$$

and j^G is M_∞ -stable. By Proposition 1.7.13 the functor $j^G \circ \text{Ind}$ is homotopy invariant and \mathbf{H} -stable. As $j^H : \mathbf{H}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^H$ is universal for this properties there exists Ind_H^G such that (52) commutes. \square

2.4. Adjointness between Ind_H^G and Res_G^H .

THEOREM 2.2.14. Let G be a group and H be a subgroup of G . The functors

$$\text{Ind}_H^G : \mathfrak{K}\mathfrak{K}^H \rightarrow \mathfrak{K}\mathfrak{K}^G \quad \text{Res}_G^H : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^H$$

are adjoint. Hence

$$\text{kk}^G(\text{Ind}_H^G(B), A) \simeq \text{kk}^H(B, \text{Res}_G^H(A)) \quad \forall B \in \mathbf{H}\text{-Alg} \quad A \in \mathbf{G}\text{-Alg}$$

PROOF: Let $A \in \mathbf{G}\text{-Alg}$ and $B \in \mathbf{H}\text{-Alg}$. We need to have natural transformations

$$\alpha_A \in \text{kk}^G(\text{Ind}_H^G \text{Res}_G^H A, A) \quad \beta_B \in \text{kk}^H(B, \text{Res}_G^H \text{Ind}_H^G B)$$

which verify the unit and counit condition.

Define $\varphi_A : \text{Ind}_H^G(\text{Res}_G^H(A)) \rightarrow M_{G/H} \otimes A$ such that

$$\varphi_A(f) = \sum_{g \in G/H} e_{gH, gH} \otimes g(f(g)) \quad \varphi_A(\xi_H(s, b)) = e_{sH, sH} \otimes s \cdot b$$

Put

$$(53) \quad \psi_B : B \rightarrow \text{Res}_G^H \text{Ind}_H^G(B) \quad \psi_B(b) = \xi_H(e, b)$$

It is easy to check that ψ_B is well-defined and is a map of \mathbf{H} -algebras. Let $[\varphi_A] \in \text{kk}^G(\text{Ind}_H^G \text{Res}_G^H A, A)$ the element represented by φ_A and $\psi_B \in \text{kk}^H(B, \text{Res}_G^H \text{Ind}_H^G B)$ the element represented by ψ_B . The composite

$$\delta : \text{Res}_G^H A \xrightarrow{\beta_{\text{Res}_G^H A}} \text{Res}_G^H \text{Ind}_H^G \text{Res}_G^H A \xrightarrow{\text{Res}_G^H(\alpha_A)} \text{Res}_G^H A$$

is represented by

$$\mathrm{Res}_G^H A \xrightarrow{\psi_{\mathrm{Res}_G^H A}} \mathrm{Res}_G^H \mathrm{Ind}_H^G \mathrm{Res}_G^H A \xrightarrow{\mathrm{Res}_G^H(\varphi_A)} \mathrm{Res}_G^H(M_{G/H} \otimes A) = M_{|G/H|} \otimes \mathrm{Res}_G^H A$$

$$a \longmapsto \xi_H(e, a) \longmapsto \varphi_A(\xi_H(e, a)) = e_{H,H} \otimes a$$

which is kk^H -equivalent to the identity in the sense of remark 1.6.2. The morphism $\chi \in \mathrm{kk}^G(\mathrm{Ind}_H^G B, \mathrm{Ind}_H^G B)$

$$\chi : \mathrm{Ind}_H^G B \xrightarrow{\mathrm{Ind}_H^G(\beta_B)} \mathrm{Ind}_H^G \mathrm{Res}_G^H \mathrm{Ind}_H^G B \xrightarrow{\alpha_{\mathrm{Ind}_H^G B}} \mathrm{Ind}_H^G B$$

is represented by

$$\gamma : \mathrm{Ind}_H^G B \xrightarrow{\mathrm{Ind}_H^G(\psi_B)} \mathrm{Ind}_H^G \mathrm{Res}_G^H \mathrm{Ind}_H^G B \xrightarrow{\varphi_{\mathrm{Ind}_H^G B}} M_{G/H} \otimes \mathrm{Ind}_H^G B$$

$$\xi_H(g, b) \longmapsto \xi_H(g, \xi_H(e, b)) \longmapsto e_{gH, gH} \otimes \xi_H(g, b)$$

The following morphism of H-algebras

$$\theta : C \rightarrow M_{G/H} \otimes C \quad \theta(c) = e_{H,H} \otimes c$$

represents to the identity in the sense of remark 1.6.2. Then $\mathrm{Ind}_H^G(\theta)$ is kk^G -equivalent to the identity. It is easy to check $\mathrm{Ind}_H^G(\theta) = \gamma$ with $C = B$. \square

Write ψ_{IR} for the isomorphism

$$(54) \quad \psi_{IR} : \mathrm{kk}^G(\mathrm{Ind}_H^G B, A) \rightarrow \mathrm{kk}^H(B, \mathrm{Res}_G^H A) \quad \psi_{IR} = \psi_B^* \circ \mathrm{Res}_G^H$$

where ψ_B is the morphism defined (53).

COROLLARY 2.2.15. Let G be a group, H be a finite subgroup of G and A be a G -algebra then

$$\mathrm{kk}^G(\ell^{(G/H)}, A) \simeq \mathrm{kk}(\ell, A \rtimes H) \simeq \mathrm{KH}(A \rtimes H)$$

PROOF: The isomorphism is the composition of ψ_{GJ} and ψ_{IR} defined in (37) and in (54). \square

3. A discrete variant of Green's imprimitivity

Let G be a group, H a subgroup of G and A an H -algebra. We consider the following left action of $\mathrm{Ind}_H^G(A) \rtimes G$ in $A^{(G)}$

$$(55) \quad ((f \rtimes g) \cdot \varphi)(t) = f(t)\varphi(g^{-1}t) \quad f \in \mathrm{Ind}_H^G(A) \quad \varphi \in A^{(G)} \quad g, t \in G$$

Observe that $\mathrm{Ind}_H^G(A)$ acts on $A^{(G)}$ by multiplication because $\mathrm{Ind}_H^G(A)$ is a subalgebra of A^G and $A^{(G)}$ is an ideal of A^G . The action of ℓG over $A^{(G)}$ is $(g \cdot \varphi)(t) = \varphi(g^{-1}t)$. Taking

the morphism obtained by the universal property of the crossed product we obtain the following commutative diagram:

$$(56) \quad \begin{array}{ccc} \text{Ind}_H^G(A) & \hookrightarrow & A^G \\ \downarrow & & \downarrow \eta \\ \Upsilon : \text{Ind}_H^G(A) \rtimes G & \hookrightarrow & A^G \rtimes G \dashrightarrow \text{End}_\ell(A^{(G)}) \\ & \uparrow & \uparrow \mu \\ & \ell G & \end{array} \quad \begin{array}{l} \eta(f)(\varphi)(t) = f(t)\varphi(t) \\ \mu(g)(\varphi)(t) = \varphi(g^{-1}t) \end{array}$$

Here $\Upsilon : \text{Ind}_H^G(A) \rtimes G \rightarrow \text{End}_\ell(A^{(G)})$ is the map asociated to the action defined in (55).

We also define on $A^{(G)}$ a right action of $A \rtimes H$ as follows

$$(57) \quad (\varphi \cdot (a \rtimes h))(t) = h^{-1} \cdot (\varphi(th^{-1})a) \quad \varphi \in A^{(G)} \quad a \in A \quad h \in H \quad t \in G$$

It is easy to prove (55) and (57) are in fact a left action and a right action respectively. One checks that these two actions satisfy

$$(f \rtimes g) \cdot [\varphi \cdot (a \rtimes h)] = [(f \rtimes g) \cdot \varphi] \cdot (a \rtimes h)$$

Hence they make $A^{(G)}$ into an $(\text{Ind}_H^G(A) \rtimes G, A \rtimes H)$ -bimodule. In particular $\text{Im } \Upsilon \subseteq \text{End}_{A \rtimes H}(A^{(G)})$. The decomposition

$$G = \coprod_{x \in G/H} x$$

induces

$$(58) \quad A^{(G)} = \bigoplus_{x \in G/H} A^{(x)}$$

and $A^{(x)} \cdot (A \rtimes H) \subseteq A^{(x)}$. Hence (58) is a direct sum of $A \rtimes H$ -right modules.

Let $T_{x,y} : A^{(y)} \rightarrow A^{(x)}$ be a morphism of $A \rtimes H$ -modules such that for each $v \in A^{(y)}$, $T_{x,y}(v) = 0$ for all but a finite number of x . Note the matrix $T = \{T_{x,y}\}_{x,y \in G/H}$ represents an element of $\text{End}_{A \rtimes H}(A^{(G)})$. Moreover, every element $T \in \text{End}_{A \rtimes H}(A^{(G)})$ can be represented in that way. If $x = gH$,

$$A \rtimes H \rightarrow A^{(gH)} \quad a \rtimes h \mapsto \chi_g \cdot (a \rtimes h) = \chi_{gh} h^{-1}(a)$$

is an isomorphism of right $A \rtimes H$ -modules. Fix a full set of representatives \mathcal{R} of G/H . Write $M_{\mathcal{R}}$ for the algebra of $\mathcal{R} \times \mathcal{R}$ -matrices with finitely many nonzero coefficients in ℓ and put

$$M_{\mathcal{R}}(A \rtimes H) = M_{\mathcal{R}} \otimes (A \rtimes H)$$

We have an homomorphism

$$(59) \quad \begin{array}{ccc} M_{\mathcal{R}}(A \rtimes H) & \longrightarrow & \text{End}_{A \rtimes H}(A^{(G)}) \\ M = \{m_{x,y}\}_{x,y \in \mathcal{R}} & \longmapsto & \sum_{x \in X} \chi_x \cdot \alpha_x \mapsto \sum_{y \in \mathcal{R}} \chi_y \cdot \sum_{x \in \mathcal{R}} m_{x,y} \alpha_x \end{array}$$

Furthermore, we have a map $G \rightarrow \mathcal{R}$ which sends each $s \in G$ to the representative $\hat{s} \in \mathcal{R}$ of sH . Observe $e_{sH,tH} \mapsto e_{\hat{s},\hat{t}}$ is an isomorphism between $M_{|G/H|}$ and $M_{\mathcal{R}}$. By composition, we obtain an algebra homomorphism

$$(60) \quad \begin{aligned} \Gamma : M_{|G/H|}(A \rtimes H) &\rightarrow \text{End}_{A \rtimes H}(A^{(G)}) \\ e_{sH,tH}(a \rtimes h) &\mapsto \sum_{y \in \mathcal{R}} \chi_y \cdot \alpha_y \mapsto \chi_{\hat{s}} \cdot (a \rtimes h) \alpha_{\hat{t}} \end{aligned}$$

THEOREM 2.3.1. Let G be a group, $H \subset G$ a subgroup, and A an H -algebra. Then there is an isomorphism

$$\alpha : \text{Ind}_H^G(A) \rtimes G \rightarrow M_{|G/H|}(A \rtimes H)$$

such that the following diagrams commute

$$\begin{array}{ccc} \text{Ind}_H^G(A) \rtimes G & \xrightarrow{\Upsilon} & \text{End}_{A \rtimes H}(A^{(G)}) \\ & \searrow \alpha & \nearrow \Gamma \\ & M_{|G/H|}(A \rtimes H) & \end{array}$$

$$\begin{array}{ccc} \text{Ind}_H^G(A) \rtimes G & \xrightarrow{\alpha} & M_{|G/H|}(A \rtimes H) \\ & \nwarrow \xi_H(e, -) \rtimes \text{id} & \nearrow e_{H,H} \otimes - \\ & A \rtimes H & \end{array}$$

The morphism Υ is defined in (56) and the morphism Γ in (60).

PROOF: For simplicity suppose $1_G \in \mathcal{R}$. For each $s \in G$ denote $\phi(s) = \hat{s}^{-1}s \in H$. Observe $\phi(sh) = \phi(s)h$ for all $s \in G$, $h \in H$. Define

$$(61) \quad \begin{aligned} \alpha : \text{Ind}_H^G(A) \rtimes G &\rightarrow M_{|G/H|}(A \rtimes H) \\ \xi_H(s, a) \rtimes g &\mapsto e_{sH, g^{-1}sH} \otimes \phi(s) \cdot a \rtimes \phi(s)\phi(g^{-1}s)^{-1} \end{aligned}$$

The map (61) is well-defined:

$$\begin{aligned} \alpha(\xi_H(sh, h^{-1} \cdot a) \rtimes g) &= e_{shH, g^{-1}shH} \otimes \phi(sh) \cdot h^{-1} \cdot a \rtimes \phi(sh)\phi(g^{-1}sh)^{-1} \\ &= e_{sH, g^{-1}sH} \otimes \phi(s) \cdot a \rtimes \phi(s)\phi(g^{-1}s)^{-1} \\ &= \alpha(\xi_H(s, a) \rtimes g) \end{aligned}$$

Note that if $g, t, s \in G$ are elements such that $t^{-1}g^{-1}s \in H$ then $\phi(gt) = \phi(s)\phi(g^{-1}s)^{-1}\phi(t)$. The map (61) is an algebra homomorphism:

$$\begin{aligned}
\alpha((\xi_H(s, a) \rtimes g)(\xi_H(t, b) \rtimes r)) &= \alpha(\xi_H(s, a)\xi_H(gt, b) \rtimes gr) \\
&= \begin{cases} \alpha(\xi_H(gt, [(t^{-1}g^{-1}s) \cdot a]b) \rtimes gr) & t^{-1}g^{-1}s \in H \\ 0 & t^{-1}g^{-1}s \notin H \end{cases} \\
&= \begin{cases} e_{gtH, r^{-1}g^{-1}gtH} \otimes \phi(gt) \cdot ([(t^{-1}g^{-1}s) \cdot a]b) \rtimes \phi(gt)\phi(r^{-1}t)^{-1} & t^{-1}g^{-1}s \in H \\ 0 & t^{-1}g^{-1}s \notin H \end{cases} \\
&= \begin{cases} e_{sH, r^{-1}tH} \otimes [\phi(s) \cdot a][\phi(gt) \cdot b] \rtimes \phi(gt)\phi(r^{-1}t)^{-1} & t^{-1}g^{-1}s \in H \\ 0 & t^{-1}g^{-1}s \notin H \end{cases} \\
&= \alpha(\xi_H(s, a) \rtimes g)\alpha(\xi_H(t, b) \rtimes r)
\end{aligned}$$

The map (61) is bijective and its inverse is the following

$$\beta : M_{|G/H|}(A \rtimes H) \rightarrow \text{Ind}_H^G(A) \rtimes G \quad e_{sH, gH}(a \rtimes h) \mapsto \xi_H(\hat{s}, a) \rtimes \hat{s}h\hat{g}^{-1}$$

Let us check that the diagrams of the theorem are commutative

$$\begin{aligned}
\Upsilon(\xi_H(s, a) \rtimes g)(\sum_{y \in \mathcal{R}} \chi_y \cdot \alpha_y) &= \chi_s \cdot [(a \rtimes \phi(g^{-1}s)^{-1})\alpha_{\widehat{g^{-1}s}}] \\
&= \chi_{\hat{s}} \cdot [(\phi(s) \cdot a \rtimes \phi(s)\phi(g^{-1}s)^{-1})\alpha_{\widehat{g^{-1}s}}] \\
&= (\Gamma \circ \alpha)(\xi_H(s, a) \rtimes g)(\sum_{y \in \mathcal{R}} \chi_y \cdot \alpha_y)
\end{aligned}$$

$$\begin{aligned}
(\alpha \circ \xi_H(1_G, -) \rtimes \text{id})(a \rtimes h) &= \alpha(\xi_H(1_G, a) \rtimes h) \\
&= e_{H, H} \otimes (a \rtimes h)
\end{aligned}$$

□

REMARK 2.3.2. The isomorphism α of the Theorem 2.3.1 is natural in A but not in the pair (G, H) , as it depends on a choice of a full set of representatives \mathcal{R} of G/H .

COROLLARY 2.3.3. Let G be a countable group, $H \subset G$ a subgroup, and A an H -algebra. Let $E : \text{Alg} \rightarrow \mathcal{C}$ be a M_∞ -stable functor. The following map is an isomorphism

$$E(\xi_H(1_G, -) \rtimes \text{id}) : E(A \rtimes H) \rightarrow E(\text{Ind}_H^G(A) \rtimes G).$$

□

4. Duality

Let $\hat{G}\text{-Alg}$ be the category of \hat{G} -algebras with homogeneous homomorphisms. In this section we define functors between the categories $G\text{-Alg}$ and $\hat{G}\text{-Alg}$. We prove that they extend to equivalences between $\mathfrak{K}\mathfrak{K}^G$ and $\mathfrak{K}\mathfrak{K}^{\hat{G}}$. In this way we obtain an algebraic duality theorem similar to the duality given by Baaĵ-Skandalis in [1].

Let A be a G -algebra. Then

$$A \rtimes G = \bigoplus_{s \in G} A \rtimes s \quad \text{and} \quad (A \rtimes s)(A \rtimes t) \subset A \rtimes st$$

thus $A \rtimes G$ is a \hat{G} -algebra. If $f : A \rightarrow B$ is a homomorphism of \hat{G} -algebras then $f \rtimes G : A \rtimes G \rightarrow B \rtimes G$ is a graded homomorphism. Hence we have a functor

$$\rtimes G : G\text{-Alg} \rightarrow \hat{G}\text{-Alg}$$

We can also define a functor

$$\rtimes \hat{G} : \hat{G}\text{-Alg} \rightarrow G\text{-Alg}$$

as follows. Let B be a \hat{G} -algebra. Let $\hat{G} \rtimes B$ be the algebra which as a module is $\ell^{(G)} \otimes B$ and the product is the following

$$(62) \quad (\chi_g \rtimes a)(\chi_h \rtimes b) := \chi_g \rtimes a_{g^{-1}h}b.$$

Recall b_g is the homogeneous element associated to g in the decomposition

$$b = \sum_{g \in G} b_g.$$

One check that the product (62) is associative and the following action of G makes it into a G -algebra

$$s \cdot (\chi_g \rtimes a) = \chi_{sg} \rtimes a$$

If $f : A \rightarrow B$ is a homogeneous homomorphism define

$$\hat{G} \rtimes f : \hat{G} \rtimes A \rightarrow \hat{G} \rtimes B \quad (\hat{G} \rtimes f)(\chi_g \rtimes a) := \chi_g \rtimes f(a)$$

It is a G -algebra homomorphism. Thus we have a functor

$$(63) \quad \rtimes \hat{G} : \hat{G}\text{-Alg} \rightarrow G\text{-Alg}$$

PROPOSITION 2.4.1. Let A be a G -algebra and let B be a \hat{G} -algebra.

a) There are natural isomorphisms of G -algebras

$$\hat{G} \rtimes (A \rtimes G) \simeq M_G \otimes A$$

b) There are natural isomorphisms of \hat{G} -algebras

$$(\hat{G} \rtimes B) \rtimes G \simeq M_{\hat{G}} \otimes B$$

PROOF:

a) Define $T : \hat{G} \rtimes (A \rtimes G) \rightarrow M_G \otimes A$ as

$$T(\chi_g \rtimes a \rtimes s) = g \cdot a \otimes e_{g,gs}.$$

It is easy to check T is an equivariant algebra isomorphism with inverse given by

$$S(a \otimes e_{r,t}) := \chi_r \rtimes r^{-1} \cdot a \rtimes r^{-1}t$$

b) Define $T : (\hat{G} \rtimes B) \rtimes G \rightarrow M_{\hat{G}} \otimes B$ as

$$T(\chi_h \rtimes b \rtimes s) = \sum_{r \in G} e_{h, s^{-1}hr} \otimes b_r.$$

It is easy to check T is a graded algebra isomorphism with inverse given by

$$(64) \quad S(e_{r,s} \otimes b_q) = \chi_r \rtimes b_q \rtimes rqs^{-1}$$

□

THEOREM 2.4.2. The functors $\rtimes G$ and $\hat{G} \rtimes$ extend to inverse equivalences

$$- \rtimes G : \mathfrak{K}\mathfrak{K}^G \longrightarrow \mathfrak{K}\mathfrak{K}^{\hat{G}} \quad \hat{G} \rtimes - : \mathfrak{K}\mathfrak{K}^{\hat{G}} \longrightarrow \mathfrak{K}\mathfrak{K}^G$$

Hence if A and B are G -algebras and C and D are \hat{G} -algebras then

$$kk^G(A, B) \simeq kk^{\hat{G}}(A \rtimes G, B \rtimes G) \quad kk^{\hat{G}}(C, D) \simeq kk^G(\hat{G} \rtimes C, \hat{G} \rtimes D)$$

PROOF: As $\rtimes G$ maps split sequences to split sequences, $j^{\hat{G}}(- \rtimes G)$ is excisive. By Proposition 2.1.2 $j^{\hat{G}}(- \rtimes G)$ is G -stable and homotopy invariant, whence it extends to $- \rtimes G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{G}}$ by universality. Similarly, as $\rtimes_{\hat{G}}$ maps split sequences to split sequences then $j^G(\hat{G} \rtimes -)$ is excisive. Because $\hat{G} \rtimes$ maps graded homotopies to equivariant homotopies and $j^G(\hat{G} \rtimes -)$ is M_{∞} -stable, $j^G(\hat{G} \rtimes -)$ extends to $\hat{G} \rtimes - : \mathfrak{K}\mathfrak{K}^{\hat{G}} \rightarrow \mathfrak{K}\mathfrak{K}^G$ by universality. To finish we must show that the maps

$$kk^G(A, B) \rightarrow kk^{\hat{G}}(A \rtimes G, B \rtimes G) \quad \text{and} \quad kk^{\hat{G}}(C, D) \rightarrow kk^G(\hat{G} \rtimes C, \hat{G} \rtimes D)$$

are isomorphisms. This is true by Proposition 2.4.1. □

5. Green-Julg theorem for $kk^{\mathcal{H}}$

In this section we consider a finite dimensional semisimple Hopf algebra \mathcal{H} and prove a version of the Green-Julg theorem for $kk^{\mathcal{H}}$.

5.1. Smash product. We shall recall the smash product for \mathcal{H} -algebras. It is a generalization for \mathcal{H} -algebras of the crossed product $A \rtimes G$.

DEFINITION 2.5.1. Let A be an \mathcal{H} -algebra. The **smash product algebra** $A \# \mathcal{H}$ is the ℓ -module $A \otimes \mathcal{H}$ with the following product

$$(a \# h)(b \# k) = \sum a(h_1 \cdot b) \# h_2 k \quad a, b \in A \quad h, k \in \mathcal{H}$$

If $f : A \rightarrow B$ is a morphism of \mathcal{H} -algebras, we put

$$f \# \mathcal{H} : A \# \mathcal{H} \rightarrow B \# \mathcal{H} \quad f \# \mathcal{H}(a \# h) = f(a) \# h$$

which is a morphism of algebras. Hence, we have a functor $\# \mathcal{H} : \mathcal{H}\text{-Alg} \rightarrow \text{Alg}$.

PROPOSITION 2.5.2. Let M be an \mathcal{H} -module and A be an \mathcal{H} -algebra. The following is an isomorphism of algebras

$$(65) \quad \phi : \text{End}_{\ell}(M) \otimes (A \# \mathcal{H}) \rightarrow (\text{End}_{\ell}(M) \otimes A) \# \mathcal{H} \quad \phi(\varphi \otimes a \# h) = \sum \varphi_{1, h_1} \otimes a \# h_2$$

Moreover, the following restriction of ϕ is also an isomorphism

$$\phi : \text{End}_{\ell}^F(M) \otimes (A \# \mathcal{H}) \rightarrow (\text{End}_{\ell}^F(M) \otimes A) \# \mathcal{H}$$

□

PROOF: Let us check (65) is an algebra morphism:

$$\begin{aligned}
\phi((\varphi \otimes a \# h)(\tilde{\varphi} \otimes \tilde{a} \# \tilde{h})) &= \sum \phi(\varphi \tilde{\varphi} \otimes a(h_1 \cdot \tilde{a}) \# h_2 \tilde{h}) \\
&= \sum (\varphi \tilde{\varphi})_{1, h_2 \tilde{h}_1} \otimes a(h_1 \cdot \tilde{a}) \# h_3 \tilde{h}_2 \\
&= \sum (\varphi(\tilde{\varphi}(S(h_3 \tilde{h}_1) \cdot \quad)) \otimes a(\epsilon(h_1) h_2 \cdot \tilde{a}) \# h_4 \tilde{h}_2 \\
&= \sum \varphi(S(h_1) h_2 \tilde{\varphi}(S(h_4 \tilde{h}_1) \cdot \quad)) \otimes a(h_3 \cdot \tilde{a}) \# h_5 \tilde{h}_2 \\
&= \sum \varphi_{1, h_1} \tilde{\varphi}_{h_2, h_4 \tilde{h}_1} \otimes a(h_3 \cdot \tilde{a}) \# h_5 \tilde{h}_2 \\
&= \sum (\varphi_{1, h_1} \otimes a)(\tilde{\varphi}_{h_2, h_4 \tilde{h}_1} \otimes h_3 \cdot \tilde{a}) \# h_5 \tilde{h}_2 \\
&= \sum (\varphi_{1, h_1} \otimes a)((\tilde{\varphi}_{1, \tilde{h}_1})_{h_2, h_4} \otimes h_3 \cdot \tilde{a}) \# h_5 \tilde{h}_2 \\
&= \sum (\varphi_{1, h_1} \otimes a) h_2 \cdot (\varphi_{1, \tilde{h}_1} \otimes \tilde{a}) \# h_3 \tilde{h}_2 \\
&= \phi(\varphi \otimes a \# h) \phi(\tilde{\varphi} \otimes \tilde{a} \# \tilde{h})
\end{aligned}$$

It is easy to check that the following map is also an algebra morphism and is the inverse of (65).

$$\psi : (\text{End}_{\ell}(M) \otimes A) \# \mathcal{H} \rightarrow \text{End}_{\ell}(M) \otimes (A \# \mathcal{H}) \quad \psi(\varphi \otimes a \# h) = \sum \varphi_{1, \overline{S}(h_1)} \otimes a \# h_2$$

By remark 1.7.22, we can restrict the homomorphisms defined above and obtain the following isomorphisms

$$\psi : (\text{End}_{\ell}^F(M) \otimes A) \# \mathcal{H} \rightarrow \text{End}_{\ell}^F(M) \otimes (A \# \mathcal{H}) \quad \phi : \text{End}_{\ell}^F(M) \otimes (A \# \mathcal{H}) \rightarrow (\text{End}_{\ell}^F(M) \otimes A) \# \mathcal{H}. \quad \square$$

PROPOSITION 2.5.3. Let \mathcal{H} a Hopf algebra with finite dimension. There exists a unique functor $\# \mathcal{H} : \mathfrak{K} \mathfrak{K}^{\mathcal{H}} \rightarrow \mathfrak{K} \mathfrak{K}$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{H}\text{-Alg} & \xrightarrow{\# \mathcal{H}} & \text{Alg} \\
j^{\mathcal{H}} \downarrow & & \downarrow j \\
\mathfrak{K} \mathfrak{K}^{\mathcal{H}} & \xrightarrow{\# \mathcal{H}} & \mathfrak{K} \mathfrak{K}
\end{array}$$

PROOF: By 1.8.3 it is enough to prove $j(-\# \mathcal{H})$ is excisive, homotopy invariant and \mathcal{H} -stable. The two first properties are straightforward. Let M_1, M_2 be \mathcal{H} modules with countable dimension and A be an \mathcal{H} -algebra. Consider the isomorphism ψ defined in the

Proposition 2.5.2. Note the following diagram is commutative

$$\begin{array}{ccc} (\mathrm{End}_\ell(M_1) \otimes A) \# \mathcal{H} & \xrightarrow{(\tilde{\iota} \otimes 1) \# \mathcal{H}} & (\mathrm{End}_\ell(M_1 \oplus M_2) \otimes A) \# \mathcal{H} \\ \psi \downarrow & & \downarrow \psi \\ \mathrm{End}_\ell(M_1) \otimes (A \# \mathcal{H}) & \xrightarrow{\tilde{\iota} \otimes 1 \# \mathcal{H}} & \mathrm{End}_\ell(M_1 \oplus M_2) \otimes (A \# \mathcal{H}) \end{array}$$

As j is M_∞ -stable we know $j(\tilde{\iota} \otimes 1 \# \mathcal{H})$ is an isomorphism. Hence $j(-\# \mathcal{H})(\tilde{\iota} \otimes 1)$ is an isomorphism and $j(-\# \mathcal{H})$ is \mathcal{H} -stable. \square

Let A be an \mathcal{H} -algebra. Consider $\mathcal{H} \otimes A$ as a left \mathcal{H} -module with the diagonal action. Also consider $\mathcal{H} \otimes A$ as a right A -module with the regular action. In other words

$$h \cdot (k \otimes a) = \sum h_1 k \otimes h_2 \cdot a \quad (k \otimes a) \cdot c = k \otimes ac$$

It is easy to check that

$$(66) \quad t \cdot ((h \otimes a) \cdot c) = \sum (t_1 \cdot (h \otimes a)) \cdot (t_2 \cdot c)$$

We define

$$\mathrm{End}_A(\mathcal{H} \otimes A) := \{\varphi \in \mathrm{End}_\ell(\mathcal{H} \otimes A) \text{ such that } \varphi(k \otimes ac) = \varphi(k \otimes a) \cdot c\}$$

The structure of \mathcal{H} -module in $\mathcal{H} \otimes A$ gives an \mathcal{H} -algebra structure in $\mathrm{End}_\ell(\mathcal{H} \otimes A)$, see Example 1.1.3. It is easy to check that $\mathrm{End}_A(\mathcal{H} \otimes A)$ is a sub- \mathcal{H} -algebra of $\mathrm{End}_\ell(\mathcal{H} \otimes A)$. Consider $\mathrm{End}_\ell(\mathcal{H}) \otimes A$ as the \mathcal{H} -algebra defined in Example 1.1.4. We have the following homomorphism of \mathcal{H} -algebras

$$(67) \quad T : \mathrm{End}_\ell(\mathcal{H}) \otimes A \rightarrow \mathrm{End}_A(\mathcal{H} \otimes A) \quad T(\varphi \otimes a)(h \otimes b) = \varphi(h) \otimes ab$$

If \mathcal{H} is finite dimensional, then (67) is an isomorphism.

THEOREM 2.5.4. Let \mathcal{H} be a semisimple Hopf ℓ -algebra. The functor given by the trivial action $\tau : \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{H}}$ is left adjoint to the functor given by the smash product $\# \mathcal{H} : \mathfrak{K}\mathfrak{K}^{\mathcal{H}} \rightarrow \mathfrak{K}\mathfrak{K}$. In particular there is a natural isomorphism

$$\mathbf{kk}^{\mathcal{H}}(A^\tau, B) \simeq \mathbf{kk}(A, B \# \mathcal{H}) \quad A \in \mathbf{Alg} \quad B \in \mathcal{H}\text{-Alg}$$

PROOF: It is enough to prove that there exist

$$\overline{\alpha}_A \in \mathbf{kk}(A, A^\tau \# \mathcal{H}) \quad \text{and} \quad \overline{\beta}_B \in \mathbf{kk}^{\mathcal{H}}((B \# \mathcal{H})^\tau, B)$$

such that

$$A^\tau \xrightarrow{\tau(\overline{\alpha}_A)} (A^\tau \# \mathcal{H})^\tau \xrightarrow{\overline{\beta}_{A^\tau}} A^\tau \quad \text{and} \quad B \# \mathcal{H} \xrightarrow{\overline{\alpha}_{B \# \mathcal{H}}} (B \# \mathcal{H})^\tau \# \mathcal{H} \xrightarrow{\#(\overline{\beta}_B)} B \# \mathcal{H}$$

are the identities in $\mathbf{kk}^{\mathcal{H}}(A^\tau, A^\tau)$ and $\mathbf{kk}(B \# \mathcal{H}, B \# \mathcal{H})$ respectively. As \mathcal{H} is semisimple there exists an element $t \in \mathcal{H}$ such that

$$\epsilon(t) = 1 \quad th = \epsilon(h)t \quad \forall h \in \mathcal{H}$$

Define

$$\alpha_A : A \rightarrow A^\tau \# \mathcal{H} = A \otimes \mathcal{H} \quad \alpha_A(a) = a \otimes t$$

It is an algebra morphism because t is idempotent. Let

$$\begin{aligned} \beta_B : (B \# \mathcal{H})^\tau &\rightarrow \text{End}_B(\mathcal{H} \otimes B) \\ b \# h &\mapsto \varphi_{b \# h} \end{aligned}$$

$$\varphi_{b \# h}(k \otimes a) = \sum k_1 \bar{S}(h_2) \otimes (k_2 \bar{S}(h_1) \cdot b) a$$

One checks that β_B is an equivariant algebra homomorphism. Consider

$$(68) \quad \begin{array}{ccccc} A^\tau & \xrightarrow{\tau(\alpha_A)} & (A^\tau \# \mathcal{H})^\tau & \xrightarrow{\beta_{A^\tau}} & \text{End}_{A^\tau}(\mathcal{H} \otimes A^\tau) \\ a & \mapsto & a \otimes t & \mapsto & \varphi_{a \otimes t} \end{array}$$

Note

$$\varphi_{a \otimes t}(k \otimes b) = \sum k_1 \bar{S}(t_2) \otimes (k_2 \bar{S}(t_1) \cdot a) b = \sum k_1 \bar{S}(t_2) \otimes \epsilon(k_2) \epsilon(t_1) \cdot a b = k \bar{S}(t) \otimes a b = \epsilon(k) t \otimes a b$$

As \mathcal{H} is unimodular we have $t = \bar{S}(t)$. We write $\langle t \rangle$ for the subspace of \mathcal{H} generated by t . Let $I = \ker \epsilon$ and note $\mathcal{H} = \langle t \rangle \oplus I$ as \mathcal{H} -modules. Let $\varphi : \mathcal{H} \rightarrow \mathcal{H}$ be the projection over $\langle t \rangle$, $\varphi(h) = \epsilon(h)t$. Because the following diagram commutes the map (68) represents the identity in $\mathbf{kk}^{\mathcal{H}}(A^\tau, A^\tau)$ in the sense of remark 1.6.2.

$$\begin{array}{ccc} A^\tau & \xrightarrow{\iota} & \text{End}_\ell(\mathcal{H}) \otimes A^\tau \\ & \searrow (68) & \downarrow (67) \\ & & \text{End}_{A^\tau}(\mathcal{H} \otimes A^\tau) \end{array} \quad \begin{array}{ccc} a & \xrightarrow{\quad} & \varphi \otimes a \\ & \searrow & \downarrow \\ & & \varphi_{a \otimes t} \end{array}$$

It remains to prove that the following morphism represents the identity in $\mathbf{kk}(B \# \mathcal{H}, B \# \mathcal{H})$

$$(69) \quad \begin{array}{ccccc} B \# \mathcal{H} & \xrightarrow{\alpha_{B \# \mathcal{H}}} & (B \# \mathcal{H})^\tau \otimes \mathcal{H} & \xrightarrow{\beta_B} & \text{End}_B(\mathcal{H} \otimes B) \# \mathcal{H} \\ b \# h & \mapsto & b \# h \otimes t & \mapsto & \varphi_{b \# h} \# t \end{array}$$

The following morphism

$$\Omega : \text{End}_B(\mathcal{H} \otimes B) \# \mathcal{H} \rightarrow \text{End}_{B \# \mathcal{H}}(\mathcal{H} \otimes B \# \mathcal{H}) \quad \Omega(\eta \# l)(x \otimes c \# y) = \sum \eta(l_1 x \otimes l_2 \cdot c) \# l_3 y$$

makes the following diagram commutative

$$\begin{array}{ccc} \text{End}_B(\mathcal{H} \otimes B) \# \mathcal{H} & \xleftarrow{T_B \# \text{id}} & (\text{End}_\ell(\mathcal{H}) \otimes B) \# \mathcal{H} \\ \Omega \downarrow & & \downarrow \psi \\ \text{End}_{B \# \mathcal{H}}(\mathcal{H} \otimes B \# \mathcal{H}) & \xleftarrow{T_{B \# \mathcal{H}}} & \text{End}_\ell(\mathcal{H}) \otimes (B \# \mathcal{H}) \end{array}$$

here ψ is the isomorphism defined in Proposition 2.5.2 and T_B and $T_{\mathcal{H} \# B}$ are the morphisms defined in (67). As \mathcal{H} is finite dimensional, T_B and $T_{\mathcal{H} \# B}$ are isomorphisms. Hence Ω is an isomorphism too. Write by $\Lambda_{b \# h} = \Omega(\varphi_{b \# h} \# t)$. We shall prove that the map $b \# h \mapsto \Lambda_{b \# h}$ represents to the identity. Define

$$\delta : \mathcal{H} \otimes B \# \mathcal{H} \rightarrow \mathcal{H} \otimes B \# \mathcal{H} \quad \gamma : \mathcal{H} \otimes B \# \mathcal{H} \rightarrow \mathcal{H} \otimes B \# \mathcal{H}$$

$$\delta(x \otimes a \# y) = \sum x_1 \otimes x_2 \cdot a \# x_3 y \quad \gamma(x \otimes a \# y) = \sum x_1 \otimes S(x_3) \cdot a \# S(x_2) y$$

It is easy to check that they are mutually inverse. By M_∞ -stability the following morphism represents the identity

$$B\#\mathcal{H} \rightarrow \mathrm{End}_\ell(\mathcal{H}) \otimes (B\#\mathcal{H}) \quad b\#h \mapsto \varphi \otimes b\#h$$

Hence

$$B\#\mathcal{H} \rightarrow \mathrm{End}_{B\#\mathcal{H}}(\mathcal{H} \otimes B\#\mathcal{H}) \quad b\#h \mapsto \delta \circ T_{B\#\mathcal{H}}(\varphi \otimes b\#h) \circ \gamma$$

also represents the identity. We finish the proof verifying that

$$\begin{aligned} (\delta \circ T_{B\#\mathcal{H}}(\varphi \otimes b\#h) \circ \gamma)(x \otimes a\#y) &= \sum (\delta \circ \Gamma_{b\#h})(x_1 \otimes S(x_3) \cdot a\#S(x_2)y) \\ &= \sum \delta(\epsilon(x_1)t \otimes b(h_1S(x_3) \cdot a)\#h_2S(x_2)y) \\ &= \sum \delta(t \otimes b(h_1S(x_2) \cdot a)\#h_2S(x_1)y) \\ &= \sum t_1 \otimes t_2(b(h_1S(x_2)) \cdot a)\#t_3h_2S(x_1)y \\ &= \sum t \cdot (1 \otimes b(h_1S(x_2)) \cdot a)\#h_2S(x_1)y \\ &= \sum t\epsilon(x_1)\epsilon(h_3) \cdot (1 \otimes b(h_1S(x_3)) \cdot a)\#h_2S(x_2)y \\ &= \sum tx_1\bar{S}(h_3) \cdot (1 \otimes b(h_1S(x_3)) \cdot a)\#h_2S(x_2)y \\ &= \sum t_1x_1\bar{S}(h_6) \otimes (t_2x_2\bar{S}(h_5) \cdot b)(t_3x_3\bar{S}(h_4)h_1S(x_6) \cdot a)\#t_4x_4\bar{S}(h_3)h_2S(x_5)y \\ &= \sum t_1x_1\bar{S}(h_4) \otimes (t_2x_2\bar{S}(h_3) \cdot b)(t_3x_3\bar{S}(h_2)h_1S(x_4) \cdot a)\#t_4y \\ &= \sum t_1x_1\bar{S}(h_2) \otimes (t_2x_2\bar{S}(h_1) \cdot b)(t_3 \cdot a)\#t_4y \\ &= \Lambda_{b\#h}(x \otimes a\#y) \end{aligned}$$

□

CHAPTER 3

Isomorphism conjectures with proper coefficients

In this chapter we study isomorphism conjectures in the sense of [8]. We consider model category structures on G -simplicial sets and G -topological spaces. If \mathcal{F} is a family of subgroups of G we consider model category structures on \mathbb{S}^G and \mathbf{Top}^G . With this structure weak equivalences and fibration are object-wise. Cofibrant object are those X such that the stabilizer subgroup G_x is a subgroup in \mathcal{F} for all $x \in X$. We prove that the following is a Quillen equivalence

$$\mathbf{Top}^G \begin{array}{c} \xrightarrow{\text{Sing}_*} \\ \xleftarrow{\parallel_*} \end{array} \mathbb{S}^G.$$

We say that a functor $H : \mathbb{S}^G \rightarrow \mathbf{Spt}$ from the category of G -simplicial set to the category of spectra satisfies the (G, \mathcal{F}) -isomorphism conjecture if for the cofibrant replacement $\pi : \mathcal{E}(G, \mathcal{F}) \rightarrow *$ in the \mathcal{F} -model category mentioned aboved, the map

$$H(\pi) : H(\mathcal{E}(G, \mathcal{F})) \rightarrow H(*)$$

is an equivalence. If $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ is a functor and R is a unital G -ring, one constructs, following Davis-Luck [8], a functor

$$H^G(-, E(R)) : \mathbb{S}^G \rightarrow \mathbf{Spt}$$

such that $H^G(*, E(R)) = E(R \rtimes G)$. The (G, \mathcal{F}, E, R) -isomorphism conjecture is the (G, \mathcal{F}) -conjecture for the functor $H^G(-, E(R))$. We show that under very mild assumptions on E , the STANDING ASSUMPTIONS 3.2.5 (which are satisfied for example when E is either K or KH , see propositions 3.4.18 and 3.5.3), $H^G(-, E(A))$ is defined not only for unital G -rings, but also for all E -excisive G -rings A , that is all G -rings on which E -satisfies excision. Moreover we show that if

$$(70) \quad 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

is an exact sequence of E -excisive rings and X is a G -simplicial set, then

$$H^G(X, E(A')) \rightarrow H^G(X, E(A)) \rightarrow H^G(X, E(A''))$$

is a homotopy fibration. This is a basic property needed to establish an algebraic analogue of the Dirac-dual method which is used to prove the Baum-Connes conjecture for some groups. Another basic property, which provide us with enough sequences (70) in which at least one of the rings satisfies the isomorphism conjecture, is Theorem 3.7.3 which shows that if E satisfies the standing assumptions and A is an E -excisive G -ring of the form

$$(71) \quad A = \bigoplus_i \text{Ind}_{K_i}^G B_i$$

with B_i a K_i -ring and $K_i \in \mathcal{F}$ for all i , then the functor $H^G(-, E(A))$ maps (G, \mathcal{F}) -equivalences to equivalences. In particular the (G, \mathcal{F}, E, A) -isomorphism conjecture holds. We use this in Section 8 to show that under some additional assumptions (which are still satisfied if $E = K$, KH, see 3.8.1), for each E -excisive G -ring A there is a functorial exact sequence of E -excisive G -rings

$$\mathfrak{F}^0 B \rightarrow \mathfrak{F}^\infty B \rightarrow \mathfrak{F}^\infty B / \mathfrak{F}^0 B$$

and a natural map $A \rightarrow \mathfrak{F}^0 B$ such that

- i) $H^G(X, E(A)) \rightarrow H^G(X, E(\mathfrak{F}^0 B))$ is an equivalence for all G -simplicial set X .
- ii) $H^G(X, E(\mathfrak{F}^\infty B)) \rightarrow *$ is an equivalence if X is (G, \mathcal{F}) -cofibrant.
- iii) $H^G(-, E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B))$ maps (G, \mathcal{F}) -equivalences to equivalences.

It follows that the assembly map

$$H^G(\mathcal{E}(G, \mathcal{F}), E(A)) \rightarrow E(A \rtimes G)$$

is an equivalence iff the connecting map

$$\Omega(E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B \rtimes G)) \rightarrow E(A \rtimes G)$$

is an equivalence. In particular all this applies when $E = K$, KH. We also show in Theorem 3.9.2 that under stronger hypothesis on E , of which the main one is that E satisfies excision (e.g. KH satisfies this but K does not), then the (G, \mathcal{F}, E, A) -isomorphism conjecture is true whenever A is (G, \mathcal{F}) -proper. If X is a locally finite simplicial set with a G -action then a G -ring A is proper over X if it is an algebra over the ring $\mathbb{Z}^{(X)}$ of finitely supported polynomial maps on X , the algebra action is compatible with the actions of G on A and on X , and $\mathbb{Z}^{(X)} \cdot A = A$. We say that A is (G, \mathcal{F}) -proper if it is proper over a locally finite simplicial set X on which G acts with all stabilizers in \mathcal{F} . For example an algebra is of the form (71) if and only if it is proper over the zero-dimensional G -simplicial set $X = \coprod G / K_i$.

We remark that the notion of (G, \mathcal{F}) -proper ring used here is the algebraic analogue of the notion of proper G - C^* -algebra, and that Theorem 3.9.2 is an algebraic version of the known fact that Baum-Connes conjecture holds for proper G - C^* -algebras [11].

1. G -simplicial sets and model category structures

1.1. G -simplicial sets. A G -simplicial set or a G -complex X is a simplicial set with a simplicial action of G . Let X be a G -complex; consider the category $(\Delta \downarrow X)_G$ defined as follows. Its objects are the triples $(G / H, [n], \sigma : G / H \times \Delta^n \rightarrow X)$ such that H is a subgroup of G and $\sigma : G / H \times \Delta^n \rightarrow X$ is a morphism such that $\sigma(gH, x) = g \cdot \sigma(H, x)$. An arrow is a pair (s, θ) , $s : G / H \rightarrow G / K$, $\theta : [n] \rightarrow [m]$, such that the following diagram commutes

$$\begin{array}{ccc} G / H \times \Delta^n & \xrightarrow{s \times \text{hom}_\Delta(-, \theta)} & G / K \times \Delta^m \\ & \searrow \sigma & \swarrow \tau \\ & X & \end{array} .$$

As in the nonequivariant case (see [10], I.2), any G -simplicial set is the colimit of its cells. Let recall this in the following lemma.

LEMMA 3.1.1. Consider

$$T_X : (\Delta \downarrow X)_G \rightarrow \mathbb{S}^G \quad (G/H, [n], \sigma : G/H \times \Delta^n \rightarrow X) \mapsto \sigma : G/H \times \Delta^n \rightarrow X$$

Then

$$X = \operatorname{colim}_{G/H \times \Delta^n \rightarrow X} G/H \times \Delta^n = \operatorname{colim} T_X$$

□

The n -skeleton X_n of a G -complex X can be obtained by attaching equivariant cells to X_{n-1} as the following pushout diagram shows

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times \partial \Delta^n & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times \Delta^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

A family \mathcal{F} of subgroups of G is a nonempty family closed under conjugation and under taking subgroups. A G -complex X is a (G, \mathcal{F}) -complex if $H_i \in \mathcal{F}$ for all $i \in I$, $n \in \mathbb{N}$.

1.2. Orbit category. Let \mathcal{F} be a family of subgroups of G . We consider the **orbit category** of G relative to \mathcal{F} and we write it $\mathbf{Or}_{\mathcal{F}} G$. Its objects are the G -sets G/H with $H \in \mathcal{F}$ and its maps are the G -equivariant maps. We write $\mathbf{Or} G$ for $\mathbf{Or}_{\mathbf{All}} G$. If $H, K \in \mathcal{F}$ and $g \in G$ is such that $g^{-1}Hg \subset K$ then

$$r_g : G/H \rightarrow G/K \quad r_g(tH) = tgK$$

is a morphism of $\mathbf{Or}_{\mathcal{F}} G$. Moreover, every morphism in $\mathbf{Or}_{\mathcal{F}} G$ is r_g for some $g \in G$.

1.3. Quillen equivalences. We consider \mathbf{Top} and \mathbb{S} with their usual cofibrantly generated closed model category structures (see Section 2.4, Section 3.2 and Definition 2.1.17 of [14]). The sets of generating cofibrations $I_{\mathbf{Top}}$, $I_{\mathbb{S}}$ and the sets of generating trivial cofibration $J_{\mathbf{Top}}$, $J_{\mathbb{S}}$ are the following

$$I_{\mathbf{Top}} = \{f : S^{n-1} \hookrightarrow D^n : n \geq 0\} \quad I_{\mathbb{S}} = \{f : \partial \Delta^n \hookrightarrow \Delta^n : n \geq 0\}$$

$$J_{\mathbf{Top}} = \{f : D^n \hookrightarrow D^n \times I : f(x) = (x, 0) \quad n \geq 0\} \quad J_{\mathbb{S}} = \{f : \Delta_k^n \hookrightarrow \Delta^n : n > 0, \quad 0 \leq k \leq n\}$$

By Theorem 3.6.7 of [14] we have that the geometric realization functor $\| : \mathbb{S} \rightarrow \mathbf{Top}$ and its right adjoint $\operatorname{Sing} : \mathbf{Top} \rightarrow \mathbb{S}$ form a Quillen equivalence (see Definition 1.3.12 in [14])

$$\begin{array}{ccc} & \operatorname{Sing} & \\ \mathbf{Top} & \xrightleftharpoons{\quad} & \mathbb{S} \\ & \parallel & \end{array}$$

If $\mathcal{C} = \mathbf{Top}$, \mathbb{S} , and I is any small category, then, by [13, Thm. 11.6.1], \mathcal{C}^I is again a cofibrantly generated closed model category, with object-wise fibrations and weak equivalences, and where generating (trivial) cofibrations are of the form

$$\coprod_{\operatorname{hom}_I(\alpha, -)} f : \coprod_{\operatorname{hom}_I(\alpha, -)} \operatorname{dom} f \rightarrow \coprod_{\operatorname{hom}_I(\alpha, -)} \operatorname{cod} f$$

with $\alpha \in I$ and $f : \text{dom} f \rightarrow \text{cod} f$ a generating (trivial) cofibration in \mathcal{C} . By [13, Thm. 11.6.5], the induced functors

$$\mathbf{Top}^I \begin{array}{c} \xrightarrow{\text{Sing}_*} \\ \parallel_* \\ \xleftarrow{\quad} \end{array} \mathbb{S}^I$$

are also Quillen equivalences.

Next fix a group G and a family \mathcal{F} of subgroups of G . By the previous discussion applied to the orbit category $\mathbf{Or}_{\mathcal{F}} G^{op}$, we have a Quillen equivalence

$$(72) \quad \mathbf{Top}^{\mathbf{Or}_{\mathcal{F}} G^{op}} \begin{array}{c} \xrightarrow{\text{Sing}_*} \\ \parallel_* \\ \xleftarrow{\quad} \end{array} \mathbb{S}^{\mathbf{Or}_{\mathcal{F}} G^{op}}$$

Let H be a subgroup of G and X an object in \mathcal{C}^G . Consider

$$X^H := \{x \in X : h \cdot x = x \quad \forall h \in H\}$$

Note $X^H = \text{map}_G(G/H, X)$.

For $\mathcal{C} = \mathbf{Top}, \mathbb{S}$, consider the functor

$$R : \mathcal{C}^G \rightarrow \mathcal{C}^{\mathbf{Or}_{\mathcal{F}} G^{op}}, \quad R(X)(G/H) = \text{map}_G(G/H, X) = X^H$$

and its left adjoint, the coend

$$L : \mathcal{C}^{\mathbf{Or}_{\mathcal{F}} G^{op}} \rightarrow \mathcal{C}^G, \quad L(Y) = \int^{G/H} Y(G/H) \times G/H$$

The Quillen equivalence (72) fits into a diagram

$$(73) \quad \begin{array}{ccc} \mathbf{Top}^{\mathbf{Or}_{\mathcal{F}} G^{op}} & \begin{array}{c} \xrightarrow{\text{Sing}_*} \\ \parallel_* \\ \xleftarrow{\quad} \end{array} & \mathbb{S}^{\mathbf{Or}_{\mathcal{F}} G^{op}} \\ \begin{array}{c} \uparrow L \\ \downarrow R \end{array} & & \begin{array}{c} \uparrow L \\ \downarrow R \end{array} \\ \mathbf{Top}^G & \begin{array}{c} \xrightarrow{\text{Sing}_*} \\ \parallel_* \\ \xleftarrow{\quad} \end{array} & \mathbb{S}^G \end{array}$$

PROPOSITION 3.1.2. Let H be a subgroup of G and X an object in \mathcal{C}^G .

(1) Let B an object in \mathcal{C} , B is also an object in \mathcal{C}^G with the trivial action, and

$$\text{hom}_{\mathcal{C}^G}(B \times G/H, X) \simeq \text{hom}_{\mathcal{C}}(B, X^H)$$

(2) Let the following be a cocartesian diagram in \mathcal{C}^G with g injective,

$$\begin{array}{ccc} A & \xrightarrow{j} & D \\ g \uparrow & & \uparrow i \\ B & \xrightarrow{f} & C \end{array}$$

Then

$$\begin{array}{ccc} A^H & \xrightarrow{j^H} & D^H \\ g^H \uparrow & & \uparrow i^H \\ B^H & \xrightarrow{f^H} & C^H \end{array}$$

is cocartesian in \mathcal{C} .

PROOF: Straightforward. \square

PROPOSITION 3.1.3. Let $\mathcal{C} = \mathbf{Top}, \mathbb{S}$.

- i) \mathcal{C}^G is a closed model category where a map f is a fibration (resp. a weak equivalence) if and only if $R(f)$ is. Moreover \mathcal{C}^G is cofibrantly generated, where the generating (trivial) cofibrations are the maps $f \times \text{id} : \text{dom} f \times G/H \rightarrow \text{cod} f \times G/H$, with f a generating (trivial) cofibration and $H \in \mathcal{F}$.
- ii) Each of the pairs of functors of diagram (73) is a Quillen equivalence.

PROOF. One can give conditions on two sets of maps and a subcategory of a category \mathcal{D} to be respectively the generating cofibrations, generating trivial cofibrations and weak equivalences in a closed model structure of \mathcal{D} ; see M. Hovey's book [14, Thm. 2.1.19]. It is straightforward that those conditions are satisfied in our case, for $\mathcal{D} = \mathcal{C}^G$. This proves i). The top pair of functors in diagram (73) is a Quillen equivalence by the discussion above the proposition. By definition of fibrations and weak equivalences in \mathcal{C}^G , these are both preserved and reflected by R . In particular (L, R) is a Quillen pair. To show that it is an equivalence, it suffices, by [14, Cor. 1.3.16], to show that if $X \in \mathcal{C}^{\mathbf{Or}_{\mathcal{F}} G^{op}}$ is cofibrant, then the unit map

$$(74) \quad X \rightarrow RLX$$

is a weak equivalence; in fact we shall see that it is an isomorphism. Because every cofibrant object is a retract of a cofibrant cell complex, it suffices to check that (74) is an isomorphism on cell complexes. By definition, the generating cofibrant cells in $\mathcal{C}^{\mathbf{Or}_{\mathcal{F}} G^{op}}$ are of the form $\coprod_{\text{map}_G(-, G/H)} \Delta^n$. But for every $T \in \mathbb{S}$, we have:

$$\begin{aligned} RL\left(\coprod_{\text{map}_G(-, G/H)} T\right)(G/K) &= R(G/H \times T)(G/K) \\ &= (G/H \times T)^K \\ &= \text{map}_{\mathbf{Or}_G}(G/K, G/H) \times T = \coprod_{\text{map}_G(G/K, G/H)} T \end{aligned}$$

Thus the unit map is an isomorphism on cells, and therefore on coproducts of cells, since taking fixed points under a subgroup preserves coproducts of G -simplicial sets. In particular (74) is an isomorphism on the zero skeleton of X . Assume by induction that

(74) is an isomorphism on the n -skeleton. The $n + 1$ -skeleton is a pushout

$$\begin{array}{ccc} \coprod_{H \in I_n} \coprod_{\text{map}_G(-, G/H)} \Delta^n & \longrightarrow & X_{n+1}(-) \\ \uparrow & & \uparrow \\ \coprod_{H \in I_n} \coprod_{\text{map}_G(-, G/H)} \partial \Delta^n & \longrightarrow & X_n(-) \end{array}$$

By Lemma 3.1.2 and the inductive hypothesis, the diagram

$$\begin{array}{ccc} \coprod_{H \in I_n} (\text{map}_G(-, G/H)) \times \Delta^n & \longrightarrow & RLX_{n+1}(-) \\ \uparrow & & \uparrow \\ \coprod_{H \in I_n} (\text{map}_G(-, G/H)) \times \partial \Delta^n & \longrightarrow & X_n(-) \end{array}$$

is again a pushout. It follows that $RLX_{n+1} \cong X_{n+1}$ and thus (74) is an isomorphism on all cell complexes, as we had to prove. We have shown that the top horizontal and both vertical pairs of functors are Quillen equivalences; by [14, Cor. 1.3.15], this implies that also the bottom pair is a Quillen equivalence. \square

1.4. Assembly map. For the model structures of Proposition 3.1.3, the functorial cofibrant replacement in \mathbf{Top}^G of the point space $*$ is a model for the classifying space of G with respect to \mathcal{F} and the cofibrant replacement of $*$ in \mathbb{S}^G is a simplicial version. Moreover because $|-|_* : \mathbb{S}^G \rightarrow \mathbf{Top}^G$ is a Quillen equivalence, it takes the simplicial version to the topological one. In particular if E is a functor from \mathbf{Top}^G to spectra and $\pi : \mathcal{E}(G, \mathcal{F}) \rightarrow *$ is the cofibrant replacement in \mathbb{S}^G , then we have a map

$$(75) \quad E(\pi) : E(|\mathcal{E}(G, \mathcal{F})|) \rightarrow E(*)$$

If

$$E(X) = F_{\%}(X) = R(X) \otimes_{\mathbf{Or} G} F := \int^{\mathbf{Or} G} X_+^H \wedge F(G/H)$$

for some functor $F : \mathbf{Or} G \rightarrow \mathbf{Spt}$, (75) is the Davis-Lück assembly map of [8, Section 5.1]. In case $F = |F'|$ is the geometric realization of a functorial spectrum in the simplicial set sense, we have further

$$|F'|_{\%}(|X|) = \left| \int^{\mathbf{Or} G} X_+^H \wedge F'(G/H) \right| = |F'_{\%}(X)|$$

and the assembly map for F is the geometric realization of that of F' . Hence we can equivalently work with assembly maps in the topological or the simplicial setting; we choose to do the latter. In particular all spectra considered henceforth are simplicial.

2. Equivariant homology

2.1. Crossed products and equivariant homology. A groupoid is a small category where all arrows are isomorphisms. Let \mathcal{G} be a groupoid, and let R be a unital ring. An action of \mathcal{G} on R is a functor $\rho : \mathcal{G} \rightarrow \mathbf{Ring}_1$ such that $\rho(x) = R$ for all $x \in \text{ob} \mathcal{G}$. For

example we may take $\rho(g) = \text{id}_R$ for all arrows $g \in \text{ar}\mathcal{G}$; this is called the **trivial action**. Whenever ρ is fixed, we omit it from our notation, and write

$$g(r) = \rho(g)(r)$$

for $g \in \text{ar}\mathcal{G}$ and $r \in R$. Given a triple (\mathcal{G}, ρ, R) , we consider a small \mathbb{Z} -linear category $R \rtimes \mathcal{G}$. The objects of $R \rtimes \mathcal{G}$ are those of \mathcal{G} , and

$$\text{hom}_{R \rtimes \mathcal{G}}(x, y) = R \otimes \mathbb{Z}[\text{hom}_{\mathcal{G}}(x, y)]$$

If $s \in R$ and $g \in \text{hom}_{\mathcal{G}}(x, y)$, we write $s \rtimes g$ for $s \otimes g$. Composition is defined by the rule

$$(76) \quad (r \rtimes f) \cdot (s \rtimes g) = rf(s) \rtimes fg$$

here $r, s \in R$, and f and g are composable arrows in \mathcal{G} . In case the action of \mathcal{G} on R is trivial, we also write $R[\mathcal{G}]$ for $R \rtimes \mathcal{G}$.

Let G be a group; consider the functor $\mathcal{G}^G : G\text{-Set} \rightarrow \mathbf{Gpd}$ which sends a G -set S to its **transport groupoid**. By definition $\text{ob}\mathcal{G}^G(S) = S$, and $\text{hom}_{\mathcal{G}^G(S)}(s, t) = \{g \in G : g \cdot s = t\}$.

NOTATION 3.2.1. If E is a functor from \mathbb{Z} -linear categories to spectra, R a unital G -ring, and X a G -space, we put

$$H^G(X, E(R)) := E(R \rtimes \mathcal{G}^G(?))_{\%}(X)$$

2.2. The ring $\mathcal{A}(\mathcal{C})$. Let \mathcal{C} be a small \mathbb{Z} -linear category. Put

$$(77) \quad \mathcal{A}(\mathcal{C}) = \bigoplus_{a, b \in \text{ob}\mathcal{C}} \text{hom}_{\mathcal{C}}(a, b)$$

The following multiplication law

$$(78) \quad (fg)_{a, b} = \sum_{c \in \text{ob}\mathcal{C}} f_{c, b} \circ g_{a, c}$$

makes $\mathcal{A}(\mathcal{C})$ into an associative ring, which is unital if and only if $\text{ob}\mathcal{C}$ is finite. Whatever the cardinal of $\text{ob}\mathcal{C}$ is, $\mathcal{A}(\mathcal{C})$ is always a ring with **local units**, i.e. a filtering colimit of unital rings.

$\mathcal{A}(?)$ and tensor products. The **tensor product** of two \mathbb{Z} -linear categories \mathcal{C} and \mathcal{D} is the \mathbb{Z} -linear category $\mathcal{C} \otimes \mathcal{D}$ with $\text{ob}(\mathcal{C} \otimes \mathcal{D}) = \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{D})$ and

$$\text{hom}_{\mathcal{C} \otimes \mathcal{D}}((c_1, d_1), (c_2, d_2)) = \text{hom}_{\mathcal{C}}(c_1, c_2) \otimes \text{hom}_{\mathcal{D}}(d_1, d_2)$$

We have

$$\mathcal{A}(\mathcal{C} \otimes \mathcal{D}) = \mathcal{A}(\mathcal{C}) \otimes \mathcal{A}(\mathcal{D})$$

EXAMPLE 3.2.2. If \mathcal{G} is a groupoid acting trivially on a unital ring R , then

$$\mathcal{A}(R[\mathcal{G}]) = \mathcal{A}(R \otimes \mathbb{Z}[\mathcal{G}]) = R \otimes \mathcal{A}(\mathbb{Z}[\mathcal{G}])$$

$\mathcal{A}(?)$ and crossed products. If A is any, not necessarily unital ring, and \mathcal{G} is a groupoid acting on A , we put

$$\mathcal{A}(A \rtimes \mathcal{G}) = \bigoplus_{x,y \in \text{ob } \mathcal{G}} A \otimes \mathbb{Z}[\text{hom}_{\mathcal{G}}(x, y)]$$

The rules (76) and (78) make $\mathcal{A}(A \rtimes \mathcal{G})$ into a ring, which in general is nonunital and does not have local units. The ring $\mathcal{A}(A \rtimes \mathcal{G})$ may also be described in terms of the unitalization \tilde{A} of A . By definition, $\tilde{A} = A \oplus \mathbb{Z}$ equipped with the trivial \mathcal{G} -action on the \mathbb{Z} -summand and the following multiplication

$$(79) \quad (a, \lambda)(b, \mu) = (ab + \lambda b + a\mu, \lambda\mu)$$

We have

$$(80) \quad \mathcal{A}(A \rtimes \mathcal{G}) = \ker(\mathcal{A}(\tilde{A} \rtimes \mathcal{G}) \rightarrow \mathcal{A}(\mathbb{Z}[\mathcal{G}]))$$

Note that $\mathcal{A}(A \rtimes \mathcal{G})$ is defined, even though $A \rtimes \mathcal{G}$ is not. One can actually define $A \rtimes \mathcal{G}$ as a nonunital category, i.e. a category without identity morphisms, but we do not go into that here.

Next we fix a group G and a subgroup $H \subset G$ and consider the ring $\mathcal{A}(A \rtimes \mathcal{G}^G(G/H))$ associated to the crossed product by the transport groupoid. Note that

$$\text{hom}_{\mathcal{G}^G(G/H)}(H, H) = H = \text{hom}_{\mathcal{G}^H(H/H)}(H, H)$$

thus there is a fully faithful functor $\mathcal{G}^H(H/H) \rightarrow \mathcal{G}^G(G/H)$. This functor induces a ring homomorphism

$$j : A \rtimes H = \mathcal{A}(A \rtimes \mathcal{G}^H(H/H)) \subset \mathcal{A}(A \rtimes \mathcal{G}^G(G/H))$$

The next lemma compares the map j with the canonical inclusion

$$\iota : A \rtimes H \rightarrow M_{G/H}(A \rtimes H), \quad x \mapsto e_{H,H} \otimes x$$

In the following lemma and elsewhere, we make use of a section $s : G/H \rightarrow G$ of the canonical projection onto the quotient by a subgroup $H \subset G$. We say that the section s is **pointed** if it is a map of pointed sets, that is, if it maps the class of H to the element $1 \in G$.

LEMMA 3.2.3. Let A be a ring, G a group acting on A , and $H \subset G$ a subgroup. Then there is an isomorphism $\alpha : \mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) \xrightarrow{\cong} M_{G/H}(A \rtimes H)$ making the following diagram commute:

$$\begin{array}{ccc} A \rtimes H & \xrightarrow{j} & \mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) \\ & \searrow \iota & \downarrow \wr^\alpha \\ & & M_{G/H}(A \rtimes H) \end{array}$$

The isomorphism α is natural in A but not in the pair (G, H) , as it depends on a choice of pointed section $s : G/H \rightarrow G$ of the projection $\pi : G \rightarrow G/H$.

PROOF. Let s be as in the lemma; put $\hat{g} = s(\pi(g))$ ($g \in G$). The isomorphism $\alpha : \mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) \xrightarrow{\cong} M_{G/H}(A \rtimes H)$ is defined as follows. For $b \in A$, $s, t \in G$, and $g \in \text{hom}_{\mathcal{G}^G(G/H)}(sH, tH)$, put

$$\alpha(b \rtimes g) = e_{tH, sH} \otimes \hat{t}^{-1}(b) \rtimes (\hat{t}^{-1}g\hat{s})$$

It is straightforward to check that α is an isomorphism and that $\alpha j = \iota$. \square

Functoriality of $\mathcal{A}(?)$. If $F : \mathcal{C} \mapsto \mathcal{D}$ is a \mathbb{Z} -linear functor which is injective on objects, then it defines a homomorphism $\mathcal{A}(F) : \mathcal{A}(\mathcal{C}) \rightarrow \mathcal{A}(\mathcal{D})$ by the rule $\alpha \mapsto F(\alpha)$. Hence we may regard \mathcal{A} as a functor

$$(81) \quad \mathcal{A} : \text{inj-}\mathbb{Z}\text{-Cat} \rightarrow \mathbf{Ring}$$

from the category of \mathbb{Z} -linear categories and functors which are injective on objects, to the category of rings. However $\mathcal{A}(F)$ is not defined for general \mathbb{Z} -linear F .

REMARK 3.2.4. The use of the prefix *inj* here differs from that in [8]. Indeed, here *inj* indicates that functors are injective on objects, whereas in [8], it refers to functors which are injective on arrows.

2.3. The nonunital case. A Milnor square is a pullback square of rings

$$(82) \quad \begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

such that either f or g is surjective. Below we shall assume f is surjective. Let $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ be a functor. If A is a not necessarily unital ring, embedded as an ideal in a unital ring R , we write $E(R : A) = \text{hofiber}(E(R) \rightarrow E(R/A))$. The functor E is said to satisfy **excision** for the Milnor square (82) if

$$\begin{array}{ccc} E(R') & \longrightarrow & E(R) \\ \downarrow & & \downarrow E(f) \\ E(S') & \longrightarrow & E(S) \end{array}$$

is homotopy cartesian. If $\ker f \cong A$, then E satisfies excision on (82) if and only if the spectrum

$$E(R', R : A) = \text{hofiber}(E(R' : A) \rightarrow E(R : A))$$

is weakly contractible. We say that the ring A is **E-excisive** if E satisfies excision on every Milnor square (82) with $\ker f \cong A$. Assume unital rings are E-excisive; if A is any, not necessarily E-excisive ring, we consider its unitalization \tilde{A} , defined in (79) above. Put

$$E(A) = \text{hofiber}(E(\tilde{A}) \rightarrow E(\mathbb{Z}))$$

Because of our assumption that unital rings are E-excisive, if A happens to be unital, the two definitions of $E(A)$ are naturally homotopy equivalent. Note that if

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

is an exact sequence of rings and A' is E-excise, then

$$E(A') \rightarrow E(A) \rightarrow E(A'')$$

is a homotopy fibration.

We have already considered rings with local units; we shall need an even weaker form of unitality, called *s-unitality*. A ring A is called *s-unital* if for every finite collection $a_1, \dots, a_n \in A$ there exists an element $e \in A$ such that $a_i e = e a_i = a_i$. Note that if we add the requirement that e be idempotent we recover the notion of ring with local units.

STANDING ASSUMPTIONS 3.2.5. From now on, we shall be primarily concerned with functors $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ that satisfy the following:

- i) Every *s-unital* ring is E-excise.
- ii) If H is a group and A an E-excise H -ring, then $A \rtimes H$ is E-excise.
- iii) If A is E-excise, X a set and $x \in X$, then $M_X A$ is E-excise, and E sends the map $A \rightarrow M_X A$, $a \mapsto e_{x,x} a$ to a weak equivalence.
- iv) There is a natural weak equivalence $E(\mathcal{A}(\mathcal{C})) \xrightarrow{\sim} E(\mathcal{C})$ of functors $\text{inj-}\mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$.
- v) Let $\{A_i : i \in I\}$ be a family of rings, and let $A = \bigoplus_{i \in I} A_i$ be their direct sum, with coordinate-wise multiplication. Then A is E-excise if and only if each A_i is. Moreover if these equivalent conditions are satisfied, then the map $\bigoplus_i E(A_i) \rightarrow E(A)$ is an equivalence.

Let G be a group. Assume E satisfies the standing assumptions above. For A an E-excise G -ring, consider the **Or** G -spectrum

$$(83) \quad G/H \mapsto E(A \rtimes \mathcal{G}^G(G/H)) = \text{hofiber}(E(\tilde{A} \rtimes \mathcal{G}^G(G/H)) \rightarrow E(\mathbb{Z}[\mathcal{G}^G(G/H)]))$$

Applying $(?)_{\%}$ to (83) defines an equivariant homology theory of G -simplicial sets, which we denote $H^G(-, E(A))$. Moreover, for each fixed G -simplicial set X , $H^G(X, E(?))$ is a functor of E-excise rings. Observe that, for unital A , we have two definitions of $E(A \rtimes \mathcal{G}^G(-))$ and two definitions of $H^G(-, E(A))$; the next proposition says that the two definitions are equivalent.

PROPOSITION 3.2.6. Let $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ be a functor and G a group. Assume that E satisfies the standing assumptions 3.2.5 above.

- a) If R is a unital G -ring, then the two definitions of $E(R \rtimes \mathcal{G}^G(-))$ and of $H^G(-, E(R))$ are equivalent.
- b) If

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

is an exact sequence of E-excise G -rings, and X is a G -simplicial set, then

$$E(A' \rtimes \mathcal{G}^G(-)) \rightarrow E(A \rtimes \mathcal{G}^G(-)) \rightarrow E(A'' \rtimes \mathcal{G}^G(-))$$

and

$$H^G(X, E(A')) \rightarrow H^G(X, E(A)) \rightarrow H^G(X, E(A''))$$

are homotopy fibrations.

PROOF. If A is E -excisive and $H \subset G$ is a subgroup, then conditions ii) and iii) together with Lemma 3.2.3 imply that $\mathcal{A}(A \rtimes \mathcal{G}^G(G/H))$ is E -excisive. Hence, by condition iv), the spectrum in (83) is equivalent to $E(\mathcal{A}(A \rtimes \mathcal{G}^G(G/H)))$. In particular, by i), $\mathcal{A}(R \rtimes \mathcal{G}^G(G/H))$ is E -excisive for R unital, and the map

$$\mathrm{hofiber}(E(\tilde{R} \rtimes \mathcal{G}^G(G/H)) \rightarrow E(\mathbb{Z}[\mathcal{G}^G(G/H)])) \rightarrow E(R \rtimes \mathcal{G}^G(G/H))$$

induced by the projection $\tilde{R} \cong R \times \mathbb{Z} \rightarrow R$ is an equivalence. This proves a). Moreover, because $\mathcal{A}(\cdot \rtimes \mathcal{G}^G(G/H))$ preserves exact sequences, then applying (83) to the exact sequence of part b) yields an object-wise homotopy fibration of **Or** G -spectra, which is the first homotopy fibration of b). Applying $(?)_{\%}$ we obtain the second one. \square

REMARK 3.2.7. Let $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ and let A be any, not necessarily E -excisive G -ring, equivariantly embedded as an ideal in a unital G -ring R . Consider the **Or** G -spectrum

$$E(R \rtimes \mathcal{G}^G(-) : A \rtimes \mathcal{G}^G(-)) = \mathrm{hofiber}(E(R \rtimes \mathcal{G}^G(-)) \rightarrow E((R/A) \rtimes \mathcal{G}^G(-)))$$

Put

$$H^G(X, E(R : A)) = E(R \rtimes \mathcal{G}^G(-) : A \rtimes \mathcal{G}^G(-))_{\%}(X).$$

Assembly gives a map of homotopy fibrations

$$\begin{array}{ccccc} H^G(\mathcal{E}(G, \mathcal{F}), E(R : A)) & \longrightarrow & H^G(\mathcal{E}(G, \mathcal{F}), E(R)) & \longrightarrow & H^G(\mathcal{E}(G, \mathcal{F}), E(R/A)) \\ \downarrow & & \downarrow & & \downarrow \\ E(R \rtimes G : A \rtimes G) & \longrightarrow & E(R \rtimes G) & \longrightarrow & E((R/A) \rtimes G) \end{array}$$

Hence if the (G, \mathcal{F}) assembly map for the functorial spectrum $E(-)$ is an equivalence on unital rings, then both the middle and right hand side vertical maps are equivalences; it follows that the same is true of the map on the left. We record a particular case of this in the following corollary.

COROLLARY 3.2.8. Let $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ be a functor; assume E satisfies the Standing assumptions 3.2.5. Further let G be a group and \mathcal{F} a family of subgroups, and assume that the assembly map $H^G(\mathcal{E}(G, \mathcal{F}), E(R)) \rightarrow E(R \rtimes G)$ is an equivalence for every unital ring R . Then $H^G(\mathcal{E}(G, \mathcal{F}), E(A)) \rightarrow E(A \rtimes G)$ is an equivalence for every E -excisive ring A .

PROPOSITION 3.2.9. Let $A \triangleleft R$ be an ideal in a unital G -ring, closed under the action of G . Let $E : \mathbf{Ring} \rightarrow \mathbf{Spt}$ be a functor satisfying the standing assumptions. If A is E -excisive then

$$E(A \rtimes \mathcal{G}^G(-)) \rightarrow E(R \rtimes \mathcal{G}^G(-) : A \rtimes \mathcal{G}^G(-))$$

is an object-wise weak equivalence of **Or** G^{op} -spectra.

PROOF. Let H be a subgroup of G . By Standing Assumption ii)

$$E(A \rtimes H) \rightarrow E(R \rtimes H : A \rtimes H)$$

is an equivalence. The proof follows from Lemma 3.2.3, using assumptions iii) and iv). \square

2.4. The ring $\mathcal{R}(\mathcal{C})$. Let \mathcal{C} be a \mathbb{Z} -linear category. Imitating a construction used by M. Joachim ([15]) in the C^* -algebra context, we shall associate to \mathcal{C} a ring $\mathcal{R}(\mathcal{C})$ which is a quotient of the tensor algebra of $\mathcal{A}(\mathcal{C})$; first we need some notation. If M is an abelian group, we write $T(M) = \bigoplus_{n \geq 1} M^{\otimes n}$ for the (unaugmented) tensor algebra. Put

$$\mathcal{R}(\mathcal{C}) = T(\mathcal{A}(\mathcal{C})) / \langle \{g \otimes f - g \circ f : f \in \text{hom}_{\mathcal{C}}(a, b), g \in \text{hom}_{\mathcal{C}}(b, c), \quad a, b, c \in \text{ob } \mathcal{C}\} \rangle$$

Note that any \mathbb{Z} -linear functor $\mathcal{C} \rightarrow \mathcal{D} \in \mathbb{Z}\text{-}\mathbf{Cat}$ defines a homomorphism $\mathcal{R}(\mathcal{C}) \rightarrow \mathcal{R}(\mathcal{D})$. Thus we may regard \mathcal{R} as a functor

$$\mathcal{R} : \mathbb{Z}\text{-}\mathbf{Cat} \rightarrow \mathbf{Ring}, \quad \mathcal{C} \mapsto \mathcal{R}(\mathcal{C})$$

Observe that the canonical surjection $T(\mathcal{A}(\mathcal{C})) \rightarrow \mathcal{A}(\mathcal{C})$ factors through a map

$$(84) \quad \mathcal{R}(\mathcal{C}) \twoheadrightarrow \mathcal{A}(\mathcal{C})$$

whose kernel is the ideal generated by the elements $g \otimes f$ for non-composable g and f . In Lemma 3.2.10 we give conditions on a functor $E : \mathbf{Ring} \rightarrow \mathbf{Spt}$ to send (84) to a weak equivalence; first we need some notation. A functor $E : \mathbf{Ring} \rightarrow \mathbf{Spt}$ is called **excisive** if every ring is E -excisive. An excisive functor which satisfies Standing Assumption iii) is called **matrix stable**. Note that standing assumptions i)-ii) are automatically satisfied if $E : \mathbf{Ring} \rightarrow \mathbf{Spt}$ is excisive and matrix stable. Any excisive functor E satisfies condition iv) for finite sums; if it satisfies it for arbitrary sums, we say that E is **additive**. For example if E is excisive and E_* commutes with filtering colimits, then E is additive.

For the proof of the next lemma we also need to recall the concept of multiplier ring which we borrow from [16]. The **multiplier ring** of a ring A is the ring $\mathcal{M}(A)$ whose elements are the pairs (f, g) of maps $A \rightarrow A$ such that f is a right A -module homomorphism, g is a left A -module homomorphism and the following compatibility condition is satisfied

$$g(a)b = af(b)$$

Multiplication in $\mathcal{M}(A)$ is defined by

$$(f_1, g_1)(f_2, g_2) = (f_1 f_2, g_2 g_1)$$

If $a \in A$ then the pair $m(a) = (L_a, R_a)$ given by left and right multiplication by a is an element of $\mathcal{M}(A)$, and $a \mapsto m(a)$ is a ring homomorphism $m : A \mapsto \mathcal{M}(A)$. The image of m is always an ideal of $\mathcal{M}(A)$; its kernel is the two-sided annihilator of A

$$\ker m = \{a \in A : aA = Aa = 0\}$$

This kernel vanishes for example if A is s -unital.

LEMMA 3.2.10. Let \mathcal{C} be a \mathbb{Z} -linear category and let $E : \mathbf{Ring} \rightarrow \mathbf{Spt}$ be an excisive and matrix stable functor. Then E_* sends (84) to a naturally split surjection. Assume in addition that E is invariant under polynomial homotopy. Then E sends (84) to a weak equivalence.

PROOF. Let $\text{ob}_+ \mathcal{C} = \text{ob } \mathcal{C} \coprod \{+\}$ be the set of objects of \mathcal{C} with a base point added. Consider the homomorphism

$$j : \mathcal{A}(\mathcal{C}) \rightarrow M_{\text{ob}_+ \mathcal{C}} \mathcal{R}(\mathcal{C}), \quad j(f) = f \otimes e_{b,a} \quad (f \in \text{hom}_{\mathcal{C}}(a, b))$$

Write p for the map (84). Left and right multiplication by each of the following two matrices

$$W = \sum_{a \in \text{ob } \mathcal{C}} 1_a \otimes e_{a,+}$$

$$V = \sum_{a \in \text{ob } \mathcal{C}} 1_a \otimes e_{+,a}$$

leave $M_{\text{ob}_+ \mathcal{C}} \mathcal{A}(\mathcal{C})$ stable, and thus define elements $m(V), m(W) \in \mathcal{M}(M_{\text{ob}_+ \mathcal{C}} \mathcal{A}(\mathcal{C}))$. Moreover the map $m : M_{\text{ob}_+ \mathcal{C}} \mathcal{A}(\mathcal{C}) \rightarrow \mathcal{M}(M_{\text{ob}_+ \mathcal{C}} \mathcal{A}(\mathcal{C}))$ is injective, since $M_{\text{ob}_+ \mathcal{C}} \mathcal{A}(\mathcal{C})$ has local units. The composite $q = M_{\text{ob}_+ \mathcal{C}}(p) \circ j$ sends $f \in \mathcal{A}(\mathcal{C})$ to

$$q(f) = W(f \otimes e_{+,+})V$$

All this together with matrix invariance imply that $E_*(q) = E_*(? \otimes e_{+,+})$ is an isomorphism [3, 2.2.6]. This proves the first assertion of the Lemma. To prove the second, it suffices to show that $r = j \circ p$ is homotopic to the inclusion $\iota(a) = a \otimes e_{+,+}$. If $f \in \text{hom}_{\mathcal{C}}(a, b)$, write $H(f) \in M_{\text{ob}_+ \mathcal{C}}(\mathcal{R}(\mathcal{C}))[t]$ for

$$H(f) = f \otimes (-t(t^3 - 2t)e_{+,+} + t(t^2 - 1)e_{+,a} + (1 - t^2)(t^3 - 2t)e_{b,+} + (1 - t^2)^2 e_{b,a})$$

Note that $\text{ev}_0 H(f) = r(f)$, $\text{ev}_1 H(f) = \iota(f)$. Further, one checks that if $g \in \text{hom}_{\mathcal{C}}(b, c)$, then $H(gf) = H(g)H(f)$. Thus H induces a homomorphism $\mathcal{R}(\mathcal{C}) \rightarrow M_{\text{ob}_+ \mathcal{C}}(\mathcal{R}(\mathcal{C}))[t]$ which is a homotopy from r to ι . This concludes the proof. \square

EXAMPLE 3.2.11. Let R, S be unital rings, and let \mathcal{C} be the \mathbb{Z} -linear category with two objects a and b such that $\text{hom}_{\mathcal{C}}(a, b) = \text{hom}_{\mathcal{C}}(b, a) = 0$, $\text{hom}_{\mathcal{C}}(a, a) = R$ and $\text{hom}_{\mathcal{C}}(b, b) = S$. Then $\mathcal{A}(\mathcal{C}) = R \oplus S$ and $\mathcal{R}(\mathcal{C}) = R \coprod S$ is the nonunital coproduct. By Lemma 3.2.10, any excisive, matrix stable, homotopy invariant functor $E : \mathbf{Ring} \rightarrow \mathbf{Spt}$ sends $p : R \coprod S \rightarrow R \oplus S$ to a weak equivalence. We remark that the hypothesis on E are necessary; in particular there are functors $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ which satisfy the standing assumptions, and which do not send p to a weak equivalence.

3. K-theory

3.1. The K -theory spectrum. Given a \mathbb{Z} -linear category \mathcal{C} , we denote by \mathcal{C}_{\oplus} the \mathbb{Z} -linear category whose objects are finite sequences of objects of \mathcal{C} , and whose morphisms are matrices of morphisms in \mathcal{C} with the obvious matrix product as composition. Concatenation of sequences yields a sum \oplus and hence we obtain, functorially, an additive category; write $\text{Idem } \mathcal{C}_{\oplus}$ for its idempotent completion. We shall also need Karoubi's cone $\Gamma(\mathcal{C})$ ([18, pp 270]). The objects of $\Gamma(\mathcal{C})$ are the sequences $x = (x_1, x_2, \dots)$ of objects of \mathcal{C} such that the set

$$(85) \quad F(x) = \{c \in \mathcal{C} : (\exists n) \quad x_n = c\}$$

is finite. A map $x \rightarrow y$ in $\Gamma(\mathcal{C})$ is a matrix $f = (f_{i,j})$ of homomorphisms $f_{i,j} : x_j \rightarrow y_i$ such that

- (1) There exists an N such that every row and every column of f has at most N nonzero entries.
- (2) The set $\{f_{i,j} : i, j \in \mathbb{N}\}$ is finite.

Interspersing of sequences defines a symmetric monoidal operation $\boxplus : \Gamma(\mathcal{C}) \times \Gamma(\mathcal{C}) \rightarrow \Gamma(\mathcal{C})$ and there is an endofunctor τ such that $1 \boxplus \tau \cong \tau$ (see [17, § III]). If \mathcal{C} has finite direct sums, e.g. if $\mathcal{C} = \mathcal{D}_\oplus$ for some \mathbb{Z} -linear category \mathcal{D} , then the interspersing operation is naturally equivalent to the induced sum $(x \oplus y)_i = x_i \oplus y_i$ ([17, Lemme 3.3]). In particular, if \mathcal{C} is additive, then $\Gamma\mathcal{C}$ is a **flasque** additive category; that is, there is an additive endofunctor $\tau : \mathcal{C} \rightarrow \mathcal{C}$ such that $\tau \oplus 1 \cong \tau$. A morphism f in $\Gamma(\mathcal{C})$ is **finite** if $f_{ij} = 0$ for all but finitely many (i, j) . Finite morphisms form an ideal, and we write $\Sigma(\mathcal{C})$ for the category with the same objects as $\Gamma(\mathcal{C})$, and morphisms taken modulo the ideal of finite morphisms. The category $\Sigma(\mathcal{C})$ is Karoubi's **suspension** of \mathcal{C} . By [27, Thm. 5.3], if \mathcal{C} is additive, we have a homotopy fibration sequence

$$(86) \quad K^Q(\text{Idem } \mathcal{C}) \rightarrow K^Q(\Gamma(\text{Idem } \mathcal{C})) \rightarrow K^Q(\Sigma(\text{Idem } \mathcal{C}))$$

Here each of the categories is regarded as a semisimple exact category, and K^Q denotes the fibrant simplicial set for its algebraic K -theory. Because $\Gamma(\text{Idem } \mathcal{C})$ is flasque, $K^Q(\Gamma(\text{Idem } \mathcal{C}))$ is contractible, whence $K^Q(\text{Idem } \mathcal{C}) \cong \Omega K^Q(\Sigma(\text{Idem } \mathcal{C}))$. Now let \mathcal{C} be any small \mathbb{Z} -linear category, possibly without direct sums. Consider the sequence of categories

$$(87) \quad \mathcal{C}^{(0)} = \text{Idem}(\mathcal{C}_\oplus), \quad \mathcal{C}^{(n+1)} = \text{Idem}(\Sigma \mathcal{C}^{(n)})$$

Then we have a spectrum $K(\mathcal{C}) = \{ {}_n K(\mathcal{C}) \}$, with

$$(88) \quad {}_n K(\mathcal{C}) \cong K^Q(\mathcal{C}^{(n)})$$

REMARK 3.3.1. If R is a unital ring, then by [18, Prop. 1.6], we have category equivalences

$$(89) \quad \text{Idem}(\Gamma(\text{proj}(R))) \cong \text{proj}(\Gamma(R)) \text{ and } \text{Idem}(\Sigma(\text{proj}(R))) \cong \text{proj}(\Sigma(R))$$

Hence the spectrum $K(R)$ defined above is equivalent to the usual, Gersten-Karoubi-Wagoner spectrum of the ring R .

LEMMA 3.3.2. Let \mathcal{C} be an additive category, and let \bullet be the only object of $\Gamma(\mathbb{Z})$. Consider the functor

$$\begin{aligned} \mu : \Gamma\mathbb{Z} \otimes \mathcal{C} &\rightarrow \Gamma(\mathcal{C}) \\ \mu(\bullet, c) &= (c, c, \dots), \quad \mu(f \otimes \alpha)_{ij} = f_{ij} \alpha \end{aligned}$$

Then

- i) The functor μ is fully faithful.
- ii) Let $F(-)$ be as in (85). For every object $x \in \Gamma(\mathcal{C})$ there exist morphisms $\phi_c : \mu(\bullet, c) \rightarrow x$ and $\psi_c : x \rightarrow \mu(\bullet, c)$, $c \in F(x)$ such that $\sum_{c \in F(x)} \phi_c \psi_c = 1_x$.
- iii) The functor μ induces a fully faithful functor $\bar{\mu} : \Sigma \otimes \mathcal{C} \rightarrow \Sigma(\mathcal{C})$.

PROOF. Part i) is proved in [5, Lemma 4.7.1] for the case when \mathcal{C} has only one object; the same argument applies in general. To prove ii), let $x \in \Gamma(\mathcal{C})$ be an object. If $c \in F(x)$, write $I(c) = \{n \in \mathbb{N} : x_n = c\}$, and let $\chi_{I(c)}$ be the characteristic function. Put

$$\phi_c : \mu(\bullet, c) \rightarrow x, \quad \psi_c : x \rightarrow \mu(\bullet, c), \quad (\phi_c)_{i,j} = (\psi_c)_{i,j} = \delta_{i,j} \chi_{I(c)}(j) 1_c$$

One checks that

$$\sum_{c \in F(x)} \phi_c \psi_c = 1_x$$

This proves ii). Next, consider the exact sequence

$$0 \rightarrow M_\infty \mathbb{Z} \rightarrow \Gamma \mathbb{Z} \xrightarrow{\pi} \Sigma \mathbb{Z} \rightarrow 0$$

As is explained in [5, pp 92], it follows from results of Nöbeling [26] that the sequence above is split as a sequence of abelian groups. Hence if $c, d \in \mathcal{C}$, then

$$\ker(\pi \otimes 1 : \text{hom}_{\Gamma \mathbb{Z} \otimes \mathcal{C}}((\bullet, c), (\bullet, d)) \rightarrow \text{hom}_{\Sigma \mathbb{Z} \otimes \mathcal{C}}((\bullet, c), (\bullet, d))) = M_\infty \mathbb{Z} \otimes \text{hom}_{\mathcal{C}}(c, d)$$

Next observe that if $\alpha \in \text{hom}_{\mathcal{C}}(c, d)$ and $f \in M_\infty \mathbb{Z}$, then $\mu(f \otimes \alpha)$ is a finite morphism. Hence μ passes to the quotient, inducing a functor $\bar{\mu} : \Sigma \mathbb{Z} \otimes \mathcal{C} \rightarrow \Sigma(\mathcal{C})$. If $c, d \in \text{B } \mathcal{C}$ and we put $x = \mu(\bullet, c)$, $y = \mu(\bullet, d)$ then we have a map of exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow M_\infty \mathbb{Z} \otimes \text{hom}_{\mathcal{C}}(c, d) & \longrightarrow & \Gamma \mathbb{Z} \otimes \text{hom}_{\mathcal{C}}(c, d) & \longrightarrow & \Sigma \mathbb{Z} \otimes \text{hom}_{\mathcal{C}}(c, d) & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow \text{hom}_{\text{Fin}(\mathcal{C})}(x, y) & \longrightarrow & \text{hom}_{\Gamma(\mathcal{C})}(x, y) & \longrightarrow & \text{hom}_{\Sigma(\mathcal{C})}(x, y) & \rightarrow & 0 \end{array}$$

Here $\text{Fin}(\mathcal{C}) \subset \Gamma(\mathcal{C})$ is the subcategory of finite morphisms. The second vertical map is an isomorphism by part i). In particular the first map is injective; furthermore, one checks that it is onto. It follows that the third vertical map is an isomorphism; this proves iii). \square

3.2. Comparing $K(\mathcal{C})$ with $K(\mathcal{A}(\mathcal{C}))$.

The operation \diamond . Let X be a set and let \mathcal{C} and \mathcal{D} be \mathbb{Z} -linear categories with $\text{ob } \mathcal{C} = \text{ob } \mathcal{D} = X$. Consider the category $\mathcal{C} \diamond \mathcal{D}$ with set of objects $\text{ob}(\mathcal{C} \diamond \mathcal{D}) = X$, homomorphisms

$$\text{hom}_{\mathcal{C} \diamond \mathcal{D}}(x, y) = \text{hom}_{\mathcal{C}}(x, y) \oplus \text{hom}_{\mathcal{D}}(x, y)$$

and coordinate-wise composition. If \mathcal{C} , \mathcal{D} and \mathcal{E} are \mathbb{Z} -linear categories, we have

$$\begin{aligned} (\mathcal{C} \diamond \mathcal{D})_\oplus &= \mathcal{C}_\oplus \diamond \mathcal{D}_\oplus \\ (90) \quad \text{Idem}((\mathcal{C} \diamond \mathcal{D})_\oplus) &= \text{Idem } \mathcal{C}_\oplus \times \text{Idem } \mathcal{D}_\oplus \\ (91) \quad (\mathcal{C} \diamond \mathcal{D}) \otimes \mathcal{E} &= (\mathcal{C} \otimes \mathcal{E}) \diamond (\mathcal{D} \otimes \mathcal{E}) \end{aligned}$$

Unitalization. We have already recalled the definition of the unitalization \tilde{A} of a not necessarily unital ring A . Now we need a version of unitalization for \mathbb{Z} -linear categories; this can be more generally defined for nonunital \mathbb{Z} -categories, but we will have no occasion for that. Let $\mathcal{C} \in \mathbb{Z} - \mathbf{Cat}$; write $\tilde{\mathcal{C}}$ for the category with $\text{ob } \tilde{\mathcal{C}} = \text{ob } \mathcal{C}$ and with homomorphisms given by

$$\text{hom}_{\tilde{\mathcal{C}}}(x, y) = \text{hom}_{\mathcal{C}}(x, y) \oplus \delta_{x,y} \mathbb{Z} = \begin{cases} \text{hom}_{\mathcal{C}}(x, y) & x \neq y \\ \text{hom}_{\mathcal{C}}(x, x) \oplus \mathbb{Z} & x = y \end{cases}$$

Composition between $(f, \delta_{x,y} n) \in \text{hom}_{\tilde{\mathcal{C}}}(x, y)$ and $(g, \delta_{y,z} m) \in \text{hom}_{\tilde{\mathcal{C}}}(y, z)$ is defined by the formula

$$(g, \delta_{y,z} m) \circ (f, \delta_{x,y} n) = (gf + \delta_{y,z} mf + \delta_{x,y} gn, \delta_{x,y} \delta_{y,z} mn)$$

Observe that if R is a ring, considered as a \mathbb{Z} -linear category with one object, then

$$\tilde{R} \rightarrow R \times \mathbb{Z} = R \diamond \mathbb{Z}, \quad (r, n) \mapsto (r + n \cdot 1, n)$$

is an isomorphism. This isomorphism generalizes to \mathbb{Z} -categories as follows. Let $\mathbb{Z}\langle \text{ob } \mathcal{C} \rangle \in \mathbb{Z} - \mathbf{Cat}$, be the \mathbb{Z} -linear category with the same objects as \mathcal{C} , homomorphisms given by

$$\text{hom}_{\mathbb{Z}\langle \text{ob } \mathcal{C} \rangle}(x, y) = \delta_{x, y} \mathbb{Z}$$

We have an isomorphism of linear categories

$$(92) \quad \mathcal{C} \diamond \mathbb{Z}\langle \text{ob } \mathcal{C} \rangle \rightarrow \tilde{\mathcal{C}}$$

which is the identity on objects, as well as on $\text{hom}_{\mathcal{C} \diamond \mathbb{Z}\langle \text{ob } \mathcal{C} \rangle}(x, y)$ for $x \neq y$, and which sends

$$\text{hom}_{\mathcal{C} \diamond \mathbb{Z}\langle \text{ob } \mathcal{C} \rangle}(x, x) \ni (f, n) \mapsto (f - n1_x, n) \in \text{hom}_{\tilde{\mathcal{C}}}(x, x)$$

The map $K(\mathcal{C}) \rightarrow K(\mathcal{A}(\mathcal{C}))$. If \mathcal{C} is a \mathbb{Z} -linear category, and $x, y \in \text{ob } \mathcal{C}$, then by definition of $\mathcal{A}(\mathcal{C})$,

$$(93) \quad \text{hom}_{\mathcal{C}}(x, y) \subset \mathcal{A}(\mathcal{C})$$

and the inclusion is compatible with composition. We also have an inclusion

$$(94) \quad \text{hom}_{\tilde{\mathcal{C}}}(x, x) \ni (f, n) \mapsto (f, n) \in \widetilde{\mathcal{A}(\mathcal{C})}$$

The inclusions (93) and (94) together with the only map $\text{ob } \tilde{\mathcal{C}} \rightarrow \text{ob } \widetilde{\mathcal{A}(\mathcal{C})} = \{\bullet\}$ define a functor

$$(95) \quad \phi : \tilde{\mathcal{C}} \rightarrow \widetilde{\mathcal{A}(\mathcal{C})}$$

Observe that $\mathbb{Z}\langle \text{ob } \mathcal{C} \rangle \subset \tilde{\mathcal{C}}$ and that $\phi(\mathbb{Z}\langle \text{ob } \mathcal{C} \rangle) \subset \mathbb{Z} \subset \widetilde{\mathcal{A}(\mathcal{C})}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{\phi} & \widetilde{\mathcal{A}(\mathcal{C})} \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathbb{Z}\langle \text{ob } \mathcal{C} \rangle & \longrightarrow & \mathbb{Z} \end{array}$$

Here the vertical maps are the obvious projections. By (92) and (90) we have an equivalence

$$K(\tilde{\mathcal{C}}) \xrightarrow{\sim} K(\mathcal{C}) \times K(\mathbb{Z}\langle \text{ob } \mathcal{C} \rangle)$$

Under this equivalence the map induced by π_1 becomes the canonical projection; hence its fiber is $K(\mathcal{C})$. On the other hand, by definition, $K(\mathcal{A}(\mathcal{C}))$ is the fiber of $K(\pi_2)$. Hence ϕ induces a map

$$(96) \quad \varphi : K(\mathcal{C}) \rightarrow K(\mathcal{A}(\mathcal{C}))$$

PROPOSITION 3.3.3. Let \mathcal{C} be a \mathbb{Z} -linear category. Then the map (96) is an equivalence.

PROOF. Because both the source and the target of (96) commute with filtering colimits, we may assume that \mathcal{C} has finitely many objects. Then $\mathcal{A}(\mathcal{C})$ is unital, and thus we have an isomorphism $\widetilde{\mathcal{A}(\mathcal{C})} \cong \mathcal{A}(\mathcal{C}) \times \mathbb{Z}$. Recall that the idempotent completion of an additive category \mathfrak{A} is the category whose objects are the idempotent endomorphisms in \mathfrak{A} and where a map $f : e_1 \rightarrow e_2$ is an element of $\text{hom}_{\mathfrak{A}}(\text{dome}_1, \text{dome}_2)$ such that $f = e_2 f e_1$. One checks that the composite

$$\begin{aligned} \mathcal{C}_{\oplus} &\rightarrow \text{Idem } \mathcal{C}_{\oplus} \xrightarrow{1 \times 0} \text{Idem } \mathcal{C}_{\oplus} \times \text{Idem } \mathbb{Z} \langle \text{ob } \mathcal{C} \rangle \cong \\ &\text{Idem}(\widetilde{\mathcal{C}_{\oplus}}) \xrightarrow{\phi} \text{Idem}(\widetilde{\mathcal{A}(\mathcal{C})}_{\oplus}) \cong \text{Idem}(\mathcal{A}(\mathcal{C})_{\oplus}) \times \text{Idem}(\mathbb{Z}_{\oplus}) \rightarrow \text{Idem}(\mathcal{A}(\mathcal{C})_{\oplus}) \end{aligned}$$

is the functor ψ which sends an object (c_1, \dots, c_n) to the idempotent $\text{diag}(1_{c_1}, \dots, 1_{c_n})$ and a map $f = (f_{i,j}) : (c_1, \dots, c_n) \rightarrow (d_1, \dots, d_m)$ to the corresponding matrix $(f_{i,j}) \in \text{hom}_{\mathcal{A}(\mathcal{C})_{\oplus}}(\bullet^n, \bullet^m)$. Because ψ is fully faithful and cofinal, it induces an equivalence $K(\mathcal{C}) \rightarrow K(\mathcal{A}(\mathcal{C}))$. It follows that (96) is an equivalence. \square

4. K-theory and the standing assumptions

In this section we prove some technical result to see that K-theory satisfy the standing assumptions.

4.1. The groups $\text{Tor}_*^{\tilde{A}}(-, A)$. Let $M = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}$. Theorems of Suslin [28] (for $M = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$) and Suslin-Wodzicki (for $M = \mathbb{Q}$) establish that a ring A is excisive for K-theory with coefficients in M if and only if

$$\text{Tor}_*^{\tilde{A}}(M, A) = 0$$

EXAMPLE 3.4.1. A ring A is said to have the **triple factorization property** if for every finite family $a_1, \dots, a_n \in A$ there exist $b_1, \dots, b_n, c, d \in A$ such that

$$a_i = cdb_i \text{ and } \{a_i : a_i d = 0\} = \{a : acd = 0\} \quad i = 1 \dots n$$

It was proved in [29, Theorem C] that rings having the triple factorization property are K-excisive. In particular, s -unital rings are K-excisive.

We shall introduce, for any abelian group M , a functorial abelian group $\bar{Q}(A, M)$ which computes $\text{Tor}_*^{\tilde{A}}(M, A)$. Consider the functor $\perp : \tilde{A} - \text{mod} \rightarrow \tilde{A} - \text{mod}$,

$$\perp M = \bigoplus_{m \in M} \tilde{A}.$$

The functor \perp is the free \tilde{A} -module cotriple [33, 8.6.6]. Let $Q(A) \rightarrow A$ be the canonical simplicial resolution by free \tilde{A} -modules associated to \perp [33, 8.7.2]; by definition, its n -th term is $Q_n(A) = \perp^n A$. Put

$$\bar{Q}(A, M) = M \otimes_{\tilde{A}} Q(A).$$

We have

$$\pi_*(\bar{Q}(A, M)) = \text{Tor}_*^{\tilde{A}}(M, A)$$

We abbreviate $\bar{Q}(A) = \bar{Q}(\mathbb{Z}, A)$. Note that

$$\bar{Q}(A, M) = M \otimes \bar{Q}(A)$$

We have

$$\bar{Q}_0(A) = \mathbb{Z}[A], \quad \bar{Q}_{n+1} = \mathbb{Z}[\tilde{A} \otimes \bar{Q}_n(A)].$$

LEMMA 3.4.2. Let $F \xrightarrow{\sim} A$ be a simplicial resolution in **Rings** and M an abelian group. Let $\text{diag } \bar{Q}(F)$ be the diagonal of the bisimplicial abelian group $\bar{Q}(F)$. Then

$$\text{Tor}_*^{\tilde{A}}(M, A) = \pi_*(M \otimes \text{diag } \bar{Q}(F))$$

PROOF. Because $F \rightarrow A$ is a simplicial resolution in **Rings**, $\bar{Q}_0(F) = \mathbb{Z}[F] \rightarrow \mathbb{Z}[A] = \bar{Q}_0(A)$ is a free simplicial resolution in **Ab** of the free abelian group $\mathbb{Z}[A]$. Observe that if $G \rightarrow N$ is a free resolution of a free abelian group N , then $\tilde{A} \otimes G \rightarrow \tilde{A} \otimes N$ is a free simplicial \tilde{A} -module resolution, and $\mathbb{Z}[\tilde{A} \otimes G] \rightarrow \mathbb{Z}[\tilde{A} \otimes N]$ is a free simplicial \mathbb{Z} -module resolution. Thus for each n , $\bar{Q}_n(F) \rightarrow \bar{Q}_n(A)$ is an equivalence of free simplicial abelian groups, and thus it remains an equivalence after tensoring by M . It follows that $M \otimes \text{diag } \bar{Q}(F)$ computes $\text{Tor}_*^{\tilde{A}}(M, A)$. \square

PROPOSITION 3.4.3. Let $F \xrightarrow{\sim} A$ be a simplicial resolution and M an abelian group. Then there is a first quadrant spectral sequence

$$E_{p,q}^2 = \pi_q(\text{Tor}_p^{\tilde{F}}(M, F)) \Rightarrow \text{Tor}_{p+q}^{\tilde{A}}(M, A)$$

PROOF. This is just the spectral sequence of the bisimplicial abelian group $([p], [q]) \mapsto \bar{Q}_p(M, F_q)$. \square

COROLLARY 3.4.4. Let $F \xrightarrow{\sim} A$ be free simplicial a resolution in **Rings**. Then

$$\pi_*(M \otimes (F/F^2)) = \text{Tor}_*^{\tilde{A}}(M, A)$$

PROOF. In view of the previous proposition, and of the fact that $\text{Tor}_0^{\tilde{B}}(M, B) = M \otimes B/B^2$ for every ring B , it suffices to show that if V is a free abelian group, and TV the tensor algebra, then $\text{Tor}_n^{TV}(M, TV) = 0$ for $n \geq 1$. But this is clear, since TV is free as a \tilde{TV} -module; indeed, the multiplication map $\tilde{TV} \otimes V \rightarrow TV$ is an isomorphism. \square

4.2. Bar complex. Let A be a ring. Consider the complex $P(A)$ given by $P_n(A) = \tilde{A} \otimes A^{\otimes n+1}$ ($n \geq 0$), with boundary map

$$b''(a_{-1} \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=-1}^{n-1} (-1)^i a_{-1} \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

The multiplication map $\mu : \tilde{A} \otimes A \rightarrow A$ gives a surjective quasi-isomorphism $\mu : P(A) \twoheadrightarrow A$ [33, 8.6.12]. A canonical \mathbb{Z} -linear section of μ is $j = 1 \otimes - : A \rightarrow \tilde{A} \otimes A$. Let $\epsilon : \tilde{A} \rightarrow A$, $\epsilon(a, n) = a$. A \mathbb{Z} -linear homotopy $j\mu \rightarrow 1$ is defined by

$$s : A^{\otimes n+1} \rightarrow A^{\otimes n+2}, \quad s(a_{-1} \otimes \cdots \otimes a_n) = 1 \otimes \epsilon(a_{-1}) \otimes a_0 \otimes \cdots \otimes a_n$$

Thus $P(A)$ is a resolution of A by \tilde{A} -modules, and moreover these \tilde{A} -modules are scalar extensions of \mathbb{Z} -modules. Hence if A is flat as \mathbb{Z} -module, then $C^{bar}(A) = \mathbb{Z} \otimes_{\tilde{A}} P(A)$ computes $\text{Tor}_*^{\tilde{A}}(\mathbb{Z}, M)$ and $M \otimes C^{bar}(A)$ computes $\text{Tor}_*^{\tilde{A}}(\mathbb{Z}, M)$. In general, the homology

of $C^{bar}(A)$ can be interpreted as the Tor groups relative to the extension $\mathbb{Z} \rightarrow \tilde{A}$. For an arbitrary ring A , one can use the natural homotopy s to give a natural map

$$L(A) \rightarrow P(A)$$

The induced map $M \otimes \bar{Q}(A) \rightarrow M \otimes C^{bar}(A)$ is a quasi-homomorphism if A is flat as a \mathbb{Z} -module. In particular, we have the following.

LEMMA 3.4.5. Let $F \xrightarrow{\sim} A$ be a simplicial resolution by flat rings, and M an abelian group. Then

$$\mathrm{Tor}_*^{\tilde{A}}(M, A) = H_*(\mathrm{Tot}(M \otimes C^{bar}(F)))$$

4.3. H -unital rings. Let

$$(97) \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be an exact sequence of rings. We say that (97) is **pure** if for every abelian group V , the sequence of abelian groups

$$0 \rightarrow A \otimes V \rightarrow B \otimes V \rightarrow C \otimes V \rightarrow 0$$

A ring A is called H -unital if for every abelian group V , the complex $C^{bar}(A) \otimes V$ is acyclic. If $X(-)$ is a functorial chain complex, then we say that A is **pure X -excisive** if for every pure exact sequence (97),

$$X(A) \rightarrow X(B) \rightarrow X(C)$$

is a distinguished triangle. The following theorem was proved by M. Wodzicki in 3.4.6.

THEOREM 3.4.6. (Wodzicki) The following conditions are equivalent for a ring A .

- i) A is H -unital.
- ii) A is pure C^{bar} -excisive.
- iii) A is pure HH -excisive.
- iv) A is pure HC -excisive.

EXAMPLE 3.4.7. Any linearly split sequence (97) is pure. In particular, any sequence (97) with A a \mathbb{Q} -algebra is pure, since any \mathbb{Q} -vectorspace is injective as an abelian group. Thus for a \mathbb{Q} -algebra A , Wodzicki's theorem remains valid if we omit the word "pure" everywhere. Furthermore, the Suslin-Wodzicki theorem cited above, for A a \mathbb{Q} -algebra then the conditions of Theorem 3.4.6 are also equivalent to A being $K^{\mathbb{Q}}$ -excisive. In fact it is well-known that for a \mathbb{Q} -algebra A , being $K^{\mathbb{Q}}$ -excisive is equivalent to being K -excisive; as explained in [4, Lemma 4.1] this well-known fact follows from the main result of [31]. See [29, Lemma 1.9] for a different proof.

4.4. Colimits. The bar complex manifestly commutes with filtering colimits, and thus H -unital rings are closed under them. The next proposition establishes the analogue of this property for K -excisive rings.

PROPOSITION 3.4.8. Let $\{A_i\}$ be a filtering system of rings, and let M be an abelian group. Write $A = \mathrm{colim} A_i$. Then

$$\mathrm{Tor}_*^{\tilde{A}}(M, A) = \mathrm{colim}_i \mathrm{Tor}_*^{\tilde{A}_i}(M, A_i)$$

PROOF. Write $\perp: \mathbf{Rings} \rightarrow \mathbf{Rings}$, $\perp B = T(\mathbb{Z}[B])$ for the cotriple associated with the forgetful functor $\mathbf{Rings} \rightarrow \mathbf{Set}$ and its adjoint. Write $F(A) \xrightarrow{\sim} A$ for the cotriple resolution $F(A)_n = \perp^{n+1} A$ ([33, Section 8/6]). We have $F(A) = \operatorname{colim}_i F(A_i)$. Thus $\operatorname{Tot}(M \otimes C^{bar} F(A)) = \operatorname{colim}_i M \otimes C^{bar} F(A_i)$. Hence we are done by Lemma 3.4.5. \square

COROLLARY 3.4.9. K -excisive rings are closed under filtering colimits.

Let M^0 and M^1 be chain complexes of abelian groups, and let $f \in [1]^n$. Put

$$T^f(M^0, M^1) = M^{f(1)} \otimes \dots \otimes M^{f(n)}$$

Let

$$M^0 \star M^1 = \bigoplus_{n \geq 0} \bigoplus_{f \in \operatorname{map}([n], [1])} T^f(M^0, M^1)$$

LEMMA 3.4.10. Let A and B be rings. Then

$$C^{bar}(A \oplus B) = (C^{bar}(A)[-1] \star C^{bar}(B)[-1])[+1]$$

PROOF. If D is a ring then $C^{bar}(D) = T(D[-1])[+1]$ as graded abelian groups. Hence for \coprod the coproduct of rings, we have

$$\begin{aligned} C^{bar}(A \oplus B) &= T(A[-1] \oplus B[-1])[+1] \\ &= (T(A[-1]) \coprod T(B[-1]))[+1] \\ &= (C^{bar}(A)[-1] \star C^{bar}(B)[-1])[+1] \end{aligned}$$

It is straightforward to check that the identifications above are compatible with boundary maps. \square

PROPOSITION 3.4.11. Let $\{A_i\}$ be a family of rings and $A = \bigoplus_i A_i$. Then A is K -excisive if and only if each A_i is, and in that case $\bigoplus_i K(A_i) \rightarrow K(A)$ is an equivalence.

PROOF. Let B and C be rings, and $F \rightarrow B$ and $G \rightarrow C$ be free simplicial resolutions in \mathbf{Rings} . Then $F \oplus G \rightarrow B \oplus C$ is a flat simplicial resolution. Fix $q \geq 0$, and put $C^0 = C^{bar}(F_q)$, $C^1 = C^{bar}(G_q)$. Let $p \geq 1$, and $f \in [1]^p$. Then by the Künneth formula

$$\begin{aligned} H_n(T^f(C^0[-1], C^1[-1])[+1]) &= \\ T^f(H_*(C^0), H_*(C^1))_{n+1} &= \begin{cases} T^f(F/F^2, G/G^2) & p = n + 1 \\ 0 & p \neq n + 1 \end{cases} \end{aligned}$$

Hence the second page of the spectral sequence for the double complex of Lemma 3.4.5 is

$$E_{p,q}^2 = \bigoplus_{f \in [1]^{p+1}} \pi_q(T^f(F/F^2, G/G^2))$$

If B and C are K -excisive, we have $E^2 = 0$, by the Künneth formula, and thus $B \oplus C$ is again K -excisive. It follows from this and from Proposition 3.4.8 that if A_i is a family

of K -excisive rings as in the proposition, then A is K -excisive. If B and C are arbitrary, then

$$E_{0,q}^2 = \mathrm{Tor}_q^{\tilde{B}}(\mathbb{Z}, B) \oplus \mathrm{Tor}_q^{\tilde{C}}(\mathbb{Z}, C)$$

$$E_{p,0}^2 = \bigoplus_{f \in [1]^{p+1}} T^f(A/A^2, B/B^2)$$

Hence if $B \oplus C$ is excisive, $E_{*,0}^2 = 0$. It follows that $E_{0,1}^2 = 0$, and therefore $E_{*,1}^2 = \pi_1(T^f(F/F^2, G/G^2))$ involves only tensor products of the form $E_{p,0}^2 \otimes E_{0,1}^2$ and its symmetric, and both of these are zero. A recursive argument shows that $E^2 = 0$, whence both B and C are K -excisive. If now A and $\{A_i\}$ are as in the proposition, A is excisive, and $j \in I$ then setting $B = A_j$ and $C = \bigoplus_{i \neq j} A_i$ above, we obtain that A_i is K -excisive. The last assertion of the proposition is well-known if each A_i is unital. More generally, assume all A_i are K -excisive, and consider the exact sequence

$$(98) \quad 0 \rightarrow A \rightarrow \bigoplus_i \tilde{A}_i \rightarrow \bigoplus_i \mathbb{Z} \rightarrow 0$$

We have a commutative diagram with homotopy fibration rows

$$\begin{array}{ccccc} \bigoplus_i K(A_i) & \longrightarrow & \bigoplus_i K(\tilde{A}_i) & \longrightarrow & \bigoplus_i K(\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ K(A) & \longrightarrow & K(\bigoplus_i \tilde{A}_i) & \longrightarrow & K(\bigoplus_i \mathbb{Z}) \end{array}$$

Because the middle and right vertical arrows are equivalences, it follows that the left one is an equivalence too. \square

PROPOSITION 3.4.12. Let $\{A_i\}$ be a family of rings and $A = \bigoplus_i A_i$. Then A is H -unital if and only if each A_i is, and in that case $\bigoplus_i HH(A_i) \rightarrow HH(A)$ and $\bigoplus_i HC(A_i) \rightarrow HC(A)$ are quasi-isomorphisms.

PROOF. The last assertion is proved by the same argument as its K -theoretic counterpart. By 3.4.6, if B and C are rings and B is H -unital, then $C^{bar}(B \oplus C) \otimes V \rightarrow C^{bar}(C) \otimes V$ is a quasi-isomorphism for every abelian group V . Thus if also C is H -unital, then so is $B \oplus C$. Using this and the fact that H -unitality is preserved under filtering colimits, it follows that if $\{A_i\}$ is a family of H -unital rings, then $A = \bigoplus_i A_i$ is H -unital. Suppose conversely that A is H -unital, and consider the pure extension (98). A similar argument as that of the proof of Proposition 3.4.11 shows that $\bigoplus_i HH(A_i) \rightarrow HH(A)$ is a quasi-isomorphism. Next fix an index j and let

$$0 \rightarrow A_j \rightarrow B \rightarrow C \rightarrow 0$$

be a pure extension. Then

$$0 \rightarrow A \rightarrow \bigoplus_{i \neq j} A_i \oplus \tilde{B} \rightarrow \bigoplus_{i \neq j} A_i \oplus \tilde{C} \rightarrow 0$$

is a pure extension. Applying HH yields a distinguished triangle quasi-isomorphic to

$$\bigoplus_i HH(A_i) \rightarrow \bigoplus_{i \neq j} A_i \oplus HH(B) \oplus HH(\mathbb{Z}) \rightarrow \bigoplus_{i \neq j} A_i \oplus HH(C) \oplus HH(\mathbb{Z})$$

Removing summands, we obtain a triangle

$$HH(A_j) \rightarrow HH(B) \rightarrow HH(C)$$

We have shown that A_j satisfies excision for pure extensions in Hochschild homology; by 3.4.6, this implies that A_j is H -unital. \square

4.5. Tensor products. It was proved by Suslin and Wodzicki [29, Theorem 7.10] that the tensor product of H -unital rings is H -unital. Here we establish a weak analogue of this property for K -excisive rings.

Let A be a ring. Put

$$L_{-1}A = A, \quad L_{n+1}A = \ker(A \otimes L_n(A) \rightarrow L_n(A)) \quad (n \geq -1)$$

LEMMA 3.4.13. Let A be a K -excisive ring, and V an abelian group. Assume both A and V are flat over \mathbb{Z} . Then

$$\mathrm{Tor}_n^{\widetilde{A \otimes TV}}(\mathbb{Z}, A \otimes TV) = L_{n-1}A \otimes V^{\otimes n+1} \quad n \geq 0$$

PROOF. If M is a left A -module such that

$$(99) \quad A \cdot M = M,$$

and $L(M) = \ker(A \otimes M \rightarrow M)$ is the kernel of the multiplication map, then we have a short exact sequence

$$0 \rightarrow L(M) \otimes T^{\geq n+1}V \rightarrow \widetilde{A \otimes TV} \otimes M \otimes V^{\otimes n} \rightarrow M \otimes T^{\geq n}V \rightarrow 0$$

By definition, $L_nA = L^{n+1}A$. By [29, Theorem 7.8 and Lemma 7.6], $M = L_nA$ satisfies (99) for all n , and moreover, it is a flat abelian group, by induction. Thus for $n \geq 1$, the sequence

$$0 \rightarrow L_{n-1}(M) \otimes T^{\geq n+1}V \rightarrow \widetilde{A \otimes TV} \otimes L_{n-2}M \otimes V^{\otimes n} \rightarrow L_{n-2}M \otimes T^{\geq n}V \rightarrow 0$$

is exact. Hence

$$\begin{aligned} \mathrm{Tor}_i^{\widetilde{A \otimes TV}}(\mathbb{Z}, A \otimes TV) &= \mathrm{Tor}_i^{\widetilde{A \otimes TV}}(\mathbb{Z}, L_{-1}A \otimes T^{\geq 1}V) \\ &= \mathrm{Tor}_0^{\widetilde{A \otimes TV}}(\mathbb{Z}, L_{i-1}A \otimes T^{\geq i+1}V) \\ &= L_{i-1}A \otimes V^{\otimes i+1} \end{aligned}$$

\square

PROPOSITION 3.4.14. Let A and B be K -excisive rings, at least one of them flat as a \mathbb{Z} -module. Then $A \otimes B$ is K -excisive.

PROOF. Assume A is flat. Let $F \xrightarrow{\sim} B$ be a simplicial resolution by free rings. Then $A \otimes F \xrightarrow{\sim} A \otimes B$ is a resolution by flat rings. By Lemma 3.4.13, the second page of the spectral sequence of Proposition 3.4.3 is

$$E_{p,q}^2 = \pi_q(A \otimes (F/F^2)^{\otimes p+1}) = A \otimes (\pi_q((F/F^2)^{\otimes p+1}))$$

which equals zero by Corollary 3.4.4 and the Künneth formula, since B is K -excisive by assumption. \square

4.6. Crossed products. Let G be a group and $\pi : \mathbb{Z}[G] \rightarrow \mathbb{Z}$ the augmentation $g \mapsto 1$. Put

$$JG = \ker \pi$$

LEMMA 3.4.15. Let V be a $\mathbb{Z}[G]$ -module, free as an abelian group. Then

$$\mathrm{Tor}_n^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV \rtimes G) = V^{\otimes n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G] \quad n \geq 0$$

PROOF. Note that the subset

$$V^{\otimes n} \oplus TV^{\geq n+1} \rtimes G \subset TV \rtimes G$$

is a left ideal, and that the map

$$(100) \quad \widetilde{TV \rtimes G} \otimes V^{\otimes n} \rightarrow V^{\otimes n} \oplus TV^{\geq n+1} \rtimes G$$

$$1 \otimes y \mapsto y$$

$$x \rtimes g \otimes y \mapsto xg(y) \rtimes g$$

(101)

is a $\widetilde{TV \rtimes G}$ -module isomorphism. Consider the map

$$V^{\otimes n} \otimes M \oplus (TV^{\geq n+1} \rtimes G) \otimes M \rightarrow TV^{\geq n}V \otimes M, \quad (x, (y \rtimes g) \otimes m) \mapsto x + y \otimes gm$$

Composing with the isomorphism (100), we obtain a \mathbb{Z} -split surjective map

$$\widetilde{TV \rtimes G} \otimes V^{\otimes n} \twoheadrightarrow TV^{\geq n} \otimes M$$

This map fits in an exact sequence

$$0 \rightarrow T^{\geq n+1}V \otimes JG \otimes M \rightarrow \widetilde{TV \rtimes G} \otimes V^{\otimes n} \rightarrow T^{\geq n}V \otimes M \rightarrow 0$$

If M is flat as an abelian group, then the middle term in the exact sequence above is a flat $\widetilde{TV \rtimes G}$ -module. Applying this successively, starting with $M = \mathbb{Z}[G]$, we obtain

$$\begin{aligned} \mathrm{Tor}_n^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV \rtimes G) &= \mathrm{Tor}_0^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV^{\geq n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G]) \\ &= V^{\otimes n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G] \end{aligned}$$

□

PROPOSITION 3.4.16. Let G be a group and $A \in G - \mathbf{Rings}$. Assume A is K -excisive. Then $A \rtimes G$ is K -excisive.

PROOF. Note that the forgetful functor from $G - \mathbf{Rings}$ to sets has a left adjoint; namely $X \mapsto T(\mathbb{Z}[G \times X])$. Hence A admits a free resolution $F \xrightarrow{\sim} A$ such that each F_n is a G -ring; for example we may take the cotriple resolution associated to the adjoint pair just described. Since F is a simplicial G -ring, we can take its crossed product with G , to obtain a \mathbb{Z} -flat resolution $F \rtimes G \xrightarrow{\sim} A \rtimes G$. Now proceed as in the proof of Proposition 3.4.14, using Lemma 3.4.15. □

PROPOSITION 3.4.17. Let G be a group and $A \in G - \mathbf{Rings}$. Assume A is H -unital. Then $A \rtimes G$ is H -unital.

PROOF. The bar resolution $B(G, M)$ ([33, S6.5]) is functorial on the G -module M . Applying it dimensionwise to $C^{bar}(A)$, we obtain a simplicial chain complex $B(G, C^{bar}(A))$. We may view the latter as a double chain complex with $A^{\otimes q+1} \otimes \mathbb{Z}[G^{p+1}]$ in the (p, q) spot. Removing the first row and the first column yields a double complex whose total chain complex we shall call $M[-1]$. Note M is a chain complex of $A \rtimes G$ -modules and homomorphisms. We have $M_0 \cong (A \rtimes G)^{\otimes 2}$, and the multiplication map $(A \rtimes G)^{\otimes 2} \rightarrow A \rtimes G$ induces a surjection onto the kernel L of the augmentation $A \rtimes G \rightarrow A$, $a \rtimes g \rightarrow a$. Note that the hypothesis that A is H -unital implies that the augmented complex

$$\dots \rightarrow M_1 \rightarrow M_0 \rightarrow L$$

is acyclic. Now proceed as in the proof of [29, Theorem 7.10]. \square

PROPOSITION 3.4.18. The functor $K : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ satisfies the standing assumptions.

PROOF. Assumption iv) was proved in Proposition 3.3.3. By Example 3.4.1, s -unital rings are K -excisive; hence K -theory satisfies i). Assumption ii) holds by Proposition 3.4.16. If A is K -excisive and X is a set, then $M_X A$ is K -excisive, by Proposition 3.4.14. Assumption iii) follows from this and the fact that K -theory is matrix stable on unital rings. Assumption v) is proved in Proposition 3.4.11. \square

5. Homotopy K-theory

If \mathcal{C} is a \mathbb{Z} -linear category, then we write $\mathcal{C}^{\Delta^\bullet}$ for the simplicial \mathbb{Z} -linear category

$$\mathcal{C}^{\Delta^\bullet} : [n] \mapsto \mathcal{C}^{\Delta^n} = \mathcal{C} \otimes \mathbb{Z}[t_0, \dots, t_n] / \langle t_0 + \dots + t_n - 1 \rangle$$

Applying the functor K dimensionwise we get a simplicial spectrum whose total spectrum is the homotopy K -theory spectrum $KH(\mathcal{C})$. In particular if R is a unital ring, then $KH(R)$ was defined by Weibel in [32]. The following theorem was proved in [32]; see also [3, Section 5].

THEOREM 3.5.1. (Weibel) The functor $KH : \mathbf{Ring} \rightarrow \mathbf{Spt}$ is excisive, matrix invariant, and invariant under polynomial homotopy.

PROPOSITION 3.5.2. There is a natural weak equivalence $KH(\mathcal{C}) \xrightarrow{\sim} KH(\mathcal{R}(\mathcal{C}))$.

PROOF. We begin by observing that the inclusions (93) and (94) lift to inclusions $\mathrm{hom}_{\mathcal{C}}(x, y) \subset \mathcal{R}(\mathcal{C})$ and $\mathrm{hom}_{\tilde{\mathcal{C}}}(x, x) \subset \widetilde{\mathcal{R}(\mathcal{C})}$. Thus we have a functor

$$\phi' : \tilde{\mathcal{C}} \rightarrow \widetilde{\mathcal{R}(\mathcal{C})}$$

Composing it with

$$\tilde{p} : \widetilde{\mathcal{R}(\mathcal{C})} \rightarrow \widetilde{\mathcal{A}(\mathcal{C})}$$

we obtain the map

$$\phi : \tilde{\mathcal{C}} \rightarrow \widetilde{\mathcal{A}(\mathcal{C})}$$

of (95) above. Tensoring with $\mathbb{Z}^{\Delta^\bullet}$ and applying $K(-)$ we obtain a commutative diagram

$$\begin{array}{ccc} KH(\tilde{\mathcal{C}}) & \longrightarrow & KH(\widetilde{\mathcal{R}(\mathcal{C})}) \\ & \searrow \phi & \downarrow p \\ & & KH(\widetilde{\mathcal{A}(\mathcal{C})}) \end{array}$$

The diagram above maps to the diagram

$$\begin{array}{ccc} KH(\mathbb{Z}\langle \text{ob } \mathcal{C} \rangle) & \xrightarrow{\phi} & KH(\mathbb{Z}) \\ & \searrow \phi & \parallel \\ & & KH(\mathbb{Z}) \end{array}$$

Taking fibers and using (90), (91) and (92), we obtain a homotopy commutative diagram

$$\begin{array}{ccc} KH(\mathcal{C}) & \xrightarrow{\varphi''} & KH(\mathcal{R}(\mathcal{C})) \\ & \searrow \varphi' & \downarrow p \\ & & KH(\mathcal{A}(\mathcal{C})) \end{array}$$

Here φ' comes from a map of simplicial spectra

$$\varphi^\bullet : K(\mathcal{C} \otimes \mathbb{Z}^{\Delta^\bullet}) \xrightarrow{\sim} K(\tilde{\mathcal{C}} \otimes \mathbb{Z}^{\Delta^\bullet} : \mathbb{Z}\langle \text{ob } \mathcal{C} \rangle \otimes \mathbb{Z}^{\Delta^\bullet}) \rightarrow K(\widetilde{\mathcal{A}(\mathcal{C})} \otimes \mathbb{Z}^{\Delta^\bullet} : \mathbb{Z} \otimes \mathbb{Z}^{\Delta^\bullet}),$$

and $\varphi^0 = \varphi$ is the map (96), which is an equivalence by Proposition 3.3.3. The same argument of the proof of Proposition 3.3.3 shows that φ^n is an equivalence for every n . On the other hand, by Theorem 3.5.1 and Lemma 3.2.10, the map $p : KH(\mathcal{R}(\mathcal{C})) \rightarrow KH(\mathcal{A}(\mathcal{C}))$ is an equivalence. It follows that φ'' is an equivalence too. \square

PROPOSITION 3.5.3. The functor $KH : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ satisfies the standing assumptions.

PROOF. This follows from Theorem 3.5.1, Proposition 3.5.2, and Lemma 3.2.10. \square

6. Proper G-rings

6.1. Extending polynomial functions and excision properties.

LEMMA 3.6.1. If X is a locally finite simplicial set, then $\mathbb{Z}^{(X)}$ is a free abelian group.

PROOF. By [5, 3.1.3] the lemma is true when X is finite. Hence if X is any simplicial set, and $\sigma \in X$ is a simplex, then $\mathbb{Z}^{<\sigma>}$ is free. If X locally finite, then by Example 2.2.4, $\mathbb{Z}^{(X)}$ is a subgroup of a free group, and therefore it is free. \square

THEOREM 3.6.2. Let X be a simplicial set, $Y \subset X$ a sub-simplicial set and A a ring. Let $\phi \in A^Y$ and $K = \text{supp } \phi$. There there exists $\psi \in A^X$ with $\text{supp } \psi \subset \overline{\text{St}}_X K$ such that $\psi|_{\text{Link}_X(K)} = 0$ and $\psi|_Y = \phi$.

PROOF. We have $K \subset \text{St}_Y K \subset \overline{\text{St}}_Y K$, whence $\phi|_{\text{Link}_Y(K)} = 0$. Note $\text{St}_X K \cap Y = \text{St}_Y K$; thus ϕ vanishes on $\text{Link}_X(K) \cap Y$. Hence we may extend ϕ to a $\phi' : Y' = Y \cup \text{Link}_X(K) \rightarrow A^{\Delta^\bullet}$ by $\phi'|_{\text{Link}_X(K)} = 0$. Put $Y'' = Y \cup \overline{\text{St}}_X K$. Because $Y' \subset Y''$ is a cofibration and $A^{\Delta^\bullet} \rightarrow 0$ is a trivial fibration, we may further extend ϕ' to a $\phi'' : Y'' \rightarrow A^{\Delta^\bullet}$. By construction, $\{\sigma \in X : \phi''(\sigma) \neq 0\} \subset \text{St}_X K$, and ϕ'' vanishes on $\text{Link}_X K$. Hence we may further extend ϕ'' to a $\psi : X \rightarrow A^{\Delta^\bullet}$, by letting $\psi(\sigma) = 0$ if $\sigma \notin \overline{\text{St}}_X K$. This concludes the proof. \square

COROLLARY 3.6.3. If X is locally finite and $Y \subset X$ is a subsimplicial set, then the restriction map $A^{(X)} \rightarrow A^{(Y)}$ is surjective.

PROOF. It follows from Theorem 3.6.2, using 2.2.1. \square

PROPOSITION 3.6.4. (Compare [7, Lemma 2.5]) Let A be a nonzero ring. The following are equivalent for a simplicial set X .

- i) For every simplex $\sigma \in X$ there exists $\phi \in A^{(X)}$ such that $\phi(\sigma) \neq 0$.
- ii) X is locally finite.

PROOF. Observe that if $\sigma, \tau \in X$ are simplices, with $\langle \tau \rangle \supset \langle \sigma \rangle$ and $\phi \in A^X$ satisfies $\phi(\sigma) \neq 0$, then $\phi(\tau) \neq 0$. If X is not locally finite, then by Lemma 2.2.1, there exists a simplex $\sigma \in X$ which is contained in infinitely many nondegenerate simplices. By the previous observation, $\phi(\sigma) = 0$ for every $\phi \in A^{(X)}$. We have proved that i) \Rightarrow ii). Assume conversely that X is locally finite, and let σ be a simplex of X . We want to show that there exists $\phi \in A^{(X)}$ such that $\phi(\sigma) \neq 0$. We may assume that σ is nondegenerate. Let $Y = \langle \sigma \rangle \subset X$ be the sub-simplicial set generated by σ ; by Corollary 3.6.3, it suffices to show that $A^Y \neq 0$. Now Y is an n -dimensional quotient of Δ^n , whence $S^n = \Delta^n / \partial \Delta^n$ is a quotient of Y . So we may further reduce to showing A^{S^n} is nonzero. Now

$$A^{S^n} = Z_n A^{\Delta^\bullet} = \bigcap_{i=0}^n \ker(d_i : A^{\Delta^n} \rightarrow A^{\Delta^{n-1}})$$

But if $0 \neq a \in A$, then $at_0 \dots t_n$ is a nonzero element of $Z_n A^{\Delta^\bullet}$. \square

PROPOSITION 3.6.5. If X is a locally finite simplicial set, then $\mathbb{Z}^{(X)}$ is s -unital.

PROOF. Let $\phi_1, \dots, \phi_n \in \mathbb{Z}^{(X)}$, and let $K = \bigcup_i \text{supp}(\phi_i)$. By Theorem 3.6.2 there is $\mu \mathbb{Z}^{(X)}$ such that $\mu|_K = 1$ is the constant map. Thus

$$(6.1) \quad \phi_i = \phi_i \mu \quad (\forall i).$$

\square

PROPOSITION 3.6.6. If A is K -excisive and X is locally finite, then $\mathbb{Z}^{(X)} \otimes A$ is K -excisive.

PROOF. Follows from Lemma 3.6.1 and Propositions 3.6.5 and 3.4.14. \square

REMARK 3.6.7. If A is a ring and X a locally finite simplicial set, then there is a natural map

$$\mathbb{Z}^{(X)} \otimes A \rightarrow A^{(X)}$$

It was proved in [5, 3.1.3] that this map is an isomorphism if X is finite.

6.2. Proper rings over a G-simplicial set. Fix a group G and consider rings equipped with an action of G by ring automorphisms. We write $G\text{-}\mathbf{Rings}$ for the category of such rings and equivariant ring homomorphisms. If $C \in G\text{-}\mathbf{Rings}$ is commutative but not necessarily unital and $A \in G\text{-}\mathbf{Rings}$, then by a compatible (G, C) -algebra structure on A we understand a C -bimodule structure on A such that the following identities hold for $a, b \in A$, $c \in C$, and $g \in G$:

$$\begin{aligned} c \cdot a &= a \cdot c \\ (6.2) \quad c \cdot (ab) &= (c \cdot a)b = a(c \cdot b) \\ g(c \cdot a) &= g(c) \cdot g(a) \end{aligned}$$

(6.3)

If X is a G -simplicial set and $A \in G\text{-}\mathbf{Rings}$, then we say that A is **proper** over X if it carries a compatible $(G, \mathbb{Z}^{(X)})$ algebra structure such that

$$(6.4) \quad \mathbb{Z}^{(X)} \cdot A = A$$

If \mathcal{F} is a family of subgroups of G , we say that A is (G, \mathcal{F}) -**proper** if it is proper over some (G, \mathcal{F}) complex X .

EXAMPLE 3.6.8. Fix a group G , a family of subgroups \mathcal{F} and a (G, \mathcal{F}) -complex X . By Proposition 3.6.5, we have $\mathbb{Z}^{(X)} \cdot \mathbb{Z}^{(X)} = \mathbb{Z}^{(X)}$; thus $\mathbb{Z}^{(X)}$ is proper. If A is proper over X , and B is any ring, then $A \otimes B$ is proper over X . In particular, $\mathbb{Z}^{(X)} \otimes B$ is proper.

Let A be a G -ring, proper over a locally finite G -simplicial set X . We write A_X for the unitalization of A as an algebra over $\mathbb{Z}^{(X)}$; this is the abelian group

$$(6.5) \quad A_X = A \oplus \mathbb{Z}^{(X)}$$

equipped with the multiplication law given by the formula 79.

LEMMA 3.6.9. Let A be a G -ring, proper over a locally finite G -simplicial set X . Then the ring A_X of (6.5) is s -unital.

PROOF. Immediate from Proposition 3.6.5 and condition (6.4). \square

Let X be a locally finite simplicial set, and $K \subset X$ a subobject. Put

$$I(K) = \{\phi : \text{supp}\phi \subset K\} \triangleleft \mathbb{Z}^{(X)}$$

If $A \in \mathbf{Ring}$ has a compatible $\mathbb{Z}^{(X)}$ structure, we put

$$A(K) = I(K) \cdot A \triangleleft A$$

LEMMA 3.6.10. Let A be a G -ring. Assume that A is (G, \mathcal{F}) -proper. Then $\{A_i\}$ of ideals of A such that $A = \cup_i A_i$ and such that each A_i is proper over a finite (G, \mathcal{F}) -complex.

PROOF. By hypothesis, there exists a (G, \mathcal{F}) -complex X such that A is proper over X . For each G -finite (G, \mathcal{F}) -subcomplex $K \subset X$, consider $I(K)$ and $A(K)$. It is clear that $\{I(K)\}$ and $\{A(K)\}$ are filtering systems of ideals and that $\cup_K I(K) = \mathbb{Z}^{(X)}$. We claim furthermore that $A = \cup_K A(K)$. By definition of $\mathbb{Z}^{(X)}$ -algebra, $A = \mathbb{Z}^{(X)} \cdot A$. Hence if $a \in A$, then there exist $\phi_1, \dots, \phi_n \in \mathbb{Z}^{(X)}$ and $a_1, \dots, a_n \in A$ such that $a = \sum_i \phi_i a_i$. Hence $a \in A(K)$ for $K = \cup_i G \cdot \text{supp}(\phi_i)$. \square

LEMMA 3.6.11. (cf. [11, pp. 51]) Let $A \in \mathbf{G-Rings}$ be proper over a locally finite G -simplicial set X , and let $f : X \rightarrow Y$ be an equivariant map with Y locally finite. Then the map $f^* : \mathbb{Z}^Y \rightarrow \mathbb{Z}^X$ induces a compatible $(G, \mathbb{Z}^{(Y)})$ -algebra structure on A which makes it proper over Y .

PROOF. We begin by showing that the compatible $(G, \mathbb{Z}^{(X)})$ -algebra structure on A extends to a compatible (G, \mathbb{Z}^X) -module structure. By the lemma above, if $a \in A$ then there exists a finite subsimplicial set $K \subset X$ such that $a \in A(K) = I(K) \cdot A$. By Theorem 3.6.2 there exists $\mu_K \in \mathbb{Z}^X$, with $\text{supp} \mu_K \subset \overline{\text{St}}(K)$ such that

$$(6.6) \quad \mu_K a = a \quad \forall a \in A(K).$$

Because X is locally finite, $\overline{\text{St}}(K)$ is finite and $\mu_K \in \mathbb{Z}^{(X)}$. Thus we have a map $A(K) \rightarrow I(\overline{\text{St}}(K)) \otimes A(K)$, $a \mapsto \mu_K \otimes a$. Now $I(\overline{\text{St}}(K))$ is an ideal in \mathbb{Z}^X by (38); using the multiplication of \mathbb{Z}^X we obtain a map

$$(6.7) \quad \mathbb{Z}^X \otimes A(K) \rightarrow A(\overline{\text{St}}(K)), \quad \phi \otimes a \mapsto (\phi \cdot \mu_K) a.$$

If $L \supset K$, and we choose an element μ_L as above, then for $a \in A(K)$ and $\phi \in \mathbb{Z}^X$ we have:

$$(\phi \cdot \mu_L) \cdot a = (\phi \cdot \mu_L) \cdot (\mu_K \cdot a) = (\phi \cdot \mu_K) a$$

This shows that (6.7) is independent of the choice of the element μ_K of (6.6), and that we have a well-defined action $\mathbb{Z}^X \otimes A \rightarrow A$. Compatibility with the G -action follows from the fact that $g \cdot \mu_K$ is the identity on $g \cdot K$. The remaining compatibility conditions are immediate. Now A becomes an $\mathbb{Z}^{(Y)}$ -module through f^* . If $K \subset X$ is a finite subsimplicial set, then $L = f(K) \subset Y$ is finite, and since Y is locally finite, there is a $\mu_L \in \mathbb{Z}^{(Y)}$ which is the identity on L , and thus $f^*(\mu_L)$ is the identity on K . It follows that the action of $\mathbb{Z}^{(Y)}$ on A satisfies (6.4). The remaining $(G, \mathbb{Z}^{(Y)})$ -compatibility conditions of (6.2) are straightforward. \square

If $C, A \in \mathbf{H-Rings}$ with C commutative and we have a compatible (H, C) -algebra structure on A , then $\text{Ind}_H^G(A)$ carries a compatible $(G, \text{Ind}_H^G(C))$ -algebra structure, given by

$$\xi_H(s, c) \cdot \xi_H(t, a) = \begin{cases} \xi_H(s, c \cdot a) & s = t \\ 0 & sH \neq tH \end{cases}$$

If moreover $C \cdot A = A$, then $\text{Ind}_H^G(C) \cdot \text{Ind}_H^G(A) = \text{Ind}_H^G(A)$. We record a particular case of this in the following

LEMMA 3.6.12. If $A \in \mathbf{H-Rings}$ is proper over an H -simplicial set X , then the G -ring $\text{Ind}_H^G(A)$ is proper over $\text{Ind}_H^G(X)$.

6.3. Compression. Let $A \in \mathbf{G-Rings}$, and $H \subset G$ a subgroup. Assume that A is proper over G/H . Let $\chi_H \in \mathbb{Z}^{(G/H)}$ be the characteristic function of H . The **compression** of A over H is the subring

$$\text{Comp}_H^G(A) = \chi_H \cdot A$$

Note the action of G on A restricts to an action of H on $\text{Comp}_H^G(A)$, which makes it into an object of $\mathbf{H-Rings}$.

PROPOSITION 3.6.13. (Compare [11] Lemma 12.3, and paragraph after 12.4)

i) If $B \in \mathbf{H-Rings}$, then $\mathrm{Ind}_H^G(B)$ is proper over G/H , and

$$B \rightarrow \mathrm{Comp}_H^G \mathrm{Ind}_H^G B, \quad b \mapsto \xi_H(1, b)$$

is an H -equivariant isomorphism.

ii) If $A \in \mathbf{G-Rings}$ is proper over G/H , then

$$\mathrm{Ind}_H^G \mathrm{Comp}_H^G(A) \rightarrow A, \quad \xi_H(s, \chi_H a) \mapsto \chi_{sH} s(a)$$

is a G -equivariant isomorphism.

PROOF. Any $B \in \mathbf{H-Rings}$ is proper over the 1-point space $*$. Hence $\mathrm{Ind}_H^G(B)$ is proper over $\mathrm{Ind}_H^G(*) = G/H$, by Lemma 3.6.12. The proof that the maps of i) and ii) are isomorphisms is straightforward; to show equivariance, one uses (44) and (46). \square

7. Induction and equivariant homology

LEMMA 3.7.1. Let G be a group, $K \subset G$ a subgroup, A a K -ring, and $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ a functor satisfying the standing assumptions. Then A is E -excisive if and only if $\mathrm{Ind}_K^G(A)$ is E -excisive.

PROOF. The map (47) gives a nonequivariant isomorphism

$$\mathrm{Ind}_K^G(A) \cong \mathbb{Z}^{(G/K)} \otimes A = \bigoplus_{x \in G/K} A$$

The equivalence of the lemma follows from assumption v). \square

Let G , K and A be as in the Lemma 3.7.1, and let X be a G -simplicial set. If A is unital, then for each subgroup $S \subset K$ we have a functor

$$\begin{aligned} A \rtimes \mathcal{G}^K(K/S) &\rightarrow \mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/S) \\ kS &\mapsto kS, \\ a \rtimes k &\mapsto \xi_K(1, a) \rtimes k \end{aligned}$$

If A is any E -excisive ring, the map above is defined for the unitalization \tilde{A} ; applying E , taking fibers relative to the augmentation $\tilde{A} \rightarrow \mathbb{Z}$, and using the standing assumptions, we get a map

$$E(A \rtimes \mathcal{G}^K(K/S)) \rightarrow E(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/S)).$$

The maps

$$X_+^S \wedge E(A \rtimes \mathcal{G}^K(K/S)) \rightarrow X_+^S \wedge E(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/S)) \rightarrow H^G(X, E(\mathrm{Ind}_K^G A))$$

assemble to

$$(7.1) \quad \mathrm{Ind} : H^K(X, E(A)) \rightarrow H^G(X, E(\mathrm{Ind}_K^G(A)))$$

PROPOSITION 3.7.2. (Compare [11, Proposition 12.9]) Let A be an E -excisive G -ring. Then the map (7.1) is an equivalence.

PROOF. Because equivariant homology satisfies excision on X [8, Section 8], and because X is obtained by gluing together cells of the form $\text{Ind}_H^G(\Delta^n)$, $H \in \mathcal{All}$, it suffices to prove the proposition for $X = \text{Ind}_H^G(T)$ where H acts trivially on T . Let \mathcal{R} be a full set of representatives of $K \backslash G/H$. We have

$$\begin{aligned} \text{Ind}_H^G(T) &= T \times G/H \\ &= \coprod_{\theta \in \mathcal{R}} T \times K \theta H \\ &\cong \coprod_{\theta \in \mathcal{R}} T \times K / K_\theta \end{aligned}$$

Here as in (49), $K_\theta = c_\theta(H) \cap K$. Thus

$$H^K(\text{Ind}_H^G(T), E(A)) = T_+ \wedge \bigvee_{\theta \in \mathcal{R}} E(A \rtimes \mathcal{G}^K(K / K_\theta))$$

On the other hand,

$$H^G(\text{Ind}_H^G(T), E(\text{Ind}_K^G(A))) = T_+ \wedge E(\text{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H))$$

We have to show that

$$\bigvee_{\theta \in \mathcal{R}} E(A \rtimes \mathcal{G}^K(K / K_\theta)) \rightarrow E(\text{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H))$$

is an equivalence. By standing assumptions iv) and v) we may replace the map above by that induced by the corresponding ring homomorphism

$$(7.2) \quad \bigoplus_{\theta \in \mathcal{R}} \mathcal{A}(A \rtimes \mathcal{G}^K(K / K_\theta)) \rightarrow \mathcal{A}(\text{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H))$$

Here $\mathcal{A}(A \rtimes \mathcal{G}^K(K / K_\theta)) \rightarrow \mathcal{A}(\text{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H))$ is induced by $\xi_K(1, -) : A \rightarrow \text{Ind}_K^G(A)$ and by the inclusions $K \subset G$ and $K / K_\theta \rightarrow G/H$, $kK_\theta \mapsto k\theta H$. One checks that the following diagram commutes

$$\begin{array}{ccc} & \mathcal{A}(\text{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H)) & \\ \nearrow \xi_K(1, -) \rtimes \text{inc} & & \searrow \sim \quad 3.2.3 \\ \mathcal{A}(A \rtimes \mathcal{G}^K(K / K_\theta)) & & M_{G/H}(\text{Ind}_K^G(A) \rtimes H) \\ \uparrow & \xrightarrow{\xi_K(\theta^{-1}, -) \rtimes c_{\theta^{-1}}} & \uparrow e_{\theta H, \theta H} \\ A \rtimes K_\theta & \xrightarrow{\quad} & \text{Ind}_K^G(A)[H\theta^{-1}K] \rtimes H \\ \downarrow \wr \quad 1 \rtimes c_{\theta^{-1}} & & \sim \uparrow 2.2.11 \\ c_\theta^*(A) \rtimes H_{\theta^{-1}} & \xrightarrow{e_{H_{\theta^{-1}}, H_{\theta^{-1}}}} M_{H/H_{\theta^{-1}}}(c_\theta^*(A) \rtimes H_{\theta^{-1}}) \xrightarrow{\sim \quad 2.3.1} & \text{Ind}_{H_{\theta^{-1}}}^H(c_\theta^*(A)) \rtimes H \end{array}$$

Because the lower rectangle commutes, $E(A \rtimes K_\theta \rightarrow \text{Ind}_K^G(A)[H\theta K] \rtimes H)$ is an equivalence, by matrix stability. Again by matrix stability and by Lemma 3.2.3, applying E to the

top left vertical arrow is an equivalence. Hence to prove that E applied to (7.2) is an equivalence, it suffices to show that E applied to

$$(7.3) \quad \text{Ind}_K^G(A) \rtimes H = \bigoplus_{\theta \in \mathcal{R}} \text{Ind}_K^G(A)[H\theta K] \rtimes H \xrightarrow{\sum_{\theta} e_{\theta H, \theta H}} M_{G/H}(\text{Ind}_K^G(A) \rtimes H)$$

is one. But another application of matrix stability (using Remark 1.7.15) shows that (7.3) induces the same map in HoSpt as the inclusion

$$e_{H,H} : \text{Ind}_K^G(A) \rtimes H \rightarrow M_{G/H}(\text{Ind}_K^G(A) \rtimes H).$$

This concludes the proof. \square

THEOREM 3.7.3. Let $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ be a functor satisfying the standing assumptions 3.2.5. Also let G be a group, \mathcal{F} a family of subgroups of G and B an E -excisive ring, proper over a 0-dimensional (G, \mathcal{F}) -complex X . Then $H^G(-, E(B))$ maps (G, \mathcal{F}) -equivalences to equivalences. In particular, the assembly map

$$H^G(\mathcal{E}(G, \mathcal{F}), E(B)) \rightarrow E(B \rtimes G)$$

is an equivalence.

PROOF. We have $X = \coprod_i G/K_i$ for some $K_i \in \mathcal{F}$, and $\mathbb{Z}^{(X)} = \bigoplus_i \mathbb{Z}^{(G/K_i)}$. The ring $B_i = \mathbb{Z}^{(G/K_i)} \cdot B$ is proper over G/K_i , and is excisive by standing assumption v). Again by standing assumption v), it suffices to prove the assertion of the theorem individually for each B_i ; in other words, we may assume $X = G/K$ for some $K \in \mathcal{F}$. Hence for $A = \text{Comp}_G^K B$ we have $B = \text{Ind}_K^G A$, by Proposition 3.6.13. Moreover, by Lemma 3.7.1, A is E -excisive. Let $Y \rightarrow Z$ be a (G, \mathcal{F}) -equivalence. We have a commutative diagram

$$\begin{array}{ccc} H^G(Y, E(B)) & \longrightarrow & H^G(Z, E(B)) \\ \text{Ind} \uparrow & & \uparrow \text{Ind} \\ H^K(Y, E(A)) & \longrightarrow & H^K(Z, E(A)) \end{array}$$

The bottom horizontal arrow is an equivalence because $K \in \mathcal{F}$. The two vertical arrows are equivalences by Proposition 3.7.2. It follows that the top horizontal arrow is an equivalence too. \square

8. Assembly as a connecting map

Throughout this section, we consider a fixed functor $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$, and – except when otherwise stated – we assume that, in addition to the standing assumptions, it satisfies the following:

SECTIONAL ASSUMPTIONS 3.8.1.

- vi) E_* commutes with filtering colimits.
- vii) If A is E -excisive and L has local units and is flat as a \mathbb{Z} -module, then $L \otimes A$ is E -excisive.

8.1. Preliminaries.

Mapping cones. Let $f : A \rightarrow B$ be a ring homomorphism, the **mapping cone** of f is defined as the pullback

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & \Gamma B \\ \downarrow & & \downarrow \\ \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \end{array}$$

LEMMA 3.8.2. Let $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ be a functor satisfying both the standing and the sectional assumptions, and $f : A \rightarrow B$ a homomorphism of strongly E-excise rings. Then

- i) $E(\Gamma B)$ is weakly contractible.
- ii) $E(\Sigma B) \xrightarrow{\sim} \Sigma E(B)$.
- ii) Assume that Γ_f is E-excise. Then the following is a distinguished triangle in \mathbf{HoSpt}

$$E(B) \rightarrow E(\Gamma_f) \rightarrow \Sigma E(A) \rightarrow \Sigma E(B)$$

PROOF. By Lemma 3.3.2, $\Gamma B = \Gamma \mathbb{Z} \otimes B$, whence it is E-excise. Part i) follows from matrix stability and the fact that $\Gamma \mathbb{Z}$ is a ring with infinite sums (see e.g. [3, Prop. 2.3.1]). Parts ii) and iii) follow from i) and excision. \square

Matrix rings and group actions.

LEMMA 3.8.3. Let G be a group, A a G -ring and X a G -set. Write $M_{\underline{X}}$ for the ring M_X equipped with the G -action

$$g(e_{x,y}) = e_{gx,gy}$$

The map

$$(M_{\underline{X}}A) \rtimes G \rightarrow M_X(A \rtimes G), (e_{x,y} \otimes a) \rtimes g \mapsto e_{x,g^{-1}y} \otimes (a \rtimes g)$$

is a G -equivariant isomorphism of rings.

8.2. Dirac extensions. Let G be a group, \mathcal{F} a family of subgroups, $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ a functor satisfying the standing assumptions, and A an E-excise ring. A **Dirac extension** for (G, \mathcal{F}, A, E) consists of an extension of E-excise G -rings

$$(8.1) \quad 0 \rightarrow B \rightarrow Q \rightarrow P \rightarrow 0$$

together with a zig-zag

$$A = Z_0 \xrightarrow{f_0} Z_1 \xleftarrow{f_2} Z_2 \xrightarrow{f_3} \cdots \quad Z_n = B$$

such that

- a) $E(f_i \rtimes H)$ is an equivalence for every subgroup $H \subset G$.
- b) $E_*(Q \rtimes H) = 0$ for every $H \in \mathcal{F}$.
- c) The assembly map $H^G(\mathcal{E}(G, \mathcal{F}), E(P)) \rightarrow E(P \rtimes G)$ is an equivalence.

PROPOSITION 3.8.4. Let $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ be a functor satisfying the standing assumptions, G a group, \mathcal{F} a family of subgroups of G , and A a G -ring. Let (8.1) be a Dirac extension for (G, \mathcal{F}, A, E) . Then there are an exact sequence

$$E_{*+1}(A \rtimes G) \rightarrow E_{*+1}(Q \rtimes G) \rightarrow E_{*+1}(P \rtimes G) \xrightarrow{\partial} E_*(A \rtimes G)$$

an isomorphism $H_*^G(\mathcal{E}(G, \mathcal{F}), E(A)) \cong E_{*+1}(P \rtimes G)$, and a commutative diagram

$$\begin{array}{ccc} H_*^G(\mathcal{E}(G, \mathcal{F}), E(A)) & \xrightarrow{\text{Assembly}} & E_*(A \rtimes G) \\ & \searrow \cong & \nearrow \partial \\ & E_{*+1}(P \rtimes G) & \end{array}$$

PROOF. Condition a) together with standing assumptions iii) and iv) and Lemma 3.2.3 imply that the zig-zag $f = \{f_i\}$ induces an equivalence

$$H^G(X, E(A)) \xrightarrow{\sim} H^G(X, E(B))$$

for every G -space X . Hence by Proposition 3.2.6 we have a distinguished triangle

$$H^G(X, E(A)) \longrightarrow H^G(X, E(Q)) \longrightarrow H^G(X, E(P)) \xrightarrow{\partial^X} \Sigma H^G(X, E(A))$$

The proposition follows by comparison of the long exact sequence of homotopy associated to the triangles for $X = \mathcal{E}(G, \mathcal{F})$, and $X = *$, and noting that condition b) implies that $H_*^G(\mathcal{E}(G, \mathcal{F}), E(Q)) = 0$. \square

8.3. A canonical Dirac extension. Let G be a group and \mathcal{F} a family of subgroups. Consider the discrete G -simplicial sets

$$X = X_{\mathcal{F}} = \coprod_{H \in \mathcal{F}} G/H, \quad Y = G/G \coprod X$$

The group G acts on Y and thus on the ring M_Y of $Y \times Y$ -matrices with finitely many nonzero integral coefficients. The point y_0 corresponding to the unique orbit of G/G is fixed by G , whence the map $\iota : \mathbb{Z} \rightarrow M_Y$, $\lambda \mapsto \lambda e_{y_0, y_0}$ is G -equivariant. In particular we have a directed system of G -rings $\{\text{id} \otimes \iota : (M_{\infty} M_Y)^{\otimes n} \rightarrow (M_{\infty} M_Y)^{\otimes n+1}\}_n$. Put

$$\mathfrak{F}^0 = \text{colim}_n (M_{\infty} M_Y)^{\otimes n}$$

Since X is discrete, the ring of finitely supported functions breaks up into a sum

$$\mathbb{Z}^{(X)} = \bigoplus_{x \in X} \mathbb{Z} \chi_x$$

Multiplication by an element of M_Y gives an \mathbb{Z} -linear endomorphism of $\mathbb{Z}^{(Y)}$. This defines a monomorphism

$$M_Y \rightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}^{(Y)})$$

whose image consists of those linear transformations T such that the matrix of T with respect to the basis $\{\chi_y : y \in Y\}$ has finitely many nonzero entries. Note that multiplication by χ_x in $\mathbb{Z}^{(X)} \subset \mathbb{Z}^{(Y)}$ is in this image. Thus we have an injective ring homomorphism

$$\rho : \mathbb{Z}^{(X)} \rightarrow M_Y$$

For each $n \geq 1$, consider the G -ring

$$\mathfrak{F}^n = \left(\bigotimes_{i=1}^n \Gamma_{\rho} \right) \otimes \mathfrak{F}^0$$

Note that $\mathfrak{F}^n = (\bigotimes_{i=1}^n \Gamma_\rho) \otimes \mathfrak{F}^0 = (\bigotimes_{i=1}^n \Gamma_\rho) \otimes M_\infty M_{\underline{Y}} \otimes \mathfrak{F}^0$. The inclusion $M_\infty M_{\underline{Y}} \rightarrow \Gamma_\rho$ induces an inclusion $\mathfrak{F}^n \subset \mathfrak{F}^{n+1}$ for each $n \geq 0$. Put

$$\mathfrak{F}^\infty = \bigcup_{n \geq 0} \mathfrak{F}^n$$

If $A \in \mathbf{Rings}$, we also write $\mathfrak{F}^n A = \mathfrak{F}^n \otimes A$ ($n \geq 0$). We have

LEMMA 3.8.5.

- i) $\mathfrak{F}^n \subset \mathfrak{F}^\infty$ is an ideal ($n < \infty$).
- ii) For each $n \geq 0$, \mathfrak{F}^n and $\mathfrak{F}^{n+1}/\mathfrak{F}^n \cong \Sigma \mathbb{Z}^{(X)} \otimes \mathfrak{F}^n$ have local units, and are (G, \mathcal{F}) -proper rings and are flat as abelian groups.
- iii) If $H \in \mathcal{F}$, $x \in G/H$, and A is a G -ring, we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{Z}^{(G/H)} \otimes \mathfrak{F}^n A) \rtimes H & \subset & (\mathbb{Z}^{(X)} \otimes \mathfrak{F}^n A) \rtimes H \xrightarrow{(\rho \otimes 1) \times \text{id}} (M_{\underline{Y}} \mathfrak{F}^n A) \rtimes H \\ \chi_x \otimes 1 \uparrow & & \downarrow \wr (3.8.3) \\ \mathfrak{F}^n A \rtimes H & \xrightarrow{e_{x,x} \otimes -} & M_Y(\mathfrak{F}^n A \rtimes H) \end{array}$$

PROOF. Part i) is clear. Because $M_{\underline{Y}}$ is proper over Y , \mathfrak{F}^n is proper over Y for all n , by 3.6.8. Similarly,

$$(8.2) \quad \mathfrak{F}^{n+1}/\mathfrak{F}^n = \Sigma \mathbb{Z}^{(X)} \otimes \mathfrak{F}^n$$

is proper. That \mathfrak{F}^n is flat is clear for $n = 0$; the general case follows by induction, using (8.2). The ring \mathfrak{F}^0 has local units because $M_{\underline{Y}}$ and M_∞ do. To prove that \mathfrak{F}^n has local units for $n \geq 1$, it suffices to show that Γ_ρ does. We may and do identify Γ_ρ with the inverse image of $\Sigma(\rho(\mathbb{Z}^{(X)}))$ under the projection $\pi : \Gamma M_{\underline{Y}} \rightarrow \Sigma M_{\underline{Y}}$; thus

$$\Gamma_\rho = \Gamma \rho(\mathbb{Z}^{(X)}) + M_\infty M_{\underline{Y}} \subset \Gamma M_{\underline{Y}}$$

One checks that if $\phi_1, \dots, \phi_r \in \Gamma_\rho$, then there are finite subsets $F_1 \subset X$ and $F_2 \subset \mathbb{N}$ such that for $y_0 = G/G \in Y$, the element

$$e = 1 \otimes \sum_{x \in F_1} e_{x,x} + \sum_{p \in F_2} e_{p,p} \otimes e_{y_0,y_0} \in \Gamma_\rho$$

satisfies $e^2 = e$ and $e\phi_i = \phi_i e = \phi_i$ for all $i = 1, \dots, r$. This proves part ii); part iii) is straightforward. \square

THEOREM 3.8.6. (Compare [23, Theorem 5.18]) Let $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ be a functor satisfying both the standing and the sectional assumptions. Let G a group, \mathcal{F} a family of subgroups, and A a G -ring. Then

$$\mathfrak{F}^0 A \rightarrow \mathfrak{F}^\infty A \rightarrow \mathfrak{F}^\infty A / \mathfrak{F}^0 A$$

is a Dirac extension for (G, \mathcal{F}, A, E) .

PROOF. The three rings in the extension of the theorem are E -excisive, by Lemma 3.8.5 ii) and the hypothesis on A . The map $E(A) \rightarrow E(\mathfrak{F}^0 A)$ is an equivalence by standing assumption iii) and the assumption that E_* commutes with filtering colimits. Next we prove that the assembly map $H^G(\mathcal{E}(G, \mathcal{F}), E(\mathfrak{F}^\infty A / \mathfrak{F}^0 A)) \rightarrow E(\mathfrak{F}^\infty A / \mathfrak{F}^0 A \rtimes G)$ is an

equivalence. By excision and the hypothesis that E_* commutes with filtering colimits, it suffices to show that

$$(8.3) \quad H^G(\mathcal{E}(G, \mathcal{F}), E(\mathfrak{F}^n A / \mathfrak{F}^0 A)) \rightarrow E(\mathfrak{F}^n A / \mathfrak{F}^0 A \rtimes G)$$

is an equivalence. Consider the extension

$$0 \rightarrow \mathfrak{F}^n A / \mathfrak{F}^0 A \rightarrow \mathfrak{F}^{n+1} A / \mathfrak{F}^0 A \rightarrow \mathfrak{F}^{n+1} A / \mathfrak{F}^n A \rightarrow 0$$

By Proposition 3.2.6, assembly gives a map of homotopy fibration sequences

$$\begin{array}{ccc} H^G(\mathcal{E}(G, \mathcal{F}), E(\mathfrak{F}^n A / \mathfrak{F}^0 A)) & \longrightarrow & E(\mathfrak{F}^n A / \mathfrak{F}^0 A \rtimes G) \\ \downarrow & & \downarrow \\ H^G(\mathcal{E}(G, \mathcal{F}), E(\mathfrak{F}^{n+1} A / \mathfrak{F}^0 A)) & \longrightarrow & E(\mathfrak{F}^{n+1} A / \mathfrak{F}^0 A \rtimes G) \\ \downarrow & & \downarrow \\ H^G(\mathcal{E}(G, \mathcal{F}), E(\mathfrak{F}^{n+1} A / \mathfrak{F}^n A)) & \longrightarrow & E(\mathfrak{F}^{n+1} A / \mathfrak{F}^n A \rtimes G) \end{array}$$

By Lemma 3.8.5 and Theorem 3.7.3, the bottom horizontal map is an equivalence. Hence (8.3) is an equivalence for each n , by induction. It remains to show that $E_*(\mathfrak{F}^\infty A \rtimes H) = 0$ for each $H \in \mathcal{F}$. Because E_* preserves filtering colimits by assumption, we may further restrict ourselves to proving that the map $j_n : E_*(\mathfrak{F}^n A \rtimes H) \rightarrow E_*(\mathfrak{F}^{n+1} A \rtimes H)$ induced by inclusion is zero for all n . By Lemma 3.8.2 we have a long exact sequence ($q \in \mathbb{Z}$)

$$\begin{array}{ccc} E_q(\mathfrak{F}^n A \rtimes H) & \xrightarrow{j_n} & E_q(\mathfrak{F}^{n+1} A \rtimes H) \longrightarrow E_{q-1}(\mathbb{Z}^{(X)} \otimes \mathfrak{F}^n A \rtimes H) \\ & & \downarrow \partial \\ & & E_{q-1}(\mathfrak{F}^n A \rtimes H) \end{array}$$

where $\partial = E_{q-1}(\rho \otimes 1 \rtimes 1)$. By Lemma 3.8.5, part iii), ∂ is a split surjection. It follows that $j_n = 0$; this concludes the proof. \square

EXAMPLE 3.8.7. The hypothesis of Theorem 3.8.6 are satisfied, for example, by the functorial spectra K and KH .

9. Isomorphism conjectures with proper coefficients

9.1. The excisive case.

THEOREM 3.9.1. Let $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ be a functor. Assume that E is excisive, additive and matrix-stable. Let $A \in G - \mathbf{Rings}$ be proper over a locally finite, finite dimensional (G, \mathcal{F}) -complex X . Then the assembly map

$$H^G(\mathcal{E}(G, \mathcal{F}), E(A)) \rightarrow E(A \rtimes G)$$

is a weak equivalence.

PROOF. If $\dim X = 0$, this follows from Theorem 3.7.3. Let $n > 0$ and assume the theorem true in dimensions $< n$. If $\dim X = n$, and $Y \subset X$ is the $n - 1$ -skeleton, we have a pushout diagram

$$\begin{array}{ccc} \coprod_i \operatorname{Ind}_{H_i}^G(\Delta^n) & \longrightarrow & X \\ \uparrow & & \uparrow \\ \coprod_i \operatorname{Ind}_{H_i}^G(\partial\Delta^n) & \longrightarrow & Y \end{array}$$

Here $H_i \in \mathcal{F}$ and the horizontal arrows are proper, since X is assumed locally finite. Hence we obtain a pullback diagram

$$\begin{array}{ccc} \bigoplus_i \mathbb{Z}^{(\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)} & \longleftarrow & \mathbb{Z}^{(X)} \\ \downarrow & & \downarrow \\ \bigoplus_i \mathbb{Z}^{(\partial\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)} & \longleftarrow & \mathbb{Z}^{(Y)} \end{array}$$

Let $I = \ker(\mathbb{Z}^{(X)} \rightarrow \mathbb{Z}^{(Y)})$ be the kernel of the restriction map; because the diagram above is cartesian, $I \cong \bigoplus_i \ker(\mathbb{Z}^{(\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)} \rightarrow \mathbb{Z}^{(\partial\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)})$. The quotient $A/I \cdot A$ is proper over Y , and $I \cdot A$ is proper over $\coprod_i \operatorname{Ind}_{H_i}^G(\Delta^n)$, whence also over the zero-dimensional $\coprod_i G/H_i$, by Lemma 3.6.11. Thus the theorem is true for both $A/I \cdot A$ and $I \cdot A$; because E is excisive by hypothesis, this implies that the theorem is also true for A . \square

THEOREM 3.9.2. Let $E : \mathbb{Z} - \mathbf{Cat} \rightarrow \mathbf{Spt}$ be a functor. Assume that E is excisive and matrix stable, and that E_* commutes with filtering colimits. Let \mathcal{F} be a family of subgroups of a group G . Let $A \in G - \mathbf{Rings}$ be proper over a locally finite (G, \mathcal{F}) -complex X . Then the assembly map

$$H_*^G(\mathcal{E}(G, \mathcal{F}), E(A)) \rightarrow E_*(A \rtimes G)$$

is an isomorphism.

PROOF. By Lemma 3.6.10, we may write A as a filtering colimit $A = \operatorname{colim}_i A_i$ such that each A_i is proper over a finite (G, \mathcal{F}) complex. Because E_* commutes with filtering colimits by hypothesis, we may therefore restrict to the case when X is a finite (G, \mathcal{F}) -complex. Now apply Theorem 3.9.1. \square

EXAMPLE 3.9.3. Homotopy K -theory satisfies the hypothesis of Theorem 3.9.2.

Conclusiones

KK-TEORÍA ALGEBRAICA EQUIVARIANTE

El siguiente diccionario establecido en [5]

KK-teoría de Kasparov	kk -teoría algebraica
K -teoría bivariante en las C^* -álgebras separables C^* -Alg	K -teoría bivariante en las ℓ -álgebras Alg_ℓ
$k : C^*\text{-Alg} \rightarrow KK$	$j : \text{Alg}_\ell \rightarrow \mathfrak{K}\mathfrak{K}$
k es estable respecto a los operadores compactos \mathcal{K}	j es estable respecto a las matrices $M_\infty = \bigcup_{n \in \mathbb{N}} M_n$
k es invariante por homotopías continuas	j es invariante por homotopías polinómicas
k es exacto partido	j es escisivo
k es universal para las propiedades descriptas anteriormente	j es universal para las propiedades descriptas anteriormente
$KK_*(\mathbb{C}, A) \simeq K_*(A)$	$kk_*(\ell, A) \simeq KH_*(A)$

admite la siguiente versión equivariante,

KK -teoría de Kasparov equivariante	kk -teoría algebraica equivariante
K -teoría bivalente en la categoría de G - C^* -álgebras separables G - C^* -Alg	K -teoría en \mathcal{X} -álgebras \mathcal{X} -Alg
$k : G\text{-}C^*\text{-Alg} \rightarrow KK^G$	$j : \mathcal{X}\text{-Alg} \rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{X}}$
k es estable respecto a $\mathcal{K}(\ell^2(G \times \mathbb{N}))$	j es \mathcal{X} -estable
k es invariante por homotopías continuas	j es invariante por homotopías polinomiales
k es exacto partido	j es escisivo
k es universal para las propiedades descriptas anteriormente	j es universal para las propiedades descriptas anteriormente

TEOREMAS DE ADJUNCIÓN EN LA KK -TEORÍA

Supongamos que las \mathcal{X} -álgebras son las G -álgebras siendo G un grupo numerable. La kk -teoría algebraica equivariante satisface los siguientes teoremas análogos a la KK -teoría de Kasparov.

- **Teorema de Green-Julg**

Si G es un grupo finito, $1/|G| \in \ell$, B es un álgebra y A es una G -álgebra entonces existe un isomorfismo

$$\psi_{GJ} : kk^G(B^\tau, A) \rightarrow kk(B, A \rtimes G).$$

Aquí B^τ indica la G -álgebra con la acción trivial de G . En particular tomando $B = \ell$,

$$kk^G(\ell, A) \simeq KH(A \rtimes G) \quad \text{para toda } A \in G\text{-Alg}.$$

- **Adjunción entre la inducción y la restricción**

Sea H un subgrupo de G . Los funtores de inducción y restricción

$$\text{Ind}_H^G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^H \quad \text{Res}_G^H : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^H$$

son funtores adjuntos. En otras palabras,

$$\mathrm{kk}^G(\mathrm{Ind}_H^G(B), A) \simeq \mathrm{kk}^H(B, \mathrm{Res}_G^H(A)) \quad \text{para } A \in G\text{-Alg y } B \in H\text{-Alg}.$$

En particular si A es una G -álgebra,

$$\mathrm{kk}^G(\ell^{(G)}, A) \simeq \mathrm{KH}(A).$$

Aquí $\ell^{(G)} = \bigoplus_{g \in G} \ell$ con la acción regular de G .

- **Teorema de imprimitividad de Green** Si $H \subset G$ es un subgrupo y A es una H -álgebra. Existe un isomorfismo

$$\alpha : \mathrm{Ind}_H^G(A) \rtimes G \rightarrow M_{|G/H|}(A \rtimes H)$$

tal que los siguientes diagramas conmutan

$$\begin{array}{ccc} \mathrm{Ind}_H^G(A) \rtimes G & \xrightarrow{\quad \Upsilon \quad} & \mathrm{End}_{A \rtimes H}(A^{(G)}) \\ & \searrow \alpha & \nearrow \Gamma \\ & M_{|G/H|}(A \rtimes H) & \\ \mathrm{Ind}_H^G(A) \rtimes G & \xrightarrow{\quad \alpha \quad} & M_{|G/H|}(A \rtimes H) \\ & \nwarrow \xi_H(e, -) \times \mathrm{id} & \nearrow e_{H, H} \otimes - \\ & A \rtimes H & \end{array}$$

- **Dualidad de Baaj-Skandalis**

La categoría de G -álgebras $\mathfrak{K}\mathfrak{K}^G$ y la categoría de álgebras G -graduadas $\mathfrak{K}\mathfrak{K}^{\hat{G}}$ son equivalentes. Esta equivalencia está dada por los funtores

$$\rtimes G : \mathfrak{K}\mathfrak{K}^G \rightarrow \mathfrak{K}\mathfrak{K}^{\hat{G}} \quad \hat{G} \rtimes : \mathfrak{K}\mathfrak{K}^{\hat{G}} \rightarrow \mathfrak{K}\mathfrak{K}^G.$$

inducidos por los productos cruzados.

Supongamos que las \mathcal{X} -álgebras son las \mathcal{H} -módulo álgebras siendo \mathcal{H} un algebra de Hopf de dimensión finita sobre un cuerpo. La kk -teoría algebraica equivariante satisface los siguientes teoremas análogos a la KK -teoría de Kasparov.

- **Teorema de Green-Julg** Existen funtores

$$\tau : \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{K}\mathfrak{K}^{\mathcal{H}} \quad \# : \mathfrak{K}\mathfrak{K}^{\mathcal{H}} \rightarrow \mathfrak{K}\mathfrak{K}$$

inducidos por la acción trivial y el producto *smash*. Cuando \mathcal{H} es un álgebra semisimple obtenemos que

$$\psi_{GJ} : \mathrm{kk}^{\mathcal{H}}(B^{\tau}, A) \rightarrow \mathrm{kk}(B, A \# \mathcal{H})$$

es un isomorfismo siendo B una ℓ -álgebra y A una \mathcal{H} -álgebra. En particular si A es un \mathcal{H} -módulo álgebra entonces

$$\mathrm{kk}^{\mathcal{H}}(\ell, A) \simeq \mathrm{KH}(A \# \mathcal{H}).$$

 CONJETURAS DE ISOMORFISMO CON COEFICIENTES PROPIOS

Consideramos estructuras de modelo en la categoría de G -conjuntos simpliciales y en la categoría de G -espacios topológicos. Dichas estructuras estan definidas en función de una familia \mathcal{F} de subconjuntos de G y son tales que las equivalencias débiles y las fibraciones son punto a punto. Los objetos cofibrantes son aquellos X tales que el estabilizador G_x pertenece a la familia \mathcal{F} para todo $x \in X$. Al probar que el siguiente par de funtores es una equivalencia de Quillen

$$\mathbf{Top}^G \begin{array}{c} \xrightarrow{\text{Sing}_*} \\ \xleftarrow{\parallel_*} \end{array} \mathbb{S}^G.$$

obtenemos que es equivalente trabajar con un modelo simplicial de espacio o con un modelo topológico.

Decimos que un funtor $H : \mathbb{S}^G \rightarrow \mathbf{Spt}$ de la categoría de G -conjuntos simpliciales en la categoría de espectros satisface la (G, \mathcal{F}) -conjetura de isomorfismo, si para el reemplazo cofibrante $\pi : \mathcal{E}(G, \mathcal{F}) \rightarrow *$ en la categoría de modelos mencionada anteriormente el morfismo

$$H(\pi) : H(\mathcal{E}(G, \mathcal{F})) \rightarrow H(*)$$

es una equivalencia. Si $E : \mathbb{Z}\text{-Cat} \rightarrow \mathbf{Spt}$ es un funtor y R es un G -anillo con unidad, podemos construir, siguiendo la línea Davis-Luck [8], un funtor

$$H^G(-, E(R)) : \mathbb{S}^G \rightarrow \mathbf{Spt}$$

tal que $H^G(*, E(R)) = E(R \rtimes G)$. La (G, \mathcal{F}, E, R) -conjetura de isomorfismo es la (G, \mathcal{F}) -conjetura para el funtor $H^G(-, E(R))$. Probamos que considerando unas condiciones mínimas sobre E , las **STANDING ASSUMPTIONS 3.2.5** (condiciones que satisfacen K y KH , ver proposiciones 3.4.18 y 3.5.3), $H^G(-, E(A))$ está definido no sólo para los G -anillos con unidad, si no que está definido también para todo G -anillo A que satisface E -escisión. Más aún, probamos que si

$$(9.1) \quad 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

es una sucesión exacta de anillos E -escisivos y X es un G -conjunto simplicial, entonces

$$H^G(X, E(A')) \rightarrow H^G(X, E(A)) \rightarrow H^G(X, E(A''))$$

es una fibración homotópica. Esta es una propiedad necesaria para establecer un análogo algebraico al método dual-Dirac que es un método usado en la prueba de la conjetura de Baum-Connes para algunos grupos. Otra propiedad básica que nos proporciona sucesiones (9.1) en las cuales al menos uno de los anillos satisface la conjetura de isomorfismos, es el teorema 3.7.3. En éste se muestra que si E satisface las *standing assumptions* y A es un anillo E -escisivo de la forma

$$(9.2) \quad A = \bigoplus_i \text{Ind}_{K_i}^G B_i$$

en donde B_i es un K_i -anillo y $K_i \in \mathcal{F}$ para todo i , entonces el funtor $H^G(-, E(A))$ lleva (G, \mathcal{F}) -equivalencias en equivalencias. En particular la (G, \mathcal{F}, E, A) -conjetura de isomorfismo se satisface. Usamos este hecho para mostrar que bajo otros supuestos adicionales

3.8.1 (los cuales siguen siendo verificados por $E = K$ y KH), para cada G -anillo E -escisivo A existe de manera funtorial una sucesión de G -anillos E -escisivos

$$\mathfrak{F}^0 B \rightarrow \mathfrak{F}^\infty B \rightarrow \mathfrak{F}^\infty B / \mathfrak{F}^0 B$$

y un morfismo natural $A \rightarrow \mathfrak{F}^0 B$ tal que

- i) $H^G(X, E(A)) \rightarrow H^G(X, E(\mathfrak{F}^0 B))$ es una equivalencia para todo G -conjunto simplicial X .
- ii) $H^G(X, E(\mathfrak{F}^\infty B)) \rightarrow *$ es una equivalencia si X es (G, \mathcal{F}) -cofibrante.
- iii) $H^G(-, E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B))$ lleva (G, \mathcal{F}) -equivalencias en equivalencias.

A partir de esta sucesión obtenemos que

$$H^G(\mathcal{E}(G, \mathcal{F}), E(A)) \rightarrow E(A \rtimes G)$$

es una equivalencia si y sólo si el morfismo de conexión

$$\Omega(E(\mathfrak{F}^\infty B / \mathfrak{F}^0 B \rtimes G)) \rightarrow E(A \rtimes G)$$

es una equivalencia. En particular lo anterior se aplica cuando $E = K, KH$. En el teorema 3.9.2 probamos que bajo hipótesis más fuertes en E , la cual la más importante es que E satisface escisión (e.g. KH satisface pero K no), entonces la (G, \mathcal{F}, E, A) -conjetura de isomorfismo es verdad para cuando A es un anillo (G, \mathcal{F}) -propio. Si X es un conjunto simplicial localmente finito con una acción de G entonces el G -anillo A es propio sobre X si este es un álgebra sobre el anillo $\mathbb{Z}^{(X)}$ de funciones polinomiales finitamente soportadas en X , la acción es compatible con la acción de G en A y en X y $\mathbb{Z}^{(X)} \cdot A = A$. Decimos que A es (G, \mathcal{F}) -propio si es propio sobre algún conjunto simplicial X localmente finito en el cual G actúa de manera que los estabilizadores pertenecen a la familia \mathcal{F} . Por ejemplo un álgebra es de la forma (9.2) si y solamente si es propia sobre algún G -conjunto simplicial cero dimensional de la forma $X = \coprod G / K_i$. La noción de anillo (G, \mathcal{F}) -propio usada aquí es la noción algebraica de la G - C^* -álgebra propia, y el teorema 3.9.2 es una versión algebraica del conocido hecho de que la conjetura de Baum-Connes es cierta para las G - C^* -álgebras [11].

Bibliography

1. Saad Baaj and Georges Skandalis, *C*-algèbres de Hopf et théorie de Kasparov équivariante (Hopf C*-algebras and equivariant Kasparov theory)*, K-Theory **2** (1989), no. 6, 683–721.
2. Arthur Bartels and Wolfgang Luck, *Isomorphism conjecture for homotopy K-theory and groups acting on trees*, J. Pure Appl. Algebra **205** (2006), no. 3, 660–696.
3. Guillermo Cortiñas, *Algebraic v. topological K-theory: a friendly match*, Springer Lecture Notes in Mathematics 2008.
4. ———, *The obstruction to excision in K-theory and in cyclic homology*, Invent. Math. **164** (2006), no. 1, 143–173.
5. Guillermo Cortiñas and Andreas Thom, *Bivariant algebraic k-theory*, J. reine angew. Math. Crelle (2007), no. 610, 71–123.
6. Joachim Cuntz, *A new look at KK-theory*, K-theory **1** (1987), 31–51.
7. ———, *Noncommutative simplicial complexes and the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002), 307–329.
8. James F. Davis and Wolfgang Luck, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory*, K-theory **15** (1998), 201–252.
9. F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic k-theory*, J. Amer. Math. Soc. **6** (1993), no. 2, 249–297.
10. Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics (Boston, Mass.), 1999.
11. Erik Guentner, Nigel Higson, and Jody Trout, *Equivariant E-theory for C*-algebras*, Mem. Am. Math. Soc. **703** (2000), 86 p.
12. Nigel Higson, *A characterization of KK-theory*, Pac. J. Math. **126** (1987), no. 2, 253–276.
13. Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs. 99. Providence, RI: American Mathematical Society (AMS). xv, 457 p, 2003.
14. Mark Hovey, *Model categories*, Mathematical Surveys and Monographs. 63. Providence, RI: American Mathematical Society (AMS). xii, 209 p. , 1999.
15. Michael Joachim, *K-homology of C*-categories and symmetric spectra representing K-homology*, Math. Ann. **327** (2003), no. 4, 641–670.
16. B.E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc.(3) **14** (1964), 299–320.
17. M. Karoubi, *Foncteurs derives et K-théorie*, Semin. Heidelberg-Saarbrücken-Strasbourg K-Theorie, Lect. Notes Math. 136, 107–186, 1970.
18. Max Karoubi and Orlando E. Villamayor, *K-théorie algébrique et K-théorie topologique. I.*, Math. Scand. **28** (1971), 265–307.
19. G.G. Kasparov, *The operator K-functor and extensions of C*-algebras*, Math. USSR, Izv. **16** (1981), 513–572.
20. ———, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201.
21. Bernhard Keller, *Derived categories and their uses*, Handbook of Algebra, vol. 1, North-Holland, 1996, pp. 671 – 701.
22. Saunders Mac Lane, *Categories for the working mathematician. 2nd ed.*, Graduate Texts in Mathematics. 5. New York, NY: Springer, 1998.

23. J. Cuntz R. Meyer and J. Rosenberg., *Topological and Bivariant K-Theory*, vol. 36, Birkhauser, Oberwolfach Seminars, 2007.
24. Ralf Meyer, *Categorical aspects of bivariant K-theory*, Cortiñas, Guillermo (ed.) et al., *K-theory and noncommutative geometry*. Proceedings of the ICM 2006 satellite conference, Valladolid, Spain, August 31–September 6, 2006. Zürich: European Mathematical Society (EMS). Series of Congress Reports, 1-39, 2008.
25. Susan Montgomery, *Hopf Algebras and Their Actions on Rings*, Regional Conference Series in Mathematics, no. 82, Conference Board of the Mathematical Sciences by the AMS, 1993.
26. G. Nobeling, *Verallgemeinerung eines satzes von herrn e. specker*, *Inventiones Mathematicae* **6** (1968), 41–55.
27. Erik K. Pedersen and Charles A. Weibel, *K-theory homology of spaces*, Algebraic topology, Proc. Int. Conf., Arcata/Calif. 1986, Lect. Notes Math. 1370, 346-361, 1989.
28. A.A. Suslin, *Excision in the integral algebraic K-theory*, Proc. Steklov Inst. Math. 208, 255-279 (1995); translation from Tr. Mat. Inst. Steklova 208, 290-317, 1995.
29. Andrei A. Suslin and Mariusz Wodzicki, *Excision in algebraic K-theory.*, Ann. of Math **136** (1992), 51–122.
30. Alain Valette, *Introduction to the Baum-Connes conjecture. With notes taken by Indira Chatterji. With an appendix by Guido Mislin*, Lectures in Mathematics, ETH Zürich. Basel: Birkhäuser. x, 104 p., 2002.
31. C.A. Weibel, *Mayer-Vietoris sequences and mod p K-theory*, Algebraic K-theory, Proc. Conf., Oberwolfach 1980, Part I, Lect. Notes Math. 966, 390-407, 1982.
32. Charles A. Weibel, *Homotopy algebraic K-theory*, Algebraic K-theory and algebraic number theory, Proc. Semin., Honolulu/Hawaii 1987, Contemp. Math. 83, 461-488, 1989.
33. ———, *An introduction to homological algebra. 1st pbk-ed.*, Cambridge Studies in Advanced Mathematics. 38. Cambridge: Cambridge Univ. Press. xiv, 450 p, 1995.