VECTOR MEASURES ON δ-RINGS AND REPRESENTATION THEOREMS OF BANACH LATTICES

Ph.D. Dissertation

Author

María Aránzazu Juan Blanco

Supervisors

Dr. José Manuel Calabuig Rodríguez
Dr. Olvido Delgado Garrido

Valencia, July 2011

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Departament de Matemàtica Aplicada

VECTOR MEASURES ON δ-RINGS AND REPRESENTATION THEOREMS OF BANACH LATTICES

Ph.D. Dissertation

Author

María Aránzazu Juan Blanco

Supervisors

Dr. José Manuel Calabuig Rodríguez
Dr. Olvido Delgado Garrido

Valencia, July 2011

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Departament de Matemàtica Aplicada

Don José Manuel Calabuig Rodríguez, profesor del Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia y Doña Olvido Delgado Garrido, profesora del Departamento de Matemática Aplicada I de la Universidad de Sevilla,

CERTIFICAN que la presente memoria **Vector Measures on δ-rings and Representation Theorems of Banach lattices** ha sido realizada bajo su dirección en el Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia, por María Aránzazu Juan Blanco y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas.

Y para que así conste, en cumplimiento de la legislación vigente, presentamos ante el Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia, la referida Tesis Doctoral, firmando el presente certificado.

En Valencia, a 15 de julio de 2011

Fdo. José Manuel Calabuig Rodríguez   Fdo. Olvido Delgado Garrido

Después de años de esfuerzo, de altibajos, de idas y venidas, finalizo una etapa y cumplo un sueño.

Gracias de corazón a mis directores José Manuel Calabuig y Olvido Delgado, por su esfuerzo, dedicación, ilusión y buen hacer para conmigo. Desde aquí expresarles mi más profunda admiración y mi agradecimiento más sincero.

A Enrique, mi cómplice, a Lluís, Edu, Orlando e Irene, mis compañeros de fatiga; a mis padres, a mis soles, David y Borja, y a todos los que me habéis 'empujado' de un modo u otro, miles de gracias. Gracias por alentarme, por creer en mí, por ayudarme a no rendirme.

Y a Andrés, por su infinita paciencia y su apoyo incondicional.

To my angel

Resumen

El espacio de funciones integrables con respecto a una medida vectorial, amén de interesante en si mismo, sirve de herramienta para aplicaciones en problemas importantes como la representación integral y el estudio del dominio óptimo de operadores lineales o la representación de retículos de Banach abstractos como espacios de funciones. Las medidas vectoriales clásicas $\nu\colon \Sigma \to X$ se definen sobre σ-álgebras y con valores en un espacio de Banach, y los espacios correspondientes $L^1(\nu)$ y $L^1_w(\nu)$ de funciones integrables y débilmente integrables respectivamente, han sido estudiados en profundidad por numerosos autores, siendo su comportamiento bien conocido, ver [11], [31, Capítulo 3] y las referencias contenidas en él.

Sin embargo, este contexto no es suficiente, por ejemplo, para aplicaciones a operadores definidos en espacios que no contienen a las funciones características de conjuntos (ver [5],[16] y [17]) o retículos de Banach sin unidad débil (ver [6, pp. 22-23]). Estos casos requieren que la medida vectorial ν esté definida en una estructura más débil que la de σ-álgebra, a saber, en un δ-anillo. Más aún, la integración con respecto a medidas vectoriales definidas en δ-anillos es la generalización vectorial natural de la integración con respecto a medidas σ-finitas positivas μ, que no está incluida en el contexto de las medidas vectoriales en σ-álgebras si μ no es finita.

En consecuencia, las medidas vectoriales definidas en un δ-anillo también juegan un rol importante y merecen ser estudiadas así como sus espacios de funciones integrables. La teoría de integración con respecto a estas medidas se debe a Lewis [25] y Masani y Niemi [28], [29].

En este trabajo estamos interesados principalmente en encontrar las propiedades que garanticen la representación de un retículo de Banach a través de un espacio de funciones integrables. El Capítulo 4 se dedica a este objetivo y contiene nuestro resultado principal (Theorem 4.1.7).

Algunas cuestiones interesantes aparecen de forma natural al intentar resolver este problema de representación abstracto. Las propiedades analíticas de una medida vectorial ν definida sobre un δ-anillo están directamente relacionadas con las propiedades reticulares del espacio $L^1(\nu)$ (ver [15]). Es también objetivo de este trabajo, estudiar el efecto de ciertas propiedades sobre ν en las propiedades reticulares del espacio $L^1_w(\nu)$ y el Capítulo 2 está dedicado a desarrollar nuestros resultados en ese contexto. Concretamente, analizamos la orden continuidad, la orden densidad y las propiedades de tipo Fatou para $L^1_w(\nu)$. Demostramos que el comportamiento de $L^1_w(\nu)$ difiere del caso en el que ν se define en una σ-álgebra cuando ν no satisface cierta propiedad de σ-finitud local.

En el Capítulo 3 estudiamos las propiedades reticulares de los retículos de Banach $L^p(\nu)$ y $L^p_w(\nu)$ para una medida vectorial ν definida en un δ-anillo. La relación entre estos dos espacios, el estudio de la continuidad y ciertas propiedades de compacidad para algunos operadores de multiplicación entre diferentes espacios $L^p(\nu)$ y/o $L^q_w(\nu)$ son el eje fundamental de esta parte del trabajo.

Resum

L'espai de funcions integrables respecte a una mesura vectorial, interessant en si mateixa, serveix d'eina per aplicacions en problemes importants com la representació i l'estudi del domini òptim d'operadors lineals o la representació de retículs de Banach abstractes com a espais de funcions. Les mesures vectorials clàssiques $\nu\colon \Sigma \to X$ es defineixen sobre σ-àlgebres i amb valors en un espai de Banach, i els espais corresponents $L^1(\nu)$ i $L^1_w(\nu)$ de funcions integrables i dèbilment integrables respectivament, han sigut estudiats en profunditat per nombrosos autors, essent el seu comportament ben conegut, veure [11], [31, Capítol 3] i les referències contingudes en ell.

No obstant això, aquest context no és suficient, per exemple, per a aplicacions a operadors definits en espais que no contenen les funcions característiques de conjunts (veure [5],[16] i [17]) o retículs de Banach sense unitat dèbil (veure [6, pp. 22-23]). Aquestos casos requereixen que la mesura vectorial siga definida en una estructura més dèbil que la de σ-àlgebra, es a dir, en un δ-anell. Més encara, la integració respecte a mesures vectorials definides en δ-anells és la generalització vectorial natural de la integració respecte a mesures σ-finites positives μ, que no està inclosa en el context de les mesures vectorials en σ-àlgebres quan μ no és finita.

En conseqüència, les mesures vectorials definides en un δ-anell també juguen un paper important i mereixen esser estudiades així com els seus espais de funcions integrables. La teoria d'integració respecte a aquestes mesures es deu a Lewis [25] i Masani i Niemi [28], [29].

En aquest treball estem interessats principalment en trobar les propietats que garanteixen la representació d'un retícul de Banach mitjançant un espai de funcions integrables. El Capítol 4 es centra en aquest objectiu i conté el nostre resultat principal (Teorema 4.1.7).

Altres qüestions apareixen d'una manera natural quan intentem resoldre aquest problema de representació abstracte. Les propietats analítiques d'una mesura vectorial definida sobre un δ-anell estan directament relacionades amb les propietats reticulars de l'espai $L^1(\nu)$ (veure [15]). És també objectiu d'aquest treball, estudiar l'efecte de certes propietats sobre ν en les propietats reticulars de l'espai $L^1_w(\nu)$ i el Capítol 2 està dedicat a desenvolupar els nostres resultats en aquest context. Concretament, analitzem la continuïtat en ordre, la densitat en ordre y les propietats de tipus Fatou per $L^1_w(\nu)$. Demostrem que el comportament de $L^1_w(\nu)$ difereix del cas en el qual ν es defineix en una σ-àlgebra quan ν no complix certa propietat de σ-finitud local.

En el Capítol 3 estudiem les propietats reticulars dels retículs de Banach $L^p(\nu)$ i $L^p_w(\nu)$ per a una mesura vectorial definida en un δ-anell. La relació entre aquestos dos espais, l'estudi de la continuïtat i certes propietats de compacitat per alguns operadors de multiplicació entre diferents espais $L^p(\nu)$ i/o $L^q_w(\nu)$ són l'eix fonamental d'aquesta part del treball.

Summary

The space of integrable functions with respect to a vector measure which is already interesting by itself, finds applications in important problems as the integral representation and the study of the optimal domain of linear operators or the representation of abstract Banach lattices as spaces of functions. Classical vector measures $\nu\colon \Sigma \to X$ are considered to be defined on a σ-algebra and with values in a Banach space, and the corresponding spaces $L^1(\nu)$ and $L^1_w(\nu)$ of integrable and weakly integrable functions respectively have been studied in depth by many authors being their behavior well understood, see [11], [31, Chapter 3] and the references therein.

However, this framework is not enough, for instance, for applications to operators on spaces which do not contain the characteristic functions of sets (see [5], [16] and [17]) or Banach lattices without weak unit (see [6, pp. 22-23]). These cases require ν to be defined on a weaker structure than σ-algebra, namely, a δ-ring. Furthermore, integration with respect to vector measures defined on δ-rings is the natural vector valued generalization of the case of integration with respect to positive σ-finite measures μ, which is not included in the frame of vector measures on σ-algebras if μ is non finite.

Consequently, vector measures defined on a δ-ring also play an important role and deserve to be studied together with their spaces of integrable functions. The integration theory with respect to these vector measures ν is due to Lewis [25] and Masani and Niemi [28], [29].

In this work we are mainly interested in providing the properties which guarantee the representation of a Banach lattice by means of an space of integrable functions. Chapter 4 is devoted to this aim and contains our main result (Theorem 4.1.7).

Some interesting questions appeared when we tried to solve this abstract representation problem. The analytic properties of a vector measure ν defined on a δ-ring are directly related to the lattice properties of the space $L^1(\nu)$ (see [15]). It will be also the aim of this work to study the effect of certain properties of ν on the lattice properties of the space $L^1_w(\nu)$ and Chapter 2 is devoted to develop our results in this context. Concretely, we analyze order continuity, order density and Fatou type properties for $L^1_w(\nu)$. We will see that the behavior of $L^1_w(\nu)$ differs from the case in which ν is defined on a σ-algebra whenever ν does not satisfies certain local σ-finiteness property.

In Chapter 3 we study the lattice properties of the Banach lattices $L^p(\nu)$ and $L^p_w(\nu)$ for a vector measure ν defined on a δ-ring. The relation between these two spaces, the study of the continuity and some kind of compactness properties of certain multiplication operators between different spaces $L^p(\nu)$ and/or $L^q_w(\nu)$ play a fundamental role.

Contents

Introduction

The classical integration theory of scalar functions with respect to a vector measure was created by Bartle, Dunford and Schwartz in 1955 for studying the vector extension of the Riesz representation theorem [3]. They developed a Lebesgue type theory which extends the ordinary space of Lebesgue integrable functions with respect to an scalar measure to the case of a countably additive measure with values lying in a Banach space, in a way that the analogous convergence theorems hold.

Later, in 1970, Lewis provided an axiomatic version for vector measures having values in a locally convex space and proved some fundamental results on integration [24],[25]. The definition of integrable function is given by duality and the theory is equivalent to the one of Bartle, Dunford and Schwartz when the space where the vector measure takes its values is a Banach space.

The Banach space properties of the space $L^1(\nu)$ of integrable functions with respect to a Banach valued vector measure ν, was firstly studied by Kluvánek and Knowles in [23] which was published in 1975. Some other authors also contributed to the development of these ideas, as for instance Ricker and Okada (see [31, Chapter 3] and the references therein). The Banach lattice properties of this space, which became specially interesting for applications, was deeply studied by Curbera in a series of three papers at the beginning of the nineties ([7],[8],[9]). After the work of Stefansson in 1993 ([33]), the space $L^1_w(\nu)$ of weakly integrable functions with respect to ν began to be considered an important element. The spaces of p-integrable functions were introduced by Sánchez-Pérez in 2002 and the corresponding spaces $L^p(\nu)$ and $L^p_w(\nu)$ have been studied in depth by many authors being their behavior well understood, see [11],[19] and [32].

In 2000 a new line of applications of these spaces started, basically in two directions: the determination of the optimal domain for classical operators and the representation of abstract Banach lattices as spaces of integrable functions. In both cases, the integration with respect to vector measures on σ-algebras is not sufficient for covering the general purposes of these research. This framework is not enough, for instance, for applications to operators on spaces which do not contain the characteristic functions of sets (see [5],[16] and [17]) or Banach lattices without weak unit (see [6, pp. 22-23]).

These cases require ν to be defined on a weaker structure than σ-algebra, namely, a δ-ring. Furthermore, integration with respect to vector measures defined on δ-rings is the natural vector valued generalization of the case of integration with respect to positive σ-finite measures μ, which is not included in the frame of vector measures on σ-algebras if μ is non finite.

Consequently, vector measures defined on a δ-ring also play an important role and deserve to be studied together with their spaces of integrable functions. The integration theory with respect to these vector measures ν is due to Lewis [25] and Masani and Niemi [28],[29] (see also [15]).

In this work we are mainly interested in providing the properties which guarantee the representation of a Banach lattice by means of an space of integrable functions. Some results were already well known when we started this project. Namely, every order continuous Banach lattice E with a weak unit can be identified with an space $L^1(\nu)$, where ν is defined on a σ-algebra (see [7, Theorem 8]). If the existence of a weak order unit is not assumed, it is still possible to represent E but using in this case a vector measure on a δ-ring (see [6, pp.22-23]). If the order continuity fails but E has the σ-Fatou property and a weak unit belonging to its order continuous part E_a, then E can be identified with an space $L^1_w(\nu)$ where ν is defined on a σ-algebra (see [10, Theorem 2.5]). Similar results are known for Banach lattices with convexity properties. That is, for $1 < p < \infty$, if E is a p-convex order continuous Banach lattice with a weak unit, then E is order isomorphic to an space $L^p(\nu)$ with ν defined on a σ-algebra ([19, Proposition 2.4]). On the other hand, if E is a p-convex Banach lattice having the σ-Fatou property and a weak unit belonging to E_a, then E is order isomorphic to an space $L^p_w(\nu)$ with ν on a σ-algebra ([12]).

The main goal of this memoir is to get a representation theorem for Banach lattices without weak unit as general as possible by using vector measures defined on a δ-ring. Chapter 4 is devoted to this aim and contains our main result (Theorem 4.1.7).

Some interesting questions appeared when we tried to solve this abstract representation problem. The analytic properties of a vector measure ν defined on a δ-ring are directly related to the lattice properties of the space $L^1(\nu)$ (see [15]). It will be also the aim of this work to study the effect of certain properties of ν on the lattice properties of the space $L^1_w(\nu)$ and Chapter 2 is devoted to develop our results in this context. Concretely, we analyze order continuity, order density and Fatou type properties for $L^1_w(\nu)$. We will see that the behavior of $L^1_w(\nu)$ differs from the case in which ν is defined on a σ-algebra whenever ν does not satisfies certain local σ-finiteness property.

In Chapter 3 we study the lattice properties of the Banach lattices $L^p(\nu)$ and $L^p_w(\nu)$ for a vector measure ν defined on a δ-ring. The relation between these two spaces, the study of the continuity and some kind of compactness properties of certain multiplication operators between different spaces L^p and/or L^q_w play a fundamental role.

Some applications of the results of this memoir on the lattice properties of $L^1(\nu)$ and $L^1_w(\nu)$ and the representation theorems for Banach lattices has been already obtained. In particular, we have used this technique in order to study the limits of the equivalence between the Komlós property for Banach functions spaces X related to μ (i.e. for every bounded sequence (f_n) in X, there exists a subsequence $(f_{n_k})_k$ and a function $f \in X$ such that for any further subsequence $(h_j)_j$ of $(f_{n_k})_k$, the Cesàro sums $\frac{1}{n}\sum_{j=1}^n h_j$ converge μ-a.e. to f) and the Fatou property in these spaces. The vector measure representation of spaces as $\ell^\infty(\Gamma)$ for a non countable set of indexes Γ as $L^1_w(\nu)$ for a vector measure ν provides for instance counterexamples to the Komlós property, although this space has still the Fatou property (see [22]).

Chapter 1

Preliminaries

In this chapter, we expose the concepts and results used throughout the memoir about Banach lattices, Banach function spaces and integration of real functions with respect to a vector measure defined on a δ-ring.

1.1 Banach lattices

We will mainly use the terminology and the notation of [27] and [34].

Let E be a *Banach lattice*, that is a real Banach space endowed with a norm $\|\cdot\|$ and a partial order $\leq$ such that

(a) if $x, y, z \in E$ with $x \leq y$, then $x + z \leq y + z$,

(b) if $x, y \in E$ with $x \leq y$, then $ax \leq ay$ for all $a \geq 0$,

(c) for $x, y \in E$, there exists the supremum of x and y with respect to the order,

(d) if $x, y \in E$ with $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| = \sup\{x, -x\}$ is the *modulo* of x.

Note that (c) implies that also there exists the infimum of every $x, y \in E$. The supremum and the infimum of two elements x and y of E are usually denoted by $x \vee y$ and $x \wedge y$ respectively. A *weak unit* of E is an element $0 \leq e \in E$ such that $x \wedge e = 0$ implies $x = 0$.

We will use the index τ to mean that (x_τ) is a net and the index n to mean that (x_n) is a sequence. A net $(x_\tau) \subset E$ is an *upwards directed system* if for every τ_1, τ_2 there exists τ_3 such that $x_{\tau_1} \leq x_{\tau_3}$ and $x_{\tau_2} \leq x_{\tau_3}$. This is denoted by $x_\tau \uparrow$. Similarly, (x_τ) is a *downwards directed system* if for every τ_1, τ_2 there exists τ_3 such that $x_{\tau_1} \geq x_{\tau_3}$ and $x_{\tau_2} \geq x_{\tau_3}$, and this is denoted by $x_\tau \downarrow$. If $x_\tau \uparrow$ and $x = \sup x_\tau$ exists in E, we will write $x_\tau \uparrow x$. If $x_\tau \downarrow$ and $x = \inf x_\tau$ exists in E, we will write $x_\tau \downarrow x$. Given a sequence $(x_n) \subset E$ we will write $x_n \uparrow$ if the sequence is increasing and $x_n \downarrow$ if it is decreasing. If $x_n \uparrow$ and $x = \sup x_n$ exists in E, we will write $x_n \uparrow x$. Similarly, $x_n \downarrow x$. An upwards directed system (x_τ) in E is said to be a *Cauchy system* if for any $\epsilon > 0$ there exists τ_0 such that $\|x_{\tau_1} - x_{\tau_2}\| < \epsilon$ for all $x_{\tau_1} \geq x_{\tau_0}$ and $x_{\tau_2} \geq x_{\tau_0}$.

A subset F of E is called *solid* if $y \in E$ with $|y| \leq |x|$ for some $x \in F$ implies $y \in F$. The *solid hull* of $F \subset E$ is the smallest solid space containing F. An *ideal* F of E is a closed solid subspace of E. An ideal F in E is said to be *order dense* in E if for every $0 \leq x \in E$ there exists an upwards directed system $(x_\tau) \subset F$ such that $0 \leq x_\tau \uparrow x$ and is said to be *super order dense* if for every $0 \leq x \in E$ there exists an increasing sequence $(x_n) \subset F$ such that $0 \leq x_n \uparrow x$.

We say that E is *order continuous* if for every $(x_\tau) \subset E$ with $x_\tau \downarrow 0$ it follows that $\|x_\tau\| \downarrow 0$ and E is *σ-order continuous* if for every $(x_n) \subset E$ with $x_n \downarrow 0$ it follows that $\|x_n\| \downarrow 0$. We denote by E_{an} the *order continuous part* of E, that is, the largest order continuous ideal in E. It can be described as

$$E_{an} = \{x \in E : \ |x| \geq x_\tau \downarrow 0 \ \text{ implies } \ \|x_\tau\| \downarrow 0\}.$$

Similarly, E_a will denote the *σ-order continuous part* of E, that is, the largest σ-order continuous ideal in E, which can be described as

$$E_a = \{x \in E : \ |x| \geq x_n \downarrow 0 \ \text{ implies } \ \|x_n\| \downarrow 0\}.$$

Of course $E_{an} \subset E_a$.

The Banach lattice E is *Dedekind complete* if every non empty subset which is bounded from above in the order of E has a supremum and is *Dedekind σ-*

complete if every non empty countable subset which is bounded from above in the order of E has a supremum.

We say that E has the *Fatou property* if for every $(x_\tau) \subset E$ with $0 \leq x_\tau \uparrow$ such that $\sup \|x_\tau\| < \infty$ it follows that there exists $x = \sup x_\tau$ in E and $\|x\| = \sup \|x_\tau\|$. Similarly, E has the *σ-Fatou property* if for every $(x_n) \subset E$ with $0 \leq x_n \uparrow$ such that $\sup \|x_n\| < \infty$ it follows that there exists $x = \sup x_n$ in E and $\|x\| = \sup \|x_n\|$.

Given $x_1, ..., x_n \in E$, it is possible to define an expression $\big(\sum_{j=1}^n |x_j|^p\big)^{1/p}$ in E for every $p \geq 1$, see [26, Chapter 1.d.]. The Banach lattice E is said to be *p-convex* if there exists a constant $M > 0$ such that

$$\Big\|\Big(\sum_{j=1}^n |x_j|^p\Big)^{\frac{1}{p}}\Big\| \leq M\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{\frac{1}{p}}$$

for all n and $x_1, \ldots, x_n \in E$. The smallest constant satisfying the previous inequality is called the *p-convexity constant* of E and is denoted by $\mathbf{M}^{(p)}(E)$.

Let $T\colon E \to F$ be a linear operator between Banach lattices. The operator T is said to be *positive* if $Tx \geq 0$ whenever $0 \leq x \in E$. Every positive linear operator between Banach lattices is continuous, see [26, p. 2] or [2, Theorem 4.3]. In particular, every inclusion $E \subset F$ of Banach lattices with the same order is continuous. We will say that T is an *order isomorphism* if it is one to one, onto and satisfies that $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$. In this case, T is continuous as it is positive and also satisfies $T(x \vee y) = Tx \vee Ty$ for all $x, y \in E$. If moreover, $\|Tx\|_F = \|x\|_E$ for all $x \in E$, we will say that T is an *order isometry*. We say that E and F are *order isomorphic* (*order isometrics*) if there exists an order isomorphism (isometry) $T\colon E \to F$.

The set consisting of all continuous linear maps from E into F will be denoted by $\mathcal{B}(E, F)$ and we will write $\|T\|$ for the usual operator norm of T. An operator $T \in \mathcal{B}(E, F)$ is called *$\mathcal{L}$-weakly compact* if $\|x_n\| \to 0$ for every disjoint sequence (x_n) contained in the solid hull of $T(B_E)$, where B_E is the unit ball of E. We denote by $\mathcal{L}(E, F)$ this class of continuous operators and by $\mathcal{W}(E, F)$ the ideal of weakly compact operators. Note that $\mathcal{L}(E, F) \subset \mathcal{W}(E, F)$, see [30, Proposition 3.6.12].

1.2 Banach function spaces

Let (Ω, Σ, μ) be a measure space without assumptions of finiteness on μ. As usual, a property holds μ-almost everywhere (briefly, μ-a.e.) if it holds except on a μ-null set. We denote by $L^0(\mu)$ the space of all measurable real functions on Ω, where functions which are equal μ-a.e. are identified. The space $L^0(\mu)$ will be endowed with the *μ-a.e. pointwise order*, that is, $f \leq g$ if and only if $f \leq g$ μ-a.e. Then, $L^0(\mu)$ is a *vector lattice*, that is a real linear space satisfying (a), (b) and (c) in the definition of Banach lattice, and it is *Archimedean*, that is, for every $0 \leq f \in L^0(\mu)$ we have that $\frac{1}{n} f \downarrow 0$. Note that for $f, f_n \in L^0(\mu)$ with $f_n \uparrow$, it follows that f_n converges to f μ-a.e. if and only if $f_n \uparrow f$ in $L^0(\mu)$, that is, the pointwise supremum coincides with the lattice supremum. It is important to emphasize that the pointwise supremum of a net of measurable functions is not measurable in general, and even if it is measurable may not coincide with the lattice supremum.

By a *Banach function space* (briefly, B.f.s.) related to μ we mean a Banach space $X \subset L^0(\mu)$ satisfying that if $|f| \leq |g|$ with $f \in L^0(\mu)$ and $g \in X$ then $f \in X$ and $\|f\|_X \leq \|g\|_X$. Every B.f.s. is a Banach lattice with the μ-a.e. pointwise order, in which convergence in norm of a sequence implies μ-a.e. convergence for some subsequence. Note that for $f, f_n \in X$ with $f_n \uparrow$, it follows that f_n converges to f μ-a.e. if and only if $f_n \uparrow f$ in X.

If X is a B.f.s. related to a complete σ-finite measure space (Ω, Σ, μ) such that for every $A \in \Sigma$ with $\mu(A) < \infty$ it follows

(a) $\chi_A \in X$, and

(b) $f\chi_A \in L^1(\mu)$ for all $f \in X$,

then we will say that X is a B.f.s. *in the sense of Lindenstrauss and Tzafriri* (briefly, LT-B.f.s.), see [26, Definition 1.b.17].

1.3 Integration with respect to vector measures on δ-rings

The integration theory with respect to a vector measure defined on a δ-ring is due to Lewis [25] and Masani and Niemi [28], [29] (see also [15]). This integration theory extends the classical one for vector measures defined on σ-algebras.

Let $\mathcal{R}$ be a *δ-ring* of subsets of an abstract set Ω, that is, a ring closed under countable intersections. We write $\mathcal{R}^{loc}$ for the σ-algebra of all subsets A of Ω such that $A \cap B \in \mathcal{R}$ for all $B \in \mathcal{R}$. Note that if $\mathcal{R}$ is a σ-algebra then $\mathcal{R}^{loc} = \mathcal{R}$. Denote by $\mathcal{M}(\mathcal{R}^{loc})$ the space of all measurable real functions on $(\Omega, \mathcal{R}^{loc})$, by $\mathcal{S}(\mathcal{R}^{loc})$ the space of all simple functions and by $\mathcal{S}(\mathcal{R})$ the space of all *$\mathcal{R}$-simple functions* (i.e. simple functions supported in $\mathcal{R}$).

Let $\lambda\colon \mathcal{R} \to \mathbb{R}$ be a countably additive measure, that is, $\sum \lambda(A_n)$ converges to $\lambda(\cup A_n)$ whenever (A_n) is a sequence of pairwise disjoint sets in $\mathcal{R}$ with $\cup A_n$ in $\mathcal{R}$. The *variation* of λ is the countably additive measure $|\lambda|\colon \mathcal{R}^{loc} \to [0,\infty]$ given by

$$|\lambda|(A) = \sup\Big\{ \sum |\lambda(A_i)| \,:\, (A_i) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A \Big\}.$$

For every $A \in \mathcal{R}$ we have that $|\lambda|(A) < \infty$. The space $L^1(\lambda)$ of integrable functions with respect to λ is defined as the space $L^1(|\lambda|)$ with the usual norm. Every $\mathcal{R}$-simple function $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$ is in $L^1(\lambda)$ and the integral of φ with respect to λ is defined as usual by $\int \varphi \, d\lambda = \sum_{i=1}^n \alpha_i \lambda(A_i)$. Moreover, the space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\lambda)$. For every $f \in L^1(\lambda)$, the integral of f with respect to λ is defined as $\int f \, d\lambda = \lim \int \varphi_n \, d\lambda$ for any sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\lambda)$.

Let $\nu\colon \mathcal{R} \to X$ be a *vector measure* with values in a real Banach space X, that is, $\sum \nu(A_n)$ converges to $\nu(\cup A_n)$ in X whenever (A_n) is a sequence of pairwise disjoint sets in $\mathcal{R}$ with $\cup A_n \in \mathcal{R}$. Denoting by X^* the topological dual of X and by B_{X^*} the unit ball of X^*, the *semivariation* of ν is the map $\|\nu\|\colon \mathcal{R}^{loc} \to [0,\infty]$ given by $\|\nu\|(A) = \sup\{|x^*\nu|(A) \,:\, x^* \in B_{X^*}\}$ for all $A \in \mathcal{R}^{loc}$, where $|x^*\nu|$ is the variation of the measure $x^*\nu\colon \mathcal{R} \to \mathbb{R}$. The semivariation of ν is monotone increasing, countably subadditive, finite on $\mathcal{R}$

and for all $A \in \mathcal{R}^{loc}$ satisfies

$$\frac{1}{2}\|\nu\|(A) \leq \sup\left\{\|\nu(B)\|_X \ : \ B \in \mathcal{R} \cap 2^A\right\} \leq \|\nu\|(A) \ . \tag{1.1}$$

In view of (1.1), the vector measure ν is bounded (i.e. its range is a bounded set in X) if and only if $\|\nu\|(\Omega) < \infty$. A set $A \in \mathcal{R}^{loc}$ is ν-*null* if $\|\nu\|(A) = 0$, or equivalently, $\nu(B) = 0$ for all $B \in \mathcal{R} \cap 2^A$. A property holds ν-*almost everywhere* (briefly, ν-a.e.) if it holds except on a ν-null set.

For every $\mathcal{R}^{loc}$-measurable function $f \colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ we can define

$$\|f\|_\nu = \sup_{x^* \in B_{X^*}} \int |f| \, d|x^*\nu| \leq \infty.$$

Note that if $\|f\|_\nu < \infty$ then $|f| < \infty$ ν-a.e.

A function $f \in \mathcal{M}(\mathcal{R}^{loc})$ is said to be *weakly integrable* with respect to ν if $f \in L^1(x^*\nu)$ for all $x^* \in X^*$, or equivalently, if $\|f\|_\nu < \infty$. Let $L^1_w(\nu)$ denote the space of all weakly integrable functions with respect to ν, where functions which are equal ν-a.e. are identified. The space $L^1_w(\nu)$ is a Banach space with the norm $\|\cdot\|_\nu$.

A function $f \in L^1_w(\nu)$ is *integrable with respect to* ν if for each $A \in \mathcal{R}^{loc}$ there exists a vector denoted by $\int_A f \, d\nu \in X$, such that

$$x^*\Big(\int_A f \, d\nu\Big) = \int_A f \, dx^*\nu \ \text{ for all } x^* \in X^*.$$

We will simply write $\int f \, d\nu$ for $\int_\Omega f \, d\nu$. Let $L^1(\nu)$ denote the space of all integrable functions with respect to ν. Then, $L^1(\nu)$ is a closed subspace of $L^1_w(\nu)$ and so it is a Banach space with the norm $\|\cdot\|_\nu$. Moreover, $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu)$. Note that for every $\mathcal{R}$-simple function $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$, we have that $\int \varphi \, d\nu = \sum_{i=1}^n \alpha_i \nu(A_i)$.

The equality $L^1_w(\nu) = L^1(\nu)$ holds whenever the space X where the vector measure takes its values does not contain a copy of c_0 ([25, Theorem 5.1]).

The integration operator $I_\nu \colon L^1(\nu) \to X$ given by $I_\nu(f) = \int f \, d\nu$ is linear and continuous with $\|I_\nu(f)\|_X \leq \|f\|_\nu$.

A vector measure $\nu\colon \mathcal{R} \to E$ with values in a Banach lattice E is *positive* if $\nu(A) \geq 0$ for all $A \in \mathcal{R}$. In this case, the integration operator $I_\nu\colon L^1(\nu) \to E$ is positive (i.e. $I_\nu(f) \geq 0$ whenever $0 \leq f \in L^1(\nu)$) and it can be checked that $\|f\|_\nu = \|I_\nu(|f|)\|_X$ for all $f \in L^1(\nu)$ (see for instance [31, Lemma 3.13] with the obvious modifications in the case of δ-rings).

From [4, Theorem 3.2], there always exists a measure $\lambda\colon \mathcal{R} \to [0,\infty]$ with the same null sets as ν. Then, $L^1(\nu)$ and $L^1_w(\nu)$ are B.f.s.' related to $|\lambda|$. Moreover, $L^1(\nu)$ is order continuous and $L^1_w(\nu)$ has the σ-Fatou property (see Section 2.3).

For any measure $\mu\colon \mathcal{R}^{loc} \to [0,\infty]$ with the same null sets as ν, since the μ-a.e. pointwise order coincides with the ν-a.e. one, we will denote $L^0(\nu) = L^0(\mu)$ and say B.f.s. related to ν for B.f.s. related to μ.

Chapter 2

Banach lattice properties of L^1_w of a vector measure on a δ-ring

The spaces $L^1(\nu)$ and $L^1_w(\nu)$ of integrable and weakly integrable functions with respect to a vector measure defined on a σ-algebra an with values in a Banach space X have been studied in depth by many authors and their behavior is well understood, see [11], [31, Chapter 3] and the references therein. In [15], there is an analysis of the space $L^1(\nu)$ with ν defined on a δ-ring which gives evidence of how large the difference can be between the δ-ring and σ-algebra cases. However, there is not a deep study of the lattice behavior of the corresponding space $L^1_w(\nu)$.

The aim of this chapter is the study of the Banach lattice properties of the space $L^1_w(\nu)$. More precisely, we study some properties related to order continuity (Section 2.1) and order density (Section 2.2), and some Fatou type properties (Section 2.3). We will see that many properties satisfied for this space when ν is defined on a σ-algebra remain true in general only in the case when ν satisfies certain local σ-finiteness property, which guarantees that every function in $L^1_w(\nu)$ is the ν-a.e. pointwise limit of a sequence of functions in $L^1(\nu)$. We end with an illustrative example (Section 2.4).

From now on in this memoir $\nu\colon \mathcal{R} \to X$ will be a vector measure defined on a δ-ring $\mathcal{R}$ of subsets of an abstract set Ω, with values in a real Banach space X. Recall that measurable functions are referred to the σ-algebra $\mathcal{R}^{loc}$.

2.1 Order continuous part of $L^1_w(\nu)$

Let us begin by noting that the σ-order continuous and the order continuous parts of $L^1_w(\nu)$ coincide. Indeed, $L^1_w(\nu)$ is Dedekind σ-complete as it has the σ-Fatou property (see [34, Theorem 113.1]), and so, since $\big(L^1_w(\nu)\big)_a$ is an ideal in $L^1_w(\nu)$, it is also Dedekind σ-complete. Then, from [34, Theorem 103.6], $\big(L^1_w(\nu)\big)_a$ is order continuous and thus $\big(L^1_w(\nu)\big)_a = \big(L^1_w(\nu)\big)_{an}$.

It was noted in [10, p. 192], that in the case when $\mathcal{R}$ is a σ-algebra, the order continuous part of $L^1_w(\nu)$ is just $L^1(\nu)$. This follows from the facts that $L^1(\nu)$ is order continuous and $\mathcal{S}(\mathcal{R}^{loc}) = \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$. In the general case, $\mathcal{S}(\mathcal{R}^{loc})$ may not be in $L^1(\nu)$, even so, we will see that $\big(L^1_w(\nu)\big)_a = L^1(\nu)$ remains true. First, let us characterize when a characteristic function of a measurable set is in $L^1(\nu)$.

Lemma 2.1.1. *The following statements are equivalent for any $A \in \mathcal{R}^{loc}$.*

(a) $\chi_A \in L^1(\nu)$.

(b) $\|\nu\|(A_n) \to 0$ *for all decreasing sequences* $(A_n) \subset \mathcal{R}^{loc} \cap 2^A$ *with* $\cap A_n$ ν*-null.*

(c) $\nu(A_n) \to 0$ *for all disjoint sequences* $(A_n) \subset \mathcal{R} \cap 2^A$.

Proof. Suppose that $\chi_A \in L^1(\nu)$ and let $(A_n) \subset \mathcal{R}^{loc} \cap 2^A$ be a decreasing sequence with $\cap A_n$ ν-null. Since $L^1(\nu)$ is order continuous and $\chi_A \geq \chi_{A_n} \downarrow 0$, then $\|\nu\|(A_n) = \|\chi_{A_n}\|_\nu \to 0$. So, (a) implies (b).

If $(A_n) \subset \mathcal{R} \cap 2^A$ is a disjoint sequence, taking $B_n = \cup_{j\geq n} A_j$ we have a decreasing sequence $(B_n) \subset \mathcal{R}^{loc} \cap 2^A$ with $\cap B_n = \emptyset$ and $\|\nu(A_n)\| \leq \|\nu\|(B_n)$. So, (b) implies (c).

Suppose that (c) holds and consider the vector measure $\nu_A\colon \mathcal{R} \to X$ defined by $\nu_A(B) = \nu(A \cap B)$ for all $B \in \mathcal{R}$. Noting that $|x^*\nu_A|(B) = |x^*\nu|(A \cap B)$ for all $B \in \mathcal{R}^{loc}$ and $x^* \in X^*$, it can be checked that $\int |f|\, d|x^*\nu_A| = \int |f|\chi_A\, d|x^*\nu|$, first for simple functions and next, by using the monotone convergence theorem, for all measurable functions. Thus, $\|f\|_{\nu_A} = \|f\chi_A\|_\nu$ for every $f \in \mathcal{M}(\mathcal{R}^{loc})$. Then, $f \in L^1_w(\nu_A)$ if and only if $f\chi_A \in L^1_w(\nu)$ and, since $\mathcal{S}(\mathcal{R})$ is dense in both $L^1(\nu)$ and $L^1(\nu_A)$, it follows that $f \in L^1(\nu_A)$ if and only if $f\chi_A \in L^1(\nu)$. By hypothesis ν_A is strongly additive, so, from [15, Corollary 3.2.b)], we have that $\chi_\Omega \in L^1(\nu_A)$ and thus $\chi_A \in L^1(\nu)$. □

Let us prove now the announced result.

Theorem 2.1.2. *The equality* $\big(L^1_w(\nu)\big)_a = L^1(\nu)$ *holds.*

Proof. Obviously $L^1(\nu) \subset \big(L^1_w(\nu)\big)_a$ as $L^1(\nu)$ is order continuous. For the converse inclusion, consider first a set $A \in \mathcal{R}^{loc}$ such that $\chi_A \in \big(L^1_w(\nu)\big)_a$. Since for every decreasing sequence $(A_n) \subset \mathcal{R}^{loc} \cap 2^A$ with $\cap A_n$ ν-null, it follows that $\chi_A \geq \chi_{A_n} \downarrow 0$ and so $\|\nu\|(A_n) = \|\chi_{A_n}\|_\nu \to 0$, then, from Lemma 2.1.1, $\chi_A \in L^1(\nu)$.

Consider now $\varphi \in \mathcal{S}(\mathcal{R}^{loc})$ such that $\varphi \in \big(L^1_w(\nu)\big)_a$. Write $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j}$ with $(A_j) \subset \mathcal{R}^{loc}$ being a disjoint sequence and $\alpha_j \neq 0$. Since $\chi_{A_j} \leq |\frac{\varphi}{\alpha_j}|$ and $\big(L^1_w(\nu)\big)_a$ is an ideal, $\chi_{A_j} \in \big(L^1_w(\nu)\big)_a$. Then, $\chi_{A_j} \in L^1(\nu)$ and so $\varphi \in L^1(\nu)$.

Finally, let $f \in \big(L^1_w(\nu)\big)_a$ and take a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ satisfying that $0 \leq \varphi_n \uparrow |f|$. Note that $\varphi_n \in \big(L^1_w(\nu)\big)_a$ as $\varphi_n \leq |f|$, and so $\varphi_n \in L^1(\nu)$. Since $|f| \geq |f| - \varphi_n \downarrow 0$, we have that $\|\,|f| - \varphi_n\|_\nu \to 0$. Then, as $L^1(\nu)$ is closed in $L^1_w(\nu)$, we have that $|f|$, and so also f, is in $L^1(\nu)$. □

2.2 Order density of $L^1(\nu)$ in $L^1_w(\nu)$

The topic in this section is trivial for the case when $\mathcal{R}$ is a σ-algebra. Indeed, for every $0 \leq f \in L^0(\nu)$ there exists $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \varphi_n \uparrow f$. Since, in this case $\mathcal{R}^{loc} = \mathcal{R}$ and $\mathcal{S}(\mathcal{R}) \subset L^1(\nu)$, obviously we have that $L^1(\nu)$ is super order dense (and so order dense) in $L^0(\nu)$ (and also in $L^1_w(\nu)$). However, this argument fails for the general case as $\mathcal{S}(\mathcal{R}^{loc})$ may not be contained in $L^1(\nu)$.

Example 2.2.1. Let Γ be an uncountable abstract set, $\mathcal{R}$ the δ-ring of finite subsets of Γ and $\nu\colon \mathcal{R} \to c_0(\Gamma)$ the vector measure defined by $\nu(A) = \chi_A$, see [15, Example 2.2]. Then, $\chi_\Gamma \in L^1_w(\nu) = \ell^\infty(\Gamma)$, but there is no sequence $(f_n) \subset L^1(\nu) = c_0(\Gamma)$ such that $0 \leq f_n \uparrow \chi_\Gamma$ as in this case, since the only ν-null set is the empty set, $\Gamma = \cup_n \mathrm{supp}(f_n)$ is countable.

Therefore, in general $L^1(\nu)$ is not super order dense in $L^1_w(\nu)$, but order dense.

Theorem 2.2.2. *The space $L^1(\nu)$ is order dense in $L^1_w(\nu)$.*

Proof. Since every Banach lattice is Archimedean, by [27, Theorem 22.3] it is enough to prove that $L^1(\nu)$ is *quasi order dense* in $L^1_w(\nu)$, i.e. for every $0 \neq f \in L^1_w(\nu)$ there exists $0 \neq g \in L^1(\nu)$ such that $|g| \leq |f|$.

Let $f \in L^1_w(\nu)$ with $\|\nu\|(\mathrm{supp}(f)) > 0$. For $A_n = \{\omega \in \Omega : |f(\omega)| > \frac{1}{n}\}$, we have that $A_n \uparrow \mathrm{supp}(f)$ and so $\|\nu\|(\mathrm{supp}(f)) = \lim_n \|\nu\|(A_n)$ (see [29, Corollary 3.5.(e)]). Take n large enough such that $\|\nu\|(A_n) > 0$. Since $\|\nu\|(A_n) = \sup_{B\in\mathcal{R}\cap 2^{A_n}} \|\nu\|(B)$ (see [29, Lemma 3.4.(g)]), there exists $B_n \in \mathcal{R} \cap 2^{A_n}$ such that $\|\nu\|(B_n) > 0$.

On the other hand, take a sequence $(\psi_j) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \psi_j \uparrow |f|$. Then, there exists a ν-null set $Z \in \mathcal{R}^{loc}$ such that $0 \leq \psi_j(\omega) \uparrow f(\omega)$ for all $\omega \in \Omega\backslash Z$. Noting that $B_n = (\cup_j B_n \cap \mathrm{supp}(\psi_j)\backslash Z) \cup (B_n \cap Z)$, since $B_n \cap \mathrm{supp}(\psi_j)\backslash Z \uparrow$, it follows that $\|\nu\|(B_n) = \|\nu\|(\cup_j B_n \cap \mathrm{supp}(\psi_j)\backslash Z) = \lim_j \|\nu\|(B_n \cap \mathrm{supp}(\psi_j)\backslash Z)$. Take j_n such that $\|\nu\|(B_n \cap \mathrm{supp}(\psi_{j_n})\backslash Z) > 0$ and consider the function $g = \psi_{j_n}\chi_{B_n} \in \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$. Then, $g \neq 0$ and $0 \leq g \leq |f|$. $\square$

Remark 2.2.3. Since $L^0(\nu)$ with the ν-a.e. pointwise order is an Archimedean vector lattice, actually in Theorem 2.2.2 we have proved that $L^1(\nu)$ is order dense in $L^0(\nu)$.

Now, the natural question is when $L^1(\nu)$ is super order dense in $L^1_w(\nu)$. It is easy to see that this happens if ν is *σ-finite*, that is, if there exist a sequence

(A_n) in $\mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\Omega = (\cup A_n) \cup N$. In this case, if $0 \leq f \in L^0(\nu)$ and $(\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ is such that $0 \leq \psi_n \uparrow f$, taking $\varphi_n = \psi_n \chi_{\cup_{j=1}^n A_j} \in \mathcal{S}(\mathcal{R})$ we have that $0 \leq \varphi_n \uparrow f$. Then, $L^1(\nu)$ is super order dense in $L^0(\nu)$ and so in $L^1_w(\nu)$. However, $L^1(\nu)$ being super order dense in $L^1_w(\nu)$ does not imply that ν is σ-finite.

Example 2.2.4. The vector measure ν in Example 2.2.1 considered with values in $\ell^1(\Gamma)$ instead of $c_0(\Gamma)$, satisfies that $L^1(\nu) = L^1_w(\nu) = \ell^1(\Gamma)$. Then, obviously $L^1(\nu)$ is super order dense in $L^1_w(\nu)$ but ν is not σ-finite.

We will characterize the super order density of $L^1(\nu)$ in $L^1_w(\nu)$ by a weaker condition on ν than σ-finiteness. Namely, ν will be said to be *locally σ-finite* if every set $A \in \mathcal{R}^{loc}$ with $\|\nu\|(A) < \infty$, can be written as $A = (\cup A_n) \cup N$, with $N \in \mathcal{R}^{loc}$ ν-null and (A_n) a sequence in $\mathcal{R}$.

Remark 2.2.5. If ν is such that $L^1(\nu) = L^1_w(\nu)$ (e.g. if X does not contain any copy of c_0), then for every $A \in \mathcal{R}^{loc}$ with $\|\nu\|(A) < \infty$, we have that $\chi_A \in L^1_w(\nu) = L^1(\nu)$ and so, from [29, Theorem 4.9.(a)], ν is locally σ-finite.

Let us see that there are plenty of locally σ-finite vector measures which are not σ-finite.

Lemma 2.2.6. *Suppose that ν is* discrete*, that is, for every $\omega \in \Omega$ it follows that $\{\omega\} \in \mathcal{R}$ and $\nu(\{\omega\}) \neq 0$. Then,*

(a) *$N \in \mathcal{R}^{loc}$ is ν-null if and only if $N = \emptyset$.*

(b) $\{A \subset \Omega :\ A$ is finite$\} \subset \mathcal{R} \subset \{A \subset \Omega :\ A$ is countable$\}$.

(c) $\mathcal{R}^{loc} = 2^\Omega$.

(d) *ν is σ-finite if and only if Ω is countable.*

Proof. (a) Suppose $N \in \mathcal{R}^{loc}$ is ν-null. If $\gamma \in N$, then $\{\gamma\} \in \mathcal{R} \cap 2^N$ and so $\|\nu(\{\gamma\})\| \leq \|\nu\|(N) = 0$ which contradicts $\nu(\{\gamma\}) \neq 0$. Hence, $N = \emptyset$. The converse is obvious.

(b) If $A \subset \Omega$ is finite then $A = \cup_{\gamma \in A}\{\gamma\}$ is a finite union of sets in $\mathcal{R}$, so the first containment holds. For the second one, consider $A \in \mathcal{R}$ and the vector measure $\nu_A \colon \mathcal{R}^{loc} \to X$ defined by $\nu_A(B) = \nu(A \cap B)$ for all $B \in \mathcal{R}^{loc}$. Note that $B \in \mathcal{R}^{loc}$ is ν_A-null if and only if $A \cap B$ is ν-null, that is, $A \cap B = \emptyset$. Since ν_A is defined on a σ-algebra we can take $x_A^* \in B_{X^*}$ such that $|x_A^* \nu_A|$ has the same null sets as ν_A (see [18, Theorem IX.2.2]). For every finite set $J \subset \Omega$ it follows that

$$\sum_{\gamma \in J} |x_A^* \nu_A|(\{\gamma\}) = |x_A^* \nu_A|(J) \leq \|\nu_A\|(J) \leq \|\nu_A\|(\Omega) < \infty.$$

Then, there exists a countable set $I \subset \Omega$ such that $|x_A^* \nu_A|(\{\gamma\}) = 0$ for all $\gamma \in \Omega \backslash I$, that is, $A \cap \{\gamma\} = \emptyset$ for all $\gamma \in \Omega \backslash I$. So, $A \subset I$ is countable.

(c) Note that $\{A \subset \Omega : A \text{ is countable}\} \subset \mathcal{R}^{loc}$ as if $A \subset \Omega$ is countable then $A = \cup_{\gamma \in A} \{\gamma\}$ is a countable union of sets in $\mathcal{R}$. Given $A \in 2^\Omega$, from (b) we have that $A \cap B$ is countable, and so is in $\mathcal{R}^{loc}$, for every $B \in \mathcal{R}$. Hence, $A \cap B = B \cap (A \cap B) \in \mathcal{R}$ for every $B \in \mathcal{R}$, that is, $A \in \mathcal{R}^{loc}$.

(d) It follows from (a) and (b). □

From Remark 2.2.5 and Lemma 2.2.6, every discrete vector measure on a δ-ring of subsets of an uncountable set, with values in a Banach space without any copy of c_0, is locally σ-finite but not σ-finite. Also, there are locally σ-finite vector measures which are not σ-finite with values in a Banach space containing a copy of c_0.

Example 2.2.7. Consider the δ-ring $\mathcal{R} = \{A \subset [0, \infty) : A \text{ is finite}\}$ of subsets of $[0, \infty)$ and the vector measure $\nu \colon \mathcal{R} \to c_0$ defined by $\nu(A) = \sum_n \frac{\sharp(A \cap [n-1, n))}{2^n} e_n$, where (e_n) is the canonical basis of c_0 and $\sharp$ denotes the cardinal of a set. Since ν is discrete, it follows that ν is not σ-finite.

Note that for all $x^* = (\alpha_n) \in c_0^* = \ell^1$ and $\omega \in [0, \infty)$ one has

$$x^* \nu(\{\omega\}) = \sum_{n=1}^{\infty} \frac{\sharp\big(\{\omega\} \cap [n-1, n)\big)}{2^n} \alpha_n = \sum_{n=1}^{\infty} \frac{\chi_{[n-1,n)}(\omega)}{2^n} \alpha_n$$

and so

$$|x^* \nu|(\{\omega\}) = |x^* \nu(\{\omega\})| = \sum_{n=1}^{\infty} \frac{\chi_{[n-1,n)}(w)}{2^n} |\alpha_n|.$$

The space $L^1_w(\nu)$ can be described as the space of functions $f\colon [0,\infty) \to \mathbb{R}$ such that $f\chi_{[n-1,n)} \in \ell^1\big([0,\infty)\big)$ for all n and $\sup_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))} < \infty$. Furthermore, $\|f\|_\nu = \sup_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))}$ for all $f \in L^1_w(\nu)$.

Indeed, if $f \in L^1_w(\nu)$, for every $n \geq 1$ and every finite set $J \subset [0,\infty)$, we have that

$$\begin{aligned}\sum_{w\in J} |f(w)|\chi_{[n-1,n)}(w) &= 2^n \sum_{w\in J} |f(w)||e_n\nu|(\{w\}) = 2^n \int |f|\chi_J \, d|e_n\nu| \\ &\leq 2^n \int |f| \, d|e_n\nu| < \infty.\end{aligned}$$

So, $f\chi_{[n-1,n)} \in \ell^1\big([0,\infty)\big)$ and

$$\sup_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))} \leq \sup_n \int |f| \, d|e_n\nu| \leq \|f\|_\nu.$$

Conversely, let $f\colon [0,\infty) \to \mathbb{R}$ be a function such that $f\chi_{[n-1,n)} \in \ell^1\big([0,\infty)\big)$ for all n and $\sup_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))} < \infty$. Take now $x^* = (\alpha_n) \in c_0^*$. Since $A_n = \mathrm{supp}(f) \cap [n-1,n)$ is countable for each n, applying the monotone convergence theorem we have that

$$\begin{aligned}\int |f|\chi_{[n-1,n)} \, d|x^*\nu| &= \sum_{\omega\in A_n} \int |f(\omega)|\chi_{\{\omega\}} \, d|x^*\nu| = \sum_{\omega\in A_n} |f(\omega)||x^*\nu|(\{\omega\}) \\ &= \sum_{\omega\in A_n} |f(\omega)|\frac{|\alpha_n|}{2^n} = \frac{|\alpha_n|}{2^n}\,\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))}.\end{aligned}$$

Then, applying again the monotone convergence theorem it follows

$$\begin{aligned}\int |f| \, d|x^*\nu| &= \sum_{n=1}^{\infty} \int |f|\chi_{[n-1,n)} \, d|x^*\nu| \\ &= \sum_{n=1}^{\infty} \frac{|\alpha_n|}{2^n}\,\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))} \qquad (2.2.1) \\ &\leq \|x^*\| \cdot \sup_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))}.\end{aligned}$$

Therefore, $f \in L^1_w(\nu)$ and $\|f\|_\nu \leq \sup_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))}$.

Let us see now that the space $L^1(\nu)$ is the space of functions $f\colon [0,\infty) \to \mathbb{R}$ such that $f\chi_{[n-1,n)} \in \ell^1\big([0,\infty)\big)$ for all n and $\lim_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))} = 0$.

Suppose $f \in L^1(\nu)$. Then $f\chi_{[n-1,n)} \in \ell^1\big([0,\infty)\big)$ for all n as $f \in L^1_w(\nu)$. From (2.2.1) and noting that $e_n\nu$ is a positive measure, we have that

$$\frac{\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))}}{2^n} \leq \int |f|\, d|e_n\nu| = \int |f|\, de_n\nu = e_n\Big(\int |f|\, d\nu\Big) \to 0$$

since $\int |f|\, d\nu \in c_0$ as $|f|$ is also in $L^1(\nu)$.

Conversely, let $f\colon [0,\infty) \to \mathbb{R}$ be a function such that $f\chi_{[n-1,n)} \in \ell^1\big([0,\infty)\big)$ for all n and $\lim_n \frac{1}{2^n}\big\|f\chi_{[n-1,n)}\big\|_{\ell^1([0,\infty))} = 0$. Clearly $f \in L^1_w(\nu)$. For each $A \in \mathcal{R}^{loc} = 2^{[0,\infty)}$, take the element

$$x_A = \left(\frac{\big\|f\chi_{A\cap[n-1,n)}\big\|_{\ell^1([0,\infty))}}{2^n}\right)_n \in c_0.$$

Then, for every $x^* = (\alpha_n) \in c_0^*$, we have that

$$x^*(x_A) = \sum_{n=1}^{\infty} \frac{\big\|f\chi_{A\cap[n-1,n)}\big\|_{\ell^1([0,\infty))}}{2^n}\, \alpha_n = \int_A |f|\, dx^*\nu,$$

where the last equality can be obtained similarly to (2.2.1) but using the order continuity of $L^1(x^*\nu)$ instead of the monotone convergence theorem. Therefore, $|f| \in L^1(\nu)$ with $\int_A |f|\, d\nu = x_A$ and so f is also in $L^1(\nu)$ with

$$\int_A f\, d\nu = \left(\frac{1}{2^n} \sum_{\omega\in A\cap[n-1,n)} f(\omega)\right)_n.$$

Note that every $f \in L^1_w(\nu)$ has countable support as $\mathrm{supp}(f) \cap [n-1,n)$ is countable for all n. If $B \in \mathcal{R}^{loc}$ is such that $\|\nu\|(B) < \infty$, that is $\chi_B \in L^1_w(\nu)$, then B is countable. Hence, ν is locally σ-finite.

Let us prove now that the super order density of $L^1(\nu)$ in $L^1_w(\nu)$ is characterized by the local σ-finiteness of ν.

Theorem 2.2.8. *The space $L^1(\nu)$ is super order dense in $L^1_w(\nu)$ if and only if ν is locally σ-finite.*

Proof. Suppose that $L^1(\nu)$ is super order dense in $L^1_w(\nu)$. For each $A \in \mathcal{R}^{loc}$ with $\|\nu\|(A) < \infty$, since $0 \leq \chi_A \in L^1_w(\nu)$, there exists a sequence $(f_n) \subset L^1(\nu)$

such that $0 \leq f_n \uparrow \chi_A$. Then, there is $Z \in \mathcal{R}^{loc}$ ν-null so that $0 \leq f_n(\omega) \uparrow \chi_A(\omega)$ for all $\omega \in \Omega \backslash Z$. Thus, $A \backslash Z = \cup_n \text{supp}(f_n) \backslash Z$.

On the other hand, since each $f_n \in L^1(\nu)$, from [29, Theorem 4.9.(a)] , there exist $(A_j^n)_j \subset \mathcal{R}$ and a ν-null set $N_n \in \mathcal{R}^{loc}$ such that $\text{supp}(f_n) = (\cup_j A_j^n) \cup N_n$. Then,

$$A = (\cup_n \cup_j A_j^n \backslash Z) \cup (\cup_n N_n \backslash Z) \cup (A \cap Z)$$

where $A_j^n \backslash Z \in \mathcal{R}$ and $(\cup_n N_n \backslash Z) \cup (A \cap Z)$ is ν-null.

Conversely, suppose that ν is locally σ-finite and let $0 \leq f \in L^1_w(\nu)$. There exists a sequence $(\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \psi_n \uparrow f$. For each n, we can write $\psi_n = \sum_{j=1}^{k_n} \alpha_j^n \chi_{B_j^n}$ with $(B_j^n)_j$ pairwise disjoint and $\alpha_j^n > 0$. Then, taking $\beta_n = \min\{\alpha_1^n, ..., \alpha_{k_n}^n\}$, it follows

$$\|\nu\|(\text{supp}(\psi_n)) = \|\chi_{\text{supp}(\psi_n)}\|_\nu \leq \frac{1}{\beta_n} \|\psi_n\|_\nu \leq \frac{1}{\beta_n} \|f\|_\nu < \infty.$$

So, there is $(A_j^n)_j \subset \mathcal{R}$ and Z_n ν-null such that $\text{supp}(\psi_n) = (\cup_j A_j^n) \cup Z_n$. Denote $\varphi_n = \psi_n \chi_{\cup_{i=1}^n \cup_{j=1}^n A_j^i} \in \mathcal{S}(\mathcal{R})$. For $\omega \notin \cup_n Z_n$ we have that $\omega \in \Omega \backslash (\cup_n \text{supp}(\psi_n))$ or $\omega \in \cup_n \cup_j A_j^n$. In any case, $\varphi_n(\omega) = \psi_n(\omega)$ for all n large enough. Then, $\varphi_n \uparrow f$. □

We have seen just before Example 2.2.4 that if ν is σ-finite then $L^1(\nu)$ is super order dense in $L^0(\nu)$. The converse also holds, indeed taking Ω instead of A in the proof of the local σ-finiteness of ν in Theorem 2.2.8, the same argument works to show $\Omega = (\cup A_n) \cup N$, with $N \in \mathcal{R}^{loc}$ ν-null and $(A_n) \subset \mathcal{R}$.

We know from [29, Theorem 4.9.(a)] that for each $f \in L^1(\nu)$ there are $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\text{supp}(f) = (\cup A_n) \cup N$. Does the same hold for functions in $L^1_w(\nu)$?

Proposition 2.2.9. *For each $f \in L^1_w(\nu)$ there exist $N \in \mathcal{R}^{loc}$ ν-null and $(A_n) \subset \mathcal{R}$ such that* $\text{supp}(f) = (\cup A_n) \cup N$ *if and only if ν is locally σ-finite.*

Proof. Suppose that ν is locally σ-finite and take $f \in L^1_w(\nu)$. From the proof of Theorem 2.2.8, there exists a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ such that $0 \leq \varphi_n \uparrow |f|$. Let $Z \in \mathcal{R}^{loc}$ be a ν-null set such that $0 \leq \varphi_n(\omega) \uparrow |f(\omega)|$ for all $\omega \in \Omega \backslash Z$.

Then,

$$\operatorname{supp}(f) = (\cup \operatorname{supp}(\varphi_n)\backslash Z) \cup (\operatorname{supp}(f) \cap Z)$$

where $\operatorname{supp}(\varphi_n)\backslash Z \in \mathcal{R}$ and $\operatorname{supp}(f) \cap Z$ is ν-null. For the converse only note that if $B \in \mathcal{R}^{loc}$ is such that $\|\nu\|(B) < \infty$, then $\chi_B \in L^1_w(\nu)$. $\square$

Let $\{\Omega_\alpha : \alpha \in \Delta\}$ be a maximal family of non ν-null sets in $\mathcal{R}$ with $\Omega_\alpha \cap \Omega_\beta$ ν-null for $\alpha \neq \beta$, see the proof of [4, Theorem 3.1] for the existence of such a family. Then, $L^1(\nu)$ is the unconditional direct sum of the spaces $L^1(\nu_\alpha)$ where $\nu_\alpha\colon \Sigma_\alpha \to X$ is the restriction of ν to the σ-algebra $\Sigma_\alpha = \{A \in \mathcal{R} : A \subset \Omega_\alpha\}$. More precisely, for each $f \in L^1(\nu)$ there exists a countable set $I \subset \Delta$ such that $f = \sum_{\alpha \in I} f\chi_{\Omega_\alpha}$ ν-a.e. and the sum converges unconditionally in $L^1(\nu)$, see [15, Theorem 3.6]. Does a similar result hold for the space $L^1_w(\nu)$? The ν-a.e. pointwise convergence of the sum for functions in $L^1_w(\nu)$ is again characterized by the local σ-finiteness of ν.

Proposition 2.2.10. *For each $f \in L^1_w(\nu)$ there exists a countable $I \subset \Delta$ such that $f = \sum_{\alpha\in I} f\chi_{\Omega_\alpha}$ ν-a.e. pointwise if and only if ν is locally σ-finite.*

Proof. Suppose that for every $f \in L^1_w(\nu)$ there exists a countable $I \subset \Delta$ such that $f = \sum_{\alpha\in I} f\chi_{\Omega_\alpha}$ ν-a.e. pointwise. Then, given $B \in \mathcal{R}^{loc}$ with $\|\nu\|(B) < \infty$, since $\chi_B \in L^1_w(\nu)$, we can write $\chi_B = \sum_{\alpha\in I} \chi_{B\cap\Omega_\alpha}$ pointwise except on a ν-null set Z, for some countable $I \subset \Delta$. So, $B = (\cup_{\alpha\in I} B \cap \Omega_\alpha) \cup (B \cap Z)$, where $B \cap \Omega_\alpha \in \mathcal{R}$ and $B \cap Z$ is ν-null.

Conversely, suppose that ν is locally σ-finite and take $f \in L^1_w(\nu)$. From Proposition 2.2.9, there exists $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\operatorname{supp}(f) = (\cup A_n) \cup N$. Since each $A_n \in \mathcal{R}$, there exists a countable set $I_n \subset \Delta$ such that $A_n \cap \Omega_\beta$ is ν-null for all $\beta \in \Delta\backslash I_n$ (see the proof of [4, Theorem 3.1]). Take $I = \cup I_n$ and $Z = \operatorname{supp}(f) \backslash \cup_{\alpha\in I} \Omega_\alpha$. Let us see that Z is a ν-null set. Given $B \in \mathcal{R} \cap 2^Z$, if $\beta \in I$ we have that $B \cap \Omega_\beta = \emptyset$. On the other hand, if $\beta \notin I$, since $B \cap \Omega_\beta \subset \operatorname{supp}(f) \cap \Omega_\beta = (\cup A_n \cap \Omega_\beta) \cup (N \cap \Omega_\beta)$ where each $A_n \cap \Omega_\beta$ is ν-null, we have that $B \cap \Omega_\beta$ is ν-null. From the maximality of the family $\{\Omega_\alpha : \alpha \in \Delta\}$ it follows that B is ν-null. Then, $f = \sum_{\alpha\in I} f\chi_{\Omega_\alpha}$ pointwise except on $Z \cup (\cup_{\beta\in I} \cup_{\alpha\in I\backslash\{\beta\}} \Omega_\alpha \cap \Omega_\beta)$ which is a ν-null set. $\square$

Since $f\chi_{\Omega_\alpha} \in L^1_w(\nu_\alpha)$ for all $\alpha \in \Delta$ whenever $f \in L^1_w(\nu)$, in the case of ν

being locally σ-finite, we can say that the space $L^1_w(\nu)$ is the ν-a.e. pointwise direct sum of the spaces $L^1_w(\nu_\alpha)$.

We cannot expect that $\sum_{\alpha\in I} f\chi_{\Omega_\alpha}$ converges unconditionally to f in $L^1_w(\nu)$ for a countable set $I \subset \Delta$. Indeed, unconditional convergence of the sum in $L^1(\nu)$ is due to the order continuity of $L^1(\nu)$. For instance, if ν is a discrete vector measure (see Lemma 2.2.6), taking $\big\{\{\gamma\} : \gamma \in \Gamma\big\}$ which is a maximal family of non ν-null sets in $\mathcal{R}$ with $\{\alpha\}\cap\{\beta\}$ ν-null for $\alpha \neq \beta$, we have that if $f \in L^1_w(\nu)$ is such that $\sum_n f\chi_{\{\gamma_n\}}$ converges to f in norm $\|\cdot\|_\nu$, then $f \in L^1(\nu)$, since $\sum_{k=1}^n f\chi_{\{\gamma_k\}} = \sum_{k=1}^n f(\gamma_k)\chi_{\{\gamma_k\}} \in \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$ and $L^1(\nu)$ is closed in $L^1_w(\nu)$.

2.3 Fatou property for $L^1_w(\nu)$

The space $L^1_w(\nu)$ always has the σ-Fatou property. We include the proof for completeness. Given $(f_n) \subset L^1_w(\nu)$ such that $0 \le f_n \uparrow$ and $\sup\|f_n\|_\nu < \infty$, there exists a ν-null set $Z \in \mathcal{R}^{loc}$ such that $0 \le f_n(\omega) \uparrow$ for all $\omega \in \Omega\backslash Z$. Taking the measurable function $g \colon \Omega \to [0,\infty]$ defined by $g(\omega) = \sup f_n(\omega)$ if $\omega \in \Omega\backslash Z$ and $g(\omega) = 0$ if $\omega \in Z$, we have that $0 \le f_n\chi_{\Omega\backslash Z} \uparrow g$ pointwise and, by the monotone convergence theorem,

$$\int g\, d|x^*\nu| = \lim_n \int f_n\chi_{\Omega\backslash Z}\, d|x^*\nu| \le \|x^*\| \sup\|f_n\|_\nu$$

for every $x^* \in X^*$. So, $\|g\|_\nu \le \sup\|f_n\|_\nu < \infty$, and then $g < \infty$ ν-a.e. (except on a ν-null set N). Taking $f = g\chi_{\Omega\backslash N}$ we have that $f \colon \Omega \to [0,\infty)$ and $\|f\|_\nu = \|g\|_\nu < \infty$, so $f \in L^1_w(\nu)$. Moreover, $0 \le f_n \uparrow f$ with $\|f\|_\nu = \sup\|f_n\|_\nu$, as $\|f_n\|_\nu \le \|f\|_\nu \le \sup\|f_n\|_\nu$ for all n.

In the case when ν is defined on a σ-algebra, it was noted in [10, p. 191] that $L^1_w(\nu)$ is the σ-*Fatou completion* of $L^1(\nu)$, that is, the minimal B.f.s. related to ν, with the σ-Fatou property and containing $L^1(\nu)$. This fact does not hold for the general case. For instance, if ν is the vector measure defined in Example 2.2.1 and $\ell^\infty_0(\Gamma)$ denotes the Banach lattice of all real bounded functions on Γ with countable support, then $L^1(\nu) \subsetneq \ell^\infty_0(\Gamma) \subsetneq L^1_w(\nu)$ where $\ell^\infty_0(\Gamma)$ has the σ-Fatou property. Note that in this case ν is not locally σ-finite, as $\chi_\Gamma \in L^1_w(\nu)$.

This is the reason for which $L^1_w(\nu)$ fails to be the σ-Fatou completion of $L^1(\nu)$. Let us denote by $[L^1(\nu)]_{\sigma\text{-F}}$ the σ-Fatou completion of $L^1(\nu)$. In general we have that $[L^1(\nu)]_{\sigma\text{-F}} \subset L^1_w(\nu)$.

Theorem 2.3.1. *The σ-Fatou completion of $L^1(\nu)$ can be described as*

$$[L^1(\nu)]_{\sigma\text{-F}} = \big\{ f \in L^1_w(\nu) : \operatorname{supp}(f) = (\cup A_n) \cup N \text{ with } (A_n) \subset \mathcal{R} \text{ and } N\, \nu\text{-null} \big\}.$$

Consequently, the space $L^1_w(\nu) = [L^1(\nu)]_{\sigma\text{-F}}$ if and only if ν is locally σ-finite.

Proof. Denote by F the space of functions $f \in L^1_w(\nu)$ for which there exist $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\operatorname{supp}(f) = (\cup A_n) \cup N$. Let us see that F is a closed subspace of $L^1_w(\nu)$. Given $f \in L^1_w(\nu)$ and $(f_n) \subset F$ such that $\|f - f_n\|_\nu \to 0$, we can take a subsequence such that $f_{n_k} \to f$ ν-a.e. That is, there exists a ν-null set $Z \in \mathcal{R}^{loc}$ such that $f_{n_k}(\omega) \to f(\omega)$ for all $\omega \in \Omega\backslash Z$. Then, $\operatorname{supp}(f)\backslash Z \subset \cup_k \operatorname{supp}(f_{n_k})$. On the other hand, each f_{n_k} satisfies that $\operatorname{supp}(f_{n_k}) = (\cup_j A_j^k) \cup N_k$ for some $(A_j^k)_j \subset \mathcal{R}$ and $N_k \in \mathcal{R}^{loc}$ ν-null. So, $\operatorname{supp}(f) = \cup_k \cup_j B_j^k \cup N$ where $B_j^k = A_j^k \cap \operatorname{supp}(f)\backslash Z \in \mathcal{R}$ and $N = (\cup_k N_k \cap \operatorname{supp}(f)\backslash Z) \cup (\operatorname{supp}(f) \cap Z)$ is ν-null, that is, $f \in F$.

Note that if $|f| \leq |g|$ with $f \in L^0(\nu)$ and $g \in F$, then $f \in F$ since $\operatorname{supp}(f)\backslash Z = (\operatorname{supp}(f)\backslash Z) \cap \operatorname{supp}(g)$ for some ν-null set Z. Therefore, F endowed with the norm $\|\cdot\|_\nu$, is a B.f.s. related to ν, which, by [29, Theorem 4.9.(a)], contains $L^1(\nu)$.

Let us see now that F has the σ-Fatou property. Given $(f_n) \subset F$ such that $0 \leq f_n \uparrow$ and $\sup \|f_n\|_\nu < \infty$, since $L^1_w(\nu)$ has the σ-Fatou property, there exists $f = \sup f_n \in L^1_w(\nu)$ with $\|f\|_\nu = \sup \|f_n\|_\nu$. Moreover, since $0 \leq f_n \uparrow f$, $\operatorname{supp}(f) = (\cup \operatorname{supp}(f_n)\backslash Z) \cup (\operatorname{supp}(f) \cap Z)$ for some ν-null set $Z \in \mathcal{R}^{loc}$. Then, it follows that $f \in F$, as each $f_n \in F$.

Finally, suppose that E is a B.f.s. related to ν, with the σ-Fatou property and containing $L^1(\nu)$. Let $f \in F$ and take a sequence $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\operatorname{supp}(f) = (\cup A_n) \cup N$. On the other hand, take a sequence $(\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \psi_n \uparrow |f|$. Denoting $\varphi_n = \psi_n \chi_{\cup_{j=1}^n A_j} \in \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$ we have that $0 \leq \varphi_n \uparrow |f|$. Since $L^1(\nu) \subset E$ continuously (see Section 1.1) and then $\sup \|\varphi_n\|_E \leq C \sup \|\varphi_n\|_\nu \leq C\|f\|_\nu < \infty$ for some positive constant

C, it follows that there exists $g = \sup \varphi_n \in E$. Then, since $0 \leq \varphi_n \uparrow g$, we have that $|f| = g \in E$ and so $f \in E$.

The consequence follows from Proposition 2.2.9. $\square$

Consider now the *Fatou completion* $[L^1(\nu)]_{\mathrm{F}}$ of $L^1(\nu)$, that is, the minimal B.f.s. related to ν, with the Fatou property and containing $L^1(\nu)$. The σ-Fatou completion $[L^1(\nu)]_{\sigma\text{-}\mathrm{F}}$ always exists since $L_w^1(\nu)$ always has the σ-Fatou property. However, we do not know if in general $L_w^1(\nu)$ has the Fatou property, so $[L^1(\nu)]_{\mathrm{F}}$ could not exist.

Remark 2.3.2. In the case when $[L^1(\nu)]_{\mathrm{F}}$ exists, we have that

$$L^1(\nu) \subset [L^1(\nu)]_{\sigma\text{-}\mathrm{F}} \subset L_w^1(\nu) \subset [L^1(\nu)]_{\mathrm{F}}.$$

Indeed, given $f \in L_w^1(\nu)$, from Remark 2.2.3, there exists $(f_\tau) \subset L^1(\nu)$ such that $0 \leq f_\tau \uparrow |f|$ in $L^0(\nu)$. Since $L^1(\nu) \subset [L^1(\nu)]_{\mathrm{F}}$ continuously, it follows that $\sup \|f_\tau\|_{[L^1(\nu)]_{\mathrm{F}}} \leq C \sup \|f_\tau\|_\nu \leq C\|f\|_\nu < \infty$ for some constant $C > 0$. Then, there exists $g = \sup f_\tau$ in $[L^1(\nu)]_{\mathrm{F}}$. Noting that $f_\tau \leq g \in L^0(\nu)$ for all τ, we have that $|f| \leq g$ and so $|f| \in [L^1(\nu)]_{\mathrm{F}}$. Hence, $f \in [L^1(\nu)]_{\mathrm{F}}$. Note that actually $|f| = g$, since $f_\tau \leq |f| \in [L^1(\nu)]_{\mathrm{F}}$ for all τ and so $g \leq |f|$.

Remark 2.3.3. If $L_w^1(\nu)$ has the Fatou property, then $[L^1(\nu)]_{\mathrm{F}}$ exists and, from Remark 2.3.2, we have that $L_w^1(\nu) = [L^1(\nu)]_{\mathrm{F}}$.

The following result we give conditions under which $L_w^1(\nu)$ has the Fatou property. These conditions are satisfied for instance if ν takes values in a Banach space without any copy of c_0.

Proposition 2.3.4. *The following statements are equivalent:*

(a) $L^1(\nu) = L_w^1(\nu)$.

(b) $L_w^1(\nu)$ *is order continuous.*

(c) $L^1(\nu)$ *has the σ-Fatou property.*

If (a)-(c) hold, then $L_w^1(\nu)$ has the Fatou property and

$$L^1(\nu) = [L^1(\nu)]_{\sigma\text{-}\mathrm{F}} = L_w^1(\nu) = [L^1(\nu)]_{\mathrm{F}}.$$

Proof. The equivalence between (a) and (b) follows from Theorem 2.1.2. Condition (a) implies (c) as $L^1_w(\nu)$ always has the σ-Fatou property. Conversely, suppose that $L^1(\nu)$ has the σ-Fatou property. Let $(f_\tau) \subset L^1(\nu)$ such that $0 \leq f_\tau \uparrow$ and $\sup \|f_\tau\|_\nu < \infty$. Since $L^1(\nu)$ is order continuous, from [34, Theorem 113.4], it follows that there exists $f = \sup f_\tau$ in $L^1(\nu)$. Moreover, $\|f - f_\tau\|_\nu \downarrow 0$ as $f - f_\tau \downarrow 0$. Since $0 \leq \|f\|_\nu - \|f_\tau\|_\nu \leq \|f - f_\tau\|_\nu$, we have that $\|f\|_\nu = \sup \|f_\tau\|_\nu$. So, $L^1(\nu)$ actually has the Fatou property. Then, $[L^1(\nu)]_{\mathrm{F}} = L^1(\nu)$ and, from Remark 2.3.2, we have that $L^1(\nu) = L^1_w(\nu)$. So, (c) implies (a) and the last part of the proposition holds. $\square$

It is an open question if in general $L^1_w(\nu)$ has the Fatou property. Comparing with the proof of the σ-Fatou property for $L^1_w(\nu)$, the problem is that for an upwards directed system $0 \leq f_\tau \uparrow$ such that $(f_\tau) \subset L^1_w(\nu)$ with $\sup \|f_\tau\|_\nu < \infty$, if we consider the pointwise supremum $f = \sup f_\tau$, firstly f may not be measurable and even if $f \in L^0(\nu)$ may be $f_\tau \uparrow f$ does not hold, that is, f may not be the lattice supremum of (f_τ).

However we can give sufficient conditions for $L^1_w(\nu)$ to have the Fatou property. First, in the following proposition we will see that ν being σ-finite is a sufficient condition. This result will be the starting point to obtain a generalization of itself.

Proposition 2.3.5. *If ν is σ-finite, then $L^1_w(\nu)$ has the Fatou property. Moreover, in this case, $[L^1(\nu)]_{\sigma\text{-F}} = L^1_w(\nu) = [L^1(\nu)]_{\mathrm{F}}$.*

Proof. If ν is σ-finite, we can take a measure of the type $|x_0^*\nu|$ (with $x_0^* \in B_{X^*}$) having the same null sets as ν, see [15, Remark 3.4]. From [34, Theorem 113.4], the space $L^1(|x_0^*\nu|)$ has the Fatou property and is super Dedekind complete. In particular, $L^1(|x_0^*\nu|)$ is order separable (see [27, Definition 23.1 and Theorem 23.2.(iii)]), that is, if $0 \leq f_\tau \uparrow f$ in $L^1(|x_0^*\nu|)$ then there exists a sequence (f_{τ_n}) such that $f_{\tau_n} \uparrow f$.

Let $(f_\tau) \subset L^1_w(\nu)$ be an upwards directed system $0 \leq f_\tau \uparrow$ with $\sup \|f_\tau\|_\nu < \infty$. Then, $(f_\tau) \subset L^1(|x_0^*\nu|)$ is such that $0 \leq f_\tau \uparrow$ and $\sup \int |f_\tau|\, d|x_0^*\nu| \leq \sup \|f_\tau\|_\nu < \infty$. Since $L^1(|x_0^*\nu|)$ has the Fatou property, there exists $f = \sup f_\tau$ in $L^1(|x_0^*\nu|)$ and, since $L^1(|x_0^*\nu|)$ is order separable, we can take a sequence

$f_{\tau_n} \uparrow f$ in $L^1(|x_0^*\nu|)$. Then, $f_{\tau_n} \uparrow f$ $|x_0^*\nu|$-a.e. (equivalently ν-a.e.) and so $|x^*\nu|$-a.e. for all $x^* \in X^*$. By using the monotone convergence theorem, we have that

$$\int |f|\, d|x^*\nu| = \lim_n \int |f_{\tau_n}|\, d|x^*\nu| \leq \|x^*\| \cdot \sup_\tau \|f_\tau\|_\nu < \infty,$$

and so $f \in L^1(|x^*\nu|)$ for all $x^* \in X^*$. Hence, $f \in L^1_w(\nu)$ and $\|f\|_\nu \leq \sup_\tau \|f_\tau\|_\nu$.

Since the $|x_0^*\nu|$-a.e. pointwise order coincides with the ν-a.e. one and $0 \leq f_\tau \uparrow f$ in $L^1(|x_0^*\nu|)$, it follows that $0 \leq f_\tau \uparrow f$ in $L^1_w(\nu)$. Indeed if $g \in L^1_w(\nu)$ is such that $f_\tau \leq g$ ν-a.e. for all τ, then $g \in L^1_w(|x_0^*\nu|)$ is such that $f_\tau \leq g$ $|x_0^*\nu|$-a.e. for all τ, and so $f \leq g$ $|x_0^*\nu|$-a.e. or equivalently ν-a.e. Moreover, since $\|f_\tau\|_\nu \leq \|f\|_\nu$ for all τ, we have that $\|f\|_\nu = \sup_\tau \|f_\tau\|_\nu$. Therefore, $L^1_w(\nu)$ has the Fatou property.

From Theorem 2.3.1 and Remark 2.3.3, it follows that $[L^1(\nu)]_{\sigma\text{-F}} = L^1_w(\nu) = [L^1(\nu)]_{\text{F}}$. $\square$

Note that from Proposition 2.3.5, we have that $L^1_w(\nu)$ has the Fatou property for every vector measure ν defined on a σ-algebra. We will give now a more general condition than the σ-finiteness of ν under which $L^1_w(\nu)$ has the Fatou property.

Definition 2.3.6. A vector measure ν will be said to be $\mathcal{R}$-*decomposable* if we can write $\Omega = (\cup_{\alpha\in\Delta}\Omega_\alpha) \cup N$ where $N \in \mathcal{R}^{loc}$ is a ν-null set and $\{\Omega_\alpha : \alpha \in \Delta\}$ is a family of pairwise disjoint sets in $\mathcal{R}$ satisfying that

(i) if $A_\alpha \in \mathcal{R} \cap 2^{\Omega_\alpha}$ for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} A_\alpha \in \mathcal{R}^{loc}$, and
(ii) for each $x^* \in X^*$, if $Z_\alpha \in \mathcal{R} \cap 2^{\Omega_\alpha}$ is $|x^*\nu|$-null for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} Z_\alpha$ is $|x^*\nu|$-null.

Note that condition (ii) implies that if $Z_\alpha \in \mathcal{R} \cap 2^{\Omega_\alpha}$ is ν-null for all $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta} Z_\alpha$ is ν-null. Also note that N can be taken to be disjoint with $\cup_{\alpha\in\Delta}\Omega_\alpha$.

Remark 2.3.7. There always exists a maximal family $\{\widetilde{\Omega}_\alpha : \alpha \in \Delta\}$ of non ν-null sets in $\mathcal{R}$ with $\widetilde{\Omega}_\alpha \cap \widetilde{\Omega}_\beta$ ν-null for $\alpha \neq \beta$ (see the comments just before Proposition 2.2.10). If this family satisfies (i) and (ii) of Definition 2.3.6, then by taking $\Omega_\alpha = \widetilde{\Omega}_\alpha \backslash (\cup_{\beta\in\Delta\backslash\{\alpha\}} \widetilde{\Omega}_\beta)$ we obtain a disjoint decomposition of Ω as in Definition 2.3.6.

There are plenty of $\mathcal{R}$-decomposable vector measures, for instance σ-finite vector measures and discrete vector measures are so. The σ-finite case is obvious. For the discrete case (see Lemma 2.2.6), for instance, we can write $\Omega = \cup_{\omega\in\Omega}\{\omega\}$. Recall that $\{A \subset \Omega : A \text{ is finite}\} \subset \mathcal{R} \subset \{A \subset \Omega : A \text{ is countable}\}$. Note that $\mathcal{R}^{loc} = 2^{\Omega}$, so condition (i) holds. Given $x^* \in X^*$, denoting $N_{x^*} = \{\omega \in \Omega : x^*\nu(\{\omega\}) = 0\}$, we have that $A \subset \Omega$ is $|x^*\nu|$-null if and only if $A \subset N_{x^*}$. Indeed, if A is $|x^*\nu|$-null, for every $\omega \in A$ it follows that $\{\omega\} \in \mathcal{R} \cap 2^A$ and so $x^*\nu(\{\omega\}) = 0$. Conversely, if $A \subset N_{x^*}$, for every $B \in \mathcal{R}\cap 2^A$ it follows that $B = \cup_{\omega\in B}\{\omega\}$ where the union is countable. Then, $x^*\nu(B) = \sum_{\omega\in B} x^*\nu(\{\omega\}) = 0$. Therefore, condition (ii) holds.

Theorem 2.3.8. *If ν is $\mathcal{R}$-decomposable, then $L^1_w(\nu)$ has the Fatou property.*

Proof. Suppose that ν is $\mathcal{R}$-decomposable and take a ν-null set $N \in \mathcal{R}^{loc}$ and a family $\{\Omega_\alpha : \alpha \in \Delta\}$ of pairwise disjoint sets in $\mathcal{R}$ satisfying conditions (i) and (ii) in Definition 2.3.6 such that $\Omega = (\cup_{\alpha\in\Delta}\Omega_\alpha) \cup N$ with disjoint union. For every finite set $I \subset \Delta$, consider $\Omega_I = \cup_{\alpha\in I}\Omega_\alpha \in \mathcal{R}$ and the vector measure $\nu_I : \mathcal{R}^{loc} \to X$ defined by $\nu(A \cap \Omega_I)$ for all $A \in \mathcal{R}^{loc}$. Given $f \in \mathcal{M}(\mathcal{R}^{loc})$, by using a similar argument as in the proof of (c) implies (a) in Lemma 2.1.1, it follows that $f \in L^1_w(\nu_I)$ if and only if $f\chi_{\Omega_I} \in L^1_w(\nu)$, and in this case $\|f\|_{\nu_I} = \|f\chi_{\Omega_I}\|_\nu$. Note that, if $f \in L^1_w(\nu)$ then $f\chi_{\Omega_I} \in L^1_w(\nu)$ and so $f \in L^1_w(\nu_I)$. From Proposition 2.3.5 we have that $L^1_w(\nu_I)$ has the Fatou property as ν_I is defined on a σ-algebra.

Let $(f_\tau) \subset L^1_w(\nu)$ be such that $0 \le f_\tau \uparrow$ and $\sup\|f_\tau\|_\nu < \infty$. Since $L^1_w(\nu) \subset L^1_w(\nu_I)$ and every $Z \in \mathcal{R}^{loc}$ ν-null is ν_I-null (as $\|\nu_I\|(Z) = \|\nu\|(Z\cap\Omega_I)$), then $0 \le f_\tau \uparrow$ in $L^1_w(\nu_I)$. Moreover, $\sup\|f_\tau\|_{\nu_I} = \sup\|f_\tau\chi_{\Omega_I}\|_\nu \le \sup\|f_\tau\|_\nu < \infty$. By the Fatou property of $L^1_w(\nu_I)$, there exists $f^I = \sup f_\tau$ in $L^1_w(\nu_I)$ and $\|f^I\|_{\nu_I} = \sup\|f_\tau\|_{\nu_I}$.

Now we consider $I = \{\alpha\}$ for each $\alpha \in \Delta$ and construct the function $f : \Omega \to \mathbb{R}$ as $f(\omega) = f^{\{\alpha\}}(\omega)$ when $\omega \in \Omega_\alpha$ and $f(\omega) = 0$ when $\omega \in N$, which is well defined since Ω is a disjoint union of $(\Omega_\alpha)_{\alpha\in\Delta}$ and N. By (i), we have that $f^{-1}(B) = \cup_{\alpha\in\Delta}(f^{\{\alpha\}})^{-1}(B) \cap \Omega_\alpha \in \mathcal{R}^{loc}$ for every Borel subset B of $\mathbb{R}$ such that $0 \notin B$. If $0 \in B$, we add to the union the set N to get $f^{-1}(B)$. So, $f \in \mathcal{M}(\mathcal{R}^{loc})$.

Let us see that $f \in L^1_w(\nu)$. First note that for each finite set $I \subset \Delta$ and $\alpha \in I$, it follows that $f^{\{\alpha\}}\chi_{\Omega_\alpha} \leq f^I\chi_{\Omega_\alpha}$ ν-a.e. and $g = g\chi_{\Omega_\alpha}$ $\nu_{\{\alpha\}}$-a.e. for every $g \in \mathcal{M}(\mathcal{R}^{loc})$. Indeed, $f_\tau\chi_{\Omega_\alpha} \uparrow f^{\{\alpha\}}\chi_{\Omega_\alpha}$ in $L^1_w(\nu_{\{\alpha\}})$ as $f_\tau \uparrow f^{\{\alpha\}}$ in $L^1_w(\nu_{\{\alpha\}})$. Since $f_\tau\chi_{\Omega_\alpha} \leq f^I\chi_{\Omega_\alpha}$ ν_I-a.e. and so also $\nu_{\{\alpha\}}$-a.e. and $f^I\chi_{\Omega_\alpha} \in L^1_w(\nu_{\{\alpha\}})$ as $f^I\chi_{\Omega_\alpha} \leq f^I\chi_{\Omega_I} \in L^1_w(\nu)$, we have that $f^{\{\alpha\}}\chi_{\Omega_\alpha} \leq f^I\chi_{\Omega_\alpha}$ $\nu_{\{\alpha\}}$-a.e. (except on a $\nu_{\{\alpha\}}$-null set Z) and so ν-a.e. (except on the ν-null set $Z \cap \Omega_\alpha$). Then, $f\chi_{\Omega_I} = \sum_{\alpha\in I} f^{\{\alpha\}}\chi_{\Omega_\alpha} \leq f^I\chi_{\Omega_I}$ ν-a.e.

Fix $x^* \in X^*$. For every finite set $I \subset \Delta$, it follows

$$\begin{aligned}\sum_{\alpha\in I}\int |f|\chi_{\Omega_\alpha}\, d|x^*\nu| &= \int |f|\chi_{\Omega_I}\, d|x^*\nu| \leq \int |f^I|\chi_{\Omega_I}\, d|x^*\nu| \\ &\leq \|x^*\|\cdot\|f^I\chi_{\Omega_I}\|_\nu = \|x^*\|\cdot\|f^I\|_{\nu_I} \\ &= \|x^*\|\cdot\sup\|f_\tau\|_{\nu_I} \leq \|x^*\|\cdot\sup\|f_\tau\|_\nu < \infty.\end{aligned}$$

Then, there exists a countable set $J \subset \Delta$ such that $\int |f|\chi_{\Omega_\alpha}\, d|x^*\nu| = 0$ for all $\alpha \in \Delta\backslash J$ and so $f\chi_{\Omega_\alpha} = 0$ $|x^*\nu|$-a.e. (except on a $|x^*\nu|$-null set $Z_\alpha \in \mathcal{R}^{loc}$ which can be taken such that $Z \subset \Omega_\alpha$) for all $\alpha \in \Delta\backslash J$. Hence, $f = \sum_{\alpha\in J} f\chi_{\Omega_\alpha}$ $|x^*\nu|$-a.e. (except on the set $\cup_{\alpha\in\Delta\backslash J} Z_\alpha \cup N \in \mathcal{R}^{loc}$ which, by (ii), is $|x^*\nu|$-null). By the monotone convergence theorem we have that

$$\int |f|\, d|x^*\nu| = \sum_{\alpha\in J}\int |f|\chi_{\Omega_\alpha}\, d|x^*\nu| \leq \|x^*\|\cdot\sup\|f_\tau\|_\nu < \infty.$$

So $f \in L^1_w(\nu)$ and $\|f\|_\nu \leq \sup\|f_\tau\|_\nu$.

Let us see now that $f_\tau \uparrow f$ in $L^1_w(\nu)$. Fixing τ, for each $\alpha \in \Delta$, there exists a $\nu_{\{\alpha\}}$-null set $Z_\alpha \in \mathcal{R}^{loc}$ such that $f_\tau(\omega) \leq f^{\{\alpha\}}(\omega)$ for all $\omega \in \Omega_\alpha\backslash Z_\alpha$. Then, $Z = \cup_{\alpha\in\Delta} Z_\alpha \cap \Omega_\alpha$ is ν-null and $f_\tau(\omega) \leq f(\omega)$ for all $\omega \in \Omega\backslash(Z \cup N)$, that is, $f_\tau \leq f$ ν-a.e. Suppose that $h \in L^1_w(\nu)$ is such that $f_\tau \leq h$ ν-a.e. (except on a ν-null set $Z \in \mathcal{R}^{loc}$) and so $\nu_{\{\alpha\}}$-a.e. (except Z which also is $\nu_{\{\alpha\}}$-null) for each τ. Since $h \in L^1_w(\nu_{\{\alpha\}})$, we have that $f^{\{\alpha\}} \leq h$ $\nu_{\{\alpha\}}$-a.e. (except on a $\nu_{\{\alpha\}}$-null set $Z_\alpha \in \mathcal{R}^{loc}$). Therefore, $f \leq h$ ν-a.e. (except on the ν-null set $(\cup_{\alpha\in\Delta} Z_\alpha \cap \Omega_\alpha) \cup N \in \mathcal{R}^{loc}$). So, $f_\tau \uparrow f$ and $\|f\|_\nu = \sup\|f_\tau\|_\nu$. □

The converse of Theorem 2.3.8 does not hold as the next example shows.

Example 2.3.9. Following [21, p. 12, Definition 211E], a measure space (X, Σ, μ)

is *decomposable* (or *strictly localizable*) if there is a disjoint family $\{X_\alpha : \alpha \in \Delta\}$ of measurable sets of finite measure such that $X = \cup_{\alpha\in\Delta} X_\alpha$ and

$$\Sigma = \{E \subset X : \ E \cap X_\alpha \in \Sigma \ \text{ for all } \ \alpha \in \Delta\}$$

with $\mu(E) = \sum_{\alpha\in\Delta} \mu(E \cap X_\alpha)$ for every $E \in \Sigma$. In [21, p. 50, 216E], Fremlin constructs a measure space which is not decomposable as follows.

Let C be an abstract set of cardinal greater than the cardinal of the continuum, $\mathcal{K} = \{K \subset 2^C : K \text{ is countable}\}$ and X the set of all functions $f\colon 2^C \to \{0,1\}$. For each $\gamma \in C$, write f_γ for the function in X defined by $f_\gamma(A) = \chi_A(\gamma)$ for all $A \in 2^C$ and $F_{\gamma,K} = \{f \in X : f_{|K} = f_{\gamma|K}\}$ for every $K \in \mathcal{K}$, where $g_{|K}$ denotes the restriction of a function g to the set K. Consider the σ-algebra $\Sigma = \cap_{\gamma\in C}\Sigma_\gamma$, where

$$\Sigma_\gamma = \{E \subset X : \exists\ K \in \mathcal{K} \text{ with } F_{\gamma,K} \subset E \ \text{ or } \ \exists\ K \in \mathcal{K} \text{ with } F_{\gamma,K} \subset X\backslash E\},$$

and the measure $\mu\colon \Sigma \to [0,\infty]$ defined by $\mu(E) = \sharp(\{\gamma \in C : f_\gamma \in E\})$ for all $E \in \Sigma$, where $\sharp$ denotes the cardinal of a set. Then, (X, Σ, μ) is not decomposable.

Taking the δ-ring $\mathcal{R} = \{E \in \Sigma : \mu(E) < \infty\}$, we will show that the measure $\widetilde{\mu}\colon \mathcal{R} \to [0,\infty)$ given by the restriction of μ to $\mathcal{R}$ is not $\mathcal{R}$-decomposable.

Let us see first that

$$\mathcal{R}^{loc} = \Sigma. \tag{2.3.1}$$

If $A \in \Sigma$, then obviously $A \cap E \in \mathcal{R}$ for every $E \in \mathcal{R}$, that is $A \in \mathcal{R}^{loc}$. Conversely, suppose that $A \in \mathcal{R}^{loc}$. For a fixed $\gamma \in C$, consider the set $G_{\{\gamma\}} = \{f \in X : f(\{\gamma\}) = 1\}$ which is in Σ and $\mu(G_{\{\gamma\}}) = \sharp(\{\gamma\}) = 1$ (see [21, 216E.(c)]). Then, $G_{\{\gamma\}} \in \mathcal{R}$ and thus $A \cap G_{\{\gamma\}} \in \mathcal{R} \subset \Sigma \subset \Sigma_\gamma$. If there exists $K \in \mathcal{K}$ such that $F_{\gamma,K} \subset A \cap G_{\{\gamma\}} \subset A$, then $A \in \Sigma_\gamma$. If there exists $K \in \mathcal{K}$ such that $F_{\gamma,K} \subset X\backslash(A \cap G_{\{\gamma\}})$, then, since $F_{\gamma,K\cup\{\gamma\}} \subset F_{\gamma,K}$ and $F_{\gamma,K\cup\{\gamma\}} \subset G_{\{\gamma\}}$, it follows that $F_{\gamma,K\cup\{\gamma\}} \subset X\backslash A$ and so $A \in \Sigma_\gamma$. Therefore, $A \in \Sigma$ and (2.3.1) holds.

Moreover, for $N \in \mathcal{R}^{loc}$ we have that

$$N \text{ is } \widetilde{\mu}\text{-null if and only if } N \text{ is } \mu\text{-null.} \tag{2.3.2}$$

Indeed, if N is μ-null, for every $E \in \mathcal{R} \cap 2^N$ we have that $\widetilde{\mu}(E) = \mu(E) \leq \mu(N) = 0$ and so N is $\widetilde{\mu}$-null. Conversely, suppose that N is $\widetilde{\mu}$-null. If $\mu(N) > 0$, then there exists $\gamma \in C$ such that $\mu(N \cap G_{\{\gamma\}}) = 1$ (see [21, 216E.(h)]), this is a contradiction as $N \cap G_{\{\gamma\}} \in \mathcal{R} \cap 2^N$ and so $\mu(N \cap G_{\{\gamma\}}) = \widetilde{\mu}(N \cap G_{\{\gamma\}}) = 0$.

Suppose that $\widetilde{\mu}$ is $\mathcal{R}$-decomposable, that is, we can write $X = \big(\cup_{\alpha \in \Delta} X_\alpha\big) \cup N$ where $\{X_\alpha : \alpha \in \Delta\}$ is a family of pairwise disjoint sets in $\mathcal{R}$ satisfying that

(i) if $A_\alpha \in \mathcal{R} \cap 2^{X_\alpha}$ for all $\alpha \in \Delta$, then $\cup_{\alpha \in \Delta} A_\alpha \in \mathcal{R}^{loc}$,

(ii) if $Z_\alpha \in \mathcal{R} \cap 2^{X_\alpha}$ is $\widetilde{\mu}$-null for all $\alpha \in \Delta$, then $\cup_{\alpha \in \Delta} Z_\alpha$ is $\widetilde{\mu}$-null,

and $N \in \mathcal{R}^{loc}$ is a $\widetilde{\mu}$-null set disjoint with each X_α. From (2.3.1) and (2.3.2), $N \in \Sigma$ is μ-null. Then, $\{X_\alpha : \alpha \in \Delta\} \cup \{N\}$ is a disjoint family of sets in Σ with $\mu(N), \mu(X_\alpha) < \infty$. Let us see that

$$\Sigma = \{E \subset X : \; E \cap N \in \Sigma \text{ and } E \cap X_\alpha \in \Sigma \text{ for all } \alpha \in \Delta\}.$$

If $E \in \Sigma$, then obviously $E \cap N \in \Sigma$ and $E \cap X_\alpha \in \Sigma$ for all $\alpha \in \Delta$. Conversely, if $E \subset X$ is such that $E \cap N \in \Sigma$ and $E \cap X_\alpha \in \Sigma$ for all $\alpha \in \Delta$, since $E \cap X_\alpha \in \mathcal{R} \cap 2^{X_\alpha}$, by (i) and (2.3.1), we have that $\cup_{\alpha \in \Delta} E \cap X_\alpha \in \Sigma$. So, $E = E \cap X = (\cup_{\alpha \in \Delta} E \cap X_\alpha) \cup (E \cap N) \in \Sigma$. Moreover, $\mu(E) = \sum_{\alpha \in \Delta} \mu(E \cap X_\alpha)$ for every $E \in \Sigma$. Indeed, if $\sum_{\alpha \in \Delta} \mu(E \cap X_\alpha) < \infty$, then $\mu(E \cap X_\alpha) = 0$ for all $\alpha \in \Delta \backslash \Gamma$ for some countable $\Gamma \subset \Delta$. Since, by (ii) and (2.3.2), $\cup_{\alpha \in \Delta \backslash \Gamma} E \cap X_\alpha$ is μ-null,

$$\mu(E) = \mu(\cup_{\alpha \in \Gamma} E \cap X_\alpha) = \sum_{\alpha \in \Gamma} \mu(E \cap X_\alpha) = \sum_{\alpha \in \Delta} \mu(E \cap X_\alpha).$$

If $\sum_{\alpha \in \Delta} \mu(E \cap X_\alpha) = \infty$ then $\mu(E) = \infty$, as $\sup_{\substack{J \subset \Delta \\ \text{finite}}} \sum_{\alpha \in J} \mu(E \cap X_\alpha) \leq \mu(E)$. Therefore, (X, Σ, μ) is decomposable which is a contradiction.

So, $\widetilde{\mu}$ is not $\mathcal{R}$-decomposable. However, since $L^1(\widetilde{\mu}) = L^1_w(\widetilde{\mu})$ as $\widetilde{\mu}$ takes values in $\mathbb{R}$, we have that $L^1_w(\widetilde{\mu})$ has the Fatou property (see Proposition 2.3.4).

Now we can say that there is no relation between the main properties used in this chapter, $\mathcal{R}$-decomposability and local σ-finiteness. Indeed, the vector measure given in the example above is locally σ-finite (see Remark 2.2.5) but not $\mathcal{R}$-decomposable, while the vector measure given in Example 2.2.1 is $\mathcal{R}$-decomposable as it is discrete but not locally σ-finite.

2.4 Example

We end this chapter by showing that there exist $\mathcal{R}$-decomposable vector measures ν which are not σ-finite nor discrete.

Let Γ be an abstract set. For each $\gamma \in \Gamma$, consider a non null vector measure $\nu_\gamma\colon \Sigma_\gamma \to X_\gamma$ defined on a σ-algebra Σ_γ of subsets of a set Ω_γ and with values in a Banach space X_γ. Consider the set $\Omega = \cup_{\gamma\in\Gamma}\big(\{\gamma\} \times \Omega_\gamma\big)$, that is

$$\Omega = \big\{(\gamma,\omega) : \gamma \in \Gamma \text{ and } \omega \in \Omega_\gamma\big\}.$$

In a similar way, we denote $\cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma = \big\{(\gamma,\omega) : \gamma \in \Gamma \text{ and } \omega \in A_\gamma\big\}$, where $A_\gamma \subset \Omega_\gamma$ for all $\gamma \in \Gamma$. For every $I \subset \Gamma$ we write $\cup_{\gamma\in I}\{\gamma\} \times A_\gamma = \cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma$ whenever $A_\gamma = \emptyset$ for all $\gamma \in \Gamma\backslash I$. Note that if $A_n = \cup_{\gamma\in\Gamma}\{\gamma\}\times A^n_\gamma$ for $n \geq 1$,

$$\bigcup_{n\geq 1} A_n = \bigcup_{\gamma\in\Gamma}\{\gamma\} \times \Big(\bigcup_{n\geq 1} A^n_\gamma\Big) \quad\text{and}\quad \bigcap_{n\geq 1} A_n = \bigcup_{\gamma\in\Gamma}\{\gamma\} \times \Big(\bigcap_{n\geq 1} A^n_\gamma\Big).$$

Also, if $A = \cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma$ and $B = \cup_{\gamma\in\Gamma}\{\gamma\} \times B_\gamma$,

$$A\backslash B = \bigcup_{\gamma\in\Gamma}\{\gamma\} \times \big(A_\gamma\backslash B_\gamma\big).$$

Then the family $\mathcal{R}$ of sets $\cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma$ satisfying that $A_\gamma \in \Sigma_\gamma$ for all $\gamma \in \Gamma$ and there exists a finite set $J \subset \Gamma$ such that A_γ is ν_γ-null for all $\gamma \in \Gamma\backslash J$, is a δ-ring of parts of Ω.

Moreover,

$$\mathcal{R}^{loc} = \big\{ \cup_{\gamma\in\Gamma} \{\gamma\} \times A_\gamma : A_\gamma \in \Sigma_\gamma \text{ for all } \gamma \in \Gamma\big\}.$$

Indeed, given $A \in \mathcal{R}^{loc}$, if we take $B_\gamma = \{\omega \in \Omega_\gamma : (\gamma,\omega) \in A\}$ we have that

$$A = \cup_{\gamma\in\Gamma}\{\gamma\} \times B_\gamma,$$

where $\{\gamma\} \times B_\gamma = A \cap (\{\gamma\} \times \Omega_\gamma) \in \mathcal{R}$ (as $\{\gamma\} \times \Omega_\gamma \in \mathcal{R}$). So, $B_\gamma \in \Sigma_\gamma$. Conversely, take $A = \cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma$ with $A_\gamma \in \Sigma_\gamma$ for every $\gamma \in \Gamma$. If $B = \cup_{\gamma\in\Gamma}\{\gamma\} \times B_\gamma \in \mathcal{R}$,

$$A \cap B = \bigcup_{\gamma\in\Gamma}\{\gamma\} \times \big(A_\gamma \cap B_\gamma\big) \in \mathcal{R}$$

and so $A \in \mathcal{R}^{loc}$.

Note that given a function $f\colon \Omega \to \mathbb{R}$, considering for each $\gamma \in \Gamma$ the sections $f(\gamma,\cdot)\colon \Omega_\gamma \to \mathbb{R}$, we have that

$$f^{-1}(B) = \cup_{\gamma\in\Gamma}\{\gamma\} \times f(\gamma,\cdot)^{-1}(B)$$

for every Borel set B on $\mathbb{R}$. Then, f is $\mathcal{R}^{loc}$-measurable if and only if $f(\gamma,\cdot)$ is Σ_γ-measurable for all $\gamma \in \Gamma$.

Denote by $c_0\big(\Gamma,(X_\gamma)_{\gamma\in\Gamma}\big)$ the Banach space of all families $(x_\gamma)_{\gamma\in\Gamma}$ such that $x_\gamma \in X_\gamma$ for every $\gamma \in \Gamma$ and $\big(\|x_\gamma\|_{X_\gamma}\big)_{\gamma\in\Gamma} \in c_0(\Gamma)$, endowed with the norm $\|(x_\gamma)_{\gamma\in\Gamma}\| = \sup_{\gamma\in\Gamma}\|x_\gamma\|_{X_\gamma}$. Note that the topological dual $c_0\big(\Gamma,(X_\gamma)_{\gamma\in\Gamma}\big)^*$ can be identified with the Banach space $\ell^1\big(\Gamma,(X^*_\gamma)_{\gamma\in\Gamma}\big)$ of families $(x^*_\gamma)_{\gamma\in\Gamma}$ such that $x^*_\gamma \in X^*_\gamma$ for every $\gamma \in \Gamma$ and $\big(\|x^*_\gamma\|_{X^*_\gamma}\big)_{\gamma\in\Gamma} \in \ell^1(\Gamma)$, endowed with the norm $\|(x^*_\gamma)_{\gamma\in\Gamma}\| = \sum_{\gamma\in\Gamma}\|x^*_\gamma\|_{X_\gamma}$. The action of any $x^* = (x^*_\gamma)_{\gamma\in\Gamma} \in \ell^1\big(\Gamma,(X^*_\gamma)_{\gamma\in\Gamma}\big)$ on $x = (x_\gamma)_{\gamma\in\Gamma} \in c_0\big(\Gamma,(X_\gamma)_{\gamma\in\Gamma}\big)$ is given by $x^*(x) = \sum_{\gamma\in\Gamma} x^*_\gamma(x_\gamma)$.

Consider the finitely additive set function $\nu\colon \mathcal{R} \to c_0\big(\Gamma,(X_\gamma)_{\gamma\in\Gamma}\big)$ given by

$$\nu\big(\cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma\big) = \big(\nu_\gamma(A_\gamma)\big)_{\gamma\in\Gamma}.$$

Let us see that ν is a vector measure. Given $A_n = \cup_{\gamma\in\Gamma}\{\gamma\} \times A^n_\gamma \in \mathcal{R}$ for $n \geq 1$ mutually disjoint sets such that $\cup_{n\geq1}A_n \in \mathcal{R}$, we have that

$$\bigcup_{n\geq1} A_n = \bigcup_{\gamma\in\Gamma}\{\gamma\} \times \Big(\bigcup_{n\geq1} A^n_\gamma\Big)$$

where $\bigcup_{n\geq1} A^n_\gamma$ is a disjoint union for every $\gamma \in \Gamma$ and there exists a finite set $J \subset \Gamma$ such that $\bigcup_{n\geq1} A^n_\gamma$ is ν_γ-null for all $\gamma \in \Gamma\backslash J$. Since for each $\gamma \in \Gamma$ the sum $\sum_{n\geq1}\nu_\gamma(A^n_\gamma)$ converges to $\nu_\gamma(\cup_{n\geq1}A^n_\gamma)$ in X_γ and moreover if $\gamma \in \Gamma\backslash J$ we have that $\|\nu_\gamma(\cup_{j>n}A^j_\gamma)\|_{X_\gamma} = 0$ for all n, we have that

$$\begin{aligned}
\Big\|\nu\Big(\bigcup_{n\geq1} A_n\Big) - \sum_{j=1}^{n}\nu(A_n)\Big\| &= \Big\|\nu\Big(\bigcup_{j>n} A_n\Big)\Big\| = \sup_{\gamma\in\Gamma}\Big\|\nu_\gamma\Big(\bigcup_{j>n} A^j_\gamma\Big)\Big\|_{X_\gamma} \\
&= \sup_{\gamma\in J}\Big\|\nu_\gamma\Big(\bigcup_{j>n} A^j_\gamma\Big)\Big\|_{X_\gamma} \to 0.
\end{aligned}$$

Note that for each $A = \cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma \in \mathcal{R}$ and $x^* = (x^*_\gamma)_{\gamma\in\Gamma} \in c_0\big(\Gamma,(X_\gamma)_{\gamma\in\Gamma}\big)^*$, we have that $x^*\nu(A) = \sum_{\gamma\in\Gamma} x^*_\gamma\nu_\gamma(A_\gamma)$. Then, a set $A = \cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma \in \mathcal{R}^{loc}$

is $|x^*\nu|$-null if and only if A_γ is $|x^*_\gamma\nu_\gamma|$-null for all $\gamma\in\Gamma$. Also, we have that A is ν-null if and only if A_γ is ν_γ-null for all $\gamma\in\Gamma$.

It is routine to show that:

(a) ν is $\mathcal{R}$-decomposable.

(b) ν is σ-finite if and only if Γ is countable.

(c) ν is discrete if and only if ν_γ is discrete for all $\gamma\in\Gamma$.

In order to describe the space $L^1_w(\nu)$, let us prove that for every $x^*=(x^*_\gamma)_{\gamma\in\Gamma}\in c_0\big(\Gamma,(X^*_\gamma)_{\gamma\in\Gamma}\big)^*$ and $f\in\mathcal{M}(\mathcal{R}^{loc})$, we have that

$$\int |f|\,d|x^*\nu| = \sum_{\gamma\in\Gamma}\int |f(\gamma,\cdot)|\,d|x^*_\gamma\nu_\gamma|. \tag{2.4.1}$$

Given $\gamma\in\Gamma$ it is direct to check that $|x^*\nu|(\{\gamma\}\times A_\gamma)=|x^*_\gamma\nu_\gamma|(A_\gamma)$ for all $A_\gamma\in\Sigma_\gamma$ (note that $x^*\nu(\{\gamma\}\times A_\gamma)=x^*_\gamma\nu_\gamma(A_\gamma)$). Then, if we take $A=\cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma\in\mathcal{R}^{loc}$, for every finite set $J\subset\Gamma$, we have that

$$\sum_{\gamma\in J}|x^*_\gamma\nu_\gamma|(A_\gamma)=\sum_{\gamma\in J}|x^*\nu|(\{\gamma\}\times A_\gamma)=|x^*\nu|\Big(\bigcup_{\gamma\in J}\{\gamma\}\times A_\gamma\Big)\le|x^*\nu|(A).$$

That is, $\sum_{\gamma\in\Gamma}|x^*_\gamma\nu_\gamma|(A_\gamma)\le|x^*\nu|(A)$. The converse inequality follows routinely. Therefore, $|x^*\nu|(A)=\sum_{\gamma\in\Gamma}|x^*_\gamma\nu_\gamma|(A_\gamma)\le\infty$ for every measurable set $A=\cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma\in\mathcal{R}^{loc}$. Given an $\mathcal{R}$-simple function $\varphi=\sum_{j=1}^n\alpha_j\chi_{A_j}$ where $A_j=\cup_{\gamma\in\Gamma}\{\gamma\}\times A^j_\gamma$ are pairwise disjoint, we can take a finite set $J\subset\Gamma$ such that for each $\gamma\in\Gamma\backslash J$ we have that A^j_γ is ν_γ-null for all j, and so, noting that $\varphi(\gamma,\cdot)=\sum_{j=1}^n\alpha_j\chi_{A^j_\gamma}$, we have that

$$\begin{aligned}\int|\varphi|\,d|x^*\nu| &= \sum_{j=1}^n|\alpha_j||x^*\nu|(A_j)=\sum_{j=1}^n|\alpha_j|\sum_{\gamma\in\Gamma}|x^*_\gamma\nu_\gamma|(A^j_\gamma)\\ &= \sum_{j=1}^n|\alpha_j|\sum_{\gamma\in J}|x^*_\gamma\nu_\gamma|(A^j_\gamma)=\sum_{\gamma\in J}\sum_{j=1}^n|\alpha_j||x^*_\gamma\nu_\gamma|(A^j_\gamma)\\ &= \sum_{\gamma\in J}\int|\varphi(\gamma,\cdot)|\,d|x^*_\gamma\nu_\gamma|=\sum_{\gamma\in\Gamma}\int|\varphi(\gamma,\cdot)|\,d|x^*_\gamma\nu_\gamma|.\end{aligned}$$

Then, (2.4.1) holds for $\mathcal{R}$-simple functions. Let now $\varphi\in\mathcal{S}(\mathcal{R}^{loc})$. Since $\varphi\chi_{\{\gamma\}\times\Omega_\gamma}\in\mathcal{S}(\mathcal{R})$ for every $\gamma\in\Gamma$, noting that $\varphi\chi_{\{\gamma\}\times\Omega_\gamma}(\gamma,\cdot)=\varphi(\gamma,\cdot)$, we have that

$$\int |\varphi|\chi_{\{\gamma\}\times\Omega_\gamma}\, d|x^*\nu| = \sum_{\beta\in\Gamma}\int |\varphi\chi_{\{\gamma\}\times\Omega_\gamma}(\beta,\cdot)|\, d|x^*_\beta\nu_\beta| = \int |\varphi(\gamma,\cdot)|\, d|x^*_\gamma\nu_\gamma|.$$

Then,

$$\begin{aligned}\sum_{\gamma\in J}\int |\varphi(\gamma,\cdot)|\, d|x^*_\gamma\nu_\gamma| &= \sum_{\gamma\in J}\int |\varphi|\chi_{\{\gamma\}\times\Omega_\gamma}\, d|x^*\nu| \\ &= \int |\varphi|\chi_{\cup_{\gamma\in J}\{\gamma\}\times\Omega_\gamma}\, d|x^*\nu| \\ &\le \int |\varphi|\, d|x^*\nu|\end{aligned}$$

for every finite set $J\subset\Gamma$, and so $\sum_{\gamma\in\Gamma}\int|\varphi(\gamma,\cdot)|\,d|x^*_\gamma\nu_\gamma|\le\int|\varphi|\,d|x^*\nu|$. On the other hand, note that $\int|\varphi|\,d|x^*\nu| = \sup_{A\in\mathcal{R}}\int_A|\varphi|\,d|x^*\nu|$, see [28, Lemma 2.30]. Given $A=\cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma\in\mathcal{R}$, since $\varphi\chi_A\in\mathcal{S}(\mathcal{R})$, noting that $\varphi\chi_A(\gamma,\cdot)=\varphi(\gamma,\cdot)\chi_{A_\gamma}$, we have that

$$\int_A |\varphi|\, d|x^*\nu| = \sum_{\gamma\in\Gamma}\int_{A_\gamma}|\varphi(\gamma,\cdot)|\, d|x^*_\gamma\nu_\gamma| \le \sum_{\gamma\in\Gamma}\int |\varphi(\gamma,\cdot)|\, d|x^*_\gamma\nu_\gamma|.$$

Therefore, (2.4.1) holds for $f\in\mathcal{S}(\mathcal{R}^{loc})$, and so also for all $f\in\mathcal{M}(\mathcal{R}^{loc})$ by using the monotone convergence theorem.

Now we can see that $L^1_w(\nu)$ is the space of functions $f\in\mathcal{M}(\mathcal{R}^{loc})$ such that $f(\gamma,\cdot)\in L^1_w(\nu_\gamma)$ for all $\gamma\in\Gamma$ with $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma}\in\ell^\infty(\Gamma)$, and moreover, $\|f\|_\nu=\sup_{\gamma\in\Gamma}\|f(\gamma,\cdot)\|_{\nu_\gamma}$ for all $f\in L^1_w(\nu)$.

Let $f\in L^1_w(\nu)$ and fix $\beta\in\Gamma$. Given $x^*_\beta\in X^*_\beta$, define the element $x^*=(x^*_\gamma)_{\gamma\in\Gamma}$ in $\ell^1\big(\Gamma,(X^*_\gamma)_{\gamma\in\Gamma}\big)$ by $x^*_\gamma=x^*_\beta$ if $\gamma=\beta$ and $x^*_\gamma=0$ in other case. Then, from (2.4.1), we have that $\int|f(\beta,\cdot)|\,d|x^*_\beta\nu_\beta|=\int|f|\,d|x^*\nu|<\infty$ and so $f(\beta,\cdot)\in L^1_w(\nu_\beta)$ with $\|f(\beta,\cdot)\|_{\nu_\beta}\le\|f\|_\nu$. Thus, $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma}\in\ell^\infty(\Gamma)$ and $\sup_{\gamma\in\Gamma}\|f(\gamma,\cdot)\|_{\nu_\gamma}\le\|f\|_\nu$.

Let now $f\in\mathcal{M}(\mathcal{R}^{loc})$ satisfying that $f(\gamma,\cdot)\in L^1_w(\nu_\gamma)$ for every $\gamma\in\Gamma$ and $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma}\in\ell^\infty(\Gamma)$. Given $x^*=(x^*_\gamma)_{\gamma\in\Gamma}\in\ell^1\big(\Gamma,(X^*_\gamma)_{\gamma\in\Gamma}\big)$, from (2.4.1), we have that

$$\begin{aligned}\int |f|\, d|x^*\nu| &= \sum_{\gamma\in\Gamma} \int |f(\gamma,\cdot)|\, d|x^*_\gamma \nu_\gamma| \leq \sum_{\gamma\in\Gamma} \|x^*_\gamma\|_{X^*_\gamma} \|f(\gamma,\cdot)\|_{\nu_\gamma} \\ &\leq \sup_{\gamma\in\Gamma} \|f(\gamma,\cdot)\|_{\nu_\gamma} \sum_{\gamma\in\Gamma} \|x^*_\gamma\|_{X^*_\gamma} < \infty.\end{aligned}$$

Then, $f \in L^1_w(\nu)$ and $\|f\|_\nu \leq \sup_{\gamma\in\Gamma} \|f(\gamma,\cdot)\|_{\nu_\gamma}$.

Therefore, $L^1_w(\nu)$ is order isometric to $\ell^\infty\big(\Gamma, (L^1_w(\nu_\gamma))_{\gamma\in\Gamma}\big)$ via the map which takes f to $\big(f(\gamma,\cdot)\big)_{\gamma\in\Gamma}$.

For describing the space $L^1(\nu)$ we need to prove that for every $x^* = (x^*_\gamma)_{\gamma\in\Gamma} \in c_0\big(\Gamma, (X^*_\gamma)_{\gamma\in\Gamma}\big)^*$ and $f \in L^1(x^*\nu)$,

$$\int_A f\, dx^*\nu = \sum_{\gamma\in\Gamma} \int_{A_\gamma} f(\gamma,\cdot)\, dx^*_\gamma \nu_\gamma \tag{2.4.2}$$

for all $A = \cup_{\gamma\in\Gamma}\{\gamma\} \times A_\gamma \in \mathcal{R}^{loc}$.

By (2.4.1), we have that $f(\gamma,\cdot) \in L^1(x^*_\gamma\nu_\gamma)$ for every $\gamma \in \Gamma$ and moreover $\int |f(\gamma,\cdot)|\, d|x^*_\gamma\nu_\gamma| = 0$ (and so $f(\gamma,\cdot) = 0$ except on a $|x^*_\gamma\nu_\gamma|$-null set Z_γ) for all $\gamma \in \Gamma\backslash J$ with J being some countable subset of Γ. Then, $f = f\chi_{\cup_{\gamma\in J}\{\gamma\}\times\Omega_\gamma}$ $x^*\nu$-a.e. (except on the $|x^*\nu|$-null set $\cup_{\gamma\in\Gamma\backslash J}\{\gamma\} \times Z_\gamma$) and so $f\chi_A = f\chi_{\cup_{\gamma\in J}\{\gamma\}\times A_\gamma}$ $|x^*\nu|$-a.e. By using the dominated convergence theorem, we have that

$$\int_A f\, dx^*\nu = \sum_{\gamma\in J} \int_{\{\gamma\}\times A_\gamma} f\, dx^*\nu.$$

Noting that $\int_{\{\gamma\}\times A_\gamma} f\, dx^*\nu = \int_{A_\gamma} f(\gamma,\cdot)\, dx^*_\gamma\nu_\gamma$ holds for $\mathcal{R}^{loc}$-simple functions and so for any $f \in L^1(x^*\nu)$ by density of the $\mathcal{R}$-simple functions in $L^1(x^*\nu)$, we conclude that (2.4.2) holds.

Now we can describe $L^1(\nu)$ as the space of functions $f \in \mathcal{M}(\mathcal{R}^{loc})$ such that $f(\gamma,\cdot) \in L^1(\nu_\gamma)$ for every $\gamma \in \Gamma$ with $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma} \in c_0(\Gamma)$.

Indeed, if $f \in L^1(\nu)$ we can take $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\nu)$. For each $\gamma \in \Gamma$, we have that $f(\gamma,\cdot) \in L^1_w(\nu_\gamma)$ (as $f \in L^1_w(\nu)$) and $(\varphi_n(\gamma,\cdot)) \subset \mathcal{S}(\Sigma_\gamma) \subset L^1(\nu_\gamma)$. Then, since $\|f(\gamma,\cdot) - \varphi_n(\gamma,\cdot)\|_{\nu_\gamma} \leq \|f - \varphi_n\|_\nu$ and $L^1(\nu_\gamma)$

is closed in $L^1_w(\nu_\gamma)$, it follows that $f(\gamma,\cdot)\in L^1(\nu_\gamma)$. On the other hand, for each n, we can write $\varphi_n=\sum_{j=1}^m \alpha_j\chi_{A_j}$ with $A_j=\cup_{\gamma\in\Gamma}\{\gamma\}\times A^j_\gamma$ and take a finite set $J\subset\Gamma$ such that for each $\gamma\in\Gamma\backslash J$ we have that A^j_γ is ν_γ-null for all j. Then, $\varphi_n(\gamma,\cdot)=\sum_{j=1}^m \alpha_j\chi_{A^j_\gamma}=0$ ν_γ-a.e. for all $\gamma\in\Gamma\backslash J$, and so $(\|\varphi_n(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma}\in c_0(\Gamma)$. Since $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma}\in\ell^\infty(\Gamma)$ and

$$\sup_{\gamma\in\Gamma}\big|\,\|f(\gamma,\cdot)\|_{\nu_\gamma}-\|\varphi_n(\gamma,\cdot)\|_{\nu_\gamma}\big|\leq\sup_{\gamma\in\Gamma}\|f(\gamma,\cdot)-\varphi_n(\gamma,\cdot)\|_{\nu_\gamma}=\|f-\varphi_n\|_\nu,$$

it follows that $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma}\in c_0(\Gamma)$.

Conversely, suppose that $f\in\mathcal{M}(\mathcal{R}^{loc})$ is such that $f(\gamma,\cdot)\in L^1(\nu_\gamma)$ for all $\gamma\in\Gamma$ and $(\|f(\gamma,\cdot)\|_{\nu_\gamma})_{\gamma\in\Gamma}\in c_0(\Gamma)$. In particular, $f\in L^1_w(\nu)$. Given an element $x^*=(x^*_\gamma)_{\gamma\in\Gamma}\in c_0\big(\Gamma,(X^*_\gamma)_{\gamma\in\Gamma}\big)^*$ and $A=\cup_{\gamma\in\Gamma}\{\gamma\}\times A_\gamma\in\mathcal{R}^{loc}$, we note that $\big(\int_{A_\gamma}f(\gamma,\cdot)\,d\nu_\gamma\big)_{\gamma\in\Gamma}\in c_0\big(\Gamma,(X_\gamma)_{\gamma\in\Gamma}\big)$ as $\|\int_{A_\gamma}f(\gamma,\cdot)\,d\nu_\gamma\|_{X_\gamma}\leq\|f(\gamma,\cdot)\|_{\nu_\gamma}$ for each $\gamma\in\Gamma$. Moreover, by (2.4.2),

$$\begin{aligned} x^*\left(\Big(\int_{A_\gamma}f(\gamma,\cdot)\,d\nu_\gamma\Big)_{\gamma\in\Gamma}\right) &= \sum_{\gamma\in\Gamma}x^*_\gamma\Big(\int_{A_\gamma}f(\gamma,\cdot)\,d\nu_\gamma\Big)\\ &= \sum_{\gamma\in\Gamma}\int_{A_\gamma}f(\gamma,\cdot)\,dx^*_\gamma\nu_\gamma=\int_A f\,dx^*\nu. \end{aligned}$$

So, $f\in L^1(\nu)$ and $\int_A f\,d\nu=\big(\int_{A_\gamma}f(\gamma,\cdot)\,d\nu_\gamma\big)_{\gamma\in\Gamma}$.

Therefore, $L^1(\nu)$ is order isometric to $c_0\big(\Gamma,(L^1(\nu_\gamma))_{\gamma\in\Gamma}\big)$ via the map which takes f to $\big(f(\gamma,\cdot)\big)_{\gamma\in\Gamma}$.

Note that if ν is locally σ-finite, since $h=\sum_{\gamma\in\Gamma}\frac{1}{\|\nu_\gamma\|(\Omega_\gamma)}\chi_{\{\gamma\}\times\Omega_\gamma}\in L^1_w(\nu)$ and $\mathrm{supp}(h)=\Omega$, from Proposition 2.2.9, it follows that ν is σ-finite. So, in this case ν is locally σ-finite if and only if ν is σ-finite if and only if Γ is countable.

In particular, consider a non atomic measure space (Θ,Σ,μ) and an order continuous B.f.s. X related to μ which does not contain any copy of c_0 and such that $\chi_\Theta\in X$, for instance $X=L^p[0,1]$ related to the Lebesgue measure for $p\geq 1$. The finitely additive set function $\eta\colon\Sigma\to X$ defined by $\eta(A)=\chi_A$ for all $A\in\Sigma$, is a vector measure as X is order continuous, and it is non discrete as μ is non atomic. For every $\varphi\in\mathcal{S}(\Sigma)$ we have that $\int\varphi\,d\eta=\varphi$ and, since η is positive, $\|\varphi\|_\eta=\|\int|\varphi|\,d\eta\|_X=\|\varphi\|_X$. In particular, $\|\nu\|(A)=\|\chi_A\|_X=0$

if and only if $\mu(A) = 0$. Since $\mathcal{S}(\Sigma)$ is dense in $L^1(\eta)$ and also in X (again by the order continuity property), then we deduce that $L^1(\eta) = X$. Even more, $L^1_w(\eta) = L^1(\eta) = X$. Taking Γ uncountable and $\nu_\gamma = \eta$ for all $\gamma \in \Gamma$, we obtain an $\mathcal{R}$-decomposable vector measure ν which is not σ-finite nor discrete. In this case, $L^1_w(\nu) = \ell^\infty(\Gamma, X)$ and $L^1(\nu) = c_0(\Gamma, X)$.

Chapter 3

Spaces of p-integrable functions with respect to a vector measure defined on a δ-ring

The spaces $L^p(\nu)$ and $L^p_w(\nu)$ of p-integrable functions and weakly p-integrable functions are nowadays well-known when the vector measure ν is defined on a σ-algebra. In fact, all the relevant (geometric, lattice, topological) properties of the spaces $L^p(\nu)$ of a vector measure ν on a σ-algebra with $1 \leq p < \infty$ has been already studied (see [19, 31, 32]), when this is not the case for the δ-ring case.

The aim of this chapter is to study the main properties of the spaces $L^p(\nu)$ and $L^p_w(\nu)$ of a vector measure ν on a δ-ring, the natural sets of multiplication operators and the inclusion relations between the spaces $L^p(\nu)$ and $L^p_w(\nu)$.

Section 3.1 is devoted to the study of the main Banach lattice properties of the spaces $L^p(\nu)$ and $L^p_w(\nu)$. The general case $0 < p < \infty$ is considered, although for $0 < p < 1$ these spaces are not necessarily Banach spaces, for instance this is the case when the vector measure is a scalar measure. However, completeness is proved also for this case but under a quasinorm.

In Section 3.2 the spaces of multiplication operators between spaces of p-integrable functions and spaces of integrable functions with respect to the same vector measure are computed, and compactness type properties of these operators are studied, generalizing in this way what is known in the case of σ-algebras (see [13]).

Finally, Section 3.3 deals with the analysis of the spaces $L^p(\nu)$ and $L^p_w(\nu)$ as intermediate spaces of $L^\infty(\nu) \cap L^1(\nu)$ and $L^\infty(\nu) + L^1(\nu)$, providing the vector measure version of the classical inclusions that hold for the Lebesgue spaces $L^p[0, \infty]$.

Let us recall that each vector measure ν defined on a σ-algebra satisfies that $\chi_\Omega \in L^1(\nu)$ and so $\|\nu\|(\Omega) = \|\chi_\Omega\|_\nu < \infty$, that is, ν is bounded. It is relevant for this chapter that this does not hold in general for vector measures defined on δ-rings ([15, Example 2.1]). Indeed, for the general case, bounded functions may be not integrable and this fact is crucial.

3.1 The spaces of p-integrable functions with respect to a vector measure on a δ-ring

Recall that we are dealing with a vector measure $\nu\colon \mathcal{R} \to X$ defined on a δ-ring $\mathcal{R}$ of subsets of an abstract set Ω, with values in a real Banach space X.

First, we introduce and study the main properties of the corresponding spaces of p-integrable functions, that is the p-power spaces $L^p(\nu)$ and $L^p_w(\nu)$ of $L^1(\nu)$ and $L^1_w(\nu)$, respectively. We show some fundamental topological and lattice properties of the spaces $L^p(\nu)$ and $L^p_w(\nu)$. Although in some cases our arguments follow the lines of the ones that prove the corresponding results for vector measures on σ-algebras (see [31, Ch.2, Ch.3] and [19, 32]), there are several technical details that make the proofs slightly different as we are *not* working in the setting of the LT-B.f.s.'. So, we will write all the proofs for the aim of completeness.

Given $0 < p < \infty$, the p-power space of $L^1_w(\nu)$ is defined as

$$L^p_w(\nu) = \big\{ f \in L^0(\nu) : \, |f|^p \in L^1_w(\nu) \big\}.$$

A function in $L^p_w(\nu)$ will be called *weakly p-integrable* with respect to ν. Similarly, the p-power space of $L^1(\nu)$ is defined as

$$L^p(\nu) = \big\{ f \in L^0(\nu) : \, |f|^p \in L^1(\nu) \big\}.$$

A function in $L^p(\nu)$ will be called *p-integrable* with respect to ν.

The following well-known inequalities involving positive real numbers will be necessary through the section (see for instance [31, Section 2.2]).

Lemma 3.1.1. *Let $a, b \in [0, +\infty)$. Then the following inequalities hold.*

$$(a+b)^r \leq a^r + b^r \ \text{ and } \ |a^r - b^r| \leq |a-b|^r, \text{ for } 0 < r \leq 1. \tag{3.1.1}$$

$$a^r + b^r \leq (a+b)^r \leq 2^{r-1}(a^r + b^r), \text{ for } r \geq 1. \tag{3.1.2}$$

$$|a^r - b^r| \leq r \cdot |a^{r-1} + b^{r-1}| \cdot |a-b|, \text{ for } r \geq 1. \tag{3.1.3}$$

From (3.1.1) and (3.1.2) we have that $L^p_w(\nu)$ and $L^p(\nu)$ are linear spaces and it is clear that $L^p(\nu) \subset L^p_w(\nu)$.

For each $f \in L^p_w(\nu)$, we denote

$$\|f\|_{p,\nu} = \big\| |f|^p \big\|_\nu^{\frac{1}{p}} = \sup_{x^* \in B_{X^*}} \Big(\int |f|^p \, d|x^*\nu| \Big)^{\frac{1}{p}}.$$

Since $\|\cdot\|_\nu$ is a norm, straightforward calculations using the previous lemma show that $\|\cdot\|_{p,\nu}$ is a *quasi-norm*, that is, it satisfies the same properties as a norm except by a constant in the triangular inequality (i.e. there exists $K > 0$ such that $\|f+g\|_{p,\nu} \leq K\big(\|f\|_{p,\nu} + \|g\|_{p,\nu}\big)$ for all $f, g \in L^p_w(\nu)$). Note that both $L^p(\nu)$ and $L^p_w(\nu)$ are solid subsets of $L^0(\nu)$ and the quasi-norm $\|\cdot\|_{p,\nu}$ is compatible with the ν-a.e. pointwise order, that is, $\|f\|_{p,\nu} \leq \|g\|_{p,\nu}$ whenever $|f| \leq |g|$. We also use the notations $\|\cdot\|_{L^p_w(\nu)}$ and $\|\cdot\|_{L^p(\nu)}$ when an explicit reference to the space is convenient.

Actually, given a B.f.s. X related to μ, the p-power space given by $X^p = \{f \in L^0(\mu) : |f|^p \in X\}$ satisfies all the above properties for $\|f\|_{X^p} = \||f|^p\|_X^{\frac{1}{p}}$. If $1 \leq p$, then $\|\cdot\|_{X^p}$ is actually a norm and the space X^p is a B.f.s. related to μ. To prove this result we need first the following lemma, that will be useful also in next sections.

Lemma 3.1.2. *Let $q, r, s > 0$ such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$ and let $f \in X^r$ and $g \in X^s$. Then, $fg \in X^q$ and $\|fg\|_{X^q} \leq \|f\|_{X^r}\|g\|_{X^s}$.*

Proof. Without loss of generality, it suffices to assume that $\|f\|_{X^r} = \|g\|_{X^s} = 1$. Note that since $\frac{q}{r} + \frac{q}{s} = 1$, the *Young's inequality* says that $ab \leq \frac{q}{r}a^{\frac{r}{q}} + \frac{q}{s}b^{\frac{s}{q}}$ for all $a, b \in [0, +\infty)$. From this it follows that $|fg|^q \leq \frac{q}{r}|f|^r + \frac{q}{s}|g|^s \in X$. So, $fg \in X^q$ and

$$\|fg\|^q_{X^q} = \big\||fg|^q\big\|_X \leq \frac{q}{r}\||f|^r\|_X + \frac{q}{s}\big\||g|^s\big\|_X = \frac{q}{r}\|f\|^r_{X^r} + \frac{q}{s}\|g\|^s_{X^s} = \frac{q}{r} + \frac{q}{s} = 1.$$

□

If $p < 1$, we will see that X^p is a *quasi-Banach function space* (briefly, q-B.f.s.) related to μ, that is, it satisfies the same properties as a B.f.s. but replacing norm by quasi-norm. Note that in this case, X^p is a quasi-Banach lattice with the μ-a.e. pointwise order and the convergence in the quasi-norm $\|\cdot\|_{X^p}$ of a sequence implies μ-a.e. convergence of some subsequence.

Proposition 3.1.3. *The space X^p is a q-B.f.s. Even more, X^p is a B.f.s. whenever $p \geq 1$.*

Proof. We only have to prove that X^p is complete for the quasi-norm $\|\cdot\|_{X^p}$ and, in the case when $p \geq 1$, that $\|\cdot\|_{X^p}$ is a norm. Let (f_n) be a Cauchy sequence in X^p. Due to the equality $|a-b| = |a^+ - b^+| + |a^- - b^-|$ for all $a, b \in \mathbb{R}$ (where a^+ and a^- denote the positive and negative parts of a respectively) and the compatibility of the quasi-norm $\|\cdot\|_{X^p}$ with the ν-a.e. pointwise order, we can assume that $f_n \geq 0$ for all n.

Suppose that $p<1$. Applying inequality (3.1.1) in Lemma 3.1.1 to f_n and f_m and taking norm $\|\cdot\|_X$, we have that

$$\|f_n^p - f_m^p\|_X \le \big\||f_n - f_m|^p\big\|_X = \|f_n - f_m\|_{X^p}^p.$$

Therefore, (f_n^p) is a Cauchy sequence in X and so there exists $f\in X$ such that (f_n^p) converges to f in norm $\|\cdot\|_X$. Note that $f\ge 0$ (as convergence in norm $\|\cdot\|_X$ of a sequence implies μ-a.e. convergence of some subsequence) and $f^{\frac{1}{p}}\in X^p$. By (3.1.3) in Lemma 3.1.1 for $r=\frac{1}{p}$ and Lemma 3.1.2 for $q=p$, $r=\frac{p}{1-p}$ and $s=1$, it follows that

$$\begin{aligned}
\big\|f_n - f^{\frac{1}{p}}\big\|_{X^p} &= \big\|(f_n^p)^{\frac{1}{p}} - f^{\frac{1}{p}}\big\|_{X^p} \le \frac{1}{p}\big\|\big((f_n^p)^{\frac{1}{p}-1} + f^{\frac{1}{p}-1}\big)\cdot(f_n^p - f)\big\|_{X^p}\\
&\le \frac{1}{p}\big\|(f_n^p)^{\frac{1}{p}-1} + f^{\frac{1}{p}-1}\big\|_{X^{\frac{p}{1-p}}}\cdot\big\|f_n^p - f\big\|_X\\
&= \frac{1}{p}\big\|\big((f_n^p)^{\frac{1}{p}-1} + f^{\frac{1}{p}-1}\big)^{\frac{p}{1-p}}\big\|_X^{\frac{1-p}{p}}\cdot\big\|f_n^p - f\big\|_X.
\end{aligned}$$

If $\frac{p}{1-p}\le 1$, applying first (3.1.1) for $r=\frac{p}{1-p}$ and then (3.1.2) for $r=\frac{1-p}{p}$, we have that

$$\begin{aligned}
\big\|\big((f_n^p)^{\frac{1}{p}-1} + f^{\frac{1}{p}-1}\big)^{\frac{p}{1-p}}\big\|_X^{\frac{1-p}{p}} &\le \big\|f_n^p + f\big\|_X^{\frac{1-p}{p}} \le \big(\|f_n^p\|_X + \|f\|_X\big)^{\frac{1-p}{p}}\\
&\le 2^{\frac{1-p}{p}-1}\big(\|f_n^p\|_X^{\frac{1-p}{p}} + \|f\|_X^{\frac{1-p}{p}}\big).
\end{aligned}$$

If $\frac{p}{1-p}>1$, applying first (3.1.2) for $r=\frac{p}{1-p}$ and then (3.1.1) for $r=\frac{1-p}{p}$, we have that

$$\begin{aligned}
\big\|\big((f_n^p)^{\frac{1}{p}-1} + f^{\frac{1}{p}-1}\big)^{\frac{p}{1-p}}\big\|_X^{\frac{1-p}{p}} &\le 2^{(\frac{p}{1-p}-1)\cdot\frac{1-p}{p}}\big\|f_n^p + f\big\|_X^{\frac{1-p}{p}}\\
&\le 2^{\frac{2p-1}{p}}\big(\|f_n^p\|_X + \|f\|_X\big)^{\frac{1-p}{p}}\\
&\le 2^{\frac{2p-1}{p}}\big(\|f_n^p\|_X^{\frac{1-p}{p}} + \|f\|_X^{\frac{1-p}{p}}\big).
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\big\|f_n - f^{\frac{1}{p}}\big\|_{X^p} &\le \frac{1}{p}\max\{2^{\frac{1-2p}{p}},2^{\frac{2p-1}{p}}\}\big(\|f_n^p\|_X^{\frac{1-p}{p}} + \|f\|_X^{\frac{1-p}{p}}\big)\big\|f_n^p - f\big\|_X\\
&\le \frac{1}{p}\max\{2^{\frac{1-2p}{p}},2^{\frac{2p-1}{p}}\}\big(\sup_{k\ge 1}\|f_k\|_{X^p}^{1-p} + \|f\|_X^{\frac{1-p}{p}}\big)\big\|f_n^p - f\big\|_X,
\end{aligned}$$

where $\sup_{k\ge 1}\|f_k\|_{X^p}^{1-p}$ is a finite constant as (f_n) is a Cauchy sequence in X^p. Hence (f_n) converges to $f^{\frac{1}{p}}$ in X^p and so X^p is a q-B.f.s.

Suppose now that $p \geq 1$. Let us see first that in this case $\|\cdot\|_{X^p}$ is a norm. Given $f, g \in X^p$, by using Lemma 3.1.2 for $q = 1$, $r = p$ and $s = \frac{p}{p-1}$, we have that

$$\begin{aligned}
\|f+g\|^p_{X^p} &= \big\||f+g|^p\big\|_X = \big\|(f+g)\cdot|f+g|^{p-1}\big\|_X \\
&\leq \big\|f\cdot|f+g|^{p-1}\big\|_X + \big\|g\cdot|f+g|^{p-1}\big\|_X \\
&\leq \|f\|_{X^p}\cdot\big\||f+g|^{p-1}\big\|_{X^{\frac{p}{p-1}}} + \|g\|_{X^p}\cdot\big\||f+g|^{p-1}\big\|_{X^{\frac{p}{p-1}}} \\
&= \big\||f+g|^{p-1}\big\|_{X^{\frac{p}{p-1}}}\cdot\big(\|f\|_{X^p}+\|g\|_{X^p}\big) \\
&= \big\||f+g|^p\big\|_X^{\frac{p-1}{p}}\cdot\big(\|f\|_{X^p}+\|g\|_{X^p}\big) \\
&= \|f+g\|^{p-1}_{X^p}\cdot\big(\|f\|_{X^p}+\|g\|_{X^p}\big)
\end{aligned}$$

and so $\|f+g\|_{X^p} \leq \|f\|_{X^p} + \|g\|_{X^p}$.

Let us see now that (f_n) converges to some function in X^p. Applying inequality (3.1.3) in Lemma 3.1.1 and Lemma 3.1.2 with $q = 1$, $r = \frac{p}{p-1}$ and $s = p$, and noting that $\|\cdot\|_{X^{\frac{p}{p-1}}}$ is a norm as $\frac{p}{p-1} \geq 1$, we have that

$$\begin{aligned}
\big\|f_n^p - f_m^p\big\|_X &\leq p\,\big\|(f_n^{p-1}+f_m^{p-1})\cdot(f_n-f_m)\big\|_X \\
&\leq p\,\big\|f_n^{p-1}+f_m^{p-1}\big\|_{X^{\frac{p}{p-1}}}\cdot\|f_n-f_m\|_{X^p} \\
&\leq p\,\Big(\big\|f_n^{p-1}\big\|_{X^{\frac{p}{p-1}}}+\big\|f_m^{p-1}\big\|_{X^{\frac{p}{p-1}}}\Big)\cdot\|f_n-f_m\|_{X^p} \\
&= p\,\big(\|f_n\|^{p-1}_{X^p}+\|f_m\|^{p-1}_{X^p}\big)\cdot\|f_n-f_m\|_{X^p} \\
&\leq 2p\,\big(\sup_{k\geq 1}\|f_k\|^{p-1}_{X^p}\big)\cdot\|f_n-f_m\|_{X^p}.
\end{aligned}$$

Therefore (f_n^p) is a Cauchy sequence in X and so there exists $f \in X$ such that (f_n^p) converges to f in norm $\|\cdot\|_X$. Note that $f \geq 0$ and $f^{\frac{1}{p}} \in X^p$. From (3.1.2) in Lemma 3.1.1 it follows that $|a-b|^r \leq |a^r - b^r|$ for every $r \geq 1$ and $a, b \geq 0$. Applying this inequality for $r = p$ we have that

$$\big\|f_n - f^{\frac{1}{p}}\big\|_{X^p} = \big\||f_n - f^{\frac{1}{p}}|^p\big\|_X^{\frac{1}{p}} \leq \big\|f_n^p - f\big\|_X^{\frac{1}{p}}.$$

Hence (f_n) converges to $f^{\frac{1}{p}}$ in X^p and so X^p is a B.f.s. □

Therefore, from Proposition 3.1.3, the spaces $L^p(\nu)$ and $L^p_w(\nu)$ are B.f.s.' related to ν for $p \geq 1$ and q-B.f.s.' for $p < 1$.

Note that $\mathcal{S}(\mathcal{R}) \subset L^p(\nu)$ as the p-power of any $\mathcal{R}$-simple function is also $\mathcal{R}$-simple. Even more, $\mathcal{S}(\mathcal{R})$ is dense in $L^p(\nu)$. Indeed, if $0 \leq f \in L^p(\nu)$, then $f^p \in L^1(\nu)$ and by the density of $\mathcal{S}(\mathcal{R})$ in $L^1(\nu)$ there exists a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f^p in $L^1(\nu)$. Note that for a general B.f.s. X and $g, h \in X^p$, from the proof of Proposition 3.1.3 it follows that

$$\|g-h\|_{X^p} \leq \|g^p - h^p\|_X^{\frac{1}{p}}, \quad \text{if } p \geq 1$$
$$\|g-h\|_{X^p} \leq K \cdot \big(\|g\|_{X^p}^{1-p} + \|h\|_{X^p}^{1-p}\big) \cdot \|g^p - h^p\|_X, \quad \text{if } p < 1$$

where $K = \frac{1}{p} \max\big\{2^{\frac{1-2p}{p}}, 2^{\frac{2p-1}{p}}\big\}$. Then, for $X = L^1(\nu)$, $g = f$ and $h = |\varphi_n|^{\frac{1}{p}}$, we have that

$$\|f - |\varphi_n|^{\frac{1}{p}}\|_{p,\nu} \leq \|f^p - |\varphi_n|\|_\nu^{\frac{1}{p}} \leq \|f^p - \varphi_n\|_\nu^{\frac{1}{p}}, \quad \text{if } p \geq 1$$
$$\|f - |\varphi_n|^{\frac{1}{p}}\|_{p,\nu} \leq \widetilde{K}\|f^p - |\varphi_n|\,\|_\nu \leq \widetilde{K}\|f^p - \varphi_n\|_\nu, \quad \text{if } p < 1$$

where $\widetilde{K} = K\big(\|f\|_{p,\nu}^{1-p} + \sup_{n\geq 1}\|\varphi_n\|_\nu^{\frac{1-p}{p}}\big)$ is a finite constant. In any case it follows that $(|\varphi_n|^{\frac{1}{p}}) \subset \mathcal{S}(\mathcal{R})$ converges to f in $L^p(\nu)$. The extension to a general $f \in L^p(\nu)$ is obtained by taking positive and negative parts of f.

The spaces $L^p_w(\nu)$ and $L^p(\nu)$ are p-convex, as for every $f_1, \ldots, f_n \in L^p_w(\nu)$ it follows that

$$\Big\|\Big(\sum_{j=1}^n |f_j|^p\Big)^{\frac{1}{p}}\Big\|_{p,\nu} = \Big\|\sum_{j=1}^n |f_j|^p\Big\|_\nu^{\frac{1}{p}} \leq \Big(\sum_{j=1}^n \big\||f_j|^p\big\|_\nu\Big)^{\frac{1}{p}} = \Big(\sum_{j=1}^n \|f_j\|_{p,\nu}^p\Big)^{\frac{1}{p}}.$$

Moreover, since $\big\|(|f|^p)^{\frac{1}{p}}\big\|_{p,\nu} = \|f\|_{p,\nu} = \big(\|f\|_{p,\nu}^p\big)^{\frac{1}{p}}$ for all $f \in L^p_w(\nu)$, that is, the inequality above is an equality for $n = 1$, both spaces have p-convexity constant $\mathbf{M}^{(p)}\big(L^p_w(\nu)\big) = \mathbf{M}^{(p)}\big(L^p(\nu)\big) = 1$.

Let us see that for $p < 1$, certain convexity property makes the spaces $L^p_w(\nu)$ and $L^p(\nu)$ to be B.f.s.'. Actually, this holds for the p-power X^p of any B.f.s. X.

Proposition 3.1.4. *Let $0 < p < 1$. If X is $\frac{1}{p}$-convex, then X^p is a B.f.s. with the norm*

$$|\!|\!| f |\!|\!|_p = \inf\left\{\sum_{j=1}^n \|f_j\|_{X^p} : |f| \leq \sum_{j=1}^n |f_j| \ \text{ with } f_1, ..., f_n \in X^p,\ n \geq 1\right\},$$

which is equivalent to the quasi-norm $\|\cdot\|_{X^p}$. If moreover $\mathbf{M}^{(\frac{1}{p})}(X) = 1$, the norm $|\!|\!|\cdot|\!|\!|_p$ coincides exactly with $\|\cdot\|_{X^p}$.

Proof. It is direct to check that $\|\!| \cdot |\!\|_p$ is a norm on X^p compatible with the order. Let us see that $\|\!| \cdot |\!\|_p$ and $\| \cdot \|_{X^p}$ are equivalent. From the definition of $\|\!| \cdot |\!\|_p$ it is clear that $\|\!| f |\!\|_p \leq \|f\|_{X^p}$ for all $f \in X^p$ (just taking $f_1 = f$). On the other hand, given $f \in X^p$ and $\epsilon > 0$, we can choose $f_1, \dots f_n \in X^p$ such that $|f| \leq \sum_{j=1}^n |f_j|$ and $\sum_{j=1}^n \|f_j\|_{X^p} \leq \|\!| f |\!\|_p + \epsilon$. Since X is $\frac{1}{p}$-convex, it follows

$$\begin{aligned}
\|f\|_{X^p} &= \big\||f|^p\big\|_X^{\frac{1}{p}} \leq \Big\|\Big(\sum_{j=1}^n |f_j|\Big)^p\Big\|_X^{\frac{1}{p}} = \Big\|\Big(\sum_{j=1}^n (|f_j|^p)^{\frac{1}{p}}\Big)^p\Big\|_X^{\frac{1}{p}} \\
&\leq \left(\mathbf{M}^{(\frac{1}{p})}(X) \cdot \Big(\sum_{j=1}^n \big\||f_j|^p\big\|_X^{\frac{1}{p}}\Big)^p\right)^{\frac{1}{p}} = \mathbf{M}^{(\frac{1}{p})}(X)^{\frac{1}{p}} \cdot \sum_{j=1}^n \|f_j\|_{X^p} \\
&\leq \mathbf{M}^{(\frac{1}{p})}(X)^{\frac{1}{p}} \cdot (\|\!| f |\!\|_p + \epsilon).
\end{aligned}$$

As ϵ is arbitrary, we have that $\|\!| f |\!\|_p \leq \|f\|_{X^p} \leq \mathbf{M}^{(\frac{1}{p})}(X)^{\frac{1}{p}} \cdot \|\!| f |\!\|_p$. Hence, X^p is a B.f.s. with the norm $\|\!| \cdot |\!\|_p$. Note that $\|\!| f |\!\|_p = \|f\|_{X^p}$ whenever $\mathbf{M}^{(\frac{1}{p})}(X) = 1$. $\square$

Therefore, for $p < 1$, if $L^1_w(\nu)$ (resp. $L^1(\nu)$) is $\frac{1}{p}$-convex, then $L^p_w(\nu)$ (resp. $L^p(\nu)$) is a B.f.s. with an equivalent norm to $\| \cdot \|_{p,\nu}$.

Note that in the case when $p < 1$, the spaces of p-integrable functions are quasi-Banach lattices. The analogous definitions related to Banach lattices apply to this case.

Proposition 3.1.5. *The following statements hold:*

(a) *The space $L^p_w(\nu)$ has the σ-Fatou property.*

(b) *The space $L^p(\nu)$ is order continuous.*

Proof. (a) Let $(f_n) \subset L^p_w(\nu)$ be a sequence such that $0 \leq f_n \uparrow$ and $\sup \|f_n\|_{p,\nu} < \infty$. Then $(f_n^p) \subset L^1_w(\nu)$ is such that $0 \leq f_n^p$ and $\sup \|f_n^p\|_\nu = \sup \|f_n\|_{p,\nu}^p < \infty$. The σ-Fatou property of $L^1_w(\nu)$ assures the existence of $g = \sup f_n^p$ in $L^1_w(\nu)$ with $\|g\|_\nu = \sup \|f_n^p\|_\nu$. Then $f = g^{\frac{1}{p}} \in L^p_w(\nu)$ is such that $f_n \uparrow f$ (as $f_n^p \uparrow g$) and $\|f\|_{p,\nu} = \|g^{\frac{1}{p}}\|_{p,\nu} = \|g\|_\nu^{\frac{1}{p}} = \sup \|f_n^p\|_\nu^{\frac{1}{p}} = \sup \|f_n\|_{p,\nu}$.

(b) Let $(f_\tau) \subset L^p(\nu)$ be a downwards directed system $f_\tau \downarrow 0$. Then, $f_\tau^p \downarrow 0$ in $L^1(\nu)$ and since $L^1(\nu)$ is order continuous, $\|f_\tau^p\|_\nu \downarrow 0$. So, $\|f_\tau\|_{p,\nu} = \|f_\tau^p\|_\nu^{\frac{1}{p}} \downarrow 0$. $\square$

It is easy to find examples of $L^p(\nu)$ spaces which have not the σ-Fatou property and $L^p_w(\nu)$ spaces which are not order continuous. For instance, considering the vector measure $\nu : \mathcal{R} \to c_0(\Gamma)$ given in Example 2.2.1, we have that $L^1(\nu) = c_0(\Gamma)$ and $L^1_w(\nu) = \ell^\infty(\Gamma)$. Then, $L^p(\nu) = c_0(\Gamma)$ which does not have the σ-Fatou property and $L^p_w(\nu) = \ell^\infty(\Gamma)$ which is not order continuous. However, if the same ν takes values in $\ell^1(\Gamma)$ (Example 2.2.4) instead of in $c_0(\Gamma)$, then $L^1(\nu) = L^1_w(\nu) = \ell^1(\Gamma)$ and so $L^p(\nu) = L^p_w(\nu) = \ell^p(\Gamma)$ which is order continuous and has the σ-Fatou property. This is just what happens in the case when $p = 1$. In fact, we can extend Proposition 2.3.4 as follows.

Proposition 3.1.6. *The following statements are equivalent:*

(a) $L^1(\nu) = L^1_w(\nu)$.

(b) $L^p(\nu) = L^p_w(\nu)$.

(c) $L^1_w(\nu)$ *is order continuous.*

(d) $L^p_w(\nu)$ *is order continuous.*

(e) $L^1(\nu)$ *has the σ-Fatou property.*

(f) $L^p(\nu)$ *has the σ-Fatou property.*

If (a)-(f) hold, then $L^1_w(\nu)$ and so $L^p_w(\nu)$ has the Fatou property.

Remark that the lattice properties of a B.f.s. involved in the previous proposition are preserved by its p-powers, so the proof is a routine. By the same reason, we have the following result.

Proposition 3.1.7. *The following statements hold:*

(a) $(L^p_w(\nu))_a = (L^p_w(\nu))_{an} = L^p(\nu)$.

(b) $L^p(\nu)$ *is order dense in* $L^p_w(\nu)$ *(also in* $L^0(\nu)$*).*

(c) *If* $L^1_w(\nu)$ *has the Fatou property, so has* $L^p_w(\nu)$.

3.2 Multiplication operators

Let $p > 1$ and suppose that ν is defined on a σ-algebra, in which case $\|\nu\|(\Omega) < \infty$. It is routine to check that $L^p_w(\nu) \subset L^1_w(\nu)$ with $\|f\|_\nu \leq \|\nu\|(\Omega)^{1/p'} \|f\|_{p,\nu}$

where p' is the conjugated exponent of p, and so, by density of the simple functions on $L^p(\nu)$ and since $L^1(\nu)$ is closed in $L^1_w(\nu)$, it follows that $L^p(\nu) \subset L^1(\nu)$. A subtler inclusion $L^p_w(\nu) \subset L^1(\nu)$ is established (see [19, Proposition 3.1 and Corollary 3.2.]). Moreover, in [19, Proposition 3.3] is proved that the inclusion is an L-weakly compact operator (and so a weakly compact operator). However, for vector measures on δ-rings these inclusions are not necessarily true. For instance, we only have to think that $L^p[0,\infty]$ is not included in $L^1[0,\infty]$. It is well-known that for a positive σ-finite measure μ, the inclusions $L^1(\mu) \cap L^\infty(\mu) \subset L^p(\mu) \subset L^1(\mu) + L^\infty(\mu)$ substitute for many purposes the inclusions $L^\infty(\mu) \subset L^p(\mu) \subset L^1(\mu)$ which hold for μ finite. In Section 3.3 we analyze the similar inclusions for the spaces $L^p_w(\nu)$ and $L^p(\nu)$. For this aim, we will need first to study some inclusion relations of the multiplication operators involving spaces of p-integrable functions.

The multiplication operators between $L^p(\nu)$ spaces have been studied recently in a series of papers for the case when ν is defined on a σ-algebra (see [31, Ch.3], [13], [14], [20] and [5]). In particular, the equality $L^p_w(\nu) \cdot L^{p'}(\nu) = L^1(\nu)$ and the compactness properties of the multiplication operators are nowadays well-known in this case. In what follows we will study multiplication operators and some of their properties in the context of vector measures defined on a δ-ring.

Lemma 3.2.1. *Let $p, p' > 1$ be conjugated exponents. Then*

(a) $L^p_w(\nu) \cdot L^{p'}_w(\nu) = L^1_w(\nu)$, *and*

(b) $L^p(\nu) \cdot L^{p'}(\nu) = L^p_w(\nu) \cdot L^{p'}(\nu) = L^1(\nu)$.

Proof. (a) Taking into account Lemma 3.1.2 we have that $L^p_w(\nu) \cdot L^{p'}_w(\nu) \subset L^1_w(\nu)$. Now if $f \in L^1_w(\nu)$, writing $f = \text{sign}(f)|f| = (\text{sign}(f)|f|^{\frac{1}{p}}) \cdot |f|^{\frac{1}{p'}}$ we have the converse inclusion.

(b) First, we will prove that $L^p_w(\nu) \cdot L^{p'}(\nu) = L^1(\nu)$. Let $f \in L^p_w(\nu)$ and $g \in L^{p'}(\nu)$. We can suppose without loss of generality that $f, g \geq 0$. Since $g^{p'}$ is in $L^1(\nu)$ there exist $(A_n) \subset \mathcal{R}$ and a ν-null set N such that $\text{supp}(g) = \text{supp}(g^{p'}) = (\cup A_n) \cup N$ (see the comments before Proposition 2.2.9). Take a sequence (φ_n) in $\mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \varphi_n \uparrow g$ and define $\xi_n = \varphi_n \chi_{\cup_{j=1}^n A_j} \in \mathcal{S}(\mathcal{R})$. Then $0 \leq \xi_n \uparrow g$ and by order continuity of $L^{p'}(\nu)$, it follows that (ξ_n) converges to g in norm. On the other hand, take a sequence (ψ_n) in $\mathcal{S}(\mathcal{R}^{loc})$ such that $0 \le \psi_n \uparrow f$. Note that $\psi_n \xi_n \in \mathcal{S}(\mathcal{R}) \subset L^1(\nu)$ and $fg \in L^p_w(\nu) \cdot L^{p'}_w(\nu) = L^1_w(\nu)$, so it suffices to prove that $\|\psi_n\xi_n - fg\|_\nu \to 0$ as $L^1(\nu)$ is closed in $L^1_w(\nu)$. Since $0 \le \frac{\psi_n}{f}\xi_n \chi_{\mathrm{supp}(f)} \le \xi_n \in L^{p'}(\nu)$, from Lemma 3.1.2 we have that

$$\begin{aligned} \|\psi_n\xi_n - fg\|_\nu &= \Big\| f\chi_{\mathrm{supp}(f)}(\frac{\psi_n}{f}\xi_n - g)\Big\|_\nu \\ &\le \|f\|_{L^p_w(\nu)} \cdot \Big\|(\frac{\psi_n}{f}\xi_n - g)\chi_{\mathrm{supp}(f)}\Big\|_{L^{p'}(\nu)}. \end{aligned}$$

Since $0 \le \frac{\psi_n}{f}\xi_n\chi_{\mathrm{supp}(f)} \uparrow g\chi_{\mathrm{supp}(f)}$, again by order continuity we have that $\Big\|(\frac{\psi_n}{f}\xi_n - g)\chi_{\mathrm{supp}(f)}\Big\|_{L^{p'}(\nu)} \to 0$.

Remark that $L^p(\nu)\cdot L^{p'}(\nu) \subset L^p_w(\nu)\cdot L^{p'}(\nu) \subset L^1(\nu)$ and by using the same arguments as in (a), we obtain $L^1(\nu) \subset L^p(\nu)\cdot L^{p'}(\nu)$. □

Remark 3.2.2. Note that in general $L^p_w(\nu)\cdot L^{p'}_w(\nu) \not\subset L^1(\nu)$. To see this, just consider a vector measure ν such that $L^1(\nu) \ne L^1_w(\nu)$ and take a function f in $L^1_w(\nu) \setminus L^1(\nu)$. Then f can be written as $f = \mathrm{sign}(f)|f| = (\mathrm{sign}(f)|f|^{\frac{1}{p}}) \cdot |f|^{\frac{1}{p'}}$, but $f \notin L^1(\nu)$. For instance, if ν is the vector measure given in Example 2.2.1 for which $L^1_w(\nu) = \ell^\infty(\Gamma)$ and $L^1(\nu) = c_0(\Gamma)$, we have that $L^p_w(\nu)\cdot L^{p'}_w(\nu) = \ell^\infty(\Gamma) \supset L^1(\nu)$.

Lemma 3.2.1 can be rewritten in terms of multiplication operators as follows. Given $g \in L^0(\nu)$ we denote by $M_g : L^0(\nu) \to L^0(\nu)$ the multiplication operator by g, that is $M_g(f) = gf$ for all $f \in L^0(\nu)$. Given two B.f.s' X, Y related to ν, if $M_g : X \to Y$ is well defined then it is automatically continuous. Indeed, consider g^+ and g^- the positive and negative parts of g respectively. Since $g^+, g^- \le |g|$ we have that $M_g = M_{g^+} - M_{g^-}$ where M_{g^+}, M_{g^-} are positive operators between Banach lattices and so they are continuous.

Lemma 3.2.3. *Let $p, p' > 1$ be conjugated exponents and $g \in L^{p'}(\nu)$. Then*

(a) *$M_g \in \mathcal{B}(L^p(\nu), L^1(\nu))$, and*
(b) *$M_g \in \mathcal{B}(L^p_w(\nu), L^1(\nu))$.*

In any case $\|M_g\|$ coincides with $\|g\|_{L^{p'}(\nu)}$.

Proof. Note that M_g is always well defined from Lemma 3.2.1. Moreover, by Lemma 3.1.2, we have that $\|M_g(f)\|_{L^1(\nu)} = \|gf\|_{L^1(\nu)} \leq \|g\|_{L^{p'}(\nu)} \cdot \|f\|_{L^p_w(\nu)}$ for all $f \in L^p_w(\nu)$, thus in both cases $\|M_g\| \leq \|g\|_{L^{p'}(\nu)}$. For the converse inequality, just take the function $f_0 = \|g\|^{-p'/p}_{L^{p'}(\nu)} |g|^{p'/p} \in B_{L^p(\nu)}$ for which $\|gf_0\|_{L^1(\nu)} = \|g\|_{L^{p'}(\nu)}$. □

The arguments used in the proof of the previous lemma also gives the next result.

Lemma 3.2.4. *Let $p, p' > 1$ be conjugated exponents and $g \in L^{p'}_w(\nu)$. Then*

(a) *$M_g \in \mathcal{B}(L^p_w(\nu), L^1_w(\nu))$ with $\|M_g\| \leq \|g\|_{L^{p'}_w(\nu)}$, and*

(b) *$M_g \in \mathcal{B}(L^p(\nu), L^1(\nu))$ with $\|M_g\| = \|g\|_{L^{p'}_w(\nu)}$.*

In the remainder of this section we will require $L^1_w(\nu)$ to have the Fatou property (e.g. if ν is $\mathcal{R}$-decomposable). Recall that in this case $L^p_w(\nu)$ has the Fatou property.

Theorem 3.2.5. *Let $p, p' > 1$ be conjugated exponents and let $g \in L^0(\nu)$. If ν is such that $L^1_w(\nu)$ has the Fatou property, then the following statements are equivalent:*

(a) $g \in L^{p'}_w(\nu)$.

(b) $M_g \in \mathcal{B}(L^p(\nu), L^1(\nu))$.

(c) $M_g \in \mathcal{B}(L^p(\nu), L^1_w(\nu))$.

(d) $M_g \in \mathcal{B}(L^p_w(\nu), L^1_w(\nu))$.

Proof. By Lemma 3.2.4 we have that $(a) \Rightarrow (b)$. Implication $(b) \Rightarrow (c)$ is obvious. Let us see $(c) \Rightarrow (d)$. Let $0 \leq f \in L^p_w(\nu)$. By order density of $L^p(\nu)$ in $L^0(\nu)$ we can take $(f_\tau) \subset L^p(\nu)$ such that $0 \leq f_\tau \uparrow f$ in $L^0(\nu)$. By (3) we have that $0 \leq |g|f_\tau \in L^1_w(\nu)$. Moreover, $|g|f_\tau \uparrow |g|f$ in $L^0(\nu)$. Indeed, if $h \in L^0(\nu)$ is such that $|g|f_\tau \leq h$ for all τ, then $f_\tau = f_\tau \chi_{\mathrm{supp}(g)} + f_\tau \chi_{\Omega \setminus \mathrm{supp}(g)} \leq \frac{h}{|g|}\chi_{\mathrm{supp}(g)} + f\chi_{\Omega \setminus \mathrm{supp}(g)}$ for all τ. So $f \leq \frac{h}{|g|}\chi_{\mathrm{supp}(g)} + f\chi_{\Omega \setminus \mathrm{supp}(g)}$, that is, $|g|f \leq h$. On the other hand, for every τ we have

$$\big\||g|f_\tau\big\|_{L^1_w(\nu)} = \|M_g(f_\tau)\|_{L^1_w(\nu)} \leq \|M_g\| \cdot \|f_\tau\|_{L^p(\nu)} \leq \|M_g\| \cdot \|f\|_{L^p_w(\nu)}.$$

The Fatou property of $L^1_w(\nu)$ yields that there exists $h \in L^1_w(\nu)$ such that $|g|f_\tau \uparrow h$ in $L^1_w(\nu)$. Then $|g|f \leq h$ as $|g|f_\tau \uparrow |g|f$ in $L^0(\nu)$ and so $|g|f \in L^1_w(\nu)$. Note that actually $|g|f = h$. For a general $f \in L^p_w(\nu)$, by taking positive and negative parts of f, it follows that $fg \in L^1_w(\nu)$.

Finally, let us see $(d) \Rightarrow (a)$. Let $M_g \in \mathcal{B}(L^p_w(\nu), L^1_w(\nu))$. By order density of $L^{p'}(\nu)$ in $L^0(\nu)$ there exists $(f_\tau) \subset L^{p'}(\nu)$ such that $0 \leq f_\tau \uparrow |g|$ in $L^0(\nu)$. Moreover, by Lemma 3.2.4.(b) and noting that $\|f_\tau h\|_{L^1_w(\nu)} \leq \|gh\|_{L^1_w(\nu)}$ for all $h \in L^p_w(\nu)$, we have that

$$\sup \|f_\tau\|_{L^{p'}_w(\nu)} = \sup \|M_{f_\tau}\| \leq \|M_g\|.$$

The Fatou property of $L^{p'}_w(\nu)$ ensures that there exists $f = \sup f_\tau \in L^{p'}_w(\nu)$. Then $f_\tau \uparrow |g|$ in $L^0(\nu)$ and $f_\tau \uparrow f$ in $L^{p'}_w(\nu)$ yield $f = |g|$ and so $g \in L^{p'}_w(\nu)$. $\square$

Theorem 3.2.6. *Let $p, p' > 1$ be conjugate exponents and let $g \in L^0(\nu)$. If ν is such that $L^1_w(\nu)$ has the Fatou property, then the following conditions are equivalent:*

(a) $g \in L^{p'}(\nu)$.

(b) $M_g \in \mathcal{B}(L^p_w(\nu), L^1(\nu))$.

Proof. By Lemma 3.2.3 we have that $(a) \Rightarrow (b)$. Let us see $(b) \Rightarrow (a)$. Suppose that $M_g \in \mathcal{B}(L^p_w(\nu), L^1(\nu))$. Then also $M_g \in \mathcal{B}(L^p(\nu), L^1(\nu))$ and so, by Theorem 3.2.5 we have that $g \in L^{p'}_w(\nu)$. Hence $|g|^{p'-1} \in L^p_w(\nu)$. Therefore, $|g|^{p'} = |g| \cdot |g|^{p'-1} \in L^1(\nu)$, that is $g \in L^{p'}(\nu)$. $\square$

We finish this section by analyzing the compactness properties of the multiplication operators.

Theorem 3.2.7. *Let $p, p' > 1$ conjugate exponents and let $g \in L^0(\nu)$. If ν is $\mathcal{R}$-decomposable then the following statements are equivalent:*

(a) $g \in L^{p'}(\nu)$.

(b) $M_g \in \mathcal{B}(L^p_w(\nu), L^1(\nu))$.

(c) $M_g \in \mathcal{L}(L^p_w(\nu), L^1(\nu))$.

(d) $M_g \in \mathcal{L}(L^p(\nu), L^1(\nu))$.

(e) $M_g \in \mathcal{L}(L^p_w(\nu), L^1_w(\nu))$.

(f) $M_g \in \mathcal{L}(L^p(\nu), L^1_w(\nu))$.

(g) $M_g \in \mathcal{W}(L^p_w(\nu), L^1(\nu))$.

(h) $M_g \in \mathcal{W}(L^p(\nu), L^1(\nu))$.

(i) $M_g \in \mathcal{W}(L^p_w(\nu), L^1_w(\nu))$.

(j) $M_g \in \mathcal{W}(L^p(\nu), L^1_w(\nu))$.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is precisely Theorem 3.2.6. The implication $(c) \Rightarrow (b)$ is evident. Let us see $(b) \Rightarrow (c)$. Let $M_g \in \mathcal{B}(L^p_w(\nu), L^1(\nu))$. We want to see that $M_g(B_{L^p_w(\nu)})$ is an $\mathcal{L}$-weakly compact set in $L^1(\nu)$, that is, $\|h_n\|_{L^1(\nu)} \to 0$ for every disjoint sequence (h_n) contained in the solid hull of $M_g(B_{L^p_w(\nu)})$. Note that the solid hull of $M_g(B_{L^p_w(\nu)})$ is itself, since $M_g(B_{L^p_w(\nu)})$ is solid in $L^1(\nu)$. In fact, let $h \in L^1(\nu)$ such that $|h| \leq |gf|$ with $f \in B_{L^p_w(\nu)}$. Then,

$$\frac{|h|}{|g|}\chi_{\mathrm{supp}(g)} \leq |f|\chi_{\mathrm{supp}(g)} \leq |f|$$

and so $\frac{h}{g}\chi_{\mathrm{supp}(g)} \in L^p_w(\nu)$ and

$$\left\|\frac{h}{g}\chi_{\mathrm{supp}(g)}\right\|_{L^p_w(\nu)} \leq \|f\|_{L^p_w(\nu)} \leq 1.$$

Hence $h = g\frac{h}{g}\chi_{\mathrm{supp}(g)} \in M_g(B_{L^p_w(\nu)})$. So we can take $(h_n) \subset M_g(B_{L^p_w(\nu)})$ and define $A_n = \cup_{j\geq n}\mathrm{supp}(h_j)$. Then, (A_n) is a decreasing sequence with $\cap A_n = \emptyset$ as (h_n) is a disjoint sequence. On the other hand, for every n there exists $f_n \in B_{L^p_w(\nu)}$ such that $h_n = M_g(f_n) = gf_n = gf_n\chi_{A_n}$. Noting that $g \in L^{p'}(\nu)$, by Lemma 3.1.2,

$$\|h_n\|_{L^1(\nu)} \leq \|f_n\|_{L^p_w(\nu)} \cdot \|g\chi_{A_n}\|_{L^{p'}(\nu)} \leq \|g\chi_{A_n}\|_{L^{p'}(\nu)}.$$

Since $g\chi_{A_n} \downarrow 0$ in the order continuous space $L^{p'}(\nu)$, then $\|g\chi_{A_n}\|_{L^{p'}(\nu)} \to 0$.

The implication $(c) \Rightarrow (d)$ is clear since $B(L^p(\nu)) \subset B(L^p_w(\nu))$. Let us show now $(d) \Rightarrow (a)$ and close the equivalences from (a) to (d). Let M_g in $\mathcal{L}(L^p(\nu), L^1(\nu))$. In particular, $M_g \in \mathcal{B}(L^p(\nu), L^1(\nu))$ and Theorem 3.2.5 yields that $g \in L^{p'}_w(\nu)$.

Let us show that $g \in L^{p'}(\nu)$. Since ν is $\mathcal{R}$-decomposable, we can take a ν-null set $N \in \mathcal{R}^{loc}$ and a family $\{\Omega_\alpha : \alpha \in \Delta\}$ of pairwise disjoint sets in $\mathcal{R}$ satisfying conditions (i) and (ii) in Definition 2.3.6 such that $\Omega = (\cup_{\alpha\in\Delta}\Omega_\alpha)\cup N$

with disjoint union. For every finite set $I \subset \Delta$, we consider $\Omega_I = \cup_{\alpha \in I} \Omega_\alpha \in \mathcal{R}$ and the vector measure $\nu_I \colon \mathcal{R}^{loc} \to X$ defined by $\nu(A \cap \Omega_I)$ for all $A \in \mathcal{R}^{loc}$. It follows that $f \in L^1_w(\nu_I)$ if and only if $f\chi_{\Omega_I} \in L^1_w(\nu)$ and in this case $\|f\|_{\nu_I} = \|f\chi_{\Omega_I}\|_\nu$, see the proof of Theorem 2.3.8. Even more, $f \in L^1(\nu_I)$ if and only if $f\chi_{\Omega_I} \in L^1(\nu)$. Indeed, let $f \in L^1(\nu_I)$ and take $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ converging in $L^1(\nu_I)$ and ν_I-a.e. pointwise (except on a ν_I-null set Z). Then, $(\varphi_n\chi_{\Omega_I}) \subset \mathcal{S}(\mathcal{R})$ converges to $f\chi_{\Omega_I}$ ν-a.e. (except on the ν-null set $Z \cap \Omega_I$). Moreover, $\|\varphi_n\chi_{\Omega_I} - \varphi_m\chi_{\Omega_I}\|_\nu = \|\varphi_n - \varphi_m\|_{\nu_I} \to 0$, as $n, m \to \infty$. So, there exists $h \in L^1(\nu)$ such that $(\varphi_n\chi_{\Omega_I})$ converges to h in $L^1(\nu)$. By taking a subsequence converging ν-a.e. to h, it follows that $f\chi_{\Omega_I} = h \in L^1(\nu)$. For the converse implication a similar argument works.

Define now $B_k = \{\omega \in \Omega : 0 \le |g(\omega)| < k\}$ for $k \in \mathbb{N}$ and consider $(|g|\chi_{B_k})_{(k,I)} \subset L^{p'}(\nu_I)$ as each $g\chi_{B_k}$ is bounded and ν_I is defined on a σ-algebra. Then $|g|\chi_{B_k}\chi_{\Omega_I} \in L^{p'}(\nu)$ and it follows that $|g|\chi_{B_k}\chi_{\Omega_I} \uparrow |g|$ in $L^0(\nu)$. We claim that the upwards directed system $(|g|\chi_{B_k}\chi_{\Omega_I})_{(k,I)}$ is a Cauchy system in $L^{p'}(\nu)$. Otherwise, there would exist a number $\epsilon > 0$ and an increasing sequence $(|g|\chi_{B_{k_n}}\chi_{\Omega_{I_n}})$ such that $\big\||g|\chi_{B_{k_{n+1}}}\chi_{\Omega_{I_{n+1}}} - |g|\chi_{B_{k_n}}\chi_{\Omega_{I_n}}\big\|_{L^{p'}(\nu)} > \epsilon$ for all n, i.e. such that $\||g|\chi_{C_n}\|_{L^{p'}(\nu)} > \epsilon$ where $C_n = (B_{k_{n+1}} \cap \Omega_{I_{n+1}}) \setminus (B_{k_n} \cap \Omega_{I_n})$ are pairwise disjoint. Let $f_n = \alpha|g|^{p'/p}\chi_{C_n} \in B_{L^p(\nu)}$ where $\alpha = \big\|g\big\|^{-p'/p}_{L^{p'}_w(\nu)}$. Then $\|M_g(f_n)\|_{L^1(\nu)} \to 0$ due to the $\mathcal{L}$-weakly compactness of M_g, whereas

$$\|M_g(f_n)\|_{L^1(\nu)} = \big\||g|^{p'}\chi_{C_n}\big\|_{L^1(\nu)}\alpha = \big\||g|\chi_{C_n}\big\|^{p'}_{L^{p'}(\nu)}\alpha > \epsilon^{p'}\alpha$$

which gives a contradiction. Therefore $(|g|\chi_{B_k})_{(k,I)}$ is convergent in norm to some $h \in L^{p'}(\nu)$. By Theorem 100.8 in [34], we have that $|g|\chi_{B_k}\chi_{\Omega_I} \uparrow h$ in $L^{p'}(\nu)$ and so $g = h$.

Clearly, $(c) \Rightarrow (e)$ since $L^1(\nu)$ is continuously included in $L^1_w(\nu)$. The implication $(e) \Rightarrow (f)$ follows by the same argument as the one used to prove $(c) \Rightarrow (d)$. We will show now that $(f) \Rightarrow (d)$. Assume $M_g \in \mathcal{L}(L^p(\nu), L^1_w(\nu))$. In particular, M_g is in $\mathcal{B}(L^p(\nu), L^1_w(\nu))$ and so, Theorem 3.2.5 yields that $M_g \in \mathcal{B}(L^p(\nu), L^1(\nu))$. Hence $M_g \in \mathcal{L}(L^p(\nu), L^1(\nu))$. We already have the equivalences (a) to (f).

Since every $\mathcal{L}$-weakly compact operator is weakly compact, $(c) \Rightarrow (g)$. Implication $(g) \Rightarrow (i)$ holds again since $L^1(\nu) \subset L^1_w(\nu)$. The same argument for

$(c) \Rightarrow (d)$ gives $(i) \Rightarrow (j)$. In the same way that $(f) \Rightarrow (d)$, we obtain $(j) \Rightarrow (h)$.

Finally, let us see $(h) \Rightarrow (a)$ and so the chain will be closed. Let M_g in $\mathcal{W}(L^p(\nu), L^1(\nu))$ and so by Theorem 3.2.5, $g \in L^{p'}_w(\nu)$. For every $k \in \mathbb{N}$, let $A_k = \{\omega \in \Omega : k-1 \leq |g(\omega)|^{p'} < k\}$ and consider $(|g|^{p'}\chi_{A_k}) \subset L^1(\nu_I)$ (we follow the notation in the proof of $(4) \Rightarrow (1)$). Then $|g|^{p'}\chi_{A_k}\chi_{\Omega_I} \in L^1(\nu)$. Define

$$S_{(n,I)} = \sum_{k=1}^{n} \int |g|^{p'}\chi_{A_k}\chi_{\Omega_I}\, d\nu.$$

Writing $f_{(n,I)} = \operatorname{sign}(g)\sum_{k=1}^{n} |g|^{p'-1}\chi_{A_k}\chi_{\Omega_I} \in L^p(\nu)$, we have that $S_{(n,I)} = \int g f_{(n,I)}\, d\nu = I_\nu \circ M_g(f_{(n,I)})$. The ideal property of the weakly compact operators gives that $I_\nu \circ M_g \in \mathcal{W}(L^p(\nu), X)$. Since $|f_{(n,I)}|^p \leq |g|^{p'}$, we have that $\| |f_{(n,I)}|^p \|_{L^1(\nu)} \leq \| |g|^{p'} \|_{L^1_w(\nu)}$ and so, $\|f_{(n,I)}\|_{L^p(\nu)} \leq \|g\|^{p'/p}_{L^{p'}_w(\nu)}$. Hence, $(f_{(n,I)})_{(n,I)} \subset \|g\|_{L^{p'}_w(\nu)} \cdot B_{L^p(\nu)}$ and then the upwards directed system $(S_{(n,I)})_{(n,I)}$ is contained in a relatively weakly compact subset of X. Consequently, there exists a subsystem $(S_{(n_\alpha,I_\alpha)})_\alpha \subset (S_{(n,I)})_{(n,I)}$ weakly convergent to some $x_0 \in X$.

Since $|g^{p'}| \in L^1_w(\nu)$, we can consider the element $x_0'' \in X^{**}$ defined by $x_0''(x^*) = \int |g|^{p'}\, dx^*\nu$ for all $x^* \in X^*$. Noting that $g f_{(n,I)} \uparrow |g|^{p'}$ in $L^0(x^*\nu)$ and so in $L^1(x^*\nu)$, due to the order continuity of $L^1(x^*\nu)$, we have that

$$x^*(S_{(n,I)}) = \int g f_{(n,I)}\, dx^*\nu \to \int |g|^{p'}\, dx^*\nu = x^*(x_0'').$$

Hence, $(S_{(n,I)})_{(n,I)}$ converges in the weak* topology of X^{**} to x_0''. Since the weak* topology of X^{**} coincides in X with the weak topology of X, it follows that $x_0'' = x_0 \in X$. Therefore, $|g|^{p'} \in L^1(\nu)$ and we conclude the proof. □

Remark 3.2.8. Following the results in [13], the previous theorem can be extended to the corresponding cases of semi-compact and $\mathcal{M}$-weakly compact operators. For the definitions we refer to [30, Definition 3.6.9] and for the proof check Theorem 7 in [13].

3.3 $L^p(\nu)$ and $L^p_w(\nu)$ as intermediate spaces

It is well-known that for a positive σ-finite measure μ, the inclusion relation $L^1(\mu) \cap L^\infty(\mu) \subset L^p(\mu) \subset L^1(\mu) + L^\infty(\mu)$ substitutes for many purposes the inclusions $L^\infty(\mu) \subset L^p(\mu) \subset L^1(\mu)$ which hold for μ finite. In this section we analyze the similar inclusions for the spaces $L^p(\nu)$ and $L^p_w(\nu)$. Note that all the inclusions involving spaces of integrable functions are continuous as we are dealing with Banach lattices (see Preliminaries).

Denote by $L^\infty(\nu)$ the space of (classes of) ν-a.e. bounded functions. Of course, $L^\infty(\nu)$ is a B.f.s. related to ν for the supremum norm $\|\cdot\|_\infty$.

Proposition 3.3.1. *Let $p > 1$. The following inclusions hold.*

(a) $L^1_w(\nu) \cap L^\infty(\nu) \subset L^p_w(\nu) \subset L^1_w(\nu) + L^\infty(\nu)$.

(b) $L^1(\nu) \cap L^\infty(\nu) \subset L^p(\nu) \subset L^1(\nu) + L^\infty(\nu)$.

Proof. (a) Consider the B.f.s.' $L^1_w(\nu) \cap L^\infty(\nu)$ and $L^1_w(\nu) + L^\infty(\nu)$ with the usual lattice norms

$$\|f\|_{L^1_w(\nu)\cap L^\infty(\nu)} = \max\big\{\|f\|_{L^1_w(\nu)}, \|f\|_\infty\big\},$$

$$\|h\|_{L^1_w(\nu)+L^\infty(\nu)} = \inf\big\{\|f\|_{L^1_w(\nu)} + \|g\|_\infty : h = f+g,\ f \in L^1_w(\nu),\ g \in L^\infty(\nu)\big\}.$$

For every $f \in L^1_w(\nu) \cap L^\infty(\nu)$ we have that $|f| \le \|f\|_\infty$. Then since $p-1>0$, we have that $|f|^{p-1} \le \|f\|_\infty^{p-1}$ and so $|f|^p \le \|f\|_\infty^{p-1}\cdot|f|$. Hence $|f|^p \in L^1_w(\nu)$, that is, $f \in L^p_w(\nu)$.

For the second containment, let $f \in L^p_w(\nu)$ and define the measurable set $A = \{\omega \in \Omega : |f(\omega)| > 1\}$. Note that $\chi_A \le |f|^p$ and so $\|\chi_A\|_\nu \le \||f|^p\|_\nu = \|f\|^p_{p,\nu}$. Then $\chi_A \in L^1_w(\nu)$ and thus $\chi_A \in L^{p'}_w(\nu)$. Writing $f = f\chi_A + f\chi_{\Omega\setminus A}$, clearly $f\chi_{\Omega\setminus A} \in L^\infty(\nu)$. Moreover, since $L^p_w(\nu)\cdot L^{p'}_w(\nu) = L^1_w(\nu)$ with p, p' conjugate exponents (see Lemma 3.2.1) we have that $f\chi_A \in L^1_w(\nu)$. Hence $f \in L^1_w(\nu) + L^\infty(\nu)$.

The inclusions $L^1(\nu) \cap L^\infty(\nu) \subset L^p(\nu) \subset L^1(\nu) + L^\infty(\nu)$ in (b) follow by the same argument. $\square$

In general these relations cannot be improved in the sense $L^1_w(\nu)\cap L^\infty(\nu)\subset L^p(\nu)$ and $L^p_w(\nu)\subset L^1(\nu)+L^\infty(\nu)$. For instance, the example 2.2.1 shows that the first inclusion may fail. The following example shows that the second inclusion also may fail.

Example 3.3.2. Let (Θ,Σ,μ) be a finite non atomic measure space and consider the vector measure $\eta\colon\Sigma\to L^1(\mu)$ given by $\eta(A)=\chi_A$ for all $A\in\Sigma$. Given an uncountable abstract set Γ, we construct the vector measure ν as in Section 2.4 for $\nu_\gamma=\eta$ for all $\gamma\in\Gamma$. Then, $L^1_w(\nu)=\ell^\infty\big(\Gamma,L^1(\mu)\big)$ and so $L^p_w(\nu)=\ell^\infty\big(\Gamma,L^p(\mu)\big)$. Moreover, we have that $L^1(\nu)=c_0\big(\Gamma,L^1(\mu)\big)$ and $L^\infty(\nu)=\ell^\infty\big(\Gamma,L^\infty(\mu)\big)$. Taking $f\in L^p(\mu)\setminus L^\infty(\mu)$, we have that $(f)_{\gamma\in\Gamma}\in\ell^\infty\big(\Gamma,L^p(\mu)\big)$ but it cannot be written as a sum of elements $(h_\gamma)_{\gamma\in\Gamma}\in c_0\big((\Gamma,L^1(\nu)\big)$ and $(g_\gamma)_{\gamma\in\Gamma}\in\ell^\infty\big(\Gamma,L^\infty(\nu)\big)$, since $h_\gamma=0$ except on countable many γ. Consequently, $L^p_w(\nu)\not\subset L^1(\nu)+L^\infty(\nu)$.

However, an improvement of $L^p_w(\nu)\subset L^1_w(\nu)+L^\infty(\nu)$ is possible by using a larger space than $L^1(\nu)+L^\infty(\nu)$. Namely,

$$L^1_{w,0}(\nu)=\overline{L^1_w(\nu)\cap L^\infty(\nu)}^{L^1_w(\nu)}.$$

Remark that $L^1(\nu)\subset L^1_{w,0}(\nu)\subset L^1_w(\nu)$ since $\mathcal{S}(\mathcal{R})\in L^1_w(\nu)\cap L^\infty(\nu)$ and $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu)$.

Proposition 3.3.3. *Let $p>1$. The inclusion $L^p_w(\nu)\subset L^1_{w,0}(\nu)+L^\infty(\nu)$ holds.*

Proof. Let $f\in L^p_w(\nu)$ and consider the set $A=\{\omega\in\Omega:|f(\omega)|>1\}$ for which $\chi_A\in L^{p'}_w(\nu)$ and $f=f\chi_A+f\chi_{\Omega\setminus A}\in L^1_w(\nu)+L^\infty(\nu)$ (see the proof of Proposition 3.3.1). For every $n\in\mathbb{N}$, define $B_n=\{\omega\in A:|f(\omega)|\le n\}$ and $f_n=f\chi_{B_n}$. Note that $|f_n|\le n\chi_{B_n}\in L^1_w(\nu)\cap L^\infty(\nu)$ and so $f_n\in L^1_w(\nu)\cap L^\infty(\nu)$. By Lemma 3.1.2,

$$\|f\chi_A-f_n\|_{L^1_w(\nu)}=\|f(\chi_A-\chi_{B_n})\|_{L^1_w(\nu)}\le\|f\|_{L^p_w(\nu)}\cdot\|\chi_A-\chi_{B_n}\|_{L^{p'}_w(\nu)},$$

where p' is the conjugate exponent of p. Remark also that $\|\chi_A-\chi_{B_n}\|_{L^{p'}_w(\nu)}=\|\chi_{A\setminus B_n}\|_{L^{p'}_w(\nu)}=\|\chi_{A\setminus B_n}\|^{\frac{1}{p'}}_{L^1_w(\nu)}=\|\nu\|(A\setminus B_n)^{\frac{1}{p'}}$. Since $\chi_{A\setminus B_n}\le\frac{1}{n}|f|$, we have that $\|\nu\|(A\setminus B_n)^{\frac{1}{p}}=\|\chi_{A\setminus B_n}\|_{L^p_w(\nu)}\le\frac{1}{n}\|f\|_{L^p_w(\nu)}\to0$ and so $\|f\chi_A-f_n\|_{L^1_w(\nu)}\to0$. Hence, $f\chi_A\in L^1_{w,0}(\nu)$. $\square$

Remark 3.3.4. Consider the case when the vector measure is defined on a σ-algebra. Then, $L^1_w(\nu) \cap L^\infty(\nu) = L^\infty(\nu) \subset L^1(\nu)$ which is closed in $L^1_w(\nu)$. Hence, $L^1_{w,0}(\nu) = L^1(\nu)$ and the inclusion in the previous proposition gives $L^p_w(\nu) \subset L^1(\nu)$. Therefore, Proposition 3.3.3 is a generalization of [13, Proposition 3.1].

Chapter 4

Representation of Banach lattices as L^1_w spaces of a vector measure defined on a δ-ring

The interplay among the properties of a vector measure ν, its range and its integration operator allows us to understand the behavior of the space $L^1(\nu)$ of integrable functions with respect to ν. This makes desirable to know which spaces can be described as such L^1-spaces.

As it was already mentioned in the Introduction, in [7, Theorem 8], Curbera proves that every order continuous Banach lattice E with a weak unit is order isometric to a space $L^1(\nu)$ where ν is a vector measure defined on a σ-algebra (see also [31, Proposition 3.9] for the complex version). The result remains true if E has not a weak unit but for ν defined on a δ-ring. This was stated in [6, pages 22-23] but the proof there is just outlined. We present here a proof of this fact in full detail.

If we think now about the space $L^1_w(\nu)$ of weakly integrable functions with respect to ν, in [10, Theorem 2.5], Curbera and Ricker show that every Banach lattice E satisfying the σ-Fatou property and with a weak unit belonging to the σ-order continuous part E_a of E is order isometric to a space $L^1_w(\nu)$ for a vector measure ν defined on a σ-algebra.

The aim of this chapter is to prove the corresponding result in the case when E has not a weak unit by using a vector measure defined on a δ-ring. We prove in Section 4.1 that every Banach lattice having the Fatou property and having its σ-order continuous part as an order dense subset, can be represented as the space $L^1_w(\nu)$ of weakly integrable functions with respect to some vector measure ν defined on a δ-ring.

In Section 4.2 we also establish a representation theorem for the class of σ-Fatou Banach lattices E with the σ-order continuous part as a super order dense ideal in E, using again vector measures on δ-rings. In this case E is order isometric to the σ-Fatou completion of $L^1(\nu)$.

Section 4.3 deals with a concrete example in order to remark the differences which can be exist when the representation of a Banach lattice is possible, by using vector measures defined on either a σ-algebra or a δ-ring.

Similar representation theorems for p-convex Banach lattices as L^p and L^p_w spaces is also given in Section 4.4.

Finally, in Section 4.5 we will see that if a Banach lattice having an algebra structure can be represented as a space of integrable functions, then this space inherit in some way the algebra structure.

4.1 Representing Fatou Banach lattices

The starting point of this section is a concrete vector measure which always can be associated to an order continuous Banach lattice. This vector measure makes possible all the representations theorems appearing in this chapter.

4.1.1 Vector measure associated to an order continuous Banach lattice

Let E be an order continuous Banach lattice. We will prove in Section 4.1.2 that there exists a vector measure ν defined on a δ-ring and with values in E, such that the space $L^1(\nu)$ of integrable functions with respect to ν is order isometric to E. More precisely, the integration operator $I_\nu\colon L^1(\nu) \to E$ is an order isometry.

The key for constructing our vector measure is the following result of Lindenstrauss and Tzafriri [26, Proposition 1.a.9]: *E can be decomposed into an unconditionally direct sum of a family of mutually disjoints ideals $\{E_\alpha\}_{\alpha\in\Delta}$, each E_α having a weak unit. That is, every $e \in E$ has a unique representation $e = \sum_{\alpha\in\Delta} e_\alpha$ with $e_\alpha \in E_\alpha$, only countably many $e_\alpha \neq 0$ and the series converging unconditionally.*

Each E_α is an order continuous Banach lattice with a weak unit. Then, from [7, Theorem 8], there exist a σ-algebra Σ_α of parts of an abstract set Ω_α and a positive vector measure $\nu_\alpha\colon \Sigma_\alpha \to E_\alpha$ such that the integration operator $I_{\nu_\alpha}\colon L^1(\nu_\alpha) \to E_\alpha$ is an order isometry.

Consider the set $\Omega = \cup_{\alpha\in\Delta}\{\alpha\} \times \Omega_\alpha$ and the δ-ring $\mathcal{R}$ of subsets of Ω given by the sets $\cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha$ satisfying that $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \Delta$ and there exists a finite set $I \subset \Delta$ such that A_α is ν_α-null for all $\alpha \in \Delta\backslash I$. Then,

$$\mathcal{R}^{loc} = \big\{ \cup_{\alpha\in\Delta} \{\alpha\} \times A_\alpha : \ A_\alpha \in \Sigma_\alpha \text{ for all } \alpha \in \Delta \big\}$$

(see Section 2.4 for the computations).

Define the finitely additive set function $\nu\colon \mathcal{R} \to E$ as

$$\nu\big(\cup_{\alpha\in\Delta} \{\alpha\} \times A_\alpha \big) = \sum_{\alpha\in\Delta} \nu_\alpha(A_\alpha).$$

Let us see that ν is a vector measure. Given $A_n = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha^n \in \mathcal{R}$ for $n \geq 1$ mutually disjoint sets such that $\cup_{n\geq1}A_n \in \mathcal{R}$, we have that

$$\bigcup_{n\geq1} A_n = \bigcup_{\alpha\in\Delta} \{\alpha\} \times \Big(\bigcup_{n\geq1} A_\alpha^n \Big)$$

where $\bigcup_{n\geq 1} A^n_\alpha$ is a disjoint union for every $\alpha\in\Delta$ and there exists a finite set $I\subset\Delta$ such that $\bigcup_{n\geq 1} A^n_\alpha$ is ν_α-null for all $\alpha\in\Delta\backslash I$. Since for each $\alpha\in\Delta$ the sum $\sum_{n\geq 1}\nu_\alpha(A^n_\alpha)$ converges to $\nu_\alpha(\cup_{n\geq 1}A^n_\alpha)$ in E_α and so in E, then we have that

$$\nu\Big(\bigcup_{n\geq 1} A_n\Big)=\sum_{\alpha\in I}\nu_\alpha\Big(\bigcup_{n\geq 1} A^n_\alpha\Big)=\sum_{\alpha\in I}\sum_{n\geq 1}\nu_\alpha(A^n_\alpha)=\sum_{n\geq 1}\sum_{\alpha\in I}\nu_\alpha(A^n_\alpha)=\sum_{n\geq 1}\nu(A_n).$$

Note that ν is positive as every ν_α is so. Also note that $\big\{\{\alpha\}\times\Omega_\alpha:\ \alpha\in\Delta\big\}$ is a family of pairwise disjoint sets in $\mathcal{R}$ satisfying that if $\{\alpha\}\times A_\alpha\in\mathcal{R}\cap 2^{\{\alpha\}\times\Omega_\alpha}$ for all $\alpha\in\Delta$, then $\cup_{\alpha\in\Delta}\{\alpha\}\times A_\alpha\in\mathcal{R}^{loc}$. Moreover, given $x^*\in X^*$, if $\{\alpha\}\times Z_\alpha\in\mathcal{R}\cap 2^{\{\alpha\}\times\Omega_\alpha}$ is $|x^*\nu|$-null for all $\alpha\in\Delta$, then $Z=\cup_{\alpha\in\Delta}\{\alpha\}\times Z_\alpha$ is $|x^*\nu|$-null. Indeed, taking $A=\cup_{\alpha\in\Delta}\{\alpha\}\times A_\alpha\in\mathcal{R}\cap 2^Z$, we have that $x^*\nu(A)=x^*\big(\sum_{\alpha\in\Delta}\nu_\alpha(A_\alpha)\big)=\sum_{\alpha\in\Delta}x^*\nu_\alpha(A_\alpha)$ (note that the sum is finite). Since $\{\alpha\}\times A_\alpha\subset\{\alpha\}\times Z_\alpha$ it follows that $x^*\nu_\alpha(A_\alpha)=x^*\nu(\{\alpha\}\times A_\alpha)=0$, so $x^*\nu(A)=0$ and then Z is $|x^*\nu|$-null. Hence, ν is $\mathcal{R}$-decomposable. Moreover, a set $A=\cup_{\alpha\in\Delta}\{\alpha\}\times A_\alpha\in\mathcal{R}^{loc}$ is ν-null if and only if A_α is ν_α-null for all $\alpha\in\Delta$.

Remark 4.1.1. Let $f\in\mathcal{M}(\mathcal{R}^{loc})$. For each $\alpha\in\Delta$, we denote by f_α the sections $f(\alpha,\cdot)\colon\Omega_\alpha\to\mathbb{R}$. Note that $f_\alpha\in\mathcal{M}(\Sigma_\alpha)$ and, if $\varphi=\sum_{j=1}^n a_j\chi_{A_j}$ with $A_j=\cup_{\alpha\in\Delta}\{\alpha\}\times A^j_\alpha\in\mathcal{R}^{loc}$, then $\varphi_\alpha=\sum_{j=1}^n a_j\chi_{A^j_\alpha}\in\mathcal{S}(\Sigma_\alpha)$.

The following lemma will allow us to give useful description of the spaces $L^1(\nu)$ and $L^1_w(\nu)$.

Lemma 4.1.2. *Let $f\in\mathcal{M}(\mathcal{R}^{loc})$ and $\alpha\in\Delta$. Then,*

(a) $f\chi_{\{\alpha\}\times\Omega_\alpha}\in L^1_w(\nu)$ *if and only if* $f_\alpha\in L^1_w(\nu_\alpha)$.

(b) $f\chi_{\{\alpha\}\times\Omega_\alpha}\in L^1(\nu)$ *if and only if* $f_\alpha\in L^1(\nu_\alpha)$. *In this case*

$$\int f\chi_{\{\alpha\}\times\Omega_\alpha}\,d\nu=\int f_\alpha\,d\nu_\alpha.$$

Proof. Let $x^*\in E^*$ and $x^*_\alpha\in E^*_\alpha$ be the restriction of x^* to E_α. For each function $\varphi=\sum_{j=1}^n a_j\chi_{A_j}\in\mathcal{S}(\mathcal{R}^{loc})$ with $A_j=\cup_{\alpha\in\Delta}\{\alpha\}\times A^j_\alpha$, we have that

$\varphi\chi_{\{\alpha\}\times\Omega_\alpha} = \sum_{j=1}^n a_j \chi_{\{\alpha\}\times A_\alpha^j} \in \mathcal{S}(\mathcal{R})$ and $\varphi_\alpha = \sum_{j=1}^n a_j \chi_{A_\alpha^j} \in \mathcal{S}(\Sigma_\alpha)$, then

$$\begin{aligned}\int \varphi\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu &= \sum_{j=1}^n a_j x^*\nu(\{\alpha\}\times A_\alpha^j) = \sum_{j=1}^n a_j x^*\nu_\alpha(A_\alpha^j) \\ &= \sum_{j=1}^n a_j x_\alpha^*\nu_\alpha(A_\alpha^j) = \int \varphi_\alpha\, dx_\alpha^*\nu_\alpha.\end{aligned}$$

It is routine to check that $|x^*\nu|(\{\alpha\}\times A_\alpha) = |x_\alpha^*\nu_\alpha|(A_\alpha)$ for every A_α in Σ_α. Then, in a similar way as for $x^*\nu$, we have that $\int \varphi\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| = \int \varphi_\alpha\, d|x_\alpha^*\nu_\alpha|$.

Let $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ be a sequence such that $0 \leq \varphi_n \uparrow |f|$. Then, $0 \leq \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha} \uparrow |f|\chi_{\{\alpha\}\times\Omega_\alpha}$ and $0 \leq (\varphi_n)_\alpha \uparrow |f_\alpha|$. Using the monotone convergence theorem, we have that

$$\begin{aligned}\int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| &= \lim_n \int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| \qquad (4.1.1)\\ &= \lim_n \int (\varphi_n)_\alpha\, d|x_\alpha^*\nu_\alpha| = \int |f_\alpha|\, d|x_\alpha^*\nu_\alpha|.\end{aligned}$$

Then, $f_\alpha \in L_w^1(\nu_\alpha)$ implies $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L_w^1(\nu)$.

Let now $y^* \in E_\alpha^*$ and define $\tilde{y}^*\colon E \to \mathbb{R}$ as $\tilde{y}^*(e) = y^*(e_\alpha)$ for $e = \sum_{\alpha\in\Delta} e_\alpha$. Then, $\tilde{y}^* \in E^*$ and the restriction of $\tilde{y}^*$ to E_α coincides with y^*. So, by (4.1.1),

$$\int |f_\alpha|\, d|y^*\nu_\alpha| = \int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|\tilde{y}^*\nu|.$$

Hence, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L_w^1(\nu)$ implies $f_\alpha \in L_w^1(\nu_\alpha)$. Therefore, (a) holds.

In the case when $\int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| < \infty$, that is, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(x^*\nu)$, there exists a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ such that $\varphi_n \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(x^*\nu)$ and so $\varphi_n\chi_{\{\alpha\}\times\Omega_\alpha} \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(x^*\nu)$. Also, by (4.1.1), we have that $(\varphi_n)_\alpha \to f_\alpha$ in $L^1(x_\alpha^*\nu_\alpha)$. Hence,

$$\begin{aligned}\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu &= \lim_n \int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu \qquad (4.1.2)\\ &= \lim_n \int (\varphi_n)_\alpha\, dx_\alpha^*\nu_\alpha = \int f_\alpha\, dx_\alpha^*\nu_\alpha.\end{aligned}$$

Suppose that $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(\nu)$. In particular, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L_w^1(\nu)$ and so, by (a), $f_\alpha \in L_w^1(\nu_\alpha)$. On the other hand, taking a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ such that $\varphi_n \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(\nu)$ and so $\varphi_n\chi_{\{\alpha\}\times\Omega_\alpha} \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(\nu)$, we have that $\int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu$ converges to $\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu$ in E. Since $\int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu = \int(\varphi_n)_\alpha\, d\nu_\alpha \in E_\alpha$ and E_α is closed in E, we have that $\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu \in E_\alpha$. Given $y^* \in E^*_\alpha$ and $\tilde{y}^* \in E^*$ defined as above, it follows

$$y^*\Big(\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu\Big) = \tilde{y}^*\Big(\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu\Big) = \int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\tilde{y}^*\nu = \int f_\alpha\, dy^*\nu_\alpha,$$

where we have used (4.1.2) in the last equality. Hence, $f_\alpha \in L^1(\nu_\alpha)$ and $\int f_\alpha\, d\nu_\alpha = \int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu$.

Suppose now that $f_\alpha \in L^1(\nu_\alpha)$. In particular, $f_\alpha \in L^1_w(\nu_\alpha)$ and so, by a), $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$. Since $\int f_\alpha\, d\nu_\alpha \in E_\alpha \subset E$, for every $x^* \in E^*$ we have that

$$x^*\Big(\int f_\alpha\, d\nu_\alpha\Big) = x^*_\alpha\Big(\int f_\alpha\, d\nu_\alpha\Big) = \int f_\alpha\, dx^*_\alpha\nu_\alpha = \int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu,$$

where $x^*_\alpha \in E^*_\alpha$ is the restriction of x^* to E_α. Then, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(\nu)$. Therefore, (b) holds. $\square$

Let us give a description of the space $L^1(\nu)$ which will be needed to prove that E is order isometric to $L^1(\nu)$.

Proposition 4.1.3. *The space $L^1(\nu)$ can be described as the space of all functions $f \in \mathcal{M}(\mathcal{R}^{loc})$ such that $f_\alpha \in L^1(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta}\int |f_\alpha|\, d\nu_\alpha$ is unconditionally convergent in E, where f_α is defined as in Remark 4.1.1. Moreover, if $f \in L^1(\nu)$ we have that*

$$\int f\, d\nu = \sum_{\alpha\in\Delta}\int f_\alpha\, d\nu_\alpha.$$

Proof. Let $f \in L^1(\nu)$. Then, for every $\alpha \in \Delta$, we have that $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(\nu)$ and so, by Lemma 4.1.2.(b), $f_\alpha \in L^1(\nu_\alpha)$. Let $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ be a sequence such that $\varphi_n \to f$ in $L^1(\nu)$ and ν-a.e. (except on a ν-null set Z). Since each φ_n is supported in $\mathcal{R}$, we can write supp $\varphi_n = \cup_{\alpha\in\Delta}\{\alpha\}\times A^n_\alpha$ where A^n_α is ν_α-null for all $\alpha \in \Delta\backslash I_n$ with $I_n \subset \Delta$ finite. Then,

$$(\Omega\setminus Z)\cap \operatorname{supp} f \subset \bigcup_{n\geq 1} \operatorname{supp}\, \varphi_n = \bigcup_{n\geq 1}\bigcup_{\alpha\in\Delta}\{\alpha\}\times A^n_\alpha = \bigcup_{\alpha\in\Delta}\{\alpha\}\times\Big(\bigcup_{n\geq 1} A^n_\alpha\Big).$$

Note that $\cup_{n\geq 1}A^n_\alpha$ is ν_α-null for every $\alpha \notin I = \cup_n I_n$. Consequently, we have that $\cup_{\alpha\in\Delta\backslash I}\{\alpha\}\times\big(\cup_{n\geq 1} A^n_\alpha\big)$ is ν-null and thus

$$f = f\chi_{\cup_{\alpha\in I}\{\alpha\}\times(\cup_{n\geq 1}A^n_\alpha)} \quad \nu\text{-a.e.}$$

For every $\alpha \in \Delta\backslash I$, from Lemma 4.1.2.(b) and since $f\chi_{\{\alpha\}\times\Omega_\alpha} = 0$ ν-a.e., we have that

$$\int |f_\alpha|\, d\nu_\alpha = \int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu = 0.$$

Write $I = \{\alpha_j\}_{j\geq 1}$ and $g_n = \sum_{j=1}^{n} |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}$. Note that $0 \leq g_n \uparrow |f| \in L^1(\nu)$. Then, since $L^1(\nu)$ is order continuous, $g_n \to |f|$ in $L^1(\nu)$ and so

$$\sum_{j=1}^{n} \int |f_{\alpha_j}|\, d\nu_{\alpha_j} = \sum_{j=1}^{n} \int |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}\, d\nu = \int g_n\, d\nu \to \int |f|\, d\nu \ \text{ in } E.$$

Therefore, $\sum_{\alpha\in\Delta} \int |f_\alpha|\, d\nu_\alpha$ is unconditionally convergent in E.

Conversely, let $f \in \mathcal{M}(\mathcal{R}^{loc})$ be a function such that $f_\alpha \in L^1(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta} \int |f_\alpha|\, d\nu_\alpha$ is unconditionally convergent in E. From this and since ν_α is positive, we have that there exists a countable set $N \subset \Delta$ such that

$$\|f_\alpha\|_{\nu_\alpha} = \Big\| \int |f_\alpha|\, d\nu_\alpha \Big\|_E = 0 \ \text{ for all } \alpha \in \Delta\backslash N.$$

That is, $f_\alpha = 0$ ν_α-a.e. for all $\alpha \in \Delta\backslash N$. So, for each $\alpha \in \Delta\backslash N$, there exists a ν_α-null set Z_α such that

$$f_\alpha(\omega) = 0 \ \text{ for all } \omega \in \Omega_\alpha\backslash Z_\alpha.$$

Note that the set $\cup_{\alpha\in\Delta\backslash N}\{\alpha\} \times Z_\alpha \in \mathcal{R}^{loc}$ is ν-null, then

$$f = \sum_{\alpha\in N} f\chi_{\{\alpha\}\times\Omega_\alpha} \ \ \nu\text{-a.e.}$$

Write $N = \{\alpha_j\}_{j\geq 1}$ and take $f_n = \sum_{j=1}^{n} f\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}$ which belongs to $L^1(\nu)$ from Lemma 4.1.2.(b). Then, for $m < n$,

$$\begin{aligned}
\|f_n - f_m\|_\nu &= \Big\| \int |f_n - f_m|\, d\nu \Big\|_E \\
&= \Big\| \sum_{j=m+1}^{n} \int |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}\, d\nu \Big\|_E \\
&= \Big\| \sum_{j=m+1}^{n} \int |f_{\alpha_j}|\, d\nu_{\alpha_j} \Big\|_E \to 0
\end{aligned}$$

as $m, n \to \infty$. Since $f_n \to f$ ν-a.e., it follows that $f \in L^1(\nu)$. Moreover, $f_n \to f$ in $L^1(\nu)$, so

$$\int f\, d\nu = \lim_n \int f_n\, d\nu = \sum_{\alpha\in\Delta} \int f_\alpha\, d\nu_\alpha.$$

$\square$

We finish this section by showing a description of $L^1_w(\nu)$ which will be used in Section 4.1.3 for the representation of Fatou Banach lattices.

Proposition 4.1.4. *The space $L^1_w(\nu)$ can be described as the space of all functions $f \in \mathcal{M}(\mathcal{R}^{loc})$ such that $f_\alpha \in L^1_w(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta}\int |f_\alpha|\, d|x^*\nu_\alpha|$ converges for all $x^* \in E^*$, where f_α is defined as in Remark 4.1.1. Moreover, if $f \in L^1_w(\nu)$ and $x^* \in E^*$, then*

$$\int f\, dx^*\nu = \sum_{\alpha\in\Delta}\int f_\alpha\, dx^*\nu_\alpha \quad \textit{and} \quad \int f\, d|x^*\nu| = \sum_{\alpha\in\Delta}\int f_\alpha\, d|x^*\nu_\alpha|.$$

Proof. Let $f \in L^1_w(\nu)$. Then, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$ and so, by Lemma 4.1.2.(a), $f_\alpha \in L^1_w(\nu_\alpha)$ for every $\alpha \in \Delta$. Take $x^* \in E^*$. For every $I \subset \Delta$ finite, by (4.1.1), we have that

$$\begin{aligned}\sum_{\alpha\in I}\int |f_\alpha|\, d|x^*\nu_\alpha| &= \sum_{\alpha\in I}\int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| \\ &= \int |f|\chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| \le \|f\|_\nu.\end{aligned}$$

So, $\sum_{\alpha\in\Delta}\int |f_\alpha|\, d|x^*\nu_\alpha|$ is convergent.

Conversely, let $f \in \mathcal{M}(\mathcal{R}^{loc})$ be such that $f_\alpha \in L^1_w(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta}\int |f_\alpha|\, d|x^*\nu_\alpha|$ converges for all $x^* \in E^*$. Fix $x^* \in E^*$. There exists a countable set $N \subset \Delta$ such that

$$\int |f_\alpha|\, d|x^*\nu_\alpha| = 0 \ \text{ for all } \alpha \in \Delta\backslash N.$$

Then, for every $\alpha \in \Delta\backslash N$, there exists a $|x^*\nu_\alpha|$-null set Z_α such that

$$f_\alpha(\omega) = 0 \ \text{ for all } \omega \in \Omega_\alpha\backslash Z_\alpha.$$

Noting that $\cup_{\alpha\in\Delta\backslash N}\{\alpha\}\times Z_\alpha$ is $|x^*\nu|$-null, it follows

$$f = \sum_{\alpha\in N} f\chi_{\{\alpha\}\times\Omega_\alpha} \quad |x^*\nu|\text{-a.e.}$$

Write $N = \{\alpha_j\}_{j\ge1}$ and take $f_n = \sum_{j=1}^n f\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}$ which, by Lemma 4.1.2.(a), is in $L^1_w(\nu)$. Then, for $m < n$, by (4.1.1),

$$\int |f_n - f_m|\, d|x^*\nu| = \sum_{j=m+1}^{n}\int |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}\, d|x^*\nu| = \sum_{j=m+1}^{n}\int |f_{\alpha_j}|\, d|x^*\nu_{\alpha_j}| \to 0$$

as $m, n \to \infty$. Note that $f_n \to f$ $|x^*\nu|$-a.e. So, $f \in L^1(|x^*\nu|)$ and $f_n \to f$ in $L^1(|x^*\nu|)$. Therefore, $f \in L^1_w(\nu)$ and, by (4.1.1) and (4.1.2),

$$\int f\, dx^*\nu = \sum_{\alpha\in\Delta} \int f_\alpha\, dx^*\nu_\alpha$$

and

$$\int f\, d|x^*\nu| = \sum_{\alpha\in\Delta} \int f_\alpha\, d|x^*\nu_\alpha| \quad \text{for all } x^* \in E^*. \quad \square$$

$\square$

4.1.2 Description of an order continuous Banach lattice as an $L^1(\nu)$

Let E be an order continuous Banach lattice and ν the associated vector measure constructed in Section 4.1.1. Let us show that $L^1(\nu)$ and E are order isometric.

Theorem 4.1.5. *The space $L^1(\nu)$ is order isometric to E. Even more, the integration operator $I_\nu\colon L^1(\nu) \to E$ is an order isometry.*

Proof. The integration operator $I_\nu\colon L^1(\nu) \to E$ is a positive (as ν is positive) continuous linear operator satisfying that $\|I_\nu(f)\|_E \le \|f\|_\nu$ for every $f \in L^1(\nu)$. Let us see that I_ν is an isometry. Fix $f \in L^1(\nu)$. From Proposition 4.1.3, it follows

$$\begin{aligned} \|f\|_\nu &= \Big\|\int |f|\, d\nu\Big\|_E = \sup_{x^*\in B_{E^*}} \Big|x^*\Big(\int |f|\, d\nu\Big)\Big| \qquad (4.1.3)\\ &= \sup_{x^*\in B_{E^*}} \Big|x^*\Big(\sum_{\alpha\in\Delta}\int |f_\alpha|\, d\nu_\alpha\Big)\Big| \\ &= \sup_{x^*\in B_{E^*}} \Big|\sum_{\alpha\in\Delta} x^*\Big(\int |f_\alpha|\, d\nu_\alpha\Big)\Big|. \end{aligned}$$

Let $x^* \in E^*$. Note that $x^* \circ I_{\nu_\alpha} \in L^1(\nu_\alpha)^*$ for all $\alpha \in \Delta$ (recall that $I_{\nu_\alpha}\colon L^1(\nu_\alpha) \to E_\alpha$ is an order isometry). Take $\xi_\alpha = \chi_{\{f_\alpha \ge 0\}} - \chi_{\{f_\alpha < 0\}}$ and note that $|f_\alpha| = \xi_\alpha \cdot f_\alpha$. Define $\tilde{x}^*\colon E \to \mathbb{R}$ by

$$\tilde{x}^*(e) = \sum_{\alpha\in\Delta} x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I^{-1}_{\nu_\alpha}(e_\alpha)\big)$$

for all $e \in E$ with $e = \sum_{\alpha\in\Delta} e_\alpha$ such that $e_\alpha \in E_\alpha$ and the sum is unconditionally convergent. Let us see that $\tilde{x}^*$ is well defined and belongs to E^*. Take an element $e = \sum_{\alpha\in\Delta} e_\alpha \in E$ as above. Then, $|e| = \sum_{\alpha\in\Delta} |e_\alpha|$ where the sum is also unconditionally convergent. Let $N \subset \Delta$ be a countable set such that $e_\alpha = 0$ for all $\alpha \in \Delta\backslash N$. Then, $\xi_\alpha I^{-1}_{\nu_\alpha}(e_\alpha) = 0$ and so $x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I^{-1}_{\nu_\alpha}(e_\alpha)\big) = 0$ for all $\alpha \in \Delta\backslash N$. Writing $N = \{\alpha_j\}_{j\geq 1}$ we have that

$$\begin{aligned}\Big|\sum_{j=n}^{m} x^* \circ I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})\big)\Big| &= \Big|x^*\Big(\sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})\big)\Big)\Big| \\ &\leq \|x^*\| \cdot \Big\|\sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})\big)\Big\|_E.\end{aligned}$$

Note that, since I_{ν_α} is an order isometry, $|I_{\nu_\alpha}(h)| = I_{\nu_\alpha}(|h|)$ for all $h \in L^1(\nu_\alpha)$ and $I_{\nu_\alpha}(\tilde{h}) \leq I_{\nu_\alpha}(h)$ whenever $\tilde{h} \leq h \in L^1(\nu_\alpha)$ (the same holds for $I^{-1}_{\nu_\alpha}$). Then,

$$\begin{aligned}\Big|\sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})\big)\Big| &\leq \sum_{j=n}^{m} \big|I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})\big)\big| \\ &= \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(|\xi_{\alpha_j} I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})|\big) \\ &\leq \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(|I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})|\big) \\ &= \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(I^{-1}_{\nu_{\alpha_j}}(|e_{\alpha_j}|)\big) = \sum_{j=n}^{m} |e_{\alpha_j}|.\end{aligned}$$

Therefore,

$$\Big|\sum_{j=n}^{m} x^* \circ I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I^{-1}_{\nu_{\alpha_j}}(e_{\alpha_j})\big)\Big| \leq \|x^*\| \cdot \Big\|\sum_{j=n}^{m} |e_{\alpha_j}|\Big\|_E \to 0$$

as $n, m \to \infty$. So, $\tilde{x}^*$ is well defined, obviously linear and continuous as $|\tilde{x}^*(e)| \leq \|x^*\| \cdot \|e\|_E$ for all $e \in E$, that is, $\tilde{x}^* \in E^*$ and $\|\tilde{x}^*\| \leq \|x^*\|$.

Moreover,

$$x^*\Big(\int |f_\alpha|\, d\nu_\alpha\Big) = x^* \circ I_{\nu_\alpha}(|f_\alpha|) = x^* \circ I_{\nu_\alpha}(\xi_\alpha f_\alpha) = x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I^{-1}_{\nu_\alpha}\big(I_{\nu_\alpha}(f_\alpha)\big)\big)$$

for all $\alpha \in \Delta$. From Proposition 4.1.3, we have that $I_\nu(f) = \sum_{\alpha\in\Delta} I_{\nu_\alpha}(f_\alpha)$ and so,

$$\tilde{x}^*\big(I_\nu(f)\big) = \sum_{\alpha\in\Delta} x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I_{\nu_\alpha}^{-1}\big(I_{\nu_\alpha}(f_\alpha)\big)\big) = \sum_{\alpha\in\Delta} x^*\Big(\int |f_\alpha|\, d\nu_\alpha\Big).$$

Hence, we have proved that for every $x^* \in B_{E^*}$ there exists $\tilde{x}^* \in B_{E^*}$ such that $\sum_{\alpha\in\Delta} x^*\Big(\int |f_\alpha|\, d\nu_\alpha\Big) = \tilde{x}^*\big(I_\nu(f)\big)$. Then, from (4.1.3), $\|f\|_\nu \le \|I_\nu(f)\|_E$. Therefore, I_ν is a linear isometry.

Let us see now that I_ν is onto. Let $e = \sum_{\alpha\in\Delta} e_\alpha \in E$. Since each $e_\alpha \in E_\alpha$, there exists $h_\alpha \in L^1(\nu_\alpha)$ such that $e_\alpha = I_{\nu_\alpha}(h_\alpha)$. Define $f\colon \Omega \to \mathbb{R}$ by $f(\alpha,\omega) = h_\alpha(\omega)$ for all $(\alpha,\omega) \in \Omega$. Then, $f \in \mathcal{M}(\mathcal{R}^{loc})$ (as $f^{-1}(B) = \cup_{\alpha\in\Delta}\{\alpha\} \times h_\alpha^{-1}(B)$ for every Borel set B on $\mathbb{R}$), $f_\alpha = h_\alpha \in L^1(\nu_\alpha)$ for all $\alpha \in \Delta$ and

$$\sum_{\alpha\in\Delta} I_{\nu_\alpha}(f_\alpha) = \sum_{\alpha\in\Delta} I_{\nu_\alpha}(h_\alpha) = \sum_{\alpha\in\Delta} e_\alpha$$

is unconditionally convergent in E. So, by Proposition 4.1.3, we have that $f \in L^1(\nu)$ and $I_\nu(f) = \sum_{\alpha\in\Delta} I_{\nu_\alpha}(f_\alpha) = e$. Note that if $e \ge 0$, that is, $e_\alpha \ge 0$ for all $\alpha \in \Delta$, then $h_\alpha \ge 0$ for all $\alpha \in \Delta$ and so $f \ge 0$. Hence, I_ν^{-1} is positive.

So, I_ν is positive, linear, one to one and onto with I_ν^{-1} positive. Then, by [26, p. 2], I_ν is an order isomorphism. □

As an example of an order continuous Banach lattice without weak unit which can be represented as an $L^1(\nu)$, we have $\ell^1(\Gamma)$ where Γ is an uncountable abstract set Γ. By Theorem 4.1.5, $\ell^1(\Gamma)$ is order isometric to $L^1(\nu)$ for some vector measure ν defined on a δ-ring, via the integration operator. The vector measure ν can be taking as the one in Example 2.2.4. That is, $\nu\colon \mathcal{R} \to \ell^1(\Gamma)$ is given by $\nu(A) = \chi_A$ for all $A \in \mathcal{R}$, for the δ-ring $\mathcal{R} = \{A \subset \Gamma :\ A \text{ is finite}\}$. In this case, the integration operator is the identity map. Note that $\ell^1(\Gamma)$ cannot be represented as $L^1(\nu)$ with ν defined on a σ-algebra, as it has no weak unit.

4.1.3 Description of a Fatou Banach lattice as an $L^1_w(\nu)$

Until now, we have considered an order continuous Banach lattice E. If we forget about the order continuity property, descriptions of E by means of a vector measure could exist. For instance, if E is a Banach lattice satisfying the σ-Fatou property with a weak unit belonging to the σ-order continuous part E_a of E, then there exists a vector measure ν defined on a σ-algebra such that E is order isometric to $L^1_w(\nu)$, see [10, Theorem 2.5] (see also [31, Proposition 3.41] for the complex version). In this case, E_a is actually order continuous and then $E_a = E_{an}$ (following the same argument as the one in the beginning of Section 2.1). The proof of the representation of E as an $L^1_w(\nu)$ consists in taking a vector measure ν such that $L^1(\nu)$ is order isometric to E_a via the integration operator I_ν, and extending I_ν to $L^1_w(\nu)$. The result is that this extension is an order isometry from $L^1_w(\nu)$ onto E. Our question now is if a similar result is possible if we forget about the weak unit and consider vector measures defined on a δ-ring, as it happens in the case when E is order continuous.

In order to prove the desired result, we will need the next Lemma. Let E be a general Banach lattice. Recall that the order continuous part E_{an} of E can be decomposed into an unconditionally direct sum of a family of mutually disjoints ideals $\{E^\alpha_{an}\}_{\alpha\in\Delta}$, each E^α_{an} having a weak unit u_α. That is, every $e \in E_{an}$ has a unique representation $e = \sum_{\alpha\in\Delta} e_\alpha$ with $e_\alpha \in E^\alpha_{an}$, only countably many $e_\alpha \neq 0$ and the series converging unconditionally (see [26, Proposition 1.a.9]).

Lemma 4.1.6. *Suppose that E_{an} is order dense in E. Then, for every $0 \le e \in E$ it follows*

$$e_{(n,I)} = \sum_{\alpha\in I} e \wedge (nu_\alpha) \uparrow e \tag{4.1.4}$$

where the indices (n, I) are such that $n \in \mathbb{N}$ and $I \subset \Delta$ is finite. Moreover, in the case when $0 \le e \in E_{an}$, there exists a countable set $\{\alpha_j\} \subset \Delta$ such that $e \wedge (nu_\alpha) = 0$ for all n and $\alpha \in \Delta\backslash\{\alpha_j\}$, and

$$e = \lim_{n,m} \sum_{j=1}^{m} e \wedge (nu_{\alpha_j}) \text{ in norm.} \tag{4.1.5}$$

Proof. Let $0 \le e \in E$ and $e_{(n,I)}$ as in (4.1.4). Then $0 \le e_{(n,I)} \uparrow$ and $e_{(n,I)} \le e$ for all (n, I). Note that $\{nu_\alpha : \alpha \in \Delta\}$ is a set of pairwise disjoint elements, so

$$e_{(n,I)} = \sum_{\alpha\in I} e \wedge (nu_\alpha) = e \wedge \Big(\sum_{\alpha\in I} nu_\alpha\Big) \tag{4.1.6}$$

(see [27, Theorem 12.5]). Let $z \in E$ be such that $e_{(n,I)} \le z$ for all (n, I). Let us see that $e \le z$. Suppose first that $e \in E_{an}$ and write $e = \sum_{j\ge1} e_{\alpha_j}$ where

$e_{\alpha_j} \in E_{an}^{\alpha_j}$ and the series converges unconditionally. Note that, since $e \geq 0$ and $\{e_{\alpha_j}\}$ is a set of pairwise disjoint elements, $e_{\alpha_j} \geq 0$ for every j. Then $\sum_{j=1}^{m} e_{\alpha_j} \uparrow e$ in the lattice order (see [34, Theorem 100.4.(i)]). For a fix j we have that $e_{\alpha_j} \wedge (nu_{\alpha_j}) \uparrow e_{\alpha_j}$ (see [26, pp. 7-8]). Then, for each m it follows that $\sum_{j=1}^{m} e_{\alpha_j} \wedge (nu_{\alpha_j}) \uparrow \sum_{j=1}^{m} e_{\alpha_j}$ (see [27, Theorem 15.2]). Since $e_{\alpha_j} \leq e$ for all j, taking $I_m = \{\alpha_1, ..., \alpha_m\}$ we have that $\sum_{j=1}^{m} e_{\alpha_j} \wedge (nu_{\alpha_j}) \leq e_{(n,I_m)} \leq z$ for all n and so $\sum_{j=1}^{m} e_{\alpha_j} \leq z$. Hence $e \leq z$. Note that actually we have proved that $\sum_{j=1}^{m} e \wedge (nu_{\alpha_j}) \uparrow e$ where the indices are (n, m). Then, by the order continuity of E_{an}, it follows that $e = \lim_{n,m} \sum_{j=1}^{m} e \wedge (nu_{\alpha_j})$ in norm. Hence, (4.1.4) and (4.1.5) hold if $e \in E_{an}$.

In the general case, since E_{an} is order dense in E, there exists $(e_\tau) \subset E_{an}$ such that $0 \leq e_\tau \uparrow e$. We now know that $\sum_{\alpha \in I} e_\tau \wedge (nu_\alpha) \uparrow e_\tau$ for every τ. Then, since $\sum_{\alpha \in I} e_\tau \wedge (nu_\alpha) \leq e_{(n,I)} \leq z$, we have that $e_\tau \leq z$ for every τ, and so $e \leq z$. $\square$

Consider the vector measure ν associated to E_{an} as in Section 4.1.1, then $I_\nu \colon L^1(\nu) \to E_{an}$ is an order isometry (Theorem 4.1.5). The question is if it is possible to extend I_ν to the space $L^1_w(\nu)$ in a way that the extension is an order isometry between $L^1_w(\nu)$ and E. Note that if this extension is possible, E must have the Fatou property since $L^1_w(\nu)$ has (recall that ν is $\mathcal{R}$-decomposable). Even more, since $L^1(\nu)$ is always order dense in $L^1_w(\nu)$ (Theorem 2.2.2), E_{an} must be order dense in E. So, we will require E to have these properties.

Now we can prove our main result by using Lemma 4.1.6. Remark that if E has the Fatou property, E has in particular the σ-Fatou property and then $E_{an} = E_a$.

Theorem 4.1.7. *If E has the Fatou property and E_a is order dense in E, then E is order isometric to $L^1_w(\nu)$.*

Proof. Let us extend I_ν to $L^1_w(\nu)$. First, consider $0 \leq f \in L^1_w(\nu)$ and choose $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \varphi_n \uparrow f$. For each n and $I \subset \Delta$ finite, we define $\xi_{(n,I)} = \varphi_n \chi_{\cup_{\alpha \in I} \{\alpha\} \times \Omega_\alpha} \in \mathcal{S}(\mathcal{R})$. Then, $(\xi_{(n,I)}) \subset L^1(\nu)$ is an upwards directed system $0 \leq \xi_{(n,I)} \uparrow f$ in $L^1_w(\nu)$. Indeed, let $g \in L^1_w(\nu)$ be such that $\xi_{(n,I)} \leq g$ for all (n, I). For each n and $\beta \in \Delta$, there exists $Z_{n,\beta} \in \mathcal{R}^{loc}$ ν-null such that

$\xi_{(n,\{\beta\})}(\alpha,\omega) \le g(\alpha,\omega)$ for all $(\alpha,\omega) \in \Omega\backslash Z_{n,\beta}$. Note that $\cup_n Z_{n,\beta} \cap \{\beta\} \times \Omega_\beta$ is ν-null and then $Z = \cup_{\beta\in\Delta} \cup_n Z_{n,\beta} \cap \{\beta\} \times \Omega_\beta$ is ν-null as ν is $\mathcal{R}$-decomposable. Moreover, for every $(\alpha,\omega) \in \Omega\backslash Z$, we have that $\varphi_n(\alpha,\omega) = \xi_{(n,\{\alpha\})}(\alpha,\omega) \le g(\alpha,\omega)$ for all n, and so $f \le g$. A similar argument gives that $0 \le \xi_{(n,I)} \uparrow f$ in $L^1_w(x^*\nu)$. Since I_ν is positive, $(I_\nu(\xi_{(n,I)})) \subset E_a \subset E$ is an upwards directed system $0 \le I_\nu(\xi_{(n,I)}) \uparrow$ and $\sup_{(n,I)} \|I_\nu(\xi_{(n,I)})\|_E = \sup_{(n,I)} \|\xi_{(n,I)}\|_\nu \le \|f\|_\nu$. Then, by the Fatou property of E, there exists $e = \sup_{(n,I)} I_\nu(\xi_{(n,I)})$ in E and $\|e\|_E = \sup_{(n,I)} \|I_\nu(\xi_{(n,I)}\|_E$. We define $T(f) = e$.

Using a similar argument to the one in [10, Theorem 2.5], we will see that T is well defined. Take another sequence $(\psi_n)_{n\ge1} \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \le \psi_n \uparrow f$. Denote $\eta_{(n,I)} = \psi_n \chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}$ and $z = \sup_{(n,I)} I_\nu(\eta_{(n,I)})$. Let $0 \le x^* \in E^*$ be fixed. Then, $x^*(e) \ge x^*\big(I_\nu(\xi_{(n,I)})\big) = \int \xi_{(n,I)}\, dx^*\nu$ for all $n \ge 1$ and $I \subset \Delta$ finite. Since $0 \le \xi_{(n,I)} \uparrow f$ in $L^1(x^*\nu)$ which has the Fatou property, we have that $\sup_{(n,I)} \int \xi_{(n,I)}\, dx^*\nu = \int f\, dx^*\nu$. Note that $\int |h|\, d|x^*\nu| = \int |h|\, dx^*\nu$ for all $h \in L^1(x^*\nu)$ as $x^*\nu$ is positive. Consequently, $x^*(e) \ge \int f\, dx^*\nu \ge x^*\big(I_\nu(\xi_{(n,I)})\big)$ for all $n \ge 1$ and $I \subset \Delta$ finite. In a similar way, $x^*(z) \ge \int f\, dx^*\nu \ge x^*\big(I_\nu(\eta_{(n,I)})\big)$ for all $n \ge 1$ and $I \subset \Delta$ finite. In particular, $x^*(e) \ge x^*\big(I_\nu(\eta_{(n,I)})\big)$ and $x^*(z) \ge x^*\big(I_\nu(\xi_{(n,I)})\big)$ for all $n \ge 1$ and $I \subset \Delta$ finite. Since this holds for all $0 \le x^* \in E^*$, we have that $e \ge I_\nu(\eta_{(n,I)})$ and $z \ge I_\nu(\xi_{(n,I)})$ for all $n \ge 1$ and $I \subset \Delta$ finite. Then, $e \ge z$ and $z \ge e$, and thus $e = z$. So, T is well defined. Moreover,

$$\|T(f)\|_E = \|e\|_E = \sup_{(n,I)} \|I_\nu(\xi_{(n,I)})\|_E = \sup_{(n,I)} \|\xi_{(n,I)}\|_\nu = \|f\|_\nu,$$

where in the last equality we have used that $L^1_w(\nu)$ has the Fatou property.

Let us see now that $T(f\wedge g) = Tf \wedge Tg$ for every $0 \le f, g \in L^1_w(\nu)$. Consider sequences $(\varphi_n), (\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ satisfying that $0 \le \varphi_n \uparrow f$ and $0 \le \psi_n \uparrow g$, and denote $\xi_{(n,I)} = \varphi_n \chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}$ and $\eta_{(n,I)} = \psi_n \chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}$. Then, $Tf = \sup_{(n,I)} I_\nu(\xi_{(n,I)})$ and $Tg = \sup_{(n,I)} I_\nu(\eta_{(n,I)})$. Note that $(\varphi_n \wedge \psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ satisfies that $0 \le \varphi_n \wedge \psi_n \uparrow f \wedge g$ (see [27, Theorem 15.3]) and also $(\varphi_n \wedge \psi_n)\chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha} = \xi_{(n,I)} \wedge \eta_{(n,I)}$. Then, since I_ν is an order isometry, we have that

$$T(f\wedge g) = \sup_{(n,I)} I_\nu(\xi_{(n,I)} \wedge \eta_{(n,I)}) = \sup_{(n,I)} I_\nu(\xi_{(n,I)}) \wedge I_\nu(\eta_{(n,I)}) = Tf \wedge Tg.$$

For a general $f \in L^1_w(\nu)$, we define $Tf = Tf^+ - Tf^-$ where f^+ and f^- are the positive and negative parts of f respectively. So, $T\colon L^1_w(\nu) \to E$ is a positive linear operator extending I_ν. For the linearity, see for instance [27, Theorem 15.8]. Moreover T is an isometry. Indeed, for $f \in L^1_w(\nu)$, since $f^+ \wedge f^- = 0$, we have that $Tf^+ \wedge Tf^- = T(f^+ \wedge f^-) = 0$. Then, it follows that $|Tf| = |Tf^+ - Tf^-| = Tf^+ + Tf^- = T|f|$, and so, $\|T(f)\|_E = \|T(|f|)\|_E = \|f\|_\nu$.

Let us prove that T is onto. Let $0 \le e \in E$. Since E_a is order dense in E, from Lemma 4.1.6 we have that $e_{(n,I)} = \sum_{\alpha\in I} e \wedge (nu_\alpha) \uparrow e$. Fix n and $\beta \in \Delta$. Since $e \wedge (nu_\beta) \in E_a^\beta$ as $0 \le e \wedge (nu_\beta) \le nu_\beta$, there exists $0 \le g_{n,\beta} \in L^1(\nu_\beta)$ such that $e \wedge (nu_\beta) = I_{\nu_\beta}(g_{n,\beta})$. Define $f_{n,\beta}\colon \Omega \to \mathbb{R}$ by $f_{n,\beta}(\alpha,\omega) = g_{n,\beta}(\omega)$ if $\alpha = \beta$ and $f_{n,\beta}(\alpha,\omega) = 0$ in other case. Then, from Proposition 4.1.3, we have that $f_{n,\beta} \in L^1(\nu)$ and $I_\nu(f_{n,\beta}) = I_{\nu_\beta}(g_{n,\beta}) = e \wedge (nu_\beta)$. Taking $\xi_{(n,I)} = \sum_{\alpha\in I} f_{n,\alpha} \in L^1(\nu)$, we have that $0 \le \xi_{(n,I)} \uparrow$ as $\xi_{(n,I)} = I_\nu^{-1}(e_{(n,I)})$ and $\sup_{(n,I)} \|\xi_{(n,I)}\|_\nu = \sup_{(n,I)} \|e_{(n,I)}\|_E \le \|e\|_E$. By the Fatou property of $L^1_w(\nu)$, there exists $f = \sup_{(n,I)} \xi_{(n,I)}$ in $L^1_w(\nu)$.

If we prove that $x^*(e) \ge \int f\, dx^*\nu$ for all $0 \le x^* \in X^*$, by the same argument used to see that T is well defined, we will have that $Tf = e$. Fix $\alpha \in \Delta$, since $0 \le \xi_{(n,I)} \uparrow f$ in $L^1_w(\nu)$, it follows that $0 \le \xi_{(n,I)}\chi_{\{\alpha\}\times\Omega_\alpha} \uparrow f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1_w(\nu)$. Indeed, if $g \in L^1_w(\nu)$ is such that $\xi_{(n,I)}\chi_{\{\alpha\}\times\Omega_\alpha} \le g$ for all (n,I), then $\xi_{(n,I)} = \xi_{(n,I)}\chi_{\{\alpha\}\times\Omega_\alpha} + \xi_{(n,I)}\chi_{\Omega\setminus(\{\alpha\}\times\Omega_\alpha)} \le g + f\chi_{\Omega\setminus(\{\alpha\}\times\Omega_\alpha)}$ for all (n,I), and so $f \le g + f\chi_{\Omega\setminus(\{\alpha\}\times\Omega_\alpha)}$. Hence, $f\chi_{\{\alpha\}\times\Omega_\alpha} \le g\chi_{\{\alpha\}\times\Omega_\alpha} \le g$. Since $\xi_{(n,I)}\chi_{\{\alpha\}\times\Omega_\alpha} = \sum_{\beta\in I} f_{n,\beta}\chi_{\{\alpha\}\times\Omega_\alpha} = f_{n,\alpha}\chi_{\{\alpha\}\times\Omega_\alpha}$, actually we deals with a sequence. Writing $h_n^\alpha = f_{n,\alpha}\chi_{\{\alpha\}\times\Omega_\alpha}$, we have that $0 \le h_n^\alpha \uparrow f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1_w(\nu)$ and so ν-a.e. (note that $I_\nu(f_{n,\alpha}) = I_{\nu_\alpha}(g_{n,\alpha}) = e \wedge (nu_\alpha) \le e \wedge ((n+1)u_\alpha) = I_\nu(f_{(n+1),\alpha})$). Fix now $0 \le x^* \in X^*$. Since $h_n^\alpha \uparrow f\chi_{\{\alpha\}\times\Omega_\alpha}$ $x^*\nu$-a.e., applying the dominate convergence theorem (see [28, Theorem 2.22]), we have that $\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu = \lim \int h_n^\alpha\, dx^*\nu$. Noting that $\int h_n^\alpha\, dx^*\nu = x^*I_\nu(f_{n,\alpha}\chi_{\{\alpha\}\times\Omega_\alpha}) \le x^*I_\nu(f_{n,\alpha}) = x^*(e \wedge (nu_\alpha))$, we obtain that

$$\begin{aligned}\sum_{\alpha\in I}\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu &= \lim \sum_{\alpha\in I}\int h_n^\alpha\, dx^*\nu \le \lim \sum_{\alpha\in I} x^*\big(e \wedge (nu_\alpha)\big) \\ &= \lim x^*(e_{(n,I)}) \le x^*(e)\end{aligned}$$

for all finite $I \subset \Delta$. Therefore, by the description of $L^1_w(\nu)$ given in Proposition

4.1.4 and (4.1.2),

$$\int f \, dx^*\nu = \sum_{\alpha \in \Delta} \int f \chi_{\{\alpha\} \times \Omega_\alpha} \, dx^*\nu \leq x^*(e).$$

For a general $e \in E$, consider e^+ and e^- the positive and negative parts of e. Let $g, h \in L_w^1(\nu)$ be such that $Tg = e^+$ and $Th = e^-$. Then, taking $f = g - h \in L_w^1(\nu)$ we have that $Tf = e$. Note that T^{-1} is positive. So, T is positive, linear, one to one and onto with inverse being positive, then T is an order isomorphism (see [26, p. 2]). □

Note that the conditions required in this theorem are necessary and sufficient for the extension of $I_\nu \colon L^1(\nu) \to E_a$ to $L_w^1(\nu)$ to be possible in the desired way (see the comments just before Lemma 4.1.6).

Finally, remark that although the converse of Theorem 2.3.8 does not hold, if $L_w^1(\widetilde{\nu})$ has the Fatou property, since Theorems 2.1.2 and 2.2.2 assure that $L_w^1(\widetilde{\nu})$ satisfies the conditions in theorem above, then there exists an $\mathcal{R}$-decomposable vector measure ν such that $L_w^1(\widetilde{\nu})$ is order isometric to $L_w^1(\nu)$.

We end the section by showing two examples of the representation of Banach lattices as $L_w^1(\nu)$ spaces.

Example 4.1.8. Consider an uncountable set Γ and the δ-ring $\mathcal{R}$ of finite subsets of Γ. The space $\ell^\infty(\Gamma)$ has the Fatou property and its σ-order continuous part $c_0(\Gamma)$ is order dense. Then, from Theorem 4.1.7, $\ell^\infty(\Gamma)$ is order isometric to $L_w^1(\nu)$ for some vector measure ν defined on a δ-ring. The vector measure $\nu \colon \mathcal{R} \to c_0(\Gamma)$ can be defined as in Example 2.2.1 and in this case, the order isometry is the identity map, see [15, Example 2.2]. Note that $\ell^\infty(\Gamma)$ cannot be represented as $L_w^1(\nu)$ with ν defined on a σ-algebra, as its σ-order continuous part has no weak unit.

Remark that if we consider a non atomic measure space (Ω, Σ, μ), the space $L^\infty(\mu)$ can not be represented as an $L_w^1(\nu)$ of any vector measure ν defined on a δ-ring, as the order continuous part of $L^\infty(\mu)$ is the trivial space.

Example 4.1.9. Also, we can find Banach lattices without weak unit satisfying the requirements of Theorem 4.1.7. Let Γ and Δ be disjoint uncountable sets and consider the Banach lattice $\ell^1(\Gamma)\times\ell^\infty(\Delta)$ endowed with the norm $\|(x,y)\| = \|x\|_{\ell^1(\Gamma)} + \|y\|_{\ell^\infty(\Delta)}$ and the order $(x,y)\le(\tilde{x},\tilde{y})$ if and only if $x\le\tilde{x}$ and $y\le\tilde{y}$ for $x,\tilde{x}\in\ell^1(\Gamma)$ and $y,\tilde{y}\in\ell^\infty(\Delta)$. This space has the Fatou property and its σ-order continuous part $\ell^1(\Gamma)\times c_0(\Delta)$ is order dense. In this case, taking the δ-ring $\mathcal{R}=\{A\subset\Gamma\cup\Delta:\ A \text{ is finite}\}$, the vector measure $\nu\colon\mathcal{R}\to\ell^1(\Gamma)\times c_0(\Delta)$ can be defined as $\nu(A)=\big(\nu_1(A\cap\Gamma),\nu_2(A\cap\Delta)\big)$ for all $A\in\mathcal{R}$, where ν_1 and ν_2 are the vector measures defined in Example 2.2.4 and Example 2.2.1 respectively. Indeed, $\big(\ell^1(\Gamma)\times c_0(\Delta)\big)^*$ is identified with $\big(\ell^1(\Gamma)\big)^*\times\big(c_0(\Delta)\big)^*$ in the way $x^*=(x_1^*,x_2^*)$ such that $x^*(a,b)=x_1^*(a)+x_2^*(b)$ for all $(a,b)\in\ell^1(\Gamma)\times c_0(\Delta)$ and with $\|x^*\|=\max\{\|x_1^*\|,\|x_2^*\|\}$. So, $x^*\nu(A)=x_1^*\nu_1(A\cap\Gamma)+x_2^*\nu_2(A\cap\Delta)$ for all $A\in\mathcal{R}$ and thus

$$|x^*\nu|(B)=|x_1^*\nu_1|(B\cap\Gamma)+|x_2^*\nu_2|(B\cap\Delta)\quad\text{for all } B\in\mathcal{R}^{loc}.$$

Then, for every $f\in\mathcal{M}(\mathcal{R}^{loc})$ we have that

$$\int|f|\,d|x^*\nu|=\int|f|\chi_\Gamma\,d|x_1^*\nu_1|+\int|f|\chi_\Delta\,d|x_2^*\nu_2|.$$

Noting that $L^1_w(\nu_1)\times L^1_w(\nu_2)=\ell^1(\Gamma)\times\ell^\infty(\Delta)$ isometrically, it follows that the operator $T\colon L^1_w(\nu)\to\ell^1(\Gamma)\times\ell^\infty(\Delta)$, defined by $Tf=(f\chi_\Gamma,f\chi_\Delta)$ for all $f\in L^1_w(\nu)$, is an order isometry. Note that T restricted to $L^1(\nu)$ is the integration operator I_ν which is and order isometry between $L^1(\nu)$ and $\ell^1(\Gamma)\times c_0(\Delta)$.

4.2 Representing σ-Fatou Banach lattices

We have represented Banach lattices E having the Fatou property and such that E_a is super order dense in E. What about if we consider the same properties but for sequences? That is, if E is a Banach lattice with the σ-Fatou property and such that E_a is super order dense in E, is it possible to give a description of E to represent E as some ideal of an space $L^1_w(\nu)$? We will give a positive answer by means of the σ-Fatou completion of $L^1(\nu)$.

Note that since $L^1(\nu)\subset[L^1(\nu)]_{\sigma\text{-F}}\subset L^1_w(\nu)$, then $\big([L^1(\nu)]_{\sigma\text{-F}}\big)_a\subset\big(L^1_w(\nu)\big)_a$ and so, from Theorem 2.1.2, we have that $\big([L^1(\nu)]_{\sigma\text{-F}}\big)_a=L^1(\nu)$ which is super order dense in $[L^1(\nu)]_{\sigma\text{-F}}$ (see the last part of the proof of Theorem 2.3.1).

The following result is proved following the same arguments as the ones in Theorem 4.1.7, but without the difficulty which the nets imply and so the proof is clearer and has an easier lecture. For this reason and for aim of completeness, we include it.

Proposition 4.2.1. *Every Banach lattice E with the σ-Fatou property such that E_a is super order dense in E is order isometric to $[L^1(\nu)]_{\sigma\text{-F}}$ for some vector measure ν defined on a δ-ring.*

Proof. Let E be a Banach lattice with the σ-Fatou property such that E_a is super order dense in E and consider the vector measure ν defined on a δ-ring such that the integration operator $I_\nu\colon L^1(\nu) \to E_a$ given by $I_\nu(f) = \int f\,d\nu$ for all $f \in L^1(\nu)$, is an order isometry, see Theorem 4.1.5. Let us extend I_ν to $[L^1(\nu)]_{\sigma\text{-F}}$. First, consider $0 \leq f \in [L^1(\nu)]_{\sigma\text{-F}}$ and take $(f_n) \subset L^1(\nu)$ such that $0 \leq f_n \uparrow f$, this is always possible since $L^1(\nu)$ is super order dense in $[L^1(\nu)]_{\sigma\text{-F}}$ as we have noted above. Since I_ν is an order isometry, the sequence $\big(I_\nu(f_n)\big) \subset E_a \subset E$ satisfies that $0 \leq I_\nu(f_n)\uparrow$ and $\sup \|I_\nu(f_n)\|_E = \sup \|f_n\|_\nu \leq \|f\|_\nu < \infty$. Then, as E has the σ-Fatou property, there exists $e = \sup I_\nu(f_n)$ in E and $\|e\|_E = \sup \|I_\nu(f_n)\|_E$. We define $T(f) = e$.

Take another sequence $(g_n) \subset L^1(\nu)$ such that $0 \leq g_n \uparrow f$ and denote $z = \sup I_\nu(g_n)$. Let $0 \leq x^* \in E^*$ be fixed. Then, $x^*(e) \geq x^*\big(I_\nu(f_n)\big) = \int f_n\,dx^*\nu$ for all n. Since $0 \leq f_n \uparrow f$ ν-a.e. and so $x^*\nu$-a.e., by using the monotone convergence theorem, we have that $x^*(e) \geq \int f\,dx^*\nu \geq x^*\big(I_\nu(f_n)\big)$ for all n. In a similar way, $x^*(z) \geq \int f\,dx^*\nu \geq x^*\big(I_\nu(g_n)\big)$ for all n. Thus, it follows that $x^*(e) \geq x^*\big(I_\nu(g_n)\big)$ and $x^*(z) \geq x^*\big(I_\nu(f_n)\big)$ for all n. Since this holds for all $0 \leq x^* \in E^*$, we have that $e \geq I_\nu(g_n)$ and $z \geq I_\nu(f_n)$ for all n. Then, $e \geq z$ and $z \geq e$, and so $e = z$. So, T is well defined. Moreover,

$$\|T(f)\|_E = \|e\|_E = \sup \|I_\nu(f_n)\|_E = \sup \|f_n\|_\nu = \|f\|_\nu,$$

where in the last equality we have used that $[L^1(\nu)]_{\sigma\text{-F}}$ has the σ-Fatou property.

Let us see now that T preserves the lattice structure, that is $T(f \wedge g) = Tf \wedge Tg$ for every $0 \leq f, g \in [L^1(\nu)]_{\sigma\text{-F}}$. Consider sequences $(f_n), (g_n) \subset L^1(\nu)$ satisfying that $0 \leq f_n \uparrow f$ and $0 \leq g_n \uparrow g$. Then, $Tf = \sup I_\nu(f_n)$ and $Tg = \sup I_\nu(g_n)$. Note that if $x_n \uparrow x$ and $y_n \uparrow y$ in a Banach lattice then

$x_n \wedge y_n \uparrow x \wedge y$, see for instance [27, Theorem 15.3]. Then, since $0 \leq f_n \wedge g_n \uparrow f \wedge g$ with $(f_n \wedge g_n) \subset L^1(\nu)$ and I_ν is an order isometry, we have that

$$T(f \wedge g) = \sup I_\nu(f_n \wedge g_n) = \sup I_\nu(f_n) \wedge I_\nu(g_n) = Tf \wedge Tg.$$

For a general $f \in [L^1(\nu)]_{\sigma\text{-F}}$, we define $Tf = Tf^+ - Tf^-$ where f^+ and f^- are the positive and negative parts of f respectively. So, $T\colon [L^1(\nu)]_{\sigma\text{-F}} \to E$ is a positive linear operator extending I_ν. For the linearity, see for instance [27, Theorem 15.2]. Moreover T is an isometry. Indeed, $Tf^+ \wedge Tf^- = T(f^+ \wedge f^-) = 0$ as $f^+ \wedge f^- = 0$, and so $|Tf| = |Tf^+ - Tf^-| = Tf^+ + Tf^- = T|f|$, see [27, Theorem 14.4]. Then, $\|T(f)\|_E = \|T(|f|)\|_E = \|f\|_\nu$ for all $f \in [L^1(\nu)]_{\sigma\text{-F}}$.

Let us prove that T is onto. Let $0 \leq e \in E$. Since E_a is super order dense in E, there exists $(e_n) \subset E_a$ such that $0 \leq e_n \uparrow e$. Let $(f_n) \subset L^1(\nu) \subset [L^1(\nu)]_{\sigma\text{-F}}$ be such that $e_n = I_\nu(f_n)$. Since I_ν^{-1} is an order isometry, we have that $0 \leq f_n \uparrow$ and $\sup \|f_n\|_\nu = \sup \|e_n\|_E \leq \|e\|_E < \infty$. Then, by the σ-Fatou property of $[L^1(\nu)]_{\sigma\text{-F}}$, there exists $f = \sup f_n$ in $[L^1(\nu)]_{\sigma\text{-F}}$. From the definition of T, we have that $Tf = \sup I_\nu(f_n) = \sup e_n = e$. For a general $e \in E$, consider e^+ and e^- the positive and negative parts of e. Let $g, h \in [L^1(\nu)]_{\sigma\text{-F}}$ be such that $Tg = e^+$ and $Th = e^-$. Then, taking $f = g - h \in [L^1(\nu)]_{\sigma\text{-F}}$ we have that $Tf = e$. Note that T^{-1} is positive. So, T is positive, linear, one to one and onto with inverse being positive, then T is an order isomorphism (see [26, p. 2]). $\square$

Note that Proposition 4.2.1 generalizes [10, Theorem 2.5] where every Banach lattice E with the σ-Fatou property having a weak unit belonging to E_a is represented by means of spaces $L^1_w(\nu)$ for a vector measure ν defined on a σ-algebra. Indeed, if $0 \leq u \in E_a$ is a weak unit, then $0 \leq e \wedge n\,u \uparrow e$ for each $0 \leq e \in E$ where $e \wedge n\,u \in E_a$, and so E_a is super order dense in E. What happens in this case is that $[L^1(\nu)]_{\sigma\text{-F}} = L^1_w(\nu)$ as ν is defined on a σ-algebra.

4.3 Identifying Banach lattices

In view of the representation theorems studied in the previous sections, there are Banach lattices which can be seen as a space $L^1(\nu)$ or $L^1_w(\nu)$ in two different ways by considering ν defined on a σ-algebra or on a δ-ring. In this section we analyze the difference between this two points of view.

Let X be an order continuous B.f.s. related to a measure space (Ω, Σ, μ), with a weak unit g. We already know that there exists an order isometry $T\colon X \to L^1(\nu)$ for some vector measure ν defined on a σ-algebra.

Suppose that $g = \chi_\Omega$ (i.e. $L^\infty(\mu) \subset X$). In this case, $\mathcal{S}(\Sigma)$ is dense in X as X is order continuous. Taking $\nu\colon \Sigma \to X$ defined by $\nu(A) = \chi_A$ for all $A \in \Sigma$, which is a vector measure by the order continuity of X, we have that $L^1(\nu) = X$ with equal norms, since $\|\varphi\|_\nu = \|\int |\varphi|\, d\nu\|_X = \|\varphi\|_X$ for all $\varphi \in \mathcal{S}(\Sigma)$ and $\mathcal{S}(\Sigma)$ is dense in both $L^1(\nu)$ and X. So, the order isometry between X and $L^1(\nu)$ is the identity map.

If χ_Ω is not a weak unit of X (i.e. $L^\infty(\mu) \not\subset X$), we can not define ν as above. However, we can consider the space $X_g = \{f \in L^0(\mu) :\ fg \in X\}$, which is an order continuous B.f.s. related to μ with the norm $\|f\|_{X_g} = \|fg\|_X$. Since $\chi_\Omega \in X_g$, we have that $X_g = L^1(\nu)$ with equal norms, where $\nu\colon \Sigma \to X_g$ is defined as above. Furthermore, the multiplication operator $M_{g^{-1}}\colon X \to X_g$ is an order isometry. Hence, the order isometry between X and $L^1(\nu)$ is just to multiply by g^{-1}.

Now, consider the δ-ring $\mathcal{R} = \{A \in \Sigma :\ \chi_A \in X\}$. We can take the vector measure $\nu\colon \mathcal{R} \to X$ defined by $\nu(A) = \chi_A$ for all $A \in \mathcal{R}$. Let us see that $\mathcal{S}(\mathcal{R})$ is dense in X. Since g is a weak unit of X, we have that $\Omega = (\cup A_n) \cup N$ where N is a μ-null set (or equivalently, ν-null) and $A_n = \{\omega \in \Omega :\ g(\omega) > 1/n\} \in \mathcal{R}$ (as $\chi_{A_n} \leq ng$), that is, ν is σ-finite. Then, if $0 \leq f \in X$ and $(\psi_n) \subset \mathcal{S}(\Sigma)$ is such that $0 \leq \psi_n \uparrow f$, taking $\varphi_n = \psi_n \chi_{\cup_{j=1}^n A_j} \in \mathcal{S}(\mathcal{R})$, we have that $0 \leq \varphi_n \uparrow f$. By the order continuity of X, it follows that (φ_n) converges to f in X. Therefore, $L^1(\nu) = X$ with equal norms, since $\|\varphi\|_\nu = \|\int |\varphi|\, d\nu\|_X = \|\varphi\|_X$ for all $\varphi \in \mathcal{S}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$ is dense in both $L^1(\nu)$ and X. So, the order isometry between X and $L^1(\nu)$ is the identity map.

The conclusion is that every order continuous B.f.s. with a weak unit can be represented as a $L^1(\nu)$ with ν defined on a σ-algebra, but the representation is an identification only in the case when $\chi_\Omega \in X$. In other case, the representation is a multiplication operator, whereas we can get an identification by considering ν in a δ-ring.

Let now X be a B.f.s. having the σ-Fatou property and a weak unit g belonging to its order continuous part X_a. Then, we know that there exists an order isometry $T\colon X \to L^1_w(\nu)$ for some vector measure ν defined on a σ-algebra.

If $g = \chi_\Omega$ (i.e. $L^\infty(\mu) \subset X_a$), then $L^1(\nu) = X_a$ with equal norms, where $\nu\colon \Sigma \to X_a$ is given by $\nu(A) = \chi_A$ for all $A \in \Sigma$. Noting that for every $0 \leq f \in L^0(\mu)$ there exists $(\varphi_n) \subset \mathcal{S}(\Sigma) \subset L^1(\nu) = X_a$ such that $0 \leq \varphi_n \uparrow f$ and $\|\varphi_n\|_\nu = \|\varphi_n\|_X$ for all n, since both spaces $L^1_w(\nu)$ and X have the σ-Fatou property, it follows that $L^1_w(\nu) = X$ with equal norms. So, the order isometry between X and $L^1_w(\nu)$ is the identity map.

In the case when $\chi_\Omega \notin X_a$ (i.e. $L^\infty(\mu) \not\subset X_a$), we can not define ν as just above. However, since $\chi_\Omega \in (X_a)_g = (X_g)_a$, we have that $L^1_w(\nu) = X_g$ with equal norms, for $\nu\colon \Sigma \to (X_g)_a$ given by the characteristic of sets. Then, since $M_{g^{-1}}\colon X \to X_g$ is an order isometry, it follows that the order isometry between X and $L^1_w(\nu)$ is just to multiply by g^{-1}.

Considering the δ-ring $\mathcal{R} = \{A \in \Sigma : \chi_A \in X_a\}$, we have seen that $X_a = L^1(\nu)$ with equal norms, where $\nu\colon \mathcal{R} \to X_a$ is σ-finite. Then, it follows that X_a is super order dense in X and $L^1(\nu)$ is super order dense in $L^1_w(\nu)$ and so $X = L^1_w(\nu)$ with equal norms. Hence, the order isometry between X and $L^1_w(\nu)$ is the identity map.

Now, the conclusion is that every B.f.s. having the σ-Fatou property and a weak unit belonging to its order continuous part can be represented as a $L^1_w(\nu)$ with ν defined on a σ-algebra, but the representation is an identification only in the case when $\chi_\Omega \in X_a$. In other case, the representation is a multiplication operator, whereas we can get an identification by considering ν in a δ-ring.

Example 4.3.1. Let $([0,\infty), \mathcal{B}[0,\infty), m)$ be the measure space where $\mathcal{B}[0,\infty)$ is the σ-algebra of the Borel sets of $[0,\infty)$ and m is the Lebesgue measure on $[0,\infty)$. The space $L^1(m)$ is an order continuous B.f.s. related to m which does not contain $\chi_{[0,\infty)}$ and so there is no vector measure ν on a σ-algebra such that $L^1(m) = L^1(\nu)$. However, for instance $g = e^{-x}$ is a weak unit in $L^1(m)$, and then we can represent $L^1(m)$ as an $L^1(\nu)$ where ν is defined on $\mathcal{B}[0,\infty)$ via $M_{g^{-1}}$

or as an $L^1(\nu)$ where ν is defined on the δ-ring $\mathcal{R} = \big\{A \in \mathcal{B}[0,\infty) : \, m(A) < \infty\big\}$ via the identity map.

Consider now the B.f.s. E related to m given by

$$E = \Big\{ f \in L^0(m) : \, \sup_n \int_{n-1}^{n} |f(x)| \, dx < \infty \Big\}$$

with norm $\|f\|_E = \sup_n \int_{n-1}^{n} |f(x)| \, dx,$. Note that E can be identified with the space $\ell^\infty\big(\mathbb{N}, \big(L^1(m_n)\big)_{n\in\mathbb{N}}\big)$, where m_n is the restriction of m to $[n-1,1)$, via the map which takes f into $\big(f\chi_{[n-1,n)}\big)_n$ (see Section 2.4). The order continuous part of E can be described as

$$E_a = \Big\{ f \in E : \, \lim_n \int_{n-1}^{n} |f(x)| \, dx = 0 \Big\},$$

which can be identified with the space $c_0\big(\mathbb{N}, \big(L^1(m_n)\big)_{n\in\mathbb{N}}\big)$.

It can be checked that E has the σ-Fatou property. Although $\chi_{[0,\infty)}$ is a weak unit in E, we have that $\chi_{[0,\infty)} \notin E_a$ and so there is no vector measure ν on a σ-algebra such that $E = L^1_w(\nu)$. However, $g = \sum_{n\geq 1} \frac{1}{n}\chi_{[n-1,n)} \in E_a$ is a weak unit of E. Then, we can represent E as an $L^1_w(\nu)$ where ν is defined on $\mathcal{B}[0,\infty)$ via $M_{g^{-1}}$ or as an $L^1_w(\nu)$ where ν is defined on the δ-ring $\mathcal{R}$ defined by $\mathcal{R} = \big\{A \in \mathcal{B}[0,\infty) : \, \lim_n m\big(A \cap [n-1,n)\big) = 0\big\}$ via the identity map.

4.4 Representing p-convex Banach lattices

We can go far away into the representation problem by means of spaces of integrable functions if we think about the spaces $L^p(\nu)$ and $L^p_w(\nu)$ defined in Chapter 3. The key will be the convexity of these spaces.

Note that every p-convex Banach lattice E can be renormed equivalently in a way that E with the new norm and the same order is a p-convex Banach lattice with p-convexity constant equal to 1 (see [26, Proposition 1.d.8]).

Theorem 4.4.1. *Let $p > 1$ and E be a p-convex order continuous Banach lattice with p-convexity constant equal to* 1*. Then there exists a positive vector measure ν defined on a δ-ring and with values in E such that $L^p(\nu)$ and E are order isometric.*

Proof. Since E is order continuous, there exists a vector measure ν_1 defined on a δ-ring $\mathcal{R}$ and with values in E, such that the space $L^1(\nu_1)$ is order isometric to E. More precisely, the integration operator $I_{\nu_1}: L^1(\nu_1) \to E$ is an order isometry (see Theorem 4.1.5). Consequently $L^1(\nu_1)$ is p-convex with p-convexity constant equal to 1 and so by Proposition 3.1.4, the space $L^{1/p}(\nu_1)$ is a B.f.s.

Consider the finitely additive set function $\nu_2 : \mathcal{R} \to L^{1/p}(\nu_1)$ given by by $\nu_2(A) = \chi_A$, for all $A \in \mathcal{R}$. Let $(A_j) \subset \mathcal{R}$ be a pairwise disjoint sequence such that $\cup A_j \in \mathcal{R}$, then by order continuity of $L^1(\nu_1)$ we have that

$$\begin{aligned}\Big\|\nu_2(\cup A_j) - \sum_{j=1}^{n} \nu_2(A_j)\Big\|_{\frac{1}{p},\nu_1} &= \Big\|\nu_2(\cup A_j) - \nu_2(\cup_{j=1}^{n} A_j)\Big\|_{\frac{1}{p},\nu_1} \\ &= \Big\|\nu_2(\cup_{j\geq n} A_j)\Big\|_{\frac{1}{p},\nu_1} = \Big\||\chi_{\cup_{j\geq n} A_j}|^{\frac{1}{p}}\Big\|_{\nu_1}^{p} \\ &= \Big\|\chi_{\cup_{j\geq n} A_j}\Big\|_{\nu_1} \to 0,\end{aligned}$$

as $n \to \infty$. Hence, ν_2 is a countably additive vector measure. It is direct to check that a set is ν_1-null if and only if is ν_2-null.

Consider now the integration operator $I_{\nu_2} : L^1(\nu_2) \to L^{1/p}(\nu_1)$. Given f in $L^1(\nu_2)$, we can take $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\nu_2)$ and ν_2-a.e. Then, $I_{\nu_2}(\varphi_n) \to I_{\nu_2}(f)$ in $L^{1/p}(\nu_1)$. Taking a subsequence converging to $I_{\nu_2}(f)$ ν_1-a.e., since $I_{\nu_2}(\varphi) = \varphi$ for every $\varphi \in \mathcal{S}(\mathcal{R})$, it follows that $I_{\nu_2}(f) = f$. Moreover, since $|I_{\nu_2}(f)| = |f| = I_{\nu_2}(|f|)$ and ν_2 is positive, we have that $\|I_{\nu_2}(f)\|_{\frac{1}{p},\nu_1} = \|I_{\nu_2}(|f|)\|_{\frac{1}{p},\nu_1} = \|f\|_{\nu_2}$. Hence, I_{ν_2} is the identity map and $L^1(\nu_2) = L^{1/p}(\nu_1)$ with equal norms. Therefore, $L^p(\nu_2) = L^1(\nu_1)$ with equal norms and so $L^p(\nu_2)$ is order isometric to E. □

Note that the previous theorem generalizes [19, Proposition 2.4] in which p-convex order continuous Banach lattices having a weak unit are represented as an $L^p(\nu)$, with ν defined on a σ-algebra (see also [31, Proposition 3.30] for the complex version).

Fatou and order density properties allow to represent a p-convex Banach lattice even it is not order continuous as an space of p-integrable functions as in the non-convex case.

Theorem 4.4.2. *Let $p > 1$ and E be a p-convex Banach lattice with p-convexity constant equal to 1, having the Fatou property and such that its order continuous part E_a is order dense in E. Then there exists a E_a-valued vector measure ν on a δ-ring such that E and $L^p_w(\nu)$ are order isometric.*

Proof. The hypothesis on E gives an $\mathcal{R}$-decomposable vector measure ν_1 on a δ-ring $\mathcal{R}$ and an order isometry $T : E \to L^1_w(\nu_1)$ (see Theorem 4.1.7). The B.f.s. $L^{1/p}_w(\nu_1)$ has the Fatou property and $L^{1/p}(\nu_1)$ is order dense in $L^{1/p}_w(\nu_1)$ (see Proposition 3.1.7). Take the vector measure $\nu_2 : \mathcal{R} \to L^{1/p}(\nu_1)$ given by $\nu_2(A) = \chi_A$, $A \in \mathcal{R}$ for which the integration operator $I_{\nu_2} : L^1(\nu_2) \to L^{1/p}(\nu_1)$ is the identity map and $L^1(\nu_2) = L^{1/p}(\nu_1)$ with equal norms (see the proof of the previous theorem). Noting that ν_1 is $\mathcal{R}$-decomposable and ν_1 and ν_2 have the same null sets, by the construction of $\mathcal{R}$ in Section 4.1.1 and since ν_2 is defined in the same $\mathcal{R}$, it can be checked that ν_2 is $\mathcal{R}$-decomposable. Hence, $L^1_w(\nu_2)$ has the Fatou property.

Let us see now that $L^1_w(\nu_2) = L^{1/p}_w(\nu_1)$ with equal norms. Take $0 \leq f \in L^1_w(\nu_2)$. Since $L^1(\nu_2)$ is order dense in $L^0(\nu_2)$ (see Remark 2.2.3), there exists an upwards directed system $(f_\tau)_\tau$ in $L^1(\nu_2)$ such that $0 \leq f_\tau \uparrow f$ in $L^0(\nu_2)$. Then $0 \leq f_\tau \uparrow$ in $L^{1/p}_w(\nu_1)$ and $\sup \|f_\tau\|_{\frac{1}{p},\nu_1} = \sup \|f_\tau\|_{\nu_2} \leq \|f\|_{\nu_2}$. Therefore, the Fatou property of $L^{1/p}_w(\nu_1)$ gives $h \in L^{1/p}_w(\nu_1)$ such that $\|h\|_{\frac{1}{p},\nu_1} = \sup_\tau \|f_\tau\|_{\frac{1}{p},\nu_1}$. Since for each τ we have that $f_\tau \leq h$ ν_1-a.e. or equivalently ν_2-a.e., then $f \leq h$ and so $f \in L^{1/p}_w(\nu_1)$. On the other hand, $f_\tau \leq f$ ν_2-a.e. (i.e. ν_1-a.e.) for all τ and thus $h \leq f$. Therefore, $\|f\|_{\frac{1}{p},\nu_1} = \|h\|_{\frac{1}{p},\nu_1} = \sup_\tau \|f_\tau\|_{\frac{1}{p},\nu_1} = \sup_\tau \|f_\tau\|_{\nu_2}$, where the last equality is due to the Fatou property of $L^1_w(\nu_2)$ as $0 \leq f_\tau \uparrow f$ also in $L^1_w(\nu_2)$. By taking positive and negative parts of a general $f \in L^1_w(\nu_2)$, we have that $L^1_w(\nu_2) \subset L^{1/p}_w(\nu_1)$ with equal norms.

The converse inclusion follows by the same arguments. Therefore, the equality $L^1_w(\nu_2) = L^{1/p}_w(\nu_1)$ holds with equal norms. Consequently $L^p_w(\nu_2) = L^1_w(\nu_1)$ with equal norms and hence E and $L^p_w(\nu_2)$ are order isometric. $\square$

A similar proof as the one in the previous theorems allow us to represent p-convex Banach lattices (whit p-convexity constante equal to 1) having the σ-Fatou property and such that E_a is super order dense in E. In this case, E is order isometric to $[L^p(\nu)]_{\sigma\text{-F}}$ for some vector measure ν defined on a δ-ring. This result generalizes [12, Theorem 4] where E has a weak unit in E_a (for the complex version see [31, Proposition 3.41]).

4.5 Representing Banach quasi-algebras

In [1, Remark 1.10] the authors introduce the notion of quasi-normed algebra, that is an algebra $\mathcal{A}$ with multiplicative law $\odot_{\mathcal{A}}$ endowed with a quasi-norm $\|\cdot\|_{\mathcal{A}}$ satisfying that there exists a constant $K > 0$ such that $\|a \odot_{\mathcal{A}} b\|_{\mathcal{A}} \leq K\|a\|_{\mathcal{A}}\|b\|_{\mathcal{A}}$, for all $a, b \in \mathcal{A}$. In our setting, $\|\cdot\|_{\mathcal{A}}$ will be a complete norm and we will say that $\mathcal{A}$ is a *Banach quasi-algebra* (*Banach algebra* if $K = 1$).

Let E be a *representable* Banach lattice, that is, a Banach lattice for which there exists a B.f.s. X related to μ such that E and X are order isomorphic. If E is a Banach quasi-algebra with the algebra product $\odot_E$ and $T : E \to X(\mu)$ is an order isomorphism, then X endowed with the algebra product $\odot_\mu$ defined by

$$f \odot_\mu g := T\big(T^{-1}(f) \odot_E T^{-1}(g)\big), \quad \text{for all } f, g \in X,$$

is a Banach quasi-algebra. If E is a Banach algebra and T is an order isometry, then X is a Banach algebra (commutative if E is so).

Therefore, if we apply the representation theorems established in the previous sections to a Banach lattice which is also a Banach quasi-algebra, we can endow the corresponding space of integrable functions with an algebra structure.

Corollary 4.5.1. *Let E be a Banach lattice which is also a Banach quasi-algebra with multiplicative law $\odot_E$.*

(a) *If E is order continuous, then E is order isometric to an L^1-space which becomes a Banach quasi-algebra.*

(b) *If E has the Fatou property and its order continuous part E_a is order dense in E, then E is representable by means of an L^1_w-space which becomes a Banach quasi-algebra.*

(c) *If E has the σ-Fatou property and its order continuous part E_a is super order dense in E, then E is order isometric to the σ-Fatou completion of an L^1-space which is also a Banach quasi-algebra.*

Note that if in the previous corollary E is actually a Banach algebra, since the representation operators are order isometries, the corresponding space of integrable functions is also a Banach algebra.

Consequently, the class of Banach quasi-algebras inside the broad class of order continuous Banach lattices is exactly the class of the L^1-spaces which are also Banach quasi-algebras. Furthermore, the class of Banach quasi-algebras inside the class of Banach lattices having the Fatou property with order continuous part dense coincides with the class of Banach quasi-algebras in the broad class of L^1_w-spaces, Also, the class of Banach lattices having the σ-Fatou property and with order continuous part as super order dense ideal which are Banach quasi-algebras is exactly the class of the Banach quasi-algebras in the class of the σ-Fatou completion of L^1-spaces.

In the case of the p-powers, it is also possible to endow the spaces $L^p(\nu)$, $L^p_w(\nu)$ and the σ-Fatou completion of $L^p(\nu)$ with an algebra structure if these spaces are representable by means of a Banach quasi-algebra.

Corollary 4.5.2. *Let $1 \leq p < \infty$ and let E be a p-convex Banach lattice which is also a Banach quasi-algebra.*

(a) *If E is order continuous, then E is order isomorphic to an L^p-space which becomes to be a Banach quasi-algebra.*

(b) *If E has the Fatou property and its σ-order continuous part E_a is order dense in E, then E is representable by means of an L^p_w-space which becomes to be a Banach quasi-algebra.*

(c) *If E has the σ-Fatou property and its σ-order continuous part E_a is super order dense in E, then E is order isomorphic to the σ-Fatou completion of an L^p-space which is also a Banach quasi-algebra.*

In Corollary 4.5.2, the representation operators are order isometries whenever E has p-convexity constant equal to one. So, in this case, if E is a Banach algebra, the corresponding space of p-integrable functions is also a Banach algebra.

Again, the class of Banach quasi-algebras inside the broad class of order continuous p-convex Banach lattices is exactly the class of the L^p-spaces which are also Banach quasi-algebras. The class of Banach quasi-algebras inside the class of p-convex Banach lattices having the Fatou property with order continuous part dense coincides with the class of Banach quasi-algebras in the broad class of L^p_w-spaces. The class of p-convex Banach lattices having the σ-Fatou property and with order continuous part as super order dense ideal which are Banach quasi-algebras is exactly the class of the Banach quasi-algebras in the class of the σ-Fatou completion of L^1-spaces.

Denote by ν and $\odot_\nu$ the corresponding vector measure and multiplicative law in the corollaries above. Remark that in all the cases, ν take values in a Banach quasi-algebra and $\odot_\nu$ depends on $\odot_E$.

References

[1] J. M. Almira and U. Luther, *Inverse closedness of approximation algebras*, J. Math. Anal. Appl. **314**, no. 1, 30-44 (2006).

[2] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, New York, 1985.

[3] R. G. Bartle, N. Dunford and J. T. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7**, 289-395 (1955).

[4] J. K. Brooks and N. Dinculeanu, *Strong additivity, absolute continuity and compactness in spaces of measures*, J. Math. Anal. Appl. **45**, 156-175 (1974).

[5] J. M. Calabuig, O. Delgado and E. A. Sánchez Pérez, *Factorizing operators on Banach function spaces through spaces of multiplication operators*, J. Math. Anal. Appl. **364**, 88-103 (2010).

[6] G. P. Curbera, *El espacio de funciones integrables respecto de una medida vectorial*, Ph. D. Thesis, Univ. of Sevilla, 1992.

[7] G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293**, 317-330 (1992).

[8] G. P. Curbera, *When L^1 of a vector measure is an AL-space*, Pacific J. Math., **162**, 287-303 (1994).

[9] G. P. Curbera, *Banach space properties of L^1 of a vector measure*, Proc. Amer. Math. Soc., **123**, 3797-3806 (1995).

[10] G. P. Curbera and W. J. Ricker, *Banach lattices with the Fatou property and optimal domains of kernel operators*, Indag. Math. (N.S.) **17**, 187-204 (2006).

[11] G. P. Curbera and W. J. Ricker, *Vector measures, integration and applications*, in: Positivity, in: Trends Math., Birkhäuser, Basel, 127-160 (2007).

[12] G. P. Curbera and W. J. Ricker, *The Fatou property in p-convex Banach lattices*, J. Math. Anal. Appl. **328**, 287-294 (2007).

[13] R. del Campo, A. Fernández, I. Ferrando, F. Mayoral and F. Naranjo, *Multiplication Operator on spaces of integrable functions with respect to a vector measure*, J. Math. Anal. Appl. **343**, 514-524 (2008).

[14] R. del Campo, A. Fernández, I. Ferrando, F. Mayoral and F. Naranjo, *Compactness of multiplication operators on spaces of integrable functions with respect to a vector measure* in *Vector measures, integration and related topics*. Operator Theory. Advances and Applications, **201**, 109-113 (2009).

[15] O. Delgado, *L^1-spaces of vector measures defined on δ–rings*, Arch. Math. **84**, 432-443 (2005).

[16] O. Delgado, *Optimal domains for kernel operators on* $[0,\infty)\times[0,\infty)$, Studia Math. **174**, 131-145 (2006).

[17] O. Delgado and J. Soria, *Optimal domain for the Hardy operator*, J. Funct. Anal. **244** (2007), 119-133.

[18] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[19] A. Fernández, F. Mayoral, F. Naranjo, C. Sáez and E. A. Sánchez-Pérez, *Spaces of p-integrable functions with respect to a vector measure*, Positivity **10**, 1-16 (2006).

[20] I. Ferrando and F. Galaz-Fontes, *Multiplication operators on vector measure Orlicz spaces*, Indag. Math. (N.S.) **20(1)**, 57-71 (2009).

[21] D. H. Fremlin, *Measure Theory*, vol. 2: Broad Foundations, Torres Fremlin, Colchester, 2001.

[22] E. Jiménez Fernández, M. A. Juan and E.A. Sánchez-Pérez, *A Komlós Theorem for abstract Banach lattices of measurable functions* J. Math. Anal. Appl. doi 10.1016/j.jmaa.2011.05.010 (2011).

[23] I. Kluvánek and G. Knowles, *Vector measures and control systems*, North Holland Math. Studies **20**, Amsterdam, 1975.

[24] D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math., **33**, 157-165 (1970).

[25] D. R. Lewis, *On integrability and summability in vector spaces*, Illinois J. Math. **16**, 294-307, (1972).

[26] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, Berlin, 1979.

[27] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland, Amsterdam, 1971.

[28] P. R. Masani and H. Niemi, *The integration theory of Banach space valued measures and the Tonelli-Fubini theorems. I. Scalar-valued measures on δ-rings*, Adv. Math. **73**, 204-241 (1989).

[29] P. R. Masani and H. Niemi, *The integration theory of Banach space valued measures and the Tonelli-Fubini theorems. II. Pettis integration*, Adv. Math. **75**, 121-167 (1989).

[30] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin-New York, 1991.

[31] S. Okada, W. J. Ricker and E. A. Sánchez Pérez, *Optimal Domains and Integral Extensions of Operators acting in Function Spaces*, Operator Theory Advances and Applications, vol.180, Birkhäuser, Basel, 2008.

[32] E. A. Sánchez Pérez, *Compactness arguments for spaces of p-integrable functions with respect to a vector measure and factorization of operators through Lebesgue-Bochner spaces.* Illinois J. Math. **45** 3, 907-923 (2001).

[33] G. F. Stefansson, *L_1 of a vector measure*, Le Matematische (Catania) **48**, 219-234 (1993).

[34] A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983.