UNIVERSIDAD CARLOS III DE MADRID

Doctoral Thesis

Volatility Models with Leverage Effect

Author:
Mª José Rodríguez Villar

Supervisor:
Esther Ruiz Ortega

Department of Statistics

Getafe, December 2010

To my husband, parents and sister.

Acknowledgments

It is a pleasure to thank the many people who have contributed making this thesis possible.

It is difficult to overstate my gratitude to my PhD supervisor, Dr. Esther Ruiz. I can only offer my sincerest gratitude to her. She has supported me throughout my thesis with patience, wisdom, encouragement and criticism. Without her, this thesis could not have been completed or written. One could not wish for a better or friendlier supervisor.

I am very grateful to the entire faculty and staff of the Department of Statistics for the support in every step of the way. A special mention to all my PhD colleagues who have always helped me each time that I need.

I wish to thank my friends, only a few but very special ones, for providing me with some memorable experiences. I would also like to show my gratitude to people from Caja Laboral. They are a friendly and cheerful group which make easier my daily work.

My final words go to my family.

I devote my most wholehearted thank to my parents for their absolute confidence in me. You have always been my biggest fans. Your encouragement and guidance have been priceless and there are no words to adequately express my appreciation and gratitude to You.

I want to thank my sister Irene for sharing with me all my fears, dreams and success and always support me on every decision I have taken in my life.

Finally, I am forever indebted to You, David. Without Your understanding, endless patience and love I would not have ever finished my PhD.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation

It is widely accepted that the volatility of financial returns evolves over time. As a consequence, the distribution of returns has higher kurtosis than if they were Gaussian, indicating that extreme returns have higher probability than expected under a Normal distribution. Another consequence of volatility movements is the clustering of large and small returns over time. This volatility clustering is reflected in positive significant autocorrelations of squared returns, which show a slow decay toward zero. Furthermore, there is also extensive empirical evidence about the existence of an asymmetric response of volatility to positive and negative past returns. In particular, increases in volatility are larger when previous returns are negative than when they have the same magnitude but are positive. This phenomenon, originally put forward by Black (1976) is known as *leverage effect*. The presence of this asymmetric behavior can be detected when analyzing the relationship between returns and future squared returns through the cross-correlations which usually are significant and negative.

In practice, the models proposed to represent the dynamic evolution of volatilities should be able to represent the empirical properties usually observed in financial returns and described above, namely, positive and persistent although small autocorrelations of squares, negative cross-correlations between returns and future squared returns and excess kurtosis.

The main objective of this thesis is to analyze and compare the ability of some popular models in Financial Econometrics to represent conditionally heteroskedastic returns with leverage effect. In empirical applications, it is important to know which of the alternative models available is more adequate or at least which are the advantages and limitations of each of them.

We consider five of the most popular conditional heteroskedastic models with leverage effect within the GARCH family, namely, the Quadratic GARCH (QGARCH) proposed independently by Engle and Ng (1993) and Sentana (1995)[1], the Threshold GARCH (TGARCH) of Zakoïan (1994), the GJR of Glosten *et al.* (1993), the Exponential GARCH (EGARCH) of Nelson (1991) and the Asymmetric Power ARCH (APARCH) of Ding *et al.* (1993). Furthermore, we only consider the simplest formulation of each of these models which specify the conditional variance as a non-linear function of one-lagged conditional variances and returns as this specification is the most popular among empirical researchers.

Previously several authors have compared the predictive power of asymmetric conditional heteroskedastic models reporting mixed results[2]. In this thesis, we focus on from two different perspectives. First, from a theoretical point of view, we study and compare how the dynamics of conditional volatilities represented by each model are restricted to guarantee the positivity of volatilities and the stationarity and finite kurtosis of returns. These restrictions come from the functional form selected to represent the volatility and are distinctive for each model. Comparing the theoretical limitations that arise from the functional form assumed by each asymmetric GARCH model considered here, is the main topic of Chapter 2.

In practice, after fitting a particular GARCH model to a time series of financial returns, it is very usual to analyse its adequacy by comparing the sample properties of the series with those implied by the fitted model, which are known as plug-in moments. In Chapter 3, we analyse whether this comparison is adequate. We show that the finite sample properties of sample and plug-in moments can be very different leading to misleading conclusions about the adequacy of a fitted model when they are compared.

The rest of this chapter is organized as follows. Section 1.2 describes the main empirical properties of several series of real financial returns that will be later used to illustrate several results along the thesis. Section 1.3 describes the main properties of asymmetric GARCH

[1]Engle and Ng (1993) and Sentana (1995) proposed the same model under two different acronyms, namely, AGARCH (Asymmetric GARCH) and QGARCH, respectively. In this thesis we use the latter because the model has been more fully worked out by Sentana (1995). Also, note that one should be careful with the use of acronyms as they have not been fully consistent in the existing literature. For example, AGARCH has been used to represent at least four different specifications.

[2]See for example, Loudon *et al.* (2000), Balaban (2004) and Alberg *et al.* (2008), Hansen and Lunde (2005), Awartani and Corradi (2004) and Chen *et al.* (2006) for other comparisons.

models. Finally, the organization of this thesis is described in Section 1.4.

1.2 Empirical characteristics of financial returns

As we mentioned before, the main sample properties that GARCH models should represent are excess kurtosis, positive and persistent although small autocorrelations of squares and negative cross-correlations between returns and future squared returns. In this section, we illustrate these properties by analyzing several time series of returns. Consider first, the series of daily returns of the SP500 index observed from January 3rd 1994 to July 31st 2007 and of the exchange rates of US Dollar against the Australian Dollar (USD/AUD) observed from January 2nd 1990 to May 9th 2006. To avoid the misleading effects of outliers on the estimation of the volatility, the series have been filtered from outliers by equaling all observations larger than $5\widehat{\sigma}_t$ to $\widehat{\sigma}_t sign(y_t)$ where $\widehat{\sigma}_t$ is an estimate of the conditional standard deviation[3]

Both series of returns have been plotted in Figure 1.1 together with their correlogram of squares, their sample cross-correlations and the Non-parametric NIC (NPN) estimator as proposed by Engle and Ng (1993)[4]. The sample kurtosis of the SP500 and AUD/USD returns are higher than in the Gaussian case, reaching values of 6.88 and 4.30, respectively. The volatility clustering observed in the series of returns is reflected in the positive and significant autocorrelations of squares. In the case of the SP500 returns, the first autocorrelation of squares is 0.20 whereas for the USD/AUD returns this value is 0.07. As usual, these autocorrelations are not very large and are highly persistent. On the other hand, the cross-correlations are also significant and negative, suggesting the presence of leverage effect which, as expected, is stronger in the SP500 returns where the first cross-correlation is -0.12, than in the exchange rate returns where it is only -0.07. A similar conclusion arises from the NPNs plotted in the last column of Figure 1.1 where the asymmetric response of volatility to positive and negative lagged returns is clear in the SP500 returns.

Daily returns of the SP500 index will also be considered in Chapter 3 but observed for a

[3]Carnero *et al.* (2008) show the large biases caused by outliers on the estimation of the underlying volatilities and advocate for using filters similar to that implemented in this chapter. Alternatively, the abnormally large movements in prices can be explained by the presence of jumps; see, for example, Eraker *et al.* (2003). Including jumps in the GARCH models considered in this chapter will complicate the analysis without changing the main results and it is beyond our objectives.

[4]It is important to note that the estimated NPNs depend heavily on the window used for their estimation.

different period of time, from January 2nd 2002 to June 25th 2010, with a kurtosis of 7.12, first autocorrelation of squares equal 0.34 and a first cross-correlation of -0.07. We also consider returns of the EUR/USD exchange rate observed during the same time period. In this case, the kurtosis is equal 4.28, there is a significant first autocorrelation of squares of 0.12 and a negligible first cross-correlation of 0.02. Figure 1.2 plots both series together with their correlograms of squares and their sample cross-correlations.

1.3 Properties of GARCH models with leverage effect.

In this section we introduce the GARCH models with leverage effect considered in this thesis and summarize the theoretical properties already known about them.

Consider the following model for returns

$$y_t = \varepsilon_t \sigma_t \tag{1.3.1}$$

where σ_t is the volatility and ε_t is a serially independent sequence with zero mean, variance one and symmetric density with finite kurtosis, κ_ε. The specification of σ_t is described bellow for each of the five models considered.

1.3.1 QGARCH model

The first model considered is the QGARCH model which is given by

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 + \delta_Q y_{t-1}. \tag{1.3.2}$$

The main statistical properties of the QGARCH model have been derived by Sentana (1995) who shows that it is stationary if $p < 1$ where $p = \alpha + \beta$ can be interpreted as the persistence of shocks to the volatility. In this case, the marginal variance of returns is given by $\sigma_y^2 = \frac{\omega}{1-p}$. Note that the stationarity of the model does not depend on the asymmetry parameter, δ_Q. However, if σ_y^2 is finite, the asymmetry parameter has to be restricted to guarantee the positivity of σ_t^2. In particular $\delta_Q^2 \leq 4\alpha\sigma_y^2(1-p)$. To avoid the dependence of the asymmetry parameter on the marginal variance, we analyze the positivity restrictions in terms of the parameter $\delta_Q^* = \frac{\delta_Q}{\sigma_y}$. Consequently, if the model is stationary, the positivity restriction is given by

$$\delta_Q^{*2} \leq 4\alpha(1-\alpha-\beta). \tag{1.3.3}$$

The restriction for the existence of the fourth order moment, derived by He and Teräsvirta (1999a), is given by

$$(k_\varepsilon - 1)\alpha^2 + p^2 < 1. \tag{1.3.4}$$

When restriction (1.3.4) is satisfied, the kurtosis of y_t is given by $k_y = k_\varepsilon \frac{1-p^2+\delta_Q^{*2}}{1-[(\kappa_\varepsilon-1)\alpha^2+p^2]}$. The dynamic properties of QGARCH models appear in the autocorrelation function (acf) of squared returns and in the cross-correlations between future squares and original returns. In particular, using the results in He and Teräsvirta (1999a), the acf of y_t^2 can be derived as follows

$$\rho_2(\tau) = \begin{cases} \frac{2\alpha(1-p+\alpha p)+\delta_Q^{*2}(k_\varepsilon\alpha+\beta)}{2(1-p^2+\alpha^2)+k_\varepsilon\delta_Q^{*2}}, & \tau = 1 \\ \\ p^{\tau-1}\rho_2(1), & \tau > 1. \end{cases} \tag{1.3.5}$$

The cross-correlations between y_t^2 and $y_{t-\tau}$, derived by Sentana (1995), are given by

$$\rho_{21}(\tau) = \begin{cases} \frac{\delta_Q^*}{(\kappa_y-1)^{1/2}}, & \tau = 1 \\ \\ (\alpha+\beta)\rho_{21}(\tau-1), & \tau > 1. \end{cases} \tag{1.3.6}$$

Finally, another useful tool to describe of the properties of GARCH models with leverage effect is the News Impact Curve (NIC) proposed by Engle and Ng (1993) which analyzes the effect of past returns on conditional variances. The NIC relates past returns and volatilities by holding constant the information up to time $t-2$ and evaluating all lagged conditional variances at the level of the marginal variance of returns. In general, the NIC captures the leverage effect by allowing either the slope of its two sides to differ or its center to be located at a point where past returns, y_{t-1}, are positive. In particular, Engle and Ng (1993) show that the NIC of the QGARCH mode is given by

$$\sigma_t^2 = A + \alpha\left(y_{t-1} + \frac{\delta_Q}{2\alpha}\right)^2 \tag{1.3.7}$$

where $A = \omega\left(\frac{1-\alpha}{1-\alpha-\beta}\right) - \frac{\delta_Q^2}{4\alpha}$.

1.3.2 TGARCH model

The TGARCH model specifies the volatility as follows

$$\sigma_t = \omega + \alpha\left|y_{t-1}\right| + \beta\sigma_{t-1} + \delta_T y_{t-1}. \tag{1.3.8}$$

The stationarity condition is given by

$$\delta_T^2 < 1 - \alpha^2 - \beta^2 - 2\alpha\beta\nu_1 \tag{1.3.9}$$

where $\nu_1 = E\left(|\varepsilon_t|\right)$ which, for example, is given by $\sqrt{\frac{2}{\pi}}$ when ε_t is Gaussian and by $\sqrt{\frac{(\nu-2)}{\pi}\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}}$ when ε_t has a Student-ν distribution; see He and Teräsvirta (1999a). When the stationary restriction in (1.3.9) is satisfied, the marginal variance of returns is given by

$$\sigma_y^2 = \omega^2\frac{1+q}{(1-q)(1-p)}$$

where $q = \alpha\nu_1 + \beta$ and $p = \delta_T^2 + \alpha^2 + \beta^2 + 2\alpha\beta\nu_1$.

The kurtosis of y_t, derived by He and Teräsvirta (1999a), is given by

$$k_y = k_\varepsilon\frac{[(1-p)(1-q)(3d+5p+3q+3dp+5dq+3pq+dpq+1]}{(1+q)^2(1-d)(1-f)} \tag{1.3.10}$$

where $d = (\alpha+\beta)^3 + \alpha^3(\nu_3-1) + 3\alpha\beta^2(\nu_1-1) + 3\delta_T^2(\alpha\nu_3+\beta)$ and $f = (\alpha+\beta)^4 + \alpha^4\left(k_\varepsilon-1\right) + 4\alpha\beta\left[\beta^2\left(\nu_1-1\right) + \alpha^2\left(\nu_3-1\right)\right] + 6\delta_T^2\left[2\alpha\beta\nu_3 + \alpha^2 k_\varepsilon + \beta^2\right]$ with $\nu_3 = E(|\varepsilon_t|^3)$.

The following expression of the acf of y_t^2 is given by He and Teräsvirta (1999a)

$$\rho_2(\tau) = \begin{cases} \frac{(1-q)(1-p)\{2\bar{q}(1-f)\Delta_3+\bar{p}\Delta_4\}-(1+q)(1-d)(1-f)\{2q+p(1-q)\}}{\Delta}, & \tau = 1 \\ p\rho_2(\tau-1) + \theta q^{\tau-1}, & \tau > 1, \end{cases} \tag{1.3.11}$$

where

$$\begin{aligned}
\Delta &= k_\varepsilon \Delta_4(1-q)(1-p) - (1+q)^2(1-d)(1-f), \\
\theta &= (1/\Delta)\{2(1-p)(1-f)[\Delta_3\bar{q}(1-q) - q(1+q)(1-d)]\} \\
\Delta_3 &= (1+p)(1+q) + 2(p+q) \\
\Delta_4 &= (1+p)(1+q)(1+d) + 2(1+p)(q+d) + 4(qd+p) \\
\bar{q} &= \alpha\nu_3 + \beta \\
\bar{p} &= \beta^2 + 2\alpha\beta\nu_3 + k_\varepsilon(\alpha^2 + \delta_T^2).
\end{aligned}$$

The cross-correlations between y_t^2 and $y_{t-\tau}$ can be derived by using the results in He et al. (2008) as follows

$$\rho_{21}(\tau) = \left\{ \frac{2\sqrt{(1-q)(1-p)}\delta_T(\sum_{j=0}^{\tau-1} q^{\tau-1-j}p^j + p^{\tau-1}\bar{q}(1+pq)(1-d)^{-1}(1+q)^{-1})}{\sqrt{(1+q)(k_y-1)}}, \quad \tau \geq 1. \right. \tag{1.3.12}$$

Finally, the NIC of the TGARCH model is given by

$$\sigma_t^2 = (A + \alpha\,|y_{t-1}| + \delta_T y_{t-1})^2 \tag{1.3.13}$$

where $A = \omega + \beta\sigma_y$; see Henstchel (1995).

1.3.3 GJR model

The GJR model specifies the conditional variance as

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta\sigma_{t-1}^2 + \delta_G I(\varepsilon_{t-1} < 0) y_{t-1}^2 \tag{1.3.14}$$

where $I(\cdot)$ is the indicator function that takes value 1 when the argument is true.

As in the QGARCH model, is necessary to restrict its parameters in order to ensure positivity. In particular, Hentschel (1995) shows that σ_t^2 is positive if

$$\omega > 0, \;\; \alpha, \beta, \delta_G \geq 0. \tag{1.3.15}$$

Furthermore, the model is stationary if

$$\delta_G < 2(1 - \alpha - \beta). \tag{1.3.16}$$

When the stationarity condition is satisfied, the marginal variance of y_t is given by $\sigma_y^2 = \frac{\omega}{1-p}$ where $p = \alpha + \beta + 0.5\delta_G$.

He and Teräsvirta (1999a) derive the following condition for the existence of the fourth order moment

$$p^2 + \alpha(\kappa_\varepsilon - 1)(\alpha + \delta_G) + 0.25\delta_G^2(2\kappa_\varepsilon - 1) < 1. \tag{1.3.17}$$

When the condition in (1.3.17) is satisfied, the kurtosis of y_t is given by

$$k_y = k_\varepsilon \frac{1 - p^2}{1 - p^2 - \alpha(k_\varepsilon - 1)(\alpha + \delta_G) - 0.25\delta_G^2(2k_\varepsilon - 1)}.$$

The acf of y_t^2, derived by He and Teräsvirta (1999a), is given by

$$\rho_2(\tau) = \begin{cases} \frac{(\beta + k_\varepsilon(\alpha+\delta_G))(1-p^2) - p(1-p^2-\alpha(k_\varepsilon-1)(\alpha+\delta_G)-0.25\delta_G^2(2k_\varepsilon-1))}{k_\varepsilon(1-p^2)-(1-p^2-\alpha(k_\varepsilon-1)(\alpha+\delta_G)-0.25\delta_G^2(2k_\varepsilon-1))} & \tau = 1 \\ \\ \rho_2(1)p^{\tau-1} & \tau > 1. \end{cases} \tag{1.3.18}$$

Using the results in He *et al.* (2008), we have derived the following expression of the cross-correlations between y_t^2 and $y_{t-\tau}$

$$\rho_{21}(\tau) = \begin{cases} \frac{\delta_G \nu_3 \gamma_3}{\sigma_y^3 \sqrt{k_y - 1}}, & \tau = 1 \\ \\ p^{\tau-1}\rho_{21}(1), & \tau > 1 \end{cases} \tag{1.3.19}$$

where $\gamma_3 = E(\sigma_t^3)$.

Finally, the NIC of the GJR model is given by

$$\sigma_t^2 = \begin{cases} A + \alpha y_{t-1}^2, & y_{t-1} > 0 \\ A + (\alpha + \delta_G) y_{t-1}^2, & y_{t-1} < 0, \end{cases} \tag{1.3.20}$$

where $A = \omega + \beta\sigma_y^2$.

1.3.4 EGARCH model

The EGARCH model specifies the conditional variance as follows

$$\log \sigma_t^2 = \omega^* + \beta \log \sigma_{t-1}^2 + \alpha \left|\varepsilon_{t-1}\right| + \delta_E \varepsilon_{t-1} \tag{1.3.21}$$

where $\omega^* = \omega - \alpha E \left|\varepsilon_t\right|$. The specification of the volatility in terms of its logarithmic transformation implies that there are not restrictions on the parameters to guarantee the positivity of the variance. Nelson (1991) establishes the conditions for covariance stationarity of the EGARCH model under particular specifications of the error distribution. Furthermore, a sufficient condition for the stationarity of the EGARCH model in (1.3.21) is $|\beta| < 1$ when ε_t has a distribution such that $E\left[\log^+\left(\omega + \alpha\left|\varepsilon_{t-1}\right| + \delta_E \varepsilon_{t-1}\right)\right] < \infty$; see Straumann and Mikosch (2006). Nelson (1991) establishes the conditions for covariance stationarity of the EGARCH model under particular specifications of the error distribution. In particular, assuming that $|\beta| < 1$, the EGARCH model is always stationary if ε_t has a Normal or a Generalized Error Distribution (GED) with parameter $\varsigma > 1$. However, when ε_t has a Student-ν or a GED distribution with parameter $\varsigma \leq 1$, y_t is stationary if $\alpha \leq -\left|\delta_E\right|$ which is a rather implausible restriction when dealing with real time series of returns. This is the reason why, in practice, many authors chose the GED distribution instead of the Student-ν distribution when dealing with heavy tailed errors in EGARCH models.

The stationarity guarantees the existence of higher order moments and, in particular, of the kurtosis. If y_t is stationary, the unconditional variance, kurtosis and acf of squares can be derived using the results in Nelson (1991) [5]. They have respectively the following expressions

$$\sigma_y^2 = \exp\left(\frac{\omega}{1-\beta}\right) \prod_{i=1}^{\infty} E(\exp(\beta^{i-1} g(\varepsilon_{t-i}))) \tag{1.3.22}$$

$$k_y = k_\varepsilon \prod_{i=1}^{\infty} \frac{E(\exp(2\beta^{i-1} g(\varepsilon_{t-i})))}{\left[E(\exp(\beta^{i-1} g(\varepsilon_{t-i})))\right]^2} \tag{1.3.23}$$

[5] He *et al.* (2002) derive the acf of power transformed returns, $\left|y_t\right|^\theta$ for the EGARCH model giving closed form expressions for some of the expectations involved in (1.3.22) to (1.3.24) when the errors are Normal or have a Generalized Errors Distribution (GED). Karanasos and Kim (2003) derive the acf of $\left|y_t\right|^\theta$ of a general EGARCH(p, q) model with Gaussian, GED and Double Exponential errors.

$$\rho_2(\tau) = \frac{E(\varepsilon_t^2 \exp(\beta^{\tau-1} g(\varepsilon_t)))P_1 P_2 - P_3}{k_\varepsilon P_4 - P_3} \tag{1.3.24}$$

where $g(\varepsilon_t) = \alpha\left(|\varepsilon_t| - E\,|\varepsilon_t|\right) + \delta_E \varepsilon_t$, $P_1 = \prod_{i=1}^{\tau-1} E(\exp(\beta^{i-1} g(\varepsilon_{t-i})))$, $P_2 = \prod_{i=1}^{\infty} E(\exp((1+\beta^\tau)\beta^{i-1} g(\varepsilon_{t-i})))$, $P_3 = \prod_{i=1}^{\infty}\left[E(\exp(\beta^{i-1} g(\varepsilon_{t-i}))\right]^2$ and $P_4 = \prod_{i=1}^{\infty} E(\exp(2\beta^{i-1} g(\varepsilon_{t-i})))$.

It is interesting to remark that the rate of decay of the autocorrelations of squares is not constant; see Carnero *et al.* (2004) who show that this rate tends to β for large lags whereas for small ones it depends on α and $\delta_{E.}$

Ruiz and Veiga (2008) derive the following expression of the cross-correlations between y_t^2 and $y_{t-\tau}$

$$\rho_{21}(\tau) = \frac{E(\varepsilon_t \exp(\beta^{\tau-1} g(\varepsilon_t)))P_1 P_5}{P_3^{1/4}[k_\varepsilon P_4 - P_3]^{1/2}} \tag{1.3.25}$$

where $P_5 = \prod_{i=1}^{\infty} E(\exp(\beta^{i+\tau-1} + \frac{1}{2}\beta^{i-1}) g(\varepsilon_{t-i}))$; see also Demos (2002) who derives the cross-correlation function for Gaussian errors and Karanasos and Kim (2003) who obtain a general expression of $\rho_{21}(\tau)$ in ARMA(r,s)-EGARCH(p,q) models from which expression (1.3.25) can be obtained as a particular case.

Finally, the NIC of the EGARCH model is given by

$$\sigma_t^2 = \begin{cases} A \exp\left[\frac{\delta_E + \alpha}{\sigma_y} y_{t-1}\right], & y_{t-1} > 0 \\ A \exp\left[\frac{\delta_E - \alpha}{\sigma_y} y_{t-1}\right], & y_{t-1} < 0, \end{cases} \tag{1.3.26}$$

where $A = \sigma_y^{2\beta} \exp\left[\omega^*\right]$.

1.3.5 APARCH model

The last model with leverage effect considered in this chapter is the APARCH that specifies the conditional variance as follows

$$\sigma_t^\lambda = \omega + \alpha_A \left(|y_{t-1}| - \delta_A y_{t-1}\right)^\lambda + \beta \sigma_{t-1}^\lambda. \tag{1.3.27}$$

The restrictions for the positivity of σ_t^λ are given by Ding *et al.* (1993) as follows:

$$\omega > 0, \lambda \geq 0, -1 < \delta_A < 1, \alpha_A \geq 0 \text{ and } \beta \geq 0. \tag{1.3.28}$$

Furthermore, the condition for the existence of $E\left(\sigma_t^{\lambda}\right)$ is given by

$$\alpha_A E\left(|\varepsilon_t| - \delta_A \varepsilon_t\right)^{\lambda} + \beta < 1 \tag{1.3.29}$$

which depends on the error distribution. Ding *et al.* (1993) derive the expression of $E\left(|\varepsilon_t| - \delta_A \varepsilon_t\right)^{\lambda}$ for Gaussian errors and Karanasos and Kim (2006) obtain it for Student-ν, GED and Double exponential distributions.

He and Teräsvirta (1999b) derived the following condition for the existence of the expectation $E(\sigma_t^{2\lambda})$

$$\frac{\alpha_A^2}{2}[(1+\delta_A)^{2\lambda}+(1-\delta_A)^{2\lambda}]E\left(|\varepsilon_t|^{2\lambda}\right)+\alpha_A\beta[(1+\delta_A)^{\lambda}+(1-\delta_A)^{\lambda}]E\left(|\varepsilon_t|^{\lambda}\right)+\beta^2<1. \tag{1.3.30}$$

A summary of the contributions and formulas detailed in this section appears in Table 1.1 which reports the contribution of the different authors that have derived conditions for the existence of moments and closed expressions of the properties considered in this chapter. Additionally, Table 1.2 summarizes analytical expressions for the positivity, stationarity and finite fourth order moment of the asymmetric GARCH models considered.

1.4 Organization of this thesis

After this chapter, where we introduce the motivation of this research, illustrate the main properties of the time series of returns considered in the empirical applications of this thesis and summarize the statistical properties of the GARCH models with leverage effect, the rest of the thesis is organized as follows.

The first part of Chapter 2 is devoted to study how the conditions for the positivity of volatility, stationarity and finite kurtosis limit the domain of the parameters of the models and therefore the dynamic of the conditional variance. In Section 2.3 we analyze whether the restriction can be violated when looking at the estimated parameters of simulated series which are generated by models that satisfy all the conditions for the existence of moments. Section 2.4 contains an empirical application that illustrates the theoretical results explained in the chapter. The chapter ends in Section 2.5 summarizing the main conclusions.

We turn our attention in Chapter 3 to the finite sample properties of the moments when asymmetric GARCH models are considered to represent the conditional volatility. We base our analysis on Monte Carlo methodologies. Section 3.2 is focused on comparing the finite sample properties of the plug-in and sample moments when TGARCH specifications are considered. Section 3.3 extends the Monte Carlo experiments to QGARCH and EGARCH models. The illustration in Section 3.4 completes this chapter before ending in Section 3.5. with a summary of conclusions.

Finally, Chapter 4 contains the main conclusions of this thesis and describes several venues for further research.

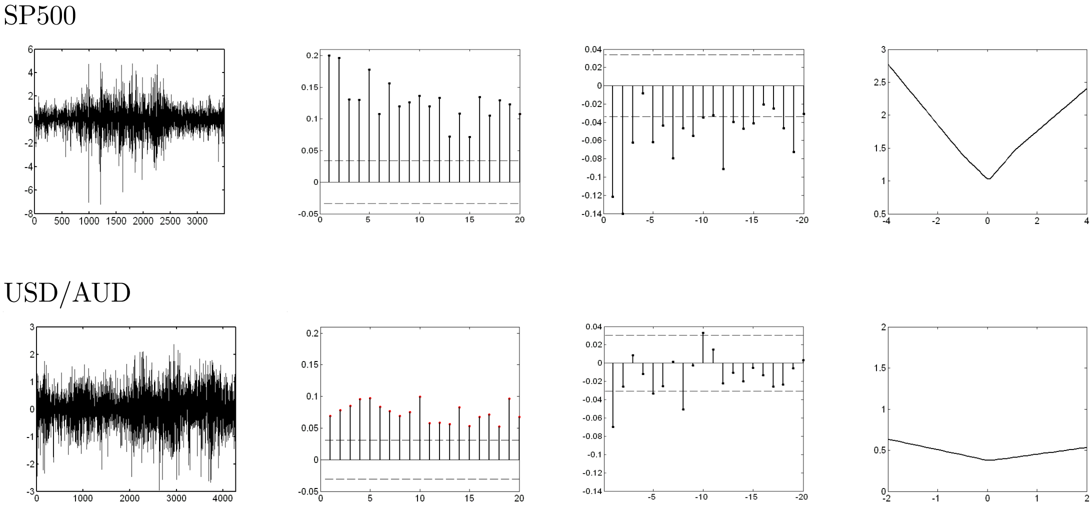


Figure 1.1: Daily returns, y_t, (first column), sample autocorrelations of y_t^2 (second column), cross-correlations between y_t and y_{t+1}^2 (third column) and PNP (fourth column) of SP500 (first row) and USD/AUD observed returns (second row).

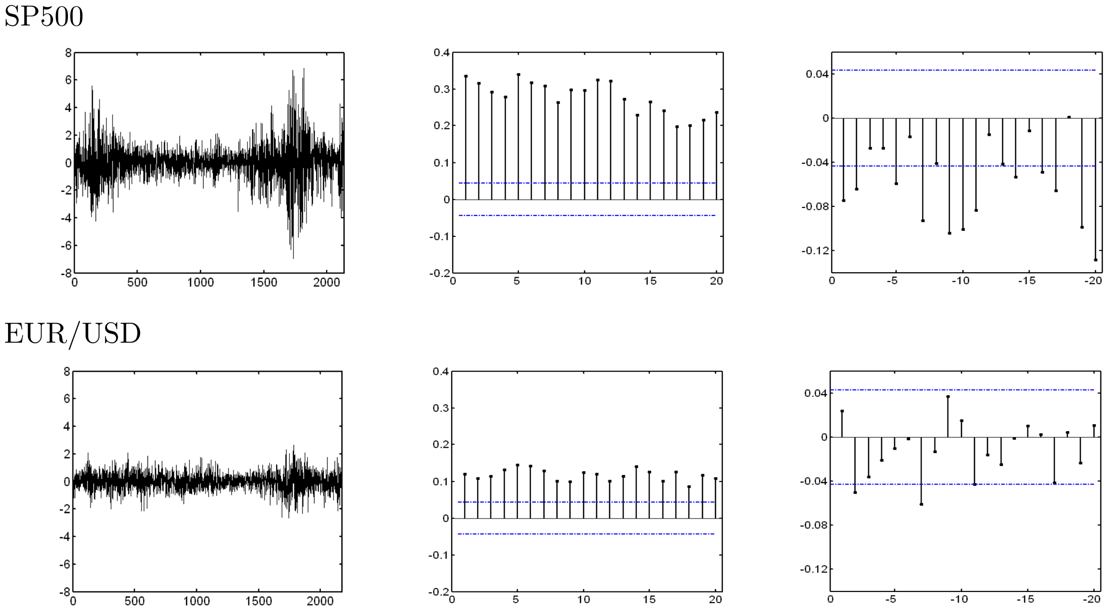


Figure 1.2: Daily returns, y_t, (first column), sample autocorrelations of y_t^2 (second column), cross-correlations between y_t and y_{t+1}^2 (third column) of SP500 (first row) and EUR/USD observed returns (second row).

	Positivity	Stationarity	Finite 4rd moment	F.A.C of squares	C.C.F	NIC
QGARCH	Sentana (1995)	Sentana (1995) He and Teräsvirta (1999a)	Sentana (1995) He and Teräsvirta (1999a)	Sentana (1995) He and Teräsvirta (1999a)	Sentana (1995) He et al. (2005)	Rodriguez and Ruiz (2008)
TGARCH	Hentschel (1995)	Hentschel (1995) He and Teräsvirta (1999a)	He and Teräsvirta (1999a) He et al. (2005)	He and Teräsvirta (1999a) Arvanitis and Demos (2004)	He et al. (2005)	Hentschel (1995)
GJR	Hentschel (1995)	Hentschel (1995)	He and Teräsvirta (1999a) He et al. (2005)	He and Teräsvirta (1999a)	He et al. (2005)	Engle and Ng (1993) Hentschel (1995)
EGARCH	Hentschel (1995)	Nelson (1991) Hentschel (1995) He et al. (2002)	Nelson (1991) He et al. (2002)	Nelson (1991) He et al. (2002) Demos (2002) Karanasos and Kim (2003)	Demos (2002) Karanasos and Kim (2003) Ruiz and Veiga (2008)	Engle and Ng (1993) Hentschel (1995)
APARCH	Hentschel (1995) Ding et al. (1993)	Hentschel (1995) Ding et al. (1993) Karanasos and Kim (2006)	He and Teräsvirta (1999b)	He and Teräsvirta (1999b)	He and Teräsvirta (1999b)	Hentschel (1995)

Table 1.1: Summary of the different authors that have derived positivity, stationarity, finite 4^{th} order moment, kurtosis, autocorrelation of squares, cross-correlation and NIC for the models considered in this chapter.

	Positivity	Stationarity	Fourth moment
QGARCH	$\delta_Q^{*2} < 4\alpha(1-p)$ $\alpha, \beta \geq 0$ $\omega > 0$	$\forall \delta_Q^{*2}$ $\alpha + \beta < 1$	$\forall \delta_Q^{*2}$ $(\kappa_\epsilon - 1)\,\alpha^2 + p^2 < 1$ $p = \alpha + \beta$
TGARCH	*Always*	$\delta_T^2 < 1 - \alpha^2 - \beta^2 - 2\alpha\beta\nu_1$	$(\alpha+\beta)^4 + \alpha^4(\kappa_\epsilon - 1) + \ldots$ $4\alpha\beta\left[\beta^2(\nu_1 - 1) + \alpha^2(\nu_3 - 1)\right] + \ldots$ $6\delta_T^2\left[2\alpha\beta\nu_3 + \alpha^2\kappa_\epsilon + \beta^2\right] < 1$
GJR	$\delta_G, \alpha, \beta \geq 0$ $\omega > 0$	$\delta_G < 2\,(1 - \alpha - \beta)$	$p^2 + \alpha\,(\kappa_\epsilon - 1)\,(\alpha + \delta_G) + 0.25\delta_G^2\,(2\kappa_\epsilon - 1) < 1$ $p = \alpha + \beta + 0.5\delta_G$
EGARCH	*Always*	$\beta < 1$	$\beta < 1$

Table 1.2: Restrictions for GARCH models with leverage effect.

Chapter 2

GARCH models with leverage effect

2.1 Introduction

In this chapter, we look at the ability of the alternative GARCH models described in Chapter 1 for representing the properties often observed in real time series of financial returns when they are restricted to satisfy the conditions that guarantee the positivity of volatilities and the covariance stationarity and finite kurtosis described in Chapter 1. Although these restrictions may unduly restrict the dynamics of the conditional variances that each of the models can represent, they are often imposed because of their interest from different points of view. For example, the asymptotic properties of many estimators usually implemented to estimate the parameters of heteroskedastic models often rely on the stationarity and/or on the existence of finite fourth order moments of returns; see, for example, Straumann and Mikosch (2006). There are also several important results on the properties of conditionally heteroskedastic models that require finite kurtosis of returns; see, for example, Drost and Nijman (1993) and Meddahi and Renault (2004) for properties under temporal aggregation of GARCH and square-root stochastic autoregressive volatility (SR-SARV) models respectively. Furthermore, it is obviously desirable that the model fitted to estimate the evolution of volatility guarantee the positivity of the estimated volatilities at all times. The positivity, stationarity and finite kurtosis restrictions are also important when predicting the future evolution of volatilities which is itself important for a wide range of financial applications; see, for example, Engle and Ng (1993) for a description of many of these applications. Therefore, in practice, one often looks for models that guarantee the positivity of the estimated conditional variances and simultaneously satisfy the stationarity and/or finite kurtosis conditions.

However, if the parameters are not adequately restricted, the positivity, stationarity and finite kurtosis restrictions are often violated when asymmetric GARCH models are fitted to real time series of financial returns. For example, the parameters of the GJR models estimated by Engle and Ng (1993), Henstchel (1995) and Engle (2003) do not satisfy the restriction for the existence of the kurtosis while Loudon *et al.* (2000) estimate TGARCH models that do not satisfy the restrictions for finite marginal variances. The violation of the restrictions can be interpreted in two different ways. First of all, it is possible that the corresponding moments are not truly defined in the returns series analyzed. However, there is also the possibility that the moments are well defined but the evolution of the volatility is such that cannot be explained by the restricted model; see Carnero *et al.* (2004) for this second possibility in the context of the symmetric GARCH(1,1) model.

The main objective of this chapter is to analyze how the restrictions imposed on the parameters to guarantee the positivity of volatilities and the covariance stationarity and finite kurtosis of returns limit the dynamics of the conditional variances that each of the models can represent. We show that in order to represent the asymmetry of volatility usually observed in real financial returns, it is not unusual that the estimated parameters of the GJR model do not satisfy the restriction for finite fourth order moment. However, this should not be in general interpreted as the series having infinite kurtosis. The parameters of the GJR model and, consequently, the volatility dynamics have to be heavily restricted to guarantee the existence of the kurtosis. Therefore, it could be expected that, in order to represent real volatilities, the finite fourth order restriction is often violated in practice. The restriction imposed on the TGARCH model to have finite fourth order moment also limits the dynamics that this model can represent but to a lesser degree. On the other hand, the asymmetry that the QGARCH model is able to represent can also be heavily restricted when the parameters satisfy the positivity restriction. This is a very serious drawback of the QGARCH model as one always looks for positive volatilities. The APARCH model is apparently more flexible although there are not general conditions for the stationarity and finite fourth order moment. Finally, even if one should be careful with the error distribution assumed in the EGARCH model, this model is the most flexible among the models considered.

In this chapter, we also analyze with simulated data whether the restrictions can be violated when looking at estimated parameters, even when they are truly satisfied. We show that when the parameters are estimated, one can often conclude that the moments do not exist even when the data is generated by Data Generating Processes (DGP) with well defined moments.

Finally, the results are illustrated by estimating the volatility of two real time series of daily financial returns, namely returns of the SP500 index and of the exchange rate of the Australian Dollar against the Dollar which have been described in Chapter 1. Taking into account that usually the final goal when fitting a conditionally heteroskedastic model is to estimate the underlying volatility of returns, we also look at the differences among the estimates of the conditional standard deviations generated by the models considered. We show that the TGARCH, EGARCH and APARCH estimates are rather similar. However, the conditional variances estimated by the QGARCH and GJR models differ between them and with respect to the other three specifications.

The rest of this chapter is organized as follows. Section 2.2 analyses the flexibility of the five models with leverage effect considered when their parameters are restricted to satisfy the positivity, stationarity and finite kurtosis conditions. In Section 2.3, we analyze with simulated data, the implications on the positivity of volatilities and the stationarity and finite kurtosis conditions when they are checked by looking at estimated parameters. With this goal, we carry out Monte Carlo experiments by fitting all the models considered to series generated by each of the other models. Section 2.4 illustrates the results by analyzing daily financial returns of the SP500 index and of the Australian-Dollar against the Dollar. Finally, Section 2.5 summarizes the main conclusions and gives some guidelines for future research.

2.2 Restrictions in GARCH-type models with leverage effect

In this section, we analyze the restrictions on the dynamics of the underlying volatilities imposed by each of the asymmetric GARCH models described in Chapter 1 when they are restricted to satisfy the positivity, stationary and finite fourth order moment parameter restrictions summarized in Section 1.3. In particular, we focus on the ability of each of the models considered to represent the combinations of kurtosis, acf of squares and cross-correlations between returns and future squared returns often observed in real returns.

2.2.1 QGARCH model

In the context of the QGARCH model defined in (1.3.2) one can observe that the stationarity of the model does not depend on the asymmetry parameter, δ_Q, and it is only determined by $\alpha+\beta$. Furthermore, when the model is stationary, the positivity restriction in (1.3.3) limits the maximum asymmetry that the model can represent. More explicitly, this restriction implies that, for fixed α, the maximum absolute asymmetry parameter decreases as β and, consequently, the persistence, increases. This result is illustrated in Figure 2.1 that plots the maximum values of $|\delta_Q^*|$ that satisfy the positivity restriction as a function of α and β when $\alpha+\beta<1$. On the other hand, for fixed β, the maximum absolute value of δ_Q^* increases with α when $\alpha<0.5(1-\beta)$ and decreases otherwise. Finally, in expression (1.3.3), it is easy to observe that if the persistence, $\alpha+\beta$, is fixed, the maximum value of $\left|\delta_Q^*\right|$ increases with α. Note that, for the parameter values usually encountered in practice, i.e. small α and large β, Figure 2.1 shows that the maximum absolute value of δ_Q^* and, consequently, the leverage effect that the QGARCH model is able to represent, is very small. Therefore, regardless of the error distribution, the positivity restriction of the QGARCH model can unduly restrict the dynamics of the conditional variance.

In the case of the restriction for the existence of the fourth order moment in (1.3.4), it does not depend on the asymmetry parameter. However, note that the restrictions imposed on α and β for the fourth order moment of returns to exist are stronger as the kurtosis of ε_t increases. Figure 2.2 shows how the parameter space of α and β reduces under different error distributions. In particular, we plot the parameter space that guarantees finite fourth order moment for Gaussian, Student-7 and Student-5 errors.

With respect to the autocorrelations (1.3.5), they decay exponentially with parameter p as in the symmetric GARCH model; see Sentana (1995). Therefore, the rate of decay of the acf of y_t^2 does not depend on the asymmetry parameter, δ_Q^*. The presence of the leverage effect only affects the first order autocorrelation which is larger than in the corresponding symmetric model. However, remember that if one wants to guarantee the positivity of σ_t^2 with the persistence, p, being close to one and α close to zero, the maximum absolute value of $|\delta_Q^*|$ is very small. Consequently, in this case, the effect of the leverage effect on the autocorrelations of squares is negligible. As an illustration, the first panel of Figure 2.3 plots the acf of squares of a QGARCH model with Gaussian errors and parameters $\omega=0.03$, $\alpha=0.1$, $\beta=0.87$ and $\delta_Q^*=-0.109$. These values have been chosen to resemble the values usually estimated when QGARCH models are fitted to real time series of financial returns;

see, for example, Engle and Ng (1993) and Henstchel (1995). In any case, note that the asymmetry parameter, δ_Q^*, has been chosen at its maximum absolute value to guarantee the positivity of σ_t^2 and the constant, ω, has a value such that the marginal variance of returns is one. In this model, the persistence is, as usually observed in practice, rather large, 0.98, while the kurtosis of returns is 5.44. Figure 2.3, which also plots the acf of the corresponding symmetric model with $\delta_Q^* = 0$, shows that, for the parameter values considered, the autocorrelations of squares are nearly indistinguishable in the QGARCH model with leverage effect with respect to the corresponding symmetric model.

When analyzing the expression of the cross-correlations in (1.3.6), we can observe that their magnitude decreases as the persistence of the volatility increases because, as we mentioned above, in this case, δ_Q^* has to decrease to guarantee the positivity of σ_t^2. Figure 2.3, which plots the cross-correlations of the same model considered above, illustrates that, in the cases of interest from the empirical point of view, in which the persistence is rather large, the QGARCH model generates very small cross-correlations between returns and future squared returns.

Finally, note that the NIC of the QGARCH model defined in (1.3.7) is a parabola shifted to the right by a distance $\frac{\delta_Q}{2\alpha}$ with respect to the corresponding symmetric GARCH model with $\delta_Q = 0$. Figure 2.3 plots the NIC of the QGARCH model chosen with illustrative purposes together with that of the corresponding symmetric model[1].

2.2.2 TGARCH model

From the definition of the TGARCH model in (1.3.8) one can observe that there are not necessary restrictions to guarantee the positivity of σ_t^2. However, the parameters of the TGARCH model have to be restricted to guarantee stationarity and the existence of the fourth order moment.

Starting with the stationarity condition in (1.3.9), Figure 2.5 represents the admissible values of $|\delta_T|$ that guarantee the stationarity of the TGARCH model with Gaussian errors as a function of positive α and β parameters. Note that although the stationarity of the TGARCH model depends on the distribution of ε_t, the values of ν_1 are not very different for the distributions of ε_t usually assumed in practice, namely the Gaussian and Student-

[1]See also Figure 2.4 which plots the autocorrelations of squares, cross-correlations and NICs of other QGARCH models with similar conclusions

ν distributions. For example, when ε_t has a Student-ν distribution with $\nu = 5$ degrees of freedom, $\nu_1 = 0.735$ while $\nu_1 = 0.798$ when ε_t is Gaussian. Consequently, given that the restrictions imposed on δ_T for the TGARCH model to be stationary are rather similar for these error distributions, Figure 2.5 only plots the stationarity restrictions for Gaussian errors. Observe that when β is close to one, the maximum asymmetry allowed is rather small. On the other hand, the asymmetry that the TGARCH model can represent decreases when α increases.

The parameters of the TGARCH model have to be further restricted to guarantee finite fourth order moment as it is clear from the expressions of d and f in (1.3.10). Given that when α and β are positive, as it is the case in the empirically relevant models, $f < 1$ implies that $d < 1$, the kurtosis is finite if

$$f < 1. \tag{2.2.1}$$

In expression (2.2.1) it is not obvious which is the relationship between the asymmetry, the parameters α and β and the error distribution. Therefore, Figure 2.5 also represents the values of the asymmetry parameter that guarantee finite kurtosis as a function of positive α and β when the errors are Gaussian and when they have Student-ν distributions with $\nu = 5$ and 7 degrees of freedom. Note that when β is close to 1, as it is often the case in many empirical applications, δ_T has to be very small. Consequently, the TGARCH model may have difficulties to represent simultaneously leverage effect with finite kurtosis and large persistence. Figure 2.5 also illustrates that for a fixed value of β, it is possible to increment the asymmetry that the TGARCH model is able to represent by decreasing the dependence of squared returns, i.e. by decreasing α. Finally, note that the restrictions imposed on $|\delta_T|$ become stronger as the degrees of freedom decrease. Therefore, as the errors are allowed to have more kurtosis, the leverage effect represented by the TGARCH model should be smaller and it loses even more flexibility. As an additional illustration, Figure 2.6 plots the parameter space for α and β that guarantee stationarity and finite kurtosis in the TGARCH model with Gaussian errors when $\delta_T = 0$ and $\delta_T = 0.15$.

The autocorrelations of squares of the TGARCH model, defined in (1.3.11) are represented in the second row of Figure 2.3 for a TGARCH model with Gaussian errors and parameters $\omega = 0.057$, $\alpha = 0.14$, $\beta = 0.825$ and $\delta_T = -0.12$. Once more, the parameter ω has been chosen in such a way that the marginal variance of returns is one and the asymmetry parameter has its maximum value to guarantee the existence of the kurtosis of y_t. The persistence is 0.88 smaller than that of the QGARCH model chosen in the previous subsection. When choosing models with larger persistence of shocks to the variance, they often violate the restriction of existence of the kurtosis for sensible values of the parameters. Finally, note that the kurtosis of returns is rather large, 10.10. For comparison shake, Figure 2.3 also plots the autocorrelations of squares of the corresponding symmetric models. Observe that in the TGARCH model, the presence of asymmetries could generate large differences in these autocorrelations.

With respect to the cross-correlations in expression in (1.3.12), Figure 2.3 plots them for the same TGARCH model considered above. These cross-correlations are rather large when compared with those of the QGARCH model. They decay exponentially. Therefore, it seems that although the TGARCH model has difficulties to represent series with finite kurtosis and persistent shocks to volatility, the restrictions imposed on its asymmetry parameter to guarantee finite kurtosis are milder than those imposed on the QGARCH model to guarantee positive conditional variances.

Finally, note that the expression of the NIC of the TGARCH model in equation (1.3.13) permits a rotation of the NIC but it does not allow for a shift. Figure 2.3 plots the NIC of the TGARCH model described above and that of the corresponding symmetric model. Once more, we can observe that the asymmetry that the TGARCH model can represent is stronger than that of the QGARCH model[2].

2.2.3 GJR model

The GJR model defined in (1.3.14) is similar to the TGARCH model but the volatility is specified in terms of σ_t^2 instead of σ_t. Consequently it is necessary to restrict its parameters in order to ensure positivity. Furthermore, the GJR model also has to be restricted to guarantee stationarity. The expressions for the corresponding conditions can be found in (1.3.15) and (1.3.16), respectively. Note that the positivity and stationarity do not depend on the conditional distribution of ε_t.

Figure 2.8, that plots the parameter space of the GJR model that guarantees stationarity, illustrates that, as in the TGARCH model, the maximum value of δ_G decreases with α and β.

[2]Figure 2.7 illustrates the autocorrelations of squares, cross-correlations and NICs of additional TGARCH models.

For small values of α and values of β close to one, the maximum values of δ_G that guarantee the stationarity are rather small.

When analyzing the condition for the existence of the fourth order moment, one can realize that the existence of this moment depends on the distribution of ε_t. For fixed values of α and β, larger kurtosis of ε_t imply more restrictive conditions on δ_G. Figure 2.8 also plots the maximum allowed values of δ_G that guarantee finite kurtosis of y_t as a function of the parameters α and β when ε_t has Normal, Student-5 and Student-7 distributions. Note that, as expected given the close relationship between the TGARCH and GJR models, the shapes of these surfaces are very similar to those plotted in Figure 2.5 for the TGARCH model. Finally note that the restriction in (1.3.17) implies that the GJR model has also difficulties to represent high persistence together with finite kurtosis.

The third row of Figure 2.3 plots the acf of y_t^2 of a GJR model given in expression in (1.3.18). We have considered a model with Gaussian errors and parameters $\omega = 0.035$, $\alpha = 0.1$, $\beta = 0.83$ and $\delta_G = 0.07$ which have been chosen to resemble the parameter estimates obtained when the GJR model is fitted to real time series of returns; see, for example, Engle (2003) and Chen *et al.* (2006). Once more, the constant, ω, has been chosen in such a way that the marginal variance is one and the asymmetry parameter has the maximum allowed value to guarantee finite kurtosis. The persistence is 0.965 and the kurtosis is 7.20. Note that the differences between the acf of squares in the GJR model and in the corresponding symmetric model are even larger than those observed in the TGARCH model described above.

The cross-correlation function of the GJR model defined in (1.3.19) has the particularity that γ_3 has to be computed by simulation as suggested by He *et al.* (2008)[3]. The corresponding cross-correlations, plotted in Figure 2.3 for the same GJR model considered above, have an exponential decay similar to that observed in the QGARCH and TGARCH models. The magnitudes of the cross-correlations are very small, in line with the values shown by the QGARCH model and far from the TGARCH case. However, note that the cross-correlations of the GJR model are less reliable as they are based on simulated moments.

The NIC in (1.3.20) is quadratic and has different slopes on either side of the origin. Given that δ_G has to be positive to guarantee positive conditional variances, the NIC has a steeper slope in its negative side than in its positive side. Figure 2.3 illustrates the shape

[3] For each model, 1500 series of size 5000 are generated. For each series the median of σ_t^3 is computed. Then, the median of $MED(\sigma_t^3)$ is computed through replicates.

of the NIC in (1.3.20) by plotting it for the particular model described above. This plot illustrates that the asymmetry that the GJR model is able to represent when it is restricted to have finite kurtosis is rather small. Unless the shocks are very large, the effect of positive and negative past returns is nearly the same[4].

2.2.4 EGARCH model

The EGARCH model, proposed by Nelson (1991) does not need any restriction on the parameters to guarantee the positivity of the variance; see expression in (1.3.21). The stationarity condition depends on the error distribution considered, as explained in Section 1.3.4, and it guarantees the existence of the kurtosis.

Using the expression in (1.3.24), the last row of Figure 2.3 plots the acf of squares for a Gaussian EGARCH model with parameters $\omega = -0.003$, $\alpha = 0.12$, $\beta = 0.99$ and $\delta_E = -0.08$. As we mentioned above, in the EGARCH model with Gaussian errors, the asymmetry parameter is not restricted to guarantee positivity, stationarity or finite kurtosis and, therefore, we have chosen it to resemble the values often estimated with real data; see Engle and Ng (1993). The marginal variance is equal to one, the kurtosis is 5.84 and the persistence is 0.99. Figure 2.3 also plots the acf of squares of the corresponding symmetric model. This figure illustrates that the EGARCH model can generate first order autocorrelations clearly larger than in the corresponding symmetric model.

Figure 2.3 also plots the cross-correlation function in equation (1.3.25) of the EGARCH model described before. The exponential decay is similar to that observed in QGARCH, TGARCH or GJR models.

Finally, consider the NIC in (1.3.26). It allows an asymmetric response of volatility to negative and positive returns. Furthermore, because the exponential curve eventually dominates the quadratic, in the EGARCH model, big returns have a greater impact on volatility than in the symmetric GARCH models. As the TGARCH model, the EGARCH model does not allow for a shift of the NIC. Figure 2.3, which plots the NIC of the EGARCH model considered before, shows that the effect of positive and negative past returns of the same magnitude can be clearly different. Also note the similarity of the NICs of the TGARCH and EGARCH models. Even more, comparing the autocorrelations of squares, the cross-

[4]Figure 2.9 illustrates the shapes of the autocorrelations of squares, cross-correlations and NICs of alternative GJR models.

correlations and the NICs of the TGARCH and EGARCH models, it is rather clear the similarity between these two models[5].

2.2.5 APARCH model

One of the main attractiveness of the APARCH model described in (1.3.27) is that it nests some of the GARCH models with leverage effect described before. For example, when $\lambda = 1$ and $\delta_T = \alpha_A \delta_A$, we obtain the TGARCH model in (1.3.8) while when $\lambda = 2$, and $\delta_G = 4\alpha_A \delta_A$ and $\alpha = \alpha_A(1 - \delta_A)$, the GJR model in (1.3.14) is obtained. Consequently, the APARCH model has been extensively implemented in the empirical analysis of financial returns. However, one of its limitations is that the conditions for the positivity of σ_t^2 and the stationarity and finite kurtosis of returns are unknown in the general case. Therefore, one cannot know the empirical properties that the APARCH model is able to represent. Nevertheless, for completeness, next we describe the available results on the properties of the APARCH model.

When analyzing the restriction for the positivity of σ_t^λ, it is not clear whether it should be positive for all λ. Consider, for example, that $\lambda = 1$, as in the TGARCH model. In this case, σ_t^2 is always positive regardless of whether σ_t is positive or negative. Therefore (1.3.28) is a sufficient but not necessary condition for the positivity of the conditional variance.

Similarly, when facing to the expression in (1.3.29) one can realize that this condition is sufficient for stationary when $\lambda \geq 2$, otherwise it guarantees the existence of moments of order $\lambda < 2$ but it does not necessarily imply the existence of the marginal variance. For example, consider, once more, $\lambda = 1$, then the restriction in equation (1.3.29) guarantees the existence of $E(\sigma_t)$ which is necessary but not sufficient for the existence of σ_y^2.

Again, condition (1.3.30) does not imply finite kurtosis for all λ. For example, if $\lambda = 1$, it only guarantees the existence of the variance as in the TGARCH model. However, when $\lambda = 2$, it reduces to $\beta^2 + \alpha_A^2(1 + 6\delta_A^2 + \delta_A^4) + 2\alpha_A\beta(1 + \delta_A^2) < 1$ which is the condition for finite kurtosis in the GJR model. Therefore, condition (1.3.30) is sufficient for stationarity when $\lambda \geq 1$ while it is sufficient for finite kurtosis when $\lambda \geq 2$. Consequently, it is not possible to carry out a comparative analysis of the maximum allowed values of δ_A when the model is stationary, has finite kurtosis and the conditional variances are positive for general

[5]Figure 2.10 plots the autocorrelations of squares, the cross-correlations and the NICs of the alternative EGARCH.

values of λ. In practice, λ is usually estimated between 1 and 2. In this case, the restriction in (1.3.30) guarantees the existence of the variance but not of the kurtosis. Therefore, we do not pursue the same kind of analysis carried out for the other four models considered in this chapter.

It is important to note that there are not closed form expressions of the variance and kurtosis of y_t. These expressions are only available when $\lambda = 1, 2$ and, in these cases, they coincide with the expressions given above for the TGARCH and GJR models respectively. Similarly, the autocorrelations of squares and cross-correlations are only available for these two particular cases.

2.3 The restrictions with estimated parameters.

In the previous section, we have analyzed the restrictions imposed on the parameter space of the QGARCH, TGARCH, GJR and EGARCH models in order to guarantee positivity of the conditional variances and stationarity and finite kurtosis of returns. However, in empirical applications, the true model is unknown and these restrictions are checked using estimated parameters. In this section, we analyze whether the conclusions obtained by checking the restrictions on estimated parameters are in concordance with the true existence of moments. The objective is twofold. First, for each model used as DGP, we assume that the true specification is fitted and the parameters are estimated by Maximum Likelihood (ML) by maximizing the likelihood corresponding to the assumed error distribution and by Quasi Maximum Likelihood (QML)[6] based on maximizing the Gaussian likelihood. The consistency of the QML estimator can be proved by using the results in Straumann and Mikosch (2006) who prove, as particular cases, the consistency of the QML estimator of the QGARCH and EGARCH models[7]. However, it is important to note that the asymptotic distributions of the ML and QML estimators of the EGARCH model are still unknown and

[6] The software for the ML and QML estimators has been developed by the first author in Matlab. The sample variance has been used as the initial value for the conditional variance; in any case, Straumann and Mikosch (2006) show that, as far as the models are stationary, the initial values for the conditional variances do not have any asymptotic effect.

[7] Deb (1996), using simulated data, suggests that the QML estimator of the EGARCH model based on maximizing the Gaussian likelihood when the true distribution of the errors is Student-11, may not even be consistent. However, the design used in the Monte Carlo experiments is not very realistic as it is based on ARCH parameters which are too large as to represent real data. Furthermore, the number of replicates is too small.

this open problem can be seen as one of the limitations of this model[8]. Once the parameters have been estimated, we check whether the restrictions of interest, which are known to be truly satisfied in the DGP, are satisfied when testing them on the estimated parameters.

Our second objective is to analyze the robustness of each fitted model when the data is generated by a different DGP. In this sense, we analyze the robustness of each model against misspecification of both the functional form of the conditional variance and the error distribution.

For these purposes, we generate $R = 1000$ series of size $T = 2000$ by each of the four GARCH models with leverage effect considered in the previous section and, additionally, by an APARCH model with $\lambda = 1.2$ and $\omega = 0.03$, $\alpha_A = 0.1$, $\beta = 0.8$ and $\delta_A = 0.05$. Furthermore, we generate the series by assuming not only a Gaussian distribution but also a standardized Student-7 and a Skewed-Student with parameters 7 and -0.15; see, for example, Giot and Laurent (2003) for a definition of the Skewed-Student distribution. Then, all five models, QGARCH, TGARCH, GJR, EGARCH and APARCH, are fitted to each series and the parameters are estimated by ML and QML. Finally, the restrictions are checked using the estimated parameters.

Table 2.1 reports the percentage of estimated models that satisfy the conditions for positivity, finite variance and kurtosis when the parameters are estimated by ML and the errors are Gaussian. Consider first the results obtained when the QGARCH model is fitted. The results in Table 2.1 show that the QGARCH model may have problems to satisfy the positivity restriction when the series are generated by the QGARCH and TGARCH models and to satisfy the finite kurtosis condition when they are generated by the GJR and EGARCH models. Remember that all the models used as DGP are such that the positivity restrictions are satisfied. However, when the DGP is the QGARCH and we fit it, only 50.6% of the estimated parameters satisfy this restriction. The situation is even worse when the DGP is the TGARCH model as, in this case, only 32.8% of the estimated QGARCH models satisfy the positivity restriction. The stationarity condition is almost always satisfied regardless of the DGP. However, when looking at the existence of the kurtosis, there are 14.9% and 8.2% of the estimated QGARCH that do not satisfy this condition when the series are generated by the GJR and EGARCH models, respectively.

[8]Straumann (2005) derive the asymptotic distribution of the ML estimator when $\beta = 0$. Alternatively, Pérez and Zaffaroni (2008) propose to use a Whittle estimator of the EGARCH model for which the asymptotic distribution is known and analyse its finite sample properties under the assumption of Gaussian errors.

Looking now at the results obtained when the TGARCH model is fitted, we can observe that both the stationarity and finite fourth order conditions are satisfied in nearly all cases. Only when the series are generated by the EGARCH model, there are 7.4% of the TGARCH models that do not satisfy the condition for finite kurtosis.

The next model fitted is the GJR model. In this case, there is a large percentage of series in which the estimated parameters do not satisfy the positivity condition mainly when the series are generated by the TGARCH, EGARCH and APARCH models. The stationarity condition is satisfied in nearly all cases. Only when the series are generated by the EGARCH model, there are 14.2% of the estimated GJR models that do not satisfy this condition. However, the percentages of estimated models that satisfy the condition for the existence of the kurtosis are very small. Note, for example, that only 19% of the GJR models fitted when the series are generated by the TGARCH model satisfy the condition for the existence of the kurtosis. Even when the series are generated by a GJR model with finite kurtosis, only 74.5% of the estimated GJR models satisfy the condition for the existence of the kurtosis.

On the other hand, the EGARCH estimates always satisfy the conditions for the existence of the kurtosis regardless of the DGP.

Finally, when fitting the APARCH model, the positivity restriction has been imposed[9]. Remember that conditions (1.3.29) and (1.3.30) are the stationarity conditions when $\lambda \geq 2$ and $\lambda \geq 1$, respectively. Therefore, we have computed the percentage of series that satisfy (1.3.29) when $\hat{\lambda} \geq 2$ and that satisfy (1.3.30) when $\hat{\lambda} \geq 1$. This is the quantity reported as the percentage of series that satisfy the stationarity condition in Table 2.1. On the other hand, (1.3.30) is the condition for the existence of kurtosis when $\lambda \geq 2$. Therefore, the percentage reported in Table 2.1 corresponds to the percentage of series that satisfy (1.3.30) among those in which $\hat{\lambda} \geq 2$. When the series are generated by the GJR and EGARCH models and the APARCH model is fitted, there is a large percentage of series in which the stationarity condition is not satisfied. This percentage is even larger when looking at the condition for finite kurtosis. In this case, with the exception of series generated by the own APARCH model, the APARCH estimates only satisfy the finite kurtosis restriction in very small percentage of series.

Table 2.2 reports the Monte Carlo results of the same experiment when the errors are Gaussian and $T = 5000$. We can observe that the conclusions are similar to those obtained from the results reported in Table 2.1 for $T = 2000$. Therefore, it seems that the problem is

[9] Without this restriction, the estimator did not converge in a large number of replicates.

not related with the precision when estimating the parameters but with the specification of the volatility assumed by each of the models. In order to analyse the effect of the distribution on the conclusions above, the series have also been generated with Student-7 and Skewed-Student-7 errors. Tables 2.3 and 2.4 report the Monte Carlo results when the parameters are estimated by ML and $T = 2000$[10]. The conclusions are similar to those obtained when the errors are Gaussian. Finally, the parameters of the models with Student-7 and Skewed-Student-7 errors have been estimated by QML by maximizing the Gaussian likelihood. Tables 2.5 and 2.6 reports respectively the results when $T = 2000$. Once more these tables illustrate that the results are similar regardless of the estimator implemented.

2.4 Empirical application

In this section, the five GARCH models with leverage effect previously described are fitted to represent the evolution of the volatility of the series of SP500 returns observed from January 3rd 1991 to July 31st 2007 and of the exchange rates USD/AUD form January 2nd 1990 to May 9th 2006 described in Section 1.2. For each model and series analyzed, we check whether the estimated parameters satisfy the positivity, finite variance and kurtosis restrictions. Furthermore, given that the final goal when fitting a conditionally heteroskedastic model is to obtain estimates of the underlying volatility, we compare the estimated volatilities obtained with the alternative models.

As we show in Section 1.2, both series have significant autocorrelations of squares and leverage effect (more pronounced in the SP500 returns). Therefore, we fit the QGARCH, TGARCH, GJR, EGARCH and APARCH models with Student-ν errors. Table 2.7 reports the estimation results for the SP500 returns[11]. The estimated degrees of freedom in the 5 models are approximately 8, so that it seems that the distribution of the standardized returns shows leptokurtosis. Furthermore, the estimated power parameter of the APARCH model is 1.280 which is closer to the TGARCH model than to the GJR formulation. In fact, the parameters estimated when the TGARCH model is fitted are almost identical to those estimated for the APARCH model[12]. In any case, the estimates of β are always very close

[10]In this illustration $\beta < 1$ is the stationarity and finite kurtosis restriction considered for the EGARCH models.

[11]The Appendix reports the results when the models are fitted assuming that the errors are Gaussian.

[12]Note that the asymmetry parameter of the TGARCH model is equal to the product of the α and δ_A parameters of the APARCH model.

to 1 while the estimates of α are small although significant. Finally, note that the estimates of the asymmetry parameter are also significant in all models.

After estimating the conditional deviations for each of the models, $\widehat{\sigma}_t$, the residuals are computed as $\hat{\varepsilon}_t = y_t/\widehat{\sigma}_t$. Table 2.7 also reports several diagnostics based on these residuals. For all the models fitted, the kurtosis have been clearly reduced with respect to the sample kurtosis of SP500 returns which is 6.88 although they are significantly different from the kurtosis implied by the estimated Student distribution when using a Kolmogorov-Smirnov test. Note, that the kurtosis implied by a Student-8 distribution is 4.5 so that the kurtosis of the residuals is only slightly larger. Furthermore, the autocorrelations of squared residuals are not significant. Therefore, it seems that all the fitted models have been successful in representing the dynamic evolution of the squares and the kurtosis observed in the SP500 returns.

Engle and Ng (1993) propose to test the adequacy of the specification of asymmetric GARCH models by implementing the sign bias, negative bias, positive bias and joint tests which are reported in Table 2.7 together with their corresponding standard deviations. We can observe that the conclusions for all models are the same and the four tests are significant. Therefore, all models fitted to represent the evolution of conditional variances of the SP500 returns fail to represent adequately the leverage effect and can be further improved.

Finally, Table 2.7 also reports whether the positivity and finite variance and kurtosis restrictions are satisfied. The estimates of the parameters of the QGARCH and GJR models are such that the positivity and stationarity conditions are not satisfied. Consequently, when looking at these estimated models, we may conclude that the marginal variance, kurtosis, autocorrelations of squares and cross-correlations are apparently not defined for SP500 returns. In the other two models, the TGARCH and the EGARCH, the estimated parameters satisfy the three conditions[13]. Finally, in the APARCH model, we can only check whether the stationarity condition is satisfied. Figure 2.11 plots the plug-in correlations of squares, cross-correlations and NIC obtained when the estimated parameters of the TGARCH and EGARCH models are substituted in the corresponding formula described in the previous section. When comparing first the three plug-in functions, we can observe that they are very close regardless of whether the model fitted is the TGARCH or the EGARCH. Furthermore, comparing the plug-in autocorrelations of squares and cross-correlations between returns and

[13]The stationarity conditions of the EGARCH model should be taken with caution as, as we mentioned above, they are unknown when the errors have a Student-ν distribution.

future squared returns with their corresponding sample counterparts plotted in Figure 1.1, we can observe that both are very close. Therefore, the estimated models represent adequately the autocorrelations of squares and cross-correlations. However, when comparing the plug-in NICs with the corresponding PNPs, we can see that the former have positive slopes which are too small when compared with those of the later. Therefore, it seems that the effect of positive returns is not adequately represented by the fitted models. This can explain why some of the adequacy test described above are rejected.

As mentioned above, the final goal when fitting conditionally heteroskedastic models is to obtain estimates of the underlying volatilities. The differences among the variances estimated by each of the models may be important when using them in financial applications as, for example, for option valuation models, rebalancing portfolios or measuring the risk by computing the Value at Risk (VaR). The main diagonal of Figure 2.12 plots estimates of the volatility obtained after fitting each of the five models considered to the SP500 returns. The lower triangle of this figure plots the differences between the volatilities estimates while its upper triangle plots scatter plots of the estimated volatilities taken two by two. The general shape of the estimated volatilities is similar. However, it is remarkable the similarity between the variances estimated by the TGARCH, EGARCH and APARCH models. The similarity between TGARCH and APARCH models could be expected given that, as we mentioned above, the parameter estimates in both models are very similar. On the other hand, the conditional variances estimated by the QGARCH and GJR models are different of the conditional variances estimated by any of the other alternative specifications and different between them. Note that, in general, when the volatility is large, conditional variances estimated by the QGARCH models are smaller than those estimated by any of the other models, whereas GJR models behaves in an opposite way[14].

The estimation results corresponding to the US Dollar/Australian Dollar exchange rates described in Section 1.2 are reported in Table 2.8. In this case, the estimated degrees of freedom are approximately 7 implying a leptokurtic distribution of the errors, regardless of the model specified to represent the evolution of the conditional variances[15]. Also, it is interesting to note that the asymmetry parameters are not significant in the QGARCH and GJR models and significant although rather small in the TGARCH and EGARCH models. This result could be expected given that exchange rates show smaller leverage

[14]Similar results can be observed for the Gaussian case represented in Figure 2.14 in the Appendix.
[15]The results under Gaussian errors are reported in Table 2.10 in the Appendix.

effects than equity indexes[16]. Again, the APARCH model suggests an specification closer to the TGARCH model, with the estimated power parameter being $\hat{\lambda} = 1.279$. Its parameters are also close to those estimated by the TGARCH model. As usual, all the estimates of α are small and those of β are close to 1. With respect to the diagnostics, the residuals have slightly smaller kurtosis than the returns which is 4.3. Note, that if the errors have a Student-7 distribution, their kurtosis is 5. Therefore, although the Kolmogorov-Smirnov test rejects that kurtosis of standardized residuals is equal to the kurtosis of the estimated Student distribution, this rejection is due to the fact that the former is smaller than the later. The autocorrelations of squared residuals and the cross-correlations of residuals are not significant. Furthermore, the sign and negative bias tests are not significant although the positive bias test is significant. Therefore, there is still room for improvement in the specification of the asymmetries.

The last part of Table 2.8 reports whether the restrictions for positivity and finite variance and kurtosis are satisfied. All the models satisfy the positivity restrictions. However, the QGARCH and GJR estimates do not satisfy the conditions for the existence of the kurtosis. Once more, by looking at these estimates we may conclude that the USD/AUD exchange rate returns do not have finite kurtosis. However, when looking at the estimates of the TGARCH and EGARCH models, the kurtosis seems to be perfectly defined.

Figure 2.11 plots the autocorrelations of squares, cross-correlations and NICs implied by the estimated TGARCH and EGARCH models. As in the case of the SP500 returns, the plug-in autocorrelations, cross-correlations and NIC are very similar regardless of whether the TGARCH or the EGARCH models are fitted. Furthermore, when comparing the plug-in functions with their sample counterparts, we can also observe that they are very close to each other.

Figure 2.13 plots the same quantities plotted in Figure 2.12 for the exchange rates volatilities. The conclusions about the relationships between the variances estimated by each of the models are very similar to those obtained from the plots in Figure 2.12 for the SP500 returns. Once more, we observe a great similarity between the conditional variances estimated by the TGARCH, EGARCH and APARCH models. In this case, there is also similarity between the QGARCH and GJR estimates.

[16]When the errors are assumed to be Gaussian, the estimated leverage effect parameters were significant. It seems that although the assumption on the error distribution does not affect the other parameters of the volatility, it has an effect on the estimated asymmetry parameter.

2.5 Conclusions

There is a large number of alternative GARCH models proposed in the financial econometrics literature to represent the dynamic evolution of volatilities with leverage effects. In this chapter, we compare the properties of five popular asymmetric GARCH models when they are restricted to guarantee positivity of conditional standard deviations, stationarity and existence of fourth order moments. In particular, we consider the QGARCH, TGARCH, GJR, EGARCH and APARCH models. We show that the QGARCH dynamics are heavily restricted when its parameters are restricted to guarantee positivity of the conditional variances. Consequently, the asymmetry that the QGARCH model is able to represent in practice is very limited. On the other hand, the TGARCH asymmetry parameter has to be restricted to guarantee stationarity and finite kurtosis. However, it seems that these restrictions do not impose strong limitations on the leverage effect as far as the persistence is not too high. With respect to the GJR model, we show that the leverage effect that this model can represent is strongly limited when the parameters are restricted to satisfy the finite kurtosis condition. The EGARCH model is more flexible although very similar to the TGARCH model. The conclusions about the strong limitations of the QGARCH and GJR models are illustrated by analyzing two time series of real daily returns. We show that when fitting these two models, one may conclude that the kurtosis is not finite while the conclusion is reversed when fitting the TGARCH and EGARCH models. Finally, in the two empirical examples considered, we also show that the conditional standard deviations estimated by the TGARCH, EGARCH and APARCH models are very similar. Therefore, if the objective is to estimate the underlying volatilities of a series of returns, choosing any of these three models seems to give the same answer.

There are many other alternative models proposed in the literature to represent the asymmetric response of volatilities to positive and negative returns. For example Brännäs and Gooijer (2004) propose an extension of the QGARCH model that allows more flexibility in the asymmetric response of volatility. However, it seems that in this model even the low order moments lack explicit analytical expressions. Also, Babsiri and Zakoïan (2001) propose a model that introduces contemporaneous asymmetry on returns and Wu and Xiao (2002) conclude that an extension of the EGARCH model with separate coefficients for large and small negative shocks is more adequate to capture the asymmetry effect than the standard EGARCH model. There are also several proposals to model the leverage effect in the context of Stochastic Volatility (SV) models. Harvey and Shephard (1996) propose to represent the asymmetric response of volatilities by introducing correlation between the level and volatility noises; see also Yu (2005) for further properties of this model. So *et al.* (2002) proposed a Threshold Stochastic Volatility model to represent simultaneously the mean and variance asymmetries. Demos (2002) has proposed a model that encompasses both the Asymmetric SV (A-SV) and the EGARCH models as particular cases[17]. Recently, Kawakatsu (2007) has proposed a new A-SV model that generalizes that proposed by Harvey and Shephard (1996). In this model, the log-volatility is a quadratic function of a latent variable. Zhang and King (2008) also propose a generalization of the SV model with leverage effect based on the Box-Cox transformation of the squared volatility. Finally, Meddahi and Renault (2004) propose a semiparametric volatility model, closely related to the SV model, that encompasses many of the asymmetric GARCH models considered in this chapter without imposing so many restrictions on the parameters. In general, the restrictions imposed on the parameters of SV models to guarantee existence of moments are less severe than those imposed in GARCH models. Comparing asymmetric GARCH and SV models is left for further research. Finally, it could be also of interest to extend the analysis to models with volatility effects in the mean as those proposed by He *et al.* (2008) or Arvanitis and Demos (2004).

[17]Carnero *et al.* (2004) show that the A-SV model is more flexible than the EGARCH model.

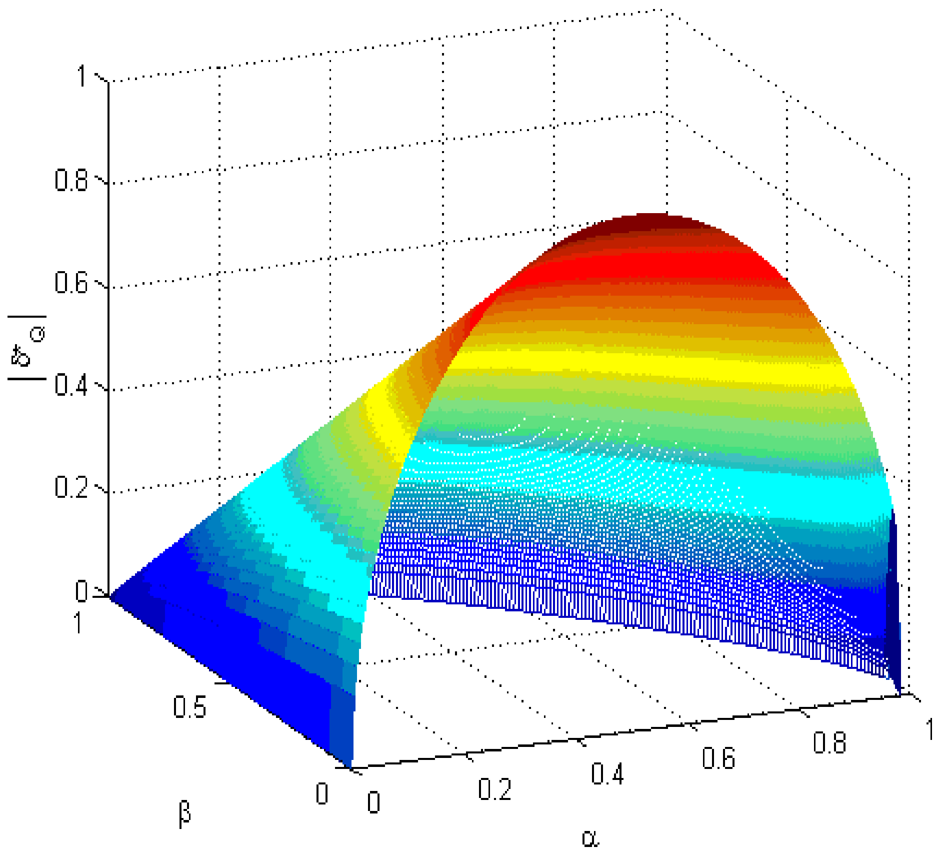


Figure 2.1: Parameter space for the stationary QGARCH model when the positivity restriction is satisfied.

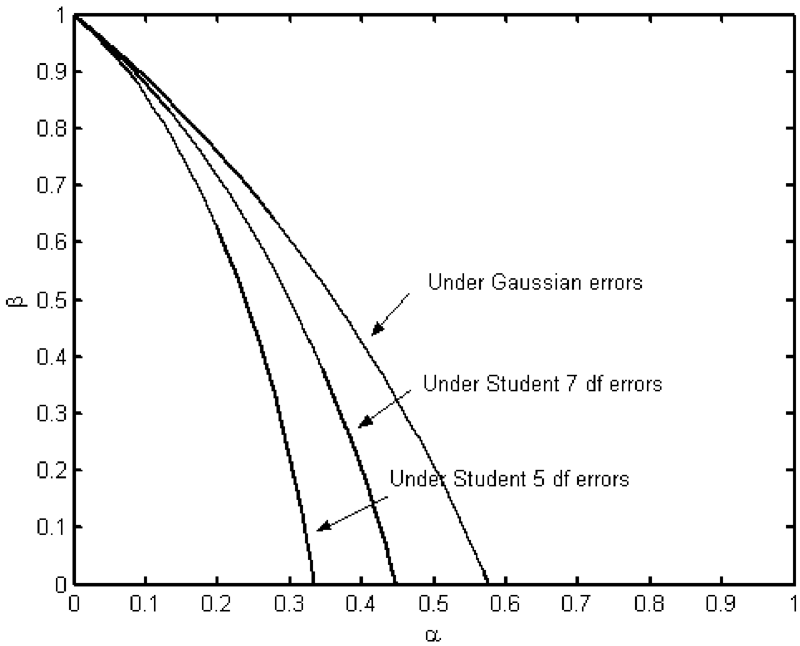


Figure 2.2: Parameter space for finite kurtosis in QGARCH models with Gaussian, Student-7 and Student-5 errors.

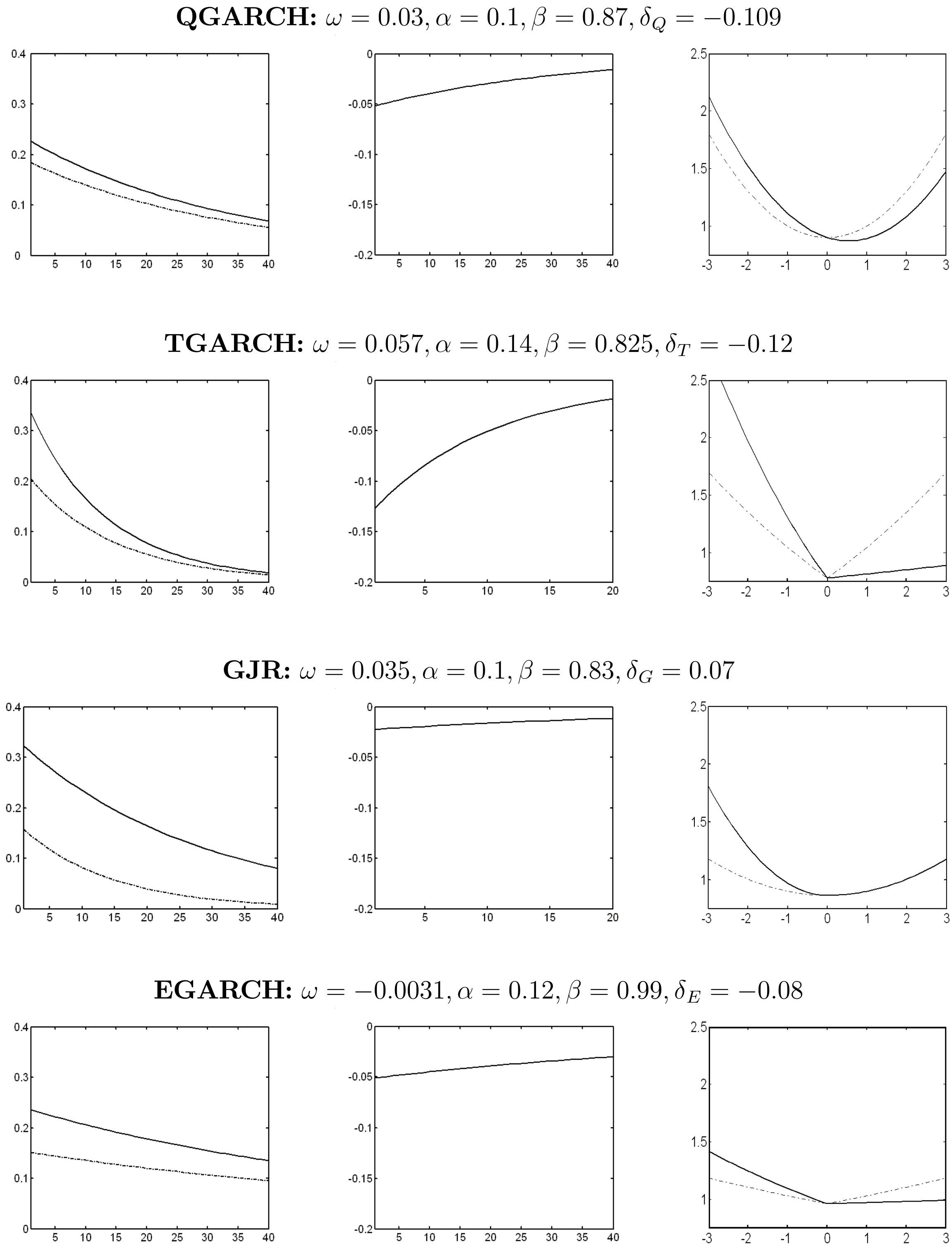


Figure 2.3: Autocorrelations of squares (first column), cross-correlations (second column) and NIC (third column) of different asymmetric GARCH models with Gaussian noises. The solid lines correspond to the functions when $\delta_{(\bullet)} \neq 0$ while the dashed lines are the corresponding functions when $\delta_{(\bullet)} = 0$.

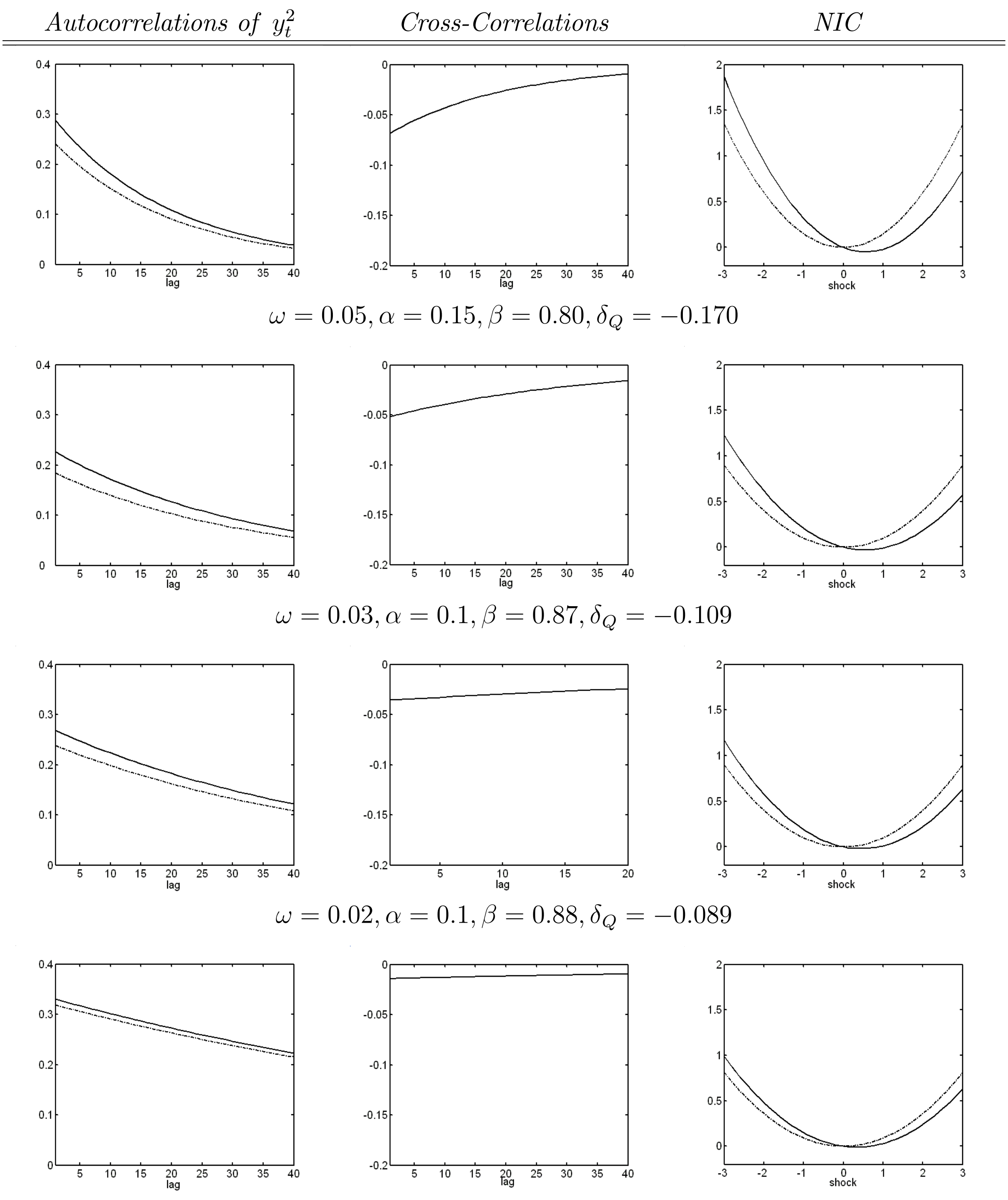


Figure 2.4: Autocorrelation function of squares, cross-correlation function and news impact curve for different QGARCH models with Gaussian noises. The solid lines correspond to the moments when $\delta_Q \neq 0$ while the dashed lines correspond to $\delta_Q = 0$.

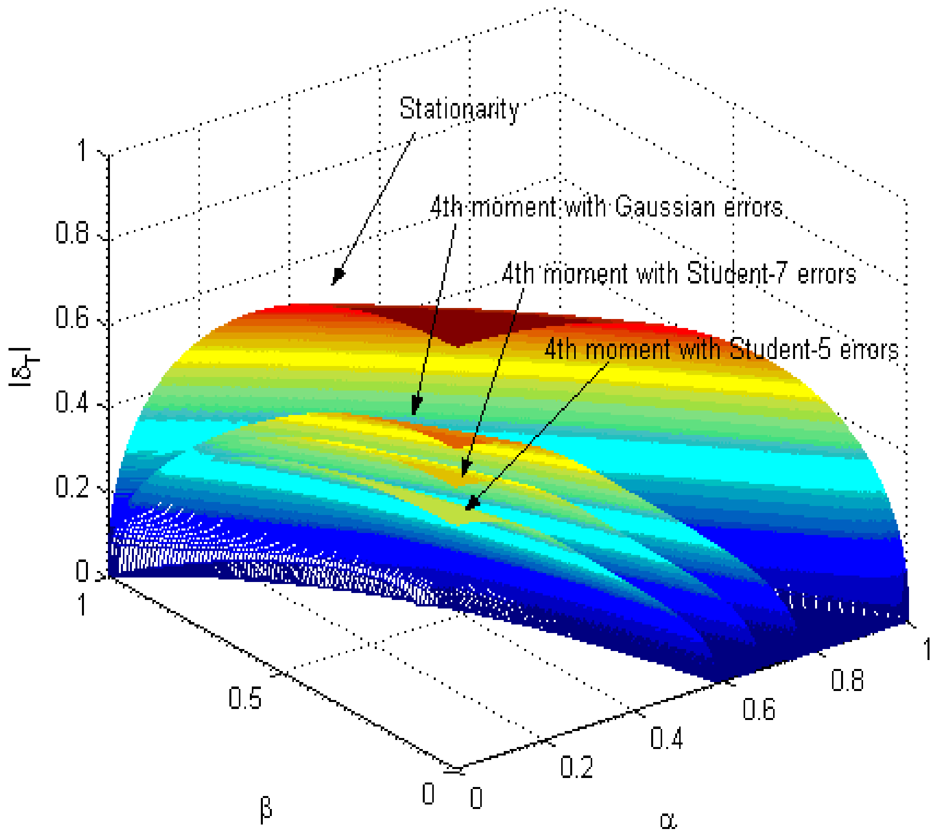


Figure 2.5: Admissible parameter space of the TGARCH model with different error distributions that satisfy the stationarity and finite fourth order moment restrictions.

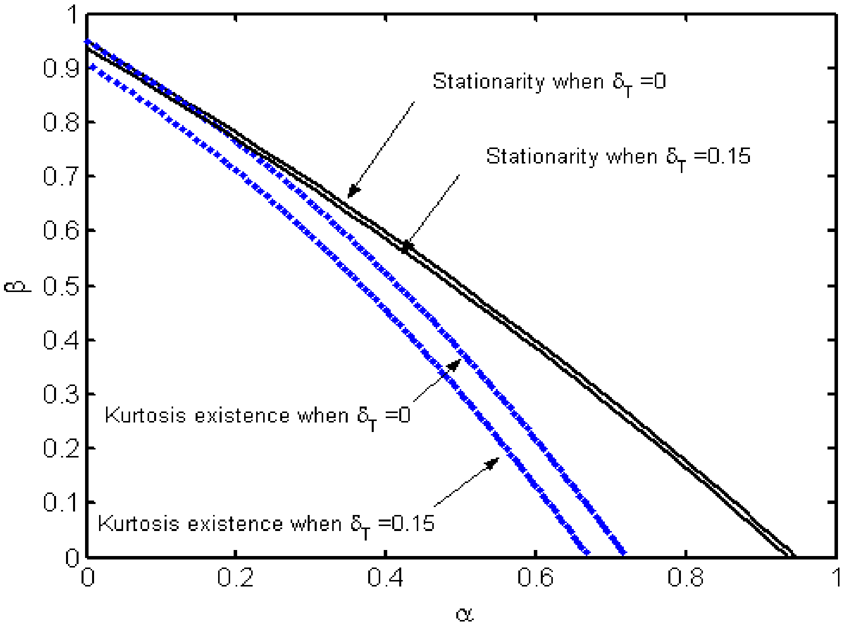


Figure 2.6: Parameter space for the parameters of TGARCH models with Gaussian errors and persistence $p = 0.90$ that satisfy the stationarity and finite fourth order moment restrictions.

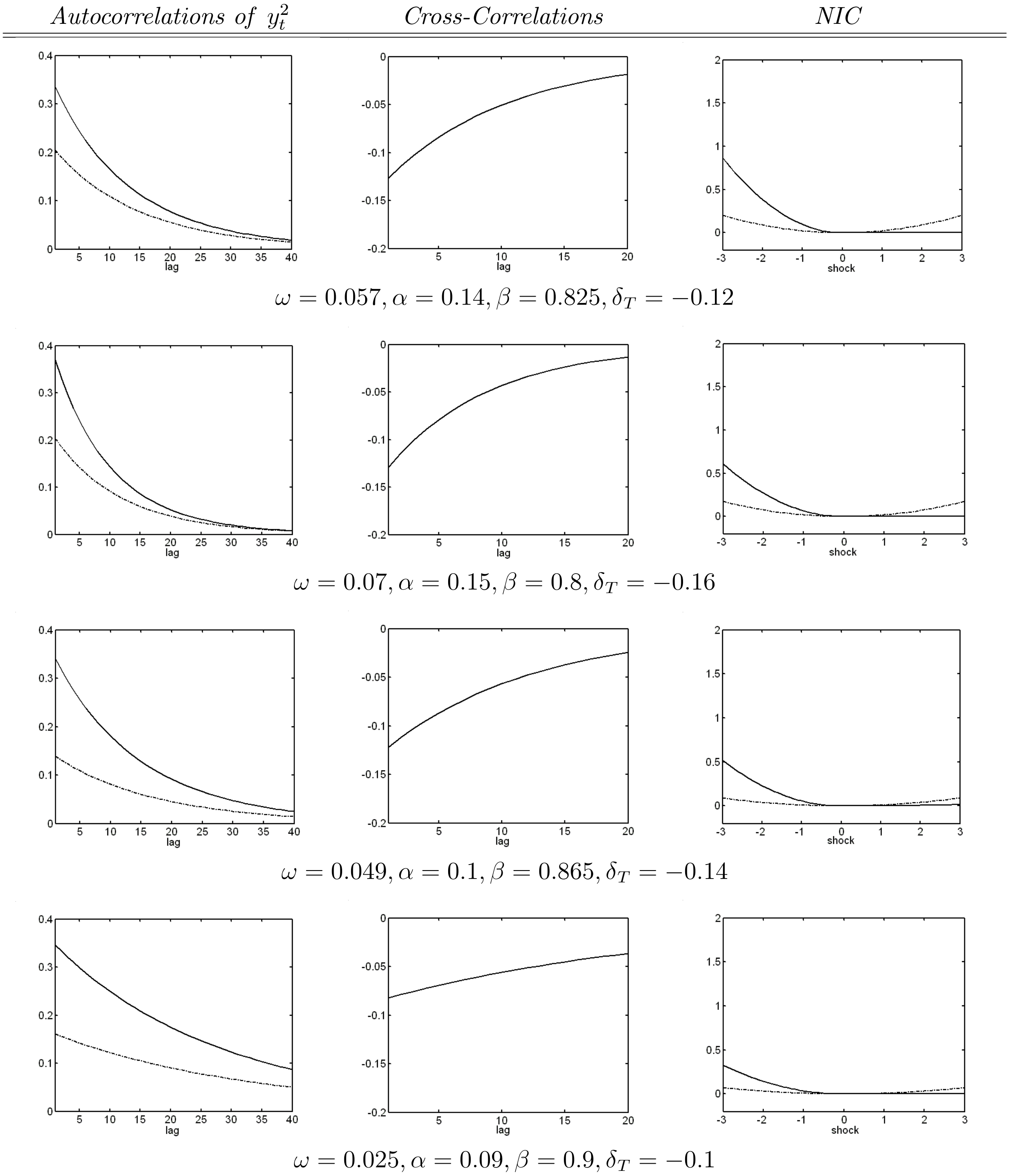


Figure 2.7: Autocorrelation function of squares, cross-correlation function and news impact curve for different TGARCH models with Gaussian noises. The solid lines correspond to the moments when $\delta_T \neq 0$ while the dashed lines correspond to $\delta_T = 0$.

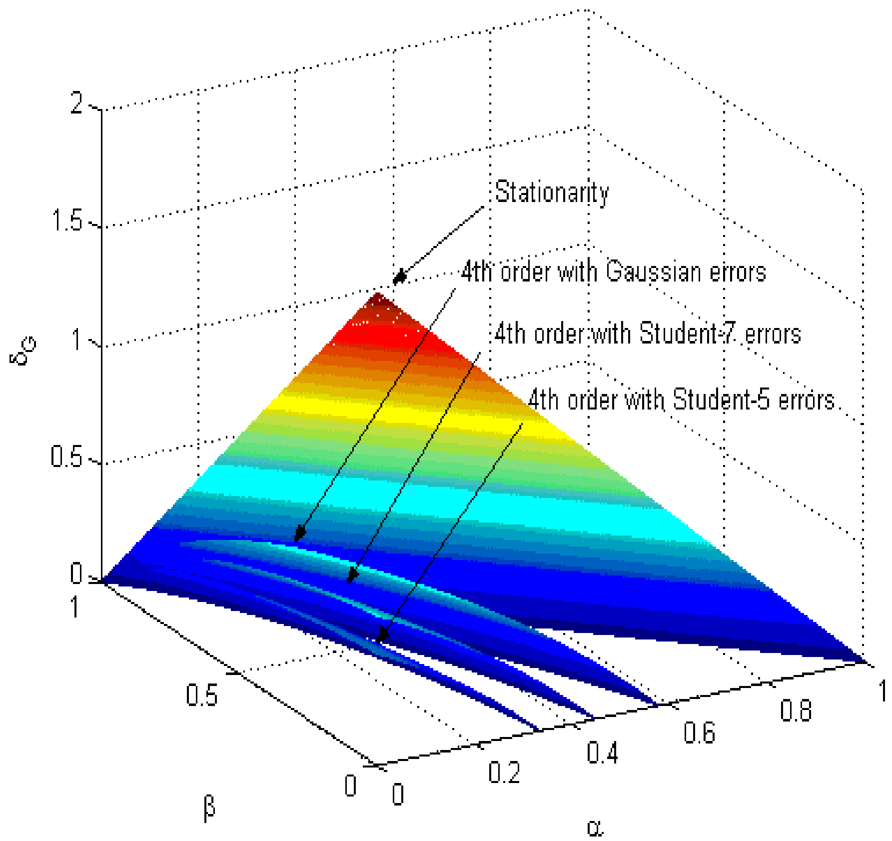


Figure 2.8: Admissible parameter space of the GJR model with different error distributions that satisfy the stationarity and finite fourth order moment restrictions.

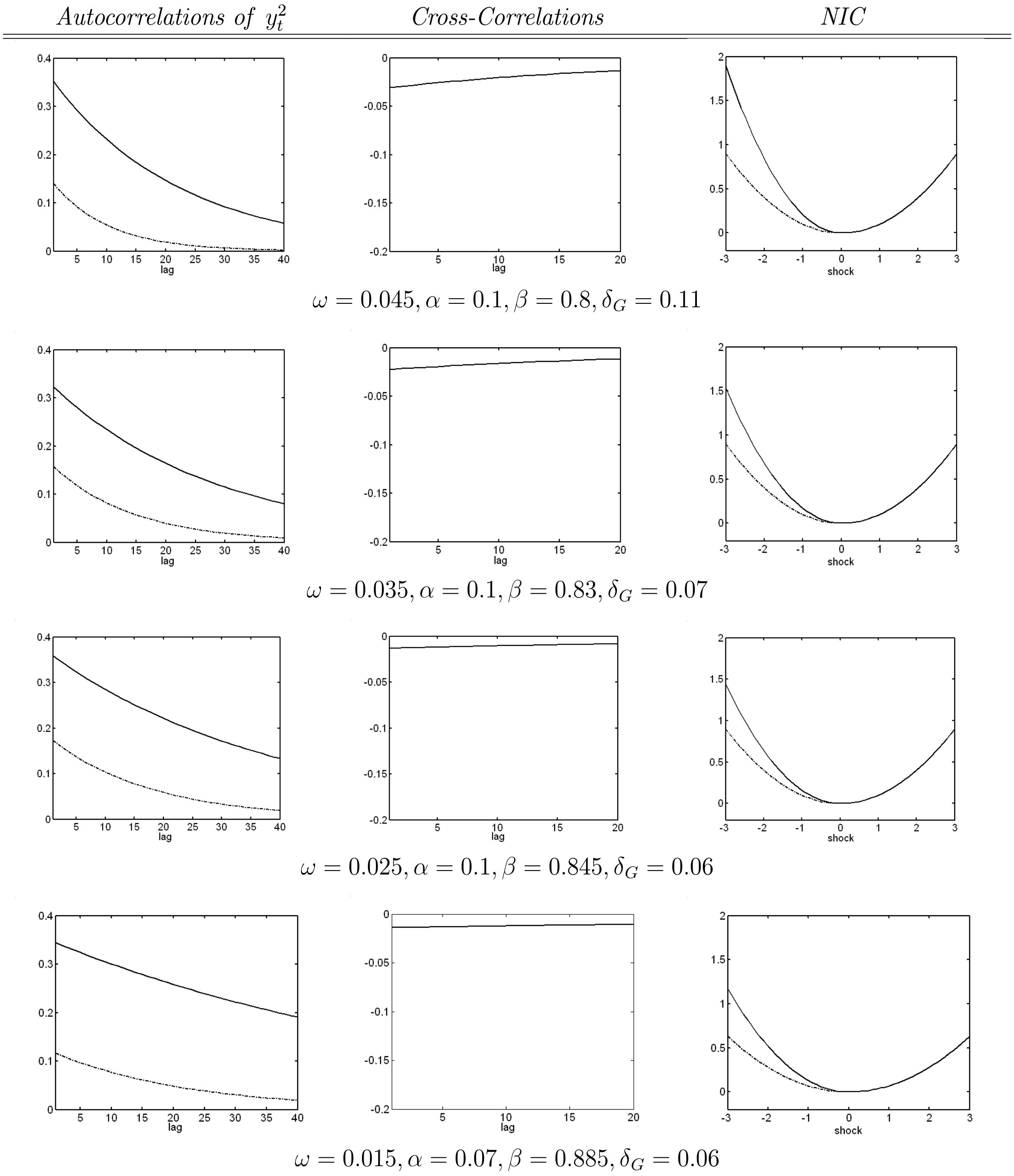
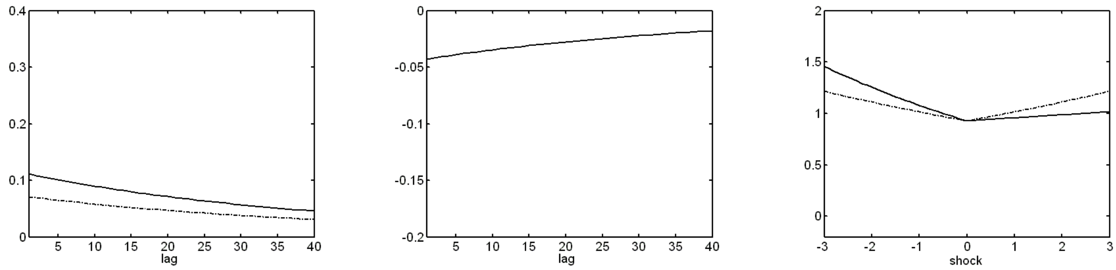
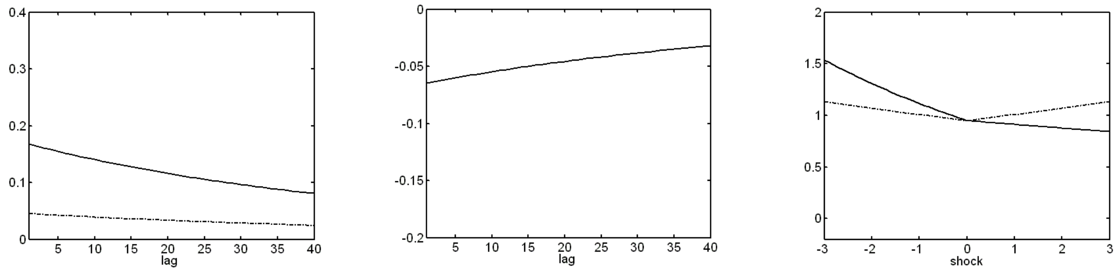


Figure 2.9: Autocorrelation function of squares, cross-correlation function and news impact curve for different GJR models with Gaussian noises. The solid lines correspond to the moments when $\delta_G \neq 0$ while the dashed line correspond to $\delta_G = 0$. Third order moments are computed numerically trough a MonteCarlo simulation.

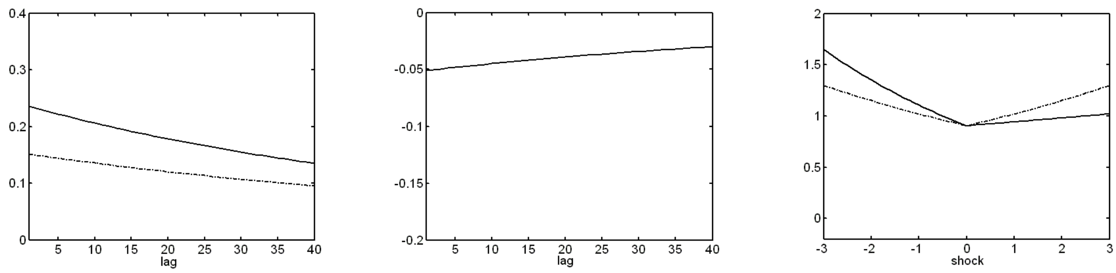
Autocorrelations of y_t^2	*Cross-Correlations*	*NIC*



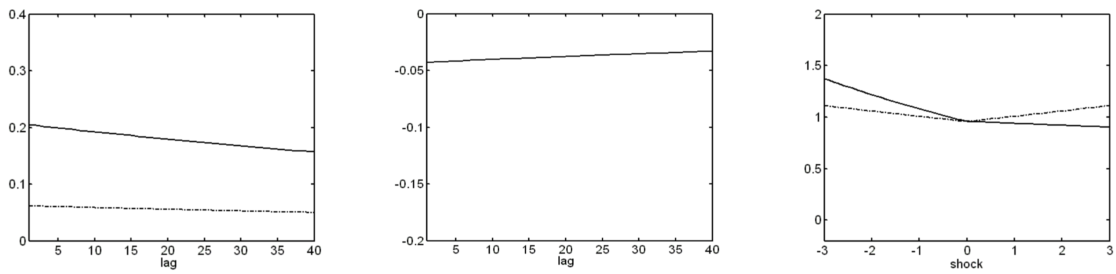
$\omega = -0.0017, \alpha = 0.09, \beta = 0.98, \delta_E = -0.06$



$\omega = -0.0029, \alpha = 0.06, \beta = 0.985, \delta_E = -0.1$



$\omega = -0.0031, \alpha = 0.12, \beta = 0.99, \delta_E = -0.08$



$\omega = -0.0015, \alpha = 0.05, \beta = 0.995, \delta_E = -0.07$

Figure 2.10: Autocorrelation function of squares, cross-correlation function and news impact curve for different EGARCH models with Gaussian noises. The solid lines correspond to the moments when $\delta_E \neq 0$ while the dashed lines correspond to $\delta_E = 0$.

Fitted	DGP	QGARCH	TGARCH	GJR	EGARCH	APARCH
QGARCH	$\sigma_t^2 > 0$	50.6	32.8	99.7	97.9	100.0
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	99.8	100.0
	$\kappa_y < \infty$	99.8	100.0	85.1	91.8	100.0
TGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100	100.0	99.8	99.6	100.0
	$\kappa_y < \infty$	100	100.0	99.4	92.6	100.0
GJR	$\sigma_t^2 > 0$	92.0	62.4	99.6	85.5	79.1
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	85.8	100.0
	$\kappa_y < \infty$	60.2	19.0	74.5	20.1	100.0
EGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	100.0	100.0
	$\kappa_y < \infty$	100.0	100.0	100.0	100.0	100.0
APARCH	$\sigma_t^2 > 0$	*Imposed*	*Imposed*	*Imposed*	*Imposed*	*Imposed*
	$\sigma_y^2 < \infty$	91.4	99.40	77.12	77.22	100
	$\kappa_y < \infty$	27.9	66.70	21.78	22.27	100

Table 2.1: Percentages of fitted Gaussian models that satisfy the restrictions for positivity and finite variance and kurtosis when $T = 2000$ and the parameters are estimated by ML.

Fitted	DGP	QGARCH	TGARCH	GJR	EGARCH	APARCH
QGARCH	$\sigma_t^2 > 0$	49.2	28.9	100	99.8	100.0
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	99.8	100.0
	$\kappa_y < \infty$	100.0	100.0	99.5	95.7	100.0
TGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	99.8	98.2	100.0
	$\kappa_y < \infty$	100.0	100.0	99.5	95.7	100.0
GJR	$\sigma_t^2 > 0$	60.1	72.2	99.8	98.3	88.7
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	91.5	100.0
	$\kappa_y < \infty$	63.4	10.0	85.3	5.0	100.0
EGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	100.0	100.0
	$\kappa_y < \infty$	100.0	100.0	100.0	100.0	100.0
APARCH	$\sigma_t^2 > 0$	*Imposed*	*Imposed*	*Imposed*	*Imposed*	*Imposed*
	$\sigma_y^2 < \infty$	90.8	98.9	74.8	72.3	100.0
	$\kappa_y < \infty$	25.4	61.5	18.5	20.1	100.0

Table 2.2: Percentages of fitted Gaussian models that satisfy the restrictions for positivity and finite variance and kurtosis when $T = 5000$ and the parameters are estimated by ML.

Fitted	DGP	QGARCH	TGARCH	GJR	EGARCH	APARCH
QGARCH	$\sigma_t^2 > 0$	49.7	25.5	100	97.1	100.0
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	98.6	100.0
	$\kappa_y < \infty$	98.1	91.3	100.0	68.8	100.0
TGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	99.8	100.0
	$\kappa_y < \infty$	99.4	97.4	100.0	86.9	100.0
GJR	$\sigma_t^2 > 0$	99.6	65.3	100.0	85.1	66.7
	$\sigma_y^2 < \infty$	99.7	99.2	100.0	80.2	100.0
	$\kappa_y < \infty$	18.1	1.0	0.1	9.1	100.0
EGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	100.0	100.0
	$\kappa_y < \infty$	100.0	100.0	100.0	100.0	100.0
APARCH	$\sigma_t^2 > 0$	*Imposed*	*Imposed*	*Imposed*	*Imposed*	*Imposed*
	$\sigma_y^2 < \infty$	90.0	99.4	99.7	96.2	97.6
	$\kappa_y < \infty$	22.0	0.0	0.9	7.1	86.6

Table 2.3: Percentages of fitted Student$-\nu$ models that satisfy the restrictions for positivity and finite variance and kurtosis when $T = 2000$ and series are generated under Student-7 errors.

Fitted	DGP	QGARCH	TGARCH	GJR	EGARCH	APARCH
QGARCH	$\sigma_t^2 > 0$	47.8	40.9	99.4	94.6	100.0
	$\sigma_y^2 < \infty$	100.0	100.0	99.0	96.6	100.0
	$\kappa_y < \infty$	81.3	85.7	35.5	58.8	100.0
TGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	99.6	99.9
	$\kappa_y < \infty$	98.9	96.4	95.5	92.6	99.9
GJR	$\sigma_t^2 > 0$	98.0	61.3	99.0	85.5	66.3
	$\sigma_y^2 < \infty$	100.0	100.0	98.9	95.0	100.0
	$\kappa_y < \infty$	28.6	1.0	34.1	22.1	100.0
EGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	100.0	100.0
	$\kappa_y < \infty$	100.0	100.0	100.0	100.0	100.0
APARCH	$\sigma_t^2 > 0$	*Imposed*	*Imposed*	*Imposed*	*Imposed*	*Imposed*
	$\sigma_y^2 < \infty$	99.8	96.5	52.0	99.7	100.0
	$\kappa_y < \infty$	29.2	0.0	22.7	5.0	80.2

Table 2.4: Percentages of fitted Skewed-Student$-\nu$ models that satisfy the restrictions for positivity and finite variance and kurtosis when $T = 2000$ and series are generate under Skewed-Student-7 errors.

Fitted	DGP	QGARCH	TGARCH	GJR	EGARCH	APARCH
QGARCH	$\sigma_t^2 > 0$	47.1	29.3	100	96.0	100.0
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	98.0	100.0
	$\kappa_y < \infty$	97.5	100.0	100.0	87.6	100.0
TGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	99.8	100.0	99.8	93.9	100.0
	$\kappa_y < \infty$	96.9	95.8	99.5	68.6	100.0
GJR	$\sigma_t^2 > 0$	99.4	60.8	100.0	81.6	66.0
	$\sigma_y^2 < \infty$	99.3	98.7	100.0	79.4	100.0
	$\kappa_y < \infty$	39.1	9.8	100.0	30.8	100.0
EGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	100.0	100.0
	$\kappa_y < \infty$	100.0	100.0	100.0	100.0	100.0
APARCH	$\sigma_t^2 > 0$	*Imposed*	*Imposed*	*Imposed*	*Imposed*	*Imposed*
	$\sigma_y^2 < \infty$	88.2	99.8	100.0	98.0	98.9
	$\kappa_y < \infty$	20.2	60.0	0	61.9	91.5

Table 2.5: Percentages of fitted Gaussian models that satisfy the restrictions for positivity and finite variance and kurtosis when $T = 2000$ and series are generated under Student-7 errors.

Fitted	DGP	QGARCH	TGARCH	GJR	EGARCH	APARCH
QGARCH	$\sigma_t^2 > 0$	50.7	48.1	99.3	86.8	100.0
	$\sigma_y^2 < \infty$	100.0	100.0	97.1	96.7	100.0
	$\kappa_y < \infty$	97.3	97.8	70.7	79.7	100.0
TGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	97.8	92.7	100.0
	$\kappa_y < \infty$	96.1	96.6	87.0	63.9	100.0
GJR	$\sigma_t^2 > 0$	97.0	58.4	97.6	83.1	66.2
	$\sigma_y^2 < \infty$	99.7	99.6	97.7	95.1	100.0
	$\kappa_y < \infty$	77.3	24.1	69.9	55.2	100.0
EGARCH	$\sigma_t^2 > 0$	*Always*	*Always*	*Always*	*Always*	*Always*
	$\sigma_y^2 < \infty$	100.0	100.0	100.0	100.0	100.0
	$\kappa_y < \infty$	100.0	100.0	100.0	100.0	100.0
APARCH	$\sigma_t^2 > 0$	*Imposed*	*Imposed*	*Imposed*	*Imposed*	*Imposed*
	$\sigma_y^2 < \infty$	93.0	98.5	79.8	93.9	98.8
	$\kappa_y < \infty$	43.0	45.5	27.5	44.9	91.4

Table 2.6: Percentages of fitted Gaussian models that satisfy the restrictions for positivity and finite variance and kurtosis when $T = 2000$ and series are generated under Skewed-Student-7 errors.

	QGARCH	TGARCH	GJR	EGARCH	APARCH
ω	0.014^* (0.000)	0.014^* (0.001)	0.009^* (0.001)	-0.085^* (0.001)	0.012^* (0.001)
α	0.055^* (0.001)	0.058^* (0.001)	-0.002 (0.008)	0.108^* (0.001)	0.050^* (0.001)
β	0.933^* (0.001)	0.942^* (0.001)	0.938^* (0.001)	0.985^* (0.001)	0.941^* (0.001)
δ	-0.075^* (0.001)	-0.049^* (0.006)	0.110^* (0.001)	-0.089^* (0.001)	-0.88^* (0.037)
λ					1.28^* (0.208)
ν	7.85	7.97	7.88	7.91	8.01
Residuals					
Mean	0.000	0.002	0.000	0.002	0.002
S.D.	0.998	0.998	0.999	0.998	0.999
Skewness	-0.437^*	-0.425^*	-0.474^*	-0.430^*	-0.444^*
Kurtosis	4.713^*	4.681^*	4.938^*	4.758^*	4.771^*
Jarque − Bera	536.2^*	516.1^*	676.7^*	557^*	570.9^*
$Q(20)$	23.56	24.25	24.12	24.87	24.45
$Q_2(20)$	10.55	9.40	11.50	10.88	9.35
$Q_{21}(20)$	45.80^*	44.36^*	45.27^*	43.84^*	44.26^*
Sign Bias	-0.212^* (0.065)	-0.201^* (0.065)	-0.223^* (0.067)	-0.201^* (0.065)	-0.208^* (0.065)
Negative Bias	-1.654^* (0.042)	-1.599^* (0.042)	-1.658^* (0.044)	-1.606^* (0.043)	-1.606^* (0.043)
Positive Bias	0.893^* (0.052)	0.878^* (0.052)	0.848^* (0.054)	0.884^* (0.052)	0.865^* (0.053)
Joint Test	1497^* (0.000)	1337^* (0.000)	1248^* (0.000)	1306^* (0.000)	1291^* (0.000)
Restrictions					
Positivity	–	*Always*	–	*Always*	*Yes*
σ_y^2	–	1.11	–	1.07	*Yes*
k_y	–	9.76	–	8.35	*Unknown*

- Means that the moment is not defined.
* Significant at 5% level.
Asymptotic standard deviations in parenthesis.

Table 2.7: Estimated models for daily SP500 returns.

	QGARCH	TGARCH	GJR	EGARCH	APARCH
ω	0.001^* (0.001)	0.004^* (0.001)	0.001^* (0.005)	-0.009^* (0.002)	0.003^* (0.002)
α	0.034^* (0.006)	0.041^* (0.006)	0.029^* (0.001)	0.134^* (0.015)	0.041^* (0.007)
β	0.963^* (0.006)	0.962^* (0.007)	0.964^* (0.004)	0.988^* (0.001)	0.962^* (0.006)
δ	-0.005 (0.004)	-0.006^* (0.004)	0.010 (0.008)	-0.013^* (0.001)	-0.132^* (0.082)
λ					1.279^* (0.282)
ν	7.21	7.31	7.20	6.81	7.29
Residuals					
Mean	-0.003	-0.002	-0.003	-0.001	-0.002
S.D.	0.996	0.996	0.996	0.969	0.996
Skewness	-0.293^*	-0.292^*	-0.293^*	-0.268^*	-0.292^*
Kurtosis	4.264^*	4.218^*	4.265^*	4.260^*	4.236^*
$Jarque - Bera$	345.1^*	324.1^*	345.8^*	333.4^*	333.3^*
$Q(20)$	18.84	19.18	19.10	18.85	19.09
$Q_2(20)$	18.28	20.47	18.39	22.51	19.83
$Q_{21}(20)$	31.23	31.76	31.41	30.89	31.60
Sign Bias	0.008 (0.055)	0.003 (0.055)	0.010 (0.056)	0.004 (0.053)	0.006 (0.056)
Negative Bias	-0.112 (0.080)	-0.099 (0.080)	0.112 (0.080)	-0.103 (0.076)	-0.104 (0.080)
Positive Bias	0.163^* (0.080)	0.155^* (0.080)	0.162^* (0.080)	0.131^* (0.076)	0.157^* (0.079)
Joint Test	3.56^* (0.013)	3.121^* (0.025)	3.560^* (0.013)	2.798^* (0.039)	3.241^* (0.021)
Restrictions					
Positivity	*Yes*	*Always*	*Yes*	*Always*	*Yes*
σ_y^2	0.57	0.46	0.50	0.57	*Yes*
k_y	–	6.47	–	6.89	*Unknown*

- Means that the moment is not defined.
* Significant at 5% level.
Asymptotic standard deviations in parenthesis.

Table 2.8: Estimated models for daily USD/AUD exchange rates returns.

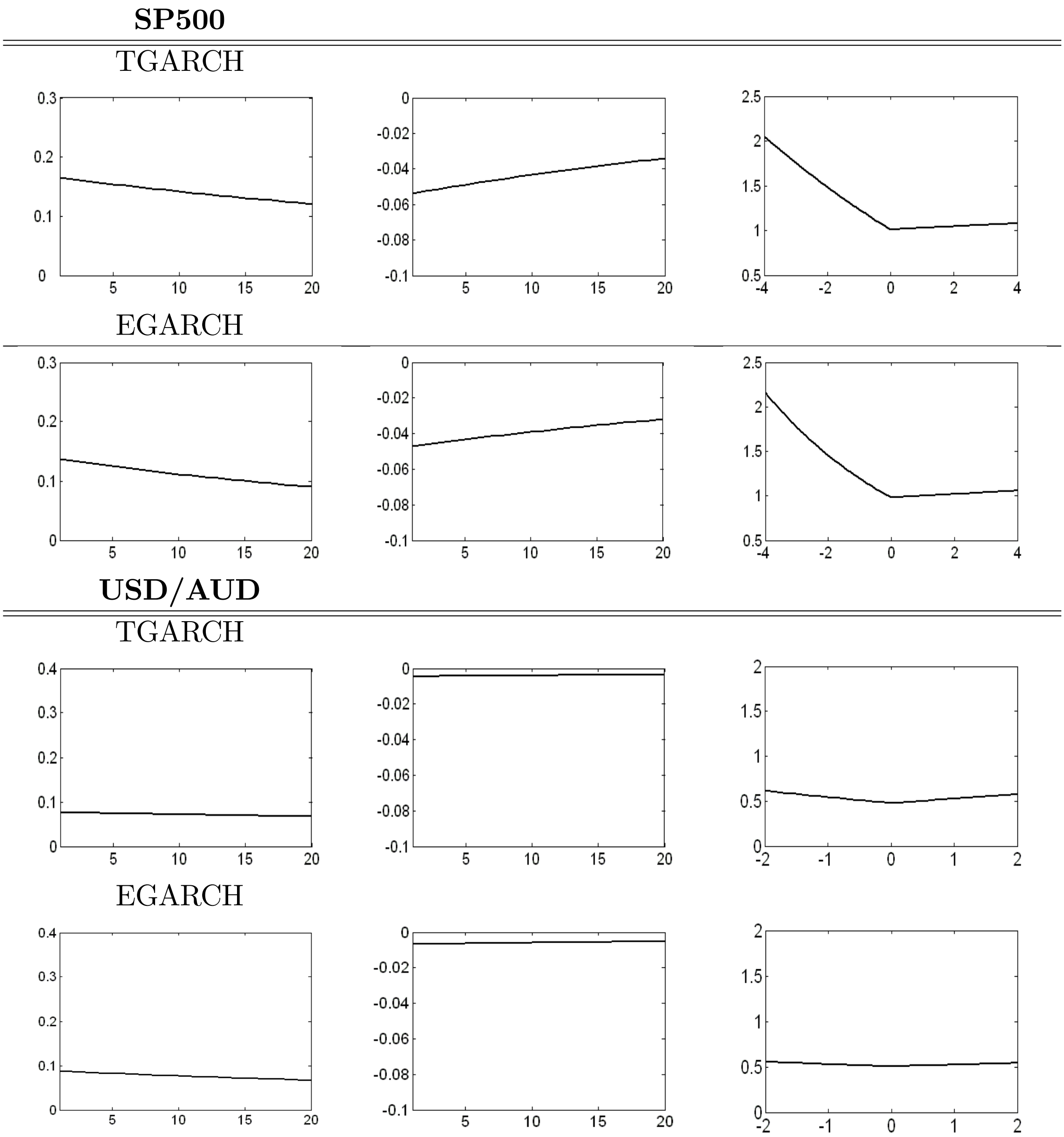


Figure 2.11: Plug-in autocorrelogram of squares, cross-correlogram and NIC under the TGARCH and EGARCH models estimated for the SP500 and USD/AUD daily returns.

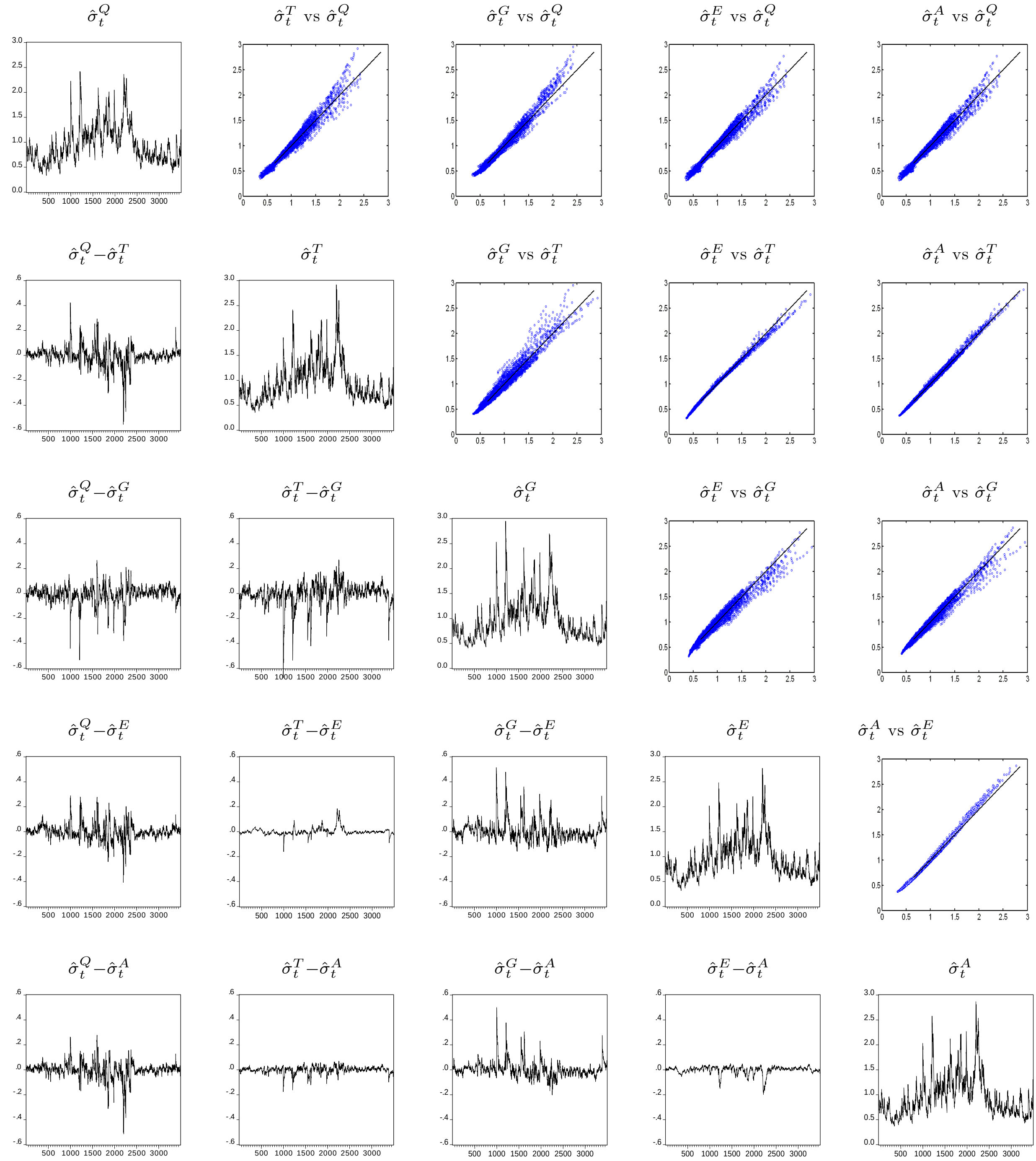


Figure 2.12: Estimated conditional standard deviations, differences between them and scatter plots between conditional standard deviations of the SP500 returns. $\hat{\sigma}_t^Q$,$\hat{\sigma}_t^T$,$\hat{\sigma}_t^G$,$\hat{\sigma}_t^E$ and $\hat{\sigma}_t^A$ are the conditional standard deviations estimated by the QGARCH, TGARCH, GJR, EGARCH and APARCH models, respectively.

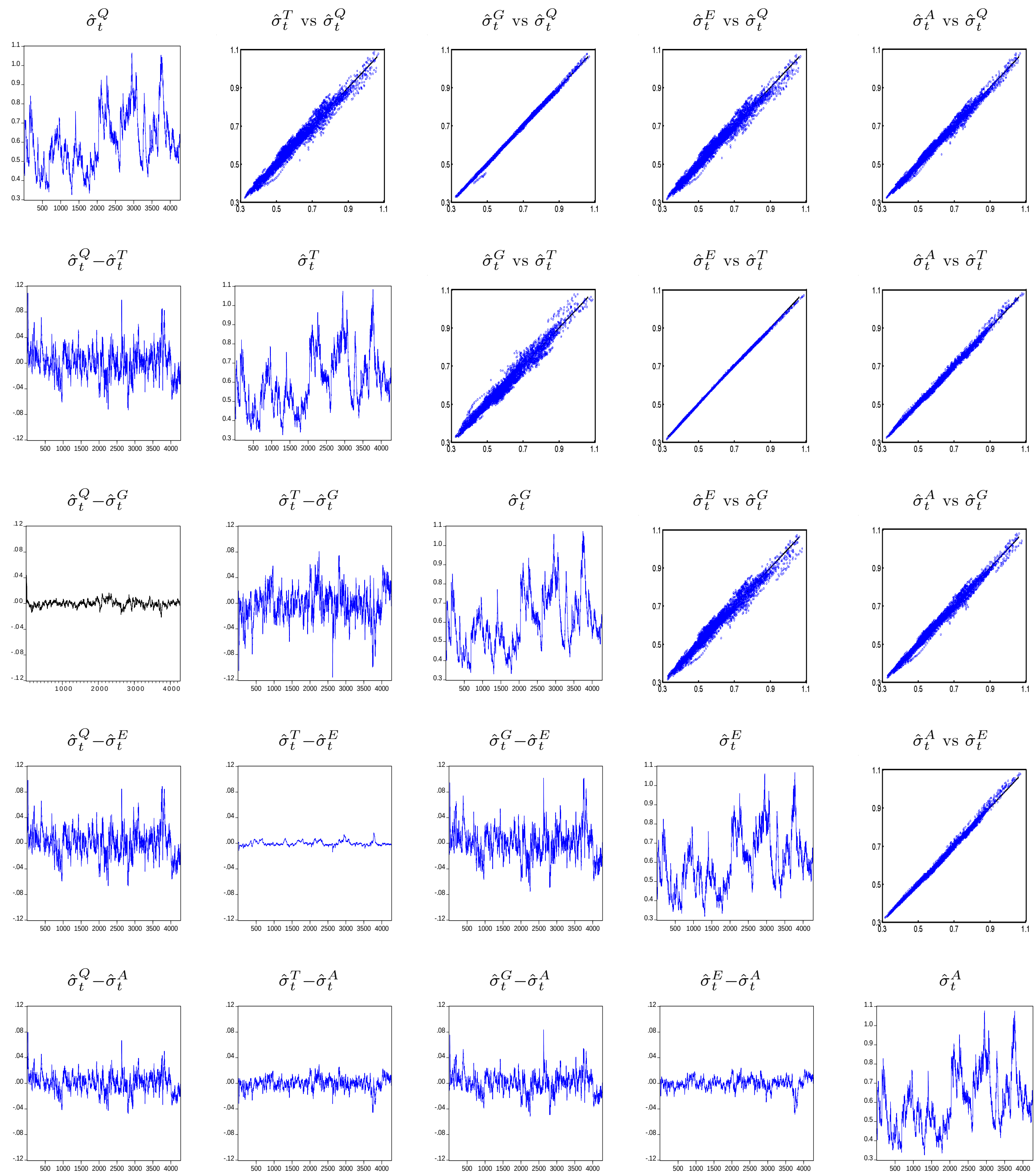


Figure 2.13: Estimated conditional standard deviations, differences between them and scatter plots between conditional standard deviations of the USD/AUD returns. $\hat{\sigma}_t^Q$,$\hat{\sigma}_t^T$,$\hat{\sigma}_t^G$,$\hat{\sigma}_t^E$ and $\hat{\sigma}_t^A$ are the conditional standard deviations estimated by the QGARCH, TGARCH, GJR, EGARCH and APARCH models, respectively.

2.6 Appendix A. Empirical results when fitting the asymmetric GARCH models with Gaussian errors

This appendix contains the tables and figures corresponding to the estimation results when the QGARCH, TGARCH, GJR, EGARCH and APARCH models are fitted to the SP500 and USD/AUD returns assuming Gaussian errors.

SP500	QGARCH	TGARCH	GJR	EGARCH	APARCH
ω	0.016^* (0.003)	0.017^* (0.003)	0.013^* (0.001)	-0.088^* (0.009)	0.017^* (0.002)
α	0.057^* (0.007)	0.059^* (0.006)	-0.001 (0.006)	0.110^* (0.012)	0.056^* (0.006)
β	0.927^* (0.009)	0.937^* (0.007)	0.930^* (0.005)	0.981^* (0.003)	0.936^* (0.005)
δ	-0.081^* (0.009)	-0.051^* (0.014)	0.115^* (0.008)	-0.110^* (0.010)	0.049^* (0.009)
λ					1.130^* (0.189)
Residuals					
$Mean$	0.000	0.002	0.000	0.002	0.001
$S.D.$	0.999	0.999	0.999	0.999	0.999
$Skewness$	-0.430^*	-0.417^*	-0.464^*	-0.418^*	-0.427^*
$Kurtosis$	4.656^*	4.613^*	4.849^*	4.659^*	4.658^*
$Jarque-Bera$	506.32^*	479.78^*	622.49^*	502.32^*	506.13^*
$r_2(1)$	-0.020	-0.020	-0.030	-0.030	-0.030
$r_{21}(1)$	-0.059	-0.056	-0.058	-0.056	-0.056
$Q(20)$	23.70	24.10	23.99	24.34	24.06
$Q_2(20)$	11.74	10.72	12.25	11.39	10.63
$Q_{21}(20)$	43.90^*	43.93^*	44.75^*	43.10^*	44.06^*
Plug-in moments					
Implied persistence	0.982	0.969	–	0.981	*Unknown*
Positivity of σ_t^2	–	*Always*	–	*Always*	*Yes*
Implied σ_y^2	–	1.264	–	1.280	*Yes*
Implied k_y	–	5.750	–	4.979	*Unknown*
Implied $\rho_2(1)$	–	0.229	–	0.205	*Unknown*
Implied $\rho_{21}(1)$	–	-0.079	–	-0.072	*Unknown*

– Means that the moment is not defined.
* Significant at 5% level.
Asymptotic standard deviations in parenthesis.

Table 2.9: Estimated models for the daily returns of SP500 index under Gaussian errors.

USD/AUD	QGARCH	TGARCH	GJR	EGARCH	APARCH
ω	0.003^* (0.000)	0.008^* (0.001)	0.003^* (0.001)	-0.077^* (0.008)	0.008^* (0.002)
α	0.034^* (0.004)	0.042^* (0.004)	0.029^* (0.005)	0.083^* (0.009)	0.045^* (0.005)
β	0.957^* (0.005)	0.954^* (0.005)	0.957^* (0.005)	0.986^* (0.002)	0.951^* (0.005)
δ	-0.005 (0.003)	-0.005^* (0.001)	0.005 (0.003)	-0.013^* (0.005)	-0.153^* (0.070)
λ					1.094^* (0.205)
Residuals					
Mean	-0.003	-0.002	-0.003	-0.003	-0.002
S.D.	1.002	1.002	1.002	1.002	1.002
Skewness	-0.298^*	-0.299^*	-0.305^*	-0.299^*	-0.296^*
Kurtosis	4.256^*	4.241^*	4.302^*	4.248^*	4.240^*
Jarque − Bera	344.03^*	337.49^*	367.63^*	340.47^*	335.85^*
$r_2(1)$	0.012	0.021	0.012	0.021	0.017
$r_{21}(1)$	-0.052	-0.054	-0.054	-0.053	-0.054
$Q(20)$	18.673	18.710	18.804	18.816	18.668
$Q_2(20)$	13.574	14.995	12.756	14.766	13.968
$Q_{21}(20)$	29.903	30.338	30.338	29.839	30.407
Plug-in moments					
Implied persistence	0.991	0.977	0.986	0.986	*Unknown*
Positivity of σ_t^2	*Yes*	*Always*	*Yes*	*Always*	*Yes*
Implied σ_y^2	0.371	0.451	0.239	0.454	*Yes*
Implied k_y	3.506	4.682	3.199	3.318	*Unknown*
Implied $\rho_2(1)$	0.080	0.180	0.058	0.068	*Unknown*
Implied $\rho_{21}(1)$	-0.008	-0.040	$-$	-0.013	*Unknown*

$-$ Means that the moment is not defined.
* Significant at 5% level.
Asymptotic standard deviations in parenthesis.

Table 2.10: Estimated models for the daily returns of the USD/AUD exchange under Gaussian errors.

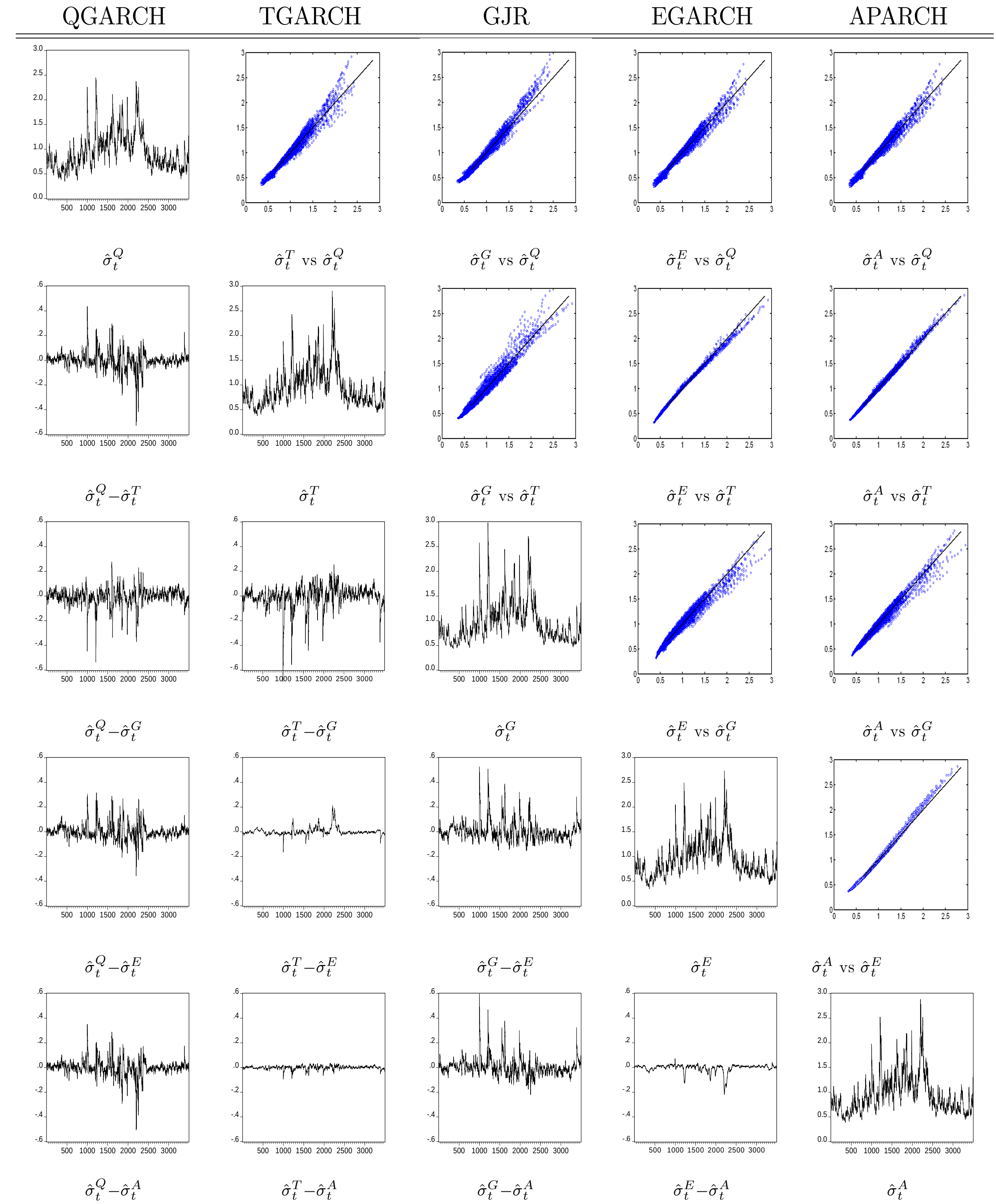


Figure 2.14: Estimated conditional standard deviation, differences between them and scatter plot between conditional standard deviation for the SP500 returns under Gaussian errors.

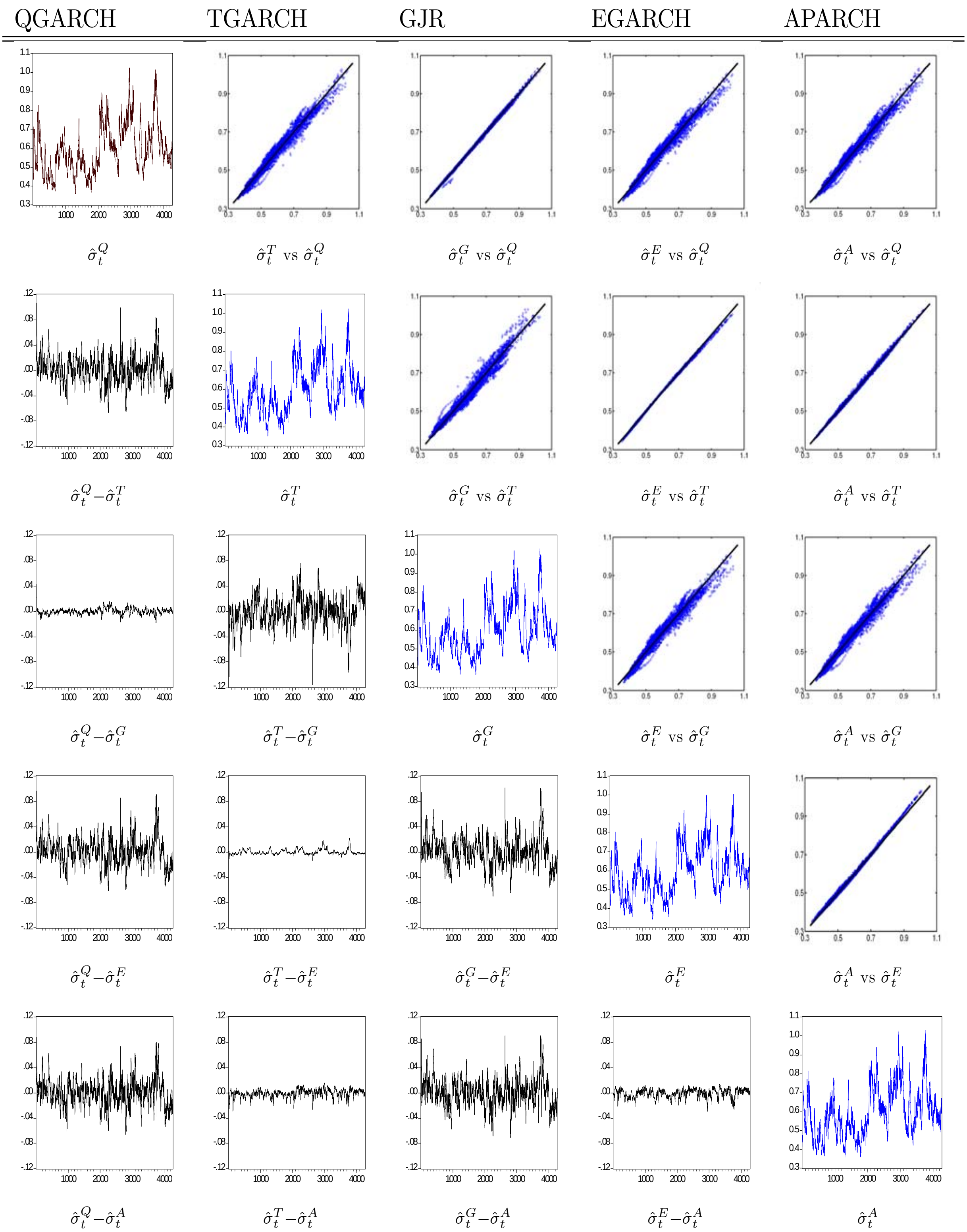


Figure 2.15: Estimated conditional standard deviation, differences between them and scatter plot between conditional standard deviation for the USD/AUD returns under Gaussian errors.

Chapter 3

Comparing sample and plug-in moments in asymmetric GARCH models.

3.1 Introduction

It is very common to analyze the adequacy of a fitted model by comparing its implied or plug-in kurtosis, autocorrelations of squares and cross-correlations with the corresponding sample moments of the original returns; see Breidt et al. (1998), Baille and Chung (2001), Karanasos and Kim (2006), Malmsten and Teräsvirta (2004) and Figà-Talamanca (2008) and the empirical application in Chapter 2 among many others. However, although the finite sample properties of the sample kurtosis and autocorrelations of squares have already been analyzed, those of the corresponding plug-in moments are unknown; see An and Ahmed (2008) for the negative finite sample biases of the sample kurtosis and Bollerslev (1998), He and Teräsvirta (1999a) and Pérez and Ruiz (2003) for the negative biases of the sample autocorrelations. In this chapter, we analyze the finite sample properties of the plug-in moments, considering not only the kurtosis and autocorrelations of squares but also the cross-correlations between returns and future squared returns. We further extend our analysis to study whether comparing these moments with the corresponding sample moments is appropriate for analysing the adequacy of a fitted model. We focus our analysis on the context of GARCH models with leverage effect. In particular, we consider the TGARCH model of Zakoïan (1994) because of its good performance when representing heteroskedastic time series with leverage effect; see previous chapter. We also consider the QGARCH model of Sentana (1995) and the EGARCH model of Nelson (1991) because of their popularity. Although in Chapter 2 we have shown the limitations of the QGARCH model to represent the empirical properties often observed in real time series, we include this model in the analysis because a large number of authors implement it following Engle and Ng (1993). The rest of this Chapter is organized as follows. Section 3.2 reports the results of several Monte Carlo experiments carried out to compare the finite sample properties of the sample and plug-in moments in the context of the TGARCH model. Section 3.3 extends these results to the EGARCH and QGARCH models. Section 3.4 illustrates the results by comparing the sample and plug-in moments of two financial series of returns. Finally, Section 3.5 concludes this chapter.

3.2 Finite sample properties of plug-in and sample moments under TGARCH specifications.

Because of the results on the adequacy of the different models described in Chapter 2, we focus on the TGARCH model defined in equations (1.3.1) and (1.3.8). The distribution of ε_t is assumed to be either Gaussian or Student-7. As explained in detail in Chapter 2, the parameters of model (1.3.8) have to be restricted to guarantee stationarity, finite fourth order moment of y_t and positive conditional variances under conditions (1.3.9) and (1.3.10), respectively.

In order to compare the finite sample properties of the plug-in and sample moments, we generate $R = 1000$ series of different sizes by alternative TGARCH models. To illustrate our findings, firstly consider the TGARCH model with parameters $\alpha = 0.17$, $\beta = 0.8$ and $\delta_T = -0.1$ and sizes $T = 500$, 2000 and 5000. The parameter ω is such that the marginal variance of y_t is one. Denote by κ, $\rho_2(1)$ and $\rho_{21}(1)$, the population kurtosis, first order autocorrelation of squares and first order cross-correlation between y_t and y_{t+1}^2, respectively which are given by $\kappa = 9.01$, $\rho_2(1) = 0.344$ and $\rho_{21}(1) = -0.112$ when the errors are Gaussian, whereas $\kappa = 16.91$, $\rho_2(1) = 0.237$ and $\rho_{21}(1) = -0.077$ when they are Student-7. The corresponding plug-in moments are denoted by $\hat{\kappa}$, $\hat{\rho}_2(1)$ and $\hat{\rho}_{21}(1)$. Finally, the sample moments are denoted by k, $r_2(1)$, $r_{21}(1)$. For each time series generated, we compute the sample and the plug-in moments[1]. Table 3.1 reports their Monte Carlo relative biases and standard deviations.

Consider first the results for the kurtosis. The plug-in kurtosis have positive relative biases which can be very large when $T = 500$. Both, biases and standard deviations, decrease with the sample size. However, the relative biases of the sample kurtosis are negative and very large regardless of the error distribution and sample size considered; see An and Ahmed (2008) for a similar conclusion. Furthermore, the biases hardly decrease with the sample size. The relative biases of $k - \hat{\kappa}$ are rather large and even larger when the errors are Student-7. Therefore, the sample and plug-in kurtosis tend to be far apart even when the model is correctly specified; see also the first column of Figure 3.1 which plots kernel densities of k, $\hat{\kappa}$ and their differences for $T = 2000$ and Gaussian errors. Comparing plug-in and sample kurtosis may lead to misleading conclusions about the adequacy of a fitted model.

When looking at the results corresponding to the plug-in first order autocorrelation of squares, we can observe that the relative biases are negative. The magnitude of the biases and standard errors are similar regardless of the error distribution. On the other hand, although the biases of the sample autocorrelations are also negative, they are much larger in magnitude; Bollerslev (1988), He and Teräsvirta (1999) and Pérez and Ruiz (2003) also report negative biases of the sample autocorrelations. As expected, both the biases and standard deviations decrease with the sample size. Therefore, we expect that the plug-in first order autocorrelations of squares would be in average larger than their sample counterparts and, obviously, closer to the population autocorrelations. Also note that the standard deviations of the sample autocorrelations are much larger than those of the plug-in. Consequently, as in the case of the kurtosis, comparing the plug-in first order autocorrelation of squares with the sample autocorrelation can lead to reject the adequacy of a well specified GARCH model; see also the second column of Figure 3.1 which plots kernel densities of $r_2(1)$, $\hat{\rho}_2(1)$ and their differences when $T = 2000$ and the errors are Gaussian.

Finally, the relative biases and standard deviations of the sample cross-correlations depend on the error distribution and sample size considered. It is also important to note that although, the biases of the sample cross-correlations have magnitudes larger than those of the corresponding plug-in cross-correlations, they are, in general, relatively small. Once more, the standard deviations of the differences are very large compared with the magnitude of the cross-correlations. Therefore, comparing $\hat{\rho}_{21}(1)$ and $r_{21}(1)$ may also be rather misleading to

[1] The parameters have been estimated by ML using software developed by the first author in Matlab.

conclude about the adequacy of an asymmetric GARCH model fitted to a given time series of returns; see the third column of Figure 3.1.

Tables 3.2 and 3.3 summarize the relative biases of alternative TGARCH specifications with Gaussian errors when the generated series have sizes equal to $T = 1000$, $T = 2000$ and $T = 5000$. The first table reports the results of models with leverage effect while the second corresponds to the same models with the asymmetry parameter equal to zero. The four models considered cover a wide range of population kurtosis values and have first autocorrelation of squares and cross-correlation values similar to those observed in practice. The parameters and the population moments of each model are summarized in the first and second column of the table, respectively.

Note that in any case, one can observe that main characteristics on the plug-in and sample moments previously mentioned in this section, remain valid for any of the alternative TGARCH models. The size of the bias when estimating the sample kurtosis increases with the population kurtosis; see, for example, the fourth TGARCH model in the table with $\kappa = 19.017$. This model has biases on the sample and plug-in kurtosis that are around ten times the corresponding biases observed in the first model of the table, whose population kurtosis is 4.412. This effect is independent of the sample size considered and is also clear when comparing the biases of the intermediate case of $\kappa = 9.012$ and $\kappa = 9.318$.

Figure 3.2 plots kernel densities of the differences between sample and plug-in moments for the four asymmetric TGARCH models when $T = 2000$. The conclusions are similar to those described in relation with figure 3.1.

3.3 Finite sample properties of plug-in and sample moments for QGARCH and EGARCH models.

In this section, we analyze the finite sample properties of the plug-in and sample moments of alternative GARCH models with leverage effect. The models considered are the QGARCH and EGARCH models described in (1.3.8) and (1.3.21), respectively. We have considered them because of their popularity in financial literature although we have already mentioned their limitations to represent the leverage effect.

For each model, we select four alternative parameter specifications with Gaussian errors.

For each specification, we carry out Monte Carlo experiments analogous to those described for the TGARCH model in the previous section. Again, the models have to be restricted to guarantee the positivity, stationarity and finite fourth moment conditions as described in Chapter 1.

First, consider the simulation results under the QGARCH models of size $T = 1000$, $T = 2000$ and $T = 5000$ which are reported in Table 3.4. It contains the relative biases between sample and plug-in moments with respect to the corresponding populations moments that appear in the second column of the table.

As in the TGARCH model, the plug-in kurtosis has positive relative biases while the biases of to the sample kurtosis are negative. In any case, both standard deviations decrease with the sample size as they do also the biases of the sample kurtosis. For example, in the second model with $\kappa = 5.446$ and parameters $\alpha = 0.1$, $\beta = 0.87$ and $\delta_Q = -0.109$, the relative biases of k go from -19.19% with standard deviation equal to 1.381 when $T = 1000$, to -10.24% with standard deviation equal 1.227 when $T = 5000$. It is worth of mention that the size of the sample kurtosis bias increases, in absolute terms, with kurtosis. Compare for example the sample kurtosis bias of -33.65% when $T = 5000$ in the fourth model considered where $\kappa = 10.235$ with respect to the values in the second model already comented.

In the case of the relative biases for $\hat{\kappa}$, it is not clear that the biases decreases with the sample size or not, it seems that there is a dependence on the population kurtosis. For example, in the first and second models were $\kappa = 4.914$ and $\kappa = 5.446$, the biases decreases with the sample size but this effect is just the opposite in the third and fourth model where the population kurtosis are 8.437 and 10.235, respectively. Therefore, it is difficult to conclude whether the plug-in bias increases or decreases with sample size and/or population kurtosis or not.

Consider now the results for $\hat{\rho}_2(1)$ and $r_2(1)$, they both show, in general, negative biases that decrease with the sample size as in the TGARCH case. The standard deviations decay also with the sample size. See for example the second model where the biases for $\hat{\rho}_2(1)$ go from -1.29% with standard deviation equal to 0.062 when $T = 1000$, to -1.04% with standard deviation equal 0.027 when $T = 5000$. For the biases of the sample first autocorrelation, they go from -12.92% with standard deviation equal to 0.076 when $T = 1000$ to 0.55% with standard deviation equal 0.051 when $T = 5000$.

The situation with the cross-correlation is the opposite to the one observed for the autocorrelations. In general, contrarily to the behavior observed in the TGARCH models, $\hat{\rho}_{21}(1)$ and $r_{21}(1)$, show positive biases that in the case of the sample cross-correlation decrease with the sample size. For the plug-in cross-correlation, the relative biases are much smaller than the corresponding sample biases and there seems to remain stable for all the sample sizes. See the values of the second model, when $T = 1000$ the bias of $r_{21}(1)$ is 6.38% whereas it is only 1.12% for the plug-in biases and when $T = 5000$ the bias of the sample cross-correlation reduces to 4.67% whereas for the plug-in case it is 1.23%, which is similar to the value for $T = 1000$.

Table 3.5 summarizes the results of the Monte Carlo experiments when the corresponding symmetric models are considered. One can observe that the biases show similar patterns to those described above for the asymmetric models.

As a final illustration, the sign of the biases and the shape of the kernel densities of the differences between sample and plug-in moments of the QGARCH models when $T = 2000$, can be observed in Figure 3.3.

Now focus on the EGARCH simulation results that are summarized in Table 3.6. As in the TGARCH and QGARCH models, the biases of the sample kurtosis are negative and they decrease when sample size increase. See for example the evolution of the biases of k in the second model that go from -23.03% when $T = 1000$ to -7.62% when $T = 5000$.

Things change when face to the biases of the plug-in kurtosis. These are positively biased, are smaller, in absolute terms, than the sample ones but they do not decrease with the sample size as the sample ones do. The plug-in kurtosis biases tend to be stable for all the sample sizes considered as can be observed in the second model where they all are around 5%, or in the last one where the biases start at 8.60% when $T = 1000$ and are still 7.39% when the sample size increases to $T = 5000$.

In the context of the autocorrelation of squares, both sample and plug-in biases are negative and the biases decrease when the sample size increases. Again, in absolute terms, the biases of the sample moments are bigger than the plug-in ones. See for example the second model when $T = 1000$, where the sample and plug-in biases are -25.46% and -6.17%, respectively. All theses characteristics goes in line with those highligted for the TGARCH and QGARCH models.

With respect to the sample cross-correlations, contrarily to the previous models considered, they show negative biases. This is not the general case for the plug-in first cross-correlations that tend to be positively biased, and which are smaller, in absolute terms, than the sample ones. In any case, both sample and plug-in cross-correlation biases decrease when the sample size increases. The second model when $T = 5000$ is a clear example of the different behavior that the biases of the sample and plug-in cross-correlation show under EGARCH specifications.

In the case of the symmetric EGARCH models, whose relative biases are summarized in Table 3.7, the conclusions are similar to those mentioned for the asymmetric models but it is worth of mention than the size of the biases for k and $\hat{\kappa}$ and the standard deviations are smaller in the symmetric model, in absolute terms, when compared to the corresponding asymmetric ones independently of the sample size considered.

Figure 3.4 shows the kernel densities of the differences between sample and plug-in moments of the EGARCH models when $T = 2000$. Again, note the effect of the negative biases on sample kurtosis and autocorrelation of squares on the shape of these densities.

3.4 Empirical application

In this section we fit the TGARCH and EGARCH models to the series of daily returns of the SP500 index and of the EUR/USD exchange rate observed from January 2nd 2002 to June 25th 2010 and described in Section 1.3. Figure 1.2 plots both series together with their corresponding sample autocorrelations of squares and cross-correlations between y_t and y_{t+h}^2. The autocorrelations of squares of both series are significant; see also Table 3.8 which reports the corresponding Box-Ljung statistic. The cross-correlations of SP500 returns are also significant and negative suggesting the presence of leverage effect. However, the cross-correlations of EUR/USD returns are not significant. Therefore, a GARCH model with leverage effect may be appropriate for the SP500 returns while the EUR/USD returns could be represented by a symmetric GARCH model. We fit TGARCH and EGARCH models with Student-ν errors to each of these series. The estimated TGARCH model for the SP500 returns is given by

$$\sigma_t = \underset{(0.002)}{0.012} + \underset{(0.009)}{0.054}\,|y_{t-1}| + \underset{(0.008)}{0.948}\sigma_{t-1} \underset{(0.006)}{-0.054}\,y_{t-1}$$

with $\widehat{\nu} = 11.95$. The estimated volatility of the EUR/USD returns is given by

$$\sigma_t = \underset{(0.002)}{0.003} + \underset{(0.007)}{0.040}\,|y_{t-1}| + \underset{(0.007)}{0.963}\sigma_{t-1} \underset{(0.005)}{-0.004}\,y_{t-1}$$

with $\widehat{\nu} = 15.08$.

Under the EGARCH specification, the estimated model for the SP500 returns is

$$\log\sigma_t^2 = \underset{(0.013)}{-0.068} + \underset{(0.017)}{0.090}\,|\varepsilon_{t-1}| + \underset{(0.002)}{0.991}\log\sigma_{t-1}^2 - \underset{(0.011)}{0.101}\varepsilon_{t-1}$$

with $\widehat{\nu} = 11.61$ and for the EUR/USD returns it is given by

$$\log\sigma_t^2 = \underset{(0.013)}{-0.067} + \underset{(0.015)}{0.080}\,|\varepsilon_{t-1}| + \underset{(0.003)}{0.994}\log\sigma_{t-1}^2 - \underset{(0.007)}{0.007}\varepsilon_{t-1}$$

with $\widehat{\nu} = 15.25$.

Note that, as expected, the asymmetry of the EUR/USD returns is not significant. Table 3.8, which reports several moments of the standardized returns, shows that they have smaller kurtosis than the original observations. Furthermore, when looking at the Box-Ljung statistic to test for the significance of the autocorrelations of squares and cross-correlations, we can observe that they are not any more significant. Therefore, it seems that the TGARCH and EGARCH models are able to explain the autocorrelations of squares and cross-correlations between returns and future squared returns.

Finally, Table 3.8 reports the plug-in moments obtained after substituting the parameter estimates in the expressions of the corresponding population moments[2]. When looking at the results for the SP500 returns, we observe that the plug-in kurtosis is much larger than the sample kurtosis. Therefore, we may think that the TGARCH and EGARCH models are not adequate to represent the SP500 kurtosis. However, according to our simulation results, the plug-in kurtosis is positively biased while the sample kurtosis has a negative bias. Therefore, in spite of the large distance between the sample and plug-in kurtosis, the TGARCH and EGARCH models could still be adequate for the SP500 returns. When comparing the plug-in and sample autocorrelations of squares and cross-correlations between returns and future squared returns, we can observe that the differences are pretty small. However, although the

[2]Again take with caution the plug-in moments under EGARCH models when the errors have a Student-ν distribution.

sample autocorrelations of squares are larger than the plug-in autocorrelations, which is in contrast with the biases observed in our Monte Carlo results, remember that the dispersion of the differences between sample and plug-in autocorrelations is very large. When comparing plug-in and sample moments of the EUR/USD returns, we can observe that all moments are very similar.

3.5 Conclusions

This chapter analyzes the suitability of comparing plug-in and sample kurtosis, autocorrelations of squares and cross-correlations between returns and future squared returns when checking the adequacy of a fitted GARCH model. We show that the biases of the sample and plug-in kurtosis have opposite sign, depend on the sample size and in the case of the sample kurtosis biases, they are also influenced by the level of leptokurtosis.

The differences between sample and plug-in autocorrelations are in general negatively biased with respect to the population value. Again they decrease with the sample size but in any case, they have very large dispersion.

Finally, when comparing the plug-in and sample cross-correlations, the sign of the biases depends on the model considered. Once more, the dispersion of the differences between plug-in and sample cross-correlations is very large even in large sample sizes.

Therefore, comparing sample and plug-in moments is not an adequate tool to decide about the adequacy of a fitted conditionally heterocedastic model.

Alternatively to the GARCH models, it could also be of interest to compare the finite sample properties of the plug-in and sample moments in the context of asymmetric stochastic volatility models; see Pérez at al. (2009) for the population moments when the asymmetry is introduced through correlation between the level and volatility disturbances. However, we left this topic for further research due to the lack of a well established estimator for the parameters of asymmetric stochastic volatility models; see, for example Harvey and Shephard (1996) and Kawakatsu (2007) for extensions of the QML estimator, while Asai (2008) propose a MCL estimator, Lee et al. (2011) propose a hierarchical-likelihood approach, Durham (2006) propose a SML estimator and Omori and Watanabe (2008), Raggi and Bordignon (2006) and Smith (2009) propose MCMC estimators.

	Gaussian			*Student-7*		
	T=500	*T=2000*	*T=5000*	*T=500*	*T=2000*	*T=5000*
$\hat{\kappa}$	37.89% (14.448)	19.06% (8.169)	4.76% (2.466)	17.38% (28.600)	15.31% (14.52)	13.25% (9.187)
k	−35.67% (3.540)	−24.41% (3.409)	−20.52% (2.944)	−56.56% (6.261)	−42.09% (5.692)	−34.65% (5.216)
k-$\hat{\kappa}$	−77.11% (14.027)	−43.62% (3.409)	−25.29% (2.943)	−65.56% (21.559)	−57.41% (14.106)	−47.90% (9.518)
$\hat{\rho}_2(1)$	−6.80% (0.061)	−1.32% (0.034)	−0.46% (0.022)	−8.00% (0.063)	−2.07% (0.086)	−0.24% (0.022)
$r_2(1)$	−27.67% (0.109)	−15.32% (0.085)	−11.85% (0.067)	−26.66% (0.106)	−11.69% (0.033)	−8.04% (0.072)
$r_2(1) - \hat{\rho}_2(1)$	−22.85% (0.096)	−14.13% (0.085)	−11.40% (0.067)	−18.81% (0.091)	−9.17% (0.084)	−7.78% (0.069)
$\hat{\rho}_{21}(1)$	−5.47% (0.023)	−1.90% (0.012)	−0.08% (0.007)	−1.72% (0.022)	−1.66% (0.011)	−1.80% (0.007)
$r_{21}(1)$	−2.18% (0.084)	4.60% (0.047)	3.70% (0.033)	19.93% (0.079)	22.81% (0.051)	23.19% (0.036)
$r_{21}(1) - \hat{\rho}_{21}(1)$	3.30% (0.096)	6.92% (0.047)	4.52% (0.033)	21.02% (0.071)	24.41% (0.049)	24.93% (0.039)

Table 3.1: Monte Carlo relative biases and standard deviations (in parenthesis) of sample and plug-in moments and their differences.

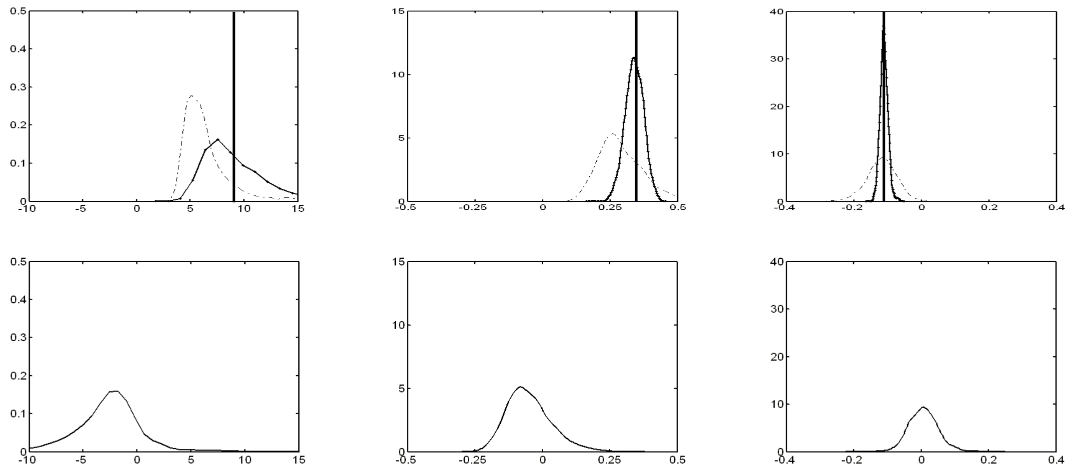


Figure 3.1: Kernel densities of the Monte Carlo sample moments (dashed), plug-in moments (continuous) (top panel) and their differences (lower panel). The vertical line represents the population moments. The first column corresponds to κ_y, the second to $\rho_2(1)$ and the third to $\rho_{21}(1)$.

Model	*Population Moments*		*T=1000*	*T=2000*	*T=5000*
$\alpha = 0.13$	$\kappa = 4.412$	$\hat{\kappa}$	1.70 (0.679)	5.00 (0.431)	0.41 (0.263)
		k	-4.72 (1.167)	-4.19 (0.851)	-0.27 (0.749)
$\beta = 0.78$	$\rho_2(1) = 0.221$	$\hat{\rho}_2(1)$	-2.15 (0.039)	-1.24 (0.027)	-0.23 (0.005)
		$r_2(1)$	-13.17 (0.076)	-8.73 (0.061)	-3.27 (0.049)
$\delta_T = -0.10$	$\rho_{21}(1) = -0.138$	$\hat{\rho}_{21}(1)$	-1.77 (0.020)	-0.97 (0.013)	0.01 (0.009)
		$r_{21}(1)$	-6.26 (0.044)	-4.84 (0.033)	-2.69 (0.023)
$\alpha = 0.17$	$\kappa = 9.012$	$\hat{\kappa}$	27.62 (10.231)	19.06 (8.169)	4.76 (2.466)
		k	-31.83 (3.494)	-24.41 (3.409)	-20.52 (2.944)
$\beta = 0.8$	$\rho_2(1) = 0.344$	$\hat{\rho}_2(1)$	-3.00 (0.046)	-1.32 (0.034)	-0.46 (0.022)
		$r_2(1)$	-21.99 (0.089)	-15.32 (0.085)	-11.85 (0.067)
$\delta_T = -0.1$	$\rho_{21}(1) = -0.112$	$\hat{\rho}_{21}(1)$	-4.09 (0.016)	-1.90 (0.012)	-0.08 (0.007)
		$r_{21}(1)$	-1.05 (0.059)	4.60 (0.047)	3.70 (0.033)
$\alpha = 0.1$	$\kappa = 9.318$	$\hat{\kappa}$	22.08 (8.357)	11.56 (5.700)	7.23 (2.775)
		k	-34.46 (3.238)	-28.95 (3.133)	-21.16 (3.043)
$\beta = 0.88$	$\rho_2(1) = 0.309$	$\hat{\rho}_2(1)$	-1.44 (0.041)	-0.83 (0.076)	0.64 (0.020)
		$r_2(1)$	-25.23 (0.092)	-18.92 (0.077)	-13.02 (0.065)
$\delta_T = -0.1$	$\rho_{21}(1) = -0.110$	$\hat{\rho}_{21}(1)$	-4.46 (0.013)	-3.21 (0.042)	-1.70 (0.006)
		$r_{21}(1)$	-0.63 (0.058)	0.65 (0.043)	1.01 (0.031)
$\alpha = 0.15$	$\kappa = 19.017$	$\hat{\kappa}$	38.08 (29.724)	46.19 (27.436)	28.63 (17.778)
		k	-57.19 (7.009)	-51.61 (6.527)	-45.77 (6.270)
$\beta = 0.8$	$\rho_2(1) = 0.369$	$\hat{\rho}_2(1)$	-1.37 (0.031)	0.57 (0.023)	1.19 (0.016)
		$r_2(1)$	-23.90 (0.102)	-18.58 (0.090)	-15.35 (0.075)
$\delta_T = -0.16$	$\rho_{21}(1) = -0.129$	$\hat{\rho}_{21}(1)$	-5.02 (0.020)	-5.75 (0.019)	-4.72 (0.014)
		$r_{21}(1)$	31.74 (0.062)	32.71 (0.091)	30.75 (0.034)

Table 3.2: Percentages of Monte Carlo biases and standard deviations in the TGARCH Gaussian models.

Model	*Population Moments*		*T=1000*	*T=2000*	*T=5000*
$\alpha = 0.13$	$\kappa = 3.393$	$\hat{\kappa}$	0.62 (0.191)	0.34 (0.129)	0.22 (0.080)
		k	-1.62 (0.317)	-0.54 (0.221)	0.00 (0.150)
$\beta = 0.78$	$\rho_2(1) = 0.151$	$\hat{\rho}_2(1)$	-0.01 (0.042)	-0.16 (0.026)	0.57 (0.016)
		$r_2(1)$	-6.79 (0.052)	-4.63 (0.041)	-1.08 (0.028)
$\delta_T = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$-0.01^{\blacklozenge}$ (0.035)	$0.02^{\blacklozenge}$ (0.017)	$-0.01^{\blacklozenge}$ (0.011)
		$r_{21}(1)$	$0.06^{\blacklozenge}$ (0.043)	$0.02^{\blacklozenge}$ (0.030)	$0.06^{\blacklozenge}$ (0.020)
$\alpha = 0.17$	$\kappa = 4.540$	$\hat{\kappa}$	2.41 (0.996)	2.07 (0.650)	0.40 (0.359)
		k	-8.40 (0.816)	-4.54 (0.811)	-2.30 (0.755)
$\beta = 0.8$	$\rho_2(1) = 0.258$	$\hat{\rho}_2(1)$	-2.26 (0.047)	-0.22 (0.036)	-0.52 (0.021)
		$r_2(1)$	-14.73 (0.074)	-8.857 (0.064)	-4.73 (0.047)
$\delta_T = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$0.02^{\blacklozenge}$ (0.022)	$0.01^{\blacklozenge}$ (0.015)	$0.04^{\blacklozenge}$ (0.010)
		$r_{21}(1)$	$-0.05^{\blacklozenge}$ (0.054)	$0.02^{\blacklozenge}$ (0.040)	$0.08^{\blacklozenge}$ (0.027)
$\alpha = 0.1$	$\kappa = 3.687$	$\hat{\kappa}$	0.10 (0.425)	0.80 (0.300)	0.06 (0.164)
		k	-4.11 (0.415)	-1.56 (0.408)	-0.53 (0.299)
$\beta = 0.88$	$\rho_2(1) = 0.158$	$\hat{\rho}_2(1)$	-3.33 (0.047)	-0.41 (0.030)	-0.64 (0.018)
		$r_2(1)$	-14.12 (0.057)	-6.62 (0.050)	-1.84 (0.035)
$\delta_T = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$-0.06^{\blacklozenge}$ (0.038)	$-0.40^{\blacklozenge}$ (0.013)	$-0.01^{\blacklozenge}$ (0.008)
		$r_{21}(1)$	$-0.02^{\blacklozenge}$ (0.040)	$-0.16^{\blacklozenge}$ (0.032)	$-0.03^{\blacklozenge}$ (0.021)
$\alpha = 0.15$	$\kappa = 3.830$	$\hat{\kappa}$	-10.23 (0.706)	-18.89 (0.030)	-19.00 (0.017)
		k	-4.08 (0.504)	-1.46 (0.480)	-0.38 (0.344)
$\beta = 0.8$	$\rho_2(1) = 0.203$	$\hat{\rho}_2(1)$	-0.84 (0.057)	-0.63 (0.012)	-0.29 (0.007)
		$r_2(1)$	-10.61 (0.062)	-6.12 (0.051)	-1.10 (0.040)
$\delta_T = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$-0.04^{\blacklozenge}$ (0.041)	$0.02^{\blacklozenge}$ (0.015)	$0.01^{\blacklozenge}$ (0.009)
		$r_{21}(1)$	$-0.16^{\blacklozenge}$ (0.045)	$-0.17^{\blacklozenge}$ (0.034)	$-0.06^{\blacklozenge}$ (0.023)

Table 3.3: Percentages of Monte Carlo biases and standard deviations in the symmetric TGARCH Gaussian models. "$\blacklozenge$" Means that the difference is absolute.

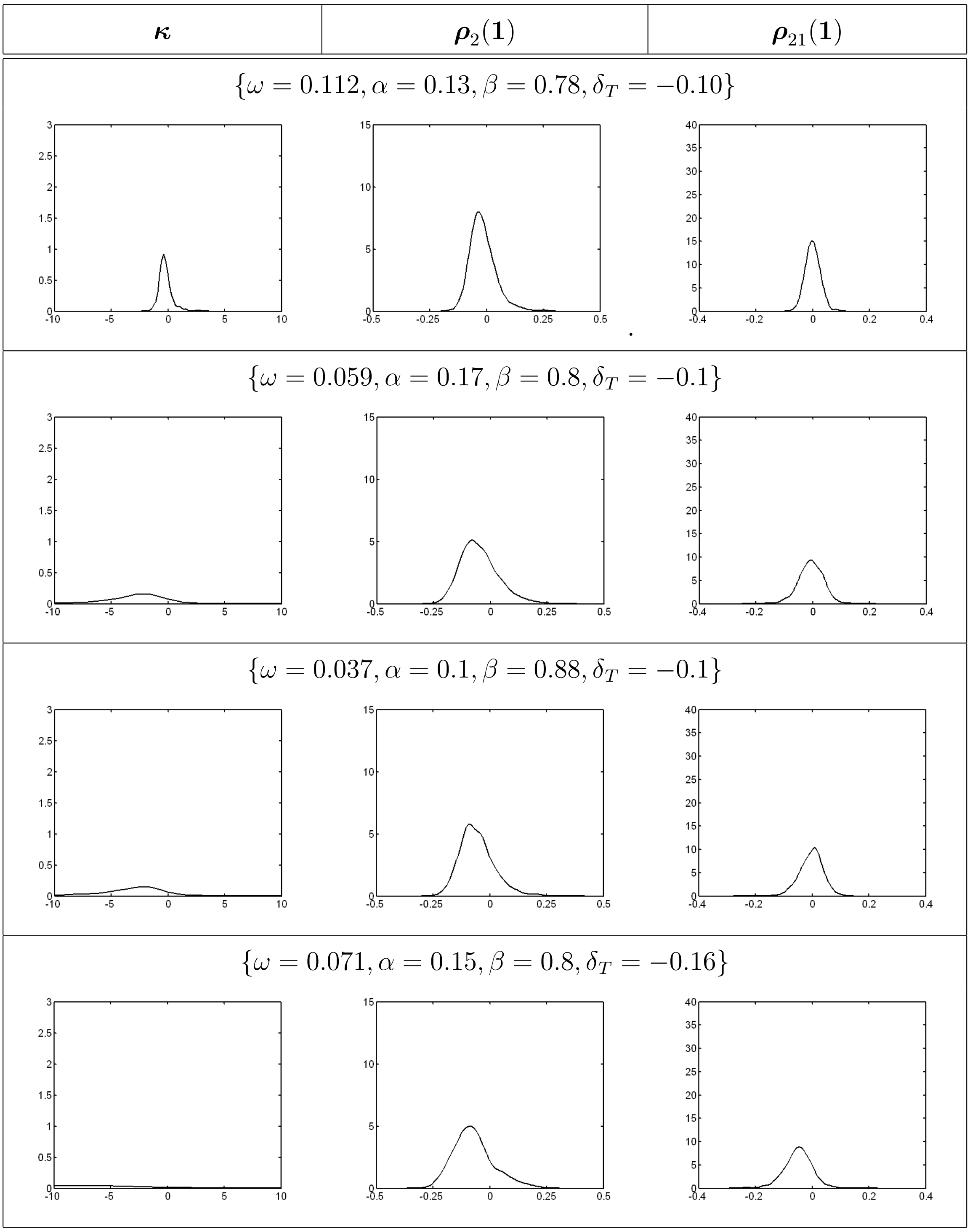


Figure 3.2: Kernel densities of the Monte Carlo differences between sample moments and plug-in moments for different TGARCH Gaussian models when $T = 2000$.

Model	*Population Moments*		*T=1000*	*T=2000*	*T=5000*
$\alpha = 0.14$	$\kappa = 4.912$	$\hat{\kappa}$	13.55 (3.026)	8.01 (2.320)	1.82 (0.648)
		k	-11.14 (1.836)	-6.64 (1.840)	-2.40 (1.774)
$\beta = 0.8$	$\rho_2(1) = 0.218$	$\hat{\rho}_2(1)$	-1.48 (0.058)	0.09 (0.042)	-0.001 (0.026)
		$r_2(1)$	1.67 (0.084)	8.99 (0.072)	13.26 (0.062)
$\delta_Q = -0.1$	$\rho_{21}(1) = -0.051$	$\hat{\rho}_{21}(1)$	-0.03 (0.016)	0.04 (0.012)	-0.01 (0.007)
		$r_{21}(1)$	-0.08 (0.055)	-0.19 (0.043)	-0.29 (0.030)
$\alpha = 0.1$	$\kappa = 5.446$	$\hat{\kappa}$	22.55 (6.954)	9.28 (6.697)	1.59 (0.959)
		k	-19.19 (1.381)	-13.52 (1.381)	-10.24 (1.227)
$\beta = 0.87$	$\rho_2(1) = 0.226$	$\hat{\rho}_2(1)$	-1.29 (0.062)	-0.9 (0.045)	-1.04 (0.027)
		$r_2(1)$	-12.92 (0.076)	-5.87 (0.065)	0.55 (0.051)
$\delta_Q = -0.109$	$\rho_{21}(1) = -0.051$	$\hat{\rho}_{21}(1)$	1.12 (0.016)	1.30 (0.011)	1.23 (0.007)
		$r_{21}(1)$	6.38 (0.048)	9.18 (0.038)	4.67 (0.027)
$\alpha = 0.14$	$\kappa = 8.437$	$\hat{\kappa}$	4.07 (5.859)	7.93 (5.265)	13.09 (4.726)
		k	-37.06 (2.390)	-32.45 (2.331)	-26.13 (2.283)
$\beta = 0.825$	$\rho_2(1) = 0.298$	$\hat{\rho}_2(1)$	-20.50 (0.056)	-12.32 (0.045)	-3.96 (0.033)
		$r_2(1)$	-16.34 (0.088)	-10.88 (0.076)	-3.63 (0.064)
$\delta_Q = -0.12$	$\rho_{21}(1) = -0.044$	$\hat{\rho}_{21}(1)$	16.84 (0.018)	26.61 (0.048)	21.66 (0.01)
		$r_{21}(1)$	26.29 (0.060)	23.69 (0.048)	20.45 (0.035)
$\alpha = 0.16$	$\kappa = 10.235$	$\hat{\kappa}$	-1.58 (7.813)	4.16 (7.080)	12.25 (6.449)
		k	-45.35 (3.055)	-40.37 (2.987)	-33.65 (2.830)
$\beta = 0.8$	$\rho_2(1) = 0.320$	$\hat{\rho}_2(1)$	-11.65 (0.057)	-6.67 (0.044)	-2.36 (0.033)
		$r_2(1)$	-16.13 (0.095)	-11.21 (0.078)	-4.86 (0.065)
$\delta_Q = -0.16$	$\rho_{21}(1) = -0.040$	$\hat{\rho}_{21}(1)$	26.30 (0.018)	14.99 (0.014)	-4.52 (0.011)
		$r_{21}(1)$	45.89 (0.065)	34.74 (0.051)	32.49 (0.037)

Table 3.4: Percentages of Monte Carlo relative biases and standard deviations in the QGARCH Gaussian models.

Model	*Population Moments*		*T=1000*	*T=2000*	*T=5000*
$\alpha = 0.14$	$\kappa = 4.523$	$\hat{\kappa}$	13.64 (4.303)	4.94 (1.230)	2.47 (0.638)
		k	-10.31 (1.286)	-6.89 (1.152)	-4.49 (0.822)
$\beta = 0.8$	$\rho_2(1) = 0.197$	$\hat{\rho}_2(1)$	1.79 (0.059)	-0.63 (0.044)	0.39 (0.030)
		$r_2(1)$	5.40 (0.081)	11.29 (0.068)	19.36 (0.057)
$\delta_Q = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$-0.03^{\blacklozenge}$ (0.016)	$-0.02^{\blacklozenge}$ (0.011)	$0.03^{\blacklozenge}$ (0.007)
		$r_{21}(1)$	$-0.48^{\blacklozenge}$ (0.054)	$-0.16^{\blacklozenge}$ (0.043)	$0.14^{\blacklozenge}$ (0.028)
$\alpha = 0.1$	$\kappa = 4.534$	$\hat{\kappa}$	9.77 (2.651)	8.44 (2.067)	2.82 (0.795)
		k	-13.12 (1.288)	-8.58 (1.305)	-4.47 (1.189)
$\beta = 0.87$	$\rho_2(1) = 0.184$	$\hat{\rho}_2(1)$	-5.77 (0.063)	-1.94 (0.051)	-0.14 (0.033)
		$r_2(1)$	-10.94 (0.074)	2.50 (0.066)	11.34 (0.054)
$\delta_Q = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$0.06^{\blacklozenge}$ (0.013)	$-0.03^{\blacklozenge}$ (0.009)	$-0.01^{\blacklozenge}$ (0.005)
		$r_{21}(1)$	$-0.02^{\blacklozenge}$ (0.047)	$-0.25^{\blacklozenge}$ (0.037)	$-0.02^{\blacklozenge}$ (0.027)
$\alpha = 0.14$	$\kappa = 6.976$	$\hat{\kappa}$	2.35 (4.721)	8.12 (3.978)	14.93 (4.030)
		k	-32.80 (2.190)	-27.97 (2.092)	-22.96 (1.966)
$\beta = 0.825$	$\rho_2(1) = 0.269$	$\hat{\rho}_2(1)$	-12.95 (0.013)	-10.17 (0.052)	-1.45 (0.041)
		$r_2(1)$	-14.82 (0.085)	-7.81 (0.073)	-1.18 (0.060)
$\delta_Q = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$0.02^{\blacklozenge}$ (0.064)	$0.05^{\blacklozenge}$ (0.008)	$0.03^{\blacklozenge}$ (0.005)
		$r_{21}(1)$	$-0.18^{\blacklozenge}$ (0.057)	$-0.01^{\blacklozenge}$ (0.045)	$0.15^{\blacklozenge}$ (0.032)
$\alpha = 0.16$	$\kappa = 8.647$	$\hat{\kappa}$	-6.18 (6.020)	1.59 (5.710)	11.52 (5.212)
		k	-42.68 (3.113)	-37.98 (3.078)	-29.82 (3.037)
$\beta = 0.8$	$\rho_2(1) = 0.298$	$\hat{\rho}_2(1)$	-15.21 (0.063)	-9.07 (0.051)	-2.95 (0.037)
		$r_2(1)$	-17.04 (0.088)	-9.94 (0.076)	-1.38 (0.067)
$\delta_Q = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$0.00^{\blacklozenge}$ (0.013)	$0.01^{\blacklozenge}$ (0.009)	$0.02^{\blacklozenge}$ (0.005)
		$r_{21}(1)$	$0.11^{\blacklozenge}$ (0.059)	$0.03^{\blacklozenge}$ (0.047)	$0.07^{\blacklozenge}$ (0.038)

Table 3.5: Percentages of Monte Carlo biases and standard deviations in the symmetric QGARCH Gaussian models. "$\blacklozenge$" Means that the difference is absolute.

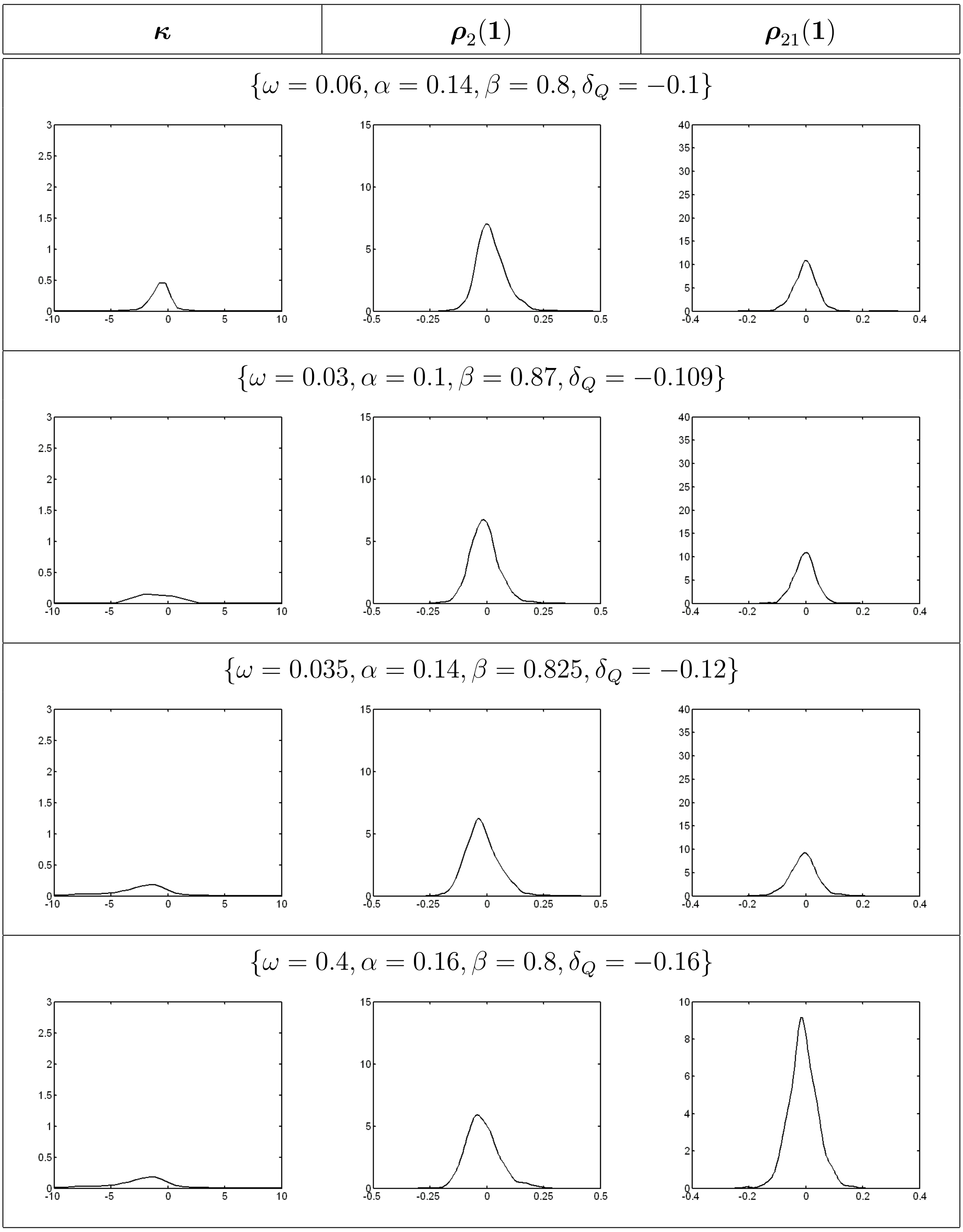


Figure 3.3: Kernel densities of the Monte Carlo differences between sample moments and plug-in moments for different QGARCH Gaussian models when $T = 2000$.

Model	*Population Moments*		*T=1000*	*T=2000*	*T=5000*
$\alpha = 0.12$	$\kappa = 3.262$	$\hat{\kappa}$	-0.06 (0.148)	0.03 (0.096)	0.09 (0.054)
		k	-0.53 (0.246)	-0.24 (0.175)	-0.13 (0.116)
$\beta = 0.93$	$\rho_2(1) = 0.079$	$\hat{\rho}_2(1)$	-5.84 (0.033)	-2.27 (0.022)	-0.17 (0.012)
		$r_2(1)$	-7.82 (0.046)	-1.64 (0.034)	-2.63 (0.021)
$\delta_E = -0.08$	$\rho_{21}(1) = -0.059$	$\hat{\rho}_{21}(1)$	-2.92 (0.028)	-1.18 (0.016)	0.31 (0.008)
		$r_{21}(1)$	-3.65 (0.037)	-2.19 (0.025)	-2.17 (0.017)
$\alpha = 0.13$	$\kappa = 4.388$	$\hat{\kappa}$	5.13 (0.722)	5.46 (0.460)	5.56 (0.287)
		k	-23.03 (1.763)	-15.36 (1.348)	-7.62 (1.2572)
$\beta = 0.99$	$\rho_2(1) = 0.236$	$\hat{\rho}_2(1)$	-6.17 (0.049)	-2.85 (0.034)	-0.91 (0.021)
		$r_2(1)$	-25.46 (0.077)	-17.28 (0.063)	-9.53 (0.050)
$\delta_E = -0.08$	$\rho_{21}(1) = -0.051$	$\hat{\rho}_{21}(1)$	2.64 (0.011)	2.50 (0.007)	0.59 (0.004)
		$r_{21}(1)$	-12.44 (0.051)	-6.73 (0.034)	-4.18 (0.025)
$\alpha = 0.3$	$\kappa = 9.590$	$\hat{\kappa}$	5.53 (3.519)	3.93 (2.147)	3.83 (1.299)
		k	-24.87 (2.852)	-15.07 (2.825)	-6.02 (2.774)
$\beta = 0.97$	$\rho_2(1) = 0.350$	$\hat{\rho}_2(1)$	-2.89 (0.036)	-1.62 (0.024)	-0.63 (0.015)
		$r_2(1)$	-18.40 (0.091)	-13.54 (0.080)	-7.96 (0.064)
$\delta_E = -0.14$	$\rho_{21}(1) = -0.091$	$\hat{\rho}_{21}(1)$	1.72 (0.014)	-3.57 (0.010)	0.54 (0.006)
		$r_{21}(1)$	-9.51 (0.064)	-2.90 (0.051)	1.87 (0.038)
$\alpha = 0.2$	$\kappa = 9.766$	$\hat{\kappa}$	8.60 (2.630)	6.89 (1.671)	7.39 (1.034)
		k	-15.15 (3.314)	-5.72 (3.276)	4.35 (3.193)
$\beta = 0.98$	$\rho_2(1) = 0.321$	$\hat{\rho}_2(1)$	-2.79 (0.035)	-1.65 (0.023)	-0.56 (0.014)
		$r_2(1)$	-21.71 (0.090)	-14.78 (0.078)	-9.46 (0.062)
$\delta_E = -0.15$	$\rho_{21}(1) = -0.090$	$\hat{\rho}_{21}(1)$	2.20 (0.012)	0.94 (0.008)	0.25 (0.005)
		$r_{21}(1)$	-7.04 (0.060)	-1.12 (0.049)	-1.05 (0.033)

Table 3.6: Percentages of Monte Carlo biases and standard deviations in the EGARCH Gaussian models.

Model	*Population Moments*		*T=1000*	*T=2000*	*T=5000*
$\alpha = 0.12$	$\kappa = 3.085$	$\hat{\kappa}$	-0.06 (0.068)	0.00 (0.051)	0.00 (0.030)
		k	-0.11 (0.184)	-0.06 (0.129)	-0.05 (0.082)
$\beta = 0.93$	$\rho_2(1) = 0.054$	$\hat{\rho}_2(1)$	-1.18 (0.036)	-3.72 (0.016)	1.34 (0.013)
		$r_2(1)$	-7.63 (0.038)	-1.50 (0.028)	-1.32 (0.018)
$\delta_E = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$-0.03^{\blacklozenge}$ (0.027)	$-0.07^{\blacklozenge}$ (0.024)	$-0.02^{\blacklozenge}$ (0.008)
		$r_{21}(1)$	$-0.07^{\blacklozenge}$ (0.034)	$-0.12^{\blacklozenge}$ (0.025)	$-0.07^{\blacklozenge}$ (0.016)
$\alpha = 0.13$	$\kappa = 3.501$	$\hat{\kappa}$	0.98 (0.296)	1.40 (0.192)	1.89 (0.125)
		k	-10.99 (0.655)	-8.30 (0.519)	-5.18 (0.431)
$\beta = 0.99$	$\rho_2(1) = 0.151$	$\hat{\rho}_2(1)$	-10.48 (0.050)	-6.00 (0.033)	-2.87 (0.023)
		$r_2(1)$	-22.31 (0.061)	-12.67 (0.052)	-7.12 (0.038)
$\delta_E = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$0.04^{\blacklozenge}$ (0.015)	$-0.06^{\blacklozenge}$ (0.008)	$0.00^{\blacklozenge}$ (0.005)
		$r_{21}(1)$	$-0.01^{\blacklozenge}$ (0.040)	$-0.11^{\blacklozenge}$ (0.040)	$-0.07^{\blacklozenge}$ (0.021)
$\alpha = 0.3$	$\kappa = 5.607$	$\hat{\kappa}$	1.40 (1.156)	1.71 (0.751)	2.56 (0.516)
		k	-7.04 (1.622)	-1.83 (1.347)	1.44 (1.161)
$\beta = 0.97$	$\rho_2(1) = 0.290$	$\hat{\rho}_2(1)$	-3.45 (0.045)	-1.60 (0.031)	-0.41 (0.020)
		$r_2(1)$	-15.60 (0.079)	-8.98 (0.071)	-4.34 (0.053)
$\delta_E = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$0.10^{\blacklozenge}$ (0.018)	$-0.01^{\blacklozenge}$ (0.012)	$0.03^{\blacklozenge}$ (0.008)
		$r_{21}(1)$	$0.00^{\blacklozenge}$ (0.057)	$0.08^{\blacklozenge}$ (0.045)	$0.05^{\blacklozenge}$ (0.031)
$\alpha = 0.2$	$\kappa = 4.467$	$\hat{\kappa}$	-0.10 (0.541)	1.02 (0.385)	2.01 (0.254)
		k	-2.11 (0.790)	1.52 (0.738)	5.46 (0.541)
$\beta = 0.98$	$\rho_2(1) = 0.212$	$\hat{\rho}_2(1)$	-6.67 (0.047)	-3.73 (0.033)	-1.47 (0.022)
		$r_2(1)$	-18.98 (0.067)	-12.60 (0.053)	-5.55 (0.040)
$\delta_E = 0$	$\rho_{21}(1) = 0$	$\hat{\rho}_{21}(1)$	$-0.01^{\blacklozenge}$ (0.016)	$0.06^{\blacklozenge}$ (0.010)	$0.01^{\blacklozenge}$ (0.006)
		$r_{21}(1)$	$0.11^{\blacklozenge}$ (0.050)	$0.00^{\blacklozenge}$ (0.035)	$0.03^{\blacklozenge}$ (0.024)

Table 3.7: Percentages of Monte Carlo biases and standard deviations in the symmetric EGARCH Gaussian models. "$\blacklozenge$" Means that the difference is absolute.

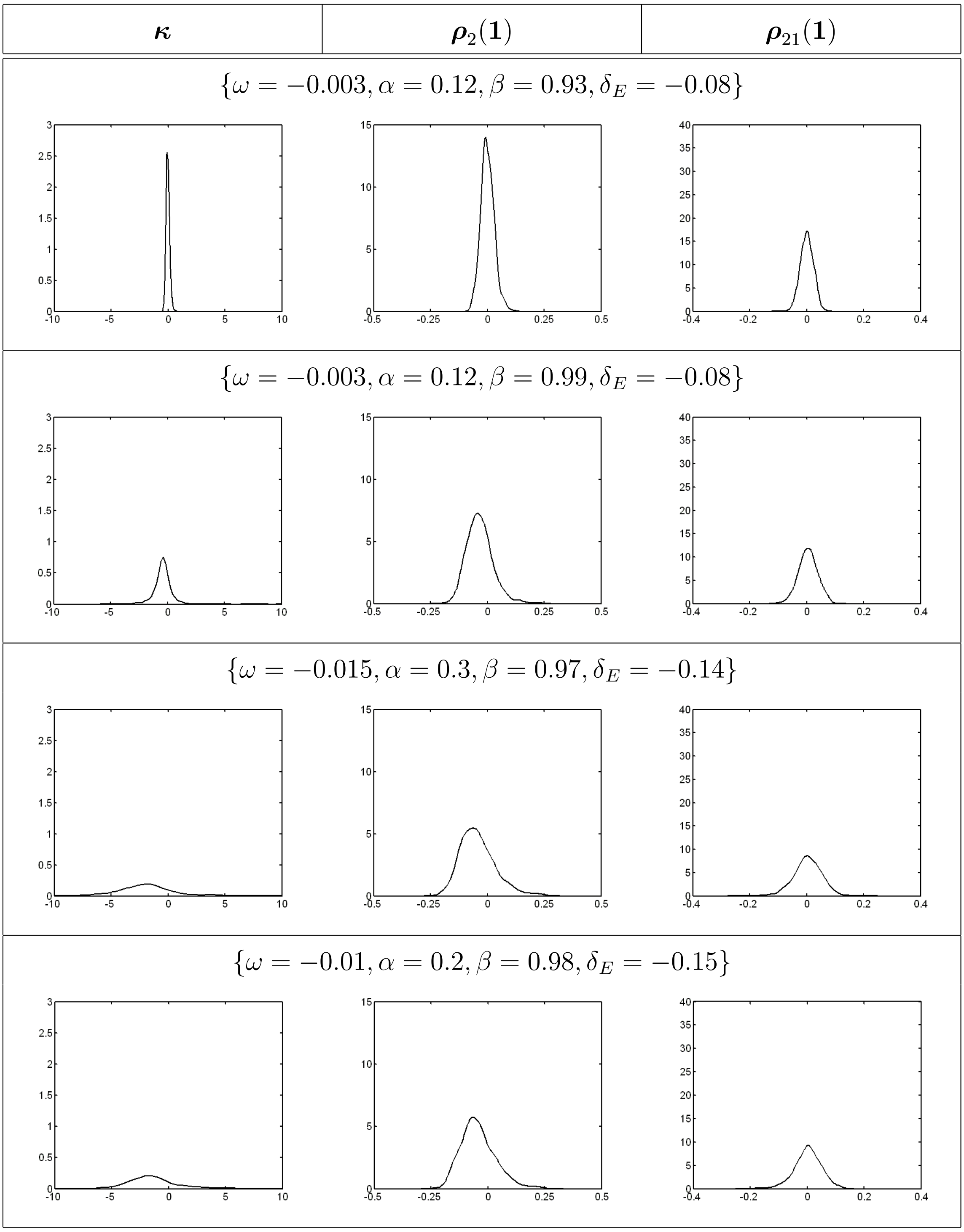


Figure 3.4: Kernel densities of the Monte Carlo differences between sample moments and plug-in moments for different EGARCH Gaussian models when $T = 2000$.

	SP500					EUR/USD				
	Sample	*Residuals*		*Plug-in*		*Sample*	*Residuals*		*Plug-in*	
		TGARCH	EGARCH	TGARCH	EGARCH		TGARCH	EGARCH	TGARCH	EGARCH
Kurtosis	7.12	4.19*	4.27*	16.82	16.13	4.28	3.50*	3.49*	4.71	4.14
$\rho_2(1)$	0.34	−0.07	−0.07	0.25	0.20	0.12	−0.03	−0.03	0.11	0.10
$Q_2(20)$	3148.50*	30.04	25.31	–	–	616.08*	15.86	15.21	–	–
$\rho_{21}(1)$	−0.07	−0.05	−0.05	−0.07	−0.05	0.02	0.02	0.02	0.00	0.00
$Q_{21}(20)$	217.15*	30.77	27.56	–	–	34.41	18.30	17.35	–	–

* Significant at 5% level

Table 3.8: Sample moments together with their corresponding diagnostic statistics and plug-in moments.

Chapter 4

Summary of conclusions and future research

In this thesis we focus on analyzing and comparing the ability of asymmetric GARCH models to represent conditional volatility when it has leverage effect.

In Chapter 2 we compare the theoretical limitations of several GARCH-type models when their parameters are restricted to satisfy positivity, stationarity and finite kurtosis restrictions. This limitations come from the functional form selected to represent the volatility under each model. We show that the QGARCH and GJR models although very popular in empirical applications, lack of flexibility to represent the asymmetry often observed in real time series of financial returns. In the first model, the limitations come mostly from the positivity restriction because it strongly limits the maximum value for the asymmetry parameter, δ_Q, when α is small, which is the case for financial time series. In the GJR model, is the finite kurtosis restriction, above all, the one that limits the asymmetry of the model. This condition is even more restrictive the larger is β and the smaller is α, which are again the usual combination of parameters when fitting real financial data. On the other hand, the TGARCH and EGARCH models show more flexibility to represent the dynamic evolution of volatilities when they are restricted to guarantee stationarity and finite kurtosis together with positive conditional variances.

In Chapter 3 we study whether comparing sample moments of a series as, for example, kurtosis, autocorrelations of squares or cross-correlations with the corresponding plug-in values implied by a fitted is appropriate in order to decide about the adequacy of the model.

We conclude that plug-in and sample moments have very different properties as estimators of their corresponding population moments. Consequently comparing them is not adequate independently of the asymmetric GARCH model considered. Within the models considered, QGARCH, TGARCH or EGARCH, the QGARCH model show the biggest dispersion between sample and plug-in moments when compared to corresponding quantities obtained when analyzing TGARCH or EGARCH models.

As we mentioned in previous chapters, it is of interest to analyze the abilities of asymmetric Stochastic Volatility Models from the same points of view considered in Chapter 2, checking their theoretical limitations, and in Chapter 3, revisiting the relationships between their sample, plug-in and population moments.

Finally, from a predictive perspective, comparing the predictive limitations that each model could show from the restrictions that arise from the volatility expressions of both asymmetric GARCH and asymmetric Stochastic Volatility models it is doubtless a question to be considered in future research.

Chapter 5

Resumen

El objetivo de esta tesis es analizar y comparar la capacidad de algunos de los modelos habituales de series temporales para representar la volatilidad de las series financieras y sus características más importantes. En concreto, una de las principales consiste en que las series suelen presentan mayor número de observaciones extremas que las esperadas bajo Gausianidad. Además, las observaciones se agrupan de tal manera que tras movimientos grandes siguen movimientos grandes, mientras que por el contrario, cuando los movimientos comienzan a ser pequeños siguen siéndolo durante cierto tiempo. Este agrupamiento de volatilidad se refleja a través de la autocorrelación de cuadrados que suele ser significativa, positiva y presenta decaimiento exponencial.

Finalmente, otra característica extensamente observada y propuesta por Black (1976) es la respuesta asimétrica de la volatilidad ante rendimientos positivos o negativos y conocida como *leverage effect*. En concreto, el incremento en la volatilidad es mayor cuando los retornos anteriores son negativos que cuando éstos son de la misma magnitud pero positivos. La presencia de este tipo de comportamiento se detecta en las correlaciones cruzadas entre rendimientos y rendimientos futuros al cuadrado, que habitualmente son significativos y negativos.

Los modelos considerados en esta tesis son algunos de los más conocidos dentro de la familia de modelos GARCH de leverage effect. Son el modelo Quadratic GARCH (QGARCH), propuesto de manera independiente por Engle y Ng (1993) y Sentana (1995), el modelo Threshold GARCH (TGARCH) de Zakoïan (1994), el modelo GJR de Glosten *et al.* (1993), el modelo GARCH Exponencial (EGARCH) de Nelson (1991) y el modelo Asymmetric Power

GARCH (APARCH) de Ding *et al.* (1993).

En esta tesis, nos centramos en el análisis bajo dos perspectivas diferentes. En primer lugar, desde un punto de vista teórico, en el Capítulo 2 estudiamos y comparamos cómo la dinámica de la volatilidad representada por cada modelo queda restringida al asumir que éste verifica las condiciones de positividad para la varianza, la estacionariedad y la existencia de kurtosis finita. Éstas condiciones se definen unívocamente para cada modelo como consecuencia de la forma funcional que lo define y limitan su espacio paramétrico. Además, en la Sección 2.3 analizamos a través de la metodología Monte Carlo, cómo incluso partiendo de series construidas bajo modelos GARCH asimétricos con momentos finitos, podría llegarse a conclusiones diferentes dependiendo del tipo de modelo elegido para ajustar las series. Finalmente, en la Sección 2.4 se consideran series financieras reales para ilustrar las capacidades descritas en el Capítulo 2 para los distintos modelos. Las conclusiones del capítulo se recogen en la Sección 2.5, e indican que los modelos se agrupan bajo dos patrones diferenciados. En primer plano estarían los modelos TGARCH, EGARCH y APARCH, para los que las restricciones sobre los parámetros no condicionan la dinámica de los modelos cuando se ajustan a series financieras. Por otro lado, la volatilidad condicional estimada por estos tres modelos es muy similar. En otro plano estarían los modelos QGARCH y GJR cuyas restricciones para la existencia de los momentos limitan fuertemente la dinámica de la volatilidad que pueden representar.

En la práctica, tras ajustar un modelo GARCH concreto a una serie temporal de rendimientos, es habitual analizarlo comparando los mometos muestrales de la serie con los plug-in inferidos por el modelo. En el Capítulo 3 estudiamos si este segundo enfoque de análisis es o no adecuado. Las secciones 3.2 y 3.3 se ocupan del estudio de los modelos mediante metodologías Monte Carlo, mientras en que la Sección 3.4 se reserva al análisis bajo casos reales de series financieras. Las conclusiones desarrolladas en la Sección 3.5 indican, entre otros aspectos, cómo el diferente comportamiento entre los momentos muestrales y plug-in con respecto a los poblaciones, puede conllevar a conclusiones inapropiadas.

Como capítulo final, en el Capítulo 4 se resumen las principales conclusiones extraídas de los capítulos anteriores y se puntualizan algunas futuras líneas de estudio que surgen como consecuencia de lo expuesto en esta tesis. Entre ellas, cabe destacar la importancia de comparar las propiedades aquí estudiadas bajo el enfoque de la predicción, analizando el impacto de imponer condiciones de existencia de momentos sobre la capacidad predictiva de los modelos. Por otro lado, también sería de interés analizar las capacidades de los modelos de volatilidad estocástica bajo los mismos enfoques de estudio que hemos planteado para los modelos GARCH de leverage effect.

Bibliography

[1] Alberg, D., H. Shalit, and R. Yosef. 2008. Estimating stock market volatility using asymmetric GARCH models. *Applied Financial Economics* 18: 1201-1208.

[2] An, L., and S.E. Ahmed. 2008. Improving the performance of kurtosis estimator.*Computational Statistics and Data Analysis*. 52: 2669-2910.

[3] Arvanitis, S., and A. Demos. 2004. Time dependence and moments of a family of time-varying parameter GARCH in mean models. *Journal of Time Series Analysis* 25: 155-171.

[4] Asai, M. 2008. Autorregressive Stochastic Volatility models with heavy-tailed distributions. A comparison with Multifactor Volatility Models. *Journal of Empirical Finance* 15: 332-341.

[5] Awartani, B.M.A., and V. Corradi. 2004. Predicting the volatility of the S&P-500 stock index via GARCH models: the role of asymmetries. *International Journal of Forecasting* 21: 167-183.

[6] Baillie, R., and H. Chung. 2001. Estimation of GARCH models from the autocorrelations of the squares of a process. *Journal of Time Series Analysis* 22(6): 631-650.

[7] Babsiri, M., and J.M. Zakoïan. 2001. Contemporaneous asymmetry in GARCH processes. *Journal of Econometrics* 101: 255-294.

[8] Balaban, E. 2004. Comparative forecasting performance of symmetric and asymmetric conditional volatility models of an exchange rate. *Economics Letters* 83: 99-105.

[9] Black R. 1976. Studies in stock price volatility changes. *Proceedings of the 1976 Business Meeting of the Business and Economics Statistics Section, American Statistical Association,* 177-181.

[10] Bollerslev, T. 1988. On the correlation structure for the Generalized Autorregresive Heteroskedastic process. *Journal of Time Series Analysis* 9: 121-131.

[11] Brännäs, K., and J.G. Gooijer. 2004. Asymmetries in conditional mean and variance: modelling stock returns by asMA-asQGARCH. *Journal of Forecasting* 23: 155-171.

[12] Breidt, F.J., N. Crato, and P.J.F. de Lima. 1998. The detection and estimation of the Long Memory in Stochastic Volatility. *Journal of Econometrics* 83: 325-348.

[13] Carnero, M.A., D. Peña, and E. Ruiz. 2004. Persistence and kurtosis in GARCH and Stochastic Volatility Models. *Journal of Financial Econometrics* 2: 319-342.

[14] Carnero, M.A., D. Peña, and E. Ruiz. 2008. Estimating and forecasting GARCH volatility in the presence of outliers. IVIE, WP-AD 2008-13.

[15] Chen, C.W., R. Gerlach, and M. So. 2006. Comparison of nonnested asymmetric heteroskedastic models. *Computational Statistics and Data Analysis* 51: 2164-2178.

[16] Deb, P. (1996). Finite sample properties of maximum likelihood and Quasi-Maximum likelihood estimators of EGARCH models, *Econometric Reviews,* 15(1): 51-68.

[17] Demos, A. 2002. Moments and dynamic structure of a time-varying-parameter stochastic volatility in mean model. *Econometrics Journal* 5: 345-357.

[18] Ding, Z., R.F. Engle, and C.W.J. Granger. 1993. A long memory property of stock market return and a new model. *Journal of Empirical Finance* 1: 83-106.

[19] Drost, F.C., and T.H.E. Nijman. 1993. Temporal aggregation of GARCH processes, *Econometrica* 61: 909-927.

[20] Durham, B.D. 2006. Monte Carlo methods for estimating, smoothing, and filtering one and two-factor stochastic volatility models. *Journal of Econometrics* 133: 273-305.

[21] Engle, R.F. 2003. Risk and volatility: Econometric models and financial practice. *Nobel Lecture.*
http://nobelprize.org/nobel_prizes/economics/laureates/2003/engle-lecture.html

[22] Engle, R.F., and V.K. Ng. 1993. Measuring and testing the impact of news on volatility. *Journal of Finance* 48: 1749-1778.

[23] Eraker, B., M. Johannes, and N. Polson. 2003. The impact of jumps in volatility and returns. *Journal of Finance* 58: 1269-1300.

[24] Figà-Talamanca, G. 2008. Testing Volatility autocorrelation in the Constant Elasticity of Variance Stochastic Volatility Model. *Computational Statistics and Data Analysis* 53: 2201-2218.

[25] Giot, P., and S. Laurent. 2003. Value-at-risk for long and short trading positions. *Journal of Applied Econometrics* 18: 641-663.

[26] Glosten, L.R., R. Jaganathan, and D.E. Runkle. 1993. Relationship between the expected value and the volatility of the nominal excess return on stocks. *The Journal of Finance* 48: 1779-1801.

[27] Hansen, P.R., and A. Lunde. 2005. A forecast comparison of volatility models: does anything beat a GARCH(1,1)?. *Journal of Applied Econometrics* 20: 873-889.

[28] Harvey, A., and N.G. Shephard. 1996. Estimation of an Asymmetric Stochastic Volatility Model for asset returns. *Journal of Business and Economic Statistics* 14: 429-439.

[29] He, C., and T. Teräsvirta. 1999a. Properties of moments of a family of GARCH processes. *Journal of Econometrics* 92: 173-192.

[30] He, C., and T. Teräsvirta. 1999b. Statistical properties of the asymmetric power ARCH process, in Engle, R.F. and H. White (eds.), *Cointegration, Causality and Forecasting. A Festschrift in Honour of Clive W.J. Granger*, Oxford University Press.

[31] He, C., T. Teräsvirta, and H. Malmsten. 2002. Fourth moment structure of a family of first-order exponential GARCH models. *Econometric Theory* 18: 868-885.

[32] He, C., A. Silvennoinen, and T. Teräsvirta. 2008. Parameterizing unconditional skewness in models for financial time series. *Journal of Financial Econometrics* 6: 208-230.

[33] Hentschel, L. 1995. All in the family: Nesting symmetric and asymmetric GARCH models. *Journal of Financial Economics* 39: 71-104.

[34] Karanasos, M., and J. Kim. 2003. Moments of the ARMA-EGARCH model. *Econometrics Journal* 6: 146-166.

[35] Karanasos, M., and J. Kim. 2006. A re-examination of the asymmetric power ARCH model. *Journal of Empirical Finance* 13: 113-128.

[36] Kawakatsu, H. 2007. Specification and estimation of discrete time quadratic stochastic volatility models. *Journal of Empirical Finance* 14: 424-442.

[37] Lee, W., J. Lim, Y. Lee, and J. Castillo. 2011. The hierarchical-likelihood approach to Autoregressive Stochastic Volatility models. *Computational Statistics and Data Analysis* 55: 248-260.

[38] Loudon, G.F., W.H. Watt, and P.K. Yadav. 2000. An empirical analysis of alternative parametric ARCH models. *Journal of Applied Econometrics* 15: 117-136.

[39] Malmsten, H., and T. Teräsvirta. 2004. Stylized facts of financial time series and three popular models of volatility. *SSE/EFI Working papers series in Economics and Finance 563*. Stockholm School of Economics.

[40] Meddahi, N., and E. Renault. 2004. Temporal aggregation of volatility models. *Journal of Econometrics* 119: 355-379.

[41] Nelson, D.B. 1991. Conditional heteroskedasticity in asset returns: A new approach. *Econometrica* 59: 347-370.

[42] Omori, Y., and T. Watanabe. 2008. Block sampler and posterior mode estimation for asymmetric stochastic volatility models. *Computational Statistics and Data Analysis* 52: 2892-2910.

[43] Pérez, A., and E. Ruiz. 2003. Properties of the Sample Autocorrelations of Nonlinear Transformations in Long-Memory Stochastic Volatility Models. *Journal of Financial Econometrics* 1: 420-444.

[44] Pérez, A., and P. Zaffaroni. 2008. Finite-sample properties of Maximum Likelihood and Whittle estimators in EGARCH and FIEGARCH models. *Quantitative and Qualitative Analysis in Social Sciences* 2(1): 78-97.

[45] Pérez, A., E. Ruiz, and H. Veiga. 2009. A note on the properties of power-transformed returns in long-memory stochastic volatility models with leverage effect. *Computational Statistics and Data Analysis.* 53: 3593-3600.

[46] Raggi, D., and S. Bordignon. 2006. Comparing stochastic volatility models through Monte Carlo simulations. *Computational Statistics and Data Analysis* 50: 1678-1699.

[47] Ruiz, E., and H. Veiga. 2008. Modelling long-memory volatilities with leverage effect: A-LMSV versus FIEGARCH. *Computational Statistics and Data Analysis* 52: 2846-2862.

[48] Sentana, E. 1995. Quadratic ARCH models. *Review of Economics Studies* 62: 639-661.

[49] Smith D.R., 2009. Asymmetry in Stochastic Volatility models: threshold or correlation? *Studies in Nonlinear Dynamics and Econometrics*: 13(3).

[50] So, M.K.P., W.K. Li, and K. Lam. 2002. A threshold stochastic volatility model. *Journal of Forecasting* 21: 473-500.

[51] Straumann, D., and T. Mikosch. 2006. Quasi-Maximum-Likelihood estimation in conditionally heteroscedastic time series: a stochastic recurrence equations approach, *The Annals of Statistics* 34(5): 2449-2495.

[52] Wu, G., and Z. Xiao. 2002. A generalized asymmetric partially linear model of asymmetric volatility. *Journal of Empirical Finance* 9: 287-319.

[53] Yu, J. 2005. On leverage in a Stochastic Volatility model. *Journal of Econometrics* 127: 165-178.

[54] Zakoïan, J.M. 1994. Threshold Heteroskedastic Models. *Journal of Economic Dynamics and Control* 18: 931-944.

[55] Zhang, X., ML. King. 2008. Box-Cox stochastic volatility models with heavy tails and correlated errors. *Journal of Empirical Finance* 15: 549-566.