Universidad de Valladolid

TESIS DOCTORAL:

A dynamical theory for monotone neutral functional differential equations with application to compartmental systems

Presentada por Víctor Muñoz Villarragut para optar al grado de doctor por la Universidad de Valladolid

Dirigida por Rafael Obaya García

Certifico que la presente memoria ha sido realizada por Víctor Muñoz Villarragut bajo mi dirección en el Departamento de Matemática Aplicada de la Universidad de Valladolid.

Valladolid, a 10 de diciembre de 2010.

Fdo.: Dr. Rafael Obaya García

Contents

Acknowledgements

Me gustaría dar las gracias a todos los que me han apoyado durante la realización de este trabajo.

En especial, quiero dar las gracias a Rafael Obaya por haber aceptado dirigir esta tesis y por la calidad del trabajo que ha resultado de la misma.

También quiero dar las gracias a Russell Johnson por haberme iniciado en diferentes aspectos de esta materia durante mi estancia en Florencia en el verano de 2008.

Agradezco su apoyo a Jorge Álvarez, Rubén Berenguel, Jesús Manuel Domínguez, Carlos Lorenzo, Carmen Núñez, Jesús Rojo y Ana Sanz. Merece un agradecimiento especial Sylvia Novo, con quien he trabajado de forma constante y me ha ido guiando por los entresijos de los sistemas dinámicos.

Doy las gracias a mis padres y a mi hermana, que me han apoyado durante toda mi vida, y a Leticia, sin quien no habría llegado hasta aquí.

Introduction

One of the main questions in the theory of non-autonomous differential equations is the description of the long-term behavior of their trajectories. When the functions defining such equations present a recurrent variation in time, their solutions naturally define a skew-product semiflow. Thanks to this skew-product semiflow, the trajectories can be analyzed in detail by means of methods of topological dynamics. In this work, the structure of omega-limit sets is studied, which allows a global view of the dynamics of the equation. It is well known that, in some cases, omega-limit sets inherit some dynamical properties of the field defining the equation; in other cases, their dynamics might be far more complex.

Functional differential equations (FDEs for short) with delay are a specific kind of differential equations which take into account not only the present state of the system, but also some of its past states. Their practical interest lies in the fact that they allow to construct mathematical models of processes for which the past has an influence over the future; some remarkable applications are models in epidemiology, population dynamics, and control engineering. Neutral functional differential equations (NFDEs for short) with delay are a very important generalization of such equations. They consider the derivative of the value of an operator rather than the derivative of the solution. Thus, models using NFDEs can represent spontaneous increments and decrements of the solution apart from the time dependence provided by FDEs.

Some of the earliest results on NFDEs are due to Hale and Meyer [HM] in the 1960s. A fast development of the theory of NFDEs ensued, and, as a result, nowadays there is a vast amount of theoretical and practical results in such theory (see Hale [Ha], Hale and Verduyn Lunel [HV], Kolmanovskii and Myshkis [KM], Salamon [Sa], and the references therein).

The study of the dynamical properties of a skew-product semiflow has been often tackled by assuming some monotonicity conditions on the semiflow. These conditions are a helpful tool when it comes to deducing the long-term behavior of the solutions. It is noteworthy that there is a great diversity of monotonicity conditions, varying from quasimonotonicity to strong monotonicity.

Monotone autonomous differential equations have been widely studied for decades (see Hirsch [Hi], Matano [M], Poláčik [P], and Smith [Sm], among many others). Under adequate hypotheses, it has been proved that the relatively compact trajectories of a strongly monotone semiflow converge generically to the set of equilibria. Subsequently, Smith and Thieme [ST, ST2] studied the dynamics of the semiflow induced by a FDE with finite delay, which is monotone for the exponential order. This order relation is technically complicated, but it allowed them to study equations which do not satisfy the usual quasimonotone condition associated to the standard order. These results were extended by Krisztin and Wu [KW], and Wu and Zhao [WZ] to scalar NFDEs with finite delay and evolutionary equations, respectively.

Recently, a big effort has been done in order to study deterministic and random monotone non-autonomous differential equations, providing a new dynamical theory for both the standard order and the exponential order (see e.g. Chueshov [Chu], Jiang and Zhao [JZ], Muñoz-Villarragut, Novo, and Obaya [MNO], Novo, Obaya, and Sanz [NOS], Novo, Obaya, and Villarragut [NOV], and Shen and Yi [SY]). Assuming some properties of boundedness, relative compactness, and uniform stability of the trajectories, this theory ensures the convergence of the orbits to solutions which reproduce the dynamics presented by the time variation of the equation. It is noteworthy that, when dealing with FDEs with infinite delay or, more generally, NFDEs with infinite delay, the property of strong monotonicity never holds, so that weaker assumptions on the monotonicity of the semiflow need to be made.

The origin of this theory goes back to the 1970s, when Sacker and Sell [SS] proved some previous results on the structure of omega-limit sets in the case of almost periodic equations. Later on, their work was followed in [SY] for the case of a distal base flow. More general results can be found in [NOS]; specifically, they studied the structure of omega-limit sets in BU, the space of bounded and uniformly continuous functions from $(-\infty, 0]$ into $\mathbb{R}^m$ endowed with the compact-open topology, when the base flow is just minimal and assuming a property of stability, which is closely connected to fiber distallity. They also deduce that it is appropriate to consider that topology when studying NFDEs with infinite delay because, under natural assumptions, the restrictions of the semiflows defined by these equations to their omega-limit sets turn out to be continuous.

An alternative study of the recurrent solutions of almost periodic FDEs can be made by means of fading memory spaces (see Hino, Murakami, and Naito [HMN] for an axiomatic definition and their main properties), though, under natural assumptions, the topology of the norm on these spaces coincides with the compact-open topology over the closure of relatively compact trajectories, which makes the approach in [NOS] seem more reasonable.

Another interesting approach to the study of NFDEs can be found in Staffans [St], where it is established that any NFDE with finite delay and autonomous stable operator can be written as a FDE with infinite delay in an appropriate fading memory space. Gripenberg, Londen, and Staffans [GLS] study the main properties of the convolution operator associated to the equation. These ideas were used in some subsequent papers (see e.g. Arino and Bourad [AB], and Haddock, Krisztin, Terjéki, and Wu [HKTW]). More general results in this line can be found in [MNO] and [NOV], where autonomous linear operators with infinite delay are considered. Many problems previously solved for FDEs have been generalized to the case of NFDEs; in turn, these extensions have raised challenging problems giving rise to the present framework. In the case of monotone NFDEs with infinite delay and autonomous operator, a transformation of both the standard order and the exponential order by means of the convolution operator associated to the equation provides the tool needed to achieve the expected results, as seen in [MNO] and [NOV].

Some of the many models consisting of NFDEs with delay are compartmental models. They are formed by several compartments linked by means of pipes; the compartments contain some material which flows between them through the pipes, which takes a non-negligible time. In turn, the compartments create and destroy material, which is represented by the neutral part of the equation. The theoretical interest of these models lies in the existence of a first integral, guaranteeing some stability properties for the semiflow which are essential in the theory. These NFDEs model physical and biological processes for which there is some non-instantaneous balance, though they have been used in other areas such as economics. Some of these applications are ecology, epidemiology, pharmacology, thermodynamics, control theory, and drug kinetics (see Eisenfeld [Ei2], and Haddad, Chellaboina, and Hui [HCH], among many others).

Compartmental systems have been used as mathematical models for the study of the dynamical behavior of many processes in biological and physical sciences (see Jacquez [Ja], Jacquez and Simon [JS, JS2], and the references therein). Some initial results for the case of FDEs with finite and infinite delay are due to Györi [G] and Györi and Eller [GE]. Later, Arino and Haourigui [AH] proved that compartmental systems described by almost periodic FDEs with finite delay give rise to certain almost periodic solutions. Györi and Wu [GW], Wu [W], Wu and Freedman [WF], [AB], and [KW] studied the case of compartmental systems represented by NFDEs with finite and infinite delay and autonomous operator. More recently, these results were extended in [MNO] and [NOV], concluding that relatively compact trajectories converge to solutions reproducing the time variation of the equation, and, moreover, a prediction on the eventual amount of material within the compartments depending on the geometry of the pipes can be done.

In this work, we study non-autonomous NFDEs with non-autonomous linear operator and infinite delay. In this situation, the main conclusions in the previous literature do not remain valid, and, thus, the extension of the theory requires the use of an alternative definition of exponential order which can be applied in the present context, preserving the dynamical properties of the preceding theory. We assume some recurrence properties on the temporal variation of the NFDE; thus, its solutions induce a skew-product semiflow with a minimal flow on the base, Ω. In particular, the almost periodic and almost automorphic cases are included in this formulation. We invert the non-autonomous convolution operator associated to the equation, generalizing previous results in this line found in [MNO]. The regularity properties of this convolution operator depend on the kind of recurrence presented by the time variation of the equation. Besides, new transformed order relations, associated to both the standard order and the exponential order, are considered; since the operator is non-autonomous, this partial order is not defined on BU, but on each fiber of the product $\Omega \times BU$ instead. Thus, we give an alternative version of the order structure introduced in [WF] which is valid in the case of non-autonomous operators. When using BU as a phase space, the standard theory of NFDEs provides existence, uniqueness, and continuous dependence of the solutions. This allows us to study the structure of the omega-limit sets of bounded trajectories when the equation satisfies some monotonicity condition, improving previous results found in [MNO], [NOV], [ST], and [ST2], among others.

The use of the transformed exponential order makes it possible to impose monotonicity conditions which do not require the differentiability of the coefficients defining the operator, but only their continuity. This makes the transformed exponential order more natural than the direct exponential order when the operator is non-autonomous. These theoretical results are applied to compartmental systems, and so we obtain conclusions under more general conditions than those presented in the previous literature, improving this way some previous results on dynamical systems which are monotone for the exponential order even in their autonomous versions. Specifically, we describe the eventual amount of material within the compartments in the case of compartmental systems defined by NFDEs with non-autonomous operator and infinite delay.

Nonetheless, in Chapters 9 and 10, we assume the differentiability of the coefficients defining the operator and study some compartmental systems which are monotone for the direct exponential order. In addition, we show that the 1-covering property of omega-limit sets holds, extending this way earlier results in [KW] to the case of NFDEs with a recurrent variation in time.

This work is organized as follows. In Chapter 1, we present some basic preliminary definitions and results of topological dynamics. Chapter 2 contains a study of the stability properties of the operator associated to the equation by means of the solutions of a difference equation for future times and another one for past times. Chapter 3 deals with the invertibility and the regularity of the convolution operator associated to the linear operator of the equation, depending on the recurrence properties of the coefficients of such equation. In Chapters 4 and 5, we present the theory concerning the transformed usual order to obtain the 1-covering property, and apply it to the case of compartmental systems. The long-term behavior of compartmental systems depending on their geometrical structure is tackled in Chapter 6; specifically, we study the eventual amount of mass within the compartments in terms of the pipes connecting them, and give some examples previously seen in the literature. In Chapters 7 and 8, we establish the 1-covering property of omega-limit sets by means of the transformed exponential order and apply this result to the study of compartmental systems, obtaining some monotonicity conditions which are different from those given in Chapter 5. Finally, in Chapters 9 and 10, we deal with the case of the direct exponential order and describe the topological structure of sets with some stability properties in order to establish the 1-covering property of omega-limit sets.

Chapter 1

Preliminaries

In this chapter, we will present some basic definitions and concepts of monotone dynamical systems and topological dynamics. There are a few works which are especially worth mentioning, namely, [Sm], [ST], and [ST2] on monotone dynamical systems; Ellis [El] on topological dynamics; and [SY] concerning skew-product semiflows. However, more detailed references will be given along the chapter.

1.1 Flows over compact metric spaces

Let (Ω, d) be a compact metric space. A real *continuous flow* $(\Omega, \sigma, \mathbb{R})$ is defined by a continuous mapping $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \sigma(t, \omega)$ satisfying the following conditions:

(i) $\sigma_0 = \mathrm{Id}$;

(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for all $t,\ s \in \mathbb{R}$;

where Id denotes the identity map of Ω, and $\sigma_t(\omega) = \sigma(t, \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. The set $\{\sigma_t(\omega) : t \in \mathbb{R}\}$ is called the *orbit* or the *trajectory* of the point ω. It is customary to denote this real continuous flow by $(\Omega, \sigma, \mathbb{R})$. Besides, this flow is usually referred to as a *dynamical system*, for it is a set of homeomorphisms $\{\sigma_t : \Omega \to \Omega\ :\ t \in \mathbb{R}\}$ with a group structure.

We say that a subset $\Omega_1 \subset \Omega$ is *σ-invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A subset $\Omega_1 \subset \Omega$ is called *minimal* if it is compact, σ-invariant, and its only nonempty compact σ-invariant subset is itself. Every compact and σ-invariant set contains a minimal subset (as a consequence of Zorn's lemma); in particular it is easy to prove that a compact σ-invariant subset is minimal if and only if every trajectory is dense. We say that the continuous flow $(\Omega, \sigma, \mathbb{R})$ is *recurrent* or *minimal* if Ω is minimal.

If $\omega_0 \in \Omega$ is a point such that the subset $\{\sigma_t(\omega_0) : t \geq t_0\} \subset \Omega$ is relatively compact for some $t_0 > 0$, then its *omega-limit set* can be defined by

$$\bigcap_{\tau \geq t_0} \mathrm{cls}\{\sigma(t+\tau, \omega_0) : t \geq 0\},$$

which is a compact and invariant subset. Analogously, given $\omega_0 \in \Omega$ such that the set $\{\sigma_t(\omega_0) : t \leq -t_0\} \subset \Omega$ is relatively compact for some $t_0 > 0$, we can consider its *alpha-limit set*, defined by

$$\bigcap_{\tau \leq -t_0} \mathrm{cls}\{\sigma(t+\tau, \omega_0) : t \leq 0\}.$$

Both omega-limit set and alpha-limit set contain minimal subsets.

The flow $(\Omega, \sigma, \mathbb{R})$ is *distal* if, for any two distinct points ω_1, $\omega_2 \in \Omega$, the orbits keep at a positive distance, that is, $\inf_{t\in\mathbb{R}} d(\sigma(t, \omega_1), \sigma(t, \omega_2)) > 0$. The flow $(\Omega, \sigma, \mathbb{R})$ is *almost periodic* when, for every $\varepsilon > 0$, there is a $\delta > 0$ such that, if ω_1, $\omega_2 \in \Omega$ with $d(\omega_1, \omega_2) < \delta$, then $d(\sigma(t, \omega_1), \sigma(t, \omega_2)) < \varepsilon$ for every $t \in \mathbb{R}$; equivalently, the flow $(\Omega, \sigma, \mathbb{R})$ is almost periodic if the family $\{\sigma_t\}_{t\in\mathbb{R}}$ is equicontinuous. If $(\Omega, \sigma, \mathbb{R})$ is almost periodic, then it is distal as well. The converse is not true; even if $(\Omega, \sigma, \mathbb{R})$ is minimal and distal, it does not need to be almost periodic. For the main properties of almost periodic and distal flows we refer the reader to Sell [Se, Se2], [El], and [SS]. If $\inf\{d(\sigma_t(\omega_1), \sigma_t(\omega_2)) : t \in \mathbb{R}\} = 0$, then the points ω_1 and ω_2 are said to be a *proximal pair*; otherwise, they are said to be a *distal pair*. Whenever $\inf\{d(\sigma_t(\omega_1), \sigma_t(\omega_2)) : t \geq 0\} = 0$ (resp. $\inf\{d(\sigma_t(\omega_1), \sigma_t(\omega_2)) : t \leq 0\} = 0$), it is said that the points ω_1 and ω_2 are a *positively* (resp. *negatively*) *proximal pair*.

Given another continuous flow $(Y, \Psi, \mathbb{R})$, a *flow homomorphism* from $(Y, \Psi, \mathbb{R})$ into $(\Omega, \sigma, \mathbb{R})$ is a continuous mapping $\pi\colon Y \to \Omega$ such that, for every $y \in Y$ and $t \in \mathbb{R}$, $\pi(\Psi(t, y)) = \sigma(t, \pi(y))$. If π is also surjective, then it is called a *flow epimorphism*; in this case, Ω is a *factor* of Y, and Y is an *extension* of Ω. If π is a flow epimorphism and there exists $k \geq 1$ such that $\mathrm{card}(\pi^{-1}(\omega)) = k$ for all $\omega \in \Omega$, then it is said that the flow $(Y, \Psi, \mathbb{R})$ is a *k-cover* or a *k-copy* of $(\Omega, \sigma, \mathbb{R})$. If $k = 1$, then the flows are isomorphic; in particular, they have the same topological properties. In such a case, we will simply say that they are covers or copies. As for homomorphisms between distal flows, now we present a relevant result (see [SY] and [SS]).

Theorem 1.1. *Let $(\Omega, \sigma, \mathbb{R})$ be a minimal and distal flow, and consider a homomorphism between distal flows $\pi : (Y, \Psi, \mathbb{R}) \to (\Omega, \sigma, \mathbb{R})$. If there is an $\omega \in \Omega$ such that* $\mathrm{card}(\pi^{-1}(\omega)) = N$ *for some $N \in \mathbb{N}$, then*

(i) *Y is an N-copy of Ω;*

(ii) $(Y, \Psi, \mathbb{R})$ *is almost periodic if and only if* $(\Omega, \sigma, \mathbb{R})$ *is almost periodic.*

Let $\pi : (Y, \Psi, \mathbb{R}) \to (\Omega, \sigma, \mathbb{R})$ be a flow epimorphism, and suppose that $(Y, \Psi, \mathbb{R})$ is a minimal flow (then, so is $(\Omega, \sigma, \mathbb{R})$, because, given $\omega = \pi(y)$ and $\omega_0 = \pi(y_0)$, there exists $\{t_n\}_n \subset \mathbb{R}$ such that $\Psi_{t_n}(y_0) \to y$ as $n \to \infty$, and, due to the continuity of π and its being a homomorphism, we have that $\pi(\Psi_{t_n}(y_0)) = \sigma_{t_n}(\omega_0) \to \omega$ as $n \to \infty$). $(Y, \Psi, \mathbb{R})$ is said to be an *almost automorphic extension* of $(\Omega, \sigma, \mathbb{R})$ if there exists $\omega \in \Omega$ such that $\mathrm{card}(\pi^{-1}(\omega)) = 1$. Furthermore, $(Y, \Psi, \mathbb{R})$ is said to be a *proximal extension* of $(\Omega, \sigma, \mathbb{R})$ if, whenever $\pi(y_1) = \pi(y_2)$ for some y_1, $y_2 \in Y$, then they are a proximal pair. An almost automorphic extension is always a proximal extension (see Veech [V]). From this last remark together with statement (i) of Theorem 1.1, it is deduced that, if $(Y, \Psi, \mathbb{R})$ is a minimal and almost periodic flow which is an almost automorphic extension of an almost periodic flow $(\Omega, \sigma, \mathbb{R})$, then it must be a copy of $(\Omega, \sigma, \mathbb{R})$.

A point $\omega_0 \in \Omega$ is said to be an *almost automorphic point* if, given any sequence $\{s_n\}_n \subset \mathbb{R}$, we can find a subsequence $\{t_n\}_n$ of it such that the limits $\lim_{n\to\infty} \sigma_{t_n}(\omega_0) = \omega_1$ and $\lim_{n\to\infty} \sigma_{-t_n}(\omega_1) = \omega_0$ exist. The flow $(\Omega, \sigma, \mathbb{R})$ is *almost automorphic* when there is an almost automorphic point which has a dense orbit. An almost automorphic flow is always minimal, that is, actually all the orbits are dense. According to a result in [V], a flow is almost automorphic if and only if it is an almost automorphic extension of an almost periodic (minimal) flow.

If $(Y, \Psi, \mathbb{R})$ is an almost automorphic flow and $(\Omega, \sigma, \mathbb{R})$ is an almost periodic (and minimal) flow satisfying that there exists a flow epimorphism $p : (Y, \Psi, \mathbb{R}) \to (\Omega, \sigma, \mathbb{R})$ such that $\mathrm{card}(p^{-1}(\omega)) = 1$ for some $\omega \in \Omega$, then the subset of Y formed by all of the almost automorphic points in Y is given by

$$\{y \in Y : p^{-1}(p(y)) = \{y\}\},$$

and it is a residual set (see Corollary 2.15 in [SY], part I).

We recall that a subset of a topological space E is said to be *residual* if its complementary is of first category in the sense of Baire, that is, its complementary is given by the union of countably many nowhere dense subsets of E.

1.2 Skew-product semiflows

Consider a complete metric space X, and let $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$. A *continuous semiflow* $(X, \Phi, \mathbb{R}^+)$ is determined by a continuous mapping

$\Phi : \mathbb{R}^+ \times X \to X,\ (t,x) \mapsto \Phi(t,x)$ which satisfies the following properties:

(i) $\Phi_0 = \mathrm{Id}$;

(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t,\ s \in \mathbb{R}^+$;

where Id denotes the identity map of X, and $\Phi_t(x) = \Phi(t,x)$ for each $x \in X$ and $t \in \mathbb{R}^+$. The set $\{\Phi_t(x) : t \geq 0\}$ is the *semiorbit* of the point x. A subset X_1 of X is *positively invariant* (or just Φ-*invariant*) if $\Phi_t(X_1) \subset X_1$ for all $t \geq 0$.

A semiflow $(X, \Phi, \mathbb{R}^+)$ admits a *flow extension* if there exists a continuous flow $(X, \widetilde{\Phi}, \mathbb{R})$ such that $\widetilde{\Phi}(t,x) = \Phi(t,x)$ for all $x \in X$ and $t \in \mathbb{R}^+$. A compact and positively invariant subset admits a flow extension if the semiflow restricted to it admits one.

Write $\mathbb{R}^- = \{t \in \mathbb{R} : t \leq 0\}$. A *backward orbit* of a point $x \in X$ in the semiflow $(X, \Phi, \mathbb{R}^+)$ is a continuous map $\psi : \mathbb{R}^- \to X$ such that $\psi(0) = x$, and, for each $s \leq 0$, it holds that $\Phi(t, \psi(s)) = \psi(s+t)$ whenever $0 \leq t \leq -s$. If $\{\Phi(t,x) : t \geq t_0\}$ is relatively compact for some $x \in X$ and $t_0 \geq 0$, we can consider the *omega-limit set* of x,

$$\bigcap_{s \geq t_0} \mathrm{cls}\{\Phi(t+s, x) : t \geq 0\}\,;$$

then all the elements y in the omega-limit set of x admit a backward orbit within that set. In fact, any compact and positively invariant set M admits a flow extension whenever all of the points in M admit a unique backward orbit which remains within M (see [SY], part II).

We will say that a compact subset K which is positively invariant for the semiflow $(X, \Phi, \mathbb{R}^+)$ is *minimal* whenever it does not include any proper, nonempty, closed, and positively invariant subset. If X itself is minimal, we will say that the semiflow $(X, \Phi, \mathbb{R}^+)$ is minimal.

A semiflow $(\Omega \times X, \tau, \mathbb{R}^+)$ is said to be a *skew-product semiflow* when it has the following form:

$$\begin{array}{rccc} \tau : & \mathbb{R}^+ \times \Omega \times X & \longrightarrow & \Omega \times X \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x)), \end{array} \tag{1.1}$$

where $(\Omega, \sigma, \mathbb{R})$ is a real continuous flow, $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t,\omega) \mapsto \omega{\cdot}t$, called *base flow* (we will assume in general that it is minimal), and (X, d) is a complete metric space. The skew-product semiflow (1.1) is *linear* if $u(t, \omega, \cdot)$ is linear for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$.

A semiorbit $\{\tau(t, \omega_0, x_0) : t \geq 0\}$ of the semiflow (1.1) is said to be *uniformly stable* if, for all $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, called *modulus of* *uniform stability*, such that, if $s \geq 0$ and $\mathsf{d}(u(s,\omega_0,x_0),x) \leq \delta$ for certain $x \in X$, then, for each $t \geq 0$,

$$\mathsf{d}(u(t+s,\omega_0,x_0),u(t,\omega_0{\cdot}s,x)) = \mathsf{d}(u(t,\omega_0{\cdot}s,u(s,\omega_0,x_0)),u(t,\omega_0{\cdot}s,x)) \leq \varepsilon.$$

Moreover, $\{\tau(t,\omega_0,x_0) : t \geq 0\}$ is *uniformly asymptotically stable* if it is uniformly stable and there is a $\delta_0 > 0$ with the following property: for each $\varepsilon > 0$ there is a $t_0(\varepsilon) > 0$ such that, if $s \geq 0$ and $\mathsf{d}(u(s,\omega_0,x_0),x) \leq \delta_0$, then

$$\mathsf{d}(u(t+s,\omega_0,x_0),u(t,\omega_0{\cdot}s,x)) \leq \varepsilon \quad \text{for each } t \geq t_0(\varepsilon)\,.$$

The reference [SY] includes results relative to the relation between the property of uniform stability and the extension of semiflows to flows.

1.3 Stability and extensibility results for omega-limit sets

Let us consider a continuous skew-product semiflow $(\Omega\times X,\tau,\,\mathbb{R}^+)$ defined as in (1.1) over a minimal base flow $(\Omega,\sigma,\mathbb{R})$ and a complete metric space (X,d). Let us remark that $(\Omega,\sigma,\mathbb{R})$ does not need to be distal. We will recall some results given in [NOS] which extend classical stability and extensibility results to this setting.

In order to do this, let us give the definitions of uniform stability and uniform asymptotic stability for a compact τ-invariant set $K \subset \Omega\times X$.

Definition 1.2. Let C be a positively invariant and closed set in $\Omega\times X$. A compact positively invariant set $K \subset C$ is *uniformly stable* (with respect to C) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, called the *modulus of uniform stability*, such that, if $(\omega,x) \in K$, $(\omega,y) \in C$ are such that $\mathsf{d}(x,y) < \delta(\varepsilon)$, then $\mathsf{d}(u(t,\omega,x),u(t,\omega,y)) \leq \varepsilon$ for all $t \geq 0$.

K is *uniformly asymptotically stable* if it is uniformly stable and, besides, there exists $\delta_0 > 0$ such that, if $(\omega,x) \in K$, $(\omega,y) \in C$ satisfy $\mathsf{d}(x,y) < \delta_0$, then $\lim_{t\to\infty} \mathsf{d}(u(t,\omega,x),u(t,\omega,y)) = 0$ uniformly in $(\omega,x) \in K$.

We will usually use this definition with either $C = \Omega\times X$ or $C = K$. If no explicit mention to C is made, then C is assumed to be the whole space, whereas if the restricted semiflow $(K,\tau,\mathbb{R}^+)$ is said to be uniformly stable, then the choice would be $C = K$. Clearly, if $C = \Omega\times X$, then all the trajectories in a uniformly (asymptotically) stable set are uniformly (asymptotically) stable as well. Conversely, if the semiorbit of certain (ω,x) is relatively compact and uniformly (asymptotically) stable, then the omega-limit set of (ω,x) is a uniformly (asymptotically) stable set with the same modulus of uniform stability as that of the semiorbit (see [Se]).

Given a compact and positively invariant set $K \subset \Omega \times X$, consider its projection over X, that is,

$$K_X = \{x \in X : \text{ there exists } \omega \in \Omega \text{ such that } (\omega, x) \in K\}.$$

The compactness of K implies that K_X is a compact subset of X as well. Let $\mathcal{P}_c(K_X)$ be the set of all closed subsets of K_X endowed with the Hausdorff metric ρ, that is, for any two sets A, $B \in \mathcal{P}_c(K_X)$,

$$\rho(A, B) = \sup\{\alpha(A, B),\ \alpha(B, A)\}\,,$$

where $\alpha(A, B) = \sup\{r(a, B) : a \in A\}$ and $r(a, B) = \inf\{\mathsf{d}(a, b) : b \in B\}$. At this point, we can consider the map

$$\begin{array}{ccl} \Omega & \longrightarrow & \mathcal{P}_c(K_X) \\ \omega & \mapsto & K_\omega = \{x \in X : (\omega, x) \in K\}\,, \end{array}$$

which is usually called *section map.* Thanks to the minimality of Ω and the compactness of K, the set K_ω is nonempty for every $\omega \in \Omega$; moreover, this map is well-defined and semicontinuous, and it has a residual set of continuity points (see Aubin and Frankowska [AF], and Choquet [Cho]).

Now we state a result relating the property of uniform stability to that of fiber distallity whenever there exists a flow extension.

Theorem 1.3. *Let $K \subset \Omega \times X$ be a compact τ-invariant set admitting a flow extension. If $(K, \tau, \mathbb{R})$ is uniformly stable as $t \to \infty$, then it is a fiber distal flow which is also uniformly stable as $t \to -\infty$. Furthermore, the section map for K, $\omega \in \Omega \mapsto K_\omega = \{x \in X : (\omega, x) \in K\} \in \mathcal{P}_c(K_X)$, is continuous at every $\omega \in \Omega$.*

The same result holds if we assume the existence of backward extensions of semiorbits.

Theorem 1.4. *Let $K \subset \Omega \times X$ be a compact positively invariant set such that every point of K admits a backward orbit. If the semiflow $(K, \tau, \mathbb{R}^+)$ is uniformly stable, then it admits a flow extension which is fiber distal and uniformly stable as $t \to -\infty$. Besides, the section map for K, $\Omega \to \mathcal{P}_c(K_X)$, $\omega \mapsto K_\omega$, is continuous at every $\omega \in \Omega$.*

The next result is a theorem on the structure of uniformly asymptotically stable sets admitting backward semiorbits. We prove that these sets are N-covers of the base flow.

Theorem 1.5. *Consider a compact positively invariant set $K \subset \Omega \times X$ for the skew-product semiflow* (1.1)*, and assume that every semiorbit in K admits a backward extension. If $(K, \tau, \mathbb{R}^+)$ is uniformly asymptotically stable, then it is an N-cover of the base flow $(\Omega, \sigma, \mathbb{R})$.*

We recall some results on the structure of omega-limit sets.

Proposition 1.6. *Let $\{\tau(t, \widetilde{\omega}, \widetilde{x}) : t \geq 0\}$ be a forward orbit of the skew-product semiflow* (1.1) *which is relatively compact, and let $\widetilde{K}$ denote the omega-limit set of $(\widetilde{\omega}, \widetilde{x})$. The following statements hold:*

(i) *if $\widetilde{K}$ contains a minimal set K which is uniformly stable, then $\widetilde{K} = K$, and it admits a fiber distal flow extension;*

(ii) *if the semiorbit is uniformly stable, then the omega-limit set $\widetilde{K}$ is a uniformly stable minimal set which admits a fiber distal flow extension;*

(iii) *if the semiorbit is uniformly asymptotically stable, then the omega-limit set $\widetilde{K}$ is a uniformly asymptotically stable minimal set which is an N-cover of the base flow.*

1.4 Infinite delay equations on the hull

We recall the basic properties of a skew-product semiflow determined by a family of functional differential equations with infinite delay. These results were presented in [NOS].

Let $(\Omega, \sigma, \mathbb{R})$ be a minimal flow over a compact metric space (Ω, d), and denote $\sigma(t, \omega) = \omega{\cdot}t$ for each $\omega \in \Omega$ and $t \in \mathbb{R}$. We consider $\mathbb{R}^m$ endowed with the maximum norm defined by $\|v\| = \max_{j=1,\ldots,m} |v_j|$ for all $v \in \mathbb{R}^m$, and the Fréchet space $X = C((-\infty, 0], \mathbb{R}^m)$ endowed with the compact-open topology, which is metrizable for a distance d.

Let $BU \subset X$ be the Banach space

$$BU = \{x \in X : x \text{ is bounded and uniformly continuous}\}$$

endowed with the supremum norm $\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|$. We will refer to this topology as the norm topology on BU. Given $r > 0$, we will denote

$$B_r = \{x \in BU : \|x\|_\infty \leq r\}\,.$$

We often consider the restriction of the compact-open topology to the subsets of BU, and we refer to this topology as the metric topology on BU. In particular, B_r is closed for the metric topology for each $r > 0$.

Given $I = (-\infty, a] \subset \mathbb{R}$, $t \in I$, and a continuous function $z : I \to \mathbb{R}^m$, z_t will denote the element of X defined by $z_t(s) = z(t+s)$ for all $s \in (-\infty, 0]$.

We are in a position to consider the family of non-autonomous infinite delay functional differential equations

$$z'(t) = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \tag{1.2$_\omega$}$$

defined by a function $F : \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$ which satisfies the following conditions:

(H1) F is continuous on $\Omega \times BU$ and locally Lipschitz continuous in x for the norm $\|\cdot\|_\infty$;

(H2) for each $r > 0$, $F(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^m$;

(H3) for each $r > 0$, $F\colon \Omega \times B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r, i.e. if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with $x \in B_r$, then $\lim_{n\to\infty} F(\omega_n, x_n) = F(\omega, x)$.

Thanks to (H1), the standard theory of infinite delay functional differential equations (see [HMN]) assures that for each $x \in BU$ and each $\omega \in \Omega$ the system $(1.2)_\omega$ locally admits a unique solution $z(\cdot, \omega, x)$ with initial value x, i.e. $z(s, \omega, x) = x(s)$ for each $s \in (-\infty, 0]$. As a result, the family $(1.2)_\omega$ induces a local skew-product semiflow, which is defined on an open subset $\mathcal{U}$ of $\mathbb{R}^+ \times \Omega \times BU$:

$$\begin{array}{rlcl} \tau & : \mathcal{U} \subset \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU \\ & \qquad (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x))\,, \end{array} \tag{1.3}$$

where $u(t, \omega, x) \in BU$ and $u(t, \omega, x)(s) = z(t+s, \omega, x)$ for all $s \in (-\infty, 0]$.

Using hypotheses (H1) and (H2), it is known that each bounded solution of $(1.2)_\omega$ gives rise to a relatively compact trajectory.

Proposition 1.7. *Let $z(\cdot, \omega_0, x_0)$ be a bounded solution of equation $(1.2)_{\omega_0}$. Then $\mathrm{cls}_X\{u(t, \omega_0, x_0) : t \geq 0\}$ is a compact subset of BU for the compact-open topology.*

From hypotheses (H1)–(H3), the continuity of the semiflow restricted to some compact subsets of $\Omega \times BU$ is deduced, when the compact-open topology is considered on BU.

Proposition 1.8. *Let $\{(\omega_n, x_n)\} \subset \Omega \times B_R$ for some $R > 0$ be such that $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ for $(\omega, x) \in \Omega \times B_R$. If it holds that*

$$\sup\{\|z(s, \omega_n, x_n)\| : s \in [0, t],\ n \geq 1\} \leq R$$

for some $t > 0$, then $u(t, \omega_n, x_n) \xrightarrow{\mathsf{d}} u(t, \omega, x)$.

Corollary 1.9. *Let $K \subset \Omega \times BU$ be a compact set for the product metric topology, and assume that there is an $r > 0$ such that $\tau_t(K) \subset \Omega \times B_r$ for all $t \geq 0$. Then the map*

$$\begin{array}{rccl} \tau & : \mathbb{R}^+ \times K & \longrightarrow & \Omega \times BU \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x))\,, \end{array}$$

is continuous when the product metric topology is considered.

It is important to note that Proposition 1.8 and Corollary 1.9 can be proved under more general assumptions. Nevertheless, the versions presented here yield Proposition 1.10, which was our aim.

Let us recall one more result from [NOS] concerning the extensibility of the semiflow to a flow on an omega-limit set, which can be defined as in Section 1.1 thanks to Proposition 1.7 by considering the compact-open topology on BU. This strong property is characteristic of the infinite delay setting.

Proposition 1.10. *Fix $(\omega_0, x_0) \in \Omega \times BU$, and let us suppose that we have $\sup_{t\geq 0} \|z(t, \omega_0, x_0)\| < \infty$. Then $K = \mathcal{O}(\omega_0, x_0)$ is a positively invariant compact subset admitting a flow extension.*

1.5 Almost periodic and almost automorphic dynamics

In order to find a link between non-autonomous differential equations with some recurrence in time and the theory of dynamical systems, we recall the basic definitions and results for the class of almost periodic and almost automorphic functions. We will give a brief explanation about the way this kind of equations give rise to skew-product flows or semiflows using the so-called hull as a base flow, which in turn will have some recurrence properties as well.

The concept of almost periodic function came up in the 1920s as an extension of the notion of periodicity. Some references like Bohr [Boh, Boh2] studied exhaustively the properties of these functions. The book by Fink [Fi] is a detailed and well written reference on this topic.

Several equivalent definitions of almost periodic function may be found in the literature. Thus, in order to study harmonic functions, it is better to choose the characterization (as adopted by Corduneanu [Co]) saying that a function is almost periodic whenever it can be approximated uniformly by a sequence of trigonometric polynomials on the whole real line, whereas, if our aim is to study differential equations, the preferred definition is the one introduced by Bohr, which is in the end the most frequently chosen one, (as seen in Amerio and Prouse [AP] and Besicovitch [Be]). A subset S of $\mathbb{R}$ is said to be *relatively dense* if there exists $l > 0$ such that every interval of length l intersects S. A complex function f, defined and continuous on $\mathbb{R}$, is *almost periodic* if, for all $\varepsilon > 0$, the set

$$T(f, \varepsilon) = \{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense. The set $T(f, \varepsilon)$ is called *ε-translation set of f*. Almost periodic functions are bounded and uniformly continuous on $\mathbb{R}$. The set formed by all these functions is an algebra over $\mathbb{C}$, which is invariant by translations and closed under conjugation and uniform limits. Moreover, if f is almost periodic and $|f(t)| \geq m > 0$ for all $t \in \mathbb{R}$, then the function $1/f$ is almost periodic as well. Besides, if f is almost periodic and differentiable, then f' is almost periodic if and only if it is uniformly continuous on $\mathbb{R}$. As for integration, if a primitive of an almost periodic function is bounded, then it is also almost periodic.

The concept of almost periodicity can be extended to continuous functions taking values in a complete metric space (E, d) in a straightforward way: for each $\varepsilon > 0$, the set

$$T(f, \varepsilon) = \{\tau \in \mathbb{R} : \mathsf{d}(f(t+\tau), f(t)) < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

must be relatively dense in $\mathbb{R}$. The reference [AP] contains a study about almost periodic functions taking values in a Banach space and their relation with the theory of functional equations.

Bochner introduced another equivalent definition in terms of sequences (adopted for instance in [Fi]): a continuous function f is almost periodic if, given any sequence $\{\alpha_n\}_n \subset \mathbb{R}$, we can find a subsequence $\{\alpha_{n_j}\}_j$ of the previous one such that $\lim_{j\to\infty} f(t + \alpha_{n_j})$ exists uniformly on $\mathbb{R}$.

Besides, Bochner pointed out that, in order to simplify the proofs involving almost periodic functions, a property satisfied by such functions with respect to a group G could be used (see Bochner [Boc]); when $G = \mathbb{R}$, this property can be stated as follows: given a complex function f, defined and continuous on $\mathbb{R}$, and given any sequence $\{\alpha_n\}_n$ of real numbers, we can find a subsequence $\{\alpha_{n_j}\}_j$ in such a manner that the following limits exist pointwise on $\mathbb{R}$:

$$\lim_{j\to\infty} f(t + \alpha_{n_j}) = g(t),$$
$$\lim_{j\to\infty} g(t - \alpha_{n_j}) = f(t)$$

for some function g. All the functions satisfying that property, whether they are almost periodic or not, are said to be *almost automorphic*. The fundamental properties of these functions with respect to groups, together with almost automorphic abstract minimal flows, were studied by Veech [V, V2, V3], among others. In principle, the function g does not need to be continuous. If the function g is continuous for all sequences, then we say that f is *almost automorphic in the sense of Bohr*. From now on, we will assume that almost automorphic functions are almost automorphic in the sense of Bohr, so that almost automorphic functions are bounded and uniformly continuous on $\mathbb{R}$ (see [V]). Almost periodic functions are always almost automorphic, but the converse is not true; several examples can be found in the foregoing references.

In the early 1940s, Fréchet defined and studied the concept of asymptotic almost periodicity. A function f continuous on $\mathbb{R}^+ = [0, \infty)$ is said to be *asymptotically almost periodic* if it can be represented as $f = f_1 + f_2$, where f_1 is an almost periodic function, and f_2 vanishes pointwise as $t \to \infty$. In fact, that representation is unique.

The relation between almost periodic functions and almost periodic flows is quite simple (see [El], Nemytskii and Stepanoff [NS], and [Fi]). First, if $(\Omega, \sigma, \mathbb{R})$ is an almost periodic continuous real flow, then all the trajectories $t \in \mathbb{R} \mapsto \sigma(t, \omega) \in \Omega$ define almost periodic functions taking values in the compact metric space Ω. It is said that an element ω of a continuous real flow $(\Omega, \sigma, \mathbb{R})$ is an *almost periodic point* if, for any $\varepsilon > 0$, the set

$$T(\omega, \varepsilon) = \{\tau \in \mathbb{R} : d(\sigma(\tau, \omega), \omega) < \varepsilon\}$$

is relatively dense in $\mathbb{R}$; such points are sometimes referred to as points with a *recurrent* orbit (see [NS]). This condition is tantamount to the fact that the closure of the trajectory of such point, $\mathrm{cls}\{\sigma(t, \omega) : t \in \mathbb{R}\}$, is a minimal subset for the flow. Notice that, if the flow is minimal, then all its points are almost periodic. As a consequence, the flow $(\Omega, \sigma, \mathbb{R})$ can be decomposed as the disjoint union of a family of minimal subsets if and only all its points are almost periodic. Clearly, if the trajectory of ω, $t \in \mathbb{R} \mapsto \sigma(t, \omega) \in \Omega$, is an almost periodic function, then ω is an almost periodic point; moreover, in this case, the closure of its orbit is an almost periodic minimal set which coincides with both the omega-limit and alpha-limit sets of ω. Specifically, almost periodic flows are decomposed as a disjoint union of almost periodic and minimal flows.

As for almost automorphic flows, we know that there is an almost automorphic point with a dense orbit. If a point $\omega \in \Omega$ is almost automorphic, then its trajectory, $t \in \mathbb{R} \mapsto \sigma(t, \omega) \in \Omega$, is an almost automorphic function taking values in Ω (as before, the definition can be extended to this case in a natural manner). However, now there is no need for all the points to be almost automorphic, though all the points in a residual subset of Ω are (as we remarked in Section 1.1).

Conversely, let us check how to obtain almost periodic and almost automorphic flows from functions with analogous properties.

Definition 1.11. A function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is said to be *admissible* if, for every compact subset $K \subset \mathbb{R}^n$, f is bounded and uniformly continuous on $\mathbb{R} \times K$. Besides, if f is of class C^r ($r \geq 1$) in $x \in \mathbb{R}^n$ and f and all its partial derivatives with respect to x up to order r are admissible, then we will say that f is either *C^r-admissible* or *admissible of class C^r*. A function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ is *uniformly almost automorphic* (resp. *almost periodic*) if it is admissible and almost automorphic (resp. almost periodic) in $t \in \mathbb{R}$.

Given an admissible function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$, we consider the family of time translated functions $\{f_\tau : \tau \in \mathbb{R}\}$, where $f_\tau(t, x) = f(t + \tau, x)$ for all τ, $t \in \mathbb{R}$, and all $x \in \mathbb{R}^n$. Hence, we can define the *hull* of f, which will be denoted by Ω or $H(f)$, as the closure within the space $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ of the set of time translated functions for the compact-open topology, that is, the topology of uniform convergence over compact subsets. Thanks to Arzelà-Ascoli's theorem, we can assure that the space $H(f)$ is compact and, furthermore, metrizable. Moreover, a continuous real flow is induced over the hull in a natural way, just by considering the mapping $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(s, h) \mapsto h_s$, h translated a time s, that is, there is a flow over the hull defined by translation.

The next result assures that the initial function f admits a unique continuous extension to the hull and shows how the properties of recurrence of f are translated to the hull (see e.g. [SY]).

Theorem 1.12. *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ be an admissible function. The following statements hold:*

(i) *all the functions $h \in H(f)$ are admissible; in fact, if f is admissible of class C^r, so are all the functions $h \in H(f)$;*

(ii) *there exists a unique function $F \in C(H(f) \times \mathbb{R}^n, \mathbb{R}^m)$ which extends f, in the sense that $F(f_t, x) = f(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$; besides, if f is C^r-admissible, then F is of class C^r in x;*

(iii) *the flow $(H(f), \sigma)$ is almost automorphic (resp. almost periodic) if f is uniformly almost automorphic (resp. almost periodic).*

It is convenient to point out that the function F is defined specifically by $F(h, x) = h(0, x)$, $(h, x) \in H(f) \times \mathbb{R}^n$. The construction of the flow on the hull is often used when dealing with differential equations, as we will see next.

Let $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a C^r-admissible function such that the flow $(H(f), \sigma)$ is minimal, and consider its unique continuous extension to the hull $\Omega = H(f)$, $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, which, according to the previous theorem, is a function of class C^r in $x \in \mathbb{R}^m$. In particular, if the initial equation is given by a uniformly almost periodic or almost automorphic function, then we are in the foregoing context. This way, from a system of non-autonomous ordinary differential equations

$$x' = f(t, x),$$

we can obtain a family of differential equations with indexes in the hull

$$x'(t) = F(\omega{\cdot}t, x(t)), \quad \omega \in \Omega,$$

where the flow on Ω is denoted by $\omega{\cdot}t = \sigma(t, \omega)$. Notice that, fixing $\omega = f$, we get the original system, i.e. $x'(t) = f(t, x(t))$.

According to the standard theory of existence, uniqueness, and continuation of solutions for this kind of equations (see e.g [Ha]), these families of systems give rise to a local skew-product flow

$$\begin{array}{rccc} \tau : & \mathcal{U} \subset \mathbb{R} \times \Omega \times \mathbb{R}^m & \longrightarrow & \Omega \times \mathbb{R}^m \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x)), \end{array}$$

where $u(t, \omega, x)$ is the value of the solution of the system corresponding to ω with initial value $x(0) = x$ at time t, for t in the interval where the solution is defined. Thanks to the classical theorems of continuous dependence with respect to the initial values, u inherits the same regularity, C^r, with respect to x.

The use of this technique, that is, of including a non-autonomous system within a family of systems linked to one another by means of the flow on the hull, is focused to the application of the methods and results of the theory of skew-product flows to the new problem, where the solutions of the systems have been considered as a part of the trajectories of a dynamical system. It is noteworthy that, in the new family of systems generated from a given system, there are just their translated systems as well as their limits, so that the flowassociated to this family is a good representation of the dynamics of the initial system and, in particular, the asymptotic behavior of its bounded solutions. The references Johnson [Jo, Jo2] contain examples of almost periodic differential equations with almost automorphic solutions which are not almost periodic (see also Jorba, Núñez, Obaya, and Tatjer [JNOT], and Yi [Y]).

That being said, it is important to mention that a similar construction involving a separable metric space rather than $\mathbb{R}^m$ yields analogous results to the ones presented above. We refer the reader to [HMN] for an extensive study on this issue. More specifically, Theorem 1.4 of Chapter 8 in [HMN] implies that the space BU introduced in Section 1.4 when endowed with the supremum norm is not a suitable choice as a phase space (a substitute for $\mathbb{R}^m$), for it is not separable. An alternative construction will be given below for a family of infinite delay differential equations of our interest.

1.6 Inclusion of a specific system in a family of systems on the hull

We now focus on a system of functional differential equations to illustrate the technique by means of which an infinite delay system of differential equations can be included in the skew-product setting. Namely, we consider

$$z_i{}'(t) = -\sum_{j=1}^{m} \widetilde{g}_{ji}(t, z_i(t)) + \sum_{j=1}^{m} \int_{-\infty}^{0} \widetilde{g}_{ij}(t+s, z_j(t+s))\, d\mu_{ij}(s)\,, \qquad (1.4)$$

$i = 1, \ldots, m$, where $\widetilde{g} = (\widetilde{g}_{ij})_{i,j} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{m\times m}$ and μ_{ij} is a regular Borel measure with finite total variation for all $i, j \in \{1, \ldots, m\}$. As mentioned in the previous section, it is not possible to use BU as a phase space because it is not separable. Our aim is to include this system of equations in a family of equations given over a minimal flow, so that we can use the skew-product formalism in the study of equation (1.4). It is clear, however, that $\mathbb{R}$ is separable, and it is $\mathbb{R}$ and not BU which takes part in the definition of $\widetilde{g}$.

In order to do so, let us assume the following hypotheses:

(E1) $\widetilde{g}$ is C^1-admissible;

(E2) $\widetilde{g}$ is a recurrent function, i.e. its hull is minimal.

Let $\Omega = H(\widetilde{g})$ be the hull of $\widetilde{g}$ endowed with the compact-open topology, that is, the topology of uniform convergence on compact sets. Thanks to (E1), Ω is a compact metric space (see [HMN]). Let $(\Omega, \sigma, \mathbb{R})$ be the continuous real flow defined on Ω by translation, $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t$, with $\omega{\cdot}t(s, v) = \omega(t + s, v)$ for all $(s, v) \in \mathbb{R}^2$. From hypothesis (E2), the flow $(\Omega, \sigma, \mathbb{R})$ is minimal. In addition, if $\widetilde{g}$ is almost periodic (resp. almost automorphic), then the flow will be almost periodic (resp. almost automorphic); both cases are included in this formulation.

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}^{m\times m}$, $(\omega, v) \mapsto \omega(0, v)$, continuous on $\Omega \times \mathbb{R}$, and denote $g = (g_{ij})_{i,j}$. It is clear that there is an $\widetilde{\omega} \in \Omega$ such that $g(\widetilde{\omega}{\cdot}t, v) = \widetilde{g}(t, v)$ for all $t \in \mathbb{R}$ and all $v \in \mathbb{R}^m$. Let $F : \Omega \times BU \to \mathbb{R}^m$ be the map defined by

$$F_i(\omega, x) = -\sum_{j=1}^{m} g_{ji}(\omega, x_i(0)) + \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}s, x_j(s))\, d\mu_{ij}(s)\,,$$

for $(\omega, x) \in \Omega \times BU$ and $i \in \{1, \ldots, m\}$. Hence, the family

$$z'(t) = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,,$$

is of the form $(1.2)_\omega$ and includes system (1.4) when $\omega = \widetilde{\omega}$.

1.7 Ordered Banach spaces

A Banach space X is said to be *ordered* if there exists a convex and closed cone, that is, a closed subset $X_+ \subset X$ such that

(i) $X_+ + X_+ \subset X_+$;

(ii) $\mathbb{R}^+ X_+ \subset X_+$;

(iii) $X_+ \cap (-X_+) = \{0\}$.

Then a (partial) *order* relation can be defined in X as follows:

$$x_2 \leq x_1 \iff x_1 - x_2 \in X_+,$$
$$x_2 < x_1 \iff x_1 - x_2 \in X_+ \text{ and } x_1 \neq x_2.$$

The positive cone X_+ is said to be *normal* if the norm $\|\cdot\|$ of the Banach space X is *semimonotone*, i.e. there is a positive constant $k > 0$ such that, if $0 \leq x \leq y$, then $\|x\| \leq k\|y\|$. The norm $\|\cdot\|$ is monotone when $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$. The positive cone is normal if and only if every ordered interval

$$[a, b] = \{x \in X : a \leq x \leq b\}$$

(which is nonempty whenever $a \leq b$) is a bounded set, which in turn is tantamount to the existence of a norm on X which is equivalent to $\|\cdot\|$ and is monotone.

The simplest example of ordered Banach space is the space $\mathbb{R}^m$ together with the positive cone $\mathbb{R}^m_+ = \{y \in \mathbb{R}^m : y_i \geq 0 \text{ for all } i \in \{1, \ldots, m\}\}$, where y_i denotes the i-th component of y. This positive cone generalizes the first quadrant of $\mathbb{R}^2$; the partial order relation which is induced this way is as follows: given y, $z \in \mathbb{R}^m$,

$$y \leq z \iff y_i \leq z_i \quad \text{for all } i \in \{1, \ldots, m\},$$
$$y < z \iff y \leq z \quad \text{and } y \neq z.$$

This cone is normal, and the usual norms on $\mathbb{R}^m$, such as the Euclidean norm, the maximum norm, and the 1-norm, are all monotone for this order.

Next, we give two basic definitions regarding monotone semiflows. We refer the reader to Amann [A] for further details.

Definition 1.13. A semiflow $(X, \Phi, \mathbb{R}^+)$ is said to be *monotone* if, whenever x, $y \in X$ with $x \leq y$ and $t \geq 0$, it holds that

$$\Phi_t(x) \leq \Phi_t(y).$$

Besides, the semiflow $(X, \Phi, \mathbb{R}^+)$ is said to be *strongly monotone* if, whenever x, $y \in X$ with $x < y$ and $t > 0$, it holds that

$$\Phi_t(y) - \Phi_t(x) \in \mathrm{Int}(X_+).$$

Chapter 2

Non-autonomous stable linear D-operators

In this chapter, we will introduce some linear operators D which appear in the definition of the neutral functional differential equations that we are going to study in subsequent chapters. Besides, we will give some interesting properties of such operators leading to the existence and uniqueness of the solutions of two linear difference equations associated to the operator D (one in the future and one in the past). Later on, we will introduce the concept of stability of an operator D, generalizing previous definitions found in [Ha] and [HV] in the case of functional differential equations with finite delay. This concept will play a fundamental role in this chapter as well as in the following ones.

To begin with, we give some notation which will be used throughout the remainder of the work.

We consider $\mathbb{R}^m$ endowed with the maximum norm, which is defined by $\|v\| = \max_{j=1,\dots,m} |v_j|$ for all $v \in \mathbb{R}^m$. For any real $m \times m$ matrix A, we denote $\|A\| = \max_{i=1,\dots,m} \sum_{j=1}^m |a_{ij}|$, the norm of A as an operator from $\mathbb{R}^m$ into $\mathbb{R}^m$ when we consider the maximum norm on $\mathbb{R}^m$. Given an $m \times m$ matrix $\mu = [\mu_{ij}]_{ij}$ of measures with finite total variation on a measurable space (Y, ζ) and a measurable subset of Y, $E \in \zeta$, $|\mu_{ij}|(E)$ will denote the total variation of μ_{ij} over E; the maximum norm of the $m \times m$ matrix $[|\mu_{ij}|(E)]_{ij}$ will be denoted by $\|\mu\|(E)$, and the $m \times m$ matrix of positive measures $[|\mu_{ij}|]_{ij}$ will be denoted by $|\mu|$. Besides, the integral of any measurable function $f : E \to \mathbb{R}^m$ over E with respect to μ is defined by

$$\int_E [d\mu]\, f = \left(\sum_{j=1}^m \int_E f_j \, d\mu_{ij} \right)_{i=1}^m .$$

Let (Ω, d) be a compact metric space, and let $\sigma : \mathbb{R} \times \Omega \to \Omega$ be a continuous real flow on Ω. We will denote $\omega{\cdot}t = \sigma(\omega, t)$, $t \in \mathbb{R}$, $\omega \in \Omega$. We will assume in the remainder of the work that the flow σ is minimal.

As in Chapter 1, let $X = C((-\infty, 0], \mathbb{R}^m)$, which is a Fréchet space when endowed with the compact-open topology, i.e. the topology of uniform convergence over compact subsets. This topology happens to be metric for the distance

$$\mathsf{d}(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}\,, \quad x,y \in X\,,$$

where $\|x\|_n = \sup_{s\in[-n,0]} \|x(s)\|$, and $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^m$. Let $BU \subset X$ be the Banach space

$$BU = \{x \in X : x \text{ is bounded and uniformly continuous}\}$$

with the supremum norm $\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|$. Given $r > 0$, we will denote

$$B_r = \{x \in BU : \|x\|_\infty \le r\}.$$

As usual, given $I = (-\infty, a] \subset \mathbb{R}$, $t \in I$ and a continuous function $x : I \to \mathbb{R}^m$, x_t will denote the element of X defined by $x_t(s) = x(t+s)$ for $s \in (-\infty, 0]$.

Let $D : \Omega \times BU \to \mathbb{R}^m$ be an operator satisfying the following hypotheses:

(D1) D is linear and continuous in its second variable, and the mapping $\Omega \to \mathcal{L}(BU, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous;

(D2) for each $r > 0$, $D\colon \Omega \times B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r, that is, if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with (ω_n, x_n), $(\omega, x) \in \Omega \times B_r$, then we have $\lim_{n\to\infty} D(\omega_n, x_n) = D(\omega, x)$.

Next, we give an integral representation of the operator D, which will turn out to be very useful.

Lemma 2.1. *For each $\omega \in \Omega$, there exists an $m\times m$ matrix $\mu(\omega) = [\mu_{ij}(\omega)]_{ij}$ of real Borel regular measures with finite total variation such that*

$$D(\omega, x) = \int_{-\infty}^{0} [d\mu(\omega)]\, x, \quad (\omega, x) \in \Omega \times BU.$$

Proof. Fix $\omega \in \Omega$. From Riesz representation theorem we obtain the above relation for each x whose components are of compact support. Moreover, if $x \in BU$, there are an $r > 0$ and a sequence of functions of compact support

$\{x_n\}_n \subset B_r$ with $\|x_n\|_\infty \leq \|x\|_\infty$ such that $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ and, from hypothesis (D2), $\lim_{n\to\infty} D(\omega, x_n) = D(\omega, x)$. Thus, as we saw before,

$$D(\omega, x_n) = \int_{-\infty}^{0} [d\mu(\omega)(s)]\, x_n(s),$$

and Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} D(\omega, x_n) = \int_{-\infty}^{0} [d\mu(\omega)(s)]\, x(s)\,,$$

which finishes the proof. □

For each $\omega \in \Omega$, let $B(\omega) = \mu(\omega)(\{0\})$. Now let $\nu(\omega) = B(\omega)\delta_0 - \mu(\omega)$, $\omega \in \Omega$, where δ_0 is the Dirac measure at 0, that is,

$$\int_{-\infty}^{0} [d\,\delta_0\, I]x = x(0)$$

for all $x \in BU$. It is clear that $|\nu_{ij}(\omega)|(\{0\}) = 0$ for all $i,\, j \in \{1, \ldots, m\}$ and all $\omega \in \Omega$. Besides, from the dominated convergence theorem, it follows that

$$\lim_{\rho\to 0^+} |\nu_{ij}(\omega)|([-\rho, 0]) = 0 \quad \text{and} \quad \lim_{\rho\to\infty} |\nu_{ij}(\omega)|((-\infty, -\rho]) = 0$$

for each $\omega \in \Omega$. These definitions allow us to have a second integral representation of the operator D:

$$D(\omega, x) = B(\omega)\, x(0) - \int_{-\infty}^{0} [d\nu(\omega)]\, x, \quad (\omega, x) \in \Omega \times BU. \tag{2.1}$$

Proposition 2.2. *The map $B : \Omega \to \mathbb{M}_m(\mathbb{R})$, $\omega \mapsto B(\omega)$ is continuous.*

Proof. For each $\rho > 0$, let $\varphi_\rho : (-\infty, 0] \to \mathbb{R}$ be the function given for $s \leq 0$ by

$$\varphi_\rho(s) = \begin{cases} 0 & \text{if } s \leq -2\rho, \\ \rho^{-1}s + 2 & \text{if } -2\rho < s \leq -\rho, \\ 1 & \text{if } -\rho < s \leq 0. \end{cases}$$

Let $i \in \{1, \ldots, m\}$, and let $\{\omega_n\}_n \subset \Omega$ be a sequence which converges to some $\omega_0 \in \Omega$. A straightforward application of Lebesgue dominated convergence theorem yields

$$\lim_{\rho\to 0^+} \int_{-\infty}^{0} [d\mu(\omega_n)]\, \varphi_\rho\, e_i = B(\omega_n)\, e_i \quad \text{for all } n \in \mathbb{N},$$

and, likewise,

$$\lim_{\rho\to 0^+} D(\omega_0,\varphi_\rho\, e_i) = B(\omega_0)\, e_i .$$

On the other hand, from (D1) we deduce that

$$\lim_{n\to\infty}\int_{-\infty}^{0}[d\mu(\omega_n)]\,\varphi_\rho\, e_i = D(\omega_0,\varphi_\rho\, e_i)$$

uniformly for $\rho > 0$, and the result follows immediately. □

Proposition 2.3. *Let $L : \Omega\times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto B(\omega)\, x(0) - D(\omega, x)$. Then the mapping $\Omega \to \mathcal{L}(BU,\mathbb{R}^m)$, $\omega \mapsto L(\omega,\cdot)$ is continuous. Equivalently, for every sequence $\{\omega_n\}_n \subset \Omega$ converging to $\omega_0 \in \Omega$ and all i, $j \in \{1,\dots,m\}$, we have that $\lim_{n\to\infty} |\nu_{ij}(\omega_n) - \nu_{ij}(\omega_0)|((-\infty, 0]) = 0$.*

Proof. Let ω_1, $\omega_2 \in \Omega$ and $x \in BU$ with $\|x\|_\infty \le 1$; then

$$\|L(\omega_1, x) - L(\omega_2, x)\| \le \|B(\omega_1) - B(\omega_2)\| + \|D(\omega_1, x) - D(\omega_2, x)\|.$$

The result follows from (D1) and Proposition 2.2. □

The next result states an important property of uniform convergence to zero for the measures defining the operator D.

Corollary 2.4. *The following statements hold:*

(i) $\lim_{\rho\to 0^+} \|\nu(\omega)\|([-\rho, 0]) = 0$ *uniformly for $\omega \in \Omega$;*

(ii) $\lim_{\rho\to\infty} \|\nu(\omega)\|((-\infty, -\rho]) = 0$ *uniformly for $\omega \in \Omega$.*

Proof. In order to prove (i), it is important to notice that, for each ω_1, $\omega_2 \in \Omega$ and each i, $j \in \{1,\dots,m\}$, we have that

$$\big|\,|\nu_{ij}(\omega_1)|([-\rho, 0]) - |\nu_{ij}(\omega_2)|([-\rho, 0])\big| \le |\nu_{ij}(\omega_1) - \nu_{ij}(\omega_2)|((-\infty, 0]).$$

Moreover, $\lim_{\rho\to 0^+} \|\nu(\omega)\|([-\rho, 0]) = 0$ for all $\omega \in \Omega$. Consequently, the family of continuous functions $\{h_\rho\}_{\rho>0}$ given by

$$h_\rho(\omega) = \|\nu(\omega)\|([-\rho, 0]), \quad \omega \in \Omega,$$

decreases to 0 when $\rho \downarrow 0$. Hence, using Dini's theorem, this family converges to 0 uniformly for $\omega \in \Omega$. The proof of (ii) is analogous. □

Let us assume one more hypothesis on the operator D, which is a natural generalization of the atomic character as seen in [Ha] and [HV]:

(D3) $B(\omega)$ is a regular matrix for all $\omega \in \Omega$.

We introduce a linear difference equation associated to the operator D: given $(\omega, \varphi) \in \Omega \times BU$ and $h \in C([0,\infty), \mathbb{R}^m)$ with $D(\omega, \varphi) = h(0)$, we try to find $x \in C(\mathbb{R}, \mathbb{R}^m)$ such that

$$\begin{cases} D(\omega{\cdot}t, x_t) = h(t), & t \geq 0, \\ x_0 = \varphi. \end{cases} \tag{2.2$_\omega$}$$

Let us check that, in order to solve this equation, we can suppose without loss of generality that $B(\omega) = I$ for all $\omega \in \Omega$. Indeed, thanks to Proposition 2.2 and (D3), it is clear that a function $x \in C(\mathbb{R}, \mathbb{R}^m)$ is a solution of $(2.2)_\omega$ if and only if it is a solution of

$$\begin{cases} D_1(\omega{\cdot}t, x_t) = B(\omega{\cdot}t)^{-1} h(t), & t \geq 0, \\ x_0 = \varphi, \end{cases}$$

where the operator D_1 is defined by

$$D_1(\omega^1, x^1) = x^1(0) - \int_{-\infty}^0 [dB(\omega^1)^{-1}\, \nu(\omega^1)]\, x^1, \quad (\omega^1, x^1) \in \Omega \times BU.$$

Consequently, in the remainder of the chapter we can assume without loss of generality that $B(\omega) = I$ for all $\omega \in \Omega$, and all the results which follow will hold in the general case. This considerations together with (2.1) yield the following integral representation for D:

$$D(\omega, x) = x(0) - \int_{-\infty}^0 [d\nu(\omega)]\, x, \quad (\omega, x) \in \Omega \times BU. \tag{2.3}$$

Next, we give a result on the existence of solutions of equation $(2.2)_\omega$.

Theorem 2.5. *For all $h \in C([0,\infty), \mathbb{R}^m)$ and all $(\omega, \varphi) \in \Omega \times BU$ with $D(\omega, \varphi) = h(0)$, there exists $x \in C(\mathbb{R}, \mathbb{R}^m)$ such that $(2.2)_\omega$ holds.*

Proof. From Corollary 2.4, it follows that there is a $\rho > 0$ satisfying that $\|\nu(\omega)\|([-t, 0]) < 1/4$ for all $t \in [0, \rho]$ and all $\omega \in \Omega$. Consider the set $V = \{x \in C([0,\rho], \mathbb{R}^m) : x(0) = 0\}$ and the mapping $T : V \to V$ defined for $x \in V$ by

$$\begin{aligned} T(x) : \quad [0,\rho] \quad &\longrightarrow \quad \mathbb{R}^m \\ t \quad &\mapsto \quad h(t) - \varphi(0) + \int_{-t}^0 d[\nu(\omega{\cdot}t)(\theta)]\, (x(t+\theta) + \varphi(0)) \\ &\qquad + \int_{-\infty}^{-t} [d\nu(\omega{\cdot}t)(\theta)]\, \varphi(t+\theta). \end{aligned}$$

It is easy to check that T is well defined. Now, let

$$R \geq 4 \max \left\{ \sup_{\tilde\omega\in\Omega} \|\nu(\tilde\omega)\|((-\infty,0])\,\|\varphi\|_\infty + \sup_{0\leq t\leq\rho} \|h(t)\|,\ \|\varphi(0)\| \right\}.$$

Let $B=\{x\in V : \|x(s)\|\leq R \text{ for all } s\in[0,\rho]\}$. Let us check that $T(B)\subset B$. Indeed, if $x\in B$, then

$$\begin{aligned}
\sup_{0\leq t\leq\rho} \|T(x)(t)\| \leq & \sup_{0\leq t\leq\rho} \|\nu(\omega{\cdot}t)\|([-t,0])(R+\|\varphi(0)\|) \\
& + \sup_{\tilde\omega\in\Omega} \|\nu(\tilde\omega)\|((-\infty,0])\,\|\varphi\|_\infty + \sup_{0\leq t\leq\rho} \|h(t)\| + \|\varphi(0)\| \\
\leq & \frac{1}{4}(R+\|\varphi(0)\|) + \sup_{\tilde\omega\in\Omega} \|\nu(\tilde\omega)\|((-\infty,0])\,\|\varphi\|_\infty \\
& + \sup_{0\leq t\leq\rho} \|h(t)\| + \|\varphi(0)\| \\
\leq & \frac{1}{4}\,R + \frac{1}{4}\,\|\varphi(0)\| + \frac{1}{4}\,R + \|\varphi(0)\| \leq R.
\end{aligned}$$

Let us prove that T is a contraction. In order to do so, fix x^1, $x^2\in V$ and $s\in[0,\rho]$; then

$$\begin{aligned}
\|T(x^1)(s)-T(x^2)(s)\| &\leq \left\| \int_{-s}^{0} [d\nu(\omega{\cdot}s)](x^1_s - x^2_s) \right\| \\
&\leq \sup_{0\leq t\leq\rho} \|\nu(\omega{\cdot}t)\|([-t,0]) \sup_{0\leq t\leq\rho} \|x^1(t)-x^2(t)\| \\
&\leq \frac{1}{4} \sup_{0\leq t\leq\rho} \|x^1(t)-x^2(t)\|.
\end{aligned}$$

As a result, T is a contraction, and, applying a fixed point theorem (see e.g. Lemma 4.1 in [Ha]), it follows that T has a fixed point $x\in B$. It remains to check that x yields a solution of $(2.2)_\omega$ on $[0,\rho]$. Let $\tilde x:(-\infty,\rho]\to\mathbb{R}^m$ be the continuous map defined for $t\in(-\infty,\rho]$ by

$$\tilde x(t) = \begin{cases} \varphi(t) & \text{if } t\leq 0, \\ x(t)+\varphi(0) & \text{if } t\geq 0. \end{cases}$$

It is clear that $\tilde x_0=\varphi$, and, for all $t\in[0,\rho]$,

$$\begin{aligned}
D(\omega{\cdot}t,\tilde x_t) =& x(t)+\varphi(0) - \int_{-t}^{0} d[\nu(\omega{\cdot}t)(\theta)]\,(x(t+\theta)+\varphi(0)) \\
&+ \int_{-\infty}^{-t} [d\nu(\omega{\cdot}t)(\theta)]\,\varphi(t+\theta) = h(t)
\end{aligned}$$

thanks to the fact that $x(t)=T(x)(t)$. Consequently, $\tilde x$ is a solution of $(2.2)_\omega$ on $[0,\rho]$, and, this way, the result follows by taking steps of length ρ. $\square$

Lemma 2.6. *Given $\rho > 0$, there are positive constants k^1_ρ, k^2_ρ such that, if x is a solution of the equation*

$$\begin{cases} D(\omega{\cdot}t, x_t) = h(t)\,, & t \ge 0\,, \\ x_0 = \varphi\,, \end{cases}$$

where $h \in C([0,\infty),\mathbb{R}^m)$, $(\omega,\varphi) \in \Omega \times BU$ and $D(\omega,\varphi) = h(0)$, then for each $t \in [0,\rho]$

$$\|x_t\|_\infty \le k^1_\rho \sup_{0\le u\le t} \|h(u)\| + k^2_\rho\, \|\varphi\|_\infty\,.$$

Proof. From Corollary 2.4, it follows that, for each i, $j \in \{1,\ldots,m\}$, there is a $\rho_0 > 0$ such that $\|\nu(\omega)\|([-t,0]) < 1/2$ if $t \in [0,\rho_0]$ and $\omega \in \Omega$. Let x be a solution of the equation. From the expression of D,

$$x(t) = h(t) + \int_{-t}^{0} [d\nu(\omega{\cdot}t)(s)]\, x(t+s) + \int_{-\infty}^{-t} [d\nu(\omega{\cdot}t)(s)]\, \varphi(t+s)$$

for each $t \ge 0$. Consequently, if $t \in [0,\rho_0]$, then

$$\|x(t)\| \le \|h(t)\| + \frac{1}{2} \sup_{0\le u\le t} \|x(u)\| + \|\varphi\|_\infty \sup_{\omega_1\in\Omega} \|\nu(\omega_1)\|((-\infty,0])\,,$$

whence we deduce that, if $t \in [0,\rho_0]$, then

$$\sup_{0\le u\le t} \|x(u)\| \le 2 \sup_{0\le u\le t} \|h(u)\| + 2\,a\,\|\varphi\|_\infty\,, \tag{2.4}$$

where $a = \sup_{\omega_1\in\Omega} \|\nu(\omega_1)\|((-\infty,0])$. Next, let $y(t) = x(t+\rho_0)$, $t \in \mathbb{R}$, which is a solution of

$$\begin{cases} D((\omega{\cdot}\rho_0){\cdot}t, y_t) = h(t+\rho_0)\,, & t \ge 0\,, \\ y_0 = x_{\rho_0}\,. \end{cases}$$

As above, we conclude that, if $t \in [0,\rho_0]$, then

$$\sup_{0\le u\le t} \|y(u)\| \le 2 \sup_{0\le u\le t} \|h(u+\rho_0)\| + 2\,a\,\|x_{\rho_0}\|_\infty\,,$$

which, together with $\|x_{\rho_0}\|_\infty \le \|\varphi\|_\infty + \sup_{0\le u\le \rho_0} \|x(u)\|$ and (2.4), yields

$$\begin{aligned} \sup_{0\le u\le t} \|x(u)\| &\le 2 \sup_{0\le u\le t} \|h(u)\| + 2\,a(\|\varphi\|_\infty + \sup_{0\le u\le \rho_0} \|x(u)\|) \\ &\le 2 \sup_{0\le u\le t} \|h(u)\| + 2\,a\,\|\varphi\|_\infty + 4\,a \sup_{0\le u\le t} \|h(u)\| + 4\,a^2\,\|\varphi\|_\infty \\ &\le b \sup_{0\le u\le t} \|h(u)\| + c\,\|\varphi\|_\infty\,, \end{aligned}$$

for $t \in [\rho_0, 2\,\rho_0]$ and some positive constants b and c independent of h, ω, and φ. This way, the result is obtained in a finite number of steps by choosing $k^2_\rho \ge 1$. □

This bound for the solution of $(2.2)_\omega$ leads us to its uniqueness. Namely, if x^1, x^2 are solutions of the equation, then for all $t \geq 0$, we have

$$\begin{cases} D(\omega{\cdot}t, x_t^1 - x_t^2) = 0\,, \quad t \geq 0\,, \\ x_0^1 - x_0^2 = 0, \end{cases}$$

Thus, given $t > 0$, $\|x^1(t) - x^2(t)\| \leq k_t^1\, 0 + k_t^2\, 0 = 0$.

Lemma 2.7. *For each $\omega \in \Omega$, there exist $\phi_1(\omega), \ldots, \phi_m(\omega) \in BU$ such that $D(\omega, \phi_i(\omega)) = e_i$ for all $i \in \{1, \ldots, m\}$ and $\sup_{\theta \leq 0} \|\Phi(\omega)(\theta)\| \leq 2$, where $\Phi(\omega) = (\phi_1(\omega), \ldots, \phi_m(\omega))$, for all $\omega \in \Omega$.*

Proof. For each $s > 0$, let $\psi_s : (-\infty, 0] \to \mathbb{R}$ be the function given for $\theta \leq 0$ by

$$\psi_s(\theta) = \begin{cases} 0 & \text{if } \theta \leq -s, \\ 1 + \frac{\theta}{s} & \text{if } -s < \theta \leq 0. \end{cases}$$

Given $\omega \in \Omega$ and $x_1, \ldots, x_m \in BU$, let us denote

$$D(\omega)(x_1, \ldots, x_m) = (D(\omega, x_1), \ldots, D(\omega, x_m)).$$

Now, for each $\omega \in \Omega$, $s > 0$ and i, $j \in \{1, \ldots, m\}$,

$$|D_j(\omega, \psi_s\, e_i) - \delta_{ji}| = \left| \int_{-s}^{0} \left(1 + \frac{\theta}{s}\right) d\nu_{ji}(\omega)(\theta) \right| \leq |\nu_{ji}(\omega)|([-s, 0]),$$

and the latter converges to 0 uniformly for $\omega \in \Omega$ as we proved in Corollary 2.4. Thus, there is an $s > 0$ such that, for all $\omega \in \Omega$,

$$\|D(\omega)(\psi_s\, I) - I\| \leq \frac{1}{2} \quad \text{and} \quad \det\left(D(\omega)(\psi_s\, I)\right) \neq 0.$$

Let $\Phi : \Omega \to (BU)^m$, $\omega \mapsto (\psi_s\, I)(D(\omega)(\psi_s\, I))^{-1}$. It is clear that

$$\begin{aligned} D(\omega)\Phi(\omega) &= D(\omega)((\psi_s\, I)(D(\omega)(\psi_s\, I))^{-1}) \\ &= (D(\omega)(\psi_s\, I))(D(\omega)(\psi_s\, I))^{-1} = I \end{aligned}$$

for all $\omega \in \Omega$.

Now, given $\omega \in \Omega$, we have that $\|I - D(\omega)(\psi_s\, I)\| \leq 1/2 < 1$, so $D(\omega)(\psi_s\, I)$ is regular, and

$$\|(D(\omega)(\psi_s\, I))^{-1}\| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Finally, for each $\omega \in \Omega$,

$$\sup_{\theta \leq 0} \|\Phi(\omega)(\theta)\| \leq \sup_{\theta \leq 0} \|\psi_s(\theta)\, I\|\, \|(D(\omega)(\psi_s\, I))^{-1}\| \leq 2\,,$$

and we are done. □

Lemma 2.8. *For each $\rho > 0$, there exists $k_\rho > 0$ such that, for all $\omega \in \Omega$ and all $h \in C([0,\infty),\mathbb{R}^m)$, the solution of*

$$\begin{cases} D(\omega{\cdot}t, x_t) = h(t)\,, & t \ge 0\,, \\ x_0 = \Phi(\omega)\, h(0)\,, \end{cases}$$

satisfies $\|x(t)\| \le k_\rho \sup_{0\le u\le t} \|h(u)\|$, $t \in [0,\rho]$.

Proof. Thanks to Lemma 2.6, there exist numbers k_ρ^1, $k_\rho^2 > 0$ such that, if $t \in [0,\rho]$, then

$$\begin{aligned} \|x_t\|_\infty &\le k_\rho^1 \sup_{0\le u\le t} \|h(u)\| + k_\rho^2\, \|\Phi(\omega)h(0)\| \\ &\le k_\rho^1 \sup_{0\le u\le t} \|h(u)\| + k_\rho^2 \sup_{s\le 0} \|\Phi(\omega)(s)\|\, \|h(0)\|. \end{aligned}$$

Now, applying Lemma 2.7, we get

$$\|x_t\|_\infty \le (k_\rho^1 + 2\,k_\rho^2) \sup_{0\le u\le t} \|h(u)\|,$$

as wanted. □

The following definition generalizes some other definitions given in [Ha] and [HV] to this setting.

Definition 2.9. The mapping D given by (2.1) is said to be *stable* if there is a continuous function $c \in C([0,\infty),\mathbb{R})$ with $\lim_{t\to\infty} c(t) = 0$ such that, for each $(\omega,\varphi) \in \Omega \times BU$ with $D(\omega,\varphi) = 0$, the solution of the homogeneous problem

$$\begin{cases} D(\omega{\cdot}t, x_t) = 0\,, & t \ge 0 \\ x_0 = \varphi\,, \end{cases}$$

satisfies $\|x(t)\| \le c(t)\, \|\varphi\|_\infty$ for each $t \ge 0$.

Let us remark that we are still assuming without loss of generality that $B(\omega) = I$ for all $\omega \in \Omega$. As before, all the results which follow hold in the general case.

Lemma 2.10. *Let us assume that D is stable. Then there exists a positive constant $d > 0$ such that, for each $h \in C([0,\infty),\mathbb{R}^m)$ with $h(0) = 0$ and each $\omega \in \Omega$, the solution of*

$$\begin{cases} D(\omega{\cdot}t, x_t) = h(t)\,, & t \ge 0\,, \\ x_0 = 0\,, \end{cases}$$

satisfies $\|x(t)\| \le d \sup\limits_{0\le u\le t} \|h(u)\|$ for each $t \ge 0$.

Proof. Let $c \in C([0,\infty),\mathbb{R}^m)$ be the function given in the definition of stability. Clearly, we can assume without loss of generality that c is decreasing. Now, we take $\rho > 0$ such that $c(\rho) < 1$. From Lemma 2.6, it follows that there exists $k_\rho^1 > 0$ such that $\|x(t)\| \leq k_\rho^1 \sup_{0\leq u\leq t} \|h(u)\|$ provided that $t \in [0,\rho]$.

If $t \geq \rho$, there is a $j \in \mathbb{N}$ such that $t \in [j\,\rho,(j+1)\,\rho]$, and, from uniqueness, it follows that $x(t) = x^1(t-(j-1)\,\rho) + x^2(t-(j-1)\,\rho)$, $t \in \mathbb{R}$, where x^1 and x^2 are the solutions of

$$\begin{cases} D(\omega{\cdot}((j-1)\,\rho + t), x_t^1) = 0\,, \quad t \geq 0\,, \\ x_0^1 = x_{(j-1)\,\rho} - \Phi(\omega{\cdot}((j-1)\,\rho))\, h((j-1)\,\rho)\,; \end{cases}$$

$$\begin{cases} D(\omega{\cdot}((j-1)\rho + t), x_t^2) = h(t + (j-1)\,\rho)\,, \quad t \geq 0\,, \\ x_0^2 = \Phi(\omega{\cdot}((j-1)\,\rho))\, h((j-1)\,\rho)\,, \end{cases}$$

respectively. From the stability of D, we deduce that

$$\begin{aligned} \|x(t)\| \leq{}& c\,(t-(j-1)\,\rho)\,\|x_{(j-1)\,\rho} - \Phi(\omega{\cdot}((j-1)\,\rho))\,h((j-1)\,\rho)\|_\infty \\ &+ k_{2\rho} \sup_{(j-1)\,\rho\leq u\leq t} \|h(u)\|\,, \end{aligned}$$

where $k_{2\rho}$ is the bound found in Lemma 2.8. In addition, since $t-(j-1)\,\rho \geq \rho$ and c is decreasing we conclude that, for $t \in [j\,\rho,(j+1)\,\rho]$,

$$\|x(t)\| \leq c(\rho)\,c_j + (c(\rho)\,b + k_{2\rho}) \sup_{0\leq u\leq t} \|h(u)\|, \tag{2.5}$$

where $c_j = \|x_{j\rho}\|_\infty = \sup_{0\leq u\leq j\rho} \|x(u)\|$ and $b = \sup_{\theta\leq 0} \|\Phi(\omega)(\theta)\|$.

Let $a_\rho = \max\{k_\rho^1,\, c(\rho)\,b + k_{2\rho}\}$. We have $c_1 \leq a_\rho \sup_{0\leq u\leq\rho} \|h(u)\|$, and, from (2.5), if $j \geq 2$, then

$$c_j \leq \max\left\{c_{j-1}, c(\rho)\,c_{j-1} + a_\rho \sup_{0\leq u\leq j\,\rho} \|h(u)\|\right\}. \tag{2.6}$$

Hence, it is clear that $c_2 \leq a_\rho\,(1+c(\rho)) \sup_{0\leq u\leq 2\,\rho} \|h(u)\|$. Assume that

$$c_j \leq a_\rho \left(1 + c(\rho) + \ldots + c(\rho)^{j-1}\right) \sup_{0\leq u\leq j\,\rho} \|h(u)\| \tag{2.7}$$

for some $j \geq 2$, and let us check that it holds for $j+1$; indeed, from inequalities (2.6) and (2.7), it follows that

$$\begin{aligned} c_{j+1} \leq \max\Bigg\{& a_\rho \left(1 + c(\rho) + \ldots + c(\rho)^{j-1}\right) \sup_{0\leq u\leq j\,\rho} \|h(u)\|, \\ & a_\rho \left(c(\rho) + \ldots + c(\rho)^{j}\right) \sup_{0\leq u\leq j\,\rho} \|h(u)\| + a_\rho \sup_{0\leq u\leq (j+1)\rho} \|h(u)\|\Bigg\}, \end{aligned}$$

whence it is obvious that (2.7) holds for $j+1$; as a result, we obtain (2.7) for all $j \geq 2$ by induction.

Again from (2.5) we deduce that for $t \geq 0$ (and hence $t \in [j\,\rho, (j+1)\,\rho]$ for some $j \geq 0$)

$$\|x(t)\| \leq a_\rho \sum_{k=0}^{j} c(\rho)^k \sup_{0\leq u\leq t} \|h(u)\| \leq \frac{a_\rho}{1-c(\rho)} \sup_{0\leq u\leq t} \|h(u)\|\,,$$

which finishes the proof. □

The following statement provides a non-homogeneous version of the concept of stability for a D-operator.

Theorem 2.11. *Let us assume that D is stable. Then there are a continuous function $c \in C([0,\infty),\mathbb{R})$ with $\lim_{t\to\infty} c(t) = 0$ and a positive constant $k > 0$ such that the solution of the equation*

$$\begin{cases} D(\omega{\cdot}t, x_t) = h(t)\,, & t \geq 0\,, \\ x_0 = \varphi\,, \end{cases}$$

where $h \in C([0,\infty),\mathbb{R}^m)$, $(\omega,\varphi) \in \Omega \times BU$ and $D(\omega,\varphi) = h(0)$, satisfies

$$\|x(t)\| \leq c(t)\,\|\varphi\|_\infty + k \sup_{0\leq u\leq t} \|h(u)\|$$

for each $t \geq 0$.

Proof. It is not hard to check that $x = x^1 + x^2$ where x^1 and x^2 are the solutions of

$$\begin{cases} D(\omega{\cdot}t, x_t^1) = \psi(t)\,h(t), & t \geq 0\,, \\ x_0^1 = \varphi\,; \end{cases}$$
$$\begin{cases} D(\omega{\cdot}t, x_t^2) = (1-\psi(t))\,h(t), & t \geq 0\,, \\ x_0^2 = 0\,, \end{cases}$$

respectively, and

$$\begin{array}{rccl} \psi\colon & [0,\infty) & \longrightarrow & \mathbb{R} \\ & t & \mapsto & \psi(t) = \begin{cases} 1-t, & 0 \leq t \leq 1\,, \\ 0\,, & 1 \leq t\,. \end{cases} \end{array}$$

Now, let $y : \mathbb{R} \to \mathbb{R}^m$, $t \mapsto x^1(t+1)$; y satisfies $D(\omega{\cdot}(t+1), y_t) = 0$, $t \geq 0$, with $y_0 = x_1^1$. Let us check how the result easily follows from this fact. First of all, we already know that there exist constants $k_1^1,\ k_1^2 > 0$ such that, if $t \in [0,1]$,

$$\|x_t^1\|_\infty \le k_1^1 \sup_{0\le u\le t} \|h(u)\| + k_1^2\, \|\varphi\|_\infty.$$

This and the definition of stability applied to y lead us to the existence of $c_1 \in C([0,\infty),\mathbb{R})$ which converges to 0 as $t \to \infty$ and such that, for all $t \ge 0$,

$$\begin{aligned}\|x^1(t+1)\| =&\|y(t)\| \le c_1(t)\, \|x_1^1\|_\infty \le c_1(t)\, k_1^1 \sup_{0\le u\le 1} \|h(u)\| + c_1(t)\, k_1^2\, \|\varphi\|_\infty \\ &\le M \sup_{0\le u\le t+1} \|h(u)\| + c(t+1)\, \|\varphi\|_\infty,\end{aligned}$$

and, for all $t \in [0,1)$,

$$\|x^1(t)\| \le \|x_t^1\|_\infty \le k_1^1 \sup_{0\le u\le t} \|h(u)\| + c(t)\, \|\varphi\|_\infty,$$

where $M > 0$ is a bound of $k_1^1\, c_1$, $c \in C([0,\infty),\mathbb{R})$ converges to 0, and c is a pointwise upper bound of k_1^2 on $[0,1)$ and of $k_1^2\, c_1(t-1)$ for $t \in [1,\infty)$. On the other hand, there is an $N > 0$ such that, if $t \ge 0$, then

$$\|x^2(t)\| \le N \sup_{0\le u\le t} \|h(u)\|.$$

Thus, if $k = \max\{k_1^1 + N, M + N\}$, then for each $t \ge 0$

$$\|x(t)\| \le c(t)\, \|\varphi\|_\infty + k \sup_{0\le u\le t} \|h(u)\|,$$

as wanted. □

We may also consider a linear difference equation for past times associated to the operator D: given $(\omega, h) \in \Omega \times BU$, we try to find $x \in BU$ such that

$$D(\omega{\cdot}t, x_t) = B(\omega{\cdot}t)\, x(t) - \int_{-\infty}^{0} [d\nu(\omega{\cdot}t)]\, x_t = h(t), \quad t \le 0.$$

As for the existence and uniqueness of the solutions of this equation, let us give an example of operator D for which there is neither of them. Let

$$\begin{array}{rccl} D: & \Omega \times BU & \longrightarrow & \mathbb{R} \\ & (\omega, x) & \mapsto & x(0) - x(-1). \end{array}$$

Then, all the constant functions are solutions of the following equation:

$$D(\omega{\cdot}t, x_t) = x(t) - x(t-1) = 0, \quad t \le 0,$$

so there is no uniqueness for these solutions. Besides, within BU, there is no solution of the equation

$$D(\omega{\cdot}t, x_t) = x(t) - x(t-1) = 1, \quad t \leq 0,$$

for a solution $\tilde{x} \in BU$ of such equation would satisfy $x(t) = n + x(t-n)$ for all $t \leq 0$ and all $n \in \mathbb{N}$, and therefore it would not be bounded.

In the following chapter, we will see how to solve these problems by assuming some additional hypotheses on the stability of D.

Chapter 3

Non-autonomous convolution operators $\widehat{D}$

The aim of this chapter is to study the regularity properties of the convolution operator $\widehat{D} : \Omega \times BU \to \Omega \times BU$ associated to D, which will be defined below. This regularity will be dealt with by using different topologies on BU, namely the compact-open and the supremum norm topologies, and it will be proved to depend heavily on the recurrence properties of the base flow $(\Omega, \sigma, \mathbb{R})$. Moreover, we will study the invertibility of $\widehat{D}$ and state a characterization of the stability of D by means of this property.

Given an operator $D : \Omega \times BU \to \mathbb{R}^m$ satisfying (D1)–(D3) as presented in Chapter 2, let us consider the convolution operator $\widehat{D}$ associated to D and defined by

$$\begin{array}{rccc} \widehat{D}: & \Omega \times BU & \longrightarrow & \Omega \times BU \\ & (\omega, x) & \mapsto & (\omega, \widehat{D}_2(\omega, x)) \end{array} \tag{3.1}$$

where $\widehat{D}_2(\omega, x) : (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto D(\omega{\cdot}s, x_s)$.

It is easy to see that $\widehat{D}_2$ has the following integral representation:

$$\widehat{D}_2(\omega, x)(s) = B(\omega{\cdot}s)\, x(s) - \int_{-\infty}^{0} [d\nu(\omega{\cdot}s)]\, x_s$$

for each $(\omega, x) \in \Omega \times BU$ and $s \in (-\infty, 0]$.

Theorem 3.1. *If $\widehat{D}$ is the operator defined by* (3.1)*, then $\widehat{D}$ is well defined, $\widehat{D}_2$ is linear and continuous for the norm in its second variable for all $\omega \in \Omega$, and, for all $r > 0$, $\widehat{D}$ is uniformly continuous on $\Omega \times B_r$ when we take the restriction of the compact-open topology to B_r. In addition, if the flow $(\Omega, \sigma, \mathbb{R})$ is almost periodic, then $\widehat{D}$ is uniformly continuous on $\Omega \times B_r$ for all $r > 0$ when we take the norm on B_r.*

Proof. Let us check that $\widehat{D}$ is well defined. Let $(\omega, x) \in \Omega \times BU$, and let $h = \widehat{D}_2(\omega, x) : (-\infty, 0] \to \mathbb{R}^m$. From (D1) and the uniform continuity of σ on, say, $[0,1] \times \Omega$, it follows that, for all $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that, if t, $s \le 0$ and $|t - s| < \delta$ then

$$\|D(\omega{\cdot}t, \cdot) - D(\omega{\cdot}s, \cdot)\| \, \|x\|_\infty \le \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{\omega_1 \in \Omega} \|D(\omega_1, \cdot)\| \, \|x_t - x_s\|_\infty \le \frac{\varepsilon}{2},$$

whence

$$\|h(t) - h(s)\| = \|D(\omega{\cdot}t, x_t) - D(\omega{\cdot}s, x_s)\| \le \varepsilon.$$

Clearly,

$$\|h\|_\infty \le \sup_{\omega_1 \in \Omega} \|D(\omega_1, \cdot)\| \, \|x\|_\infty,$$

and, consequently, $h \in BU$. This way, $\widehat{D}$ is well defined.

The linearity of $\widehat{D}_2$ in its second variable is clear. Besides, for all $\omega \in \Omega$, the continuity of $\widehat{D}_2(\omega, \cdot)$ for the norm on BU is a straightforward consequence of (D1).

Let us check the uniform continuity of $\widehat{D}$ on $\Omega \times B_r$, $r \ge 0$, when we take the restriction of the compact-open topology to B_r. In order to do so, let us fix $\rho > 0$ and $\varepsilon > 0$; there exists $\delta > 0$ such that, for all ω^1, $\omega^2 \in \Omega$ with $d(\omega^1, \omega^2) < \delta$ and all $s \in [-\rho, 0]$, it holds that

$$\|D(\omega^1{\cdot}s, \cdot) - D(\omega^2{\cdot}s, \cdot)\| < \frac{\varepsilon}{2\,r}.$$

Thanks to Corollary 2.4, there exists $\rho_0 > 0$ such that

$$\sup_{\omega \in \Omega} \|\nu(\omega)\|((-\infty, -\rho_0]) < \frac{\varepsilon}{12\,r}.$$

Now, let (ω^1, x^1), $(\omega^2, x^2) \in \Omega \times B_r$ such that $d(\omega^1, \omega^2) < \delta$ and satisfying

$$\sup_{\omega \in \Omega} \|\nu(\omega)\|((-\infty, 0]) \, \|x^1 - x^2\|_{[-\rho-\rho_0, 0]} < \frac{\varepsilon}{6} \quad \text{and}$$

$$\sup_{\omega \in \Omega} \|B(\omega)\| \, \|x^1 - x^2\|_{[-\rho, 0]} < \frac{\varepsilon}{6}.$$

If $s \in [-\rho, 0]$, then we have

$$\begin{aligned}
\|D(\omega^1{\cdot}s, x^1_s) - D(\omega^2{\cdot}s, x^2_s)\| \le \\
&\le \|D(\omega^1{\cdot}s, \cdot) - D(\omega^2{\cdot}s, \cdot)\| \, r + \|D(\omega^2{\cdot}s, (x^1 - x^2)_s)\| \\
&\le \frac{\varepsilon}{2r}\, r + \|B(\omega^2{\cdot}s)(x^1(s) - x^2(s))\| + \left\| \int_{-\infty}^{-\rho_0} [d\nu(\omega^2{\cdot}s)](x^1 - x^2)_s \right\| \\
&\quad + \left\| \int_{-\rho_0}^{0} [d\nu(\omega^2{\cdot}s)](x^1 - x^2)_s \right\| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{12\,r}\, 2\,r + \frac{\varepsilon}{6} = \varepsilon.
\end{aligned}$$

This inequality yields the expected result.

Finally, assume that $(\Omega,\sigma,\mathbb{R})$ is almost periodic. Let us prove that $\widehat{D}_2$ is uniformly continuous on $\Omega\times B_r$, $r>0$, when we take the norm on B_r. From (D1) and the almost periodicity of $(\Omega,\sigma,\mathbb{R})$, it follows that, for all $\varepsilon>0$, there exists $\delta>0$ such that, if ω^1, $\omega^2\in\Omega$, $s\le 0$ and $d(\omega^1,\omega^2)<\delta$, then

$$\|D(\omega^1{\cdot}s,\cdot)-D(\omega^2{\cdot}s,\cdot)\|<\frac{\varepsilon}{2\,r}.$$

Thanks to (D1), taking (ω^1,x^1), $(\omega^2,x^2)\in\Omega\times B_r$ with $d(\omega^1,\omega^2)<\delta$ and such that

$$\sup_{\omega\in\Omega}\|D(\omega,\cdot)\|\,\|x^1-x^2\|_\infty<\frac{\varepsilon}{2},$$

we have that, for all $s\in(-\infty,0]$,

$$\begin{aligned}\|D(\omega^1{\cdot}s,x^1_s)-D(\omega^2{\cdot}s,x^2_s)\|\le&\\ \le&\|D(\omega^1{\cdot}s,x^1_s)-D(\omega^2{\cdot}s,x^1_s)\|+\|D(\omega^2{\cdot}s,(x^1-x^2)_s)\|\\ \le&\frac{\varepsilon}{2\,r}\,r+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

As a result, we obtain the desired property. □

Notice that the kernel and the rank of the operator $\widehat{D}$ are obtained by solving the difference equation in the past which was introduced in Chapter 2. In the remainder of this chapter, we will study the properties of $\widehat{D}$ assuming the stability of the operator D.

Proposition 3.2. *Let us assume that D is stable. Then there is a positive constant $k>0$ such that $\|x^h\|_\infty\le k\,\|h\|_\infty$ for all $h\in BU,\ \omega\in\Omega$ and $x^h\in BU$ satisfying $D(\omega{\cdot}s,x^h_s)=h(s)$ for $s\le 0$.*

Proof. Let $x(t)$ be the solution of

$$\begin{cases} D(\omega{\cdot}t,x_t)=h(0)\,, & t\ge 0\,,\\ x_0=x^h\,, \end{cases}$$

let $\widetilde{h}:\mathbb{R}\to\mathbb{R}^m$ be the function defined for $t\in\mathbb{R}$ by

$$\widetilde{h}(t)=\begin{cases} h(t)\,, & \text{if } t\le 0\,,\\ h(0)\,, & \text{if } t\ge 0\,,\end{cases}$$

and, for $s\le 0$, define $y^s:\mathbb{R}\to\mathbb{R}^m$ for $t\in\mathbb{R}$ by

$$y^s(t)=\begin{cases} x(t+s)\,, & \text{if } t+s\ge 0\,,\\ x^h(t+s)\,, & \text{if } t+s\le 0\,.\end{cases}$$

Then

$$\begin{cases} D(\omega{\cdot}(s+t), y_t^s) = \widetilde{h}(t+s)\,, \quad t \geq 0\,, \\ y_0^s = x_s^h\,, \end{cases}$$

and Theorem 2.11 yields

$$\|y^s(t)\| \leq c(t)\,\|x_s^h\|_\infty + k \sup_{0\leq u\leq t} \|\widetilde{h}(u+s)\|_\infty \leq c(t)\,\|x^h\|_\infty + k\,\|h\|_\infty\,,$$

for all $t \geq 0$ and $s \leq 0$. Hence,

$$\|x^h(s)\| = \|y^{s-t}(t)\| \leq c(t)\,\|x^h\|_\infty + k\,\|h\|_\infty,$$

and, as $t \to \infty$, we prove the result. $\square$

It is important to mention that, in Chapter 2, it was checked that $\widehat{D}$ is neither injective nor surjective in general. We are now in a position to state a result which assures the invertibility of $\widehat{D}$ and specifies the regularity of its inverse when the linear operator D is stable.

Theorem 3.3. *Under hypotheses* (D1)–(D3)*, if D is stable, then $\widehat{D}$ is invertible, $\widehat{D}_2^{-1}(\omega,\cdot)$ is linear and continuous on BU for all $\omega \in \Omega$ when we consider the norm on BU, and $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$ for all $r > 0$ when we take the restriction of the compact-open topology to B_r. In addition, if the flow $(\Omega, \sigma, \mathbb{R})$ is almost periodic, $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$ for all $r > 0$ when we take the norm on B_r.*

Proof. First, $\widehat{D}$ is injective because, if we have (ω^1, x^1), $(\omega^2, x^2) \in \Omega \times BU$ with $\widehat{D}(\omega^1, x^1) = \widehat{D}(\omega^2, x^2)$, then $\omega^1 = \omega^2$, and, from Proposition 3.2 and the fact that $D(\omega^1{\cdot}s, x_s^1 - x_s^2) = 0$ for $s \leq 0$, we get $x^1 = x^2$.

In order to show that $\widehat{D}$ is surjective, let $(\omega, h) \in \Omega \times BU$ and $\{h_n\}_n \subset B_r$, for some $r > 0$, be a sequence of continuous functions whose components are of compact support such that $h_n \xrightarrow{\mathsf{d}} h$ as $n \to \infty$. Moreover, it is easy to choose them with the same modulus of uniform continuity as h. Let us check that, for each $n \in \mathbb{N}$, there is an $x^n \in BU$ such that $\widehat{D}_2(\omega, x^n) = h_n$, that is, $D(\omega{\cdot}s, x_s^n) = h_n(s)$ for $s \leq 0$ and $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and $\rho_n > 0$ such that $\operatorname{supp}(h_n) \subset [-\rho_n, 0]$. Let $\widetilde{h}_n : [0,\infty) \to \mathbb{R}^m$ be the function defined for $t \geq 0$ by

$$\widetilde{h}_n(t) = \begin{cases} h_n(t-\rho_n) & \text{if } t \in [0, \rho_n], \\ h_n(0) & \text{if } t \geq \rho_n. \end{cases}$$

Since $\widetilde{h}_n(0) = 0$, from Theorem 2.5, it follows that there exists $\widetilde{x}^n \in C(\mathbb{R}, \mathbb{R}^m)$ such that $D(\omega{\cdot}(t-\rho_n), \widetilde{x}_t^n) = \widetilde{h}_n(t)$ for all $t \geq 0$ and $\widetilde{x}_0^n = 0$. Now, let

$x^n : (-\infty, 0] \to \mathbb{R}^m$ be the function defined by $x^n(s) = \widetilde{x}^n(s+\rho_n)$ for $s \le 0$. Clearly, the function x^n is continuous and of compact support. If $s \in [-\rho_n, 0]$, then

$$D(\omega{\cdot}s, x^n_s) = D(\omega{\cdot}(-\rho_n + (s+\rho_n)), \widetilde{x}^n_{s+\rho_n}) = \widetilde{h}_n(s+\rho_n) = h_n(s),$$

and, if $s \le -\rho_n$, then

$$D(\omega{\cdot}s, x^n_s) = D(\omega{\cdot}(-\rho_n + (s+\rho_n)), \widetilde{x}^n_{s+\rho_n}) = D(\omega{\cdot}s, 0) = h_n(s) = 0,$$

as wanted.

From Proposition 3.2, there exists $k > 0$ such that

$$\|x^n\|_\infty \le k\|h_n\|_\infty \le k\,r.$$

Let us fix $\varepsilon > 0$; since the restriction of σ to $[-1, 0] \times \Omega$ is uniformly continuous, we can fix $\delta > 0$ such that, for all $\tau \in [-\delta, 0]$ and all $s \le 0$,

$$\|h_n - (h_n)_\tau\|_\infty < \frac{\varepsilon}{2\,k} \quad \text{and} \quad \|D(\omega{\cdot}(s+\tau), \cdot) - D(\omega{\cdot}s, \cdot)\| < \frac{\varepsilon}{2\,k^2\,r}.$$

For each $n \in \mathbb{N}$ and each $\tau \in [-\delta, 0]$, let

$$\begin{array}{rccl} g^\tau_n : & (-\infty, 0] & \longrightarrow & \mathbb{R}^m \\ & s & \mapsto & D(\omega{\cdot}s, (x^n - x^n_\tau)_s). \end{array}$$

Then, for all $s \le 0$, all $\tau \in [-\delta, 0]$ and all $n \in \mathbb{N}$,

$$\begin{aligned} \|g^\tau_n(s)\| \le& \|D(\omega{\cdot}s, x^n_s) - D(\omega{\cdot}(s+\tau), x^n_{s+\tau})\| \\ &+ \|D(\omega{\cdot}(s+\tau), x^n_{s+\tau}) - D(\omega{\cdot}s, x^n_{s+\tau})\| \\ \le& \frac{\varepsilon}{2\,k} + \frac{\varepsilon}{2\,k^2\,r}\,\|x^n\|_\infty \le \frac{\varepsilon}{k}. \end{aligned}$$

From Proposition 3.2, we deduce again that

$$\|x^n - x^n_\tau\|_\infty \le k\,\|g^\tau_n\|_\infty \le k\,\frac{\varepsilon}{k} = \varepsilon$$

for all $n \in \mathbb{N}$ and all $\tau \in [-\delta, 0]$. Thus $\{x^n\}_n$ is equicontinuous and, consequently, relatively compact for the compact-open topology. Hence, there is a convergent subsequence of $\{x^n\}_n$, let us assume the whole sequence, i.e. there is a continuous function x such that $x^n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$. Therefore, we have that $\|x\|_\infty \le k\,r$ and $\|x^n(s) - x^n(s+t)\| \to \|x(s) - x(s+t)\|$ as $n \to \infty$ for all $s, t \le 0$, which implies that $x \in BU$. From this, $x^n_s \xrightarrow{\mathsf{d}} x_s$ for each $s \le 0$, and the expression of D yields $D(\omega{\cdot}s, x^n_s) = h_n(s) \to D(\omega{\cdot}s, x_s)$, i.e. $D(\omega{\cdot}s, x_s) = h(s)$ for $s \le 0$ and $\widehat{D}_2(\omega, x) = h$. Then $\widehat{D}$ is surjective, as claimed.

Let us check that $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$, $r > 0$, when we take the restriction of the compact-open topology to B_r. Fix $\varepsilon > 0$ and $\rho > 0$; using Theorem 2.11, it is clear that we can find a $\rho_0 > 0$ such that $c(t) < \varepsilon/(4\,k\,r)$ for all $t \ge \rho_0$. Besides, there is a $\delta > 0$ such that, if ω_1, $\omega_2 \in \Omega$ and $d(\omega_1, \omega_2) < \delta$, then

$$\|D(\omega_1{\cdot}s, x) - D(\omega_2{\cdot}s, x)\| < \frac{\varepsilon}{4\,k}$$

for all $s \in [-\rho_0 - \rho, 0]$ and all $x \in B_{kr}$, thanks to the uniform continuity of σ on $[-\rho_0 - \rho, 0] \times \Omega$ and (D1). Let (ω_1, h_1), $(\omega_2, h_2) \in \Omega \times B_r$ such that $d(\omega_1, \omega_2) < \delta$ and $\|h_1 - h_2\|_{[-\rho_0-\rho,0]} \le \varepsilon/(4\,k)$; clearly, we have that $x_1 = \widehat{D}_2^{-1}(\omega_1, h_1)$, $x_2 = \widehat{D}_2^{-1}(\omega_2, h_2) \in B_{kr}$ thanks to Proposition 3.2. Then, for all $t \in [0, \rho_0 + \rho]$,

$$D(\omega_i{\cdot}(-\rho_0 - \rho + t), (x_i)_{-\rho_0-\rho+t}) = h_i(-\rho_0 - \rho + t), \quad i = 1, 2,$$

whence it follows that, for all $t \in [0, \rho_0 + \rho]$,

$$\begin{aligned}
&\|D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_1)_{-\rho_0-\rho+t}) - D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t})\| \le \\
&\quad \le \|D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_1)_{-\rho_0-\rho+t}) - D(\omega_2{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t})\| \\
&\qquad + \|D(\omega_2{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t}) - D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t})\| \\
&\quad \le \|h_1(-\rho_0 - \rho + t) - h_2(-\rho_0 - \rho + t)\| + \frac{\varepsilon}{4\,k} \le \frac{\varepsilon}{4\,k} + \frac{\varepsilon}{4\,k} = \frac{\varepsilon}{2\,k}.
\end{aligned}$$

Now, the stability of D and Theorem 2.11 imply that, for all $t \in [\rho_0, \rho_0 + \rho]$,

$$\begin{aligned}
\|x_1(-\rho_0 - \rho + t) - x_2(-\rho_0 - \rho + t)\| &\le \\
&\le \sup_{t \in [\rho_0, \rho_0+\rho]} c(t)\, \|x_1 - x_2\|_\infty + k\, \frac{\varepsilon}{2\,k} \\
&\le \frac{\varepsilon}{4\,k\,r}\, 2\,k\,r + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

that is, $\|x_1 - x_2\|_{[-\rho,0]} \le \varepsilon$, and the expected result holds.

Finally, we prove that, provided that $(\Omega, \sigma, \mathbb{R})$ is almost periodic, $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$, $r > 0$, when the norm is considered on B_r. Let $\varepsilon > 0$; from Theorem 3.1 and the almost periodicity of $(\Omega, \sigma, \mathbb{R})$, it follows that there exists $\delta > 0$ such that, for all ω_1, $\omega_2 \in \Omega$ with $d(\omega_1, \omega_2) < \delta$ and all $x \in B_{kr}$,

$$\|\widehat{D}_2(\omega_1, x) - \widehat{D}_2(\omega_2, x)\|_\infty < \frac{\varepsilon}{2\,k}.$$

Now, let us fix (ω_1,h_1), $(\omega_2,h_2)\in\Omega\times B_r$ such that $d(\omega_1,\omega_2)<\delta$ and also $\|h_1-h_2\|_\infty\le\varepsilon/(2\,k)$. As before, thanks to Proposition 3.2, we have that $x_1=(\widehat{D}^{-1})_2(\omega_1,h_1)$, $x_2=(\widehat{D}^{-1})_2(\omega_2,h_2)\in B_{kr}$. Let $y=(\widehat{D}^{-1})_2(\omega_1,h_2)$; using Proposition 3.2,

$$\begin{aligned}\frac{1}{k}\|y-x_2\|_\infty\le&\|\widehat{D}_2(\omega_1,y)-\widehat{D}_2(\omega_1,x_2)\|_\infty\\&=\|\widehat{D}_2(\omega_2,x_2)-\widehat{D}_2(\omega_1,x_2)\|_\infty\le\frac{\varepsilon}{2\,k},\end{aligned}$$

which, together with Proposition 3.2 again, yields

$$\begin{aligned}\|x_1-x_2\|_\infty\le&\|x_1-(\widehat{D}^{-1})_2(\omega_1,h_2)\|_\infty+\|(\widehat{D}^{-1})_2(\omega_1,h_2)-x_2\|_\infty\\\le&k\,\frac{\varepsilon}{2\,k}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

as desired. □

The following result relates the concepts of stability and continuity of the operator D for the metric topology.

Theorem 3.4. *Let $\widehat{D}$ be the mapping defined in* (3.1)*. The following statements are equivalent:*

(i) *D is stable;*

(ii) *for each $r>0$ and each sequence $\{(\omega_n,x_n)\}_n\subset\Omega\times BU$ such that $\|\widehat{D}_2(\omega_n,x_n)\|_\infty\le r$, $\omega_n\to\omega\in\Omega$ and $\widehat{D}_2(\omega_n,x_n)\xrightarrow{\mathsf{d}}0$ as $n\to\infty$, it holds that $x_n(0)\to 0$ as $n\to\infty$.*

Proof. Theorem 3.3 assures that (i) implies (ii). Conversely, let us consider the set $C_D=\{(\omega,\varphi)\in\Omega\times BU: D(\omega,\varphi)=0\}$. For each $\rho>0$, we define $\mathcal{L}_\rho: C_D\to\mathbb{R}^m$, $(\omega,\varphi)\mapsto x(\rho)$, where x is the solution of

$$\begin{cases} D(\omega{\cdot}t,x_t)=0\,, & t\ge 0\,,\\ x_0=\varphi\,.\end{cases}$$

It is important to notice that $C_D=\{(\omega,\varphi):\omega\in\Omega,\ \varphi\in C_D(\omega)\}$, where $C_D(\omega)=\{\varphi\in BU: D(\omega,\varphi)=0\}$ is a vector space for each $\omega\in\Omega$.

From the uniqueness of the solution of $(2.2)_\omega$, it is easy to check that $\mathcal{L}_\rho$ is well defined and linear in its second variable. In addition, from Theorem 2.5, we deduce that $\|\mathcal{L}_\rho(\omega,\varphi)\|=\|x(\rho)\|\le k_\rho^2\,\|\varphi\|_\infty$ for all $(\omega,\varphi)\in C_D$, whence $\|\mathcal{L}_\rho(\omega,\cdot)\|\le k_\rho^2$ for all $\omega\in\Omega$.

Next, we check that $\sup_{\omega\in\Omega}\|\mathcal{L}_\rho(\omega,\cdot)\|_\infty\to 0$ as $\rho\to\infty$; this fact shows the stability of D because $\|x(\rho)\|\le c(\rho)\,\|\varphi\|_\infty$ for all $(\omega,\varphi)\in C_D$, where

$c(\rho) = \sup_{\omega\in\Omega} \|\mathcal{L}_\rho(\omega,\cdot)\|_\infty$. Let us assume, on the contrary, that there exist $\delta > 0$, a sequence $\rho_n \uparrow \infty$, and a sequence $\{\varphi_n\}_n$ such that $\varphi_n \in C_D(\omega_n)$, $\|\varphi_n\|_\infty \le 1$ and $\|\mathcal{L}_{\rho_n}(\omega_n,\varphi_n)\| \ge \delta$ for each $n\in\mathbb{N}$. That is, $\|x^n(\rho_n)\| \ge \delta$ where x^n is the solution of

$$\begin{cases} D(\omega_n{\cdot}t, x^n_t) = 0\,, \quad t \ge 0\,, \\ x^n_0 = \varphi_n\,. \end{cases}$$

Therefore,

$$\begin{cases} D(\omega_n{\cdot}(\rho_n+s), (x^n_{\rho_n})_s) = D(\omega_n{\cdot}(\rho_n+s), x^n_{\rho_n+s}) = 0 & \text{if } s\in[-\rho_n, 0]\,, \\ D(\omega_n{\cdot}(\rho_n+s), (x^n_{\rho_n})_s) = D(\omega_n{\cdot}(\rho_n+s), (\varphi_n)_{\rho_n+s}) & \text{if } s \le -\rho_n\,, \end{cases}$$

and taking $r = \sup_{\omega\in\Omega} \|D(\omega,\cdot)\|$, the sequence $\{x^n_{\rho_n}\}_{n\in\mathbb{N}} \subset BU$ satisfies that $\|\widehat{D}(\omega_n{\cdot}\rho_n, x^n_{\rho_n})\|_\infty \le r$ and $\widehat{D}(\omega_n{\cdot}\rho_n, x^n_{\rho_n}) \xrightarrow{\mathsf{d}} 0$ as $n\to\infty$. Now, we can assume without loss of generality that $\omega_n{\cdot}\rho_n \to \omega\in\Omega$. Consequently, $x^n_{\rho_n}(0) = x^n(\rho_n) \to 0$ as $n\to\infty$, which clearly contradicts the fact that $\|x^n(\rho_n)\| \ge \delta$, and finishes the proof. □

As a consequence, the operator D is stable if and only if $\widehat{D}$ is invertible and $\widehat{D}^{-1}$ is uniformly continuous on $\Omega\times B_r$ for the compact-open topology for all $r>0$.

The following statement provides a symmetric theory for the operators $\widehat{D}$ and $\widehat{D}^{-1}$. In particular, $\widehat{D}^{-1}$ is generated by a linear operator D^* which satisfies (D1)–(D3).

Proposition 3.5. *Suppose that D is stable, and define*

$$\begin{array}{rccc} D^*: & \Omega\times BU & \longrightarrow & \mathbb{R}^m \\ & (\omega,x) & \mapsto & (\widehat{D}^{-1})_2(\omega,x)(0). \end{array}$$

Then D^* also satisfies (D1)–(D3) *and is stable. Moreover, for all $s\le 0$ and all $(\omega,x)\in\Omega\times BU$, it holds that $(\widehat{D}^{-1})_2(\omega,x)(s) = D^*(\omega{\cdot}s, x_s)$. In particular, for all $\omega\in\Omega$, there is an $m\times m$ matrix $\mu^*(\omega) = [\mu^*_{ij}(\omega)]_{ij}$ of real Borel regular measures with finite total variation such that*

$$(\widehat{D}^{-1})_2(\omega,x)(s) = \int_{-\infty}^0 [d\mu^*(\omega{\cdot}s)]x_s, \quad (\omega,x)\in\Omega\times BU,\ s\le 0.$$

Proof. It is clear from Theorem 3.3 that $\Omega\to\mathcal{L}(BU,\mathbb{R}^m)$, $\omega\mapsto D^*(\omega,\cdot)$ is well defined. Let us check that it is continuous as well. Let $\varepsilon>0$; since $\widehat{D}^{-1}$

is uniformly continuous on $\Omega \times B_1$, there exists $\delta > 0$ such that, if $\omega_1,\, \omega_2 \in \Omega$ with $d(\omega_1,\omega_2) < \delta$ and $x \in B_1$, then

$$\|(\widehat{D}^{-1})_2(\omega_1,x)(0) - (\widehat{D}^{-1})_2(\omega_2,x)(0)\| \le \varepsilon.$$

As a result, D^* satisfies property (D1).

From Theorem 3.3, we deduce that D^* satisfies (D2), and, hence, from Lemma 2.1, for each $\omega \in \Omega$, there is an $m \times m$ matrix $\mu^*(\omega) = [\mu^*_{ij}(\omega)]_{ij}$ of real Borel regular measures with finite total variation such that

$$D^*(\omega,x) = \int_{-\infty}^0 [d\mu^*(\omega)]x, \quad (\omega,x) \in \Omega \times BU.$$

We claim that (D3) holds for D^*, i.e. $\det B^*(\omega) \neq 0$ for all $\omega \in \Omega$, where $B^*(\omega) = \mu^*(\omega)(\{0\})$. Let us assume, on the contrary, that there exist $\omega \in \Omega$ and $v \in \mathbb{R}^m$ with $\|v\| = 1$ and $B^*(\omega)v = 0$. Then, for each $\varepsilon > 0$, we take $\varphi_\varepsilon \in C((-\infty,0],\mathbb{R})$ with $\|\varphi_\varepsilon\|_\infty = \varphi_\varepsilon(0) = 1$ and $\varphi_\varepsilon(s) = 0$ for each $s \in (-\infty,-\varepsilon]$. Let $x^\varepsilon \in BU$ be defined by $x^\varepsilon(s) = \varphi_\varepsilon(s)\, v$, $s \le 0$. It is now clear that

$$1 = \|x^\varepsilon\|_\infty \le \sup_{\omega_1} \|D(\omega_1,\cdot)\|\, \|(\widehat{D}^{-1})_2(\omega,x^\varepsilon)\|_\infty\,.$$

However, for each $s \in (-\infty,0]$,

$$\begin{aligned}(\widehat{D}^{-1})_2(\omega,x^\varepsilon)(s) &= D^*(\omega{\cdot}s,x^\varepsilon_s)\\ &= \varphi_\varepsilon(s)\, B^*(\omega{\cdot}s)\, v - \int_{-\infty}^0 [d\nu^*(\omega{\cdot}s)(\theta)]\, \varphi_\varepsilon(\theta+s)\, v.\end{aligned} \tag{3.2}$$

As D^* satisfies (D1) and (D2), Corollary 2.4 yields

$$\lim_{\varepsilon\to 0^+} \sup_{s\le 0} \left\| \int_{-\infty}^0 [d\nu^*(\omega{\cdot}s)(\theta)]\, \varphi_\varepsilon(\theta+s)\, v \right\| = 0.$$

That is, given $\eta > 0$, there exists $\varepsilon_0 > 0$ such that, given $\varepsilon \in [0,\varepsilon_0]$, we have

$$\|(\widehat{D}^{-1})_2(\omega,x^\varepsilon)(s)\| \le \|\varphi_\varepsilon(s)\, B^*(\omega{\cdot}s)\, v\| + \frac{\eta}{2}$$

for all $s \le 0$ and also

$$\|B^*(\omega{\cdot}s)\, v\| \le \frac{\eta}{2}$$

for all $s \in [-\varepsilon,0]$. Thanks to the choice of φ_ε and (3.2), this implies that $\|(\widehat{D}^{-1})_2(\omega,x^\varepsilon)\|_\infty \le \eta$, i.e. $\lim_{\varepsilon\to 0^+} \|(\widehat{D}^{-1})_2(\omega,x^\varepsilon)\|_\infty = 0$, a contradiction.

Clearly, D^* is stable as a consequence of Theorem 3.4. Now, let s, $\theta \le 0$. For all $(\omega, h) \in \Omega \times BU$, we have

$$\widehat{D}_2(\omega{\cdot}s, h_s)(\theta) = D(\omega{\cdot}(s+\theta), h_{s+\theta}) = \widehat{D}_2(\omega, h)(s+\theta) = \widehat{D}_2(\omega, h)_s(\theta). \quad (3.3)$$

Fix $s \le 0$. Let $(\omega, x) \in \Omega \times BU$ and $h_1 = (\widehat{D}^{-1})_2(\omega{\cdot}s, x_s)$, $h_2 = (\widehat{D}^{-1})_2(\omega, x)_s$. Let us check that $h_1 = h_2$. Indeed, thanks to (3.3),

$$\widehat{D}_2(\omega{\cdot}s, h_1) = x_s = \widehat{D}_2(\widehat{D}^{-1}(\omega, x))_s = \widehat{D}_2(\omega{\cdot}s, (\widehat{D}^{-1})_2(\omega, x)_s) = \widehat{D}_2(\omega{\cdot}s, h_2)$$

which, together with the injectivity of $\widehat{D}$, yields $h_1 = h_2$.

Hence, if $s \le 0$ and $(\omega, x) \in \Omega \times BU$, then

$$D^*(\omega{\cdot}s, x_s) = (\widehat{D}^{-1})_2(\omega{\cdot}s, x_s)(0) = (\widehat{D}^{-1})_2(\omega, x)_s(0) = (\widehat{D}^{-1})_2(\omega, x)(s),$$

and the result is proved. □

In the remainder of this chapter, we will study an example of D-operator and check that it satisfies properties (D1)–(D3), and that it is stable as well. Namely, consider the map $D : \Omega \times BU \to \mathbb{R}^m$ defined for $(\omega, x) \in \Omega \times BU$ by

$$D(\omega, x) = \left(\sum_{j=1}^{m} \left[b_{ij}(\omega)\, x_j(0) - \int_{-\infty}^{0} x_j \, d\nu_{ij}(\omega) \right] \right)_{i=1}^{m} \quad (3.4)$$

where $b_{ij} : \Omega \to \mathbb{R}$ are continuous functions, and $\nu_{ij}(\omega)$, $\omega \in \Omega$, are regular Borel measures with finite total variation for all i, $j \in \{1, \ldots, m\}$. We denote by $B(\omega)$ and $\nu(\omega)$ the matrices $[b_{ij}(\omega)]_{ij}$ and $[\nu_{ij}(\omega)]_{ij}$, $\omega \in \Omega$, respectively. This way, D has the following integral representation:

$$D(\omega, x) = B(\omega)\, x(0) - \int_{-\infty}^{0} [d\nu(\omega)]\, x$$

for each $(\omega, x) \in \Omega \times BU$.

Given two $m \times m$ matrices $A = [a_{ij}]_{ij}$ and $B = [b_{ij}]_{ij}$, we will write $A \le B$ if $a_{ij} \le b_{ij}$ for all i, j.

Let us assume the following hypotheses on $B(\omega)$ and $\nu(\omega)$, $\omega \in \Omega$:

(L1) $\nu_{ij}(\omega)(\{0\}) = 0$ for all $\omega \in \Omega$ and all i, $j \in \{1, \ldots, m\}$, and the mapping $\nu : \Omega \to \mathcal{M}$, $\omega \mapsto \nu(\omega)$ is continuous, where $\mathcal{M}$ is the Banach space of $m \times m$ matrices of Borel measures on $(-\infty, 0]$ with the supremum norm defined from the total variation of the measures;

(L2) $B(\omega)$ is a regular matrix for all $\omega \in \Omega$, and the map $B : \Omega \to \mathbb{M}_m(\mathbb{R})$, $\omega \mapsto B(\omega)$ is continuous; moreover, $B(\omega)^{-1} \geq 0$, $B(\omega)^{-1}\nu(\omega)$ is a matrix of positive measures for all $\omega \in \Omega$, and

$$\|B(\omega)^{-1}\nu(\omega)\|((-\infty, 0]) < 1.$$

Conditions (L1) and (L2) yield the following proposition, which is a converse result for Propositions 2.2 and 2.3.

Proposition 3.6. *The mapping D defined in* (3.4) *satisfies* (D1)–(D3).

Proof. Let us prove (D1). First, it is clear that $D(\omega, \cdot)$ is linear for all $\omega \in \Omega$. Now let us check that $\Omega \to \mathcal{L}(BU, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is well defined. In order to do this, fix $\omega \in \Omega$ and $\{x_n\}_n \subset BU$ with $\|x_n - x\|_\infty \to 0$ as $n \to \infty$ for some $x \in BU$. It is clear that

$$\left\| \int_{-\infty}^{0} [d\nu(\omega)](x_n - x) \right\| \leq \sup_{\omega_1 \in \Omega} \|\nu(\omega_1)\|((-\infty, 0])\, \|x_n - x\|_\infty,$$

and the latter goes to 0 as $n \to \infty$ due to (L1). This fact implies that the aforementioned mapping is well defined. Let us prove that $\Omega \to \mathcal{L}(BU, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous. Let $\{\omega_n\}_n \subset \Omega$ with $\lim_{n\to\infty} \omega_n = \omega$ for some $\omega \in \Omega$. Now

$$\begin{aligned} &\sup_{x \in B_1} \|D(\omega_n, x) - D(\omega, x)\| \leq \\ &\qquad \leq \|B(\omega_n) - B(\omega)\| + \sup_{x \in B_1} \left\| \int_{-\infty}^{0} [d(\nu(\omega_n) - \nu(\omega))]\, x \right\|, \end{aligned}$$

which converges to 0 as $n \to \infty$ thanks to (L1) and (L2).

Let us check (D2). It is noteworthy that Corollary 2.4 holds here thanks to (L1). Fix $r > 0$, and let $\{(\omega^n, x^n)\}_n \subset \Omega \times B_r$ be a sequence such that $\omega^n \to \omega^0$ and $x^n \xrightarrow{\mathsf{d}} x^0$ as $n \to \infty$ for some $(\omega^0, x^0) \in \Omega \times B_r$. Now, fix $\varepsilon > 0$ and $\rho > 0$ such that

$$\sup_{\omega \in \Omega} \|\nu(\omega)\|((-\infty, -\rho]) < \frac{\varepsilon}{8\,r}.$$

Thanks to (L1), there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{\omega \in \Omega} \|\nu(\omega)\|((-\infty, 0])\, \|x^n - x^0\|_{[-\rho, 0]} < \frac{\varepsilon}{4}$$

for all $n \geq n_0$, and

$$\left\| \int_{-\infty}^{0} [d(\nu(\omega^n) - \nu(\omega^0))]\, x^0 \right\| < \frac{\varepsilon}{2}.$$

Thus, if $n \geq n_0$, then

$$\begin{aligned}
&\left\| \int_{-\infty}^{0} [d\nu(\omega^n)]x^n - \int_{-\infty}^{0} [d\nu(\omega^0)]x^0 \right\| \leq \\
&\qquad \leq \left\| \int_{-\infty}^{0} [d\nu(\omega^n)](x^n - x^0) \right\| + \left\| \int_{-\infty}^{0} [d(\nu(\omega^n) - \nu(\omega^0))]x^0 \right\| \\
&\qquad \leq \left\| \int_{-\infty}^{-\rho} [d\nu(\omega^n)](x^n - x^0) \right\| + \left\| \int_{-\rho}^{0} [d\nu(\omega^n)](x^n - x^0) \right\| + \frac{\varepsilon}{2} \\
&\qquad \leq 2\,r\frac{\varepsilon}{8\,r} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Applying (L2), we obtain (D2), as wanted. As for (D3), it is immediate from (L1) and (L2). □

Now, let us study the relation between the stability of the operator D and the invertibility of its convolution operator $\widehat{D}$.

In order to do so, define the map $\widehat{D} : \Omega \times BU \to \Omega \times BU$ as in (3.1). For each $\omega \in \Omega$, let us define $\widehat{B}_\omega : BU \to BU$, $\widehat{B}_\omega(x)(s) = B(\omega{\cdot}s)\,x(s)$. It is easy to check that $\widehat{B}_\omega$ is a linear isomorphism of BU, and it is continuous for the norm; in addition, the mapping $(\omega, x) \mapsto \widehat{B}_\omega(x)$ is uniformly continuous on each set of the form $\Omega \times B_r$ for all $r > 0$ when the compact-open topology is considered. We denote $(\widehat{B}_\omega)^{-1} = (\widehat{B^{-1}})_\omega$, $\omega \in \Omega$.

Theorem 3.7. *For each $\omega \in \Omega$, consider the continuous linear operator $\widehat{L}_\omega : BU \to BU$ defined for $x \in BU$ and $s \leq 0$ by*

$$\widehat{L}_\omega(x)(s) = B(\omega{\cdot}s)^{-1} \int_{-\infty}^{0} [d\nu(\omega{\cdot}s)]x_s.$$

Then the following statements hold:

(i) $\sup_{\omega\in\Omega} \|\widehat{L}_\omega\| < 1$, *and, for all $(\omega, x) \in \Omega \times BU$,*

$$\widehat{D}_2(\omega, x) = [\widehat{B}_\omega \circ (I - \widehat{L}_\omega)](x);$$

(ii) $\widehat{D}$ *is invertible, and*

$$(\widehat{D}^{-1})_2(\omega, x) = \sum_{n=0}^{\infty} (\widehat{L}^n_\omega \circ (\widehat{B}_\omega)^{-1})(x)$$

for every $(\omega, x) \in \Omega \times BU$;

(iii) $\widehat{D}^{-1}$ *is* positive, *i.e. $(\widehat{D}^{-1})_2(\omega, x) \geq 0$ for all $(\omega, x) \in \Omega \times BU$ with $x \geq 0$;*

(iv) *the map* $\Omega \times B_r \to B_r$, $(\omega, x) \mapsto \widehat{L}_\omega(x)$ *is uniformly continuous for the compact-open topology for each* $r > 0$;

(v) D *is stable.*

Proof. From condition (L2), we conclude that $\sup_{\omega\in\Omega} \|\widehat{L}_\omega\| < 1$. It is immediate to check that $\widehat{D}_2(\omega, x) = [\widehat{B}_\omega \circ (I - \widehat{L}_\omega)](x)$ for each $(\omega, x) \in \Omega \times BU$. This way, (i) is proved, whence $\widehat{D}$ is invertible, and

$$(\widehat{D}^{-1})_2(\omega, x) = \sum_{n=0}^{\infty} (\widehat{L}_\omega^n \circ (\widehat{B}_\omega)^{-1})(x)$$

for every $(\omega, x) \in \Omega \times BU$, which proves (ii). Now, using (ii) and hypothesis (L2), a simple calculation yields (iii).

In addition, for all $r_1 > 0$ and all $(\omega, x) \in \Omega \times B_{r_1}$, it is clear that $(\widehat{B}_\omega)^{-1}(x) \in \Omega \times B_{r_2}$ and $\widehat{L}_\omega(x) \in \Omega \times B_{r_1}$, where $r_2 = \sup_{\omega_1\in\Omega} \|B(\omega_1)^{-1}\| r_1$; besides, if (ω_1, x_1), $(\omega_2, x_2) \in \Omega \times B_{r_1}$ and $s \in [-\rho, 0]$ for some $\rho > 0$, then

$$\begin{aligned}
&\|\widehat{L}_{\omega_1}(x_1)(s) - \widehat{L}_{\omega_2}(x_2)(s)\| \le \\
&\quad \le \|\widehat{L}_{\omega_1}(x_1)(s) - \widehat{L}_{\omega_1}(x_2)(s)\| + \|\widehat{L}_{\omega_1}(x_2)(s) - \widehat{L}_{\omega_2}(x_2)(s)\| \\
&\quad \le \sup_{\omega\in\Omega} \|B(\omega)^{-1}\| \sup_{\omega\in\Omega} \|\nu(\omega)\|((-\infty, -\rho])\, 2\, r_1 + \|x_1 - x_2\|_{[-2\rho,0]} \\
&\qquad + \|B(\omega_1{\cdot}s)^{-1} - B(\omega_2{\cdot}s)^{-1}\| \sup_{\omega\in\Omega} \|\nu(\omega)\|((-\infty, 0])\, r_1 \\
&\qquad + \sup_{\omega\in\Omega} \|B(\omega)^{-1}\| \, \|\nu(\omega_1{\cdot}s) - \nu(\omega_2{\cdot}s)\|((-\infty, 0])\, r_1.
\end{aligned}$$

This proves (iv). Let us check that, given $r > 0$, $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$ for the compact-open topology. Let us fix $\varepsilon > 0$ and $\rho > 0$. There is an $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \|\widehat{L}_\omega^n \circ (\widehat{B}_\omega)^{-1}\| < \frac{\varepsilon}{3\,r}.$$

From (iv) and the uniform continuity of $(\omega, x) \mapsto (\widehat{B}_\omega)^{-1}(x)$ for the product metric topology on each set of the form $\Omega \times B_r$, $r > 0$, it follows that there exist $\rho_0 > 0$ and $\delta > 0$ such that, if (ω_1, x_1), $(\omega_2, x_2) \in \Omega \times B_r$ with $d(\omega_1, \omega_2) < \delta$ and $\|x_1(s) - x_2(s)\| < \delta$ for all $s \in [-\rho_0, 0]$, then

$$\|\widehat{L}_{\omega_1}^j \circ (\widehat{B}_{\omega_1})^{-1}(x_1)(s) - \widehat{L}_{\omega_2}^j \circ (\widehat{B}_{\omega_2})^{-1}(x_2)(s)\| \le \frac{\varepsilon}{3\,n_0}$$

for all $j \in \{0, \dots, n_0 - 1\}$ and all $s \in [-\rho, 0]$. As a consequence,

$$\left\| \sum_{n=0}^{\infty} \widehat{L}_{\omega_1}^n \circ (\widehat{B}_{\omega_1})^{-1}(x_1)(s) - \sum_{n=0}^{\infty} \widehat{L}_{\omega_2}^n \circ (\widehat{B}_{\omega_2})^{-1}(x_2)(s) \right\| \le \varepsilon$$

for every $s \in [-\rho, 0]$, which proves that $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$, $r > 0$, for the compact-open topology, and, therefore, according to Theorem 3.4, that D is stable. This completes the proof. $\square$

Chapter 4

Transformed usual order

Throughout this chapter, we will introduce an adequate order relation and study the monotone skew-product semiflow generated by a family of NFDEs with infinite delay and stable non-autonomous D-operator. In particular, we establish the 1-covering property of omega-limit sets under the componentwise separating property and uniform stability. This result was proved in [JZ] for FDEs with finite delay and a distal base. Afterward, it was extended in [NOS] to the case of infinite delay and a minimal flow on the base; this approach does not require strong monotonicity, which is a fundamental advantage due to the impossibility to obtain such property when dealing with infinite delay FDEs. The aim of this chapter is to transfer the dynamical structure obtained in [NOS] to the case of NFDEs; specifically, the 1-covering property of omega-limit sets will hold. The main tool in the proof of the result is the transformation of the initial family of NFDEs into a family of FDEs with infinite delay in whose study the results in [NOS] turn out to be useful. This transformation is done by means of the convolution operator $\widehat{D}$ studied in Chapter 3. It is noteworthy that we transform both finite and infinite delay NFDEs into infinite delay FDEs; thus, we lose the advantage to count on finite delays, but, on the other hand, no strong monotonicity is needed, which allows us to apply the aforementioned theory.

Let $(\Omega, \sigma, \mathbb{R})$ be a minimal flow over a compact metric space (Ω, d), and denote $\sigma(t, \omega) = \omega{\cdot}t$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. In $\mathbb{R}^m$, we take the maximum norm $\|v\| = \max_{j=1,\ldots,m} |v_j|$ and the usual partial order relation:

$$
\begin{aligned}
v \le w \quad &\Longleftrightarrow \quad v_j \le w_j \quad \text{for } j = 1, \ldots, m\,,\\
v < w \quad &\Longleftrightarrow \quad v \le w \quad \text{and} \quad v_j < w_j \quad \text{for some } j \in \{1, \ldots, m\}\,.
\end{aligned}
$$

As in Chapter 2, we consider the Fréchet space $X = C((-\infty, 0], \mathbb{R}^m)$ endowed with the compact-open topology, i.e. the topology of uniform convergence over compact subsets, and $BU \subset X$ the Banach space of bounded and uniformly continuous functions with the supremum norm

$$\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|.$$

Let $D\colon \Omega \times BU \to \mathbb{R}^m$ be a non-autonomous and stable linear operator satisfying hypotheses (D1)–(D3) and given by relation (2.3). The subset

$$\mathcal{P}_D = \{(\omega, x) \in \Omega \times BU : D(\omega{\cdot}s, x_s) \geq 0 \ \text{ for each } \ s \in (-\infty, 0]\}$$

will be called *positive bundle*. It is defined by translating the usual order on BU, which in turn is given by the positive cone

$$BU^+ = \{x \in BU : x(s) \geq 0 \text{ for all } s \leq 0\},$$

using the convolution operator $\widehat{D}$; specifically, $\mathcal{P}_D = \widehat{D}^{-1}(BU^+)$. In the case that D is autonomous, i.e. it does not depend on its first variable, $\mathcal{P}_D$ turns out to be a trivial positive bundle:

$$\mathcal{P}_D = \Omega \times \{x \in BU : D(x_s) \geq 0 \text{ for all } s \leq 0\},$$

where $D : BU \to \mathbb{R}^m$ denotes the autonomous linear operator. Besides, it induces a partial order relation on $\Omega \times BU$ fiberwise; this relation is given by

$$\begin{aligned}
(\omega, x) \leq_D (\omega, y) \quad &\Longleftrightarrow \quad D(\omega{\cdot}s, x_s) \leq D(\omega{\cdot}s, y_s) \quad \text{for each } s \in (-\infty, 0]\,,\\
(\omega, x) <_D (\omega, y) \quad &\Longleftrightarrow \quad (\omega, x) \leq_D (\omega, y) \quad \text{and} \quad x \neq y\,.
\end{aligned}$$

Remark 4.1. Notice that, if we denote the usual partial order of BU

$$x \leq y \quad \Longleftrightarrow \quad x(s) \leq y(s) \quad \text{for each } s \in (-\infty, 0]\,,$$

we have that $(\omega, x) \leq_D (\omega, y)$ if and only if $\widehat{D}_2(\omega, x) \leq \widehat{D}_2(\omega, y)$, where $\widehat{D}$ is defined by relation (3.1). Although in some cases they may coincide, this new order is different from the one given in [WF].

We consider the family of non-autonomous NFDEs with infinite delay and stable and non-autonomous D-operator

$$\frac{d}{dt} D(\omega{\cdot}t, z_t) = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \tag{4.1$_\omega$}$$

defined by a function $F\colon \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$. Let us assume the following hypothesis:

(F1) $F : \Omega \times BU \to \mathbb{R}^m$ is continuous on $\Omega \times BU$, and its restriction to $\Omega \times B_r$ is Lipschitz continuous in its second variable when the norm is considered on B_r for all $r > 0$.

As seen in Wang and Wu [WW] and [W], for each $\omega \in \Omega$, the local existence and uniqueness of the solutions of equation $(4.1)_\omega$ follow from (F1). Moreover, given $(\omega, x) \in \Omega \times BU$, if $z(\cdot, \omega, x)$ represents the solution of equation $(4.1)_\omega$ with initial datum x, then the map $u(t, \omega, x) : (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto z(t+s, \omega, x)$ is an element of BU for all $t \geq 0$ where $z(\cdot, \omega, x)$ is defined.

Therefore, a local skew-product semiflow on $\Omega \times BU$ can be defined on an open subset $\mathcal{U}$ of $\mathbb{R}^+ \times \Omega \times BU$ as follows:

$$\begin{array}{rccl} \tau : & \mathcal{U} \subset \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x)). \end{array} \tag{4.2}$$

Let $(\omega, y) \in \Omega \times BU$. For each $t \geq 0$ where $u(t, \widehat{D}^{-1}(\omega, y))$ is defined, we define $\widehat{u}(t, \omega, y) = \widehat{D}_2(\omega{\cdot}t, u(t, \widehat{D}^{-1}(\omega, y)))$. Let us check that

$$\widehat{z}(\cdot, \omega, y) : t \mapsto \begin{cases} y(t) & \text{if } t \leq 0, \\ \widehat{u}(t, \omega, y)(0) & \text{if } t \geq 0, \end{cases}$$

is the solution of

$$\widehat{z}\,'(t) = G(\omega{\cdot}t, \widehat{z}_t),\ t \geq 0,\ \omega \in \Omega \tag*{$(4.3)_\omega$}$$

through (ω, y), where $G = F \circ \widehat{D}^{-1}$. Let $x = (\widehat{D}^{-1})_2(\omega, y)$; if $t \geq 0$, then

$$\begin{aligned} \frac{d}{dt}\widehat{z}(t, \omega, y) =& \frac{d}{dt}\left[\widehat{D}_2(\omega{\cdot}t, u(t, \omega, x))(0)\right] = \frac{d}{dt} D(\omega{\cdot}t, u(t, \omega, x)) \\ =& F(\omega{\cdot}t, u(t, \omega, x)) = F \circ \widehat{D}^{-1}(\widehat{D}(\omega{\cdot}t, u(t, \omega, x))) \\ =& F \circ \widehat{D}^{-1}(\omega{\cdot}t, \widehat{u}(t, \omega, x)). \end{aligned}$$

It only remains to notice that clearly $\widehat{z}(\cdot, \omega, y)_t = \widehat{u}(t, \omega, y)$ for all $t \geq 0$. Let us assume some more hypotheses concerning the map F:

(F2) $F(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^m$ for all $r > 0$;

(F3) the restriction of F to $\Omega \times B_r$ is continuous when the compact-open topology is considered on B_r, for all $r > 0$;

(F4) if (ω, x), $(\omega, y) \in \Omega \times BU$ with $(\omega, x) \leq_D (\omega, y)$ and $D_j(\omega, x) = D_j(\omega, y)$ holds for some $j \in \{1, \ldots, m\}$, then $F_j(\omega, x) \leq F_j(\omega, y)$.

Proposition 4.2. *Under hypotheses* (F1)–(F4)*, the following assertions hold:*

(i) *G is continuous on $\Omega \times BU$, and its restriction to $\Omega \times B_r$ is Lipschitz continuous in its second variable when the norm is considered on B_r for all $r > 0$;*

(ii) *$G(\Omega\times B_r)$ is a bounded subset of $\mathbb{R}^m$ for all $r>0$;*

(iii) *the restriction of G to $\Omega\times B_r$ is continuous when the compact-open topology is considered on B_r, for all $r>0$;*

(iv) *if $x,\,y\in BU$ with $x\le y$ and it holds that $x_j(0)=y_j(0)$ for some $j\in\{1,\dots,m\}$, then $G_j(\omega,x)\le G_j(\omega,y)$ for each $\omega\in\Omega$.*

Proof. First, let us check (i). Let $\{(\omega_n,x_n)\}_n\subset\Omega\times BU$ be a sequence with $\omega_n\to\omega$ and $\|x_n-x\|_\infty\to 0$ as $n\to\infty$ for some $(\omega,x)\in\Omega\times BU$. Then $x_n\stackrel{\mathsf{d}}{\to}x$ as $n\to\infty$, and there is an $r>0$ such that $x_n,\,x\in B_r$ for all $n\in\mathbb{N}$. Consequently, $\widehat{D}^{-1}(\omega_n,x_n)\stackrel{\mathsf{d}}{\to}\widehat{D}^{-1}(\omega,x)$ as $n\to\infty$, and, from Proposition 3.2, $\widehat{D}^{-1}(\omega_n,x_n)$, $\widehat{D}^{-1}(\omega,x)\in B_{kr}$ for all $n\in\mathbb{N}$. Thanks to (F3), $G(\omega_n,x_n)\to G(\omega,x)$ as $n\to\infty$. As for the Lipschitz continuity, let $r>0$ and fix (ω,y_1), $(\omega,y_2)\in\Omega\times B_r$; let $x_i=(\widehat{D}^{-1})_2(\omega,y_i)$, $i=1,\,2$. From Proposition 3.2, it follows that $\|x_i\|_\infty\le k\|y_i\|_\infty\le k\,r$, $i=1,\,2$. Let $L>0$ be the Lipschitz constant of F on $\Omega\times B_{kr}$. Again Proposition 3.2 yields

$$\|G(\omega,y_1)-G(\omega,y_2)\|=\|F(\omega,x_1)-F(\omega,x_2)\|\le L\|x_1-x_2\|_\infty\le L\,k\|y_1-y_2\|_\infty,$$

and the result is proved.

As for (ii), let $r>0$; from Proposition 3.2, it follows that

$$G(\Omega\times B_r)=F(\widehat{D}^{-1}(\Omega\times B_r))\subset F(\Omega\times B_{kr}),$$

and the latter is bounded thanks to (F2).
Let us focus on (iii). Let $r>0$; once more, Proposition 3.2 implies that $\widehat{D}^{-1}(\Omega\times B_r)\subset\Omega\times B_{kr}$, and F is continuous there when we consider the compact-open topology.
Finally, (iv) is a straightforward consequence of (F4). □

We may now define another local skew-product semiflow on an open subset $\widehat{\mathcal{U}}$ of $\mathbb{R}^+\times\Omega\times BU$ from the solutions of the equations of the family $(4.3)_\omega$ (see [HMN]) in the following manner:

$$\begin{array}{rlcl}\widehat{\tau}: & \widehat{\mathcal{U}}\subset\mathbb{R}^+\times\Omega\times BU & \longrightarrow & \Omega\times BU\\ & (t,\omega,x) & \mapsto & (\omega{\cdot}t,\widehat{u}(t,\omega,x)).\end{array}\tag{4.4}$$

Notice that an argument similar to the one given in Proposition 4.1 in [NOS] and Proposition 4.2 in [MNO] ensures that, for all $(\omega_0,x_0)\in\Omega\times BU$ giving rise to a bounded solution, $\mathrm{cls}_X\{u(t,\omega_0,x_0):t\ge 0\}$ is a compact subset of BU for the metric topology.

Proposition 4.3. *Let $z(\cdot,\omega_0,x_0)$ be a bounded solution of equation $(4.1)_{\omega_0}$, that is, $r=\sup_{t\in\mathbb{R}}\|z(t,\omega_0,x_0)\|<\infty$. Then $\mathrm{cls}_X\{u(t,\omega_0,x_0)\,:\,t\ge 0\}$ is a compact subset of BU for the compact-open topology.*

As a result, the omega-limit set of (ω_0,x_0) can be defined as

$$\begin{aligned}\mathcal{O}(\omega_0,x_0)=\{(\omega,x)\in\Omega\times BU:\ &\text{there exists } \{t_n\}_n\subset\mathbb{R} \text{ with } t_n\uparrow\infty\\ &\text{and } \omega_0{\cdot}t_n\to\omega,\ u(t_n,\omega_0,x_0)\xrightarrow{\mathsf{d}}x\}.\end{aligned}\tag{4.5}$$

It is easy to see that $\mathcal{O}(\omega_0,x_0)$ is obtained from the omega-limit set of $\widehat{D}(\omega_0,x_0)$ by means of $\widehat{D}^{-1}$:

$$\begin{aligned}\mathcal{O}(\omega_0,x_0)=\widehat{D}^{-1}\big(\{(\omega,x)\in\Omega\times BU:\ &\text{there exists } \{t_n\}_n\subset\mathbb{R} \text{ with } t_n\uparrow\infty\\ &\text{and } \omega_0{\cdot}t_n\to\omega,\ \widehat{u}(t_n,\widehat{D}(\omega_0,x_0))\xrightarrow{\mathsf{d}}x\}\big).\end{aligned}$$

Notice that the omega-limit set of a pair $(\omega_0,x_0)\in\Omega\times BU$ makes sense whenever $\mathrm{cls}_X\{u(t,\omega_0,x_0)\,:\,t\ge 0\}$ is a compact set, because then the set $\{u(t,\omega_0,x_0)(0)=z(t,\omega_0,x_0)\,:\,t\ge 0\}$ is obviously bounded. The approach used in this work is to study the bounded trajectories included within the set $C=\mathrm{cls}_{\Omega\times X}\{\tau(t,\omega_0,x_0):t\ge 0\}$ using the compact-open topology. Notice that (C,d) is a compact metric space, and $\tau:\mathbb{R}^+\times C\to C$ is a semiflow which is continuous for the metric and clearly may be considered of skew-product type, thanks to the form of C. Our aim is to study the dynamical properties of $(\tau,C,\mathbb{R}^+)$ and characterize its minimal subsets.

Now we give a technical result analogous to Proposition 4.2 in [NOS].

Lemma 4.4. *Let $\{(\omega_n,x_n)\}_n\subset\Omega\times B_r$ for some $r>0$ be such that $\omega_n\to\omega$ and $x_n\xrightarrow{\mathsf{d}}x$ for $(\omega,x)\in\Omega\times B_r$ as $n\to\infty$. If*

$$\sup\{\|z(s,\omega_n,x_n)\|\,:\,s\in[0,t],\ n\ge 1\}\le r$$

for some $t>0$, then $u(t,\omega_n,x_n)\xrightarrow{\mathsf{d}}u(t,\omega,x)$ as $n\to\infty$.

Proof. It is clear that $\{\widehat{D}_2(\omega_n,x_n)\}_n\subset\Omega\times B_{r'}$ with $r'=\sup_{\omega_1\in\Omega}\|D(\omega_1,\cdot)\|r$. Besides, from Theorem 3.1, it follows that $\widehat{D}(\omega_n,x_n)\to\widehat{D}(\omega,x)$ as $n\to\infty$ for the product metric topology. Now, for all $s\in[0,t]$ and all $n\in\mathbb{N}$,

$$\begin{aligned}\|\widehat{u}(s,\widehat{D}(\omega_n,x_n))(0)\|=&\|\widehat{D}_2(\tau(s,\omega_n,x_n))(0)\|\\ &\le\sup_{\omega_1\in\Omega}\|D(\omega_1,\cdot)\|\|u(s,\omega_n,x_n)\|_\infty\le r',\end{aligned}$$

and hence, from Proposition 4.2 in [NOS], $\widehat{u}(t,\widehat{D}(\omega_n,x_n))\xrightarrow{\mathsf{d}}\widehat{u}(t,\widehat{D}(\omega,x))$ as $n\to\infty$. Theorem 3.3 yields the expected result. $\square$

The restriction of τ to $\mathcal{O}(\omega_0, x_0)$ is continuous when the compact-open topology is considered on BU. The following proposition provides a more general result in this line, ensuring certain continuity of the semiflow when the compact-open topology is considered on BU. Its proof is analogous to that of Corollary 4.3 in [NOV].

Proposition 4.5. *Let $K \subset \Omega \times BU$ be a compact set for the product metric topology, and assume that there is an $r > 0$ such that $\tau_t(K) \subset \Omega \times B_r$ for all $t \geq 0$. Then the map*

$$\begin{array}{rcl} \tau \quad : \mathbb{R}^+ \times K & \longrightarrow & \Omega \times BU \\ (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x))\,, \end{array}$$

is continuous when the product metric topology is considered.

Our next goal is to transfer the dynamical structure of the skew-product semiflow $(\Omega \times BU, \widehat{\tau}, \mathbb{R}^+)$ to $(\Omega \times BU, \tau, \mathbb{R}^+)$.

The following result is a consequence of Proposition 4.4 in [NOS] and assures that $\mathcal{O}(\omega_0, x_0)$ admits a flow extension

Proposition 4.6. *Fix $(\omega_0, x_0) \in \Omega \times BU$ with $\sup_{t\geq 0} \|z(t, \omega_0, x_0)\| < \infty$. Then $\mathcal{O}(\omega_0, x_0)$ is a positively invariant compact subset admitting a flow extension.*

From hypothesis (F4), the monotone character of the semiflow (4.2) is deduced.

Proposition 4.7. *For all $\omega \in \Omega$ and $x, y \in BU$ such that $(\omega, x) \leq_D (\omega, y)$, it holds that*

$$\tau(t, \omega, x) \leq_D \tau(t, \omega, y)$$

whenever they are defined.

Proof. From $(\omega, x) \leq_D (\omega, y)$ we know that $\widehat{D}_2(\omega, x) \leq \widehat{D}_2(\omega, y)$, and, from Proposition 4.2 and from Proposition 4.5 in [NOS], it can be deduced that $\widehat{u}(t, \widehat{D}(\omega, x)) \leq \widehat{u}(t, \widehat{D}(\omega, y))$ whenever they are defined, that is,

$$\tau(t, \omega, x) = \widehat{D}^{-1}(\widehat{\tau}(t, \widehat{D}(\omega, x))) \leq_D \widehat{D}^{-1}(\widehat{\tau}(t, \widehat{D}(\omega, y))) = \tau(t, \omega, y)\,,$$

as stated. □

Let us define the concept of uniform stability of a forward orbit of the transformed skew-product semiflow $\widehat{\tau}$ with respect to a subset which does not need to be positively invariant.

Definition 4.8. Given $B \subset \Omega \times BU$, a forward orbit $\{\tau(t,\omega_0,x_0) : t \geq 0\}$ of the skew-product semiflow τ is said to be *uniformly stable in* B if, for every $\varepsilon > 0$, there is a $\delta > 0$, called the *modulus of uniform stability*, such that, if $s \geq 0$ and $\mathsf{d}(u(s,\omega_0,x_0),x) \leq \delta$ for certain $(\omega_0{\cdot}s, x) \in B$, then, for each $t \geq 0$,

$$\mathsf{d}(u(t+s,\omega_0,x_0),u(t,\omega_0{\cdot}s,x)) = \mathsf{d}(u(t,\omega_0{\cdot}s,u(s,\omega_0,x_0)),u(t,\omega_0{\cdot}s,x)) \leq \varepsilon.$$

We establish the 1-covering property of omega-limit sets when, in addition to hypotheses (F1)–(F4), uniform stability and the componentwise separating property for the semiflow $\widehat{\tau}$ are assumed:

(F5) there is an $r > 0$ such that all the trajectories with initial data in $\widehat{D}^{-1}(\Omega \times B_r)$ are uniformly stable in $\widehat{D}^{-1}(\Omega \times B_{r'})$ for each $r' > r$, and relatively compact for the product metric topology;

(F6) if $\omega \in \Omega$, $x, y \in BU$ with $(\omega,x) \leq_D (\omega,y)$ and $D_i(\omega,x) < D_i(\omega,y)$ holds for some $i \in \{1,\dots,m\}$, then $D_i(\tau(t,\omega,x)) < D_i(\tau(t,\omega,y))$ for all $t \geq 0$.

From these conditions, we deduce that the transformed skew-product semiflow (4.4) satisfies the properties analogous to (F5) and (F6).

Proposition 4.9. *The semiflow $\widehat{\tau}$ satisfies the following properties:*

(i) *there is an $r > 0$ such that all the trajectories with initial data in B_r are uniformly stable in $\Omega \times B_{r'}$ for each $r' > r$, and relatively compact for the product metric topology;*

(ii) *if we have $x, y \in BU$ such that $x \leq y$ and $x_i(0) < y_i(0)$ for some $i \in \{1,\dots,m\}$, then $\widehat{z}_i(t,\omega,x) < \widehat{z}_i(t,\omega,y)$ for all $t \geq 0$ and $\omega \in \Omega$.*

Finally, from Theorem 5.3 in [NOS] applied to the skew-product semiflow (4.4), we obtain the main result of this chapter, which establishes the 1-covering property of omega-limit sets for NFDEs with infinite delay.

Theorem 4.10. *Assume that hypotheses* (F1)–(F6) *are satisfied, and fix $(\omega_0,x_0) \in \widehat{D}^{-1}(\Omega \times B_r)$ such that $K = \mathcal{O}(\omega_0,x_0) \subset \widehat{D}^{-1}(\Omega \times B_r)$. Then $K = \{(\omega,c(\omega)) : \omega \in \Omega\}$ is a copy of the base, and*

$$\lim_{t\to\infty} \mathsf{d}(u(t,\omega_0,x_0),c(\omega_0{\cdot}t)) = 0\,,$$

where $c : \Omega \to BU$ is a continuous equilibrium, *i.e. $c(\omega{\cdot}t) = u(t,\omega,c(\omega))$ for all $\omega \in \Omega$, $t \geq 0$, and it is continuous for the compact-open topology on BU.*

Chapter 5

Applications of the transformed usual order

We consider compartmental models for the mathematical description of processes in which the transport of material between compartments takes a non-negligible length of time, and each compartment produces or swallows material. We provide a non-autonomous version, without strong monotonicity assumptions, of previous autonomous results in [WF] and [W].

As we pointed out in Section 1.6, we are going to study a family of recurrent compartmental systems driven by a minimal real flow $(\Omega, \sigma, \mathbb{R})$. First, we introduce the model with which we are going to deal as well as some notation. Let us suppose that we have a system formed by m compartments $C_1, \ldots, C_m$, denote by C_0 the environment surrounding the system, and by $z_i(t)$ the amount of material within compartment C_i at time t for each $i \in \{1, \ldots, m\}$. Material flows from compartment C_j into compartment C_i through a pipe P_{ij} having a transit time distribution given by a positive regular Borel measure μ_{ij} with finite total variation $\mu_{ij}((-\infty, 0]) = 1$, for each i, $j \in \{1, \ldots, m\}$. Let $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ be the so-called *transport function* determining the volume of material flowing from C_j to C_i given in terms of the time t and the value of $z_j(t)$ for $i \in \{0, \ldots, m\}, j \in \{1, \ldots, m\}$. For each $i \in \{1, \ldots, m\}$, we will assume that there exists an incoming flow of material I_i from the environment into compartment C_i which does not depend on z, and an outgoing flow of material toward the environment given by $g_{0i} : \Omega \times \mathbb{R} \to \mathbb{R}$ in terms of the time t and the value of $z_i(t)$. For each $i \in \{1, \ldots, m\}$, at time $t \geq 0$, the compartment C_i produces material itself at a rate $\sum_{j=1}^{m} \int_{-\infty}^{0} z_j'(t+s)\, d\nu_{ij}(\omega{\cdot}t)(s)$, where $\nu_{ij}(\omega)$ is a positive regular Borel measure with finite total variation and $\nu_{ij}(\omega)(\{0\}) = 0$, for all i, $j \in \{1, \ldots, m\}$ and all $\omega \in \Omega$.

Once the destruction and creation of material are taken into account, the change of the amount of material of any compartment C_i, $1 \leq i \leq m$, equals the difference between the amount of total inflow into and total outflow out of C_i, and we obtain a model governed by the following family of infinite delay NFDEs:

$$\begin{aligned}\frac{d}{dt}\left[z_i(t) - \sum_{j=1}^{m}\int_{-\infty}^{0} z_j(t+s)\, d\nu_{ij}(\omega{\cdot}t)(s)\right] = \\ = -\, g_{0i}(\omega{\cdot}t, z_i(t)) - \sum_{j=1}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) \qquad (5.1)_\omega \\ + \sum_{j=1}^{m}\int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), z_j(t+s))\, d\mu_{ij}(s) + I_i(\omega{\cdot}t),\end{aligned}$$

for $i \in \{1,\dots,m\}$. For the sake of simplicity, for each $i \in \{1,\dots,m\}$, we denote $g_{i0} : \Omega\times\mathbb{R} \to \mathbb{R}^+,\ (\omega,v) \mapsto I_i(\omega)$; let $g = (g_{ij})_{i,j} : \Omega\times\mathbb{R} \to \mathbb{R}^{m(m+2)}$.

In this chapter, we apply Theorem 4.10, and, assuming that there exists a bounded solution, we check that the properties (F5) and (F6) presented in Chapter 4 hold for this setting, so that we are able to describe the structure of minimal sets when the compact-open topology is considered on BU.

We will assume hypotheses (L1) and (L2) with $B(\omega) = I$ for all $\omega \in \Omega$ together with the following ones:

(C1) g_{ij} is C^1 and non-decreasing in its second variable; moreover, for all $\omega \in \Omega$, all $i \in \{0,\dots,m\}$, and all $j \in \{1,\dots,m\}$, $g_{ij}(\omega,0) = 0$;

(C2) $\mu_{ij}((-\infty,0]) = 1$ and $\int_{-\infty}^0 |s|\, d\mu_{ij}(s) < \infty$;

(C3) for each $\omega \in \Omega$, the measures $\eta_{ij}(\omega) = l_{ij}^-\,\mu_{ij} - \sum_{k=0}^m l_{ki}^+\,\nu_{ij}(\omega)$ are positive, where

$$l_{ij}^- = \inf_{(\omega,v)\in\Omega\times\mathbb{R}} \frac{\partial g_{ij}}{\partial v}(\omega,v) \quad \text{and} \quad l_{ij}^+ = \sup_{(\omega,v)\in\Omega\times\mathbb{R}} \frac{\partial g_{ij}}{\partial v}(\omega,v)\,.$$

In some practical cases, the solutions with physical interest belong to the positive cone, and the functions g_{ij} are only defined on $\Omega\times\mathbb{R}^+$; we can extend them to $\Omega\times\mathbb{R}$ by $g_{ij}(\omega,-v) = -g_{ij}(\omega,v)$ for all $\omega \in \Omega$ and all $v \in \mathbb{R}^+$. Note that (C3) is a condition for controlling the material produced in the compartments in terms of the material transported through the pipes.

The above formulation includes some particular interesting cases. When the measures $\nu_{ij}(\omega)$, $\omega \in \Omega$, and μ_{ij} are concentrated on a compact set, then $(5.1)_\omega$ is a NFDE with finite delay. When the measures $\nu_{ij}(\omega) \equiv 0$, then $(5.1)_\omega$ is a family of FDE with finite or infinite delay.

Let $F:\Omega\times BU\to\mathbb{R}^m$ be the map defined by

$$F_i(\omega,x)=-\sum_{j=0}^{m} g_{ji}(\omega,x_i(0))+\sum_{j=1}^{m}\int_{-\infty}^{0} g_{ij}(\omega{\cdot}s,x_j(s))\,d\mu_{ij}(s)+I_i(\omega),$$

for $(\omega,x)\in\Omega\times BU$ and $i\in\{1,\dots,m\}$. Hence, the family of equations $(5.1)_\omega$ can be written as

$$\frac{d}{dt}D(\omega{\cdot}t,z_t)=F(\omega{\cdot}t,z_t)\,,\quad t\geq 0\,,\ \omega\in\Omega\,, \tag*{$(5.2)_\omega$}$$

where the linear operator D is defined as in expression (3.4) with $B(\omega)=I$ for all $\omega\in\Omega$. Conditions (L1) and (L2) allow us to state a result concerning the properties of D.

Proposition 5.1. *The operator D satisfies* (D1)–(D3) *and is stable. Moreover, if $\widehat{D}$ is its associated convolution operator, defined as in* (3.1)*, then $\widehat{D}^{-1}$ is positive, that is, if $(\omega,x)\in\Omega\times BU$ with $x\geq 0$, then $\widehat{D}_2^{-1}(\omega,x)\geq 0$.*

Proof. Properties (D1)–(D3) follow from (L1), (L2), Proposition 3.6, and Theorem 3.7. As for the fact that $\widehat{D}^{-1}$ is positive, it is a consequence of Theorem 3.7 as well. □

The following result is an immediate consequence of conditions (C1) and (C2).

Proposition 5.2. *The mapping F satisfies* (F1)–(F3).

The following lemma will be useful when proving (F4) and (F6). Its proof is in the line of Proposition 5.1 in [WF] for the autonomous case with finite delay.

Lemma 5.3. *For all $\omega\in\Omega$, all x, $y\in BU$ with $(\omega,x)\leq_D(\omega,y)$ and all $i\in\{1,\dots,m\}$,*

$$\begin{aligned}F_i(\omega,y)-F_i(\omega,x)\geq{}&-\sum_{j=0}^{m} l_{ji}^{+}\,[D_i(\omega,y)-D_i(\omega,x)]\\&+\sum_{j=1}^{m}\int_{-\infty}^{0}(y_j(s)-x_j(s))\,d\eta_{ij}(\omega)(s)\end{aligned}\tag{5.3}$$

where the measures $\eta_{ij}(\omega)$, $\omega\in\Omega$, are defined in (C3).

Proof. Let $\omega \in \Omega$, x, $y \in BU$ with $(\omega, x) \leq_D (\omega, y)$ and $i \in \{1, \ldots, m\}$. From Proposition 5.1, it is clear that $x(s) \leq y(s)$ for all $s \leq 0$, whence

$$
\begin{aligned}
F_i(\omega, y) - F_i(\omega, x) &= -\sum_{j=0}^{m} (g_{ji}(\omega, y_i(0)) - g_{ji}(\omega, x_i(0))) \\
&\quad + \sum_{j=1}^{m} \int_{-\infty}^{0} (g_{ij}(\omega{\cdot}s, y_j(s)) - g_{ij}(\omega{\cdot}s, x_j(s)))\, d\mu_{ij}(s) \\
&\geq -\sum_{j=0}^{m} l_{ji}^{+}(y_i(0) - x_i(0)) + \sum_{j=1}^{m} l_{ij}^{-} \int_{-\infty}^{0} (y_j(s) - x_j(s)) d\mu_{ij}(s) \\
&= -\sum_{j=0}^{m} l_{ji}^{+}[D_i(\omega, y) - D_i(\omega, x)] + \sum_{j=1}^{m} l_{ij}^{-} \int_{-\infty}^{0} (y_j(s) - x_j(s))\, d\mu_{ij}(s) \\
&\quad - \sum_{j=0}^{m} l_{ji}^{+} \int_{-\infty}^{0} \sum_{k=1}^{m} (y_k(s) - x_k(s)) d\nu_{ik}(\omega)(s) \\
&\geq -\sum_{j=0}^{m} l_{ji}^{+}\, [D_i(\omega, y) - D_i(\omega, x)] \\
&\quad + \sum_{j=1}^{m} \int_{-\infty}^{0} (y_j(s) - x_j(s))\, d \left[l_{ij}^{-}\, \mu_{ij} - \sum_{k=0}^{m} l_{ki}^{+}\, \nu_{ij}(\omega) \right] (s),
\end{aligned}
$$

as wanted. □

Condition (C3) is essential to prove the monotone character of the semiflow. It can be improved in certain cases (see [AB] for the scalar one), some of which will be explained below.

Proposition 5.4. *Under assumptions* (L1), (L2), *and* (C1)–(C3), *the family* $(5.2)_\omega$ *satisfies hypotheses* (F4), (F6), *and* $\mathcal{P}_D$ *is positively invariant.*

Proof. Let (ω, x), $(\omega, y) \in \Omega \times BU$ with $(\omega, x) \leq_D (\omega, y)$ and such that $D_i(\omega, x) = D_i(\omega, y)$ for some $i \in \{1, \ldots, m\}$. We saw in Proposition 5.1 that $\widehat{D}^{-1}$ is positive. Hence, from $(\omega, x) \leq_D (\omega, y)$, that is, $\widehat{D}(\omega, x) \leq \widehat{D}(\omega, y)$, we also deduce that $x \leq y$, which, together with $D_i(\omega, x) = D_i(\omega, y)$, relation (5.3), and hypothesis (C3), yields $F_i(\omega, y) \geq F_i(\omega, x)$, that is, hypothesis (F4) holds.

Next, we check hypothesis (F6). Let (ω, x), $(\omega, y) \in \Omega \times BU$ be such that $(\omega, x) \leq_D (\omega, y)$ and $D_i(\omega, x) < D_i(\omega, y)$ for some $i \in \{1, \ldots, m\}$. Since (F4) holds, from Proposition 4.7, $\tau(t, \omega, x) \leq_D \tau(t, \omega, y)$, and, as before, we deduce in this case that $u(t,\omega,x)\le u(t,\omega,y)$ for all $t\ge 0$ and all $\omega\in\Omega$. Let $h(t)=D_i(\tau(t,\omega,y))-D_i(\tau(t,\omega,x))$. From equation $(5.2)_\omega$ and Lemma 5.3,

$$\begin{aligned} h'(t)=&F_i(\omega{\cdot}t,u(t,\omega,y))-F_i(\omega{\cdot}t,u(t,\omega,x))\\ &\ge -\sum_{j=0}^m l_{ji}^+h(t)+\sum_{j=1}^m\int_{-\infty}^0 (z_j(t+s,\omega,y)-z_j(t+s,\omega,x))\,d\eta_{ij}(\omega{\cdot}t)(s)\,, \end{aligned}$$

and, again from hypothesis (C3), it is easy to deduce that $h'(t)\ge -dh(t)$ for some $d\ge 0$, which, together with the fact that $h(0)>0$, yields

$$h(t)=D_i(\tau(t,\omega,y))-D_i(\tau(t,\omega,x))>0$$

for each $t\ge 0$, and (F6) holds. Finally, since $I_i(\omega)\ge 0$ for each $\omega\in\Omega$ and $i\in\{1,\dots,m\}$, and the semiflow is monotone, a comparison argument shows that $\mathcal{P}_D$ is positively invariant, as stated. □

Next we will study some cases in which hypothesis (F5) is satisfied. In order to do this, we define $M\colon\Omega\times BU\to\mathbb{R}$, the *total mass* of the system $(5.2)_\omega$ as

$$M(\omega,x)=\sum_{i=1}^m D_i(\omega,x)+\sum_{i=1}^m\sum_{j=1}^m\int_{-\infty}^0\left(\int_s^0 g_{ji}(\omega{\cdot}\tau,x_i(\tau))\,d\tau\right)d\mu_{ji}(s)\,,\quad (5.4)$$

for all $\omega\in\Omega$ and $x\in BU$. M is well defined due to (C2) and because, if $i,j\in\{1,\dots,m\}$ and $(\omega,x)\in\Omega\times BU$, then

$$\left|\int_s^0 g_{ji}(\omega{\cdot}\tau,x_i(\tau))d\tau\right|\le c_1|s|,$$

where c_1 is a bound of g_{ji} on $\Omega\times[-\|x\|_\infty,\|x\|_\infty]$.

The next proposition is in the line of some results found in [MNO], [NOV], and [WF] and shows some continuity properties of M and its variation along the flow.

Proposition 5.5. *The total mass M is a uniformly continuous function on all the sets of the form $\Omega\times B_r$ with $r>0$ for the product metric topology. Moreover, for each $(\omega,x)\in\Omega\times BU$ and each $t\ge 0$,*

$$\frac{d}{dt}M(\tau(t,\omega,x))=\sum_{i=1}^m\left[I_i(\omega{\cdot}t)-g_{0i}(\omega{\cdot}t,z_i(t,\omega,x))\right]\,.\qquad (5.5)$$

Proof. The uniform continuity of M follows from (D2), (C1), and (C2), as we check now. Let $r>0$ and $\varepsilon>0$. From (D2), there exists $\delta>0$ such that, if (ω^1,x^1), $(\omega^2,x^2)\in\Omega\times BU$, $d(\omega^1,\omega^2)<\delta$ and $\mathsf{d}(x^1,x^2)<\delta$, then

$$|D_i(\omega^1,x^1)-D_i(\omega^2,x^2)|<\frac{\varepsilon}{3\,m}$$

for $i\in\{1,\dots,m\}$. Besides, g is continuous on $\Omega\times[-r,r]$, so it is bounded on such set by some $c_1>0$. From (C2), there is a $k_0>0$ such that

$$\int_{-\infty}^{-k_0}|s|\,d\mu_{ij}(s)<\frac{\varepsilon}{6\,c_1\,m^2}$$

for $i,j\in\{1,\dots,m\}$. Since g is uniformly continuous on $\Omega\times[-r,r]$, and $[-\tau,0]\times\Omega\to\Omega$, $(t,\omega)\mapsto\omega{\cdot}t$ is uniformly continuous too, there exists $\delta_1\in(0,\delta)$ such that, if (ω^1,x^1), $(\omega^2,x^2)\in\Omega\times BU$, $d(\omega^1,\omega^2)<\delta_1$ and $\mathsf{d}(x^1,x^2)<\delta_1$, then, given $\tau\in[-k_0,0]$,

$$\left(\int_{-\infty}^{0}|s|\,d\mu_{ij}(s)\right)|g_{ij}(\omega^1{\cdot}\tau,x_i^1(\tau))-g_{ij}(\omega^2{\cdot}\tau,x_i^2(\tau))|\le\frac{\varepsilon}{3\,m^2}$$

for all $i,j\in\{1,\dots,m\}$. Therefore, if (ω^1,x^1), $(\omega^2,x^2)\in\Omega\times BU$ with $d(\omega^1,\omega^2)<\delta_1$ and $\mathsf{d}(x^1,x^2)<\delta_1$, then

$$\begin{aligned}
|M(\omega^1,x^1)-M(\omega^2,x^2)|\le{}&\sum_{i=1}^{m}|D_i(\omega^1,x^1)-D_i(\omega^2,x^2)|\\
&+\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{0}\left(\int_{s}^{0}|g_{ji}(\omega^1{\cdot}\tau,x_i^1(\tau))-g_{ji}(\omega^2{\cdot}\tau,x_i^2(\tau))|\,d\tau\right)d\mu_{ji}(s)\\
\le{}&\frac{\varepsilon}{3}+\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{-k_0}\left(\int_{s}^{0}2\,c_1\,d\tau\right)d\mu_{ji}(s)\\
&+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\varepsilon}{3\,m^2\int_{-\infty}^{0}|s|\,d\mu_{ij}(s)}\int_{-k_0}^{0}|s|\,d\mu_{ji}(s)\\
\le{}&\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}$$

Thus, M is uniformly continuous on $\Omega\times B_r$ when we consider the compact-open topology on B_r.

As for relation (5.5), given $(\omega,x)\in\Omega\times BU$ and $t\ge0$ where $z(\cdot,\omega,x)$ is defined, we add all the components of $(5.1)_\omega$ and integrate the resulting sum between 0 and t, obtaining

$$\begin{aligned}
&\sum_{i=1}^{m} D_i(\omega{\cdot}t, u(t,\omega,x)) - \sum_{i=1}^{m} D_i(\omega,x) = \\
&\quad = -\sum_{i=1}^{m}\int_0^t g_{0i}(\omega{\cdot}\tau, z_i(\tau,\omega,x))\,d\tau - \sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^t g_{ji}(\omega{\cdot}\tau, z_i(\tau,\omega,x))\,d\tau \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^t \left(\int_{-\infty}^0 g_{ij}(\omega{\cdot}(\tau+s), z_j(\tau+s,\omega,x))\,d\mu_{ij}(s)\right)\,d\tau \\
&\qquad + \sum_{i=1}^{m}\int_0^t I_i(\omega{\cdot}\tau)\,d\tau.
\end{aligned}$$

Now, Fubini's theorem and a simple change of variables yield

$$\begin{aligned}
&\int_0^t \left(\int_{-\infty}^0 g_{ij}(\omega{\cdot}(\tau+s), z_j(\tau+s,\omega,x))\,d\mu_{ij}(s)\right)\,d\tau = \\
&\qquad = \int_{-\infty}^0 \left(\int_s^{t+s} g_{ij}(\omega{\cdot}u, z_j(u,\omega,x))\,du\right)\,d\mu_{ij}(s)
\end{aligned}$$

for all $i,\ j \in \{1,\dots,m\}$. Thanks to (C2), for all $i,\ j \in \{1,\dots,m\}$,

$$\int_0^t g_{ji}(\omega{\cdot}\tau, z_i(\tau,\omega,x))\,d\tau = \int_{-\infty}^0 \left(\int_0^t g_{ji}(\omega{\cdot}\tau, z_i(\tau,\omega,x))\,d\tau\right)\,d\mu_{ji}(s).$$

Consequently,

$$\begin{aligned}
&\sum_{i=1}^{m} D_i(\omega{\cdot}t, u(t,\omega,x)) - \sum_{i=1}^{m} D_i(\omega,x) = \\
&\quad = -\sum_{i=1}^{m}\int_0^t g_{0i}(\omega{\cdot}\tau, z_i(\tau,\omega,x))\,d\tau + \sum_{i=1}^{m}\int_0^t I_i(\omega{\cdot}\tau)\,d\tau \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{m}\left[\int_{-\infty}^0 \left(\int_s^{t+s} g_{ij}(\omega{\cdot}\tau, z_j(\tau,\omega,x))\,d\tau\right)\,d\mu_{ij}(s)\right. \\
&\qquad\qquad \left. - \int_{-\infty}^0 \left(\int_0^t g_{ji}(\omega{\cdot}\tau, z_i(\tau,\omega,x))\,d\tau\right)\,d\mu_{ji}(s)\right],
\end{aligned}$$

whence it follows that

$$\begin{aligned}
&\sum_{i=1}^{m} D_i(\omega{\cdot}t, u(t,\omega,x)) + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{0}\left(\int_{t+s}^{t} g_{ij}(\omega{\cdot}\tau, z_j(\tau,\omega,x))\,d\tau\right) d\mu_{ij}(s) = \\
&\qquad = \sum_{i=1}^{m} D_i(\omega,x) + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{0}\left(\int_{s}^{0} g_{ij}(\omega{\cdot}\tau, z_j(\tau,\omega,x))\,d\tau\right) d\mu_{ij}(s) \\
&\qquad\quad + \sum_{i=1}^{m}\int_0^t I_i(\omega{\cdot}s)\,ds - \sum_{i=1}^{m}\int_0^t g_{0i}(\omega{\cdot}s, z_i(s,\omega,x))\,ds.
\end{aligned}$$

As a result, we obtain

$$M(\omega{\cdot}t, u(t,\omega,x)) = M(\omega,x) + \sum_{i=1}^{m}\int_0^t \left[I_i(\omega{\cdot}s) - g_{0i}(\omega{\cdot}s, z_i(s,\omega,x))\right] ds\,, \quad (5.6)$$

from which (5.5) is deduced. □

The following lemma is essential in the proof of the stability of solutions.

Lemma 5.6. *Let* (ω,x), $(\omega,y)\in\Omega\times BU$ *with* $(\omega,x)\le_D(\omega,y)$. *Then*

$$0 \le D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x)) \le M(\omega,y) - M(\omega,x)$$

for each $i=1,\dots,m$ *and whenever* $z(t,\omega,x)$ *and* $z(t,\omega,y)$ *are defined.*

Proof. From Propositions 5.4 and 4.7, the skew-product semiflow induced by $(5.2)_\omega$ is monotone. Hence, if $(\omega,x)\le_D(\omega,y)$, then $\tau(t,\omega,x)\le_D\tau(t,\omega,y)$ whenever they are defined. From this, as before, since $\widehat{D}^{-1}(\omega,\cdot)$ is positive for all $\omega\in\Omega$, we also deduce that $x\le y$ and $u(t,\omega,x)\le u(t,\omega,y)$. Therefore, $D_i(\tau(t,\omega,x))\le D_i(\tau(t,\omega,y))$ and $z_i(t,\omega,x)\le z_i(t,\omega,y)$ for each $i=1,\dots,m$. In addition, the monotonicity of transport functions yields $g_{ij}(\omega,z_j(t,\omega,x))\le g_{ij}(\omega,z_j(t,\omega,y))$ for each $\omega\in\Omega$. From all these inequalities, (5.4), and (5.6), we deduce that

$$\begin{aligned}
0 \le& D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x)) \le \sum_{i=1}^{m}\left[D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x))\right] \\
\le& M(\omega{\cdot}t, u(t,\omega,y)) - M(\omega{\cdot}t, u(t,\omega,x)) \le M(\omega,y) - M(\omega,x)\,,
\end{aligned}$$

as stated. □

Next, we give a result concerning the stability of the solutions of the family of equations $(5.1)_\omega$. This result is essential in what follows.

Proposition 5.7. *Fix $r > 0$. Then given $\varepsilon > 0$ there exists $\delta = \delta(r) > 0$ such that if (ω, x), $(\omega, y) \in \Omega \times B_r$ with $\mathsf{d}(x, y) < \delta$ then $\|z(t, \omega, x) - z(t, \omega, y)\| \leq \varepsilon$ whenever they are defined.*

Proof. From (L1) and (L2), it is clear that

$$\begin{aligned}\xi &= \sup \left\{ \sum_{j=1}^{m} \nu_{ij}(\omega)((-\infty, 0]) : \omega \in \Omega,\ i \in \{1, \dots, m\} \right\} \\ &= \sup_{\omega \in \Omega} \|\nu(\omega)\|((-\infty, 0]) < 1.\end{aligned}$$

From the uniform continuity of M on $\Omega \times B_r$, given $\varepsilon_0 = \varepsilon\,(1 - \xi) > 0$ there exists $0 < \delta < \varepsilon_0$, such that if (ω, x), $(\omega, y) \in \Omega \times B_r$ with $\mathsf{d}(x, y) < \delta$ then $|M(\omega, y) - M(\omega, x)| < \varepsilon_0$. Therefore, if (ω, x), $(\omega, y) \in \Omega \times B_r$ are such that $(\omega, x) \leq_D (\omega, y)$, from Lemma 5.6 we deduce that

$$0 \leq D_i(\tau(t, \omega, y)) - D_i(\tau(t, \omega, x)) < \varepsilon_0$$

whenever $\mathsf{d}(x, y) < \delta$. The definition of D_i yields

$$\begin{aligned}0 &\leq z_i(t, \omega, y) - z_i(t, \omega, x) \\ &< \varepsilon_0 + \sum_{j=1}^{m} \int_{-\infty}^{0} [z_j(t + s, \omega, y) - z_j(t + s, \omega, x)]\, d\nu_{ij}(\omega{\cdot}t)(s) \\ &\leq \varepsilon_0 + \|u(t, \omega, y) - u(t, \omega, x)\|_\infty \sup_{\omega_1 \in \Omega} \sum_{j=1}^{m} \nu_{ij}(\omega_1)((-\infty, 0])\,,\end{aligned}$$

from which we deduce that $\|u(t, \omega, y) - u(t, \omega, x)\|_\infty (1 - \xi) < \varepsilon_0 = \varepsilon(1 - \xi)$, that is, $\|z(t, \omega, x) - z(t, \omega, y)\| \leq \varepsilon$ whenever they are defined. The case in which x and y are not ordered follows easily from this one. □

As a consequence, from the existence of a bounded solution for one of the systems of the family, the boundedness of all solutions is inferred, and this is the case in which hypothesis (F5) holds.

In the following theorem, we study the structure of omega-limit sets when the compact-open topology is considered on BU.

Theorem 5.8. *If there exists $\omega_0 \in \Omega$ such that $(5.2)_{\omega_0}$ has a bounded solution, then all the solutions of $(5.2)_\omega$ are bounded as well, hypothesis (F5) holds, and all omega-limit sets are copies of the base when we consider the compact-open topology on BU.*

Proof. In order to prove the boundedness of all the solutions, let $x_0 \in BU$ be such that $z(\cdot, \omega_0, x_0)$ is bounded, and consider $K = \mathcal{O}(\omega_0, x_0)$. Let $r_1 > 0$ be such that $K \subset \Omega \times B_{r_1}$. Fix $(\omega, x) \in \Omega \times BU$ and let $y_s = (1-s)x_0 + s\,x$ for each $s \in [0,1]$; evidently, $y_s \le y_t$ for all $0 \le s \le t \le 1$, and there exists $r > 0$ such that $\{y_s\}_{s\in[0,1]} \subset B_r$. An application of Proposition 5.7 for $\varepsilon = 1$ implies that there are a $\delta = \delta(r) > 0$ and a partition $0 = s_0 \le s_1 \le \cdots \le s_n = 1$ of $[0,1]$ such that $\mathsf{d}(y_{s_j}, y_{s_{j+1}}) < \delta$ for all $j \in \{1, \ldots, n-1\}$, and therefore $\|z(t, \omega, y_{s_j}) - z(t, \omega, y_{s_{j+1}})\| \le 1$ for all $t \ge 0$ wherever they are defined. As a result, for each $j \in \{0, \ldots, n\}$, the solution $z(\cdot, \omega, y_{s_j})$ is globally defined, and $\|z(t, \omega, x) - z(t, \omega, x_0)\| \le n$ for all $t \ge 0$, which implies that $z(\cdot, \omega, x)$ is bounded, as desired.

As for (F5), let $(\omega, x) \in \Omega \times BU$ and $r' > 0$ such that $u(t, \omega, x) \in B_{r'}$ for all $t \ge 0$. Then also from Proposition 5.7, we deduce that given $\varepsilon > 0$ there exists $\delta = \delta(r') > 0$ such that

$$\|z(t+s, \omega, x) - z(t, \omega{\cdot}s, y)\| = \|z(t, \omega{\cdot}s, z_s(\omega, x)) - z(t, \omega{\cdot}s, y)\| < \varepsilon$$

for all $t \ge 0$ whenever $y \in B_{r'}$ and $\mathsf{d}(u(s, \omega, x), y) < \delta$, which shows the uniform stability of the trajectories in $\Omega \times B_{r'}$ for each $r' > 0$. Moreover, for each $r > 0$ there is an $r' > 0$ such that $\widehat{D}^{-1}(\Omega \times B_r) \subset \Omega \times B_{r'}$. Hence, hypothesis (F5) holds for all $r > 0$, and Theorem 4.10 applies for all initial data, which finishes the proof. □

Concerning the solutions of the original compartmental system, we obtain the following result providing a non-trivial generalization of the autonomous case, in which the asymptotically constant character of the solutions was shown (see [WF]). Although the theorem is stated in the almost periodic case, similar conclusions are obtained changing almost periodicity for periodicity, almost automorphy or recurrence, that is, all solutions are asymptotically of the same type as the transport functions.

Theorem 5.9. *In the almost periodic case, if there is a bounded solution of* $(5.1)_\omega$*, then there is at least an almost periodic solution, and all the solutions are asymptotically almost periodic. For closed systems, i.e. systems with* $I_i \equiv 0$ *and* $g_{0i} \equiv 0$ *for each* $i = 1, \ldots, m$*, there are infinitely many almost periodic solutions, and the rest of them are asymptotically almost periodic.*

Proof. The first statement is an easy consequence of the previous theorem. The omega-limit of each solution $z(\cdot, \omega_0, x_0)$ is a copy of the base,

$$\mathcal{O}(\omega_0, x_0) = \{(\omega, x(\omega)) : \omega \in \Omega\},$$

whence $t \mapsto z(t, \omega_0, x(\omega_0)) = x(\omega_0{\cdot}t)(0)$ is an almost periodic solution of $(5.1)_\omega$, because the mapping $\Omega \to \mathbb{R}^m$, $\omega \mapsto x(\omega)(0)$ is continuous, and

$$\lim_{t\to\infty} \|z(t, \omega_0, x_0) - z(t, \omega_0, x(\omega_0))\| = 0\,.$$

The statement for closed systems follows in addition from (5.5), which implies that the mass is constant along the trajectories. Hence, there are infinitely many minimal subsets because, from the definition of the mass and (C4), given $c > 0$ there is an $(\omega_0, x_0) \in \mathcal{P}_D$ such that $M(\omega_0, x_0) = c$, and hence $M(\omega, x) = c$ for each $(\omega, x) \in \mathcal{O}(\omega_0, x_0)$. $\square$

In order to illustrate the foregoing results, let us focus on the study of a family of compartmental systems with finite delay and diagonal D-operator; specifically, we consider

$$\begin{aligned} \frac{d}{dt}[z_i(t) - c_i(\omega{\cdot}t) z_i(t-\alpha_i)] = &- \sum_{j=1}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) \\ &+ \sum_{j=1}^{m} g_{ij}(\omega{\cdot}(t-\rho_{ij}), z_j(t-\rho_{ij})) \end{aligned} \tag*{$(5.7)_\omega$}$$

for $i \in \{1, \ldots, m\}$, $t \geq 0$ and $\omega \in \Omega$, where $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $c_i : \Omega \to \mathbb{R}$ are continuous functions, and α_i, $\rho_{ij} \in \mathbb{R}$ for i, $j \in \{1, \ldots, m\}$. Let us assume the following conditions on equation $(5.7)_\omega$:

(G1) g_{ij} is C^1 and non-decreasing in its second variable; moreover, for all $\omega \in \Omega$, all $i \in \{0, \ldots, m\}$ and all $j \in \{1, \ldots, m\}$, $g_{ij}(\omega, 0) = 0$;

(G2) $\alpha_i > 0$, $\rho_{ij} \geq 0$, and $0 \leq c_i(\omega) < 1$ for all $i, j \in \{1, \ldots, m\}$ and all $\omega \in \Omega$.

Notice that (G1) coincides with (C1). Besides, the family of equations $(5.7)_\omega$ corresponds to a closed compartmental system, that is, a system where there is no incoming material from the environment, and there is no outgoing material toward the environment either. As a result, the total mass of the system is invariant along the trajectories, and 0 is a constant bounded solution of all the equations of the family.

The following result is a straightforward consequence of hypotheses (G1) and (G2).

Proposition 5.10. *Under hypotheses* (G1) *and* (G2)*, the family of equations* $(5.7)_\omega$ *satisfies conditions* (C1), (C2), (L1)*, and* (L2).

As pointed out before, the scalar version of the family of NFDEs $(5.7)_\omega$ with autonomous linear operator and $\rho = \alpha$ was studied in [AB] using the standard ordering. In particular, in this paper it was established that all the minimal sets are copies of the base. Our next result obtains the same conclusion when the linear operator is non-autonomous and condition (G3) holds, by using the transformed usual ordering. It is noteworthy that the same result can be obtained by means of a totally different approach, as seen in Proposition 8.7.

For each $\omega \in \Omega$ and each i, $j \in \{1, \ldots, m\}$, let

$$\begin{gathered} l_{ij}^{-}(\omega) = \inf_{v\in\mathbb{R}} \frac{\partial g_{ij}}{\partial v}(\omega, v), \quad l_{ij}^{+}(\omega) = \sup_{v\in\mathbb{R}} \frac{\partial g_{ij}}{\partial v}(\omega, v) \\ L_i^{+}(\omega) = \sum_{j=1}^{m} l_{ji}^{+}(\omega). \end{gathered} \tag{5.8}$$

In this chapter, we will assume that $L_i^+(\omega) < \infty$ for all $\omega \in \Omega$.

Theorem 5.11. *Assume conditions* (G1) *and* (G2). *Consider the family* $(5.7)_\omega$, *and assume that, for all* $i \in \{1, \ldots, m\}$, *if* $c_i \not\equiv 0$, *then* $\alpha_i = \rho_{ii}$ *and the following assertion holds for all* $\omega \in \Omega$*:*

(G3) $l_{ii}^{-}(\omega{\cdot}(-\rho_{ii})) - L_i^{+}(\omega) c_i(\omega) \geq 0.$

Then, for all $(\omega, x) \in \Omega \times BU$, *the trajectory* $\{\tau(t, \omega, x) : t \geq 0\}$ *is bounded, and its omega-limit set is a copy of the base.*

Proof. First, it is important to notice that, for all $i \in \{1, \ldots, m\}$ and all (ω, x), $(\omega, y) \in \Omega \times BU$ with $(\omega, x) \leq_D (\omega, y)$, we have

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) &= -\sum_{j=1}^{m} (g_{ji}(\omega, y_i(0)) - g_{ji}(\omega, x_i(0))) \\ &\quad + \sum_{j=1}^{m} (g_{ij}(\omega{\cdot}(-\rho_{ij}), y_j(-\rho_{ij})) - g_{ij}(\omega{\cdot}(-\rho_{ij}), x_j(-\rho_{ij}))) \\ &\geq -\sum_{j=1}^{m} l_{ji}^{+}(\omega)(y_i(0) - x_i(0)) + \sum_{j=1}^{m} l_{ij}^{-}(\omega{\cdot}(-\rho_{ij}))(y_j(-\rho_{ij}) - x_j(-\rho_{ij})) \\ &\geq -\sum_{j=1}^{m} l_{ji}^{+}(\omega)(D_i(\omega, y) - D_i(\omega, x)) - L_i^{+}(\omega)\, c_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i)) \\ &\quad + l_{ii}^{+}(\omega{\cdot}(-\rho_{ii}))(y_i(-\rho_{ii}) - x_i(-\rho_{ii})) \end{aligned}$$

thanks to (G1) and the fact that Proposition 5.1 clearly holds here. Hence, since 0 is a solution of the system, and we have (G3), it is easy to check that Proposition 5.4 holds as well. This way, Theorem 5.8 is satisfied, as desired. $\square$

Chapter 6

Compartmental geometry

This chapter deals with the long-term behavior of the amount of material within the compartments of the compartmental system $(5.1)_\omega$ satisfying hypotheses (L1), (L2), and (C1)–(C3). As we will see next, such behavior is a consequence of the geometry of the pipes connecting the compartments, that is, the way in which compartments are connected by means of pipes. We are able to find some subsystems irrespective of the rest of the system, in the sense that they are in turn compartmental systems from which there is no outflow of material to the rest of the system, and to which there may be some inflow of material from the rest of the system acting as though it were some inflow from the environment of this subsystem. These subsystems determine the geometry of the whole system completely.

As in the previous chapter, the study of the minimal sets for the corresponding skew-product semiflow (4.2) induced by the family $(5.2)_\omega$ will be essential. In addition to the aforementioned hypotheses, we will assume the following one:

(C4) given $i \in \{0, \dots, m\}$ and $j \in \{1, \dots, m\}$ either $g_{ij} \equiv 0$ on $\Omega \times \mathbb{R}^+$, i.e. *there is not a pipe from compartment C_j to compartment C_i*, or $g_{ij}(\omega, v) > 0$ for all $\omega \in \Omega$ and all $v > 0$. In this case we will say that the pipe P_{ij} *carries material (or that there is a pipe from compartment C_j to compartment C_i).*

Let $I = \{1, \dots, m\}$. $\mathcal{P}(I)$ denotes, as usual, the set of all the subsets of I.

Definition 6.1. Consider the mapping

$$\begin{array}{rccl} \zeta: & \mathcal{P}(I) & \longrightarrow & \mathcal{P}(I) \\ & J & \mapsto & \displaystyle\bigcup_{j \in J} \{i \in I : P_{ij} \text{ carries material}\}. \end{array}$$

A subset J of I is said to be *irreducible* if $\zeta(J) \subset J$ and no proper subset of J has that property. The system $(5.1)_\omega$ is *irreducible* if the whole set I is irreducible.

Note that $\zeta(I) \subset I$, so there is always some irreducible subset of I. Irreducible sets detect the occurrence of dynamically independent subsystems. Our next result gives a useful property of irreducible sets with more than one element.

Proposition 6.2. *If a subset J of I is irreducible, then, for all $i, j \in J$ with $i \neq j$, there exist $p \in \mathbb{N}$ and $i_1, \dots, i_p \in J$ such that $P_{i_1 i}$, $P_{i_2 i_1}, \dots, P_{i_p i_{p-1}}$ and $P_{j i_p}$ carry material.*

Proof. Let us assume, on the contrary, that we have $j \notin \cup_{n=1}^{\infty} \zeta^n(\{i\}) = \widetilde{J}_i$. Then $\widetilde{J}_i \subsetneq J$, and, obviously, $\zeta(\widetilde{J}_i) \subset \widetilde{J}_i$, which contradicts the fact that J is irreducible. □

Let $J_1, \dots, J_k$ be all the irreducible subsets of I, and let $J_0 = I \setminus \cup_{l=1}^{k} J_l$. These sets reflect the geometry of the compartmental system in a good enough way as to describe the long-term behavior of the solutions, as we will see below.

Let K be any minimal subset of $\Omega \times BU$ for the skew-product semiflow induced by the family of equations $(5.2)_\omega$. From Theorem 5.8, K is of the form $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ where x is a continuous map from Ω into BU. The subsequent results give qualitative information about the long-term behavior of the solutions. Let us see that, provided that we are working on a minimal set K, if there is no inflow from the environment, then the total mass is constant on K, all compartments out of an irreducible subset are empty, and, in an irreducible subset, either all compartments are empty or all are never empty. In particular, in any irreducible subset *with some outflow of material*, all compartments are empty.

Theorem 6.3. *Assume that $I_i \equiv 0$ for all $i \in I$. Let $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ be a minimal subset of $\Omega \times BU$ with $K \subset \mathcal{P}_D$. Then the following statements hold:*

(i) *there exists $c \geq 0$ such that $M|_K \equiv c$;*

(ii) *$x_i \equiv 0$ for each $i \in J_0$;*

(iii) *if, for some $l \in \{1, \dots, k\}$, there exists $j_l \in J_l$ such that $x_{j_l} \equiv 0$, then $x_i \equiv 0$ for each $i \in J_l$. In particular, this happens if there is a $j_l \in J_l$ such that there is outflow of material from C_{j_l}.*

Proof. In order to prove this result, we suppose in the first place that the system is *closed*, i.e. $g_{0i} \equiv 0,\ I_i \equiv 0$ for all $i \in I$.

(i) From (5.5), the total mass M is constant along the trajectories, and, hence, $M(\omega{\cdot}t, x(\omega{\cdot}t)) = M(\omega, x(\omega))$ for all $t \geq 0$ and $\omega \in \Omega$, which together with the fact that Ω is minimal and M continuous, shows the statement.

(ii) Let $i \in J_0$. The set $\widetilde{J}_i = \cup_{n=1}^{\infty} \zeta^n(\{i\})$ satisfies $\zeta(\widetilde{J}_i) \subset \widetilde{J}_i$, and, hence, contains an irreducible set J_l for some $l \in \{1, \ldots, k\}$. Consequently, there are $i_1, \ldots, i_p \in J_0$ and $j_l \in J_l$ such that $P_{i_1 i},\ P_{i_2 i_1}, \ldots, P_{i_p i_{p-1}}$ and $P_{j_l i_p}$ carry material.

It is easy to prove that there is an $r > 0$ such that $\|x(\omega)\|_\infty \leq r$ for each $\omega \in \Omega$. We define $M_l \colon \Omega \times BU \to \mathbb{R}$, the *mass restricted to* J_l, as

$$M_l(\omega, y) = \sum_{i \in J_l} D_i(\omega, y) + \sum_{i,j \in J_l} \int_{-\infty}^{0} \left(\int_{s}^{0} g_{ji}(\omega{\cdot}\tau, y_i(\tau))\, d\tau \right) d\mu_{ji}(s) \quad (6.1)$$

for all $(\omega, y) \in \Omega \times BU$, which is a uniformly continuous function on $\Omega \times B_r$ for the compact-open topology. From $(\omega, x(\omega)) \geq_D (\omega, 0)$, which also implies $x(\omega) \geq 0$, and, from (C1), we have $0 \leq M_l(\omega, x(\omega)) \leq M(\omega, x(\omega)) = c$ for each $\omega \in \Omega$.

Since J_l is irreducible, for all $i \in J_l$ and all $\omega \in \Omega$,

$$\begin{aligned} \frac{d}{dt} D_i(\omega{\cdot}t, x(\omega{\cdot}t)) = &-\sum_{j \in J_l} g_{ji}(\omega{\cdot}t, x_i(\omega{\cdot}t)(0)) \\ &+ \sum_{j \in J_l \cup J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) \end{aligned}$$

because the rest of the terms vanish. Consequently,

$$\frac{d}{dt} M_l(\omega{\cdot}t, x(\omega{\cdot}t)) = \sum_{i \in J_l} \sum_{j \in J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) \geq 0 \quad (6.2)$$

for all $\omega \in \Omega$ and all $t \in \mathbb{R}$. Now, we claim that the mapping $\Omega \to \mathbb{R}$, $\omega \mapsto M_l(\omega, x(\omega))$ is constant. Let us assume, on the contrary, that we can find $\omega_1,\ \omega_2 \in \Omega$ such that $M_l(\omega_1, x(\omega_1)) < M_l(\omega_2, x(\omega_2))$, and let $t_n \uparrow \infty$ such that $\lim_{n\to\infty} \omega_2{\cdot}t_n = \omega_1$. From (6.2), we deduce that, for each $n \in \mathbb{N}$, $M_l(\omega_2, x(\omega_2)) \leq M_l(\omega_2{\cdot}t_n, x(\omega_2{\cdot}t_n))$, and taking limits as $n \to \infty$ we conclude that $M_l(\omega_2, x(\omega_2)) \leq M_l(\omega_1, x(\omega_1))$, a contradiction. Hence $M_l(\omega, x(\omega))$ is constant, and, from (6.2),

$$\sum_{i \in J_l} \sum_{j \in J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) = 0\,. \quad (6.3)$$

Next we check that $x_{i_p} \equiv 0$. From (6.3) we deduce that, for each $\omega \in \Omega$,

$$\int_{-\infty}^{0} g_{j_l i_p}(\omega{\cdot}s, x_{i_p}(\omega)(s))\, d\mu_{j_l i_p}(s) = 0\,. \tag{6.4}$$

Assume that there is an $\omega_0 \in \Omega$ such that $x_{i_p}(\omega_0)(0) > 0$. Hence there is an $\varepsilon > 0$ with $x_{i_p}(\omega_0)(s) > 0$ for each $s \in (-\varepsilon, 0]$, and, since $P_{j_l i_p}$ carries material, $g_{j_l i_p}(\omega_0{\cdot}s, x_{i_p}(\omega_0)(s)) > 0$ for $s \in (-\varepsilon, 0]$. In addition, from $\mu_{j_l i_p}((-\infty, 0]) = 1$, it follows that there is a $b \le 0$ such that $\mu_{j_l i_p}((b-\varepsilon, b]) > 0$. Hence, denoting $\omega_0{\cdot}(-b) = \omega_1$ we deduce that

$$\int_{b-\varepsilon}^{b} g_{j_l i_p}(\omega_1{\cdot}s, x_{i_p}(\omega_1)(s))\, d\mu_{j_l i_p}(s) > 0\,,$$

which contradicts (6.4) and shows that $x_{i_p} \equiv 0$, as claimed. Now, since $(\omega, x(\omega)) \ge_D (\omega, 0)$, we have $D_{i_p}(\omega, x(\omega)) \ge 0$, and from the definition of D_{i_p} we deduce that $D_{i_p}(\omega, x(\omega)) = 0$ for each $\omega \in \Omega$. Therefore,

$$0 = \frac{d}{dt} D_{i_p}(\omega{\cdot}t, x(\omega{\cdot}t)) = \sum_{j=1}^{m} \int_{-\infty}^{0} g_{i_p j}(\omega{\cdot}(t+s), x_j(\omega{\cdot}t)(s))\, d\mu_{i_p j}(s)\,,$$

from which $\displaystyle\int_{-\infty}^{0} g_{i_p i_{p-1}}(\omega{\cdot}s, x_{i_{p-1}}(\omega)(s))\, d\mu_{i_p i_{p-1}}(s) = 0$, and, as before, we have $x_{i_{p-1}} \equiv 0$. In a finite number of steps we check that $x_i \equiv 0$, as stated.

(iii) From Proposition 6.2, it follows that given i, $j_l \in J_l$ there exist $p \in \mathbb{N}$ and $i_1, \ldots, i_p \in J_l$ such that $P_{i_1 i}$, $P_{i_2 i_1}, \ldots, P_{i_p i_{p-1}}$ and $P_{j_l i_p}$ carry material. If $x_{j_l} \equiv 0$, the same argument given in the last part of (ii) shows that $x_i \equiv 0$, which finishes the proof for closed systems.

Next we deal with the case when $I_i \equiv 0$ for each $i \in I$ but the system is not necessarily closed. From (5.5) we deduce that the total mass M is decreasing along the trajectories. In particular, for all $\omega \in \Omega$ and all $t \in \mathbb{R}$,

$$\frac{d}{dt} M(\omega{\cdot}t, x(\omega{\cdot}t)) = -\sum_{i=1}^{m} g_{0i}(\omega{\cdot}t, x_i(\omega{\cdot}t)(0)) \le 0\,. \tag{6.5}$$

Assume that there are ω_1, $\omega_2 \in \Omega$ such that $M(\omega_1, x(\omega_1)) < M(\omega_2, x(\omega_2))$, and let $t_n \uparrow \infty$ such that $\lim_{n\to\infty} \omega_1{\cdot}t_n = \omega_2$. From relation (6.5) we deduce that $M(\omega_1{\cdot}t_n, x(\omega_1{\cdot}t_n)) \le M(\omega_1, x(\omega_1))$ for each $n \in \mathbb{N}$, and taking limits as $n \uparrow \infty$ we conclude that $M(\omega_2, x(\omega_2)) \le M(\omega_1, x(\omega_1))$, a contradiction, which shows that M is constant on K, as stated in (i). Consequently, the derivative in (6.5) vanishes, and $g_{0i}(\omega{\cdot}t, x_i(\omega{\cdot}t)(0)) = 0$ for all $i \in I$, $\omega \in \Omega$

and $t \geq 0$. This means that $t \mapsto z(t,\omega,x(\omega)) = x(\omega{\cdot}t)(0)$ is a solution of a closed system, and (ii) and the first part of (iii) follow from the previous case.

Finally, let $j_l \in J_l$ be such that there is outflow of material from C_{j_l}, that is, $g_{0j_l}(\omega,v) > 0$ for all $\omega \in \Omega$ and $v > 0$. Moreover, as before, $g_{0j_l}(\omega,x_{j_l}(\omega)(0)) = 0$ for each $\omega \in \Omega$, which implies that $x_{j_l} \equiv 0$ and completes the proof. □

Remark 6.4. Notice that, concerning the solutions of the family of systems $(5.2)_\omega$, we deduce that, in the case of no inflow from the environment, $\lim_{t\to\infty} z_i(t,\omega,x_0) = 0$ for all $i \in J_0$ and all $i \in J_l$ for compartments J_l with some outflow, and for each $(\omega,x_0) \geq_D (\omega,0)$.

Remark 6.5. If there is no inflow from the environment toward the compartments and, for all $l \in \{1,\dots,k\}$, there is a $j_l \in J_l$ such that there is outflow of material from C_{j_l}, then the only minimal set in $\mathcal{P}_D$ is $K = \{(\omega,0) : \omega \in \Omega\}$, and all the solutions $z(\cdot,\omega,x_0)$ with initial data $(\omega,x_0) \geq_D (\omega,0)$ converge to 0 as $t \to \infty$.

In a non-closed system, that is, a system which may have any inflow and any outflow of material, if there exists a bounded solution, i.e. all solutions are bounded as shown above, and an irreducible set which has *some inflow*, then, working on a minimal set, all compartments of that irreducible set are nonempty, and there must be some outflow from the irreducible set.

Theorem 6.6. *Assume that there exists a bounded solution of family* $(5.1)_\omega$, *and let* $K = \{(\omega,x(\omega)) : \omega \in \Omega\}$ *be a minimal subset of* $\mathcal{P}_D$. *If, for some* $l \in \{1,\dots,k\}$, *there is a* $j_l \in J_l$ *such that* $I_{j_l} \not\equiv 0$, *i.e. there is some inflow into* C_{j_l}, *then*

(i) $x_i \not\equiv 0$ *for each* $i \in J_l$;

(ii) *there is a* $j \in J_l$ *such that there is outflow of material from* C_j.

Proof. (i) Let us assume, on the contrary, that there is an $i \in J_l$ such that $x_i \equiv 0$. Then, since $(\omega,x(\omega)) \geq_D (\omega,0)$ we have that $0 \leq D_i(\omega,x(\omega))$, and from the definition of D_i we deduce that $D_i(\omega,x(\omega)) = 0$ for each $\omega \in \Omega$. Therefore,

$$\begin{aligned} 0 &= \frac{d}{dt} D_i(\omega{\cdot}t, x(\omega{\cdot}t)) \\ &= \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) + I_i(\omega{\cdot}t)\,, \end{aligned} \tag{6.6}$$

for all $\omega \in \Omega$, $t \geq 0$, and, as in (ii) of Theorem 6.3, we check that $x_{j_l} \equiv 0$. However, since $I_{j_l} \not\equiv 0$, there is an $\omega_0 \in \Omega$ such that $I_{j_l}(\omega_0) > 0$, which contradicts (6.6) for $\omega = \omega_0$, $i = j_l$ at $t = 0$.

(ii) Assume on the contrary that $g_{0j} \equiv 0$ for each $j \in J_l$. Then, if we consider (6.1), the restriction of the mass to J_l, we check that

$$\frac{d}{dt} M_l(\omega{\cdot}t, x(\omega{\cdot}t)) = \sum_{i \in J_l} \left[I_i(\omega{\cdot}t) + \sum_{j \in J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) \right],$$

which is greater or equal to 0 for all $\omega \in \Omega$ and $t \geq 0$. A similar argument to the one given in (ii) of Theorem 6.3 shows that $M_l(\omega, x(\omega))$ is constant for each $\omega \in \Omega$, which contradicts the fact that the preceding derivative is strictly positive for $\omega = \omega_0$ at $t = 0$ and proves the statement. □

Finally, we will change hypothesis (C4) for the following one, which is slightly stronger:

(C4)* given $i \in \{0, \dots, m\}$ and $j \in \{1, \dots, m\}$ either $g_{ij} \equiv 0$ on $\Omega \times \mathbb{R}^+$, i.e. *there is not a pipe from compartment C_j to compartment C_i*, or $\frac{\partial}{\partial v} g_{ij}(\omega, v) > 0$ for all $\omega \in \Omega$ and $v \geq 0$. In this case we will say that the pipe P_{ij} *carries material strictly.*

In this case, we are able to prove that, if there exists a bounded solution then all the minimal sets coincide both on irreducible sets having *some outflow* and out of irreducible sets

Theorem 6.7. *Let us assume that there exists a bounded solution of system* $(5.1)_\omega$. *Let* $K_1 = \{(\omega, x(\omega)) : \omega \in \Omega\}$ *and* $K_2 = \{(\omega, y(\omega)) : \omega \in \Omega\}$ *be two minimal subsets of* $\mathcal{P}_D$. *Then*

(i) $x_i \equiv y_i$ *for each* $i \in J_0$*;*

(ii) *if, for some* $l \in \{1, \dots, k\}$*, there is a* $j_l \in J_l$ *such that there is outflow of material from* C_{j_l} *then* $x_i \equiv y_i$ *for each* $i \in J_l$.

Proof. For each $i \in \{0, \dots, m\}$ and each $j \in \{1, \dots, m\}$ we define the map $h_{ij} \colon \Omega \to \mathbb{R}^+$ as

$$h_{ij}(\omega) = \int_0^1 \frac{\partial g_{ij}}{\partial v}(\omega, s\, x_j(\omega)(0) + (1-s)\, y_j(\omega)(0))\, ds \geq 0\,, \quad \omega \in \Omega,$$

and we consider the family of monotone linear compartmental systems

$$\begin{aligned} \frac{d}{dt} D_i(\omega{\cdot}t, \widetilde{z}_t) = &- h_{0i}(\omega{\cdot}t)\, \widetilde{z}_i(t) - \sum_{j=1}^{m} h_{ji}(\omega{\cdot}t)\, \widetilde{z}_i(t) \\ &+ \sum_{j=1}^{m} \int_{-\infty}^{0} h_{ij}(\omega{\cdot}(s+t))\, \widetilde{z}_j(t+s)\, d\mu_{ij}(s)\,, \quad \omega \in \Omega\,. \end{aligned} \tag{6.7$_\omega$}$$

satisfying the corresponding hypotheses (L1), (L2), (C1)–(C3), and (C4). Moreover, condition (C3) for the family of systems $(6.7)_\omega$ follows from

$$\inf_{\omega\in\Omega} h_{ij}(\omega) \geq \inf_{v\geq 0\,,\,\omega\in\Omega} \frac{\partial g_{ij}}{\partial v}(\omega,v)\,, \quad \sup_{\omega\in\Omega} h_{ij}(\omega) \leq \sup_{v\geq 0\,,\,\omega\in\Omega} \frac{\partial g_{ij}}{\partial v}(\omega,v)$$

and (C3) for $(5.1)_\omega$. From the definition of h_{ij} and (C4)* we deduce that the irreducible sets for the families $(6.7)_\omega$ and $(5.2)_\omega$ coincide. Consequently, Theorem 6.3 (see Remark 6.4) applies to this case, and we deduce that, if $(\omega,z_0)\geq_D(\omega,0)$ and J_l is an irreducible set with some outflow of material, then

$$\lim_{t\to\infty} \widetilde{z}_i(t,\omega,z_0)=0 \qquad \text{for each } i\in J_0\cup J_l\,.$$

The same happens for $(\omega,z_0)\leq_D(\omega,0)$ because the systems are linear.

Let $z(\omega)=x(\omega)-y(\omega)$ for each $\omega\in\Omega$. Now, it is easy to check that $\widetilde{z}(t,\omega,z(\omega))=z(\omega{\cdot}t)(0)$ for all $\omega\in\Omega$ and $t\geq 0$. Moreover, we can find z_0, $z_1\in BU$ such that, for each $\omega\in\Omega$, $(\omega,z_1)\leq_D(\omega,0)$, $(\omega,z_0)\geq_D(\omega,0)$, and $(\omega,z_1)\leq_D(\omega,z(\omega))\leq_D(\omega,z_0)$. Hence, the monotonicity of the induced skew-product semiflow and the positivity of $\widehat{D}^{-1}(\omega,\cdot)$ yield

$$\widetilde{z}(t,\omega,z_1)\leq z(\omega{\cdot}t)(0)\leq \widetilde{z}(t,\omega,z_0)\,, \quad \text{for all } \omega\in\Omega\,,\ t\geq 0\,,$$

from which we deduce that $z_i\equiv 0$ for all $i\in J_0\cup J_l$, and statements (i) and (ii) follow. □

As a consequence, under the same assumptions of the previous theorem, when for all $l\in\{1,\dots,k\}$ there is outflow of material from one of the compartments in J_l, there is a unique minimal set $K=\{(\omega,x(\omega)) : \omega\in\Omega\}$ in $\mathcal{P}_D$ attracting all the solutions with initial data in $\mathcal{P}_D$, i.e.

$$\lim_{t\to\infty} \|z(t,\omega,x_0)-x(\omega{\cdot}t)(0)\|=0 \qquad \text{whenever } \ (\omega,x_0)\geq_D 0\,.$$

Moreover, $x\not\equiv 0$ if and only if there is some $j\in\{1,\dots,m\}$ such that $I_j\not\equiv 0$, i.e. there is some inflow into one of the compartments C_j.

For the next result, we will assume the following hypothesis:

(C5) if $K_1=\{(\omega,x(\omega)) : \omega\in\Omega\}$ and $K_2=\{(\omega,y(\omega)) : \omega\in\Omega\}$ are two minimal subsets of $\mathcal{P}_D$ such that $(\omega,x(\omega))\leq_D(\omega,y(\omega))$ and, besides, $D_i(\omega_0,x(\omega_0))=D_i(\omega_0,y(\omega_0))$ for some $\omega_0\in\Omega$ and $i\in\{1,\dots,m\}$, then $x(\omega)=y(\omega)$ for each $\omega\in\Omega$, i.e. $K_1=K_2$.

Hypothesis (C5) is relevant when it applies to closed systems, and it holds in many cases studied in the literature. Systems with a unique compartment, studied in [AB] and [KW], satisfy (C5). It follows from Theorem 6.3 that irreducible closed systems described by FDEs (see [AH]) satisfy (C5). Closed systems given in [W] and [WF] in the strongly ordered case also satisfy (C5).

Definition 6.8. Let $K_1 = \{(\omega, x(\omega)) : \omega \in \Omega\}$ and $K_2 = \{(\omega, y(\omega)) : \omega \in \Omega\}$ be two minimal subsets. It is said that $K_1 <_D K_2$ if $(\omega, x(\omega)) <_D (\omega, y(\omega))$ for each $\omega \in \Omega$.

Hypothesis (C5) allows us to classify the minimal subsets in terms of the value of their total mass, as shown in the next result.

Theorem 6.9. *Assume that the system* $(5.1)_\omega$ *is closed (i.e.* $I_i \equiv 0$ *and* $g_{0i} \equiv 0$ *for each* $i \in \{1, \dots, m\}$*), and hypotheses* (L1), (L2), (C1)–(C3), (C4)*, *and* (C5) *hold. Then for each* $c > 0$ *there is a unique minimal subset* K_c *such that* $M|_{K_c} = c$*. Moreover,* $K_c \subset \mathcal{P}_D$*, and* $K_{c_1} <_D K_{c_2}$ *whenever* $c_1 < c_2$*.*

Proof. Since all of the minimal subsets are copies of the base, and the total mass (5.4) is constant along the trajectories and increasing for the D-order because $\widehat{D}^{-1}(\omega, \cdot)$ is positive for all $\omega \in \Omega$, it is easy to check that given $c > 0$ there is a minimal subset $K_c \subset \mathcal{P}_D$ such that $M|_{K_c} = c$.

Let $\widehat{D}$ be the homeomorphism of $\Omega \times BU$ defined by the relation (3.1). For each $(\omega, x) \in \Omega \times BU$ we define $(\omega, x)^+ = \widehat{D}^{-1}(\omega, \sup(0, \widehat{D}_2(\omega, x))$. Hence $(\omega, 0) \le_D (\omega, x)^+$, $(\omega, x) \le_D (\omega, x)^+$, and, if $y \in BU$ with $(\omega, x) \le_D (\omega, y)$ and $(\omega, 0) \le_D (\omega, y)$ then $(\omega, x)^+ \le_D (\omega, y)$.

Since the semiflow is monotone, from $(\omega, x) \le_D (\omega, x)^+$ we deduce that $\tau(t, \omega, x) \le_D \tau(t, (\omega, x)^+)$. Besides, since the system is closed, $u(t, \omega, 0) = 0$, and, from $(\omega, 0) \le_D (\omega, x)^+$, we check that $(\omega{\cdot}t, 0) \le_D \tau(t, (\omega, x)^+)$. Consequently $\tau(t, \omega, x)^+ \le_D \tau(t, (\omega, x)^+)$ for each $t \ge 0$.

Next we check that if $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ is minimal, the same holds for $K^+ = \{(\omega, x(\omega))^+ : \omega \in \Omega\}$. Since $x(\omega{\cdot}t) = u(t, \omega, x)$ for each $t \ge 0$, we deduce that $(\omega{\cdot}t, x(\omega{\cdot}t))^+ = \tau(t, \omega, x(\omega))^+ \le_D \tau(t, (\omega, x(\omega))^+)$, and the fact that $\widehat{D}^{-1}(\omega{\cdot}t, \cdot)$ is positive yields $[(\omega{\cdot}t, x(\omega{\cdot}t))^+]_2 \le u(t, (\omega, x(\omega))^+)$ for each $t \ge 0$. In addition, since the total mass (5.4) is constant along the trajectories and it is increasing for the D-order, we deduce

$$\begin{aligned} M((\omega, x(\omega))^+) =& M(\omega{\cdot}t, u(t, (\omega, x(\omega))^+)) \\ \ge& M((\omega{\cdot}t, u(t, \omega, x(\omega)))^+) = M((\omega{\cdot}t, x(\omega{\cdot}t))^+) \end{aligned}$$

for each $t \ge 0$. Moreover, since $(\omega, x(\omega))^+$ is a continuous function in ω and Ω is minimal, an argument similar to the one given in statement (ii) of Theorem 6.3 shows that $\omega \mapsto M((\omega, x(\omega))^+)$ is constant on Ω, and, consequently,

$$M(\omega{\cdot}t, u(t, (\omega, x(\omega))^+) = M((\omega{\cdot}t, x(\omega{\cdot}t))^+)$$

for each $\omega \in \Omega$ and $t \ge 0$. Hence, from (5.4) we conclude that

$$0 = \sum_{i=1}^{m} \left[D_i(\tau(t, (\omega, x(\omega))^+)) - D_i((\omega{\cdot}t, x(\omega{\cdot}t))^+) \right] ,$$

that is, $D(\tau(t,(\omega,x(\omega))^+)) = D((\omega{\cdot}t, x(\omega{\cdot}t))^+)$ for each $\omega \in \Omega$ and $t \geq 0$. Furthermore, it is easy to check that $[(\omega{\cdot}s,\varphi_s)^+]_2 = [((\omega,\varphi)^+)_2]_s$ whenever $(\omega,\varphi) \in \Omega \times BU$ and $s \leq 0$, from which we deduce that

$$D(\omega{\cdot}(t+s), u(t,(\omega,x(\omega))^+)_s) = D(\omega{\cdot}t, [((\omega{\cdot}t, x(\omega{\cdot}t))^+)_2]_s)$$

for each $s \leq 0$, $t \geq 0$, and $\omega \in \Omega$. That is, we have

$$\widehat{D}(\omega{\cdot}t, u(t,(\omega,x(\omega))^+)) = \widehat{D}((\omega{\cdot}t, x(\omega{\cdot}t))^+),$$

and hence it follows that $\tau(t,(\omega,x(\omega))^+) = (\omega{\cdot}t, x(\omega{\cdot}t))^+$ for each $t \geq 0$ and $\omega \in \Omega$, which in turn shows that K^+ is a minimal subset. Now, let $K_1 = \{(\omega,x(\omega)) : \omega \in \Omega\}$ and $K_2 = \{(\omega,y(\omega)) : \omega \in \Omega\}$ be two minimal subsets such that $M|_{K_i} = c$ for $i = 1,2$. Fix $\omega \in \Omega$; the change of variable $\widehat{z}(t) = z(t) - y(\omega{\cdot}t)$ takes $(5.2)_\omega$ to

$$\frac{d}{dt}D(\omega{\cdot}t, \widehat{z}_t) = G(\omega{\cdot}t, \widehat{z}_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,,$$

where $G(\omega{\cdot}t, \widehat{z}_t) = F(\omega{\cdot}t, \widehat{z}_t + y(\omega{\cdot}t)) - F(\omega{\cdot}t, y(\omega{\cdot}t))$. It is not hard to check that this is a new family of compartmental systems satisfying the corresponding hypotheses (L1), (L2), (C1)–(C3), and (C4)*, and

$$\widehat{K} = \{(\omega, x(\omega) - y(\omega)) : \omega \in \Omega\}$$

is one of its minimal subsets. As before

$$\widehat{K}^+ = \{(\omega,(x(\omega) - y(\omega)))^+ : \omega \in \Omega\}$$

is also a minimal subset, and hence

$$K^+ = \{(\omega, y(\omega) + [(\omega, x(\omega) - y(\omega))^+]_2) : \omega \in \Omega\} = \{(\omega, z(\omega)) : \omega \in \Omega\}$$

is a minimal subset for the initial family.

For each $\omega \in \Omega$, we have that $(\omega, z(\omega)) \geq_D (\omega, y(\omega))$. Assume that $D_i(\omega, z(\omega)) > D_i(\omega, y(\omega))$ for all $\omega \in \Omega$ and $i \in \{1,\ldots,m\}$. We know that

$$D_i(\omega{\cdot}s, [((\omega, x(\omega) - y(\omega))^+)_2]_s) = D_i((\omega{\cdot}s, x(\omega{\cdot}s) - y(\omega{\cdot}s))^+) > 0$$

for all $s \leq 0$ and all $i \in \{1,\ldots,m\}$, because $z(\omega) = [(\omega, x(\omega) - y(\omega))^+]_2$. Hence, $[\sup(0, \widehat{D}_2(\omega, x(\omega) - y(\omega)))(s)]_i > 0$ for all $s \leq 0$ and all $i \in \{1,\ldots,m\}$, which in turn implies that $(\omega, x(\omega)) >_D (\omega, y(\omega))$ for all $\omega \in \Omega$. Consequently, $M(\omega, x(\omega)) > M(\omega, y(\omega))$ for all $\omega \in \Omega$, a contradiction. As a result, there are an $\omega_0 \in \Omega$ and an $i \in \{1,\ldots,m\}$ such that $D_i(\omega_0, z(\omega_0)) = D_i(\omega_0, y(\omega_0))$, and from hypothesis (C5) it follows that $z(\omega) = y(\omega)$ for each $\omega \in \Omega$. That is, $(\omega, x(\omega) - y(\omega))^+ \equiv (\omega, 0)$ or, equivalently, $(\omega, x(\omega) - y(\omega)) \leq_D (\omega, 0)$ for each $\omega \in \Omega$. Finally, as before, from $M|_{K_1} = M|_{K_2}$ we conclude by contradiction that $x(\omega) = y(\omega)$ for each $\omega \in \Omega$, and the minimal set K_c is unique, as stated. The same argument shows that $K_{c_1} <_D K_{c_2}$ whenever $c_1 < c_2$ and finishes the proof. □

Let us illustrate the above results by means of some examples found in the literature about which we can draw some interesting conclusions.

Example 6.10. Consider the following non-autonomous compartmental system with finite delay:

$$\begin{aligned}
x_1'(t) =& - h_{11}(t, x_1(t)) - h_{21}(t, x_1(t)) - h_{31}(t, x_1(t)) + h_{11}(t, x_1(t-1)) \\
&+ h_{13}(t, x_3(t-2)), \\
x_2'(t) =& - h_{32}(t, x_2(t)) + h_{21}(t, x_1(t-1)), \\
x_3'(t) =& - h_{13}(t, x_3(t)) + h_{31}(t, x_1(t-2)) + \frac{1}{2}\, h_{32}(t, x_2(t)) \\
&+ \frac{1}{4}\, h_{32}(t, x_2(t-1)) + \frac{1}{4}\, h_{32}(t, x_2(t-2)),
\end{aligned} \tag{6.8}$$

for $t \geq 0$, where h_{11}, h_{21}, h_{31}, h_{13}, h_{32} are non-zero functions satisfying conditions (E1), (E2), (C1), and (C4). It is a straightforward generalization of the example studied for the autonomous case in Krisztin [K]. Notice that (L1), (L2), (C2), and (C3) are trivially satisfied by this system. This system corresponds to Figure 6.1, where compartments are represented by circles, and pipes carrying material are represented by arrows.

It is clear that the whole system is irreducible, that is, we have $k = 1$, $J_0 = \emptyset$, and $J_1 = \{1, 2, 3\}$; and it is closed. As a result, from Theorem 6.3, we can conclude that the total mass of the system converges to a constant, and, if some compartment empties, then all of them do as well. Moreover, assuming (C4)*, it follows that the value to which the total mass converges determines uniquely the initial amount of material and establishes an order in the eventual amount of material within the compartments in the sense of Theorem 6.9.

Example 6.11. Chains are a class of compartmental systems widely studied in the literature. There are linear chains (see [Ja]), linear chains set in parallel (see [JS2]), and two-way chains (see Smith [Sm2] and [JS]); they have been

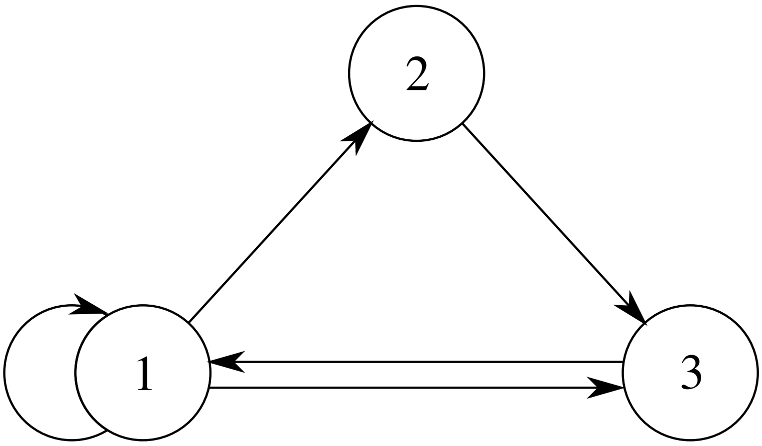

Figure 6.1: Compartmental system associated to equation (6.8).

previously studied in their autonomous versions. We present here an example of the latter. Consider the system

$$\begin{aligned} x_1'(t) =& - h_{01}(t, x_1(t)) - h_{21}(t, x_1(t)) + h_{12}(t, x_2(t-1)) + I_1(t), \\ x_i'(t) =& - h_{i-1\,i}(t, x_i(t)) - h_{i+1\,i}(t, x_i(t)) \\ &+ \frac{2}{3}\, h_{i\,i-1}(t, x_{i-1}(t-i)) + \frac{1}{3}\, h_{i\,i-1}(t, x_{i-1}(t-(i-1))) \\ &+ h_{i\,i+1}(t, x_{i+1}(t-i)), \quad i \in \{2, \ldots, m-1\} \\ x_m'(t) =& - h_{0m}(t, x_m(t)) - h_{m-1\,m}(t, x_m(t)) \\ &+ \frac{2}{3}\, h_{m\,m-1}(t, x_{m-1}(t-m)) \\ &+ \frac{1}{3}\, h_{m\,m-1}(t, x_{m-1}(t-(m-1))) + I_m(t), \end{aligned} \tag{6.9}$$

for $t \geq 0$, where, for all i, j, h_{ij} is a non-zero function satisfying conditions (E1), (E2), (C1), and (C4). It represents m compartments set in a linear array with two-way pipes connecting each compartment to its two closest neighbors; besides, the first and last compartments have inflow and outflow from and to the environment (see Figure 6.2).

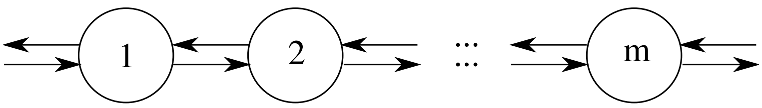

Figure 6.2: Two-way linear chain associated to equation (6.9).

Once more, it is clear that (L1), (L2), (C2), and (C3) are satisfied by this system. Again, the whole system is irreducible, that is, $k = 1$, $J_1 = \{1, \ldots, m\}$

and $J_0 = \emptyset$. Theorem 6.3 allows us to state that, if there is no inflow of material from the environment to compartments C_1 and C_m, then the total mass converges to some constant, and all the compartments get empty eventually. On the other hand, assume that there is some inflow of material either to compartment C_1 or to compartment C_m, and that there is a bounded solution; then, thanks to Theorem 6.6, there is some outflow from compartment C_1 or C_m (as it can be seen in Figure 6.2), and no compartments get empty. Now, suppose that there is a bounded solution, and (C4)* is satisfied; from Theorem 6.7, it follows that there is only one possible eventual amount of material, irrespective of the initial datum, within the compartments.

Example 6.12. Let us focus now on another class of compartmental systems which have been widely studied: cycles. They appear in their autonomous versions in Audoly and D'Angio [AD], Ashizawa and Miyazaki [AM], and Eisenfeld [Ei, Ei2], among others. Cycles are compartmental systems where m compartments are set on a circle, and they are connected to their two closest neighbors, as it can be seen in Figure 6.3. Consider the following example, which, in addition, has a pipe going from compartment C_1 to itself and some inflow and outflow of material between the environment and compartment C_2:

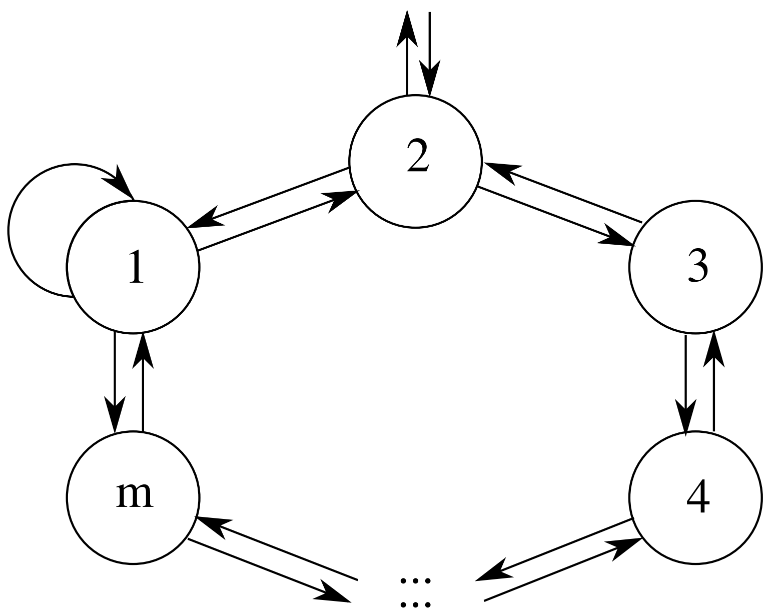

Figure 6.3: Cycle associated to equation (6.10).

$$
\begin{aligned}
x_1'(t) =& - h_{m1}(t, x_1(t)) - h_{21}(t, x_1(t)) - h_{11}(t, x_1(t)) \\
&+ h_{1m}(t, x_m(t-1)) + h_{12}(t, x_2(t-1)) + h_{11}(t, x_1(t-1)), \\
x_2'(t) =& - h_{02}(t, x_2(t)) - h_{12}(t, x_2(t)) - h_{32}(t, x_2(t)) \\
&+ h_{21}(t, x_1(t-1)) + h_{23}(t, x_3(t-1)) + I_2(t), \\
x_i'(t) =& - h_{i-1\, i}(t, x_i(t)) - h_{i+1\, i}(t, x_i(t)) + h_{i\, i-1}(t, x_{i-1}(t-1)) \\
&+ h_{i\, i+1}(t, x_{i+1}(t-1)), \quad i \in \{3, \ldots, m-1\} \\
x_m'(t) =& - h_{1m}(t, x_m(t)) - h_{m-1\, m}(t, x_m(t)) + h_{m\, m-1}(t, x_{m-1}(t-1)) \\
&+ h_{m1}(t, x_1(t-1)),
\end{aligned}
\tag{6.10}
$$

for $t \geq 0$, where, for all i, j, h_{ij} is a non-zero function satisfying conditions (E1), (E2), (C1), and (C4).

Clearly, this system satisfies (L1), (L2), (C2), and (C3), and the whole system is irreducible. Consequently, thanks to Theorem 6.3, we know that if there is no inflow of material from the environment to compartment C_2, then the total mass converges to a constant value, and all the compartments get empty. Furthermore, assuming that there is some inflow of material to compartment C_2, and there is a bounded solution, Theorem 6.6 implies that there is some outflow from compartment C_2 (as we already knew), and no compartment empties. Now, if there is a bounded solution and (C4)* holds, then, from Theorem 6.7, it follows that there is only one possible eventual amount of material, whatever the initial datum may be, in the compartments.

Example 6.13. In this example, we analyze the effects of a drug injection by means of a model presented in [JS2] for the autonomous case. Such model corresponds to Figure 6.4. In turn, this figure is associated to the system of equations

$$
\begin{aligned}
x_1'(t) =& - h_{y_1\, x_1}(t, x_1(t)) - h_{x_2\, x_1}(t, x_1(t)) \\
&+ h_{x_1\, y_1}(t, y_1(t-1)) + I_{x_1}(t), \\
x_i'(t) =& - h_{y_i\, x_i}(t, x_i(t)) - h_{x_{i+1}\, x_i}(t, x_i(t)) + h_{x_i\, x_{i-1}}(t, x_{i-1}(t-1)) \\
&+ h_{x_i\, y_i}(t, y_i(t-1)), \quad i \in \{2, \ldots, m-1\}, \\
x_m'(t) =& - h_{y_m\, x_m}(t, x_m(t)) - h_{0\, x_m}(t, x_m(t)) \\
&+ h_{x_m\, x_{m-1}}(t, x_{m-1}(t-1)) + h_{x_m\, y_m}(t, y_m(t-1)), \\
y_i'(t) =& - h_{x_i\, y_i}(t, y_i(t)) - h_{z_i\, y_i}(t, y_i(t)) + h_{y_i\, x_i}(t, x_i(t-1)) \\
&+ h_{y_i\, z_i}(t, z_i(t-1)), \quad i \in \{1, \ldots, m\}, \\
z_i'(t) =& - h_{y_i\, z_i}(t, z_i(t)) + h_{z_i\, y_i}(t, y_i(t-1)), \quad i \in \{1, \ldots, m\},
\end{aligned}
\tag{6.11}
$$

for $t \geq 0$, where, for all u, $v \in \{x_i, y_i, z_i : i = 1, 2, 3\}$, h_{uv} is a non-zero function satisfying conditions (E1), (E2), (C1), and (C4). Moreover, it is

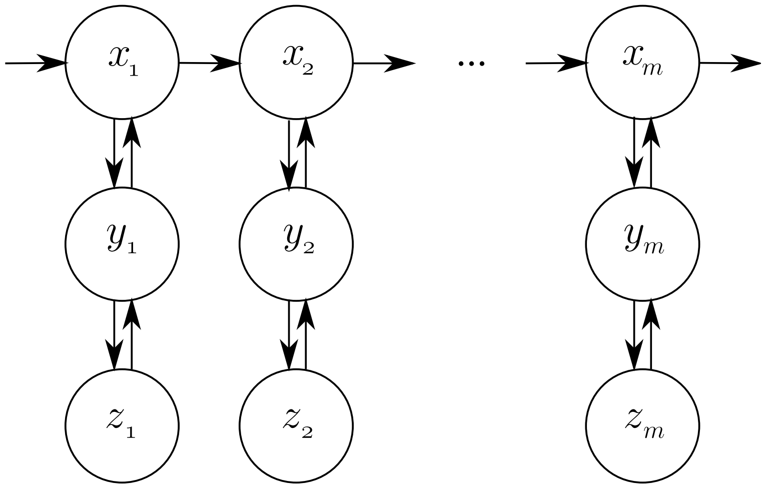

Figure 6.4: Compartmental system associated to equation (6.11).

easy to check that (L1), (L2), (C2), and (C3) hold as well. Clearly, there is only one irreducible subset, namely $J_1 = \{x_m, y_m, z_m\}$, and then

$$J_0 = \bigcup_{i=1}^{m-1} \{x_i, y_i, z_i\}.$$

As a result, from Theorem 6.3, we can conclude that, if there is no inflow of material into compartment x_1, then the total mass has a limit, and all the compartments eventually get empty due to the existence of some outflow of material from compartment x_m. Besides, whatever the inflow of material into compartment x_1 is, if there is a bounded solution of (6.11) and (C4)* holds, then there is only one possible eventual amount of material within the compartments of the system, thanks to Theorem 6.7.

Example 6.14. Finally, we present a more complex compartmental system which can be found in Foster and Jacquez [FJ] in the autonomous case. Such model corresponds to Figure 6.5, and it is governed by the system of equations

$$\begin{aligned}
x_1'(t) &= -\, h_{21}(t, x_1(t)) + h_{13}(t, x_3(t-1)) \\
x_2'(t) &= -\, h_{42}(t, x_2(t)) + h_{21}(t, x_1(t-1)) + h_{24}(t, x_4(t-1)) \\
x_3'(t) &= -\, h_{13}(t, x_3(t)) - h_{43}(t, x_3(t)) - h_{53}(t, x_3(t)) \\
&\quad + h_{38}(t, x_8(t-1))
\end{aligned}$$

$$
\begin{aligned}
x_4'(t) &= -\,h_{24}(t, x_4(t)) + h_{42}(t, x_2(t-1)) + h_{43}(t, x_3(t-1)) \\
x_5'(t) &= -\,h_{75}(t, x_5(t)) - h_{85}(t, x_5(t)) + h_{53}(t, x_3(t-1)) \\
&\quad + h_{58}(t, x_8(t-1)) \\
x_6'(t) &= h_{68}(t, x_8(t-1)) + h_{69}(t, x_9(t-1)) \\
x_7'(t) &= -\,h_{07}(t, x_7(t)) + h_{75}(t, x_5(t-1)) + h_{78}(t, x_8(t-1)) \\
x_8'(t) &= -\,h_{38}(t, x_8(t)) - h_{58}(t, x_8(t)) - h_{68}(t, x_8(t)) \\
&\quad - h_{78}(t, x_8(t)) + h_{85}(t, x_5(t-1)) \\
x_9'(t) &= -\,h_{69}(t, x_9(t)) + h_{93}(t, x_3(t-1))
\end{aligned}
\tag{6.12}
$$

for $t \geq 0$, where, for all i, j, h_{ij} is a non-zero function which satisfies conditions (E1), (E2), (C1), and (C4). It is not hard to check that (L1), (L2), (C2), and (C3) also hold. This system has three irreducible subsets: $J_1 = \{6\}$, $J_2 = \{7\}$, and $J_3 = \{2, 4\}$; clearly, $J_0 = \{1, 3, 5, 8, 9\}$, and there is some outflow of material from a compartment in J_2.

In this situation, thanks to Theorem 6.3, we know that the total mass has a limit, and compartments 1, 3, 5, 7, 8, and 9 eventually get empty. In other words, the only compartments which eventually contain some material are compartments 2, 4, and 6.

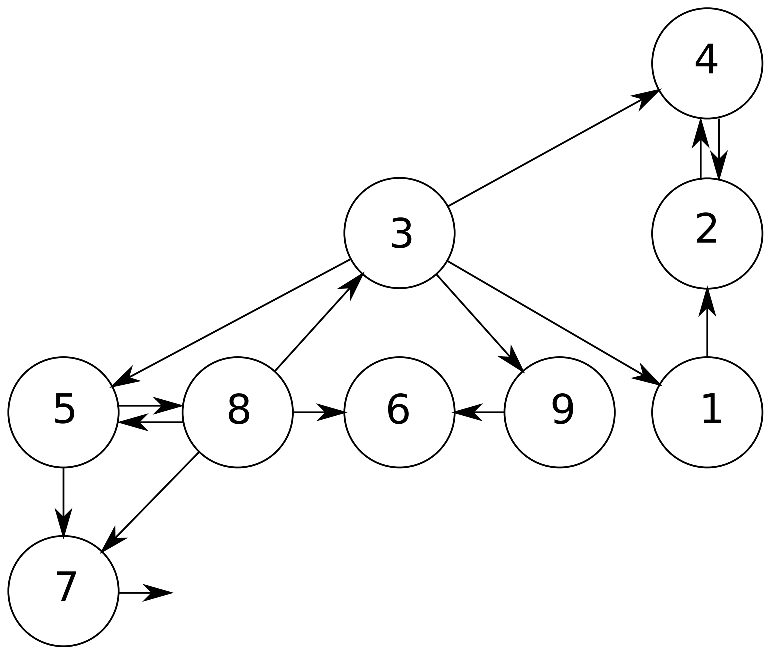

Figure 6.5: Compartmental system associated to equation (6.12).

Chapter 7

Transformed exponential order

In this chapter, we will study a skew-product semiflow generated by a family of NFDEs with infinite delay and stable non-autonomous D-operator. This semiflow will turn out to be monotone for a special order relation, namely the exponential order, which is different from the one introduced in Chapter 4. The exponential order relation was introduced in [NOV] and is in the line of those presented in [ST2]. Specifically, the 1-covering property of omega-limit sets is established under the componentwise separating property and uniform stability for semiorbits with ordered initial data.

It was proved in [NOV] that, provided that D is an autonomous operator (i.e. an operator which does not depend on its first component), and that the NFDE satisfies a componentwise separating property together with a property of uniform stability for semiorbits with initial data ordered with respect to the exponential order, then the 1-covering property holds. It is clear that such setting includes the case of FDEs with infinite delay, that is, the case $D(\omega, x) = x(0)$ for all $(\omega, x) \in \Omega \times BU$.

In this situation, the dynamical structure obtained in [NOV] can be transferred to the problem of NFDEs with non-autonomous D-operator by means of the convolution operator $\widehat{D}$ associated to D, as seen in Chapter 3. In particular, the 1-covering property of omega-limit sets holds. As we noted in Chapter 4, we transform finite and infinite delay NFDEs into infinite delay FDEs.

Let $F : \Omega \times BU \to \mathbb{R}^m$, let (Ω, d) be a compact metric space, and let $\mathbb{R} \times \Omega \to \Omega$, $\omega \mapsto \omega{\cdot}t$ be a minimal real flow on Ω. Let $D : \Omega \times BU \to \mathbb{R}^m$ be a stable operator satisfying hypotheses (D1)–(D3).

Let us consider the family of equations

$$\frac{d}{dt} D(\omega{\cdot}t, z_t) = F(\omega{\cdot}t, z_t), \quad t \geq 0,\, \omega \in \Omega. \tag{7.1$_\omega$}$$

We take the componentwise partial order relation on $\mathbb{R}^m$, as we did in Chapter 4. Let us recall that, as we saw in Chapter 3, we write $A \leq B$ for

$m\times m$ matrices $A=[a_{ij}]_{ij}$ and $B=[b_{ij}]_{ij}$ whenever $a_{ij}\le b_{ij}$ for all i,j. Let A be an $m\times m$ quasipositive matrix, i.e. a matrix such that there exists $\lambda>0$ with $A+\lambda I\ge 0$. Let $\rho>0$; let us recall the definitions of exponential ordering on BU given in [NOV]. If $x,\,y\in BU$, then

$$\begin{aligned}
x\le_{A,\rho} y &\iff x\le y,\ y(t)-x(t)\ge e^{A(t-s)}(y(s)-x(s))\,,-\rho\le s\le t\le 0\,,\\
x<_{A,\rho} y &\iff x\le_{A,\rho} y \quad\text{and}\quad x\ne y\,,\ \text{and}\\
x\le_{A,\infty} y &\iff x\le y,\ y(t)-x(t)\ge e^{A(t-s)}(y(s)-x(s))\,,-\infty< s\le t\le 0\,,\\
x<_{A,\infty} y &\iff x\le_{A,\infty} y \quad\text{and}\quad x\ne y\,.
\end{aligned}$$

In what follows, $\le_A$ will denote any of the order relations $\le_{A,\rho}$ and $\le_{A,\infty}$. However, in the case of $\le_{A,\infty}$, we will assume without further notice that all the eigenvalues of A have strictly negative real parts. The theory will provide different dynamical conclusions for each choice. The aforementioned relations define positive cones in BU, $BU_A^+=\{x\in BU : x\ge_A 0\}$, in the sense that they are closed subsets of BU and satisfy $BU_A^+ + BU_A^+\subset BU_A^+$, $\mathbb{R}^+BU_A^+\subset BU_A^+$, and $BU_A^+\cap(-BU_A^+)=\{0\}$. Note that, if $\le_A=\le_{A,\rho}$, then a smooth function (resp. a Lipschitz continuous function) x belongs to BU_A^+ if and only if $x\ge 0$ and $x'(s)\ge Ax(s)$ for each (resp. a.e.) $s\in[-\rho,0]$, and, if $\le_A=\le_{A,\infty}$, then it belongs to BU_A^+ if and only if $x\ge 0$ and $x'(s)\ge Ax(s)$ for each (resp. a.e.) $s\in(-\infty,0]$.

On each fiber of the product $\Omega\times BU$, we define the following *transformed exponential order* relation: if (ω,x), $(\omega,y)\in\Omega\times BU$, then

$$(\omega,x)\le_{D,A}(\omega,y) \iff \widehat{D}_2(\omega,x)\le_A \widehat{D}_2(\omega,y).$$

As in Chapter 4, in the case that D is autonomous, the positive cone defined from $\le_{D,A}$ on each fiber is independent of the fiber, that is, for all $\omega\in\Omega$, it is given by $\widehat{D}^{-1}(BU_A^+)$, where $D: BU\to\mathbb{R}^m$ denotes the autonomous linear operator, and $\widehat{D}$ denotes its associated convolution operator.

Let us assume hypothesis (F1) as seen in Chapter 4. Again, as seen in [WW] and [W], for each $\omega\in\Omega$, the local existence and uniqueness of the solutions of equation $(7.1)_\omega$ follow from (F1). Once more, given $(\omega,x)\in\Omega\times BU$, $z(\cdot,\omega,x)$ will denote the solution of equation $(7.1)_\omega$ with initial datum x, and a local skew-product semiflow τ on $\Omega\times BU$ can be defined as in (4.2).

Let $(\omega,y)\in\Omega\times BU$. For each $t\ge 0$ where $u(t,\widehat{D}^{-1}(\omega,y))$ is defined, we define $\widehat{u}(t,\omega,y)=\widehat{D}_2(\omega{\cdot}t,u(t,\widehat{D}^{-1}(\omega,y)))$. In Chapter 4, it was shown that the mapping

$$\widehat{z}(\cdot,\omega,y): t\mapsto\begin{cases} y(t) & \text{if } t\le 0,\\ \widehat{u}(t,\omega,y)(0) & \text{if } t\ge 0,\end{cases}$$

is the solution of $(4.3)_\omega$ through (ω, y). Let us assume hypotheses (F2) and (F3) from Chapter 4 together with the following one:

(F4)* if (ω, x), $(\omega, y) \in \Omega \times BU$ are such that $(\omega, x) \leq_{D,A} (\omega, y)$, then $F(\omega, y) - F(\omega, x) \geq A(D(\omega, y) - D(\omega, x))$.

Then statements (i), (ii), and (iii) of Proposition 4.2 hold, and we obtain the following result as a direct consequence of (F4)*.

Proposition 7.1. *If (ω, x), $(\omega, y) \in \Omega \times BU$ and $x \leq_A y$, then it holds that $G(\omega, y) - G(\omega, x) \geq A(y(0) - x(0))$.*

As in Chapter 4, we may define another local skew-product semiflow $\widehat{\tau}$ on $\Omega \times BU$ from the solutions of the equations of the family $(4.3)_\omega$ as in (4.4).

Notice that Proposition 4.3 holds here, and, therefore, the omega-limit set of $(\omega_0, x_0) \in \Omega \times BU$ can be defined as in (4.5). Furthermore, Propositions 4.5 and 4.6 hold as well.

In this situation, we can obtain the following result on monotonicity as a consequence of (F4)*.

Theorem 7.2. *Fix (ω, x), $(\omega, y) \in \Omega \times BU$ such that $(\omega, x) \leq_{D,A} (\omega, y)$. Then*

$$\tau(t, \omega, x) \leq_{D,A} \tau(t, \omega, y)$$

for all $t \geq 0$ where they are defined.

Proof. It is clear that $\widehat{D}_2(\omega, x) \leq_A \widehat{D}_2(\omega, y)$. Now, from Theorem 3.5 in [NOV] and Proposition 7.1, it follows that $\widehat{u}(t, \widehat{D}(\omega, x)) \leq_A \widehat{u}(t, \widehat{D}(\omega, y))$ or, equivalently, $\widehat{D}_2(\omega{\cdot}t, u(t, \omega, x)) \leq_A \widehat{D}_2(\omega{\cdot}t, u(t, \omega, y))$ whenever they are defined. Therefore, we have $\tau(t, \omega, x) \leq_{D,A} \tau(t, \omega, y)$ for all $t \geq 0$ where they are defined, as wanted. □

We introduce the concept of uniform stability for semiorbits with ordered initial data, which holds in more general situations than the one introduced in Definition 4.8.

Definition 7.3. Given $r > 0$, a forward orbit $\{\tau(t, \omega_0, x_0) : t \geq 0\}$ of the skew-product semiflow τ is said to be *uniformly stable for the order $\leq_A$ in $\Omega \times B_r$* if, for every $\varepsilon > 0$, there is a $\delta > 0$, called the *modulus of uniform stability*, such that, if $s \geq 0$ and $\mathsf{d}(u(s, \omega_0, x_0), x) \leq \delta$ for certain $x \in B_r$ with $x \leq_A u(s, \omega_0, x_0)$ or $u(s, \omega_0, x_0) \leq_A x$, then for each $t \geq 0$,

$$\mathsf{d}(u(t+s, \omega_0, x_0), u(t, \omega_0{\cdot}s, x)) = \mathsf{d}(u(t, \omega_0{\cdot}s, u(s, \omega_0, x_0)), u(t, \omega_0{\cdot}s, x)) \leq \varepsilon.$$

If this happens for each $r > 0$, the forward orbit is said to be *uniformly stable for the order $\leq_A$ in bounded sets.*

Let us assume two more hypotheses concerning F and the semiflow $\widehat{\tau}$. The fact that we are imposing a condition on the semiflow $\widehat{\tau}$ seems to suggest that such condition should be difficult to verify when studying specific systems of equations. As it will be shown later on, this kind of condition arises naturally in some systems and is easier to check.

(F5)* There exists $r_0 > 0$ such that all the trajectories for $\widehat{\tau}$ with a Lipschitz continuous initial datum within $B_{\widehat{r}_0}$ are relatively compact for the product metric topology and uniformly stable for the order $\leq_A$ in bounded sets, where

$$\widehat{r}_0 = \|A^{-1}\|(\sup\{\|F(\omega,x)\| : (\omega,x) \in \widehat{D}^{-1}(\Omega \times B_{r_0})\} + \|A\|r_0).$$

(F6)* If (ω,x), $(\omega,y) \in \Omega \times BU$ admit a backward orbit extension for the semiflow τ, $(\omega,x) \leq_{D,A} (\omega,y)$, and there exists $J \subset \{1,\ldots,m\}$ such that

$$\widehat{D}_2(\omega,x)_i = \widehat{D}_2(\omega,y)_i \text{ for all } i \notin J \text{ and}$$

$$\widehat{D}_2(\omega,x)_i(s) < \widehat{D}_2(\omega,y)_i(s) \text{ for all } i \in J \text{ and all } s \leq 0,$$

then $F_i(\omega,y) - F_i(\omega,x) - [A(D(\omega,y) - D(\omega,x))]_i > 0$ for all $i \in J$.

The next result is an immediate consequence of these two properties, and so we give it without a proof.

Proposition 7.4. *Under hypotheses* (F5)* *and* (F6)*, *the following assertions hold:*

(i) *there exists $r_0 > 0$ such that all the trajectories for $\widehat{\tau}$ with a Lipschitz continuous initial datum within $B_{\widehat{r}_0}$ are relatively compact for the product metric topology and uniformly stable for the order $\leq_A$ in bounded sets, where*

$$\widehat{r}_0 = \|A^{-1}\|(\sup\{\|G(\omega,x)\| : (\omega,x) \in \Omega \times B_{r_0}\} + \|A\|r_0)\,; \tag{7.2}$$

(ii) *if (ω,x), $(\omega,y) \in \Omega \times BU$ admit a backward orbit extension for the semiflow $\widehat{\tau}$, $x \leq_A y$ and there exists $J \subset \{1,\ldots,m\}$ such that*

$$x_i = y_i \textit{ for all } i \notin J \textit{ and}$$

$$x_i(s) < y_i(s) \textit{ for all } i \in J \textit{ and all } s \leq 0,$$

then $G_i(\omega,y) - G_i(\omega,x) - [A(y(0) - x(0))]_i > 0$ for all $i \in J$.

Relation (7.2) is an improved version of formula (5.1) in [NOV]. Following that paper, we come now to the main result of this chapter, which establishes the 1-covering property of omega-limit sets.

Theorem 7.5. *Assume that conditions* (D1)–(D3) *are satisfied, and that D is stable; furthermore, assume conditions* (F1)–(F3) *and* (F4)*–(F6)*. *Fix $(\omega_0, x_0) \in \widehat{D}^{-1}(\Omega \times B_{r_0})$ such that $\{\widehat{\tau}(t, \widehat{D}(\omega_0, x_0)) : t \geq 0\}$ is relatively compact for the product metric topology and uniformly stable for $\leq_A$ in bounded sets, and such that $K = \mathcal{O}(\omega_0, x_0) \subset \widehat{D}^{-1}(\Omega \times B_{r_0})$. If $\leq_A = \leq_{A,\infty}$, then we will further assume that $\widehat{D}_2(\omega_0, x_0)$ is Lipschitz continuous. Then $K = \{(\omega, c(\omega)) : \omega \in \Omega\}$, and*

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0,$$

where $c : \Omega \to BU$ is a continuous equilibrium, i.e. $c(\omega{\cdot}t) = u(t, \omega, c(\omega))$ for all $t \geq 0$ and all $\omega \in \Omega$, considering the compact-open topology on BU.

Proof. As seen in Theorem 5.6 in [NOV], from Propositions 4.2, 7.1, and 7.4, it follows that $\mathcal{O}(\widehat{D}(\omega_0, x_0)) = \{(\omega, \widehat{c}(\omega)) : \omega \in \Omega\}$, and

$$\lim_{t\to\infty} \mathsf{d}(\widehat{u}(t, \omega_0, x_0), \widehat{c}(\omega_0{\cdot}t)) = 0,$$

where $\widehat{c} : \Omega \to BU$ is a continuous equilibrium considering the compact-open topology on BU. Notice that

$$\begin{aligned} K =& \widehat{D}^{-1}(\mathcal{O}(\widehat{D}(\omega_0, x_0))) = \widehat{D}^{-1}(\{(\omega, \widehat{c}(\omega)) : \omega \in \Omega\}) \\ =& \{(\omega, (\widehat{D}^{-1})_2(\omega, \widehat{c}(\omega))) : \omega \in \Omega\}. \end{aligned}$$

Let us define $c : \Omega \to BU, \omega \mapsto (\widehat{D}^{-1})_2(\omega, \widehat{c}(\omega))$. The continuity of c, when we consider the compact-open topology on BU, is a consequence of Theorem 3.3 and the fact that $\mathcal{O}(\widehat{D}(\omega_0, x_0)) \subset \Omega \times B_{r_0}$. Moreover, c is an equilibrium, for its graph defines an omega-limit set. Finally, from Theorem 3.3 again and the boundedness of the trajectory and of $\widehat{c}(\Omega)$, we conclude that

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0,$$

and the proof is finished. □

Chapter 8

Applications of the transformed exponential order

Let (Ω, d) be a compact metric space, and let $\mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t$ be a minimal real flow on Ω. In this chapter, we apply the results given in Chapter 7 to the study of compartmental models, as presented in Chapter 5. However, in this chapter, we will assume that the compartment C_i produces or swallows material itself at a rate given by some regular Borel measures $\nu_{ij}(\omega)$, $\omega \in \Omega$, and some functions $b_{ij} : \Omega \to \mathbb{R}$, $j \in \{1, \dots, m\}$, which yields a slightly more general model than the one presented in Chapter 5.

This way, we obtain a model governed by the following system of NFDEs:

$$\begin{aligned}\frac{d}{dt}\sum_{j=1}^{m}\left[b_{ij}(\omega{\cdot}t)z_j(t) - \int_{-\infty}^{0} z_j(t+s)\,d\nu_{ij}(\omega{\cdot}t)(s)\right] = \\ = - g_{0i}(\omega{\cdot}t, z_i(t)) - \sum_{j=1}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) \qquad\qquad (8.1)_\omega \\ + \sum_{j=1}^{m}\int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), z_j(t+s))\,d\mu_{ij}(s) + I_i(\omega{\cdot}t),\end{aligned}$$

for $t \geq 0$, $i = 1, \dots, m$ and $\omega \in \Omega$, where $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$, $I_i : \Omega \to \mathbb{R}$, and μ_{ij} are positive regular Borel measures with finite total variation. For the sake of simplicity, let us denote $g_{i0} : \Omega \times \mathbb{R} \to \mathbb{R}$, $(\omega, v) \mapsto I_i(\omega)$, and let $g = (g_{ij})_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}^{m^2+2m}$. We denote by $B(\omega)$ and $\nu(\omega)$ the matrices $[b_{ij}(\omega)]_{ij}$ and $[\nu_{ij}(\omega)]_{ij}$, $\omega \in \Omega$, respectively.

Let $F : \Omega \times BU \to \mathbb{R}^m$ be the map defined for $(\omega, x) \in \Omega \times BU$ by

$$F_i(\omega, x) = -\sum_{j=0}^{m} g_{ji}(\omega, x_i(0)) + \sum_{j=1}^{m}\int_{-\infty}^{0} g_{ij}(\omega{\cdot}s, x_j(s))\,d\mu_{ij}(s) + I_i(\omega),$$

for each $i \in \{1,\dots,m\}$, and let $D : \Omega \times BU \to \mathbb{R}^m$ be the map defined for $(\omega,x) \in \Omega \times BU$ by

$$\begin{aligned} D(\omega,x) = & \left(\sum_{j=1}^{m} \left[b_{ij}(\omega)\, x_j(0) - \int_{-\infty}^{0} x_j \, d\nu_{ij}(\omega) \right] \right)_{i=1}^{m} \\ = & B(\omega)\, x(0) - \int_{-\infty}^{0} [d\nu(\omega)] x. \end{aligned}$$

With this notation, the family of equations $(8.1)_\omega$ can be written as

$$\frac{d}{dt} D(\omega{\cdot}t, z_t) = F(\omega{\cdot}t, z_t), \quad t \geq 0,\, \omega \in \Omega. \tag{8.2}_\omega$$

We will assume hypotheses (L1) and (L2) together with (C1) and (C2). We will deduce that at least the trajectories transformed by the operator $\widehat{D}$ with Lipschitz continuous initial data are uniformly stable for $\leq_A$ in bounded sets.

Thanks to (C1) and (C2), Proposition 5.2 holds here. We will assume now that (F4)* is satisfied. Some sufficient conditions for (F4)* to hold will be studied below.

Consider the *total mass* of the system $(8.1)_\omega$, $M : \Omega \times BU \to \mathbb{R}$, defined as in (5.4). Again, M is well defined, and Proposition 5.5 holds.

Lemma 8.1. *Fix (ω,x), $(\omega,y) \in \Omega \times BU$ with $(\omega,x) \leq_{D,A} (\omega,y)$. Then*

$$0 \leq D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x)) \leq M(\omega,y) - M(\omega,x)$$

for each $i = 1,\dots,m$ and whenever $z(t,\omega,x)$ and $z(t,\omega,y)$ are defined.

Proof. It follows from (F4)* and Theorem 7.2 that the skew-product semiflow induced by $(8.1)_\omega$ is monotone. The rest of the proof is analogous to that of Lemma 5.6. □

Proposition 8.2. *Fix $r > 0$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that, if (ω,x), $(\omega,y) \in \Omega \times B_r$ satisfy $\mathsf{d}(x,y) < \delta$ and $x \leq_A y$, then it holds that $\|\widehat{z}(t,\omega,x) - \widehat{z}(t,\omega,y)\| \leq \varepsilon$ whenever they are defined.*

Proof. Let $r_1 = r \sup_{\omega\in\Omega} \|(\widehat{D}^{-1})_2(\omega,\cdot)\|$. Fix $\varepsilon > 0$; it follows from Proposition 5.5 that there is a $\delta_1 > 0$ such that, if (ω,x), $(\omega,y) \in \Omega \times B_{r_1}$ with $\mathsf{d}(x,y) < \delta_1$, then $|M(\omega,y) - M(\omega,x)| \leq \varepsilon$. Now, thanks to Theorem 3.7, there is a $\delta > 0$ such that, if (ω,x), $(\omega,y) \in \Omega \times B_r$ with $\mathsf{d}(x,y) < \delta$, then

$$\mathsf{d}((\widehat{D}^{-1})_2(\omega,x), (\widehat{D}^{-1})_2(\omega,y)) < \delta_1.$$

Altogether, using Lemma 8.1, if (ω, x), $(\omega, y) \in \Omega \times B_r$ with $\mathsf{d}(x, y) < \delta$ and $x \leq_A y$, then

$$\begin{aligned} 0 \leq & D_i(\tau(t, \widehat{D}^{-1}(\omega, y))) - D_i(\tau(t, \widehat{D}^{-1}(\omega, x))) \\ \leq & M(\widehat{D}^{-1}(\omega, y)) - M(\widehat{D}^{-1}(\omega, x)) \leq \varepsilon, \end{aligned}$$

whence

$$0 \leq \widehat{z}_i(t, \omega, y) - \widehat{z}_i(t, \omega, x) \leq \varepsilon$$

for all $t \geq 0$ where they are defined and all $i \in \{1, \ldots, m\}$. The result is proved. □

In the following result, the boundedness and uniform stability with respect to the order $\leq_A$ of the trajectories for the semiflow $\widehat{\tau}$ are deduced, whenever there is some bounded solution.

Proposition 8.3. *Let us assume hypotheses* (C1), (C2), (L1), (L2), *and* (F4)*. *Suppose that there exists* $(\omega_0, x_0) \in \Omega \times BU$ *such that* $\widehat{\tau}(\cdot, \omega_0, y_0)$ *is bounded, where* $y_0 = \widehat{D}_2(\omega_0, x_0)$. *The following statements hold:*

(i) *when the order* $\leq_A$ *associated to* $\leq_{A,\rho}$ *is considered, then we have that, for all* $(\omega, x) \in \Omega \times BU$, $\widehat{\tau}(\cdot, \omega, y)$ *is bounded and uniformly stable for* $\leq_A$ *in bounded subsets, where* $y = \widehat{D}_2(\omega, x)$;

(ii) *when the order* $\leq_A$ *associated to* $\leq_{A,\infty}$ *is considered, if* y_0 *is Lipschitz continuous, then, for all* $(\omega, x) \in \Omega \times BU$ *such that* $y = \widehat{D}_2(\omega, x)$ *is Lipschitz continuous, it holds that* $\widehat{\tau}(\cdot, \omega, y)$ *is bounded and uniformly stable for* $\leq_A$ *in bounded subsets.*

Proof. To prove (i), let us define $\widetilde{y} \in BU$ as the solution of

$$\begin{cases} y'(t) = A\, y(t) + \mathbf{1}, & t \in [-\rho, 0], \\ y_{-\rho} \equiv \mathbf{1}, \end{cases} \tag{8.3}$$

where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^m$. Since the matrix A is quasipositive, the system of ordinary differential equations (8.3) is cooperative. The standard comparison theory for its solutions allows us to conclude that there exists $k_0 > 0$ such that $\widetilde{y}(t) \geq k_0\, \mathbf{1}$ for all $t \leq 0$ and $\widetilde{y} \geq_{A,\rho} 0$. Let us fix $z \in BU$, and suppose that z is Lipschitz continuous on $[-\rho, 0]$; let us check that there is a $\lambda_0 > 0$ such that, for all $\lambda \geq \lambda_0$, $-\lambda\, \widetilde{y} \leq_{A,\rho} z \leq_{A,\rho} \lambda\, \widetilde{y}$. Since $\widetilde{y}(t) \geq k_0\, \mathbf{1}$ for all $t \leq 0$, it is clear that there is a $\lambda_1 > 0$ such that $-\lambda\, \widetilde{y} \leq z \leq \lambda\, \widetilde{y}$ for all $\lambda \geq \lambda_1$. In addition, there is a $\lambda_0 \geq \lambda_1$ such that $(\lambda\, \widetilde{y} - z)' - A(\lambda\, \widetilde{y} - z) = \lambda\, \mathbf{1} - (z' - A\, z)$ is greater or equal to 0 a.e. in $[-\rho, 0]$ for all $\lambda \geq \lambda_0$, and we are done.

Now, let K be the omega-limit set of (ω_0, y_0) for the semiflow $\widehat{\tau}$, and let $r_1 > 0$ be such that $K \subset \Omega \times B_{r_1}$. We will prove that there is a $\lambda_2 > 0$ such that $\widehat{z}(\cdot, \omega, \lambda\,\widetilde{y})$ and $\widehat{z}(\cdot, \omega, -\lambda\,\widetilde{y})$ are bounded for all $\omega \in \Omega$ and all $\lambda \geq \lambda_2$. In order to do this, fix $\omega \in \Omega$ and $z \in K_\omega$. We know that there is a $\lambda_2 > 0$ (irrespective of ω) such that, for all $\lambda \geq \lambda_2$, $-\lambda\,\widetilde{y} \leq_{A,\rho} z \leq_{A,\rho} \lambda\,\widetilde{y}$. For each $s \in [0,1]$, let $y_s = (1-s)z + s\,\lambda\,\widetilde{y}$; clearly, $y_s \leq_{A,\rho} y_t$ for all $0 \leq s \leq t \leq 1$. Besides, there exists $r > 0$ such that $\{y_s\}_{s\in[0,1]} \subset B_r$. An application of Proposition 8.2 for $\varepsilon = 1$, implies that there are a $\delta > 0$ and a partition $0 = s_0 \leq s_1 \leq \cdots \leq s_n = 1$ of $[0,1]$ such that $\mathsf{d}(y_{s_j}, y_{s_{j+1}}) < \delta$ for all $j \in \{1, \ldots, n-1\}$, and therefore $\|\widehat{z}(t,\omega,y_{s_j}) - \widehat{z}(t,\omega,y_{s_{j+1}})\| \leq 1$ for all $t \geq 0$ where they are defined. Consequently, for each $j \in \{0, \ldots, n\}$, the solution $\widehat{z}(\cdot,\omega,y_{s_j})$ is globally defined, and $\|\widehat{z}(t,\omega,z) - \widehat{z}(t,\omega,\lambda\,\widetilde{y})\| \leq n$ for all $t \geq 0$, which implies that $\widehat{z}(\cdot,\omega,\lambda\,\widetilde{y})$ is bounded. Analogously, $\widehat{z}(\cdot,\omega,-\lambda\,\widetilde{y})$ is bounded as well.
Finally, let $(\omega, x) \in \Omega \times BU$ and $y = \widehat{D}_2(\omega, x)$. Let $z = \widehat{u}(\rho,\omega,y)$; since z is Lipschitz continuous on $[-\rho, 0]$, we have that there exists $\lambda \geq \lambda_2$ such that $-\lambda\,\widetilde{y} \leq_{A,\rho} z \leq_{A,\rho} \lambda\,\widetilde{y}$, which implies that $\widehat{z}(\cdot,\omega{\cdot}\rho,z)$ is bounded thanks to the fact that $\widehat{z}(\cdot,\omega,\lambda\,\widetilde{y})$ and $\widehat{z}(\cdot,\omega,-\lambda\,\widetilde{y})$ are bounded and to the monotonicity of $\widehat{\tau}$. Consequently, the trajectory through (ω, y) for $\widehat{\tau}$ is bounded. The remainder of the proof follows from Proposition 8.2.

Now we deal with statement (ii). Notice that the fact that all the eigenvalues of A have a negative real part implies that A is a hyperbolic matrix. Let us consider the cooperative system of ordinary differential equations $y' = A\,y + \mathbf{1}$. It is well known that there exists a unique solution of the aforementioned system which is bounded and exponentially stable when $t \to \infty$, namely $\widetilde{y} \equiv -A^{-1}\mathbf{1}$. Since 0 is a strong sub-equilibrium of the system, as seen in Novo, Núñez, and Obaya [NNO], there exists $k_0 > 0$ such that $\widetilde{y} \geq k_0\,\mathbf{1}$. Denote again by $\widetilde{y}$ its restriction to $(-\infty, 0]$. The rest of the proof is analogous to that of statement (i). □

This proposition proves condition (F5)* stated in Chapter 7. The next result is a direct consequence of Theorem 7.5 and establishes the 1-covering property of omega-limit sets in this setting.

Theorem 8.4. *Under the hypotheses of* Proposition 8.3*, it holds that, for all* $(\omega, x) \in \Omega \times BU$ *satisfying the conditions of statements* (i) *or* (ii) *respectively and provided that* (F6)* *is also satisfied, then* $\mathcal{O}(\omega, x)$ *is a copy of the base.*

Some sufficient conditions under which (F6)* holds will be given next together with those guaranteeing (F4)*. In order to give such conditions, let us focus on the study of the family of compartmental systems with finite delay and diagonal D-operator $(5.7)_\omega$. As in Chapter 5, we assume that conditions (G1) and (G2) are satisfied.

Notice that, again, the family of equations $(5.7)_\omega$ corresponds to a closed compartmental system. Consequently, the total mass of the system is invariant along the trajectories, and 0 is a constant bounded solution of all the equations of the family. Besides, as in Chapter 5, Proposition 5.10 follows from hypotheses (G1) and (G2).

Next, we analyze some situations where all the previous theory can be applied. They are chosen to describe different types of conditions which assure the monotonicity of the semiflow for some transformed exponential order leading to different dynamical implications. In the next statement, we take $\rho_{ii} = 2\,\alpha_i$, $i = 1, \dots, m$. Note that condition (G4.2), independently of the matrix A, is always required. The conclusion of the theorem also guarantees that all the trajectories are relatively compact.

For each i, $j \in \{1, \dots, m\}$, consider l_{ii}^-, l_{ij}^+, and L_i^+ defined as in (5.8). Again, we will assume that $L_i^+(\omega) < \infty$ for all $\omega \in \Omega$. For each $i \in \{1, \dots, m\}$, let $c_i^{[0]}(\omega) = 1$, and, for each $n \in \mathbb{N}$, define

$$c_i^{[n]}(\omega) = \prod_{j=0}^{n-1} c_i(\omega{\cdot}(-j\alpha_i)),\; \omega \in \Omega.$$

Theorem 8.5. *Let us assume hypotheses* (G1)–(G2) *together with*

(G4) *for each $i \in \{1, \dots, m\}$, if $c_i \not\equiv 0$, then $\rho_{ii} = 2\,\alpha_i$, and there exists $a_i \in (-\infty, 0]$ such that, for all $\omega \in \Omega$, the following conditions hold:*

(G4.1) $(-a_i - L_i^+(\omega))e^{a_i\alpha_i} - L_i^+(\omega)c_i(\omega) \geq 0,$

(G4.2) $l_{ii}^-(\omega{\cdot}(-\rho_{ii})) - L_i^+(\omega)c_i^{[2]}(\omega) \geq 0,$

where at least one of the inequalities is strict. Then all the trajectories of the family $(5.7)_\omega$ *are bounded, and their omega-limit sets are copies of the base.*

Proof. Proposition 5.10 guarantees that conditions (C1), (C2), (L1), and (L2) are satisfied. For each $i \in \{1, \dots, m\}$ with $c_i \equiv 0$, let $a_i = -\sup_{\omega\in\Omega} L_i^+(\omega) - 1$. Let A be the $m \times m$ diagonal matrix with diagonal elements $a_1, \dots, a_m$. We consider the order $\leq_{D,A}$ associated to $\leq_{A,\rho}$ for $\rho = \max\{\rho_{11}, \dots, \rho_{mm}\}$. Let us check that the family of equations $(5.7)_\omega$ satisfies conditions (F4)* and (F6)*. First, let us focus on condition (F4)*. If $c_i \equiv 0$, then

$$[AD(\omega, z)]_i = a_i z_i(0) \leq -L_i^+(\omega) z_i(0) \leq F_i(\omega, y) - F_i(\omega, x),$$

and we are done. Let us suppose that $c_i \not\equiv 0$. Fix $i \in \{1, \dots, m\}$. Let (ω, x), $(\omega, y) \in \Omega \times BU$ with $(\omega, x) \le_{D,A} (\omega, y)$, and denote $z = y - x$. Then we have that $D_i(\omega, z) \ge e^{a_i \alpha_i} D_i(\omega{\cdot}(-\alpha_i), z_{-\alpha_i})$, whence

$$z_i(0) - c_i(\omega) z_i(-\alpha_i) \ge e^{a_i\alpha_i}(z_i(-\alpha_i) - c_i(\omega{\cdot}(-\alpha_i)) z_i(-\rho_{ii})). \tag{8.4}$$

From Theorem 3.7, it follows that $z \ge 0$. Thus

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) &\ge -L_i^+(\omega) z_i(0) + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) z_i(-\rho_{ii}) \\ &\quad + \sum_{j \ne i} l_{ij}^-(\omega{\cdot}(-\rho_{ij})) z_j(-\rho_{ij}) \ge -L_i^+(\omega) z_i(0) + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) z_i(-\rho_{ii}). \end{aligned}$$

Note that (G4.1) implies that $-a_i - L_i^+(\omega) \ge 0$ for all $\omega \in \Omega$. Then, from (8.4), it follows that

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i &\ge (-L_i^+(\omega) - a_i) z_i(0) + a_i c_i(w) z_i(-\alpha_i) \\ &\quad + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) z_i(-\rho_{ii}) \ge [(-L_i^+(\omega) - a_i) e^{a_i\alpha_i} - L_i^+(\omega) c_i(\omega)] z_i(-\alpha_i) \\ &\quad + [l_{ii}^-(\omega{\cdot}(-\rho_{ii})) + (L_i^+(\omega) + a_i) c_i(\omega{\cdot}(-\alpha_i)) e^{a_i\alpha_i}] z_i(-\rho_{ii}). \end{aligned}$$

On the other hand, we have that $D_i(\omega{\cdot}(-\alpha_i), z_{-\alpha_i}) \ge 0$, which in turn implies that $z_i(-\alpha_i) - c_i(\omega{\cdot}(-\alpha_i)) z_i(-\rho_{ii}) \ge 0$. Consequently, thanks to (G4.1) and (G4.2),

$$\begin{aligned} &F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i \ge \\ &\quad \ge [l_{ii}^-(\omega{\cdot}(-\rho_{ii})) - L_i^+(\omega)\, c_i(\omega)\, c_i(\omega{\cdot}(-\alpha_i))] z_i(-\rho_{ii}) \ge 0. \end{aligned} \tag{8.5}$$

As a result, (G4.2) is a sufficient condition for (F4)* to hold.

As for (F6)*, if we had that $D_i(\omega{\cdot}s, x_s) < D_i(\omega{\cdot}s, y_s)$ for all $s \le 0$, then $z_i(s) > c_i(\omega{\cdot}s) z_i(s - \alpha_i) \ge 0$, $s \le 0$. It is clear that, if condition (G4.1) is strict, then the first inequality in (8.5) is strict; on the other hand, if condition (G4.2) is strict, then the second inequality in (8.5) is strict. This way, the fact that at least one of the inequalities in (G4) is strict together with the choice of a_i when $c_i \equiv 0$ and an argument similar to the foregoing one yield (F6)*, as expected. The remainder of the theorem follows from Theorem 8.4. □

Let us focus on a different approach to the study of family $(5.7)_\omega$; we will give another valid monotonicity condition for the transformed exponential order defined on the whole interval $(-\infty, 0]$. For each $i \in \{1, \dots, m\}$, each $\omega \in \Omega$ and each $a \in (-\infty, 0]$, we consider $q_{i0}(\omega, a) = -L_i^+(\omega) - a$ and

$$p_{in}(\omega, a) = -L_i^+(\omega)\, c_i^{[n]}(\omega) + e^{a(\alpha_i - \rho_{ii})} l_{ii}^-(\omega{\cdot}(-\rho_{ii}))\, c_i^{[n-1]}(\omega{\cdot}(-\rho_{ii})),$$

$$q_{in}(\omega, a) = q_{in-1}(\omega, a)\, e^{a\alpha_i} + p_{in}(\omega, a),\ n \in \mathbb{N}.$$

In the conditions of the next statement, an infinite sequence of consecutive inequalities is required. If c is Lipschitz continuous, then the conclusions hold when dealing with Lipschitz continuous initial data.

Theorem 8.6. *Assume hypotheses* (G1)–(G2) *together with the following one:*

(G5) *for each* $i \in \{1, \dots, m\}$*, if* $c_{ii} \not\equiv 0$*, then* $\rho_{ii} \le \alpha_i$ *and there exists* $a_i \in (-\infty, 0)$ *such that, for all* $\omega \in \Omega$*, there is an* $n_0(\omega) \in \mathbb{N} \cup \{0\}$ *such that*

$$q_{in}(\omega, a_i) \ge 0 \quad \textit{for all } n \in \{0, \dots, n_0(\omega) - 1\} \textit{ if } n_0(\omega) \ge 1,$$

$$q_{in_0(\omega)}(\omega, a_i) > 0 \quad \textit{and}$$
$$p_{in}(\omega, a_i) \ge 0 \quad \textit{for all } n > n_0(\omega).$$

Then, for all $(\omega, x) \in \Omega \times BU$ *such that* $\widehat{D}_2(\omega, x)$ *is Lipschitz continuous, the trajectory* $\{\tau(t, \omega, x) : t \ge 0\}$ *is bounded, and its omega-limit set is a copy of the base.*

Proof. First, Proposition 5.10 yields conditions (C1), (C2), (L1), and (L2). For each $i \in \{1, \dots, m\}$ such that $c_i \equiv 0$, let $a_i = -\sup_{\omega \in \Omega} L_i^+(\omega) - 1$. Let A be the $m \times m$ diagonal matrix with diagonal elements $a_1, \dots, a_m$, and consider the order $\le_{D,A}$ associated to $\le_{A,\infty}$. Let us check that the family of equations $(5.7)_\omega$ satisfies conditions (F4)* and (F6)*. In order to do so, let $(\omega, x), (\omega, y) \in \Omega \times BU$ with $(\omega, x) \le_{D,A} (\omega, y)$, and let $z = y - x$, $\widehat{z} = \widehat{D}_2(\omega, z)$. Fix $i \in \{1, \dots, m\}$. From Theorem 3.7, it follows that

$$z_i(s) = \sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}s)\widehat{z}_i(s - n\,\alpha_i).$$

Now, if $c_i \equiv 0$, then we have that

$$F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i \ge (-L_i^+(\omega) - a_i)\widehat{z}_i(0),$$

and (F4)* and (F6)* hold. Assume now that $c_i \not\equiv 0$; then we have that

$$\begin{aligned}
F_i(\omega, y) &- F_i(\omega, x) - [AD(\omega, z)]_i \ge \\
&\ge - L_i^+(\omega) z_i(0) + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) z_i(-\rho_{ii}) - a_i \widehat{z}_i(0) \\
&= - L_i^+(\omega) \sum_{n=0}^{\infty} c_i^{[n]}(\omega)\widehat{z}_i(-n\,\alpha_i) \\
&\quad + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) \sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}(-\rho_{ii}))\widehat{z}_i(-\rho_{ii} - n\,\alpha_i) - a_i \widehat{z}_i(0).
\end{aligned}$$

Notice that $\widehat{z} \ge_A 0$ and $\rho_{ii} \le \alpha_i$; hence, we have

$$\widehat{z}_i(-\rho_{ii} - n\,\alpha_i) \ge e^{a_i(\alpha_i - \rho_{ii})}\widehat{z}_i(-(n+1)\alpha_i),$$

and it follows that

$$\begin{aligned}
F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i \geq \\
&\geq (-L_i^+(\omega) - a_i)\widehat{z}_i(0) - L_i^+(\omega) \sum_{n=1}^{\infty} c_i^{[n]}(\omega)\widehat{z}_i(-n\,\alpha_i) \\
&\quad + e^{a_i(\alpha_i - \rho_{ii})} l_{ii}^-(\omega{\cdot}(-\rho_{ii})) \sum_{n=1}^{\infty} c_i^{[n-1]}(\omega{\cdot}(-\rho_{ii}))\widehat{z}_i(-n\,\alpha_i) \\
&= q_{i0}(\omega, a_i)\widehat{z}_i(0) + \sum_{n=1}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\,\alpha_i).
\end{aligned} \tag{8.6}$$

If $n_0(\omega) = 0$, we are done. Otherwise, notice that, for all $n \in \mathbb{N} \cup \{0\}$, $\widehat{z}_i(-n\,\alpha_i) \geq e^{a_i \alpha_i}\widehat{z}_i(-(n+1)\alpha_i)$. Hence,

$$\begin{aligned}
F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i \geq \\
&\geq (q_{i0}(\omega, a_i)e^{a_i\alpha_i} + p_{i1}(\omega, a_i))\widehat{z}_i(-\alpha_i) + \sum_{n=2}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\,\alpha_i) \\
&= q_{i1}(\omega, a_i)\widehat{z}_i(-\alpha_i) + \sum_{n=2}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\,\alpha_i) \\
&\geq \cdots \geq\ q_{in_0(\omega)}(\omega, a_i)\widehat{z}_i(-n_0(\omega)\alpha_i) + \sum_{n=n_0(\omega)+1}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\,\alpha_i).
\end{aligned}$$

This way, (F4)* and (F6)* follow easily from (G5). The desired result is a consequence of Theorem 8.4. □

As a consequence of this result, we obtain Theorem 5.11 again by using the transformed exponential ordering.

Proposition 8.7. *Assume conditions* (G1) *and* (G2). *Consider the family* $(5.7)_\omega$, *and assume that, for all* $i \in \{1, \ldots, m\}$, *if* $c_i \not\equiv 0$, *then* $\alpha_i = \rho_{ii}$, *and the following assertion holds for all* $\omega \in \Omega$:

(G3) $l_{ii}^-(\omega{\cdot}(-\rho_{ii})) - L_i^+(\omega)c_i(\omega) \geq 0.$

Then, for all $(\omega, x) \in \Omega \times BU$, *the trajectory* $\{\tau(t, \omega, x) : t \geq 0\}$ *is bounded, and its omega-limit set is a copy of the base.*

Proof. Let $\rho > 0$. For each $i \in \{1, \ldots, m\}$, let $a_i = -\sup_{\omega\in\Omega} L_i^+(\omega) - 1$, and let A be the $m \times m$ diagonal matrix with entries $a_1, \ldots, a_m$. Let us take the order $\leq_{D,A}$ associated to $\leq_{A,\rho}$. Let $n_0(\omega) = 0$; then $q_{i0}(\omega, a_i) > 0$ for all $\omega \in \Omega$, and

$$p_{in}(\omega, a) = c_i^{[n-1]}(\omega{\cdot}(-\alpha_i))(l_{ii}^-(\omega{\cdot}(-\rho_{ii})) - L_i^+(\omega)c_i(\omega)) \geq 0$$

for all $a \leq 0$ and all $n \in \mathbb{N}$. A proof analogous to that of Theorem 8.6 yields relation (8.6) again. Now, it is immediate from (G3) that condition (G5) holds, which implies the monotonicity of the semiflow. The remainder of the proof follows from Theorem 8.4. $\square$

Chapter 9

Direct exponential order and its applications

From now on, we are going to consider a different approach to the study of compartmental systems and their corresponding NFDEs. The aim of this chapter is to introduce and study the direct exponential order and its properties in order to apply the new conclusions to the family of compartmental systems $(5.7)_\omega$, leaving the proof of the main result for Chapter 10. This order was presented in [NOV] for the case of NFDEs with autonomous linear D-operator. In the present situation, the non-autonomous character of D raises new technical problems which we tackle in what follows. In addition, in this chapter we compare the conditions obtained by means of this order relation with those presented in Chapter 8.

Let us assume the setting and the notations used in Chapter 8 to study this family of equations. Specifically, we will assume hypothesis (G1).

First, we present a definition concerning the regularity properties of a mapping from Ω into $\mathbb{R}^m$. This will allow us to give some interesting conditions on the compartmental systems in order to obtain a monotone semiflow.

Definition 9.1. Let us consider a function $\mathfrak{c} : \Omega \to \mathbb{R}^m$.

(i) $\mathfrak{c}$ is said to be *Lipschitz continuous along the flow* if, for some $\omega \in \Omega$, the mapping $\mathbb{R} \to \mathbb{R}^m$, $t \mapsto \mathfrak{c}(\omega{\cdot}t)$ is Lipschitz continuous;

(ii) $\mathfrak{c}$ is *continuously differentiable along the flow* if, for all $\omega \in \Omega$, the mapping

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^m \\ \omega & \mapsto & \frac{d}{dt}\mathfrak{c}(\omega{\cdot}t)\Big|_{t=0} \end{array}$$

is well-defined and continuous. We will refer to this mapping as the derivative of $\mathfrak{c}$.

It is easy to check that, due to the density of all the trajectories within Ω, if $\mathfrak{c} : \Omega \to \mathbb{R}^m$ is Lipschitz continuous along the flow, then all the mappings $\mathbb{R} \to \mathbb{R}^m$, $t \mapsto \mathfrak{c}(\widetilde{\omega}{\cdot}t)$, $\widetilde{\omega} \in \Omega$, are Lipschitz continuous and have the same Lipschitz constant. From now on, $c : \Omega \to \mathbb{R}^m$ will denote the mapping $c = (c_i)_{i=1}^m : (\omega, x) \mapsto (c_i(\omega, x))_{i=1}^m$. It is noteworthy that, given $\omega \in \Omega$, if c is a Lipschitz continuous function, then $x \in BU$ is Lipschitz continuous if and only if $\widehat{D}_2(\omega, x)$ is Lipschitz continuous.

Proposition 9.2. *Assume that c is continuously differentiable along the flow. Suppose that $(\omega, x) \in \Omega \times BU$ admits a backward orbit extension, and that there is an $r_1 > 0$ such that $u(t, \omega, x) \in B_{r_1}$ for each $t \in \mathbb{R}$. Then the solution of $(5.7)_\omega$ with initial value x, $z = z(\cdot, \omega, x)$, belongs to $C^1(\mathbb{R}, \mathbb{R}^m)$.*

Proof. Let $\widehat{z} = \widehat{z}(\cdot, \widehat{D}(\omega, x))$. It is clear that $\widehat{z}$ is of class C^1, and it is bounded by $r_1 \sup_{\omega_1 \in \Omega} \|D(\omega_1, \cdot)\|$. Then, for all $t \in \mathbb{R}$ and all $i \in \{1, \ldots, m\}$, thanks to Theorem 3.7, it holds that

$$z_i(t) = \sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}t)\widehat{z}_i(t - n\,\alpha_i).$$

From (G2), it follows that this series converges uniformly on $\mathbb{R}$. Analogously, the formal derivative of the former series, namely

$$\sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}t)\widehat{z}_i{}'(t - n\,\alpha_i) + \sum_{n=0}^{\infty} \frac{d}{ds} c_i^{[n]}(\omega{\cdot}(t+s))\big|_{s=0}\,\widehat{z}_i(t - n\,\alpha_i), \tag{9.1}$$

converges uniformly on $\mathbb{R}$ thanks to (G2). Consequently, z_i is continuously differentiable on $\mathbb{R}$. □

Note that, in the conditions of the previous proposition, the derivative of z is given by (9.1).

We assume now the following hypothesis:

(G6) c is continuously differentiable along the flow, $\alpha_i > 0$, $\rho_{ij} \ge 0$, $0 \le c_i(\omega)$ for all $i, j \in \{1, \ldots, m\}$, and $\sum_{i=1}^m c_i(\omega) < 1$ for all $\omega \in \Omega$.

Notice that (G6) is significantly stronger than (G2), which was required for the transformed exponential order. Let $\gamma : \Omega \to \mathbb{R}^m$ denote the derivative of c. We will give an alternative condition which provides the monotonicity for the direct exponential order of the semiflow τ associated to the family $(5.7)_\omega$. The following result extends the conclusions in [KW] for the scalar periodic case to the m-dimensional system of recurrent NFDEs $(5.7)_\omega$. Note that we provide precise conditions which assure the monotonicity of the semiflow on $\Omega \times BU$.

Theorem 9.3. *Assume that conditions* (G1) *and* (G6) *hold, and, moreover, the following condition is satisfied:*

(G7) *for each $i \in \{1,\dots,m\}$, there exists $a_i \in (-\infty,0)$ such that, if A is the $m\times m$ diagonal matrix with diagonal elements $a_1,\dots,a_m$ and we consider the order $\le_A = \le_{A,\infty}$, then the inequality*

$$F_i(\omega,y)-F_i(\omega,x)-a_iD_i(\omega,y-x)+\gamma_i(\omega)(y_i(-\alpha_i)-x_i(-\alpha_i))\ge 0$$

holds for all (ω,x), $(\omega,y)\in\Omega\times BU$ with $x\le_A y$.

Fix (ω,x), $(\omega,y)\in\Omega\times BU$ such that $x\le_A y$. Then

$$u(t,\omega,x)\le_A u(t,\omega,y)$$

for all $t\ge 0$ where they are defined.

Proof. Proposition 5.10 guarantees that conditions (C1), (C2), (L1), and (L2) are satisfied. Fix (ω,x), $(\omega,y)\in\Omega\times BU$ such that $x\le_A y$ and $\rho>0$ such that $u(t,\omega,x)$, $u(t,\omega,y)$ are defined on $[0,\rho]$. Let $\varepsilon>0$, and denote by y^ε the solution of

$$\begin{cases} \frac{d}{dt}D(\omega{\cdot}t,z_t)=F(\omega{\cdot}t,z_t)+\varepsilon\mathbf{1}, & t\ge 0,\\ z_0=y, \end{cases}$$

where, $\mathbf{1}=(1,\dots,1)^T\in\mathbb{R}^m$. There exists $\varepsilon_0>0$ such that, for all $\varepsilon\in[0,\varepsilon_0)$, $z=z(\cdot,\omega,x)$ and y^ε are defined on $[0,\rho]$. Let $z^\varepsilon=y^\varepsilon-z$, and denote by t_1 the greatest element of $[0,\rho]$ such that $z^\varepsilon_{t_1}\ge_A 0$. Suppose that $t_1<\rho$. Since $z^\varepsilon_{t_1}\ge_A 0$, we have that

$$z_i^\varepsilon(t_1)\ge e^{a_i\alpha_i}z_i^\varepsilon(t_1-\alpha_i)$$

for all $i\in\{1,\dots,m\}$, and, from (G7), it follows that

$$\begin{aligned}\frac{d}{dt}&(z_i^\varepsilon(t)-c_i(\omega{\cdot}t)\,z_i^\varepsilon(t-\alpha_i))\big|_{t=t_1}-a_i(z_i^\varepsilon(t_1)-c_i(\omega{\cdot}t_1)\,z_i^\varepsilon(t_1-\alpha_i))\\ &+\gamma_i(\omega{\cdot}t_1)\,z_i^\varepsilon(t_1-\alpha_i)=F_i(\omega{\cdot}t_1,y^\varepsilon_{t_1})-F_i(\omega{\cdot}t_1,z_{t_1})\\ &-a_i(z_i^\varepsilon(t_1)-c_i(\omega{\cdot}t_1)\,z_i^\varepsilon(t_1-\alpha_i))+\gamma_i(\omega{\cdot}t_1)\,z_i^\varepsilon(t_1-\alpha_i)+\varepsilon\ge\varepsilon.\end{aligned}$$

Hence, taking $\alpha=\min\{\alpha_1,\dots,\alpha_m\}$, there exists $h\in(0,\alpha)$ such that, if $i\in\{1,\dots,m\}$ and $t\in[t_1,t_1+h]$, then

$$\frac{d}{dt}(z_i^\varepsilon(t)-c_i(\omega{\cdot}t)z_i^\varepsilon(t-\alpha_i))-a_i(z_i^\varepsilon(t)-c_i(\omega{\cdot}t)z_i^\varepsilon(t-\alpha_i))+\gamma_i(\omega{\cdot}t)z_i^\varepsilon(t-\alpha_i)\ge 0.$$

Let us fix $i \in \{1, \dots, m\}$; for $t_1 \le s \le t \le t_1 + h$, integrating between s and t, we have

$$\begin{aligned} z_i^\varepsilon(t) - c_i(\omega{\cdot}t) z_i^\varepsilon(t-\alpha_i) \ge\, & e^{a_i(t-s)}(z_i^\varepsilon(s) - c_i(\omega{\cdot}s) z_i^\varepsilon(s-\alpha_i)) \\ & - \int_s^t \gamma_i(\omega{\cdot}u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\, du. \end{aligned} \tag{9.2}$$

Fix $\eta > 0$; there exists an analytic function $\widetilde{c}_i : [t_1, t_1+h] \to \mathbb{R}$ (for instance, a polynomial) such that

$$0 \le \widetilde{c}_i(t) - c_i(\omega{\cdot}t) \le \eta \quad \text{and} \quad |\widetilde{c}_i\,'(t) - \gamma_i(\omega{\cdot}t)| \le \eta$$

for every $t \in [t_1, t_1+h]$. As a consequence, from (9.2), it follows that, for all $t_1 \le s \le t \le t_1 + h$,

$$\begin{aligned} & z_i^\varepsilon(t) - \widetilde{c}_i(t)\, z_i^\varepsilon(t-\alpha_i) \ge \\ & \quad \ge e^{a_i(t-s)}(z_i^\varepsilon(s) - \widetilde{c}_i(s)\, z_i^\varepsilon(s-\alpha_i)) - \int_s^t \widetilde{c}_i\,'(u)\, e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\, du \\ & \quad - \eta\, \|z_{t_1}^\varepsilon\|_\infty (1 + e^{a_i(t-s)} + (t-s)). \end{aligned} \tag{9.3}$$

Now, as $\widetilde{c}_i$ is analytic, there exist $s = s_0 \le s_1 \le s_2 \le \cdots \le s_J = t$ such that, for all $j \in \{1, \dots, J\}$, either $\widetilde{c}_i\,'(u) \ge 0$ or $\widetilde{c}_i\,'(u) < 0$ for all $u \in (s_{j-1}, s_j)$. Let $j \in \{0, \dots, J-1\}$, and fix $N \in \mathbb{N}$ such that $N \ge 3$; we define

$$s_{j0}^N = s_j, \quad s_{j1}^N = \left(1 - \frac{1}{N}\right) s_j + \frac{1}{N} s_{j+1},$$

$$s_{j2}^N = \frac{1}{N} s_j + \left(1 - \frac{1}{N}\right) s_{j+1} \quad \text{and} \quad s_{j3}^N = s_{j+1}.$$

Then we have that

$$\begin{aligned} \int_{s_j}^{s_{j+1}} \widetilde{c}_i\,'(u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\, du &= \sum_{l=1}^3 \int_{s_{j\,l-1}^N}^{s_{jl}^N} \widetilde{c}_i\,'(u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\, du \\ &= \sum_{l=1}^3 (\widetilde{c}_i(s_{jl}^N) - \widetilde{c}_i(s_{j\,l-1}^N)) e^{a_i(t-u_{jl}^N)} z_i^\varepsilon(u_{jl}^N - \alpha_i) \\ &= \widetilde{c}_i(s_{j+1}) e^{a_i(t-u_{j3}^N)} z_i^\varepsilon(u_{j3}^N - \alpha_i) - \widetilde{c}_i(s_j) e^{a_i(t-u_{j1}^N)} z_i^\varepsilon(u_{j1}^N - \alpha_i) \\ &\quad + \sum_{l=1}^2 \widetilde{c}_i(s_{jl}^N) \left(e^{a_i(t-u_{jl}^N)} z_i^\varepsilon(u_{jl}^N - \alpha_i) - e^{a_i(t-u_{j\,l+1}^N)} z_i^\varepsilon(u_{j\,l+1}^N - \alpha_i) \right) \\ &\le \widetilde{c}_i(s_{j+1}) e^{a_i(t-u_{j3}^N)} z_i^\varepsilon(u_{j3}^N - \alpha_i) - \widetilde{c}_i(s_j) e^{a_i(t-u_{j1}^N)} z_i^\varepsilon(u_{j1}^N - \alpha_i) \end{aligned}$$

where the points $u^N_{jl} \in [s^N_{j\,l-1}, s^N_{jl}]$ for $l = 1, 2, 3$. As a result, $u^N_{jl} - \alpha_i \le t_1$ for $l = 1, 2, 3$.

Taking limits when $N \to \infty$, we obtain

$$\int_{s_j}^{s_{j+1}} \widetilde{c}_i{}'(u)\, e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i) du \le$$
$$\le \widetilde{c}_i(s_{j+1}) e^{a_i(t-s_{j+1})}\, z_i^\varepsilon(s_{j+1}-\alpha_i) - \widetilde{c}_i(s_j)\, e^{a_i(t-s_j)} z_i^\varepsilon(s_j-\alpha_i).$$

Consequently,

$$\begin{aligned} &\int_s^t \widetilde{c}_i{}'(u)\, e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\, du \le \\ &\qquad \le \sum_{j=0}^{J-1} \int_{s_j}^{s_{j+1}} \widetilde{c}_i{}'(u)\, e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\, du \\ &\qquad \le \widetilde{c}_i(t)\, z_i^\varepsilon(t-\alpha_i) - \widetilde{c}_i(s)\, e^{a_i(t-s)} z_i^\varepsilon(s-\alpha_i), \end{aligned}$$

and, using (9.3), it yields

$$z_i^\varepsilon(t) - e^{a_i(t-s)} z_i^\varepsilon(s) \ge -\eta\, \|z^\varepsilon_{t_1}\|_\infty (1 + e^{a_i(t-s)} + (t-s)).$$

Letting $\eta \to 0$, we obtain $z_i^\varepsilon(t) - e^{a_i(t-s)} z_i^\varepsilon(s) \ge 0$ for all $i \in \{1, \ldots, m\}$. This is not possible due to the choice of t_1. Thus $z_t^\varepsilon \ge_A 0$ for all $t \in [0,\rho]$, and, taking limits as $\varepsilon \to 0$, $u(t,\omega,x) \le_A u(t,\omega,y)$ for all $t \in [0,\rho]$ and hence for all $t \ge 0$ where they are defined. $\square$

We clarify the hypotheses of Theorem 9.3 in some specific situations. For each $t \in \mathbb{R}$, let $\mathfrak{n}(t) = \min\{t, 0\}$. Next, we give a condition on the coefficients of the equation implying condition (G7).

Proposition 9.4. *Assume that conditions* (G1) *and* (G6) *together with the following one are satisfied:*

(G8) *for each $i \in \{1, \ldots, m\}$, there exists $a_i \in (-\infty, 0)$ such that, for all $\omega \in \Omega$, the following inequality holds:*

$$-L_i^+(\omega) - a_i + \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega)) e^{-a_i \alpha_i} \ge 0.$$

Let A be the $m \times m$ diagonal matrix with diagonal elements $a_1, \ldots, a_m$, and consider the order $\le_A = \le_{A,\infty}$. Then, given (ω, x), $(\omega, y) \in \Omega \times BU$ such that $x \le_A y$,

$$u(t,\omega,x) \le_A u(t,\omega,y)$$

for all $t \ge 0$ where they are defined.

Proof. It is clear that, if (ω, x), $(\omega, y) \in \Omega \times BU$ and $x \leq_A y$, then

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) - a_i(y_i(0) - x_i(0) - c_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i))) \\ + \gamma_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i)) \geq \\ \geq (-L_i^+(\omega) - a_i + \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega))e^{-a_i\alpha_i})(y_i(0) - x_i(0)). \end{aligned}$$

Consequently, property (G8) guarantees (G7), and Theorem 9.3 yields the expected result. □

Note that, if we take $\rho_{ii} = 2\,\alpha_i$ under the assumptions of Proposition 9.4, then condition (G4.2) is not required. However, if it holds, then (G4.1) is less restrictive than (G8) even in their autonomous versions.

Finally, we turn to the study of $(5.7)_\omega$ when another monotonicity condition is considered, which improves the conclusions of the previous statement.

Proposition 9.5. *Assume hypotheses* (G1) *and* (G6) *together with the following one:*

(G9) *for each* $i \in \{1, \dots, m\}$, *if* $c_i \not\equiv 0$, *then* $\rho_{ii} \leq \alpha_i$ *and there exists* $a_i \in (-\infty, 0)$ *such that, for all* $\omega \in \Omega$,

(G9.1) $-a_i - L_i^+(\omega) \geq 0$,

(G9.2) $e^{a_i\rho_{ii}}(-a_i - L_{ii}^+(\omega)) + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) + e^{a_i(\rho_{ii}-\alpha_i)}\mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega)) \geq 0.$

For each $i \in \{1, \dots, m\}$ *such that* $c_i \equiv 0$, *let* $a_i = -\sup_{\omega\in\Omega} L_i^+(\omega) - 1$. *Now, let* A *be the* $m \times m$ *diagonal matrix with diagonal elements* $a_1, \dots, a_m$, *and consider the order* $\leq_A = \leq_{A,\infty}$. *Then given* (ω, x), $(\omega, y) \in \Omega \times BU$ *such that* $x \leq_A y$,

$$u(t, \omega, x) \leq_A u(t, \omega, y)$$

for all $t \geq 0$ *where they are defined.*

Proof. First, Proposition 5.10 yields conditions (C1), (C2), (L1), and (L2). Let (ω, x), $(\omega, y) \in \Omega \times BU$ with $x \leq_A y$, and let $z = y - x$. Fix $i \in \{1, \dots, m\}$. If $c_i \equiv 0$, then

$$F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i + \gamma_i(\omega) z_i(-\alpha_i) \geq (-L_i^+(\omega) - a_i) z_i(0),$$

whence (G7) holds. Let us assume that $c_i \not\equiv 0$; in this case,

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i + \gamma_i(\omega) z_i(-\alpha_i) \geq -L_i^+(\omega) z_i(0) \\ + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) z_i(\omega{\cdot}(-\rho_{ii})) - a_i(z_i(0) - c_i(\omega) z_i(-\alpha_i)) + \gamma_i(\omega) z_i(-\alpha_i) \\ = (-a_i - L_i^+(\omega)) z_i(0) + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) z_i(-\rho_{ii}) + \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega)) z_i(-\alpha_i). \end{aligned}$$

Since $z \geq_A 0$ and $\rho_{ii} \leq \alpha_i$, we have that

$$z_i(-\rho_{ii}) \geq e^{a_i(\alpha_i - \rho_{ii})} z_i(-\alpha_i) \quad \text{and} \quad z_i(0) \geq e^{a_i \rho_{ii}} z_i(-\rho_{ii});$$

hence, from condition (G9.1) and (G9.2), it follows that

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i + \gamma_i(\omega) z_i(-\alpha_i) \geq \\ \geq [e^{a_i \rho_{ii}}(-a_i - L_i^+(\omega)) + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) \\ + e^{a_i(\rho_{ii} - \alpha_i)} \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega))] z_i(-\rho_{ii}) \geq 0. \end{aligned} \tag{9.4}$$

As a result, (G9.2) is a sufficient condition for (G7) to hold. The remainder of the proof follows from Theorem 9.3. □

Notice that, under the hypotheses of Proposition 9.5, only two supplementary conditions are required, instead of the infinite sequence needed in Theorem 8.6 to apply the transformed exponential order.

It is important to mention that, in the present situation, the conclusions in [NOV] remain valid. From now on, hypothesis (G7) and the following property will be assumed:

(G10) if (ω, x), $(\omega, y) \in \Omega \times BU$ admit a backward orbit extension, $x \leq_A y$ and there exists $J \subset \{1, \ldots, m\}$ such that $x_i = y_i$ for all $i \notin J$ and $x_i(s) < y_i(s)$ for all $i \in J$ and all $s \leq 0$, then we have

$$F_i(\omega, y) - F_i(\omega, x) - a_i D_i(\omega, y - x) + \gamma_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i)) > 0$$

for all $i \in J$.

As we will see in Chapter 10, condition (G10) is essential in the proof of the 1-covering property for the current setting, where it plays a role similar to the one played by hypothesis (F6)* in the proof of Theorem 7.5. Besides, let us remark that, regarding hypothesis (G10), we have results analogous to those regarding (G7) given in Propositions 9.4 and 9.5.

Proposition 9.6. *Let us assume that hypotheses* (G1) *and* (G6) *hold. Moreover, suppose that at least one of the following statements holds:*

(i) *hypothesis* (G8) *in its strict version is satisfied;*

(ii) *hypothesis* (G9) *is satisfied, and at least one of its inequalities is strict.*

Then property (G10) *holds as well.*

Proof. If statement (i) holds, then (G10) follows easily from the proof of Proposition 9.4. Assume now that statement (ii) is satisfied. Following the proof of Proposition 9.5, if we have that $D_i(\omega{\cdot}s, x_s) < D_i(\omega{\cdot}s, y_s)$ for all $s \leq 0$, then it is clear that $z_i(s) > c_i(\omega{\cdot}s)z_i(s-\alpha_i) \geq 0$, $s \leq 0$. This way, if condition (G9.1) is strict, then the first inequality in (9.4) is strict; on the other hand, if condition (G9.2) is strict, then the second inequality in (9.4) is strict. As a result, the fact that at least one of the inequalities in (G9) is strict together with an argument similar to the previous one yield (G10), as desired. □

In this setting, the 1-covering property of omega-limit sets holds. This result will be proved throughout Chapter 10.

Theorem 9.7. *Assume conditions* (G1), (G6), (G7), *and* (G10), *and let us consider the monotone skew-product semiflow* (4.2) *induced by the family* $(8.2)_\omega$. *Fix* $(\omega_0, x_0) \in \Omega \times BU$ *such that* x_0 *is Lipschitz continuous. Then* $K = \mathcal{O}(\omega_0, x_0) = \{(\omega, b(\omega)) : \omega \in \Omega\}$ *is a copy of the base, and*

$$\lim_{t\to\infty} \mathsf{d}(u(t,\omega_0,x_0), b(\omega_0{\cdot}t)) = 0\,,$$

where $b : \Omega \to BU$ *is a continuous equilibrium.*

According to the previous theory, we should remark that the application of the direct exponential order requires the differentiability along the flow of the vector function c, instead of just the continuity of this map, only needed by the transformed exponential order. Thus, the transformed exponential order becomes more natural in the study of NFDEs with non-autonomous linear D-operator.

Even in the periodic case, it is well known that, given an open subset $U \subset C(\Omega, \mathbb{R}^m)$, the subset of differentiable functions is dense, has an empty interior, and

$$\sup\{\|\mathfrak{c}'\|_\infty : \mathfrak{c} \in U \text{ and } \mathfrak{c} \text{ is differentiable along the flow}\} = \infty$$

(see Schwartzman [Sc]). As a consequence, the transformed exponential order is also more advantageous when dealing with rapidly oscillating differentiable coefficients c_i, $i \in \{1, \ldots, m\}$. In practice, the conditions which allow us to apply the direct exponential order or the transformed exponential order are often quite different; the particular problem to be studied will determine the advantages and disadvantages of each order.

Chapter 10

A topological theory for the direct exponential order. Proof of Theorem 9.7

The aim of this chapter is to give a rigorous proof of the 1-covering property in the setting presented in Chapter 9.

Let us turn to the study of the omega-limit sets of the relatively compact trajectories for the monotone skew-product semiflows introduced in the previous chapter. We assume the setting and notations of that chapter. Specifically, we assume hypotheses (G1), (G6), (G7), and (G10).

In this chapter, we will use the concept of uniform stability given in Definition 1.2 together with the one for semiorbits of a semiflow relative to the order relation $\leq_A$ with respect to subsets of the form $\Omega \times B_r$, $r > 0$, as presented in Definition 7.3. Besides, we will use the following definition of uniform stability for positively invariant subsets relative to $\leq_A$ with respect to bounded subsets.

Definition 10.1. Given $r > 0$, a positively τ-invariant set $K \subset \Omega \times BU$ is *uniformly stable for the order* $\leq_A$ *in* $\Omega \times B_r$ if for any $\varepsilon > 0$ there exists $\delta > 0$, called the *modulus of uniform stability*, such that, if $(\omega, x) \in K$, $(\omega, y) \in \Omega \times B_r$ are such that $\mathsf{d}(x, y) < \delta$ with $x \leq_A y$ or $y \leq_A x$, then $\mathsf{d}(u(t, \omega, x), u(t, \omega, y)) \leq \varepsilon$ for all $t \geq 0$. If this happens for each $r > 0$, K is said to be *uniformly stable for the order* $\leq_A$ *in bounded sets*.

Notice that Proposition 4.3 holds here, which allows us to define omega-limit sets as in Chapter 7. Moreover, Propositions 4.5 and 4.6 also remain valid, which implies that, for each $(\omega_0, x_0) \in \Omega \times BU$ giving rise to a bounded trajectory, the restriction of τ to $\mathcal{O}(\omega_0, x_0)$ is continuous when the compact-open topology is considered on BU, and $\mathcal{O}(\omega_0, x_0)$ admits a flow extension.

The following statement shows that the omega-limit set inherits and improves the stability properties of certain relatively compact trajectories. Let

$c^+ = \sup\{c_i(\omega) : \omega \in \Omega,\, i \in \{1,\ldots,m\}\}$ and $L_c = \sup\{\|\gamma(\omega)\| : \omega \in \Omega\}$.

Proposition 10.2. *Let $(\omega_0, x_0) \in \Omega \times BU$ be such that its forward orbit $\{\tau(t,\omega_0,x_0) : t \geq 0\}$ is relatively compact for the product metric topology and uniformly stable for $\leq_A$ in bounded sets. Let K denote the omega-limit set of (ω_0, x_0). Then we have that*

(i) *K is uniformly stable for $\leq_A$ in bounded sets;*

(ii) *$(K, \tau, \mathbb{R}^+)$ is uniformly stable.*

Proof. (i) Let $r_0 > 0$ be such that $\mathrm{cls}_{\Omega\times X}\{\tau(t,\omega_0,x_0) : t \geq 0\} \subset \Omega \times B_{r_0}$. Given $r > 0$, we check that K is uniformly stable for $\leq_A$ in $\Omega \times B_r$. Thus, we fix an $\varepsilon > 0$ and take $\delta > 0$ such that, if $s \geq 0$ and $\mathsf{d}(u(s,\omega_0,x_0), x) \leq \delta$ for certain $x \in B_{2r_0+r}$ with $x \leq_A u(s,\omega_0,x_0)$ or $u(s,\omega_0,x_0) \leq_A x$, then, for each $t \geq 0$,

$$\mathsf{d}(u(t+s,\omega_0,x_0), u(t,\omega_0{\cdot}s,x)) = \mathsf{d}(u(t,\omega_0{\cdot}s,u(s,\omega_0,x_0)), u(t,\omega_0{\cdot}s,x)) \leq \varepsilon,$$

and $\|z(t+s,\omega_0,x_0) - z(t,\omega_0{\cdot}s,x)\| \leq 1$. Let $(\omega,x) \in K$ and $(\omega,y) \in \Omega \times B_r$ with $\mathsf{d}(x,y) < \delta$ and $x \leq_A y$ or $y \leq_A x$, and take a sequence $\{t_n\}_n$ such that $\lim_{n\to\infty} \tau(t_n,\omega_0,x_0) = (\omega,x)$. Since it holds that

$$u(t_n,\omega_0,x_0) \leq_A u(t_n,\omega_0,x_0)+y-x \quad \text{or} \quad u(t_n,\omega_0,x_0)+y-x \leq_A u(t_n,\omega_0,x_0),$$

$$u(t_n,\omega_0,x_0) + y - x \in B_{2r_0+r} \quad \text{and}$$
$$\mathsf{d}(u(t_n,\omega_0,x_0), u(t_n,\omega_0,x_0) + y - x) < \delta,$$

we deduce that, for each $n \in \mathbb{N}$ and $t \geq 0$,

$$\mathsf{d}(u(t,\omega_0{\cdot}t_n,u(t_n,\omega_0,x_0)), u(t,\omega_0{\cdot}t_n,u(t_n,\omega_0,x_0)+y-x))) \leq \varepsilon$$

and also

$$\|z(t+t_n,\omega_0,x_0) - z(t,\omega_0{\cdot}t_n,u(t_n,\omega_0,x_0)+y-x)\| \leq 1.$$

Finally, from Lemma 4.4 together with the continuity of τ on the trajectory through (ω_0,x_0) and the fact that

$$\sup\{\|z(s,\omega_0{\cdot}t_n,u(t_n,\omega_0,x_0)+y-x)\| : s \in [0,t],\, n \geq 1\} \leq r_0 + 1$$

for each $t \geq 0$, as $n \to \infty$ we get $\mathsf{d}(u(t,\omega,x), u(t,\omega,y)) \leq \varepsilon$ for each $t \geq 0$, and (i) is proved.

(ii) First of all, let us check that there is a positive constant $L > 0$ such that x is Lipschitz continuous with constant L for all $(\omega,x) \in K$. From statement (ii) of Proposition 4.2, it follows that, for all $(\omega, y) \in \widehat{D}(K)$, y is Lipschitz continuous with constant

$$\widehat{L} = \sup\{\|G(\omega, y)\| : (\omega, y) \in \widehat{D}(\Omega \times B_{r_0})\}.$$

Let us define

$$L = \frac{\widehat{L}}{1 - c^+} + \frac{L_c}{(1 - c^+)^2}\,;$$

we claim that x is Lipschitz continuous with constant L for all $(\omega, x) \in K$. Clearly, (ω, x) admits a backward orbit extension because it is an element of K, and, hence, $x \in C^1((-\infty, 0], \mathbb{R}^m)$, and x' is given by (9.1) thanks to Proposition 9.2. Now it is obvious that x' is bounded by L, and the result follows.

Let us check that statement (ii) holds. Given (ω, x) and $(\omega, y) \in K$, Proposition 9.2 implies again that x, $y \in C^1((-\infty, 0], \mathbb{R}^m)$, and, since all the eigenvalues of A have a negative real part, we can define

$$\begin{aligned} a_{x,y} : (-\infty, 0] &\longrightarrow \mathbb{R}^m \\ s \quad &\mapsto \int_{-\infty}^{s} e^{A(s-\tau)} \inf\{x'(\tau) - A\,x(\tau), y'(\tau) - A\,y(\tau)\}\,d\tau. \end{aligned} \tag{10.1}$$

It is clear that $a_{x,y}$ is well defined and bounded due to the Lipschitz continuity of x and y. Moreover,

$$a'_{x,y}(s) = A\,a_{x,y}(s) + \inf\{x'(s) - A\,x(s), y'(s) - A\,y(s)\}, \quad s \in (-\infty, 0].$$

From this fact, we deduce that $a_{x,y} \in BU$; besides, it is easy to check that $a_{x,y} \leq_A x$ and $a_{x,y} \leq_A y$. Now, from the uniformity of the constant L over $K_X = \{x \in X :$ there exists $\omega \in \Omega$ such that $(\omega, x) \in K\}$ and (10.1), we deduce that there is an $r_1 > 0$ such that $a_{x,y} \in B_{r_1}$ for each $(\omega, x), (\omega, y) \in K$.

Fix $\varepsilon > 0$, and let $\delta_1 > 0$ be the modulus of uniform stability of K for $\leq_A$ in $\Omega \times B_{r_1}$ for $\varepsilon/2$. Let us check that there is a $\delta > 0$ such that $\mathsf{d}(a_{x,y}, x) \leq \delta_1$ and $\mathsf{d}(a_{x,y}, y) \leq \delta_1$ provided that $\mathsf{d}(x, y) \leq \delta$. Assume on the contrary that there are a $\delta_1 > 0$ and sequences $\{(\omega_n, x_n)\}_n$, $\{(\omega_n, y_n)\}_n \subset K$ such that $\lim_{n\to\infty} \mathsf{d}(x_n, y_n) = 0$ and $\mathsf{d}(a_{x_n,y_n}, x_n) > \delta_1$ for all $n \in \mathbb{N}$. However, $\|a_{x_n,y_n}(s) - x_n(s)\| \leq \|a_{x_n,y_n}(s) - b_{x_n,y_n}(s)\|$ with

$$b_{x_n,y_n}(s) = \int_{-\infty}^{s} e^{A(s-\tau)} \sup\{x'_n(\tau) - A\,x_n(\tau), y'_n(\tau) - A\,y_n(\tau)\}\,d\tau\,, \tag{10.2}$$

from which we deduce that

$$\begin{aligned} &\|a_{x_n,y_n}(s) - x_n(s)\| \leq \\ &\qquad \leq \int_{-\infty}^{s} \big\|e^{A(s-\tau)}\big\| (\|x'_n(\tau) - y'_n(\tau)\| + \|A\|\,\|x_n(\tau) - y_n(\tau)\|)\,d\tau \end{aligned} \tag{10.3}$$

for all $s \in (-\infty, 0]$. Moreover, let us fix $\rho > 0$ and $\xi > 0$, and check that the integral in (10.3) is bounded by ξ for all $s \in [-\rho, 0]$ and all sufficiently large $n \in \mathbb{N}$. Thanks to (G7), we can define $m_A = -\max\{a_1, \ldots, a_m\} > 0$; then $\|e^{At}\| \le e^{-m_A t}$ for all $t \ge 0$. There exists $s_0 \le -\rho$ such that

$$\int_{-\infty}^{s_0} e^{m_A(\rho+\tau)}(2\,\widehat{L} + 2\,\|A\|\,r_0)\,d\tau \le \frac{\xi}{2}\,.$$

On the other hand, let us prove that $\|x_n' - y_n'\|_{[s_0,0]} \to 0$ as $n \to \infty$. Let $\widehat{x}_n = \widehat{D}_2(\omega, x_n)$ and $\widehat{y}_n = \widehat{D}_2(\omega, y_n)$ for each $n \in \mathbb{N}$. Fix $\eta > 0$; there exists $l_0 \in \mathbb{N}$ such that

$$\sum_{l=l_0}^{\infty}(c^+)^l\,2\,\widehat{L} \le \frac{\eta}{4} \quad \text{and} \quad \sum_{l=l_0}^{\infty} l\,(c^+)^{l-1}\,2\,L_c \sup_{\omega_1\in\Omega}\|D(\omega_1,\cdot)\| \le \frac{\eta}{4}.$$

From relation (9.1), it is clear that it suffices to prove that, given $\rho_1 > 0$, $\|\widehat{x}_n - \widehat{y}_n\|_{[-\rho_1,0]}$ and $\|\widehat{x}_n{}' - \widehat{y}_n{}'\|_{[-\rho_1,0]}$ are bounded by $\eta/(4\,L_c\,l_0^2)$ and $\eta/(4\,l_0)$, respectively, but this holds due to the uniform continuity of $\widehat{D}$ on $\Omega \times B_{r_0}$ and the compactness of $\widehat{D}(K)$ for the product metric topology, together with statement (iii) of Proposition 4.2 and the fact that $\widehat{x}_n{}'(s) = G(\omega{\cdot}s, (\widehat{x}_n)_s)$ and $\widehat{y}_n{}'(s) = G(\omega{\cdot}s, (\widehat{y}_n)_s)$ for all $s \in [-\rho_1, 0]$ and all $n \in \mathbb{N}$. Consequently, there is an $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$, $\|x_n' - y_n'\|_{[s_0,0]} \le \eta$, whence $\|x_n' - y_n'\|_{[s_0,0]} \to 0$ as $n \to \infty$, as desired.

As a result, if $n_1 \in \mathbb{N}$ is such that, for all $n \ge n_1$,

$$(-s_0)(\|x_n' - y_n'\|_{[s_0,0]} + \|A\|\|x_n - y_n\|_{[s_0,0]}) \le \frac{\xi}{2}\,,$$

then we have that

$$\int_{s_0}^{s}(\|x_n'(\tau) - y_n'(\tau)\| + \|A\|\,\|x_n(\tau) - y_n(\tau)\|)\,d\tau \le \frac{\xi}{2}$$

for all $n \ge n_1$ and all $s \in [-\rho, 0]$.

Altogether, from (10.3) it follows that, for all $n \ge n_1$ and all $s \in [-\rho, 0]$,

$$\begin{aligned}\|a_{x_n,y_n}(s) - x_n(s)\| &\le \int_{-\infty}^{s_0} e^{m_A(\rho+\tau)}(2\widehat{L} + 2\|A\|\,r_0)\,d\tau \\ &\quad + \int_{s_0}^{s}(\|x_n'(\tau) - y_n'(\tau)\| + \|A\|\,\|x_n(\tau) - y_n(\tau)\|)\,d\tau \\ &\le \frac{\xi}{2} + \frac{\xi}{2} = \xi,\end{aligned}$$

which contradicts the fact that $\mathsf{d}(a_{x_n,y_n}, x_n) > \delta_1$ for all $n \in \mathbb{N}$. Therefore, there is a $\delta > 0$ such that $\mathsf{d}(a_{x,y}, x) \leq \delta_1$ and $\mathsf{d}(a_{x,y}, y) \leq \delta_1$ provided that $\mathsf{d}(x, y) \leq \delta$.

Consequently, whenever $\mathsf{d}(x, y) \leq \delta$, since $a_{x,y} \leq_A x$, $a_{x,y} \leq_A y$, and $a_{x,y} \in B_{r_1}$, the uniform stability of K for the order $\leq_A$ yields

$$\begin{aligned} &\mathsf{d}(u(t,\omega,x), u(t,\omega,y)) \leq \\ &\qquad \leq \mathsf{d}(u(t,\omega,x), u(t,\omega,a_{x,y})) + \mathsf{d}(u(t,\omega,a_{x,y}), u(t,\omega,y)) \\ &\qquad \leq \varepsilon, \end{aligned}$$

and, this way, $(K, \tau, \mathbb{R}^+)$ is uniformly stable. The proof of (ii) is complete. □

The next result shows that, under the previous assumptions, the omega-limit set K is a minimal subset provided that the initial datum has a certain regularity.

First, we recall the definition of the section map of a compact subset $M \subset \Omega \times X$. We introduce the projection set of M into the fiber space

$$M_X = \{x \in X : \text{ there exists } \omega \in \Omega \text{ such that } (\omega, x) \in M\} \subset X\,.$$

From the compactness of M, it is immediate to show that also M_X is a compact subset of X. Let $\mathcal{P}_c(M_X)$ denote the set of closed subsets of M_X, endowed with the Hausdorff metric ρ, that is, for any two closed sets C, $B \in \mathcal{P}_c(M_X)$,

$$\rho(C, B) = \sup\{\alpha(C, B),\ \alpha(B, C)\}\,,$$

where $\alpha(C, B) = \sup\{r(c, B) : c \in C\}$ and $r(c, B) = \inf\{\mathsf{d}(c, b) : b \in B\}$. Then, define the so-called *section map*:

$$\begin{array}{rcl} \Omega & \longrightarrow & \mathcal{P}_c(M_X) \\ \omega & \mapsto & M_\omega = \{x \in X : (\omega, x) \in M\}\,. \end{array} \tag{10.4}$$

Due to the minimality of Ω and the compactness of M, the set M_ω is nonempty for every $\omega \in \Omega$; besides, this map is trivially well-defined.

Lemma 10.3. *If $(\omega, x) \in \Omega \times BU$ and x is Lipschitz continuous, then $\widehat{D}_2(\omega, x)$ is Lipschitz continuous as well.*

Proof. Let $L > 0$ be the Lipschitz constant of x, and let t, $s \in (-\infty, 0]$; then, for all $i \in \{1, \ldots, m\}$,

$$\begin{aligned} |\widehat{D}_2(\omega,x)_i(t) - \widehat{D}_2(\omega,x)_i(s)| &= |D_i(\omega{\cdot}t, x_t) - D_i(\omega{\cdot}s, x_s)| \\ &\leq \|x(t) - x(s)\| + |(c_i(\omega{\cdot}t) - c_i(\omega{\cdot}s))\, x_i(t-\alpha_i)| \\ &\quad + |c_i(\omega{\cdot}s)\,(x_i(t-\alpha_i) - x_i(s-\alpha_i))| \leq (2\,L + L_c\,\|x\|_\infty)|t-s|, \end{aligned}$$

and the result follows. □

Theorem 10.4. *Let $(\omega_0, x_0) \in \Omega \times BU$ with x_0 Lipschitz continuous and forward orbit $\{\tau(t, \omega_0, x_0) : t \geq 0\}$ relatively compact for the product metric topology and uniformly stable for $\leq_A$ in bounded sets. Let K denote the omega-limit set of (ω_0, x_0). Then, K is a minimal subset.*

Proof. Let M be a minimal subset of K. We just need to show that $K \subset M$. To do this, let us take an element $(\omega, x) \in K$ and prove that $(\omega, x) \in M$. As M is in particular closed, it suffices to see that for any fixed $\varepsilon > 0$ there exists $(\omega, x^*) \in M$ such that $\mathsf{d}(x, x^*) \leq \varepsilon$.

First of all, there exists $s_n \uparrow \infty$ such that

$$\lim_{n\to\infty} (\omega_0{\cdot}s_n, u(s_n, \omega_0, x_0)) = (\omega, x).$$

Now, take a pair $(\omega, \widetilde{x}) \in M \subset K$. Then, there exists a sequence $t_n \uparrow \infty$ such that

$$(\omega, \widetilde{x}) = \lim_{n\to\infty} (\omega_0{\cdot}t_n, u(t_n, \omega_0, x_0))\,.$$

Since $(M, \tau, \mathbb{R}^+)$ is uniformly stable thanks to Proposition 10.2, Theorem 3.4 in [NOS] can be applied so that the section map (10.4) turns out to be continuous at all points. As $\omega_0{\cdot}t_n \to \omega$, we deduce that $M_{\omega_0\cdot t_n} \to M_\omega$ for the Hausdorff metric. Therefore, for each $\widetilde{x} \in M_\omega$, there exists $x_n \in M_{\omega_0\cdot t_n}$, $n \geq 1$, so that we have $x_n \to \widetilde{x}$ as $n \to \infty$. From Proposition 9.2 we deduce that $x_n \in C^1((-\infty, 0], \mathbb{R}^m)$ for each $n \in \mathbb{N}$, and, denoting $y_n = u(t_n, \omega_0, x_0)$, we have $\mathsf{d}(x_n, y_n) \to 0$ as $n \to \infty$.

Let us remark that, from relation (9.1) and the fact that M is minimal and included in $\Omega \times B_{r_0}$ together with statement (ii) of Proposition 4.2, it follows that there exists $L_x > 0$ such that x_n is Lipschitz continuous with Lipschitz constant L_x for all $n \in \mathbb{N}$.

Let us check that $y_n = u(t_n, \omega_0, x_0)$ is also Lipschitz continuous with Lipschitz constant L_y irrespective of n. Since x_0 is Lipschitz continuous, it is clear that y_n is Lipschitz continuous on $(-\infty, -t_n]$ with the same Lipschitz constant as x_0. Now, for all t, $s \in [-t_n, 0]$ and all $i \in \{1, \ldots, m\}$,

$$\begin{aligned}(y_n)_i(t) - (y_n)_i(s) = &\sum_{l=0}^{\infty} \Big(c_i^{[l]}(\omega_0{\cdot}(t + t_n)) - c_i^{[l]}(\omega_0{\cdot}(s + t_n))\Big)\, (\widehat{y}_n)_i(t - l\alpha_i) \\ &+ \sum_{l=0}^{\infty} c_i^{[l]}(\omega_0{\cdot}(s + t_n))\, ((\widehat{y}_n)_i(t - l\alpha_i) - (\widehat{y}_n)_i(s - l\alpha_i))\,,\end{aligned}$$

where $\widehat{y}_n = \widehat{D}_2(\omega_0{\cdot}t_n, y_n)$. First of all, it is easy to check that the first addend is bounded by

$$\sum_{l=0}^{\infty} l\, (c^+)^{l-1} L_c\, r_0 \sup_{\omega_1\in\Omega} \|D(\omega_1, \cdot)\|\, |t - s|,$$

where $r_0 > 0$ is a bound of $z(\cdot, \omega_0, x_0)$. As for the second addend, since $c_i^{[l]}$ is bounded by c^+ for all $l \in \mathbb{N}$ and all $i \in \{1, \ldots, m\}$, it suffices to prove that $\widehat{y}_n$ is Lipschitz continuous on $(-\infty, 0]$ with Lipschitz constant irrespective of n. As before, from Lemma 10.3, $\widehat{D}_2(\omega_0, x_0)$ is Lipschitz continuous, and so is $\widehat{y}_n$ on $(-\infty, -t_n]$ as it is given by that function. On the other hand, $\widehat{y}_n$ is Lipschitz continuous on $[-t_n, 0]$ with Lipschitz constant irrespective of n thanks to statement (ii) of Proposition 4.2, since it yields a bound for the derivative of $\widehat{y}_n$ on $[-t_n, 0]$ by using the boundedness of such function. As a result, $\widehat{y}_n$ is Lipschitz continuous with Lipschitz constant $\widehat{L}_y > 0$ irrespective of n, and so is y_n.

As a consequence,

$$\inf\{x_n'(\tau) - A\,x_n(\tau), y_n'(\tau) - A\,y_n(\tau)\}$$

is defined for almost every $\tau \in (-\infty, 0]$, and we can define a_{x_n,y_n} as in (10.1). As in Proposition 10.2, it can be checked that $a_{x_n,y_n} \in BU$, $a_{x_n,y_n} \leq_A x_n$, $a_{x_n,y_n} \leq_A y_n$, and there is an $r_1 > 0$ such that $a_{x_n,y_n} \in B_{r_1}$ for each $n \in \mathbb{N}$.

Let us check that $\lim_{n\to\infty} \mathsf{d}(a_{x_n,y_n}, x_n) = 0$ and $\lim_{n\to\infty} \mathsf{d}(a_{x_n,y_n}, y_n) = 0$. Fix $\xi > 0$ and $\rho > 0$. It is clear that

$$\|a_{x_n,y_n}(s) - x_n(s)\| \leq \|a_{x_n,y_n}(s) - b_{x_n,y_n}(s)\|,$$

where b_{x_n,y_n} is defined as in (10.2), from which we deduce that

$$\begin{aligned} \|a_{x_n,y_n}(s) - x_n(s)\| \leq \int_{-\infty}^{s} &\big\|e^{A(s-\tau)}\big\|(\|x_n'(\tau) - y_n'(\tau)\| \\ &+ \|A\|\,\|x_n(\tau) - y_n(\tau)\|)\,d\tau \end{aligned} \tag{10.5}$$

for all $s \in (-\infty, 0]$. Let $\rho_0 > \rho$ be such that

$$\int_{-\infty}^{-\rho_0} e^{m_A(-\tau)}(L_x + L_y + 2\,\|A\|\,r_0)\,d\tau \leq \frac{\xi}{2}\,,$$

where m_A is defined as in Proposition 10.2. Now, from (10.5), it follows that, for all $s \in [-\rho, 0]$,

$$\begin{aligned} \|a_{x_n,y_n}(s) - x_n(s)\| &\leq \int_{-\infty}^{-\rho_0} e^{m_A(-\tau)}(L_x + L_y + 2\,\|A\|\,r_0)\,d\tau \\ &+ \int_{-\rho_0}^{s} \big\|e^{A(s-\tau)}\big\|(\|x_n'(\tau) - y_n'(\tau)\| + \|A\|\,\|x_n(\tau) - y_n(\tau)\|)\,d\tau \\ &\leq \frac{\xi}{2} + \int_{-\rho_0}^{s} \big\|e^{A(s-\tau)}\big\|(\|x_n'(\tau) - y_n'(\tau)\| + \|A\|\,\|x_n(\tau) - y_n(\tau)\|)\,d\tau\,, \end{aligned} \tag{10.6}$$

thanks to the choice of ρ_0. Let us prove that, as $n \to \infty$, we have that $\|x_n'(\tau) - y_n'(\tau)\| \to 0$ for all $\tau \in [-\rho_0, 0]$ wherever they are defined, that is, a.e. on $[-\rho_0, 0]$. In order to do this, let us fix $\eta > 0$, and notice that $(\omega_0{\cdot}t_n, x_n) \in K \subset \Omega \times B_{r_0}$ and $(\omega_0{\cdot}t_n, \widehat{x}_n) \in \widehat{D}(K) \subset \Omega \times B_{r_1}$ for all $n \in \mathbb{N}$, where $\widehat{x}_n = \widehat{D}_2(\omega_0{\cdot}t_n, x_n)$ and $r_1 = r_0 \sup_{\omega\in\Omega} \|D(\omega, \cdot)\|$. Hence, $(\omega_0{\cdot}t_n, x_n)$ and $(\omega_0{\cdot}t_n, \widehat{x}_n)$ have a backward extension and a bounded trajectory for all $n \in \mathbb{N}$. Consequently, thanks to statement (ii) of Proposition 4.2, $\widehat{x}_n$ is Lipschitz continuous with Lipschitz constant $\widehat{L}_x > 0$ irrespective of n. Fix $n \in \mathbb{N}$, and let $\omega = \omega_0{\cdot}t_n$. Let $s \in [-\rho_0, 0)$ and $h \in (0, -s]$. Let us assume without loss of generality that $t_n \geq \rho_0$. Then, for all $i \in \{1, \dots, m\}$,

$$\begin{aligned}
&\left| \frac{(y_n)_i(s+h) - (y_n)_i(s)}{h} - \frac{(x_n)_i(s+h) - (x_n)_i(s)}{h} \right| \leq \\
&\quad \leq \sum_{l=1}^{\infty} \left| \frac{c_i^{[l]}(\omega{\cdot}(s+h)) - c_i^{[l]}(\omega{\cdot}s)}{h} \right| |(\widehat{y}_n)_i(s+h-l\alpha_i) - (\widehat{x}_n)_i(s+h-l\alpha_i)| \\
&\quad\quad + \sum_{l=0}^{\infty} \left| c_i^{[l]}(\omega{\cdot}s) \right| \left| \frac{(\widehat{y}_n)_i(s+h-l\alpha_i) - (\widehat{y}_n)_i(s-l\alpha_i)}{h} \right. \\
&\quad\quad\quad \left. - \frac{(\widehat{x}_n)_i(s+h-l\alpha_i) - (\widehat{x}_n)_i(s-l\alpha_i)}{h} \right| .
\end{aligned}$$

Let us focus on the first addend. There exists $l_0 \in \mathbb{N}$ such that

$$\sum_{l=l_0}^{\infty} l\,(c^+)^{l-1} L_c\, 2\, r_1 \leq \frac{\eta}{4};$$

therefore, it is enough to check that there is an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have

$$\|\widehat{y}_n(t) - \widehat{x}_n(t)\| \leq \frac{\eta}{L_c\, 4\, l_0^2}$$

for all $t \in [-\rho_0 - l_0\,\alpha, 0]$ and all $l \in \{0, \dots, l_0 - 1\}$, but this holds thanks to Theorem 3.1 and the fact that $\mathsf{d}(x_n, y_n) \to 0$ as $n \to \infty$ with x_n, $y_n \in B_{r_0}$. Altogether, the first addend is bounded by $\eta/4 + \eta/4 = \eta/2$ for all $n \geq n_0$. As for the second addend, there exists $l_1 \in \mathbb{N}$ such that

$$\sum_{l=l_1}^{\infty} (c^+)^l (\widehat{L}_x + \widehat{L}_y) \leq \frac{\eta}{4}\,;$$

on the other hand, there is an $n_1 \geq n_0$ such that, for all $n \geq n_1$ and all

$i \in \{1, \dots, m\}$,

$$\left| \frac{(\widehat{y}_n)_i(s+h-l\,\alpha_i) - (\widehat{y}_n)_i(s-l\,\alpha_i)}{h} - \frac{(\widehat{x}_n)_i(s+h-l\,\alpha_i) - (\widehat{x}_n)_i(s-l\,\alpha_i)}{h} \right| \leq$$

$$\begin{aligned} &\leq \int_0^1 |(\widehat{y}_n)'_i(s+t\,h-l\,\alpha_i) - (\widehat{x}_n)'_i(s+t\,h-l\,\alpha_i)|\,dt \\ &\leq \int_0^1 |G_i(\omega{\cdot}(s+t\,h-l\,\alpha_i), (\widehat{y}_n)_{s+t\,h-l\,\alpha_i}) \\ &\qquad - G_i(\omega{\cdot}(s+t\,h-l\,\alpha_i), (\widehat{x}_n)_{s+t\,h-l\,\alpha_i})|\,dt \leq \frac{\eta}{4\,l_0}\,, \end{aligned}$$

thanks to statement (iii) of Proposition 4.2 and the fact that

$$\{(\widehat{x}_n)_t : t \in [-\widetilde{r}, 0]\} \cup \{(\widehat{y}_n)_t : t \in [-\widetilde{r}, 0]\}$$

is a relatively compact subset of B_{r_1} for all $\widetilde{r} > 0$. Consequently, the second addend is bounded by $\eta/4 + \eta/4 = \eta/2$ for all $n \geq n_1$. Hence, we have that $\|x'_n(s) - y'_n(s)\| \leq \eta$ for all $n \geq n_1$ and all $s \in [-\rho_0, 0]$ where they are defined. Now, from (10.6), it follows that there is an $n_2 \in \mathbb{N}$ such that, for all $n \geq n_2$ and all $s \in [-\rho, 0]$,

$$\|a_{x_n,y_n}(s) - x_n(s)\| \leq \frac{\xi}{2} + \frac{\xi}{2} = \xi\,,$$

whence $\lim_{n\to\infty} \mathsf{d}(a_{x_n,y_n}, x_n) = 0$, and, analogously, $\lim_{n\to\infty} \mathsf{d}(a_{x_n,y_n}, y_n) = 0$, as wanted.

Consequently, if $\delta > 0$ is the modulus of uniform stability of K and the trajectory $\{\tau(t, \omega_0, x_0) : t \geq 0\}$ for $\leq_A$ in $\Omega \times B_{r_1}$ for $\varepsilon/2$, there is an $n_1 \in \mathbb{N}$ such that $\mathsf{d}(x_n, a_{x_n,y_n}) < \delta$ and $\mathsf{d}(y_n, a_{x_n,y_n}) < \delta$ for each $n \geq n_1$, and, hence, the uniform stability of K and the trajectory $\{\tau(t, \omega_0, x_0) : t \geq 0\}$ for the order $\leq_A$ in $\Omega \times B_{r_1}$ yields

$$\mathsf{d}(u(t+t_{n_1}, \omega_0, x_0), u(t, \omega_0{\cdot}t_{n_1}, x_{n_1})) = \mathsf{d}(u(t, \omega_0{\cdot}t_{n_1}, y_{n_1}), u(t, \omega_0{\cdot}t_{n_1}, x_{n_1})) \leq \varepsilon$$

for each $t \geq 0$. In particular, if $n_2 \in \mathbb{N}$ is such that $s_n - t_{n_1} \geq 0$ for $n \geq n_2$, we obtain

$$\mathsf{d}(u(s_n, \omega_0, x_0), u(s_n - t_{n_1}, \omega_0{\cdot}t_{n_1}, x_{n_1})) \leq \varepsilon \quad \text{for each } \ n \geq n_2\,. \tag{10.7}$$

Now, it remains to notice that, since $(\omega_0{\cdot}t_{n_1}, x_{n_1}) \in M$, then also

$$\tau(s_n - t_{n_1}, \omega_0{\cdot}t_{n_1}, x_{n_1}) = (\omega_0{\cdot}s_n, u(s_n - t_{n_1}, \omega_0{\cdot}t_{n_1}, x_{n_1})) \in M$$

for all $n \geq n_2$. Therefore, there is a convergent subsequence toward a pair $(\omega, x^*) \in M$, and, taking limits in (10.7), we deduce that $\mathsf{d}(x, x^*) \leq \varepsilon$, as desired. $\square$

Now, we turn to the study of the structure of omega-limit sets. As before, we consider the monotone skew-product semiflow (4.2) induced by $(7.1)_\omega$. First, we extend to this setting results in [NNO] and [NOS] ensuring the presence of almost automorphic dynamics from the existence of a semicontinuous semi-equilibrium.

Definition 10.5. A map $a : \Omega \to BU$ such that $u(t, \omega, a(\omega))$ is defined for any $\omega \in \Omega$, $t \geq 0$ is

(i) an *equilibrium* if $a(\omega{\cdot}t) = u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geq 0$;

(ii) a *super-equilibrium* if $a(\omega{\cdot}t) \geq_A u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geq 0$;

(iii) a *sub-equilibrium* if $a(\omega{\cdot}t) \leq_A u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geq 0$.

We will call *semi-equilibrium* to either a super or a sub-equilibrium.

Definition 10.6. A super-equilibrium (resp. sub-equilibrium) $a : \Omega \to BU$ is *semicontinuous* if the following properties hold:

(i) $\Gamma_a = \mathrm{cls}_X\{a(\omega) : \omega \in \Omega\}$ is a compact subset of X for the compact-open topology;

(ii) $C_a = \{(\omega, x) : x \leq_A a(\omega)\}$ (resp. $C_a = \{(\omega, x) : x \geq_A a(\omega)\}$) is a closed subset of $\Omega \times X$ for the product metric topology.

An equilibrium is *semicontinuous* in any of these cases.

As shown in Proposition 4.8 in [NOS], a semicontinuous equilibrium always has a residual subset of continuity points. This theory requires topological properties of semicontinuous maps stated, for instance, in [AF] and [Cho]. The next result shows that a semicontinuous semi-equilibrium provides an almost automorphic extension of the base if a relatively compact trajectory exists. We omit its proof, analogous to the one of Proposition 4.9 in [NOS] once Proposition 10.2 is proved.

Proposition 10.7. *Let $a : \Omega \to BU$ be a semicontinuous semi-equilibrium, and assume that there is an $\omega_0 \in \Omega$ such that $\mathrm{cls}_X\{u(t, \omega_0, a(\omega_0)) : t \geq 0\}$ is a compact subset of X for the compact-open topology. Then*

(i) *the omega-limit set $\mathcal{O}(\omega_0, a(\omega_0))$ contains a unique minimal set, which is an almost automorphic extension of the base flow;*

(ii) *if the orbit $\{\tau(t, \omega_0, a(\omega_0)) : t \geq 0\}$ is uniformly stable for $\leq_A$ in bounded sets, then $\mathcal{O}(\omega_0, a(\omega_0))$ is a copy of the base.*

If the semicontinuous semi-equilibrium satisfies some supplementary and somehow natural compactness conditions, a semicontinuous equilibrium is obtained. As shown in Proposition 4.10 in [NOS] for infinite delay, provided that $\Gamma_a \subset BU$ and $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty < \infty$, it can be proved that the following conditions are equivalent:

(T1) $\Gamma = \mathrm{cls}_X\{u(t,\omega,a(\omega)) : t \geq 0,\ \omega \in \Omega\}$ is a compact subset of BU for the compact-open topology;

(T2) for each $\omega \in \Omega$, the set $\mathrm{cls}_X\{u(t,\omega,a(\omega)) : t \geq 0\}$ is a compact subset of BU for the compact-open topology;

(T3) there is an $\omega_0 \in \Omega$ such that the set $\mathrm{cls}_X\{u(t,\omega_0,a(\omega_0)) : t \geq 0\}$ is a compact subset of BU for the compact-open topology.

Consequently, an easy adaptation of the proof of Theorem 4.11 in [NOS] proves the following result.

Theorem 10.8. *Let us assume the existence of a semicontinuous semi-equilibrium $a : \Omega \to BU$ satisfying $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty < \infty$, $\Gamma_a \subset BU$, and one of the equivalent conditions* (T1)–(T3)*. Then the following statements hold:*

(i) *there exists a semicontinuous equilibrium $b : \Omega \to BU$ with $b(\omega) \in \Gamma$ for any $\omega \in \Omega$;*

(ii) *if ω_1 is a continuity point for b, then the restriction of the semiflow τ to the minimal set*

$$K^* = \mathrm{cls}_{\Omega\times X}\{(\omega_1{\cdot}t, b(\omega_1{\cdot}t)) : t \geq 0\} \subset C_a$$

is an almost automorphic extension of the base flow $(\Omega,\sigma,\mathbb{R})$;

(iii) *K^* is the only minimal set included in the omega-limit set $\mathcal{O}(\widehat{\omega}, a(\widehat{\omega}))$ for each point $\widehat{\omega} \in \Omega$;*

(iv) *if there is a point $\widetilde{\omega} \in \Omega$ such that the trajectory $\{\tau(t,\widetilde{\omega},a(\widetilde{\omega})) : t \geq 0\}$ is uniformly stable for $\leq_A$ in bounded sets, then, for each $\widehat{\omega} \in \Omega$,*

$$\mathcal{O}(\widehat{\omega}, a(\widehat{\omega})) = K^* = \{(\omega, b(\omega)) : \omega \in \Omega\}\,,$$

i.e. it is a copy of the base determined by the equilibrium b of (i)*, which is a continuous map.*

Now we present a result regarding the stability of the trajectories with initial data in a given ball.

Proposition 10.9. *Fix $r > 0$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all (ω, x), $(\omega, y) \in \Omega \times B_r$ with $\mathsf{d}(x, y) < \delta$ and $x \leq_A y$, it holds that $\|z(t, \omega, x) - z(t, \omega, y)\| \leq \varepsilon$ whenever they are defined.*

Proof. Consider the *total mass* M as defined in (5.4). From Proposition 5.5, it is constant along the trajectories because from $(5.7)_\omega$

$$\frac{d}{dt} M(\tau(t, \omega, x)) = \frac{d}{dt} M(\omega{\cdot}t, u(t, \omega, x)) = 0\,,$$

that is, $M(\omega{\cdot}t, u(t, \omega, x)) = M(\omega, x)$ for each $t \geq 0$ where $z(t, \omega, x)$ is defined. Let (ω, x), $(\omega, y) \in \Omega \times BU$ be such that $x \leq_A y$. Now, we can apply Theorem 9.3 to deduce that the induced semiflow is monotone, and, hence, $u(t, \omega, x) \leq_A u(t, \omega, y)$ whenever they are defined. Thus, since the transport functions g_{ji} are monotone,

$$\int_{-\rho_{ji}}^{0} (g_{ji}(\omega{\cdot}(t+\tau), z_i(t+\tau, \omega, y)) - g_{ji}(\omega{\cdot}(t+\tau), z_i(t+\tau, \omega, x)))\, d\tau \geq 0$$

, and we deduce that

$$\begin{aligned}\sum_{i=1}^{m} (D_i(\tau(t, \omega, y)) - D_i(\tau(t, \omega, x))) &\leq M(\tau(t, \omega, y)) - M(\tau(t, \omega, x)) \\ &= M(\omega, y) - M(\omega, x)\,.\end{aligned}$$

Next, we check that, given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all (ω, x), $(\omega, y) \in \Omega \times B_r$ such that $x \leq_A y$ and $\mathsf{d}(x, y) < \delta$, it holds that $\|z(t, \omega, y) - z(t, \omega, x)\| \leq \varepsilon$ whenever they are defined. We define

$$\xi = \sup_{\omega_1 \in \Omega} \sum_{i=1}^{m} c_j(\omega_1) < 1.$$

From the uniform continuity of M, it follows that, given $\varepsilon_0 = \varepsilon\,(1 - \gamma) > 0$, there exists $\delta \in (0, \varepsilon_0)$ such that, if (ω, x), $(\omega, y) \in \Omega \times B_r$ are such that $\mathsf{d}(x, y) < \delta$, then $|M(\omega, y) - M(\omega, x)| < \varepsilon_0$. Moreover, if $x \leq_A y$, then we have

$$\sum_{i=1}^{m} (D_i(\tau(t, \omega, y)) - D_i(\tau(t, \omega, x))) < \varepsilon_0.$$

Consequently, from the definition of D_i,

$$\begin{aligned}0 \le z_i(t,\omega,y) - z_i(t,\omega,x) &\le \sum_{j=1}^{m}(z_j(t,\omega,y) - z_j(t,\omega,x))\\ &< \sum_{j=1}^{m}\left(D_j(\tau(t,\omega,y)) - D_j(\tau(t,\omega,x))\right)\\ &\quad + \sum_{j=1}^{m} c_j(\omega{\cdot}t)(z_j(t-\alpha_j,\omega,y) - z_j(t-\alpha_j,\omega,x))\\ &\le \varepsilon_0 + \xi\,\|u(t,\omega,y)-u(t,\omega,x)\|_\infty\,,\end{aligned}$$

from which we deduce that $\|u(t,\omega,y)-u(t,\omega,x)\|_\infty(1-\xi) < \varepsilon_0 = \varepsilon\,(1-\xi)$, that is, $\|z(t,\omega,y)-z(t,\omega,x)\| \le \varepsilon$ whenever they are defined, as claimed. $\square$

Proposition 10.10. *For all $(\omega,x) \in \Omega \times BU$, if x is Lipschitz continuous, then $\tau(\cdot,\omega,x)$ is defined on $\mathbb{R}$, and it is bounded.*

Proof. Thanks to Proposition 10.9, the proof of this result is analogous to that of Proposition 8.3. $\square$

As in Chapter 7, Proposition 4.3 holds, and then the omega-limit set of $(\omega_0,x_0) \in \Omega \times BU$ can be defined as in (4.5). Moreover, Propositions 4.5 and 4.6 are also satisfied.

The following result provides a continuous super-equilibrium for every compact, positively invariant set included in $\Omega \times B_r$ for some $r>0$.

Theorem 10.11. *Let us consider $(\omega_0,x_0) \in \Omega \times BU$ such that its forward orbit, $\{\tau(t,\omega_0,x_0) : t \ge 0\}$, is relatively compact for the product metric topology and uniformly stable for $\le_A$ in bounded sets. Let $K = \mathcal{O}(\omega_0,x_0)$ be its omega-limit set. For each $\omega \in \Omega$ we define the map $a(\omega)$ on $(-\infty,0]$ by*

$$a(\omega)(s) = \int_{-\infty}^{s} e^{A(s-\tau)}\,h(\omega)(\tau)\,d\tau\,, \quad s \le 0\,,$$

where

$$\begin{array}{rcl} h(\omega)\colon (-\infty,0] & \longrightarrow & \mathbb{R}^m \\ \tau & \mapsto & \inf\{x'(\tau) - A\,x(\tau) : (\omega,x) \in K\}\,. \end{array}$$

Then $a(\omega)$ is Lipschitz continuous for every $\omega \in \Omega$, $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty < \infty$, and the map $a\colon \Omega \to BU$, $\omega \mapsto a(\omega)$ is well-defined, it is a continuous super-equilibrium, and it satisfies $\Gamma_a \subset BU$ and the equivalent conditions (T1)–(T3).

Proof. Since K is a positively invariant compact subset admitting a flow extension, from Proposition 9.2 we deduce that $x \in C^1((-\infty,0],\mathbb{R}^m)$ for each $(\omega,x) \in K$. We claim that there is a positive constant $L>0$ such that x is a Lipschitz continuous function with constant L for each $(\omega,x)\in K$. Let $r_0>0$ be a bound of $z(\cdot,\omega_0,x_0)$. Then it is clear that

$$\widehat{z}\,'(t,\widehat{D}(\omega_0,x_0)) = G(\omega_0{\cdot}t,\widehat{u}(t,\widehat{D}(\omega_0,x_0))),$$

and it also holds that $\widehat{u}(t,\widehat{D}(\omega_0,x_0)) \in \widehat{D}_2(\Omega\times B_{r_0}) \subset B_{r_1}$ for all $t\geq 0$, where $r_1 = \sup_{\omega\in\Omega}\|D(\omega,\cdot)\|\,r_0$. Hence, from statement (ii) of Proposition 4.2, it follows that $\widehat{z}(\cdot,\widehat{D}(\omega_0,x_0))$ is Lipschitz continuous with Lipschitz constant

$$\widehat{L} = \sup\{\|G(\omega,y)\| : (\omega,y)\in \widehat{D}(\Omega\times B_{r_0})\}.$$

Fix $(\omega,x)\in K$, and let us check that x is Lipschitz continuous with uniform Lipschitz constant $L>0$ over K. First of all, x admits a backward orbit extension, and its trajectory is bounded by r_0, whence, thanks to Proposition 9.2, x is continuously differentiable, and x' is given by (9.1). Hence, if $\widehat{x} = \widehat{D}_2(\omega,x)$, then x' is bounded, since the first addend in (9.1) is bounded by $\sum_{n=0}^{\infty}(c^+)^n\widehat{L}$, and the second one is bounded by $\sum_{n=1}^{\infty} n\,(c^+)^{n-1}\,L_c$. Thus

$$L = \frac{\widehat{L}}{1-c^+} + \frac{L_c}{(1-c^+)^2}$$

satisfies the desired property.

Since $\|x\|_\infty \leq r_0$ and $\|x'\|_\infty \leq L$ for each $x\in K_\omega$, the function $h(\omega)$ is well defined for each $\omega\in\Omega$. Moreover, let us check that $\{y'|_{[-k,0]} : y\in\widehat{D}(K)_\omega\}$ is uniformly equicontinuous for all $k>0$. Suppose on the contrary that there is an $\varepsilon>0$ such that, for all $n\in\mathbb{N}$, there exist $y_n\in\widehat{D}(K)_\omega$ and $s_n,\, s_n'\in[-k,0]$ with $|s_n-s_n'|<1/n$ and $\|y_n'(s_n)-y_n'(s_n')\|>\varepsilon$. Due to the compactness of $\widehat{D}(K)_\omega$ and $[-k,0]$, there are subsequences $\{y_{n_j}\}_j$, $\{s_{n_j}\}_j$, and $\{s_{n_j}'\}_j$ that converge to some $y_0\in\widehat{D}(K)_\omega$, $s_0\in[-k,0]$, and $s_0'=s_0$, respectively. Now,

$$\|y_{n_j}'(s_{n_j}) - y_{n_j}'(s_{n_j}')\| = \|G(\omega{\cdot}s_{n_j},\widehat{u}(s_{n_j},\omega,y_{n_j})) - G(\omega{\cdot}s_{n_j}',\widehat{u}(s_{n_j}',\omega,y_{n_j}))\|,$$

and, using statement (iii) of Proposition 4.2 together with Corollary 4.3 in [NOS], we know that, as $j\to\infty$, the latter term converges toward

$$\|G(\omega{\cdot}s_0,\widehat{u}(s_0,\omega,y_0)) - G(\omega{\cdot}s_0,\widehat{u}(s_0,\omega,y_0))\| = 0,$$

a contradiction. As a result, $\{y'|_{[-k,0]} : y\in\widehat{D}(K)_\omega\}$ is uniformly equicontinuous.

Let us prove now that $\{x' : x \in K_\omega\}$ is equicontinuous. In order to do so, fix $s_0 \in (-\infty, 0]$ and $\varepsilon > 0$. Let $x \in K_\omega$ and $\widehat{x} = \widehat{D}_2(\omega, x)$. Thanks to (9.1), for all $s \in (-\infty, 0]$ and all $i \in \{1, \dots, m\}$,

$$\begin{aligned}|x_i'(s) - x_i'(s_0)| \le &\left|\sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}s)\,\widehat{x}\,'_i(s - n\,\alpha_i) - \sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}s_0)\,\widehat{x}\,'_i(s_0 - n\,\alpha_i)\right| \\ &+ \left|\sum_{n=0}^{\infty} \frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s+t))\big|_{t=0}\,\widehat{x}_i(s - n\,\alpha_i)\right. \\ &\qquad\left. - \sum_{n=0}^{\infty} \frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s_0+t))\big|_{t=0}\,\widehat{x}_i(s_0 - n\,\alpha_i)\right|.\end{aligned}$$

Let us focus on the first addend. For each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}&|c_i^{[n]}(\omega{\cdot}s)\,\widehat{x}\,'_i(s - n\,\alpha_i) - c_i^{[n]}(\omega{\cdot}s_0)\,\widehat{x}\,'_i(s_0 - n\,\alpha_i)| \le \\ &\qquad \le |c_i^{[n]}(\omega{\cdot}s) - c_i^{[n]}(\omega{\cdot}s_0)|\,|\widehat{x}\,'_i(s - n\,\alpha_i)| \\ &\qquad\quad + |c_i^{[n]}(\omega{\cdot}s_0)|\,|\widehat{x}\,'_i(s - n\,\alpha_i) - \widehat{x}\,'_i(s_0 - n\,\alpha_i)| \\ &\qquad \le n\,(c^+)^{n-1}\,L_c\,|s - s_0|\,\widehat{L} + (c^+)^n\,|\widehat{x}\,'_i(s - n\,\alpha_i) - \widehat{x}\,'_i(s_0 - n\,\alpha_i)|.\end{aligned}$$

Hence, thanks to the uniform equicontinuity of $\{y'|_{[-k,0]} : y \in \widehat{D}(K)_\omega\}$ for all $k > 0$ and the fact that the first addend is bounded by $\sum_{n=0}^{\infty} 2\,(c^+)^n\,\widehat{L}$, there exists $\delta_1 > 0$ such that, if $|s - s_0| < \delta_1$, then this addend is bounded by $\varepsilon/2$. As for the second addend, for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}&\left|\frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s+t))\big|_{t=0}\,\widehat{x}_i(s - n\,\alpha_i) - \frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s_0+t))\big|_{t=0}\,\widehat{x}_i(s_0 - n\,\alpha_i)\right| \le \\ &\qquad \le \left|\frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s+t))\big|_{t=0} - \frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s_0+t))\big|_{t=0}\right|\,|\widehat{x}_i(s - n\,\alpha_i)| \\ &\qquad\quad + \left|\frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s_0+t))\big|_{t=0}\right|\,|\widehat{x}_i(s - n\,\alpha_i) - \widehat{x}_i(s_0 - n\,\alpha_i)| \\ &\qquad \le \left|\frac{d}{dt} c_i^{[n]}(\omega{\cdot}(s_0+t))\big|_{t=0}\right|\,r_1 + n(c^+)^{n-1} L_c\,\widehat{L}|s - s_0|.\end{aligned}$$

This together with the continuous differentiability of c along the trajectories and the fact that the second addend is bounded by $\sum_{n=1}^{\infty} 2\,n\,(c^+)^{n-1}\,L_c\,r_1$ implies that there exists $\delta \in (0, \delta_1)$ such that, if $|s - s_0| < \delta$, then this addend is bounded by $\varepsilon/2$. Altogether, if $|s - s_0| < \delta$, then $|x_i'(s) - x_i'(s_0)| \le \varepsilon$, which proves the equicontinuity of $\{x' : x \in K_\omega\}$.

As a consequence, $\{x' - A\,x : x \in K_\omega\}$ is equicontinuous. It is easy to check that this implies that $h(\omega)$ is continuous on $(-\infty, 0]$ for all $\omega \in \Omega$ and $\sup_{\omega\in\Omega} \|h(\omega)\|_\infty \le L + \|A\|\, r_0$.

As a result, $a(\omega)$ is well defined for each $\omega \in \Omega$, $a(\omega) \in X$, and we have $\sup_{\omega\in\Omega} \|a(\omega)\|_\infty < \infty$. In addition, $a(\omega) \in C^1((-\infty, 0], \mathbb{R}^m)$ with

$$a(\omega)'(s) = A\,a(\omega)(s) + h(\omega)(s),\ s \le 0\,.$$

From this fact and the uniform boundedness of $a(\omega)$ and $h(\omega)$, we deduce that there is a positive constant $\widehat{L}_a > 0$ such that $a(\omega)$ is a Lipschitz continuous function with constant $\widehat{L}_a$ for each $\omega \in \Omega$. Hence, $a(\omega)$ belongs to BU, i.e. a is well defined. Moreover, $\Gamma_a = \mathrm{cls}_X\{a(\omega) : \omega \in \Omega\}$ is a compact subset of X, and actually $\Gamma_a \subset BU$.

Let us check that a defines a super-equilibrium. From Proposition 10.10, it follows that $u(t, \omega, a(\omega))$ exists for any $\omega \in \Omega$ and $t \ge 0$. It is easy to prove, as in Proposition 10.2, that $a(\omega) \le_A x$ for each $x \in K_\omega$. Next we claim that if $z \in BU$ with $z \le_A x$ for each $x \in K_\omega$ then $z \le_A a(\omega)$ provided that z is Lipschitz continuous on $(-\infty, 0]$. From the definition of $a(\omega)$, it is not hard to check that $z(s) \le a(\omega)(s)$ for each $s \le 0$. Moreover, since $x \in C^1((-\infty, 0], \mathbb{R}^m)$, $z \le_A x$, and z is Lipschitz continuous, we deduce that

$$z'(s) - A\,z(s) \le x'(s) - A\,x(s)$$

for almost every $s \in (-\infty, 0]$ and every $x \in K_\omega$. Hence, the definition of $h(\omega)$ provides at these points

$$z'(s) - A\,z(s) \le h(\omega)(s) = a(\omega)'(s) - A\,a(\omega)(s)\,,$$

and we conclude that $z \le_A a(\omega)$, as claimed.

Fix $\omega \in \Omega$, $t \ge 0$, and consider any $y \in K_{\omega\cdot t}$, i.e. $(\omega{\cdot}t, y) \in K$. Since we have a flow on K, $\tau(-t, \omega{\cdot}t, y) = (\omega, u(-t, \omega{\cdot}t, y)) \in K$, and, therefore, $a(\omega) \le_A u(-t, \omega{\cdot}t, y)$. Applying the monotonicity, $u(t, \omega, a(\omega)) \le_A y$. As this happens for any $y \in K_{\omega\cdot t}$ and $u(t, \omega, a(\omega))$ is Lipschitz continuous on $(-\infty, 0]$, we get that $u(t, \omega, a(\omega)) \le_A a(\omega{\cdot}t)$, and a is a super-equilibrium, as stated.

Now let us prove that a is continuous on Ω. From the definition of a it is enough to check the continuity of $h : \Omega \to X$. Fix $\omega \in \Omega$, and assume that $\omega_n \to \omega$ and $h(\omega_n) \xrightarrow{\mathsf{d}} y$ as $n \to \infty$. Notice that $\{h(\omega)\}_{\omega\in\Omega}$ is relatively compact for the compact-open topology. First we check that $h(\omega) \le y$. Let us fix $s \in (-\infty, 0]$ and $i \in \{1, \dots, m\}$. From the definition of h there are $(\omega_n, x_n) \in K$, depending on s and i, although dropped from the notation, such that

$$|h(\omega_n)_i(s) - (x_n')_i(s) - (A\,x_n)_i(s))| < \frac{1}{n}\,.$$

This implies that $\lim_{n\to\infty}((x'_n)_i(s) - (A\,x_n)_i(s)) = y_i(s)$. Moreover, from the compactness of K, an adequate subsequence (ω_{n_j}, x_{n_j}) converges to some $(\omega, x) \in K$ for the product metric topology. Let us check that

$$\lim_{j\to\infty} (x'_{n_j})_i(s) = x'_i(s).$$

Using Proposition 9.2 and the expression of x'_{n_j} and x' given by (9.1), we obtain that, for all $j \in \mathbb{N}$,

$$\begin{aligned}
|x'_i(s) - (x'_{n_j})_i(s)| \le {} & \left| \sum_{l=0}^{\infty} c_i^{[l]}(\omega{\cdot}s)(\widehat{x}\,'_i(s - l\,\alpha_i) - (\widehat{x}\,'_{n_j})_i(s - l\,\alpha_i)) \right| \\
& + \left| \sum_{l=0}^{\infty} \Big(c_i^{[l]}(\omega{\cdot}s) - c_i^{[l]}(\omega_{n_j}{\cdot}s) \Big) (\widehat{x}\,'_{n_j})_i(s - l\,\alpha_i) \right| \\
& + \left| \sum_{l=1}^{\infty} \frac{d}{dt} c_i^{[l]}(\omega{\cdot}(s+t))\big|_{t=0} (\widehat{x}_i(s - l\,\alpha_i) - (\widehat{x}_{n_j})_i(s - l\,\alpha_i)) \right| \\
& + \left| \sum_{l=1}^{\infty} \left(\frac{d}{dt} c_i^{[l]}(\omega{\cdot}(s+t))\big|_{t=0} - \frac{d}{dt} c_i^{[l]}(\omega_{n_j}{\cdot}(s+t))\big|_{t=0} \right) (\widehat{x}_{n_j})_i(s - l\,\alpha_i) \right|,
\end{aligned}$$

where $\widehat{x} = \widehat{D}_2(\omega, x)$ and $\widehat{x}_n = \widehat{D}_2(\omega_n, x_n)$ for each $n \in \mathbb{N}$. Fix $\varepsilon > 0$, and let us focus on the first addend. First of all, it is bounded by $\sum_{l=0}^{\infty} 2\,(c^+)^l\,\widehat{L}$; besides, for all $j \in \mathbb{N}$ and all $l \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\left| c_i^{[l]}(\omega{\cdot}s)\ (\widehat{x}\,'_i(s - l\,\alpha_i) - (\widehat{x}\,'_{n_j})_i(s - l\,\alpha_i)) \right| \le \\
&\qquad \le (c^+)^l\, \|G(\omega{\cdot}(s - l\,\alpha_i), \widehat{x}_{s - l\,\alpha_i}) - G(\omega_{n_j}{\cdot}(s - l\,\alpha_i), (\widehat{x}_{n_j})_{s - l\,\alpha_i})\|,
\end{aligned}$$

whence the relative compactness of $\{(\widehat{x}_{n_j})_{s-l\,\alpha_i} : j \in \mathbb{N}\}$ together with statement (iii) of Proposition 4.2 imply that there exists $j_0 \in \mathbb{N}$ such that, for all $j \ge j_0$, this first addend is bounded by $\varepsilon/4$. As for the second addend, it is clearly bounded by $\sum_{l=0}^{\infty} 2\,(c^+)^l\,\widehat{L}$; this fact and the continuity of c prove that there is a $j_1 \ge j_0$ such that, for all $j \ge j_1$, the second addend is bounded by $\varepsilon/4$. Regarding the third addend, note that it is bounded by $\sum_{l=1}^{\infty} l\,(c^+)^{l-1}\,2\,r_1$; moreover, the boundedness of the derivative of c, the fact that $\lim_{j\to\infty}(\omega_{n_j}, x_{n_j}) = (\omega, x)$ for the product metric topology, and statement (iii) of Proposition 4.2 show that there is a $j_2 \ge j_1$ such that the third addend is bounded by $\varepsilon/4$ for all $j \ge j_2$. Finally, the fourth addend is bounded by $\sum_{l=1}^{\infty} 2\,l\,(c^+)^{l-1}\,r_1$, which together with the continuous differentiability of c along the flow implies that there exists $j_3 \ge j_2$ such that, the fourth addend is bounded by $\varepsilon/4$ for all $j \ge j_3$. Altogether, $|x'_i(s) - (x'_{n_j})_i(s)| \le \varepsilon$ for all $j \ge j_3$. As a result, $\lim_{j\to\infty}(x'_{n_j})_i(s) = x'_i(s)$, as desired.

Hence, $y_i(s) = x_i'(s) - (A\,x)_i(s)$, and, again, from the definition of $h(\omega)$, we conclude that $h(\omega)_i(s) \le y_i(s)$. As this happens for each $s \in (-\infty, 0]$ and each $i \in \{1, \ldots, m\}$, we deduce that $h(\omega) \le y$. On the other hand, from Proposition 10.2 we know that $(K, \tau, \mathbb{R}^+)$ is uniformly stable, and then Theorem 3.4 in [NOS] asserts that the section map for K, $\omega \in \Omega \mapsto K_\omega$, is continuous at every $\omega \in \Omega$, which implies that $K_{\omega_n} \to K_\omega$ in the Hausdorff metric. Therefore, for any $z \in K_\omega$ there exist $z_n \in K_{\omega_n}$, $n \ge 1$, such that $z_n \xrightarrow{\mathsf{d}} z$. Then, $(\omega_n, z_n) \in K$ implies that $h(\omega_n)(s) \le z_n'(s) - A\,z_n(s)$, and, taking limits, $y(s) \le z'(s) - A\,z(s)$ for each $s \in (-\infty, 0]$). As this happens for any $z \in K_\omega$, we conclude that $y \le h(\omega)$. In all, $h(\omega) = y$, as wanted. Notice that $h(\omega)(\tau) = h(\omega{\cdot}\tau)(0)$ for all $\omega \in \Omega$ and all $\tau \le 0$; hence, $h(\omega) \in BU$ for all $\omega \in \Omega$. In addition, we have $\Gamma_a = \{a(\omega) : \omega \in \Omega\}$.

Finally, from Proposition 10.10 and Proposition 4.3, we deduce that the equivalent conditions (T1)–(T3) hold, and the proof is complete. Notice that $a(\omega)$ is the infimum among the Lipschitz continuous functions of the set K_ω. □

In the next theorem, we establish the 1-covering property of omega-limit sets. Notice that this result was stated in Chapter 9 as Theorem 9.7.

Theorem 10.12. *Assume conditions* (G1), (G6), (G7), *and* (G10), *and let us consider the monotone skew-product semiflow* (4.2) *induced by the family* $(8.2)_\omega$. *Fix* $(\omega_0, x_0) \in \Omega \times BU$ *such that* x_0 *is Lipschitz continuous. Then* $K = \mathcal{O}(\omega_0, x_0) = \{(\omega, b(\omega)) : \omega \in \Omega\}$ *is a copy of the base, and*

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), b(\omega_0{\cdot}t)) = 0\,,$$

where $b : \Omega \to BU$ *is a continuous equilibrium.*

Proof. We apply Propositions 10.9 and 10.10, Proposition 4.3, and Theorem 10.11 to obtain a continuous super-equilibrium a satisfying (T1)–(T3) with $a(\omega)$ Lipschitz continuous for each $\omega \in \Omega$. Then, from Theorem 10.11, we deduce that there is a continuous equilibrium $b : \Omega \to BU$ such that for each $\widehat\omega \in \Omega$,

$$\mathcal{O}(\widehat\omega, a(\widehat\omega)) = K^* = \{(\omega, b(\omega)) : \omega \in \Omega\}\,. \tag{10.8}$$

The definition of a yields $a(\omega) \le_A x$ for each $(\omega, x) \in K$, and hence $b(\omega) \le_A x$ by the construction of b. As in [JZ] and [NOS], we prove that there is a subset $J \subset \{1, \ldots, m\}$ such that

$$b(\omega)_i = x_i \quad \text{for all } (\omega, x) \in K \text{ and all } i \notin J\,,$$

$$b(\omega)_i(s) < x_i(s) \quad \text{for all } (\omega, x) \in K,\ \text{all } s \in (-\infty, 0] \text{ and } i \in J\,.$$

It is enough to check that if $b(\widetilde{\omega})_i(0)=\widetilde{x}_i(0)$ for some $i\in\{1,\dots,m\}$ and $(\widetilde{\omega},\widetilde{x})\in K$, then $b(\omega)_i=x_i$ for any $(\omega,x)\in K$. First, notice that $b(\widetilde{\omega})_i=\widetilde{x}_i$. Otherwise, there would be an $s_0\in(-\infty,0]$ with $b(\widetilde{\omega})_i(s_0)<\widetilde{x}_i(s_0)$.

From $b(\widetilde{\omega})\leq_A\widetilde{x}$ we know that

$$\widetilde{x}(0)-b(\widetilde{\omega})(0)\geq e^{-As_0}(x(s_0)-b(\widetilde{\omega})(s_0))$$

which implies that $b_i(\widetilde{\omega})(0)<\widetilde{x}_i(0)$ thanks to the fact that e^{At} is a non-negative matrix with strictly positive entries in the main diagonal for all $t\geq 0$, a contradiction.

Therefore, $b(\widetilde{\omega})_i=\widetilde{x}_i$. Next, from Theorem 10.4 we know that K is minimal. Thus we take $(\omega,x)\in K$, and a sequence $s_n\downarrow-\infty$ such that $\widetilde{\omega}{\cdot}s_n\to\omega$ and $u(s_n,\widetilde{\omega},\widetilde{x})\xrightarrow{\mathsf{d}}x$. Then,

$$\begin{aligned}x_i(0)&=\lim_{n\to\infty}u(s_n,\widetilde{\omega},\widetilde{x})_i(0)=\lim_{n\to\infty}\widetilde{x}_i(s_n)\\&=\lim_{n\to\infty}b(\widetilde{\omega})_i(s_n)=\lim_{n\to\infty}b(\widetilde{\omega}{\cdot}s_n)_i(0)=b(\omega)_i(0)\,,\end{aligned}$$

and as before this implies that $b(\omega)_i=x_i$, as wanted.

Let $(\omega,x)\in K$, and define $x_\alpha=(1-\alpha)\,a(\omega)+\alpha\,x\in B_{\widehat{k}_0}\subset BU$ for $\alpha\in[0,1]$, and

$$L=\{\alpha\in[0,1]:\mathcal{O}(\omega,x_\alpha)=K^*\}\,.$$

If we prove that $L=[0,1]$, then $K=K^*$, and the proof is finished. From the monotone character of the semiflow and since $\mathcal{O}(\omega,a(\omega))=K^*$, it is immediate to check that if $0<\alpha\in L$ then $[0,\alpha]\subset L$.

Next we show that L is closed, that is, if $[0,\alpha)\subset L$ then $\alpha\in L$. Since $\{\tau(t,\omega,x_\alpha):t\geq 0\}$ is uniformly stable for $\leq_A$ in bounded sets, let $\delta(\varepsilon)>0$ be the modulus of uniform stability for $\varepsilon>0$ in $\Omega\times B_{\widehat{k}_0}$. Thus, we take $\beta\in[0,\alpha)$ with $\mathsf{d}(x_\alpha,x_\beta)<\delta(\varepsilon)$, and we obtain $\mathsf{d}(u(t,\omega,x_\alpha),u(t,\omega,x_\beta))<\varepsilon$ for each $t\geq 0$. Moreover, $\mathcal{O}(\omega,x_\beta)=K^*$, and, hence, there is a t_0 such that $\mathsf{d}(u(t,\omega,x_\beta),b(\omega{\cdot}t))<\varepsilon$ for each $t\geq t_0$. Then, it is clear that we have $\mathsf{d}(u(t,\omega,x_\alpha),b(\omega{\cdot}t))<2\,\varepsilon$ for each $t\geq t_0$ and $\mathcal{O}(\omega,x_\alpha)=K^*$, i.e. $\alpha\in L$, as claimed.

Finally, we prove that the case $L=[0,\alpha]$ with $0\leq\alpha<1$ is impossible. For each $i\in J$ we consider the continuous maps

$$K\longrightarrow(0,\infty)\,,\quad(\widetilde{\omega},\widetilde{x})\mapsto\widetilde{x}_i(0)-b(\widetilde{\omega})_i(0)\,,$$

$$\begin{aligned}K\longrightarrow(0,\infty)\,,\quad(\widetilde{\omega},\widetilde{x})\mapsto\;&F_i(\widetilde{\omega},\widetilde{x})-F_i(\widetilde{\omega},b(\widetilde{\omega}))-(A\,D(\widetilde{\omega},\widetilde{x}-b(\widetilde{\omega})))_i\\&+\gamma_i(\widetilde{\omega})(\widetilde{x}_i(-\alpha_i)-b(\widetilde{\omega})_i(-\alpha_i))\,.\end{aligned}$$

As explained above $\widetilde{x}_i(0) - b(\widetilde{\omega})_i(0) > 0$. Besides, since $(\widetilde{\omega}{\cdot}s, \widetilde{x}_s) \in K$ for all $s \le 0$, $\widetilde{x}_i(s) - b(\widetilde{\omega})_i(s) > 0$. Moreover, from $b(\widetilde{\omega}) \le_A \widetilde{x}$ and (G10), we deduce that

$$F_i(\widetilde{\omega},\widetilde{x}) - F_i(\widetilde{\omega}, b(\widetilde{\omega})) - (A\,D(\widetilde{\omega}, \widetilde{x} - b(\widetilde{\omega})))_i + \gamma_i(\widetilde{\omega})(\widetilde{x}_i(-\alpha_i) - b(\widetilde{\omega})_i(-\alpha_i)) > 0.$$

Hence, there is an $\varepsilon > 0$ such that $\widetilde{x}_i(0) - b(\widetilde{\omega})_i(0) \ge \varepsilon$ and

$$F_i(\widetilde{\omega},\widetilde{x}) - F_i(\widetilde{\omega}, b(\widetilde{\omega})) - (A\,D(\widetilde{\omega}, \widetilde{x} - b(\widetilde{\omega})))_i + \gamma_i(\widetilde{\omega})(\widetilde{x}_i(-\alpha_i) - b(\widetilde{\omega})_i(-\alpha_i)) \ge \varepsilon$$

for each $(\widetilde{\omega},\widetilde{x}) \in K$. Besides, since $(\widetilde{\omega}{\cdot}s, u(s,\widetilde{\omega},\widetilde{x})) \in K$, $u(s,\widetilde{\omega},\widetilde{x})(0) = \widetilde{x}(s)$ for each $s \le 0$ because K admits a flow extension, and $b(\widetilde{\omega})(s) = b(\widetilde{\omega}{\cdot}s)(0)$, we deduce that

$$\begin{aligned} &\widetilde{x}_i(s) - b(\widetilde{\omega})_i(s) \ge \varepsilon \quad \text{and} \\ F_i(\widetilde{\omega}{\cdot}s,\widetilde{x}_s) &- F_i(\widetilde{\omega}{\cdot}s, b(\widetilde{\omega}{\cdot}s)) - (A\,D(\widetilde{\omega}{\cdot}s, \widetilde{x}_s - b(\widetilde{\omega}{\cdot}s)))_i \\ &+ \gamma_i(\widetilde{\omega}{\cdot}s)(\widetilde{x}_i(s-\alpha_i) - b(\widetilde{\omega})_i(s-\alpha_i)) \ge \varepsilon \end{aligned} \tag{10.9}$$

for all $s \in (-\infty, 0]$ and all $(\widetilde{\omega},\widetilde{x}) \in K$.

It is not hard to check that $\cup_{\beta\in[0,1]} \mathrm{cls}_{\Omega\times X}\{\tau(t,\omega,x_\beta) : t \ge 0\}$ is a compact set. Hence, since Proposition 10.9 implies that $\{\tau(t,\omega,x_\alpha) : t \ge 0\}$ is uniformly stable for $\le_A$ in bounded sets, $\{(\omega, x_\beta) : \beta \in [0,1]\}$ is clearly included in a ball, and hypotheses (D2) and (F3) hold, we deduce that there is a $\delta > 0$ such that

$$\begin{aligned} &\|u(t,\omega,x_\gamma)(0) - u(t,\omega,x_\alpha)(0)\| < \frac{\varepsilon}{4} \quad \text{and} \\ |F_i(\omega{\cdot}t, &u(t,\omega,x_\gamma)) - F_i(\omega{\cdot}t, u(t,\omega,x_\alpha)) \\ &- (A\,D(\omega{\cdot}t, u(t,\omega,x_\gamma) - u(t,\omega,x_\alpha)))_i \\ &+ \gamma_i(\omega{\cdot}t)(u(t,\omega,x_\gamma)_i(-\alpha_i) - u(t,\omega,x_\alpha)_i(-\alpha_i))| < \frac{\varepsilon}{4} \end{aligned} \tag{10.10}$$

for each $t \ge 0$ and $\gamma \in (\alpha, 1]$ with $\mathsf{d}(x_\alpha, x_\gamma) < \delta$. Besides, $\alpha \in L$, i.e. $\mathcal{O}(\omega, x_\alpha) = K^*$, and there is a $t_0 \ge 0$ such that

$$\begin{aligned} &\|u(t,\omega,x_\alpha)(0) - b(\omega{\cdot}t)(0)\| < \frac{\varepsilon}{4} \quad \text{and} \\ |F_i(\omega{\cdot}t, &u(t,\omega,x_\alpha)) - F_i(\omega{\cdot}t, b(\omega{\cdot}t)) \\ &- (A\,D(\omega{\cdot}t, u(t,\omega,x_\alpha) - b(\omega{\cdot}t)))_i \\ &+ \gamma_i(\omega{\cdot}t)(u(t,\omega,x_\alpha)_i(-\alpha_i) - b(\omega{\cdot}t)_i(-\alpha_i))| < \frac{\varepsilon}{4} \end{aligned} \tag{10.11}$$

for each $t \geq t_0$. Consequently, for each $t \geq t_0$, equations (10.10) and (10.11) yield

$$\begin{aligned}
&\|u(t,\omega,x_\gamma)(0) - b(\omega{\cdot}t)(0)\| < \frac{\varepsilon}{2} \quad \text{and} \\
&|F_i(\omega{\cdot}t, u(t,\omega,x_\gamma)) - F_i(\omega{\cdot}t, b(\omega{\cdot}t)) \\
&\qquad - (A\,D(\omega{\cdot}t, u(t,\omega,x_\gamma) - b(\omega{\cdot}t)))_i \\
&\qquad + \gamma_i(\omega{\cdot}t)(u(t,\omega,x_\gamma)_i(-\alpha_i) - b(\omega{\cdot}t)_i(-\alpha_i))| < \frac{\varepsilon}{2}\,.
\end{aligned} \tag{10.12}$$

Let $(\widetilde{\omega},\widetilde{x}) \in \mathcal{O}(\omega,x_\gamma)$, that is, $(\widetilde{\omega},\widetilde{x}) = \lim_{n\to\infty}(\omega{\cdot}t_n, u(t_n,\omega,x_\gamma))$ for some $t_n \uparrow \infty$. The monotonicity and $b(\omega) \leq_A x_\gamma$ imply that $b(\omega{\cdot}t_n) \leq_A u(t_n,\omega,x_\gamma)$, which yields $b(\widetilde{\omega}) \leq_A \widetilde{x}$. From (10.12) there is an n_0 such that, for each $i \in \{1,\ldots,m\}$,

$$\begin{aligned}
&0 \leq u(t_n,\omega,x_\gamma)_i(0) - b(\omega{\cdot}t_n)_i(0) < \frac{\varepsilon}{2} \quad \text{and} \\
&0 \leq F_i(\omega{\cdot}t_n, u(t_n,\omega,x_\gamma)) - F_i(\omega{\cdot}t_n, b(\omega{\cdot}t_n)) \\
&\qquad - (A\,D(\omega{\cdot}t_n, u(t_n,\omega,x_\alpha) - b(\omega{\cdot}t_n)))_i \\
&\qquad + \gamma_i(\omega{\cdot}t_n)(u(t_n,\omega,x_\alpha)_i(-\alpha_i) - b(\omega{\cdot}t_n)_i(-\alpha_i)) < \frac{\varepsilon}{2}
\end{aligned}$$

for each $n \geq n_0$. Hence,

$$\begin{aligned}
&0 \leq \widetilde{x}_i(0) - b(\widetilde{\omega})_i(0) \leq \frac{\varepsilon}{2} \quad \text{and} \\
&0 \leq F_i(\widetilde{\omega},\widetilde{x}) - F_i(\widetilde{\omega}, b(\widetilde{\omega})) - (A\,D(\widetilde{\omega}, \widetilde{x} - b(\widetilde{\omega})))_i \\
&\qquad + \gamma_i(\widetilde{\omega})(\widetilde{x}_i(-\alpha_i) - b(\widetilde{\omega})_i(-\alpha_i)) \leq \frac{\varepsilon}{2}\,.
\end{aligned}$$

As before, since this is true for each $(\widetilde{\omega},\widetilde{x}) \in \mathcal{O}(\omega,x_\gamma)$ admitting a flow extension, and $(\widetilde{\omega}{\cdot}s, \widetilde{x}_s) \in \mathcal{O}(\omega,x_\gamma)$, we deduce that

$$\begin{aligned}
&0 \leq \widetilde{x}_i(s) - b(\widetilde{\omega})_i(s) \leq \frac{\varepsilon}{2} \quad \text{and} \\
&0 \leq F_i(\widetilde{\omega}{\cdot}s, \widetilde{x}_s) - F_i(\widetilde{\omega}{\cdot}s, b(\widetilde{\omega}{\cdot}s)) - (A\,D(\widetilde{\omega}{\cdot}s, \widetilde{x}_s - b(\widetilde{\omega}{\cdot}s)))_i \\
&\qquad + \gamma_i(\widetilde{\omega}{\cdot}s)(\widetilde{x}_i(s-\alpha_i) - b(\widetilde{\omega})_i(s-\alpha_i)) \leq \frac{\varepsilon}{2}
\end{aligned} \tag{10.13}$$

for each $s \in (-\infty, 0]$. Given $(\widetilde{\omega}, z) \in K$, $(\widetilde{\omega},\widetilde{x}) \in \mathcal{O}(\omega,x_\gamma)$ and $s \in (-\infty, 0]$, equations (10.9) and (10.13) yield

$$\begin{aligned}
&\widetilde{x}_i(s) \leq z_i(s) \quad \text{and} \\
&F_i(\widetilde{\omega}{\cdot}s, z_s) - F_i(\widetilde{\omega}{\cdot}s, \widetilde{x}_s) - (A\,D(\widetilde{\omega}{\cdot}s, z_s - \widetilde{x}_s))_i \\
&\qquad + \gamma_i(\widetilde{\omega}{\cdot}s)(z_i(s-\alpha_i) - \widetilde{x}_i(s-\alpha_i)) \geq 0
\end{aligned} \tag{10.14}$$

for all $i \in J$. Let us check that the equality holds in (10.14) when $i \notin J$. First of all, we know that $\widetilde{x}_i = b(\widetilde{\omega})_i = z_i$, whence

$$(z_i(s) - \widetilde{x}_i(s)) - c_i(\widetilde{\omega}{\cdot}s)(z_i(s-\alpha_i) - \widetilde{x}_i(s-\alpha_i)) = 0$$

for all $s \le 0$, and, therefore,

$$\begin{aligned} 0 =& \frac{d}{ds}(z_i(s) - c_i(\widetilde{\omega}{\cdot}s)\, z_i(s-\alpha_i)) - \frac{d}{ds}(\widetilde{x}_i(s) - c_i(\widetilde{\omega}{\cdot}s)\, \widetilde{x}_i(s-\alpha_i)) \\ =& F_i(\widetilde{\omega}, \widetilde{x}) - F_i(\widetilde{\omega}, z)\,, \end{aligned}$$

$$0 = a_i((z_i(0) - \widetilde{x}_i(0)) - c_i(\widetilde{\omega})(z_i(-\alpha_i) - \widetilde{x}_i(-\alpha_i))) = (A\, D(\widetilde{\omega}, z - \widetilde{x}))_i$$
$$\text{and} \quad 0 = \gamma_i(\widetilde{\omega}{\cdot}s)(z_i(s-\alpha_i) - \widetilde{x}_i(s-\alpha_i))\,.$$

Consequently, this fact together with (10.14) implies that, for all $(\widetilde{\omega}, z) \in K$, all $(\widetilde{\omega}, \widetilde{x}) \in \mathcal{O}(\omega, x_\gamma)$ and all $s \in (-\infty, 0]$,

$$\begin{aligned} & z \ge \widetilde{x} \quad \text{and} \\ & \frac{d}{ds} D_i(\widetilde{\omega}{\cdot}s, z_s - \widetilde{x}_s) - (A\, D(\widetilde{\omega}{\cdot}s, z_s - \widetilde{x}_s))_i \\ & + \gamma_i(\widetilde{\omega}{\cdot}s)(z_i(s-\alpha_i) - \widetilde{x}_i(s-\alpha_i)) \ge 0 \end{aligned} \tag{10.15}$$

for all $i \in \{1, \dots, m\}$. Let $y = z - \widetilde{x}$ and $\widehat{y} = \widehat{D}_2(\widetilde{\omega}, y)$. It is clear that y is continuously differentiable because so are z and $\widetilde{x}$. As a result, (10.15) can be written as

$$\begin{aligned} & y \ge 0 \quad \text{and} \\ & \widehat{y}\,'_i(s) - a_i\, \widehat{y}_i(s) + \gamma_i(\widetilde{\omega}{\cdot}s)\, y_i(s-\alpha_i) \ge 0, \quad s \le 0 \end{aligned}$$

for all $i \in \{1, \dots, m\}$. This inequality and (9.1) yield the following telescopic series

$$\begin{aligned} y_i'(s) \ge & a_i \sum_{n=0}^{\infty} c_i^{[n]}(\widetilde{\omega}{\cdot}s)\, \widehat{y}_i(s - n\,\alpha_i) \\ & - \sum_{n=0}^{\infty} \gamma_i(\widetilde{\omega}{\cdot}(s - n\,\alpha_i))\, c_i^{[n]}(\widetilde{\omega}{\cdot}t)\, y_i(s - (n+1)\,\alpha_i) \\ & + \sum_{n=0}^{\infty} \frac{d}{dt} c_i^{[n]}(\widetilde{\omega}{\cdot}(t+s))\big|_{t=0}\, \widehat{y}_i(s - n\,\alpha_i) \\ =& a_i\, y_i(t). \end{aligned}$$

As a consequence, $y \ge 0$ and $y' \ge A\, y$, and we conclude that $\widetilde{x} \le_A z$.

Since this holds for each $(\widetilde{\omega}, z) \in K$, the definition of a implies that $b(\widetilde{\omega}) \leq_A \widetilde{x} \leq_A a(\widetilde{\omega})$. From (10.8) we know that $\mathcal{O}(\widetilde{\omega}, a(\widetilde{\omega})) = K^*$, and therefore $\mathcal{O}(\widetilde{\omega}, \widetilde{x}) = K^* \subset \mathcal{O}(\omega, x_\gamma)$. Finally, from Propositions 10.9 and 10.10, Proposition 4.3, and Theorem 10.4, we know that $\mathcal{O}(\omega, x_\gamma)$ is a minimal set, and we conclude that $\mathcal{O}(\widetilde{\omega}, \widetilde{x}) = \mathcal{O}(\omega, x_\gamma) = K^*$, that is, $\gamma \in L$, a contradiction. Therefore, $L = [0, 1]$ and $\mathcal{O}(\omega_0, x_0) = K^*$, as stated. $\square$

Conclusions

In this work, a quite general type of equations has been studied: neutral functional differential equations with infinite delay and non-autonomous operator. Likewise, we have presented some monotonicity results for such equations with respect to different order relations which generalize and, in some cases, improve other order relations in the previous literature.

In addition, we have introduced the concepts of stability associated to an order relation, which are necessary when it comes to studying the qualitative behavior of the solutions of these equations. Besides, under some monotonicity assumptions for each of the aforementioned order relations, the 1-covering property of omega-limit sets has been established.

Finally, we have studied compartmental systems in the foregoing framework, and we have given a global view of the long-term behavior of such systems. In order to do this, we have introduced a new approach to the geometry presented by the pipes of such models, and this tool has been used to tackle the study of the systems.

References

[A] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[AP] L. Amerio, G. Prouse, *Almost-Periodic Functions and Functional Equations*, The University Series in Higher Mathematics, Van Nostrand Reinhold Company, New York, 1971.

[AB] O. Arino, F. Bourad, *On the asymptotic behavior of the solutions of a class of scalar neutral equations generating a monotone semiflow*, J. Differential Equations **87** (1990), 84–95.

[AH] O. Arino, E. Haourigui, *On the asymptotic behavior of solutions of some delay differential systems which have a first integral*, J. Math. Anal. Appl. **122** (1987), 36–46.

[AM] K. Ashizawa, R. Miyazaki, *Asymptotic constancy for a linear differential system with multiple delays*, Appl. Math. Lett. **19** (2006), 1390–1394.

[AF] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, Basel, Berlin, 1990.

[AD] S. Audoly, L. D'Angiò, *On the identifiability of linear compartmental systems: a revisited transfer function approach based on topological properties*, Math. Biosci. **66** (1983), 201–228.

[Be] A.S. Besicovitch, *Almost Periodic Functions*, Dover, New York, 1954.

[Boc] S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S. **48** (1962), 2039–2043.

[Boh] H. Bohr, *Zur Theorie der fastperiodische Funktionen*, Acta Math. **46** (1925).

[Boh2] H. Bohr, *Almost Periodic Functions*, Chelsea, New York, 1947.

[Cho] G. Choquet, *Lectures on Analysis*, Vol. II, Benjamin, Mass., 1969.

[Chu] I.D. Chueshov, *Monotone Random Systems. Theory and Applications*, Lecture Notes in Math. **1779**, Springer-Verlag, Berlin, Heidelberg, 2002.

[Co] C. Corduneanu, *Almost Periodic Functions*, Chelsea Publishing Company, New York, 1968.

[Ei] J. Eisenfeld, *Partial Identification of Underdetermined Compartmental Models: A Method Based on Positive Linear Lyapunov Functions*, Math. Biosci. **132** (1996), 111–140.

[Ei2] J. Eisenfeld, *Sharp interval bounds and conditions for unique identifiability in linear compartmental systems with emphasis on the role of exits*, Nonlinear Anal. **37** (1999), 567–587.

[El] R. Ellis, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.

[Fa] J. Favard, *Lectures on Analysis*, Vol. II, Benjamin, Mass., 1969.

[Fi] A.M. Fink, *Almost periodic differential equations*, Lecture Notes in Math. **377**, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[FJ] D.M. Foster, J.A. Jacquez, *Multiple zeros for eigenvalues and the multiplicity of traps of a linear compartmental system*, Math. Biosci. **26** (1975), 89–97.

[GLS] G. Gripenberg, S.O. Londen, O. Staffans, *Volterra Integral and Functional Equations*, Encyclopedia Math. Appl., Cambridge University Press, Cambridge, New York, 1990.

[G] I. Györi, *Connections between compartmental systems with pipes and integro-differential equations*, Math. Modelling **7** (1986), 1215–1238.

[GE] I. Györi, J. Eller, *Compartmental systems with pipes*, Math. Biosci. **53** (1981), 223–247.

[GW] I. Györi, J. Wu, *A neutral equation arising from compartmental systems with pipes*, J. Dynam. Differential Equations **3** No.2 (1991), 289–311.

[HCH] W.M. Haddad, V. Chellaboina, Q. Hui, *Nonnegative and Compartmental Dynamical Systems*, Princeton University Press, 2010.

[HKTW] J.R. Haddock, T. Krisztin, J. Terjéki, J. Wu, *An invariance principle of Lyapunov-Razumikhin type for neutral functional differential equations*, J. Differential Equations **107** (1994), 395–417.

[Ha] J.K. Hale, *Theory of Functional Differential Equations*, Applied Mathematical Sciences **3**, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[HM] J.K. Hale, K.R. Meyer, *A class of functional equations of neutral type*, Mem. Amer. Math. Soc. **76**, Amer. Math. Soc., Providence, 1967.

[HV] J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences **99**, Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[HMN] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Math. **1473**, Springer-Verlag, Berlin, Heidelberg, 1991.

[Hi] M.W. Hirsch, *Stability and convergence in strongly monotone dynamical systems*, J. Reine Angew. Math., **383** (1988), 1–53.

[Ja] J.A. Jacquez, *Compartmental Analysis in Biology and Medicine*, Third Edition, Thomson-Shore Inc., Ann Arbor, Michigan, 1996.

[JS] J.A. Jacquez, C.P. Simon, *Qualitative theory of compartmental systems*, SIAM Review **35** No.1 (1993), 43–79.

[JS2] J.A. Jacquez, C.P. Simon, *Qualitative theory of compartmental systems with lags*, Math. Biosci. **180** (2002), 329–362.

[JZ] J. Jiang, X.-Q. Zhao, *Convergence in monotone and uniformly stable skew-product semiflows with applications*, J. Reine Angew. Math **589** (2005), 21–55.

[Jo] R. Johnson, *A linear, almost periodic equation with an almost automorphic solution*, Proc. Amer. Math. Soc. **82** No.2 (1981), 199–205.

[Jo2] R. Johnson, *On almost-periodic linear differential systems of Millionščikov and Vinograd*, J. Math. Anal. Appl. **85** (1982), 452–460.

[JNOT] À. Jorba, C. Núñez, R. Obaya, J.C. Tatjer, *Old and new results on strange nonchaotic attractors*, Int. J. Bifurcation Chaos **17** No.11 (2007), 3895–3928.

[KM] V. Kolmanovskii, A. Myshkis, *Introduction to the Theory and Applicactions of Functional Differential Equations*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1999.

[K] T. Krisztin, *Convergence of solutions of a nonlinear integrodifferential equation arising in compartmental systems*, Acta Scientiarum Mathematicarum **47** (1984), 471–485.

[KW] T. Krisztin, J. Wu, *Asymptotic periodicity, monotonicity, and oscillation of solutions of scalar neutral functional-differential equations*, J. Math. Anal. Appl. **199** (1996), 502–525.

[M] H. Matano, *Existence of nontrivial unstable sets for equilibriums of strongly order preserving systems*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1984), 645–673.

[MNO] V. Muñoz-Villarragut, S. Novo, R. Obaya, *Neutral functional differential equations with applications to compartmental systems*, SIAM J. Math. Anal. **40** No.3 (2008), 1003–1028.

[NS] V. Nemytskii, V. Stepanoff, *Qualitative Theory of Differential Equations*, Princeton University Press, Princeton, NJ, 1960.

[NNO] S. Novo, C. Núñez, R. Obaya, *Almost automorphic and almost periodic dynamics for quasimonotone non-autonomous functional differential equations*, J. Dynam. Differential Equations **17** No.3 (2005), 589–619.

[NOS] S. Novo, R. Obaya, A.M. Sanz, *Stability and extensibility results for abstract skew-product semiflows*, J. Differential Equations **235** No.2 (2007), 623–646.

[NOV] S. Novo, R. Obaya, V.M. Villarragut, *Exponential ordering for nonautonomous neutral functional differential equations*, SIAM J. Math. Anal. **41** No.3 (2009), 1025–1053.

[P] P. Poláčik, *Convergence in smooth strongly monotone flows defined by semilinear parabolic equations*, J. Differential Equations, **79** (1989), 89–110.

[SS] R.J. Sacker, G.R. Sell, *Lifting properties in skew-products flows with applications to differential equations*, Mem. Amer. Math. Soc. **190**, Amer. Math. Soc., Providence, 1977.

[Sa] D. Salamon, *Control and Observation of Neutral Systems*, Pitman, London, 1984.

[Sc] S. Schwartzman, *Asymptotic cycles*, Ann. Math. **66** No.2 (1957), 270–284.

[Se] G. Sell, *Nonautonomous differential equations and topological dynamics. I. The basic theory*, Trans. Amer. Math. Soc. **127** (1967), 241–262.

[Se2] G. Sell, *Nonautonomous differential equations and topological dynamics. II. Limiting equations*, Trans. Amer. Math. Soc. **127** (1967), 263–283.

[SY] W. Shen, Y. Yi, Almost Automorphic and Almost Periodic Dynamics in Skew-Product Semiflows, *Mem. Amer. Math. Soc.* **647**, Amer. Math. Soc., Providence, 1998.

[Sm] H.L. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*, Amer. Math. Soc., Providence, 1995.

[Sm2] H.L. Smith, *Linear Compartmental Systems - The Basics*, available at `http://math.la.asu.edu/~halsmith/compartment.pdf` (2006).

[ST] H.L. Smith, H.R. Thieme, *Monotone semiflows in scalar non-quasi monotone functional differential equations*, J. Math. Anal. Appl. **150** (1990), 289–306.

[ST2] H.L. Smith, H.R. Thieme, *Strongly order preserving semiflows generated by functional differential equations*, J. Differential Equations **93** (1991), 332–363.

[St] O.J. Staffans, *A Neutral FDE with Stable D-operator is Retarded*, J. Differential Equations **49** (1983), 208–217.

[V] W.A. Veech, *Almost automorphic functions on groups*, Amer. J. Math. **87** (1965), 719–751.

[V2] W.A. Veech, *Properties of minimal functions on abelian groups*, Amer. J. Math. **91** (1969), 415–441.

[V3] W.A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc. **83** (1977), 775–830.

[WW] Z. Wang, J. Wu, *Neutral Functional Differential Equations with Infinite delay*, Funkcial. Ekvac. **28** (1985), 157–170.

[W] J. Wu, *Unified treatment of local theory of NFDEs with infinite delay*, Tamkang J. Math. **22** No.1 (1991), 51–72.

[WF] J. Wu, H.I. Freedman, *Monotone semiflows generated by neutral functional differential equations with application to compartmental systems*, Can. J. Math. **43**(5) (1991), 1098–1120.

[WZ] J. Wu, X.-Q. Zhao, *Diffusive monotonicity and threshold dynamics of delayed reaction diffusion equations*, J. Differential Equations **186** (2002), 470–484.

[Y] Y. Yi, *On almost automorphic oscillations*, Difference and differential equations. Proceedings of the 7th ICDEA, Fields Institute Communications 42, 75–99. Amer. Math. Soc., Providence, 2004.

Appendix

Índice

Resumen de la tesis

Una de las cuestiones más importantes en la teoría de ecuaciones diferenciales no autónomas es la descripción a largo plazo de sus trayectorias. Cuando las funciones que definen tales ecuaciones presentan una variación recurrente en el tiempo, sus soluciones definen de manera natural un semiflujo triangular. Gracias a este semiflujo triangular, se pueden analizar en detalle las trayectorias por medio de métodos de dinámica topológica. En este trabajo, se estudia la estructura de los conjuntos omega-límite, lo cual proporciona una visión global de la dinámica de la ecuación. Es bien sabido que, en algunos casos, los conjuntos omega-límite heredan algunas de las propiedades del campo que define la ecuación; en otros casos, su dinámica

puede ser mucho más compleja.

Las ecuaciones diferenciales funcionales (abreviado FDEs) con retardo son un tipo concreto de ecuaciones diferenciales que tienen en cuenta no solo el estado actual del sistema, sino también algunos de sus estados pasados. Su interés práctico reside en el hecho de que permiten construir modelos matemáticos en los que el pasado influye en el futuro; algunas aplicaciones dignas de mención son los modelos en epidemiología, la dinámica de poblaciones y la ingeniería de control. Las ecuaciones diferenciales funcionales neutrales (abreviado NFDEs) con retardo son una generalización muy importante de tales ecuaciones. En ellas, se considera la derivada del valor de un operador en lugar de la derivada de la solución. Así, los modelos que utilizan NFDEs pueden representar incrementos y decrementos espontáneos de la solución aparte de la dependencia temporal proporcionada por las FDEs.

El estudio de las propiedades dinámicas de los semiflujos triangulares se ha abordado a menudo asumiendo ciertas condiciones de monotonía para el semiflujo. Estas condiciones son una herramienta útil a la hora de deducir el comportamiento a largo plazo de las soluciones. Cabe mencionar que hay una gran diversidad de condiciones de monotonía, que varían de la quasi-monotonía a la monotonía fuerte.

Durante décadas, se han estudiado ampliamente ecuaciones diferenciales autónomas monótonas (véanse Hirsch [Hi], Matano [M], Poláčik [P] y Smith [Sm], entre muchos otros). Bajo hipótesis adecuadas, se ha demostrado que las trayectorias relativamente compactas de un semiflujo fuertemente monótono convergen genéricamente hacia el conjunto de equilibrios. Posteriormente, Smith y Thieme [ST, ST2] estudiaron la dinámica del semiflujo inducido por una FDE con retardo finito que es monótona para el orden exponencial. Esta relación de orden es técnicamente complicada, pero les permitió estudiar ecuaciones que no satisfacen la condición quasi-monótona habitual asociada al orden estándar. Krisztin y Wu [KW], y Wu y Zhao [WZ] extendieron estos resultados para NFDEs con retardo finito y ecuaciones de evolución, respectivamente.

Últimamente, se ha hecho un gran esfuerzo para estudiar ecuaciones diferenciales no autónomas monótonas deterministas y aleatorias, lo cual ha proporcionado una teoría dinámica tanto para el orden estándar como para el orden exponencial (véanse por ejemplo Chueshov [Chu], Jiang y Zhao [JZ], Muñoz-Villarragut, Novo y Obaya [MNO], Novo, Obaya y Villarragut [NOV], [NOS] y [SY]). Asumiendo ciertas propiedades de acotación, compacidad relativa y estabilidad uniforme de las trayectorias, esta teoría asegura la convergencia de las órbitas hacia soluciones que reproducen la dinámica exhibida por la variación temporal de la ecuación. Cabe destacar que, cuando se trata de FDEs con retardo infinito o, más en general, NFDEs con retardo infinito, la propiedad de monotonía fuerte nunca es cierta, con lo que se deben hacer suposiciones más débiles acerca de la monotonía del semiflujo.

El origen de esta teoría se remonta a los años 70, cuando Sacker y Sell [SS] demostraron algunos resultados previos sobre la estructura de los conjuntos omega-límite en el caso de ecuaciones casi periódicas. Más adelante, Shen y Yi [SY] continuaron con su trabajo en el caso de un flujo distal en la base. Se pueden encontrar resultados más generales en Novo, Obaya y Sanz [NOS]; concretamente, estudiaron la estructura de los conjuntos omega-límite en BU, el espacio de funciones de $(-\infty, 0]$ en $\mathbb{R}^m$ que son acotadas y uniformemente continuas, dotado de la topología compacto-abierta, cuando el flujo de la base es solo minimal y asumiendo una propiedad de estabilidad que está íntimamente relacionada con la distalidad en la fibra. También dedujeron que es apropiado considerar esa topología al estudiar NFDEs con retardo infinito pues, bajo hipótesis naturales, las restricciones de los semiflujos definidos por estas ecuaciones a sus conjuntos omega-límite resultan ser continuas.

Se puede hacer un estudio alternativo de las soluciones recurrentes de FDEs casi periódicas utilizando espacios de memoria evanescente (véase Hino, Murakami y Naito [HMN] para una definición axiomática y algunas sus propiedades más importantes), aunque, bajo hipótesis naturales, la topología de la norma en estos espacios coincide con la topología compacto-abierta en la adherencia de las trayectorias relativamente compactas, lo cual hace que el enfoque de [NOS] parezca más razonable.

Se puede encontrar otro planteamiento interesante del estudio de NFDEs en Staffans [St], donde se establece que cualquier NFDE con retardo finito y operador estable y autónomo se puede escribir como una FDE con retardo infinito en un espacio de memoria evanescente adecuado. Gripenberg, Londen y Staffans [GLS] estudian las propiedades fundamentales del operador de convolución asociado a la ecuación. Estas ideas fueron utilizadas en algunos artículos posteriores (véanse por ejemplo Arino y Bourad [AB], y Haddock, Krisztin, Terjéki y Wu [HKTW]). Se pueden encontrar resultados más generales en esta línea en [MNO] y [NOV], donde se consideran operadores lineales autónomos con retardo infinito. Muchos problemas que habían sido resueltos anteriormente para FDEs se han generalizado al caso de NFDEs; a su vez, estas extensiones han planteado interesantes problemas que dan lugar al marco actual. En el caso de NFDEs con retardo infinito y operador autónomo, una transformación tanto del orden estándar como del orden exponencial por medio del operador de convolución asociado a la ecuación proporciona la herramienta necesaria para lograr los resultados esperados, como se puede ver en [MNO] y [NOV].

Algunos de los muchos modelos que consisten en NFDEs con retardo son los modelos compartimentales. Están formados por varios compartimentos unidos por medio de tuberías; los compartimentos contienen cierto material que fluye entre ellos a través de las tuberías, y esto ocurre en una cantidad de tiempo no despreciable. A su vez, los compartimentos crean y destruyen material, lo cual queda representado por la parte neutral de la ecuación. El interés teórico de estos modelos reside en la existencia de una integral primera que garantiza ciertas propiedades de estabilidad para el semiflujo que son esenciales en la teoría. Estas NFDEs modelan procesos físicos y biológicos para los que hay un balance que no es instantáneo, aunque se han utilizado en otras áreas como la economía. Algunas de estas aplicaciones son la ecología, la epidemiología, la farmacología, la termodinámica, la teoría de control y la cinemática de medicamentos (véanse Eisenfeld [Ei2], y Haddad, Chellaboina y Hui [HCH], entre muchos otros).

Los sistemas compartimentales se han utilizado como modelos matemáticos para el estudio del comportamiento dinámico de muchos procesos en las ciencias biológicas y físicas (véanse Jacquez [Ja], Jacquez y Simon [JS, JS2], y las referencias que allí aparecen). Algunos resultados iniciales para el caso de FDEs con retardo finito e infinito se deben a Györi [G], y Györi y Eller [GE]. Más adelante, Arino y Haourigui [AH] demostraron que los sistemas compartimentales descritos por FDEs casi periódicas con retardo finito dan lugar a ciertas soluciones casi periódicas. Györi y Wu [GW], Wu [W], Wu y Freedman [WF], [AB] y [KW] estudiaron el caso de sistemas compartimentales representados por NFDEs con retardo finito e infinito y operador autónomo. Más recientemente, estos resultados fueron extendidos en [MNO] y [NOV], concluyéndose que las trayectorias relativamente compactas convergen a soluciones que reproducen la variación temporal de la ecuación y, lo que es más, se puede predecir cuál será la cantidad final de material dentro de los compartimentos en función de la geometría de las tuberías.

En este trabajo, estudiamos NFDEs no autónomas con operador lineal no autónomo y retardo infinito. Es esta situación, las conclusiones principales que hay en la literatura previa no siguen siendo válidas y, así, la extensión de la teoría requiere el uso de una definición alternativa de orden exponencial que se pueda aplicar en el contexto actual, preservando las propiedades dinámicas de la teoría anterior. Asumimos algunas propiedades de recurrencia en la variación temporal de la NFDE; así, sus soluciones inducen un semiflujo triangular con flujo minimal en la base, Ω. En concreto, los casos casi periódico y casi automórfico quedan incluidos en esta formulación. Invertimos el operador de convolución asociado a la ecuación, generalizando resultados previos en esta línea encontrados en [MNO]. Las propiedades de regularidad de este operador de convolución dependen del tipo de recurrencia presentada por la variación temporal de la ecuación. Asimismo, se consideran nuevas relaciones transformadas de orden, asociadas tanto al orden estándar como al orden exponencial; como el operador es no autónomo, este orden parcial no está definido en BU, sino en cada una de las fibras del producto $\Omega \times BU$. De este modo, damos una versión alternativa a la estructura de orden introducida en [WF] que es válida en el caso de operadores no autónomos. Cuando se utiliza BU como espacio de fase, la teoría estándar de NFDEs proporciona existencia, unicidad y dependencia continua de las soluciones. Esto nos permite estudiar la estructura de los conjuntos omega-límite de las trayectorias acotadas cuando la ecuación satisface ciertas propiedades de monotonía, mejorando resultados previos que aparecen en [MNO], [NOV], [ST] y [ST2], entre otros.

El uso del orden exponencial transformado hace posible imponer condiciones de monotonía que no requieren la diferenciabilidad de los coeficientes que definen el operador, sino solo su continuidad. Esto hace que el orden exponencial transformado sea más natural que el orden exponencial directo cuando el operador es no autónomo. Estos resultados teóricos se aplican a sistemas compartimentales y, de este modo, obtenemos conclusiones bajo condiciones más generales que las presentadas en la literatura previa, mejorando así algunos resultados previos para sistemas dinámicos que son monótonos para el orden exponencial incluso en sus versiones autónomas. Concretamente, describimos la cantidad final de material dentro de los compartimentos en el caso de sistemas compartimentales definidos por NFDEs con operador no autónomo y retardo infinito.

No obstante, en los Capítulos 9 y 10, asumimos la diferenciabilidad de los coeficientes que definen el operador y estudiamos algunos sistemas compartimentales que son monótonos para el orden exponencial directo. Además, mostramos que la propiedad de 1-recubrimiento de los conjuntos omega-límite es cierta, extendiendo de esta forma resultados anteriores de [KW] al caso de NFDEs con variación recurrente en el tiempo.

Resumen por capítulos

Capítulo 1

En este capítulo, se introducen los conceptos básicos de la teoría de sistemas dinámicos. A saber, se definen los flujos reales y se dan sus propiedades básicas, que permiten definir adecuadamente el concepto de conjunto omega-límite asociado a estos flujos. Tras esto, se definen los semiflujos y se presenta una clase especial de semiflujos que tendrán una relevancia especial en el trabajo: los semiflujos triangulares; se define también el concepto de conjunto omega-límite para estos semiflujos. Se recuerdan los resultados básicos sobre estabilidad y extensibilidad del artículo [NOS], que serán de gran utilidad. Se presenta a continuación el marco adecuado para el estudio de ecuaciones diferenciales con retardo infinito y que tienen por base un flujo minimal, y se dan las propiedades fundamentales de las funciones casi periódicas y casi automórficas, que darán lugar a este tipo de flujos en la base. Asimismo, se da un ejemplo concreto de cómo incluir una ecuación definida a priori en un espacio no separable dentro de una familia de ecuaciones de forma que pueda ser estudiada con un semiflujo triangular. Finalmente, se presentan los espacios de Banach ordenados y las definiciones de semiflujo monótono y fuertemente monótono.

Capítulo 2

A lo largo de este capítulo, se estudian las propiedades de un operador lineal no autónomo y de las ecuaciones en diferencias en el pasado y en el futuro asociadas a tal operador. Para esto, se consideran dos topologías en el espacio BU: la topología inducida por la norma del supremo y la topología compacto-abierta. Se supone también que el operador es atómico, propiedad clave para su invertibilidad. Además, se estudia la estabilidad de este operador por medio del comportamiento asintótico de las soluciones de las citadas ecuaciones en diferencias. Estos operadores serán los que aparezcan en las ecuaciones diferenciales funcionales neutrales en capítulos posteriores.

Capítulo 3

Presentamos aquí la teoría que respecta al operador de convolución asociado al operador estudiado en el Capítulo 2. Concretamente, se deducen su invertibilidad y la continuidad tanto de este operador como de su operador inverso para la topología compacto-abierta. Si la variación temporal del operador inicial resulta ser casi periódica, entonces se deduce que tanto este operador como su inverso son también continuos cuando se considera la topología de la norma. Esto supone una diferencia importante con el caso autónomo. Por último, se presenta un operador concreto y se estudia bajo qué condiciones satisface todas las hipótesis hechas a lo largo de los Capítulos 2 y 3. Estos resultados serán útiles a la hora de estudiar las aplicaciones que se presentan en los siguientes capítulos.

Capítulo 4

En este capítulo, se aborda el estudio de ecuaciones diferenciales funcionales neutrales con operador no autónomo. Para ello, se hace uso de los resultados obtenidos en los Capítulos 2 y 3. Además, se presenta una estructura de orden nueva, que es la transformada por medio del operador de convolución del Capítulo 3 de la relación de orden usual. Esta relación de orden no había sido utilizada nunca en la literatura y define un orden diferente en cada fibra del producto sobre el que está definido el semiflujo con el que estudiamos el sistema de ecuaciones. Asumiendo que el semiflujo es monótono para el nuevo orden, se pueden aplicar los resultados presentados en el Capítulo 1 y deducir la propiedad de 1-recubrimiento para este escenario.

Capítulo 5

Introducimos aquí los sistemas compartimentales como modelos de fenómenos físicos y biológicos que se adaptan perfectamente al marco teórico desarrollado en los capítulos anteriores. Aplicamos los resultados obtenidos en el Capítulo 4 al estudio de un sistema compartimental. Para ello, definimos el concepto de masa total y esta resulta ser una integral primera para el sistema, proporcionando así los resultados de estabilidad uniforme necesarios. Finalmente, concluimos que, bajo condiciones muy generales, los conjuntos omega-límite son copias del flujo de la base.

Capítulo 6

Se aborda en este capítulo el estudio de la cantidad de material dentro de los compartimentos de un sistema compartimental a largo plazo. Para ello, se define el concepto de conjunto irreducible de compartimentos, que resulta ser un subsistema compartimental del inicial para el que la dinámica es conocida y que permite, de este modo, dibujar el comportamiento a largo plazo del sistema global. Así pues, se dan resultados acerca de la cantidad final de material dentro de los compartimentos en el caso de sistemas que pueden tener o no entradas y salidas de material desde y hacia el entorno. También se da un teorema de etiquetado para los conjuntos omega-límite en el caso cerrado. Finalmente, se presentan varios ejemplos concretos que han despertado un gran interés en la literatura y sobre los que podemos dar información nueva.

Capítulo 7

Se estudian aquí ecuaciones diferenciales funcionales neutrales con operador no autónomo que no satisfacen las propiedades de monotonía exigidas en el Capítulo 4, pero sí cumplen ciertas propiedades de monotonía con respecto al orden exponencial transformado, que se obtiene a partir del orden exponencial por medio del operador de convolución estudiado en el Capítulo 3. Nuevamente, este operador no había sido utilizado en la literatura y proporciona un orden distinto en cada fibra del producto en que está definido el semiflujo. Definimos además un concepto novedoso de estabilidad uniforme con respecto a la relación de orden citada que es mucho menos restrictivo que el utilizado comúnmente. En estas condiciones, establecemos la propiedad de

1-recubrimiento para los conjuntos omega-límite.

Capítulo 8

En este capítulo, se estudian de nuevo los modelos compartimentales. A diferencia de las hipótesis hechas en el Capítulo 5, ahora las condiciones de monotonía son las asociadas al orden exponencial transformado, lo cual permite estudiar sistemas que no quedaban incluidos en la teoría del Capítulo 5. Esto generaliza a su vez resultados previos de la literatura.

Capítulo 9

Abordamos en este capítulo el estudio de ecuaciones diferenciales funcionales neutrales con operador no autónomo, asumiendo ciertas propiedades de diferenciabilidad del operador y que son monótonas para el orden exponencial. Específicamente, se estudian las propiedades de monotonía del semiflujo asociado a estas ecuaciones y se dan condiciones suficientes para que tal semiflujo sea monótono. Asimismo, se deduce bajo qué condiciones se puede asegurar que el semiflujo satisface una propiedad de ignición componente a componente.

Capítulo 10

Finalmente, en el marco del Capítulo 9, describimos la estructura topológica de conjuntos con ciertas propiedades de estabilidad, extendiendo así resultados previos. Esto permite establecer la propiedad de 1-recubrimiento para conjuntos omega-límite, obteniéndose así el resultado esperado.

Conclusiones

En esta memoria, se ha estudiado un tipo muy general de ecuaciones, como son las ecuaciones diferenciales funcionales neutrales con retardo infinito y operador no autónomo. Asimismo, se han presentado resultados de monotonía para dichas ecuaciones con respecto a diferentes relaciones de orden que generalizan y, en algunos casos, mejoran otras relaciones de orden que aparecen en la literatura previa. También se han presentado conceptos de estabilidad asociados a una relación de orden, que son necesarios para el estudio del comportamiento cualitativo de las soluciones de estas ecuaciones. Además, se ha establecido la propiedad de 1-recubrimiento para conjuntos omega-límite bajo condiciones de monotonía para cada uno de los órdenes antes citados.

Por último, se han estudiado sistemas compartimentales dentro del marco anterior, dándose una visión global del comportamiento de tales sistemas en largos periodos de tiempo. Para ello, se ha presentado una visión novedosa de la geometría que presentan las tuberías de tales modelos y se ha utilizado esta herramienta para abordar su estudio.