The structure of p-local compact groups

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CERTIFICO que la present Memòria ha estat realitzada per n'Alex González de Miguel,sota la direcció del Dr. Carles Broto Blanco.

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Preface

The importance of Lie groups in all areas of mathematics is indisputable. They have been studied from several points of view, and the richness of their structure makes them to be present not only in all areas of mathematics, but also in other areas of science.

The theory of Lie groups was completely developed with the classification of all simple connected compact Lie groups, which cannot be attributed to a single mathematician but was achieved thanks to the contributions of several people, like É. Cartan, W. Killing, A. Borel, E. B. Dynkin and H. S. M. Coxeter among others.

As algebraic topologists, the natural step to take from that point was then to develop models isolating some of the topological properties of compact Lie groups. There is not a single way to proceed, and several models could be defined, depending on the properties we want to study.

Our particular interest then lies on the topological properties of p-completions of classifying spaces of compact Lie groups. Roughly speaking, by p-completing a space, we isolate the information that we may obtain from the cohomology of the space at the prime p from other information that may make the picture more difficult to understand.

From this point of view, a first generalization was introduced in [DW94]: the well-known p-compact groups. In fact, this model does not generalizes all compact Lie groups, but only those compact Lie groups whose group of components is a p-group. This new class of spaces was completely classified in [AGMV08] at the prime p odd, and in [AG09] and [Møl07] at the prime $p = 2$. In [DW94], the authors proved that the mod p cohomology of a finite loop space is always a finitely generated algebra.

More recently, p-local compact groups were introduced in [BLO07] as such a more general model. It has been proved in [BLO07] that indeed p-local compact groups include all (p-completions of classifying spaces of) compact Lie groups, as well as all p-compact groups. However, the more general the model is, the more difficult its study becomes. In this sense, we are far from describing all p-local compact groups in terms of a (smaller) well-understood list of p-local compact groups, and several basic properties of them have to be "explored" before this can be done.

In this work, then, we try to cover some gaps in the newborn theory of p-local compact groups, such as the definition and (some) properties of connected p-local compact groups, and the mod p cohomology rings of p-local compact groups. While the hopes are high that most of the constructions and results in this work will sooner or later be proved to hold in the general case, the theory of p-local compact groups is rather evasive, and it will not be an easy task.

Below we summarize briefly the work done in this memory, and we refer the reader to each chapter for further details on a specific subject.

The first chapter introduces the notion of a p-local compact group as a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, where S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S (roughly speaking, a category whose objects are the subgroups of S and such that the morphisms among objects are actual group monomorphisms satisfying some set of axioms(I), (II) and (III)), and where $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$ (roughly speaking, a category whose objects are the $\mathcal{F}$-centric subgroups of S, and such that the morphism sets are extensions of the corresponding morphism sets in $\mathcal{F}$, again satisfying a set of axioms (A), (B) and (C)). We also summarize all the results from [BLO07] that we will use at some point in this memory. We complement the definitions and results from this source with some useful properties of p-local compact groups which do not appear in [BLO07], like the fact that the normalizer fusion subsystem of a fully normalized subgroup is always saturated (a feature which was proved to hold for p-local finite groups in [BLO03b]), and an equivalent set of axioms for the saturation of a fusion system, inspired in the paper [KS08].

Each of the following chapters introduces original work on p-local compact groups and saturated fusion systems.

The second chapter studies the realization of saturated fusion systems by infinite groups, in the same way as this was done for finite fusion systems in [Rob07], that is, using amalgams of locally finite artinian groups. Solving this problem in particular requires extending the results on constrained saturated fusion systems done in [BCG⁺05] §4, which turns out not to be difficult in the exercise in the case of saturated fusion systems over discrete p-toral groups, thanks to the results on higher limits developed in [BLO07] §5. While we were originally planning to apply the results on realization of saturated fusion systems to avoid assuming the existence of a centric linking system, in general the groups one obtains using the results in chapter §2 are rather difficult to work with (this is not at all surprising, in view of the results in [Rob07], where it is already shown that in the case of finite fusion systems one already will find the resulting groups difficult to work with).

The third chapter introduces a notion of connectivity of saturated fusion systems and p-local compact groups, and studies the existence of connected components in the case the group S has rank 1. In this sense, we first give a list of all connected p-local compact groups of rank 1, and also show that each connected fusion system over a rank 1 p-local compact group has a unique associated linking system. This list just confirms what was initially expected, i.e., that rank 1 connected p-local compact groups correspond to connected compact Lie groups of rank 1 (that is, S^1, $SO(3)$ and S^3). Note that the same was proved to happen when classifying p-compact groups.

In addition to this list, we also prove that each saturated fusion system over a rank 1 discrete p-toral group determined a unique connected saturated fusion subsystem over a discrete p-toral subgroup of rank 1, which can be then considered as the connected component of the original fusion system. The corresponding result on rank 1 p-local compact groups is also proved, although needs a better explaining. In this case, such a p-local compact group $\mathcal{G}$ determines a unique connected rank 1 p-local compact group $\mathcal{G}_0$, which we think of as the connected component of $\mathcal{G}$. However, to properly consider $\mathcal{G}_0$ as a p-local compact subgroup of $\mathcal{G}$, some kind of inclusion, at least at the level of linking systems, had to be constructed. Such an inclusion functor has indeed been constructed, but in a rather *ad hoc* way. This inclusion, in fact, provides an example of a morphism (i.e., functor) between linking systems which does not send centric subgroups to centric subgroups. More examples of such functors are provided in chapter 5. It remains to be solved the problem of a proper classification of all rank 1 p-local compact groups in terms of the list of connected rank 1 p-local compact groups that we provide.

The fourth chapter introduces unstable Adams operations for p-local compact groups. The first two sections in this chapter contain the construction of such operations on p-local compact groups originally done in [Jun09], and the third section uses this ideas to construct unstable Adams operations on the groups realizing saturated fusion systems which we previously studied in chapter 2. There is in fact not much difficulty in doing so, since both the construction of such groups and the construction of unstable Adams operations in [Jun09] are somehow similar.

The fifth chapter studies the action of unstable Adams operations on a given p-local compact group, and is probably one of the most important chapters in this memory, together with chapter 3. More concretely, we study the fixed points of a given p-local compact group $\mathcal{G}$ under the action of families of unstable Adams operations $\{\Psi_i\}$.

This problem can be approached from different points of view, and we have chosen an algebraic way of treating the problem. Thus, we propose a definition of the invariants of $\mathcal{G}$ under the action of each Ψ_i which is not the obvious one (since it seems not to work in general) and we prove that there always exists some M such that, for all $i \geq M$, we obtain triples $\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i)$ which are almost p-local finite groups. However, the last condition to show in order to prove that the $\mathcal{G}_i$ are p-local finite groups is rather technical and quite difficult to prove in general.

Nevertheless, we prove that indeed the $\mathcal{G}_i$ are p-local finite groups for some specific situations, the most important being when $\mathcal{G}$ is a rank 1 p-local compact group.

This study allows us to conclude, for instance, that the classifying space of $\mathcal{G}$ is the (p-completion of the) homotopy colimit of the classifying spaces of the $\mathcal{G}_i$, which in fact come equipped with inclusions $\mathcal{G}_i \hookrightarrow \mathcal{G}_{i+1}$. In particular, when the $\mathcal{G}_i$ are proved to be p-local finite groups, this again provides examples of functors between linking systems which do not send centric subgroups to centric subgroups. Furthermore, if the $\mathcal{G}_i$ are p-local finite groups, then one can prove a version of the Stable Elements theorem for p-local compact groups, using the same result on p-local finite groups (proved in [BLO03b]).

The two appendices at the end contain technical results needed all along this memory. We have chosen to place them at the end and in different chapters due to the extension of the contents in each of them.

The first appendix deals with extensions of p-local compact groups by discrete p-toral groups, using the more general setting of transporter systems introduced for p-local finite groups in [OV07]. In fact, in this chapter we just prove that the results in the former paper extend as expected to the compact case.

The second chapter deals with saturated fusion subsystems of p-power index and index prime to p of a given saturated fusion system. Again, we extend the known results from [BCG+07] for p-local finite groups to the compact case, while in this case we cannot extend the whole of the contents in the former paper since a better understanding of quasicentric subgroups would be needed first. In particular, we prove the existence of a unique minimal saturated subsystem of index prime to p, a result which can be extended to a result on p-local compact groups, and which is fundamental in the definition of connectivity for p-local compact groups.

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Certainly, starting a PhD has been a great experience, and before concluding these aknowledgements, I would simply like to express again my gratitude to everyone who, directly or indirectly, has participated in. Many thanks to everybody!

Chapter 1

Background on p-local compact groups

In this chapter we introduce p-local compact groups and list the properties which will be used in the rest of this work. Most of the notions and results are taken from the main source about p-local compact groups, that is, the paper by C. Broto, R. Levi and B. Oliver [BLO07], and hence we have organized this chapter following the same order as the original paper.

More specifically, the first section is devoted to discrete p-toral groups, which will play the role of Sylow p-subgroups in p-local compact groups. Then, in the second section we introduce fusion systems over discrete p-toral groups, and in the third section we present a rather powerful construction on saturated fusion systems which will allow us to reduce computations on a finite number of (conjugacy classes of) subgroups. The fourth section introduces centric linking systems associated to saturated fusion systems, as well as classifying spaces of p-local compact groups. In this section we have also included some results from sections §8, 9 and 10 [BLO07]. The fifth section contains some results on higher limits which will be useful when showing the vanishing of some obstructions in later chapters. The sixth section studies normalizer fusion subsystems, in the same way as they were studied in appendix §A of [BLO03b]. Finally, the seventh section provides an alternative definition for saturation of fusion systems which is inspired in definition 2.4 [KS08].

This chapter is to be understood as a list of basic definitions and properties to use in later chapters on this work. We do not provide proofs of results already proved in other sources (mainly from [BLO07]) for the sake of simplicity. The reader is then refered to the corresponding source for further details.

1.1 Discrete p-toral groups

Discrete p-toral groups are the natural replacement for Sylow p-subgroups in order to extend the definitions of fusion and linking systems from [BLO03b]. They were already proved to be a resourceful tool to use on the study of p-compact groups, as shown in [DW94].

Let $\mathbb{Z}/p^\infty$ be the union of the cyclic groups $\mathbb{Z}/p^n$ under the obvious inclusions. We will think of $\mathbb{Z}/p^\infty$ as a multiplicative group for convinience. Note that $\mathbb{Z}/p^\infty$ is an infinitely p-divisible group, that is, for each $x \in \mathbb{Z}/p^\infty$ there exists $y \in \mathbb{Z}/p^\infty$ such that $y^p = x$.

Definition 1.1.1. *A **discrete** p**-toral group** is a group P which contains a normal subgroup $T_P \lhd P$, isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$, and such that the quotient P/T_P is a finite p-group.*

*For such a group, we say that T_P is the **connected component** or **maximal torus** of P and P/T_P is the **group of components** of P. We say also that the discrete p-toral group P is connected if $P = T_P$.*

*The **rank** of a discrete p-toral group P is the rank of the maximal torus T_P. That is, if $T_P \cong (\mathbb{Z}/p^\infty)^r$, then we will say that $rk(P) = r$.*

The connected component of a discrete p-toral group P can be characterized as the subgroup of all infinitely p-divisible elements of P, that is, for all $x \in P$, there exists some $y \in P$ such that $y^p = x$. This is a property that will be used repeatedly along this work. If $T_P \cong (\mathbb{Z}/p^\infty)^k$, then we can define $rk(P) = k$, and

$$|P| \stackrel{def}{=} (rk(P), |P/T_P|). \tag{1.1}$$

The order of a discrete p-toral group is then considered as an element in $\mathbb{N}^2$ with the lexicographical order. Thus, we say that $|P| \leq |P'|$ if and only if $rk(P) < rk(P')$, or $rk(P) = rk(P')$ and $|P/T_P| \leq |P'/T_{P'}|$. In particular, $P \leq P'$ implies $|P| \leq |P'|$.

This class of groups will play the role of Sylow p-subgroups for p-local compact groups. As explained in [BLO07], the reason for choosing discrete p-toral groups for such a mission is that they enjoy certain finiteness properties, necessary for the theory to work. We describe these properties below, since we will also make (implicit or explicit) use of them.

A group G is **locally finite** if every finitely generated subgroup of G is finite, and is a **locally finite p-group** if every finitely generated subgroup is a finite p-group. These two classes of groups are closed under taking subgroups, quotients and extensions.

A group G is called **artinian** if every nonempty set of subgroups, partially ordered by inclusion, has a minimal element. Equivalently, G is artinian if its subgroups satisfy the descending chain condition. Again, this class of groups is closed under subgroups, quotients and extensions.

Proposition 1.1.2. *(1.2 [BLO07]). A group is a discrete p-toral group if and only if it is artinian and a locally finite p-group.*

Below we list several properties of discrete p-toral groups.

Lemma 1.1.3. *(1.4 [BLO07]). The following hold for each discrete p-toral group P.*

(i) For each $n \geq 0$, P contains finitely many conjugacy classes of subgroups of order p^n.

(ii) P contains finitely many conjugacy classes of elementary abelian subgroups.

Lemma 1.1.4. *(1.8 [BLO07]). If $P \lneqq P'$ are distinct discrete p-toral groups, then $P \lneqq N_{P'}(P)$.*

1.2 Fusion systems over discrete p-toral groups

Now we introduce fusion systems over the class of groups described in the previous section. Both their definition and properties are similar to those for the finite case ([BLO03b]), and in fact finite fusion systems are a particular case of the definition we give here.

Definition 1.2.1. *A **fusion system** $\mathcal{F}$ over a discrete p-toral group S is a category whose objects are the subgroups of S and whose morphism sets $Hom_{\mathcal{F}}(P,P')$ satisfy the following conditions:*

(i) $Hom_S(P,P') \subseteq Hom_{\mathcal{F}}(P,P') \subseteq Inj(P,P')$ *for all* $P,P' \leq S$.

(ii) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

*Given a fusion system $\mathcal{F}$ over a discrete p-toral group S, the **rank** is the rank of the discrete p-toral group S.*

*Two subgroups P,P' are called **$\mathcal{F}$-conjugate** if $Iso_{\mathcal{F}}(P,P') \neq \emptyset$. For a subgroup $P \leq S$, we denote*

$$\langle P\rangle_{\mathcal{F}} = \{P' \leq S | P' \text{ is } \mathcal{F}\text{-conjugate to } P\}.$$

Definition 1.2.2. *Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. A subgroup $P \leq S$ is called **fully $\mathcal{F}$-normalized**, respect. **fully $\mathcal{F}$-centralized**, if $|N_S(P')| \leq |N_S(P)|$, respect. $|C_S(P')| \leq |C_S(P)|$, for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.*

*The fusion system $\mathcal{F}$ is called **saturated** if the following three conditions hold:*

(I) For each $P \leq S$ which is fully $\mathcal{F}$-normalized, P is fully $\mathcal{F}$-centralized, $Out_{\mathcal{F}}(P)$ is finite and $Out_S(P) \in Syl_p(Out_{\mathcal{F}}(P))$.

(II) If $P \leq S$ and $f \in Hom_{\mathcal{F}}(P,S)$ is such that $P' = f(P)$ is fully $\mathcal{F}$-centralized, then there exists $\widetilde{f} \in Hom_{\mathcal{F}}(N_f,S)$ such that $f = \widetilde{f}_{|P}$, where

$$N_f = \{g \in N_S(P) | f \circ c_g \circ f^{-1} \in Aut_S(P')\}.$$

(III) If $P_1 \leq P_2 \leq P_3 \leq \ldots$ is an increasing sequence of subgroups of S, with $P = \cup_{n=1}^{\infty} P_n$, and if $f \in Hom(P,S)$ is any homomorphism such that $f_{|P_n} \in Hom_{\mathcal{F}}(P_n,S)$ for all n, then $f \in Hom_{\mathcal{F}}(P,S)$.

Let $P \leq S$. Then, $\langle P\rangle_{\mathcal{F}}$ contains (at least) a fully $\mathcal{F}$-normalized element P'. Thus, $Out_{\mathcal{F}}(R)$ is finite for all $R \leq S$. Also, the group $Aut_{\mathcal{F}}(R)$ is artinian and locally finite, being an extension of a finite group by an artinian locally finite p-group (which is $Inn(R)$). Actually, as is shown in Proposition 2.3 [BLO07], the condition that $Out_{\mathcal{F}}(P)$ be finite in the definition above is not necessary, but helps in simplifying the definition.

When $\mathcal{F}$ is a saturated fusion system, we may think of $T = T_S$, the connected component of S, as the "maximal torus" of $\mathcal{F}$, and of $Aut_{\mathcal{F}}(T)$ as the "Weyl group" of T.

Lemma 1.2.3. *(2.4 [BLO07]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, with maximal torus T. Then, the following hold for all $P \leq T$.*

(i) For every $P' \leq S$ which is $\mathcal{F}$-conjugate to P and fully $\mathcal{F}$-centralized, $P' \leq T$, and there exists $\omega \in Aut_{\mathcal{F}}(T)$ such that $\omega_{|P} \in Iso_{\mathcal{F}}(P,P')$.

(ii) Every $f \in Hom_{\mathcal{F}}(P,T)$ is the restriction of some $\omega \in Aut_{\mathcal{F}}(T)$.

As noted above, the groups $Out_{\mathcal{F}}(R)$ are all finite. The following result can be understood as an extension of this fact.

Lemma 1.2.4. *(2.5 [BLO07]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then, for all $P, P' \leq S$, the set*

$$Rep_{\mathcal{F}}(P,P') \stackrel{def}{=} Inn(P') \setminus Hom_{\mathcal{F}}(P,P')$$

is finite.

We next define $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups. They play a similar role as in the finite case.

Definition 1.2.5. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group. A subgroup $P \leq S$ is called **$\mathcal{F}$-centric** if all the elements of $\langle P \rangle_{\mathcal{F}}$ contain their S-centralizers, that is,*

$$C_S(P') = Z(P') \textit{ for all } P' \in \langle P \rangle_{\mathcal{F}}.$$

*A subgroup $P \leq S$ is called **$\mathcal{F}$-radical** if $Out_{\mathcal{F}}(P)$ contains no nontrivial normal p-subgroup, that is,*

$$O_p(Out_{\mathcal{F}}(P)) = \{1\}.$$

Clearly, $\mathcal{F}$-centric subgroups are fully $\mathcal{F}$-centralized, and conversely, if P is fully $\mathcal{F}$-centralized and centric in S, then it is $\mathcal{F}$-centric.

Proposition 1.2.6. *(2.7 [BLO07]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $P \leq P' \leq S$ be such that P is $\mathcal{F}$-centric. Then, P' is $\mathcal{F}$-centric.*

Given a saturated fusion system $\mathcal{F}$, there is another class of subgroups, which contains the class of $\mathcal{F}$-centric subgroups, and that will sporadically appear all along this work.

Definition 1.2.7. *A subgroup $H \leq S$ is said to be **$\mathcal{F}$-quasicentric** is, for all $H' \in conjH\mathcal{F}$, the centralizer fusion system $C_{\mathcal{F}}(H')$ is the fusion system of the discrete p-toral group $C_S(H')$.*

While we cannot prove all the results that hold about $\mathcal{F}$-quasicentric subgroups in the finite case, some key results can indeed be extended to the compact case, mainly when no quasicentric linking system is involved. In this sense, appendix B is devoted to study some of those situations which are of interest for this work.

We finish this section introducing some notions about the $\mathcal{F}$-conjugacy classes in a (saturated) fusion system $\mathcal{F}$. This will not be used here but later on, but we find it better to include it here.

Definition 1.2.8. *Let $\mathcal{F}$ be a (saturated) fusion system over a discrete p-toral group S, and let $A \leq S$. Then,*

- *we say that A is **weakly $\mathcal{F}$-closed** if, for all $P \leq S$ containing A and all $f \in Hom_{\mathcal{F}}(A,S)$, $f(A) \leq A$;*
- *we say that A is **strongly $\mathcal{F}$-closed** if, for all $P \leq S$ and all $f \in Hom_{\mathcal{F}}(P,S)$, $f(P \cap A) \leq A$;*
- *we say that A is **$\mathcal{F}$-normal** if, for all $P \leq S$ and all $f \in Hom_{\mathcal{F}}(P,S)$, there exists $\gamma \in Hom_{\mathcal{F}}(PA,S)$ such that $\gamma_{|P} = f$ and $\gamma_{|A} \in Aut_{\mathcal{F}}(A)$.*

Note that, in any of the three cases above A is normal in S. In definition 1.6.1 we will introduce the normalizer fusion subsystem of a subgroup A in a (saturated) fusion system $\mathcal{F}$, denoted by $N_{\mathcal{F}}(A)$. It will follow by definition of this fusion subsystem that A is $\mathcal{F}$-normal if and only if $N_{\mathcal{F}}(A) = \mathcal{F}$.

Lemma 1.2.9. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $A \leq S$. If A is $\mathcal{F}$-normal then it is strongly $\mathcal{F}$-closed, and if A is strongly $\mathcal{F}$-closed then it is weakly $\mathcal{F}$-closed.*

1.3 A finite retract of a saturated fusion system

When working with saturated fusion systems over discrete p-toral groups, one has, in principle, to deal with infinitely many conjugacy classes of objects. For instance, it is easy to find examples of saturated fusion systems which contain infinitely many conjugacy classes of centric subgroups. This is, of course, a great inconvenience when trying to show certain properties for saturated fusion systems. Fortunately, it was developed in [BLO07] a construction which retracts a saturated fusion system to a certain subcategory containing finitely many conjugacy classes of objects, among which all centric radical subgroups are contained.

This construction will play a central role all along this work, and the reader will be constantly refered to the properties listed this section.

Set then, for simplicity, $W = Aut_{\mathcal{F}}(T)$, the Weyl group of the maximal torus T in $\mathcal{F}$.

Definition 1.3.1. *Set the following*

(i) The exponent of S/T,

$$e = exp(S/T) = min\{p^k | x^{p^k} \in T \text{ for all } x \in S\}.$$

(ii) For each $P \leq T$,

$$I(P) = \{t \in T | \omega(t) = t \text{ for all } \omega \in W \text{ such that } \omega_{|P} = id_P\},$$

and let $I(P)_0$ denote its connected component.

(iii) For each $P \leq S$, set $P^{[e]} = \{x^{p^e} | x \in P\} \leq T$, and set

$$P^{\bullet} = P \cdot I(P^{[e]})_0 = \{xt | x \in P, t \in I(P^{[e]})_0\}.$$

(iv) Set $\mathcal{H}^{\bullet} = \{P^{\bullet} | P \in \mathcal{F}\}$. Also let $\mathcal{F}^{\bullet} \subseteq \mathcal{F}$ and $\mathcal{L}^{\bullet} \subseteq \mathcal{L}$ the full subcategories with object sets $Ob(\mathcal{F}^{\bullet}) = \mathcal{H}^{\bullet}$ and $Ob(\mathcal{L}^{\bullet}) = \mathcal{H}^{\bullet} \cap Ob(\mathcal{L})$.

Lemma 1.3.2. *(3.2 [BLO07]). The following hold for every saturated fusion system $\mathcal{F}$ over a discrete p-toral group S.*

(i) The set $\mathcal{H}^{\bullet}$ contains finitely many S-conjugacy classes of subgroups of S.

(ii) For all $P \leq S$, $(P^{\bullet})^{\bullet} = P^{\bullet}$.

(iii) If $P \leq P' \leq S$, then $P^{\bullet} \leq (P')^{\bullet}$.

(iv) If $P \leq S$ is $\mathcal{F}$-centric, then $Z(P^\bullet) = Z(P)$.

As a consequence of the above lemma, we have the following result.

Proposition 1.3.3. *(3.3 [BLO07]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group. Fix $P, P' \leq S$ and $f \in Hom_{\mathcal{F}}(P, P')$. Then f extends to a unique homomorphism $f^\bullet \in Hom_{\mathcal{F}}(P^\bullet, (P')^\bullet)$, and this makes $P \mapsto P^\bullet$ into a functor from $\mathcal{F}$ to itself.*

Corollary 1.3.4. *(3.4 and 3.5 [BLO07]). The functor $(\_)^\bullet$ is a left adjoint to the inclusion of $\mathcal{F}^\bullet$ as a full subcategory of $\mathcal{F}$.*

All $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S are in $\mathcal{H}^\bullet$. In particular, there are only finitely many $\mathcal{F}$-conjugacy classes of such subgroups.

Finally, we state Alperin's fusion theorem for saturated fusion systems over discrete p-toral groups.

Theorem 1.3.5. *(3.6 [BLO07]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then, for each $f \in Iso_{\mathcal{F}}(P, P')$ there exist sequences of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P' \quad \text{and} \quad Q_1, \ldots, Q_k,$$

and elements $f_j \in Aut_{\mathcal{F}}(Q_j)$ such that

(i) for each j, Q_j is fully normalized in $\mathcal{F}$, $\mathcal{F}$-centric and $\mathcal{F}$-radical;

(ii) also for each j, $P_{j-1}, P_j \leq Q_j$ and $f_j(P_{j-1}) = P_j$; and

(iii) $f = f_k \circ f_{k-1} \circ \ldots \circ f_1$.

1.4 Linking systems over discrete p-toral groups

Linking systems are the third and last piece needed to form a p-local compact groups. They provide some simplicial information in order to recover classifying spaces (when the p-local compact groups comes induced by an actual group) which cannot be obtained from the fusion system.

Definition 1.4.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. A **centric linking system associated to** $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor*

$$\rho : \mathcal{L} \longrightarrow \mathcal{F}^c$$

and "distinguished" monomorphisms $\delta_P : P \to Aut_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) ρ is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, P' \in \mathcal{L}$, $Z(P)$ acts freely on $Mor_{\mathcal{L}}(P, P')$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq Aut_{\mathcal{L}}(P)$), and ρ induces a bijection

$$Mor_{\mathcal{L}}(P, P')/Z(P) \xrightarrow{\cong} Hom_{\mathcal{F}}(P, P').$$

(B) For each $\mathcal{F}$-centric subgroup $P \le S$ and each $g \in P$, ρ sends $\delta_P(g) \in Aut_{\mathcal{L}}(P)$ to $c_g \in Aut_{\mathcal{F}}(P)$.

(C) For each $\varphi \in Mor_{\mathcal{L}}(P, P')$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\delta_P(g)"'] & P' \arrow[d, "\delta_{P'}(h)"] \\ P \arrow[r, "\varphi"'] & P', \end{tikzcd}$$

where $h = \rho(\varphi)(g)$.

Definition 1.4.2. *A p-**local compact group** is a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, where S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The **classifying space** of $\mathcal{G}$ is the p-completed nerve*

$$B\mathcal{G} \overset{def}{=} |\mathcal{L}|^{\wedge}_p.$$

*Given a p-local compact group $\mathcal{G}$, the **rank** of $\mathcal{G}$ is the rank of the discrete p-toral group S.*

We will often denote a p-local compact group by $\mathcal{G}$, assuming that S is its Sylow p-subgroup, $\mathcal{F}$ is the corresponding fusion system, and $\mathcal{L}$ is the corresponding linking system.

Lemma 1.4.3. *(4.3 [BLO07]). Fix a p-local compact group $\mathcal{G}$, together with the projection $\rho : \mathcal{L} \to \mathcal{F}^c$, and let $P, Q, R \in \mathcal{L}$. Then the following hold.*

(i) Fix any sequence $P \xrightarrow{f} Q \xrightarrow{g} R$ of morphisms in $\mathcal{F}^c$, and let $\widetilde{g} \in \rho^{-1}(g)$ and $\widetilde{gf} \in \rho^{-1}(gf)$ be arbitrary liftings. Then there is a unique morphism $\widetilde{f} \in Mor_{\mathcal{L}}(P, Q)$ such that

$$\widetilde{g} \circ \widetilde{f} = \widetilde{gf},$$

and furthermore $\rho(\widetilde{f}) = f$.

(ii) If $\varphi, \varphi' \in Mor_{\mathcal{L}}(P, Q)$ are such that $\rho(\varphi') = c_g \circ \rho(\varphi)$ for some $g \in Q$. Then there is a unique $h \in Q$ such that $\varphi' = \delta_Q(h) \circ \varphi$ in $Mor_{\mathcal{L}}(P, Q)$.

As expected, the classifying space of a p-local compact group behaves nicely.

Proposition 1.4.4. *(4.4 [BLO07]). Let $\mathcal{G}$ be a p-local compact group. Then, $|\mathcal{L}|$ is p-good. Also, the composite*

$$S \xrightarrow{\pi_1(\theta)} \pi_1(|\mathcal{L}|) \longrightarrow \pi_1(B\mathcal{G})$$

induced by the inclusion $BS \xrightarrow{\theta} |\mathcal{L}|$, factors through a surjection

$$S/T \twoheadrightarrow \pi_1(B\mathcal{G}).$$

Let $\mathcal{G}$ be a p-local compact group, and let $(\_)^{\bullet}$ be the functor defined on $\mathcal{F}$ in 1.3.1. It was shown in [Jun09] that this functor can be in fact extended to a functor on $\mathcal{L}$ with similar properties, although this will not be used in this work.

We finish this section justifying (somehow) the interest of p-local compact groups: both compact Lie groups and p-compact groups induce p-local compact groups, and they are not the only examples.

Theorem 1.4.5. *(8.10, 9.10 and 10.7 [BLO07]). The following holds in general.*

(i) Fix a linear torsion group G, a prime p different from the defining characteristic of G and a Sylow subgroup $S \in Syl_p(G)$. Then, G induces a p-local compact group $\mathcal{G} = (S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ such that
$$B\mathcal{G} \simeq BG_p^\wedge.$$

(ii) Fix a compact Lie group G and a maximal discrete p-toral group $S \in Syl_p(G)$. Then, G induces a p-local compact group $\mathcal{G} = (S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ such that
$$B\mathcal{G} \simeq BG_p^\wedge.$$

(iii) Let (X, BX, e) be a p-compact group, and let $BS \xrightarrow{f} BX$ be a Sylow p-subgroup. Then, there is an induced p-local compact group $\mathcal{G} = (S, \mathcal{F}_{S,f}(BX), \mathcal{L}_{S,f}^c(BX))$ such that
$$B\mathcal{G} \simeq BX.$$

1.5 Some results on higher limits

Obstruction theory is implicit in the definition of p-local compact groups, and many results and calculations depend on the vanishin of certain obstructions. We collect certain results on higher limits which will be needed later on and also show the obstructions for the existence and uniqueness of associated centric linking systems. We start by introducing the orbit category, which plays a central role in this section.

Definition 1.5.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. The **orbit category** $\mathcal{O}(\mathcal{F})$ of $\mathcal{F}$ is defined as the category with objects the subgroups of S and morphism sets*
$$Mor_{\mathcal{O}(\mathcal{F})}(P, Q) = Rep_{\mathcal{F}}(P, Q).$$

Let $\mathcal{O}(\mathcal{F}^c)$ and $\mathcal{O}(\mathcal{F}^{\bullet c})$ be the full subcategories of $\mathcal{O}(\mathcal{F})$ with object sets the $\mathcal{F}$-centric and $\mathcal{F}$-centric subgroups in $\mathcal{H}^\bullet$ of S respectively. The orbit category has the same problem as had the fusion system: the category has "far too many objects". The results on higher limits used for p-local finite groups made use of the subcategory $\mathcal{O}(\mathcal{F}^c)$, but in the compact case this subcategory may still have inifinitely many conjugacy classes of objects. This is not the case in $\mathcal{O}(\mathcal{F}^{\bullet c})$, and the following result from [BLO07] allows us to work on $\mathcal{O}(\mathcal{F}^{\bullet c})$ without loss of generality.

Lemma 1.5.2. *(5.1 [BLO07]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then, there are well-defined functors*

$$\begin{tikzcd} \mathcal{O}(\mathcal{F}^c) \arrow[r, shift left, "(\_)^\bullet"] & \mathcal{O}(\mathcal{F}^{\bullet c}), \arrow[l, shift left, "incl"] \end{tikzcd}$$

such that $(\_)^\bullet$ is a left adjoint to the inclusion.

As a consequence, the following is proved in [BLO07].

Proposition 1.5.3. *(5.2 [BLO07]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group. Then, for any functor $F : \mathcal{O}(\mathcal{F}) \to \mathbb{Z}_{(p)}$ there is an isomorphism*

$$\varprojlim{}^{*}_{\mathcal{O}(\mathcal{F})}(F) \cong \varprojlim{}^{*}_{\mathcal{O}(\mathcal{F}^{\bullet c})}(F_{|\mathcal{F}^{\bullet}}).$$

Another important tool from §5 in [BLO07] is the following proposition.

Proposition 1.5.4. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Let also*

$$\Phi : \mathcal{O}(\mathcal{F}^c)^{op} \longrightarrow \mathbb{Z}_{(p)} - mod$$

be any functor which vanishes except on the isomorphism class of some fixed $\mathcal{F}$-centric subgroup $Q \leq S$. Then,

$$\varprojlim{}^{*}_{\mathcal{O}(\mathcal{F}^c)}(\Phi) \cong \Lambda^*(Out_{\mathcal{F}}(Q); \Phi(Q)).$$

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. In Proposition 3.1 [BLO03b] it was shown that the existence and uniqueness of centric linking systems associated to $\mathcal{F}$ depended on the vanishing of certain classes in some higher limits. Below we give a similar result for saturated fusion systems over discrete p-toral groups. As usual, let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S.

The following functor will play a central role in all calculations of higher limits that will be done in this group.

Definition 1.5.5. *We define the **center functor** as the functor $\mathcal{Z}_{\mathcal{F}} : \mathcal{O}(\mathcal{F}^c) \to Ab$ defined by $\mathcal{Z}_{\mathcal{F}}(P) = Z(P)$ and*

$$\mathcal{Z}_{\mathcal{F}}(P \xrightarrow{f} Q) = (Z(Q) \xrightarrow{incl} Z(f(P)) \xrightarrow{f^{-1}} Z(P))$$

(note that $Z(Q) \leq Z(f(P))$ since both P and Q are $\mathcal{F}$-centric).

Proposition 1.5.6. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then there is an element $\eta(\mathcal{F}) \in \varprojlim{}^{3}_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}})$ such that $\mathcal{F}$ has an associated centric linking system if and only if $\eta(\mathcal{F}) = 0$.*

Also, the group $\varprojlim{}^{2}_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}})$ acts freely and transitively on the set of all isomorphism classes of centric linking systems associated to $\mathcal{F}$.

The proof to this result is identical to the corresponding result in the finite case, Proposition 3.1 in [BLO03b].

1.6 The K-normalizer fusion subsystem of a subgroup

Given a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S and a subgroup $P \leq S$ which is fully $\mathcal{F}$-normalized, one can define rather naturally the normalizer fusion subsystem of P in $\mathcal{F}$, the saturation of which is of great importance for some constructions we will do later on. Hence, in this section we extend the results in appendix §A [BLO03b] to the compact case.

We start by introducing some notation (which already comes from appendix §A in [BLO03b]). Let $A \leq S$ be any subgroup, $K \leq Aut(A)$, and set

$$Aut^K_{\mathcal{F}}(A) = K \cap Aut_{\mathcal{F}}(A) \quad \text{and} \quad Aut^K_S(A) = K \cap Aut_S(A).$$

Define also the K-**normalizer** of A as

$$N_S^K(A) = \{x \in N_S(A) | c_x \in K\}.$$

Note that $Aut_S^K(A) = N_S^K(A)/C_S(A)$, since for all $x \in C_S(A)$, $c_x = id \in K$. We can also think about the K-normalizer of A as the pull-back of $K \overset{incl}{\to} Aut(A) \leftarrow N_S(A)$, where the right-side arrow is the morphism sending $x \in N_S(A)$ to $c_x \in Aut_S(A) \leq Aut(A)$. There are two cases of this construction that will be of special interest for us: $K = Aut(A)$ and $K = \{id\}$. In these cases, we have

$$N_S^{Aut(A)}(A) = N_S(A) \quad \text{and} \quad N_S^{\{id\}}(A) = C_S(A).$$

For $f \in Hom_{\mathcal{F}}(A, Q)$, we will write

$$fKf^{-1} = \{f \circ \gamma \circ f^{-1} | \gamma \in K\},$$

and say that A is fully K-normalized if, for all $f \in Hom_{\mathcal{F}}(A, S)$,

$$|N_S^K(A)| \geq |N_S^{fKf^{-1}}(f(A))|.$$

Definition 1.6.1. *For a fixed $A \leq S$ and a fixed $K \leq Aut(A)$, we define the K-**normalizer fusion subsystem** of A in $\mathcal{F}$, $N_{\mathcal{F}}^K(A)$ as the fusion system over $N_S^K(A)$ with morphism sets*

$$Hom_{N_{\mathcal{F}}^K(A)}(P, P') = \{f \in Hom_{\mathcal{F}}(P, P') | \exists \gamma \in Hom_{\mathcal{F}}(PA, P'A) : \gamma_{|P} = f, \gamma_{|A} \in K\}.$$

*In particular, we will refer to the **normalizer fusion system** and the **centralizer fusion system** when $K = Aut(A)$ and $K = \{id\}$ respectively. This two cases will be of great importance for us.*

The main purpose of this section is then proving the following proposition.

Proposition 1.6.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $A \leq S$, $K \leq Aut(A)$ be such that A is fully K-normalized in $\mathcal{F}$. Then, the fusion system $N_{\mathcal{F}}^K(A)$ is saturated.*

Some technical lemmas are needed in order to prove this result, just as well as they are needed in [BLO03b] for the very same purpose. We start by showing an analog of Lemma 1.4 in [BLO03b]. This will allow us in turn to follow the same proof for Proposition A.6 in [BLO03b] in the compact case.

Lemma 1.6.3. *Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S such that axioms (II) and (III) in definition 1.2.2 are satisfied, and such that the next condition is also satisfied*

(I′) Each subgroup $P \leq S$ is $\mathcal{F}$-conjugate to a fully $\mathcal{F}$-centralized subgroup P' such that $Out_{\mathcal{F}}(P')$ is finite and such that

$$Out_S(P') \in Syl_p(Out_{\mathcal{F}}(P')).$$

Then, $\mathcal{F}$ is saturated.

Proof. We have to check that axiom (I) in 1.2.2 is satisfied, that is

(I) If $P \leq S$ is fully $\mathcal{F}$-normalized, then it is fully $\mathcal{F}$-centralized, $Out_{\mathcal{F}}(P)$ is finite, and

$$Out_S(P) \in Syl_p(Out_{\mathcal{F}}(P)).$$

Thus, let $P \leq S$ be fully $\mathcal{F}$-normalized, and let P' be as in condition (I′), and $f \in Iso_{\mathcal{F}}(P, P')$. Since P and P' are $\mathcal{F}$-conjugate, there is an explicit isomorphism

$$\begin{aligned} Aut_{\mathcal{F}}(P) &\xrightarrow{f_*} Aut_{\mathcal{F}}(P') \\ \varphi &\longmapsto f \circ \varphi \circ f^{-1}. \end{aligned}$$

Furthermore, this isomorphism restricts to an isomorphism between the corresponding inner automorphism subgroups, and hence it induces an isomorphism $Out_{\mathcal{F}}(P) \cong Out_{\mathcal{F}}(P')$.

Now, since $Out_S(P') \in Syl_p(Out_{\mathcal{F}}(P'))$, and since $Out_S(P)$ is a finite p-subgroup of $Out_{\mathcal{F}}(P)$, it follows that we can choose f so that $f_*(Out_S(P))$ is a subgroup of $Out_S(P') \in Syl_p(Out_{\mathcal{F}}(P'))$.

Now, we can apply axiom (II) to the morphism f. If we set

$$N_f = \{g \in N_S(P) | f \circ c_g \circ f^{-1} \in Aut_S(P')\},$$

then it follows that $N_f = N_S(P)$ by the above, and hence P' is also fully $\mathcal{F}$-normalized. This in turn implies that P is fully $\mathcal{F}$-centralized and that condition (I) above holds. □

Proposition 1.6.4. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and $A \leq S$, $K \leq Aut(A)$. Then, the following properties hold:*

(i) There is a subgroup $A' \leq S$ and an isomorphism $f \in Iso_{\mathcal{F}}(A, A')$ such that A' is fully $\mathcal{F}$-centralized and
$$Aut_S^{fKf^{-1}}(A') \in Syl_p(Aut_{\mathcal{F}}^{fKf^{-1}}(A')).$$

(ii) A is fully K-normalized in $\mathcal{F}$ if and only if A is fully $\mathcal{F}$-centralized in $\mathcal{F}$ and

$$Aut_S^K(A) \in Syl_p(Aut_{\mathcal{F}}^K(A)).$$

(iii) Fix $f \in Hom_{\mathcal{F}}(A, S)$, and $A' = f(A)$, $K' = fKf^{-1}$. If A' is fully K'-normalized in $\mathcal{F}$, then there are homomorphisms

$$\gamma \in Hom_{\mathcal{F}}(N_S^K(A)A, S) \quad and \quad \chi \in K$$

such that $\gamma_{|A} = f \circ \chi$.

Proof. (i) Choose $f_0 \in Hom_{\mathcal{F}}(A, S)$ such that $A' = f_0(A)$ is fully normalized in $\mathcal{F}$. Then, by condition (I) of saturation, A' is fully $\mathcal{F}$-centralized and $Out_S(A') \in Syl_p(Out_{\mathcal{F}}(A'))$. Hence, $f_0 Aut_S^K(A) f_0^{-1}$ is a subgroup of a Sylow p-subgroup in $Aut_{\mathcal{F}}^K(A')$. That is, there exists $\alpha \in Aut_{\mathcal{F}}(A')$ such that

$$f_0 Aut_S^K(A) f_0^{-1} \leq (\alpha^{-1} Aut_S(A')\alpha) \cap (f_0 Aut_{\mathcal{F}}^K(A) f_0^{-1}) \in Syl_p(f_0 Aut_{\mathcal{F}}^K(A) f_0^{-1}).$$

Set $f = \alpha \circ f_0$, $K' = fKf^{-1}$. It follows that

$$Aut_S^{K'}(A') = Aut_S(A') \cap (f Aut_{\mathcal{F}}^K(A) f^{-1}) \in Syl_p(Aut_{\mathcal{F}}^{K'}(A')).$$

(ii) Suppose first that A is fully $\mathcal{F}$-centralized and

$$Aut_S^K(A) \in Syl_p(Aut_{\mathcal{F}}^K(A)).$$

In particular, $|C_S(A)| \geq |C_S(f(A))|$, for all $f \in Hom_{\mathcal{F}}(A, S)$, and we want to check that $|N_S^K(A)| \geq |N_S^{fKf^{-1}}(f(A))|$. This follows immediately from the extensions

$$C_S(A) \longrightarrow N_S^K(A) \longrightarrow Aut_S^K(A)$$
$$C_S(f(A)) \longrightarrow N_S^{fKf^{-1}}(f(A)) \longrightarrow Aut_S^{fKf^{-1}}(f(A)),$$

and the fact that f^{-1} induces an inclusion $Aut_S^{fKf^{-1}}(f(A)) \hookrightarrow Aut_S^K(A)$.

Conversely, suppose that A is fully K-normalized in $\mathcal{F}$. By (i), there exists $f \in Hom_{\mathcal{F}}(A, S)$ such that $f(A)$ is fully $\mathcal{F}$-centralized and

$$Aut_S^{fKf^{-1}}(f(A)) \in Syl_p(Aut_{\mathcal{F}}^{fKf^{-1}}(f(A))).$$

The following inequalities hold by hypothesis

- $|C_S(A)| \leq |C_S(f(A))|$,
- $|N_S^K(A)| \geq |N_S^{fKf^{-1}}(f(A))|$, and
- $|Aut_S^K(A)| \leq |Aut_S^{fKf^{-1}}(f(A))|$.

By the same arguments in the proof of Lemma 1.6.3, all these inequalities have to be equalities, and (ii) holds.

(iii) Assume that $f \in Hom_{\mathcal{F}}(A, S)$ is such that $A' = f(A)$ is fully $K' = fKf^{-1}$-normalized in $\mathcal{F}$.

First note that A' is fully K'-normalized if and only if it is fully $K' \cdot Inn(A')$-normalized in $\mathcal{F}$. Hence, we can replace K by $K \cdot Inn(A)$ without loss of generality, and thus $A \leq N_S^K(A)$ (similarly for A').

Since A' is fully K'-normalized, by (i) it is fully $\mathcal{F}$-centralized and

$$Aut_S^{K'}(A') = Aut_S^{fKf^{-1}}(A') \in Syl_p(Aut_{\mathcal{F}}^{fKf^{-1}}(A')).$$

Furthermore, there is some $\chi \in Aut_{\mathcal{F}}^K(A)$ such that

$$f(\chi Aut_S^K(A) \chi^{-1}) f^{-1} \leq Aut_S^{K'}(A').$$

Now, since A' is fully $\mathcal{F}$-centralized, we can apply axiom (II) to the morphism $f \circ \chi \in Hom_{\mathcal{F}}(A, S)$ to see that it extends to some $\gamma \in Hom_{\mathcal{F}}(N, S)$, where

$$N = N_{f\circ\chi} = \{g \in N_S(A) | (f \circ \chi) c_g (f \circ \chi)^{-1} \in Aut_S(A')\}.$$

Note that if $g \in N_S^K(A)$, then $g \in N$, and (iii) follows.

$\square$

Lemma 1.6.5. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $A \leq S$, $K \leq Aut(A)$ be such that A is fully K-normalized in $\mathcal{F}$. Then, the following holds:*

(i) For any $f \in Hom_{\mathcal{F}}(N_S^K(A)A, S)$, $f(A)$ is fully fKf^{-1}-normalized in $\mathcal{F}$.

(ii) For any $H \lhd K$, A is fully H-normalized in $\mathcal{F}$.

Proof. Point (i) corresponds to Lemma A.4 in [BLO03b] and point (ii) to Lemma A.5 in [BLO03b]. Both proofs apply here.

□

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $A \leq S$, $K \leq Aut(A)$ be such that A is fully K-normalized in $\mathcal{F}$. Consider the K-normalizer fusion subsystem $N_{\mathcal{F}}^K(A)$. We introduce some notation before the proof of 1.6.2.

For each $P \leq N_S^K(A)$, $I \leq Aut(P)$, set

$$I \cdot K = \{\alpha \in Aut(PA) | \alpha_{|P} \in I, \alpha_{|A} \in K\} \leq Aut(PA).$$

Then, the following holds

$$N^I_{N_S^K(A)}(P) = N_S^{I \cdot K}(PA) (\leq N_S^K(A)). \tag{1.2}$$

Furthermore, the following restriction map is surjective

$$Aut_{\mathcal{F}}^{I \cdot K}(PA) \twoheadrightarrow Aut^I_{N_{\mathcal{F}}^K(A)}(P). \tag{1.3}$$

Let $P \leq N_S^K(A)$, and $f \in Hom_{N_{\mathcal{F}}^K(A)}(P, N_S^K(A))$. By definition, we can see f as a morphism $f' \in Hom_{\mathcal{F}}(PA, S)$ such that $f'_{|A} \in K$, $f'_{|P} = f$, $f'(P) \leq N_S^K(A)$.

Lemma 1.6.6. *Let $P \leq N_S^K(A)$, $I \leq Aut(P)$. Then the following holds:*

(i) There exists a morphism $f \in Hom_{\mathcal{F}}(PA, S)$ such that $f(PA)$ is fully $f(I \cdot K)f^{-1}$-normalized in $\mathcal{F}$ and $f_{|P}$ is a morphism in $N_{\mathcal{F}}^K(A)$.

(ii) If P is fully I-normalized in $N_{\mathcal{F}}^K(A)$, then PA is fully $(I \cdot K)$-normalized in $\mathcal{F}$.

Proof. (i) Choose $f_0 \in Hom_{\mathcal{F}}(PA, S)$ such that $f_0(PA)$ is fully $f_0(I \cdot K)f_0^{-1}$-normalized in $\mathcal{F}$, and set $f_1 = (f_0)_{|A}$. Since A is fully K-normalized in $\mathcal{F}$, we can apply Proposition 1.6.4 (iii) to the morphism $f_1^{-1} : f_1(A) \to A$. Thus, there exist

$$\phi \in Hom_{\mathcal{F}}(N_S^{f_1 K f_1^{-1}}(f_1(A)) f_1(A), S) \quad \text{and} \quad \chi \in f_1 K f_1^{-1}$$

such that $\phi_{|f_1(A)} = f_1^{-1} \circ \chi = \chi' \circ f_1^{-1}$, where $\chi' = f_1^{-1} \chi f_1 \in K$.

If we set now $f = \phi \circ f_0$, then it satisfies

- $f_{|A} = (\phi \circ f_0)_{|A} = (\chi' \circ f_1^{-1} \circ f_1) = \chi' \in K$;
- $f(P) = \chi(f_0(P)) \leq N_S^K(A)$,

where the second inequality holds since $f_0(P) \leq N_S^{f_1Kf_1^{-1}}(f_1(A))$ and because $\phi(N_S^{f_1Kf_1^{-1}}(f_1(A))) \leq N_S^K(A)$. Thus, $f_{|P}$ is a morphism in $N_{\mathcal{F}}^K(A)$ and $f(PA)$ is fully $f(I\cdot K)f^{-1}$-normalized in by Lemma 1.6.5 (i).

(ii) Assume P is fully I-normalized in $N_{\mathcal{F}}^K(A)$. Using point (i), we can choose $f \in Hom_{\mathcal{F}}(PA,S)$ such that $f(PA)$ is fully $f(I\cdot K)f^{-1}$-normalized in $\mathcal{F}$, and $f_{|P}$ is a morphism in $N_{\mathcal{F}}^K(A)$.

Then, we have a chain of inequalities

$$|N_S^{I\cdot K}(PA)| = |N^I_{N_S^K(A)}(P)| \geq |N^{fIf^{-1}}_{N_S^K(A)}(f(P))| = |N_S^{f(I\cdot K)f^{-1}}(f(PA))|,$$

where the two equalities hold by (1.2) and because $f_{|A} \in K$, and the inequality holds because P is fully I-normalized in $N_{\mathcal{F}}^K(A)$.

Hence, PA is fully $(I\cdot K)$-normalized in $\mathcal{F}$ since $f(PA)$ is fully $f(I\cdot K)f^{-1}$-normalized in $\mathcal{F}$.

□

We are ready to prove Proposition 1.6.2 above.

Proof. (of Proposition 1.6.2). We will prove condition (I′) in 1.6.3 and conditions (II) and (III) from 1.2.2. For the sake of an easier reading, we will recall each of the statements here.

(I′) For each $P \leq N_S^K(A)$, there exists $f \in Hom_{N_{\mathcal{F}}^K(A)}(P, N_S^K(A))$ such that $f(P)$ is fully centralized in $N_{\mathcal{F}}^K(A)$, and

$$Aut_{N_S^K(A)}(f(P)) \in Syl_p(Aut_{N_{\mathcal{F}}^K(A)}(f(P))).$$

Let $K_P = Aut(P)\cdot K$. By Lemma 1.6.6 (i), there exists $\gamma \in Hom_{\mathcal{F}}(PA,S)$ such that $\gamma(PA)$ is fully $\gamma K_P\gamma^{-1}$-normalized in $\mathcal{F}$, and such that $f = \gamma_{|P}$ is a morphism in $N_{\mathcal{F}}^K(A)$.

Then, by (1.2), $C_{N_S^K(A)}(f(P)) = N_S^{1\cdot K}(\gamma(PA)) = N_S^{\gamma(1\cdot K)\gamma^{-1}}(\gamma(PA))$. Let $f' \in Hom_{N_{\mathcal{F}}^K(A)}(P, N_S^K(A))$ be any other morphism in $N_{\mathcal{F}}^K(A)$, and let $\gamma' \in Hom_{\mathcal{F}}(PA,S)$ be such that $\gamma'_{|P} = f'$. Then, again by (1.2), $C_{N_S^K(A)}(f'(P)) = N_S^{\gamma'(1\cdot K)(\gamma')^{-1}}(\gamma'(PA))$, and hence

$$|C_{N_S^K(A)}(f(P))| = |N_S^{\gamma(1\cdot K)\gamma^{-1}}(\gamma(PA))| \geq |N_S^{\gamma'(1\cdot K)(\gamma')^{-1}}(\gamma'(PA))| = |C_{N_S^K(A)}(f'(P))|,$$

because $\gamma(PA)$ is fully $\gamma(1\cdot K)\gamma^{-1}$-normalized in $\mathcal{F}$.

It remains to check the Sylow condition. Since the subgroup $\gamma(PA)$ is fully $\gamma K_P\gamma^{-1}$-normalized in $\mathcal{F}$, by Proposition 1.6.4 (ii) it follows that

$$Aut_S^{\gamma K_P\gamma^{-1}}(\gamma(PA)) \in Syl_p(Aut_{\mathcal{F}}^{\gamma K_P\gamma^{-1}}(\gamma(PA))).$$

Also, $N_{N_S^K(A)}(f(P)) = N_S^{\gamma K_P\gamma^{-1}}(\gamma(PA))$ by (1.2) and also because $\gamma K_P\gamma^{-1} = Aut(f(P))\cdot K$. Hence, by (1.3),

$$Aut_{N_S^K(A)}(f(P)) \in Syl_p(Aut_{N_{\mathcal{F}}^K(A)}(f(P))).$$

(II) If $f \in Hom_{N_{\mathcal{F}}^K(A)}(P, N_S^K(A))$ is such that $P' = f(P)$ is fully centralized, then there exists $\widetilde{f} \in Hom_{N_{\mathcal{F}}^K(A)}(N_f, N_S^K(A))$ extending f, where

$$N_f = \{g \in N_{N_S^K(A)}(P) | fc_gf^{-1} \in Aut_{N_S^K(A)}(P')\}.$$

First, we construct a candidate to extend f. Let $I = Aut_{N_f}(P)$, that is, all the automorphisms of P which are conjugation by an element in N_f. Then, $I' = fIf^{-1} \leq Aut_{N_S^K(A)}(P')$, and

$$Aut^{I'}_{N_S^K(A)}(P') = I' = Aut^{I'}_{Aut_{N_{\mathcal{F}}^K(A)}}(P'),$$

since the groups on the right and on the left are intersections with I'. Thus $Aut^{I'}_{N_S^K(A)}(P') \in Syl_p(Aut^{I'}_{N_{\mathcal{F}}^K(A)}(P'))$.

This, together with the fact that P' is fully centralized in $N_{\mathcal{F}}^K(A)$ and Proposition 1.6.4(ii), implies that P' is fully I'-normalized, and hence, by Lemma 1.6.6(ii), $f(P)A$ is fully $I' \cdot K$-normalized in $\mathcal{F}$.

By definition of $N_{\mathcal{F}}^K(A)$, there exists $\gamma \in Hom_{\mathcal{F}}(PA, S)$ such that $\gamma_{|P} = f$, $\gamma_{|A} \in K$ (thus $\gamma(PA) = f(P)A$). Also, by Proposition 1.6.4(iii), there exist

$$\phi \in Hom_{\mathcal{F}}(N_S^{I \cdot K}(PA)A, S) \quad \text{and} \quad \chi \in I \cdot K$$

such that $\phi_{|PA} = \gamma \circ \chi$.

Now, by construction and by (1.2), $N_f \leq N_S^{I \cdot K}(PA) = N^I_{N_S^K(A)}(P)$, and hence $f_0 = \phi_{|N_f} \in Hom_{\mathcal{F}}(N_f, S)$. This is our candidate to be an extension of f. Namely, f_0 has to satisfy

- $(f_0)_{|A} \in K$,
- $f_0(N_f) \leq N_S^K(A)$, and
- $(f_0)_{|P} = f$.

The first condition is satisfied since $(f_0)_{|A} = \phi_{|A} = (\gamma \circ \chi)_{|A}$, and both $\gamma_{|A}$ and $\chi_{|A}$ are in K, and the second condition follows then by construction.

The third condition, however, may fail: $(f_0)_{|P} = \gamma_{|P} \circ \chi_P$, where $\chi_{|P} \in I = Aut_{N_f}(P)$. This is not a problem since this means that $\chi_{|P} = c_g$ for some $g \in N_f$, and we can modify f_0 to satisfy also the third condition (withou losing the other conditions): f extends to $f_0 \circ c_g^{-1} \in Hom_{N_{\mathcal{F}}^K(A)}(N_f, N_S^K(A))$.

(III) Let $P_1 \leq P_2 \leq \ldots$ be an ascending chain of subgroups of $N_S^K(A)$, and let $P = \cup P_n$. If $f \in Hom(P, N_S^K(A))$ is such that $f_n = f_{|P_n}$ is a morphism in $N_{\mathcal{F}}^K(A)$ for all n, then so is f.

For each n, since f_n is a morphism in $N_{\mathcal{F}}^K(A)$, there exists a morphism $\gamma_n \in Hom_{\mathcal{F}}(P_nA, S)$ such that $(\gamma_n)_{|P_n} = f_n$, $(\gamma_n)_{|A} \in K$. Let $\phi_n = (\gamma_n)_{|A}$. For each n, since $\phi_n \in K$, there exists $\omega_n \in K$ such that

$$\phi_n = \phi_{n+1} \circ \omega_n.$$

Let $i_n : P_nA \to P_{n+1}A$ be the natural inclusion, and let $i'_n : P_nA \to P_{n+1}A$ be defined by $i'_{(}x) = x$ if $x \in P_n$ and $i'_n(x) = \omega_n(x)$ if $x \in A$. Note that if $x \in P_n \cap A$, then

$$\omega_n(x) = (\phi_{n+1}^{-1} \circ \phi_n)(x) = x,$$

and hence i'_n is a well-defined group homomorphism which makes the following diagrama commutative:

$$\begin{tikzcd}
P_{n+1}A \arrow[dr, "\gamma_{n+1}"] & \\
& S. \\
P_nA \arrow[uu, "i'_n"] \arrow[ur, "\gamma_n"'] &
\end{tikzcd}$$

Thus, $\{\gamma_n\}$ is an element of a (non-empty) inverse system, and thus, by Proposition 1.1.4 [RZ00], there exists an element γ in the limit satisfying $\gamma_{|P_n} = f_n$, $\gamma_{|P} = f$, $\gamma_{|A} \in K$. Thus, γ is a morphism in $\mathcal{F}$, and so f is a morphism in $N^K_{\mathcal{F}}(A)$.

□

1.7 An alternative definition of saturation

The definition of a saturated fusion system is rather technical, and usually proving saturation becomes quite a colossal task. This is why in the various papers about fusion systems that have appeared since they were defined different strategies to deal with saturation have been developed. We have already seen one of this situations in Lemma 1.6.3, in the previous section.

In this section we study an equivalent definition for saturation of fusion systems over discrete p-toral groups which was originally developed for fusion systems of blocks in [KS08] (see Definition 2.4 in [KS08]). Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S, and consider the following conditions:

(I′) $Aut_S(S) \in Syl_p(Aut_{\mathcal{F}}(S))$.

(II′) Let $f : P \to S$ be a morphism in $\mathcal{F}$ such that $P' = f(P)$ is fully $\mathcal{F}$-normalized. Then, f extends to a morphism $\widetilde{f} : N_f \to S$ in $\mathcal{F}$, where

$$N_f = \{g \in N_S(P) | f \circ c_g \circ f^{-1} \in Aut_S(P')\}.$$

The main result of this section is the following proposition.

Proposition 1.7.1. *The fusion system $\mathcal{F}$ is saturated (in the sense of definition 1.2.2) if and only if it satisfies axioms (I′), (II′) and (III) (the last one being the same as in 1.2.2).*

Before proving this, we need some technical results, in order to compare the axioms from definition 1.2.2 to this new set of axioms.

Lemma 1.7.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral groups S. Then, a subgroup $P \leq S$ is fully $\mathcal{F}$-normalized if and only if it is fully $\mathcal{F}$-centralized and $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$.*

Proof. The "only if" part is clear by axiom (I) for saturated fusion systems, and we have to prove the "if" part. Assume then that P is fully $\mathcal{F}$-centralized and $Aut_S(P)$ is a Sylow p-subgroup of $Aut_{\mathcal{F}}(P)$. Let also $P' \in \langle P \rangle_{\mathcal{F}}$ be a fully $\mathcal{F}$-normalized subgroup. Then, there is a morphism $f \in Hom_{\mathcal{F}}(N_S(P), N_S(P'))$ such that $f(P) = P'$, and hence also $f(C_S(P)) \leq C_S(P')$.

In fact, since P is fully $\mathcal{F}$-normalized, it follows that $f_{|C_S(P)}$ is an isomorphism, and thus f induces a monomorphism

$$\bar{f} : Aut_S(P) \longrightarrow Aut_S(P').$$

The statement follows now because $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P)$, $Aut_S(P') \in Syl_p(Aut_{\mathcal{F}}(P')$, and f induces an isomorphism

$$\begin{array}{ccc} Aut_{\mathcal{F}}(P) & \xrightarrow{f^*} & Aut_{\mathcal{F}}(P') \\ \gamma & \longmapsto & f \circ \gamma \circ f^{-1}. \end{array}$$

□

Now we can prove Proposition 1.7.1 above.

Proof. (of Proposition 1.7.1). Clearly, if $\mathcal{F}$ is saturated (in the sense of definition 1.2.2), then in particular it satisfies axioms (I′) and (II′). Thus, we are left to prove the converse.

Assuem then that $\mathcal{F}$ satisfies axioms (I′), (II′) and (III), and we have to prove that $\mathcal{F}$ satisfies axioms (I) and (II) (since (III) is already granted). First we show that axiom (I) holds on $\mathcal{F}$. Let $P \leq S$ be any subgroup, and let $f \in Hom_{\mathcal{F}}(P, S)$ be such that $P' = f(P)$ is fully $\mathcal{F}$-normalized. Then, by axiom (II′), f extends to some $\widetilde{f} \in Hom_{\mathcal{F}}(N_f, S)$, where

$$N_f = \{g \in N_S(P) | f c_g f^{-1} \in Aut_S(P)\},$$

and, in particular, $C_S(P) \leq N_f$. Thus, if follows that $\widetilde{f}(C_S(P)) \leq C_S(P')$, and hence, since this holds for all $P \in \langle P' \rangle_{\mathcal{F}}$, P' is fully $\mathcal{F}$-centralized.

Suppose now that P is fully $\mathcal{F}$-normalized, and we want to prove that $Aut_S(P)$ is a Sylow p-subgroup of $Aut_{\mathcal{F}}(P)$. Actually, by axiom (I′), there is nothing to prove if $P = S$, so we can assume that P is a proper subgroup of S. Assume also that P is maximal in order among all (fully $\mathcal{F}$-normalized) subgroups of S such that $Aut_S(P)$ is not a Sylow p-subgroup of $Aut_{\mathcal{F}}(P)$, and let $H \in Syl_p(Aut_{\mathcal{F}}(P)$ be such that $Aut_S(P) \lneq H$. Let also $f \in H \setminus Aut_S(P)$ be a morphism normalizing $Aut_S(P)$ (since both $Aut_S(P)$ and H are discrete p-toral groups, such a morphism exists).

It follows then that for each $x \in N_S(P)$, there exists some $y \in N_S(P)$ such that $f(xgx^{-1}) = yf(g)y^{-1}$, for all $g \in P$, and hence $N_f = N_S(P)$, and hence by axiom (II′), f extends to some $\gamma \in Aut_{\mathcal{F}}(N_S(P))$. Furthermore, by taking an appropriate power of γ, we may assume that γ has p-power order.

Now, let $f' \in Hom_{\mathcal{F}}(N_S(P), S)$ be such that $N' = f'(N_S(P))$ is fully $\mathcal{F}$-normalized. Since $P \lneq S$, it follows by Lemma 1.1.4 that $P \lneq N_S(P)$, and hence $Aut_S(N') \in Syl_p(Aut_{\mathcal{F}}(N'))$. In particular, $\gamma' = f'\gamma(f')^{-1}$ is conjugated in $Aut_{\mathcal{F}}(N')$ to an element in $Aut_S(N')$, and hence we can assume that f' has been chosen such that $\gamma' = c_h \in Aut_S(N')$, for some $h \in N_S(N')$.

Since $\gamma_{|P} = f$, the automorphism γ' restricts to an automorphism of $f'(P)$, and hence $y \in N_S(f'(P))$. It follows that $f'(N_S(P)) \leq N_S(f'(P))$, and since P is fully $\mathcal{F}$-normalized, the last inequality is in fact an equality, and

$$\gamma(g) = (f'(h))u(f'(h))^{-1}$$

for all $g \in N_S(P)$, and thus $f \in Aut_S(P)$, in contradiction with the hypothesis of $f \in H \setminus Aut_S(P)$.

Finally, we prove that axiom (II) holds in $\mathcal{F}$. Let $f \in Hom_{\mathcal{F}}(P, S)$ be such that $P' = f(P)$ is fully $\mathcal{F}$-centralized, and we have to prove that f extends to some $\widetilde{f} \in Hom_{\mathcal{F}}(N_f, S)$. Choose then some $\gamma \in Hom_{\mathcal{F}}(P', S)$ such that $P'' = \gamma(P')$ is fully $\mathcal{F}$-normalized and such that, in the notation of axiom (II′), $N_\gamma = N_S(P')$, and let $f' = \gamma \circ f$.

Since P'' is fully $\mathcal{F}$-normalized, by (II′) it follows that both γ and f' extend to morphisms $\widetilde{\gamma} \in Hom_{\mathcal{F}}(N_\gamma, S)$ and $\widetilde{f'} \in Hom_{\mathcal{F}}(N_{f'}, S)$ respectively. We want then to see that

(i) $N_f \leq N_{f'}$, and

(ii) $\widetilde{f'}(N_f) \leq \widetilde{\gamma}(N_\gamma)$.

Were it the case, the composition $(\widetilde{\gamma}^{-1} \circ \widetilde{f'})_{|N_f}$ would then be the extension of f we are looking for.

Let first $g \in N_f$. It follows then that there is some $h \in N_S(P')$ such that $fc_gf^{-1} = c_h$. Furthermore, since $N_\gamma = N_S(P')$, it follows then that there is some $x \in N_S(P'')$ such that

$$c_x = \gamma c_h \gamma^{-1} = \gamma(fc_gf^{-1})\gamma^{-1} = (\gamma f)c_g(\gamma f)^{-1} = (f')c_g(f')^{-1}.$$

Since this holds for all $g \in N_f$, point (i) above follows.

Let now $g \in N_f$, and let $x = \widetilde{f'}(g)$. Let also $h \in N_S(P')$ be such $f(gyg^{-1}) = hf(y)h^{-1}$ for all $y \in P$. Since $x = \widetilde{f'}(g)$, it follows that $f'(gyg^{-1}) = xf'(y)x^{-1}$, and hence

$$\gamma(hf(y)h^{-1}) = f'(gyg^{-1}) = xf'(y)x^{-1},$$

which in turn implies that $x = \widetilde{\gamma}(h)c$ for some $c \in C_S(P'')$. Now, note that $C_S(P) \leq N_{f'}$, $\widetilde{\gamma}(C_S(P')) \leq C_S(P'')$ and, since P' is fully $\mathcal{F}$-centralized, we deduce that $\widetilde{\gamma}(C_S(P')) = C_S(P'')$. Thus, $c \in \widetilde{\gamma}(N_\gamma)$, and hence $x \in \widetilde{\gamma}(N_\gamma)$. It follows then that $\widetilde{f'}(N_f) \leq \widetilde{\gamma}(N_\gamma)$.

□

1.8 Further notation

Along this work, some assorted notation will be used. Thus, this section is to be understood as a summary of the main conventions we will follow.

The first notion to introduce is that of a Sylow p-subgroup for infinite groups., whenever it makes sense.

Definition 1.8.1. *Let G be a group, and let S be an artinian locally finite p-subgroup of G. We say that S is a **Sylow p-subgroup** of G if any finite p-subgroup $P \leq G$ is G-subconjugate to S.*

The following lemma from §8 [BLO07] provides a tool to decide whether a group has Sylow p-subgroups.

Lemma 1.8.2. *(8.1 [BLO07]). Fix a group G, a normal discrete p-toral group $Q \lhd G$, and a group $K \leq G$ such that $G = QK$. Assume that K has Sylow p-subgroups. Then, G has Sylow p-subgroups, and*

$$Syl_p(G) = \{QS | S \in Syl_p(G)\}.$$

This is the case, for instance, of the automorphism groups $Aut_{\mathcal{F}}(P)$ and $Aut_{\mathcal{L}}(P)$ in a (saturated) fusion system or a linking system, since these groups are locally finite. Furthermore, since these groups have a normal discrete p-toral group of finite index ($Inn(P)$ and P respectively), their Sylow p-subgroups turn out to be discrete p-toral groups again. This fact will be used repeatedly and implicitely in this work.

The following notions, too, will used frequently. Let G be a group, and p a prime number. Some of the following subgroups have already appeared in this chapter.

- $O_p(G)$ is the maximal normal p-subgroup of G;
- $O_{p'}(G)$ is the maximal normal subgroup of G of order prime to p;
- $O^p(G)$ is the minimal normal subgroup of G of p power index;

- $O^{p'}(G)$ is the minimal normal subgroup of G of p' index.

Definition 1.8.3. *Let G be a group and p a prime number. We say that G is p-**reduced** if $O_p(G) = \{1\}$, and that G is p'-**reduced** if $O_{p'}(G) = \{1\}$.*

The following lemma needs no proof.

Lemma 1.8.4. *Let G be a group and let p be a prime number. Then, the following holds:*

(i) $O^p(G)$ is the subgroup of G generated by all infinitely p-divisible elements of G, and

(ii) $O^{p'}(G)$ is the subgroup of G generated by all elements of p-power order.

Chapter 2

Groups realizing fusion systems

When dealing with p-local compact groups, and with saturated fusion systems in general, it is sometimes useful to have a group realizing the fusion system. The problem of constructing such group models for finite fusion systems has been dealt with from different points of view, see for instance [Rob07] and [LS07]. In this chapter we extend (some of) the results in [Rob07] for fusion systems over discrete p-toral groups. That is, given such a saturated fusion system $\mathcal{F}$ over S, we show the existence of a group G such that $\mathcal{F}_S(G) = \mathcal{F}$.

While the results in [Rob07] hold for a larger class of fusion systems than the class of saturated ones (and it is possible to extend the results in this chapter to the generality of [Rob07]), we are actually interested only on saturated fusion systems, and hence restrict the statements here to this class of fusion systems for simplicity. In this sense, we depend, in a previous step, on the work on constrained fusion systems done in [BCG+05]. Equivalent results for the compact case are needed if we want to extend the results from the first paper. Thus, this chapter is divided in two sections, the first one studying constrained fusion systems over discrete p-toral groups, and the second one realizing saturated fusion systems over discrete p-toral groups.

We have chosen to extend Robinson's models for fusion systems because of their resemblance with the construction of unstable Adams operations for p-local compact groups done in [Jun09], since we intend to combine both constructions in later chapters. We have not, however, explored the possibilities of other group models for fusion systems, which will certainly be of interest in the theory of p-local compact groups.

2.1 Constrained fusion systems

As happened already in the case of saturated fusion systems over finite p-groups, constrained fusion systems are really well behaved, and this translates in the fact that, for such a fusion system, there exists a unique associated centric linking system. Actually, there is not much work to do in this section since most of the work on higher limits that we need here has already been done in [BLO07].

Let G be a locally finite, artinian p'-reduced group (definition 1.8.3). The group G is then said to be ***p*-constrained** if there exists some normal p-subgroup $P \triangleleft G$ which is centric in G (i.e., $C_G(P) \leq P$).

Definition 2.1.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. We say that $\mathcal{F}$ is* ***constrained*** *if it contains an object P which is $\mathcal{F}$-centric and $\mathcal{F}$-normal.*

We now show the equivalent to Proposition 4.2 in [BCG+05] about the obstructions to existence and uniqueness of associated linking systems. For a saturated fusion system $\mathcal{F}$, recall the center functor $\mathcal{Z}_{\mathcal{F}}$ on $\mathcal{O}(\mathcal{F}^c)$ defined in 1.5.5 by $\mathcal{Z}_{\mathcal{F}}(P) = Z(P)$ for all $\mathcal{F}$-centric subgroups.

Proposition 2.1.2. *Let $\mathcal{F}$ be a constrained saturated fusion system over a discrete p-toral group S. Then,*

$$\varprojlim{}^i{}_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}}) = 0$$

for all $i > 0$. In particular, there exists a unique (up to isomorphism) centric linking system associated to $\mathcal{F}$.

Proof. As a first reduction, by Proposition 1.5.3, there is an isomorphism $\varprojlim{}^*{}_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}}) \cong \varprojlim{}^*{}_{\mathcal{O}(\mathcal{F}^{\bullet c})}(\mathcal{Z}_{\mathcal{F}})$, where the orbit subcategory $\mathcal{O}(\mathcal{F}^{\bullet c})$ contains finitely many conjugacy classes. This allows us now to follow the strategy for the proof of Proposition 4.2 in [BCG+05].

Fix a subgroup $Q \lhd S$ which is $\mathcal{F}$-centric and normal in $\mathcal{F}$. Let also $P_1, P_2, \dots P_m$ be representatives of the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric subgroups $P \leq S$ such that $P \ngeq Q$. We can order these representatives by their order, that is, $|P_j| \leq |P_{j+1}|$ for all j. We can then filtrate the functor $\mathcal{Z}_{\mathcal{F}}$ as follows. For $j = 0, 1, \dots, m$, let $\mathcal{Z}_| \subseteq \mathcal{Z}_{\mathcal{F}}$ be the subfunctor

$$\mathcal{Z}_j(P) = \begin{cases} Z(P) & \text{, if } P \text{ is } \mathcal{F}\text{-conjugate to } P_j \text{ for some } k > j, \\ 0 & \text{, otherwise.} \end{cases}$$

This filtration satisfies then that for each j, $\mathcal{Z}_{j-1}/\mathcal{Z}_j$ vanishes except on the single $\mathcal{F}$-conjugacy class of P_j. By Proposition 5.4 in [BLO07], it follows that

$$\varprojlim{}^*{}_{\mathcal{O}(\mathcal{F}^{\bullet c})}(\mathcal{Z}_{j-1}/\mathcal{Z}_j) \cong \Lambda^*(Out_{\mathcal{F}}(P_j); Z(P_j)).$$

Since $P_j \ngeq Q$ (and since P_j is $\mathcal{F}$-centric), $N_{P_jQ}(P_j)/P_j \cong Out_Q(P_j)$ is a non-trivial normal subgroup of $Out_{\mathcal{F}}(P_j)$, and since $Out_{\mathcal{F}}(P_j)$ is finite, it follows by 6.1 (ii) in [JMO92b] that $\Lambda^*(Out_{\mathcal{F}}(P_j); Z(P_j)) = 0$. This in turn implies that $\varprojlim{}^*(\mathcal{Z}_j) = 0$ for all j, and in particular for $j = 0$. Thus, there is an isomorphism

$$\varprojlim{}^*{}_{\mathcal{O}(\mathcal{F}^{\bullet c})}(\mathcal{Z}_{\mathcal{F}}) \cong \varprojlim{}^*{}_{\mathcal{O}(\mathcal{F}^{\bullet c})}(\mathcal{Z}_{\mathcal{F}}/\mathcal{Z}_0), \tag{2.1}$$

where $\mathcal{Z}_{\mathcal{F}}/\mathcal{Z}_0$ is the quotient functor

$$(\mathcal{Z}_{\mathcal{F}}/\mathcal{Z}_0)(P) = \begin{cases} Z(P) = Z(Q)^P & \text{, if } P \geq Q, \\ 0 & \text{, if } P \ngeq Q. \end{cases} \tag{2.2}$$

Let $\Gamma = Out_{\mathcal{F}}(Q)$ and $S' = Out_S(Q) \cong S/Q$. Since $\mathcal{F}$ is saturated (and Q is $\mathcal{F}$-normal), it follows that $S' \in Syl_p(\Gamma)$. Consider the $\mathbb{Z}_{(p)}[\Gamma]$-module $M = Z(Q)$. Note that, since Q is $\mathcal{F}$-normal and $\mathcal{F}$-centric, $rk(Q) = rk(S)$ (since $Out_S(Q)$ has to be finite), and hence S' is a finite p-group. Thus, we can consider the fixed-point functor on $\mathcal{O}_{S'}(\Gamma)$, H^0M, defined by $H^0M(P) = M^P$, and this turns out to be acyclic by Proposition 5.2 in [JMO92b]. Hence, by (2.1), the proof will be finished if we show that

$$\varprojlim{}^*{}_{\mathcal{O}(\mathcal{F}^{\bullet c})}(\mathcal{Z}_{\mathcal{F}}/\mathcal{Z}_0) \cong \varprojlim{}^*{}_{\mathcal{O}_{S'}(\Gamma)}(H^0M).$$

As in the finite case (using the functor $(\_)^{\bullet}$), it follows now, since Q is $\mathcal{F}$-normal and $\mathcal{F}$-centric, that $\mathcal{O}_{S'}(\Gamma)$ is isomorphic to the full subcategory of $\mathcal{O}(\mathcal{F}^{\bullet c})$ with objects the subgroups of S containing Q. Under this identification, H^0M is the restriction of $\mathcal{Z}_{\mathcal{F}}/\mathcal{Z}_0$ by (2.2), and the isomorphism above follows since $(\mathcal{Z}_{\mathcal{F}}/\mathcal{Z}_0)(P) = 0$ for all $P \not\geq Q$, and since there are no morphisms in $\mathcal{O}(\mathcal{F}^{\bullet c})$ from an object in the subcategory to an object not in it.

Finally, existence and uniqueness of a centric linking $\mathcal{L}$ associated to $\mathcal{F}$ follow by Proposition 1.5.6.

□

Finally, we prove the main result about constrained saturated fusion systems.

Proposition 2.1.3. *Let $\mathcal{F}$ be a constrained saturated fusion system over a discrete p-toral group S. Then, there exists a unique (up to isomorphism) p'-reduced p-constrained artinian locally finite group G such that*

(i) $S \in Syl_p(G)$,

(ii) $\mathcal{F} = \mathcal{F}_S(G)$, *and*

(iii) if $\mathcal{L}$ is a (the) centric linking system associated to $\mathcal{F}$, then $G \cong Aut_{\mathcal{L}}(P)$ for any $\mathcal{F}$-centric subgroup P which is normal in $\mathcal{F}$, and $\mathcal{L} \cong \mathcal{L}_S(G)$.

Proof. By Proposition 2.1.2, let $\mathcal{L}$ be the linking system associated to $\mathcal{F}$, and let $\rho : \mathcal{L} \to \mathcal{F}^c$ be the projection functor. Fix also a compatible set of inclusions $\{\iota_{P,Q}\}$ in $\mathcal{L}$.

Let $Q\triangleleft$ be a $\mathcal{F}$-centric $\mathcal{F}$-normal subgroup, and let $G = Aut_{\mathcal{L}}(Q)$. Then, since $\mathcal{L}$ is a transporter system (as showed in Proposition A.2.5) there is an inclusion

$$S = N_S(Q) \xrightarrow{\delta_Q} Aut_{\mathcal{L}}(Q) = G.$$

Since Q is $\mathcal{F}$-centric and $\mathcal{F}$-normal, $S/Q \cong Out_S(Q) \in Syl_p(Out_{\mathcal{F}}(Q))$, and hence there is a commutative diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d, "\delta_Q"] & Out_S(Q) \arrow[d] \\ Q \arrow[r, "\delta_Q"'] & G \arrow[r] & Out_{\mathcal{F}}(Q). \end{tikzcd}$$

This implies, in particular, that S is a Sylow p-subgroup of G in the sense that each p-subgroup of G is subconjugate to a subgroup of S (since $Out_{\mathcal{F}}(Q)$ is finite and hence the Sylow theorems apply on it). Note also that, since G is the automorphism group of Q in $\mathcal{L}$, it is an artinian locally finite group.

The rest of properties in the statement above now hold using the same arguments used to prove Proposition 4.2 in [BCG+05], but we reproduce here the prove for the sake of clarity.

We next prove that G is p'-reduced and p-constrained. Let $P, P' \leq S$ be any pair of subgroups containing Q. By Lemma 3.2 [OV07], for any $\varphi \in Mor_{\mathcal{L}}(P, P')$, there is a unique restriction of φ to Q, in the sense that there is a unique $\gamma(\varphi) \in G$ such that $\iota_{Q,S} \circ \gamma(\varphi) = \varphi \circ \iota_{Q,P}$. Furthermore, these restrictions satisfy that

(a) $\gamma(\varphi' \circ \varphi) = \gamma(\varphi') \cdot \gamma(\varphi) \in G$, for any pair of composable morphisms (containing Q), and

(b) $\gamma(\delta(x)) = x$ for all $x \in N_S(P, P')$.

In addition, by axiom (C) of linking systems, it follows that, for each $g \in P$,

$$\delta(\rho(\varphi))(g) \circ \varphi = \varphi \circ \delta(g) \in Mor_{\mathcal{L}}(P, S),$$

which, after restricting to Q, yields the relation

$$\delta(\rho(\varphi)(g)) \circ \gamma(\varphi) = \gamma(\varphi) \circ \delta(g) \in G,$$

or, equivalently,

(c) $\gamma(\varphi) \in N_G(P, P')$ and $c_{\gamma(\varphi)} = \rho(\varphi) \in Hom_{\mathcal{F}}(P, P')$.

Now, there are equalities

$$C_G(Q) = Ker(Aut_{\mathcal{L}}(Q) \xrightarrow{\rho} Aut_{\mathcal{F}}(Q)) = Z(Q),$$

where the first equality holds from property (c) (applied with $P = P' = Q$), and the second holds by axiom (A) of linking systems. Hence, Q is centric in G. This also shows that G is p'-reduced and p-constrained.

Next we prove that $\mathcal{F} = \mathcal{F}_S(G)$. The inclusion $Hom_{\mathcal{F}}(P, P') \subseteq Hom_G(P, P')$ is easily seen to hold by lifting morphisms to $\mathcal{L}$ and using property (c) above. On the other hand, let $P, P' \leq S$ (and assume Q is contained in both P and P'), and let $g \in N_G(P, P')$, and c_g be the induced morphism. Then, $f = c_g$ restricts to an automorphism in $\mathcal{F}$ of Q by (c) above, and since Q is fully $\mathcal{F}$-centralized, this morphism extends to some $\widetilde{f} \in Hom_{\mathcal{F}}(N_f, S)$, where

$$N_f = \{x \in S = N_S(Q) | f c_x f^{-1} \in Aut_S(Q)\}$$

as usual. Furthermore, it is clear by hypothesis that $P \leq N_f$. In particular, f extends to $\widetilde{f} \in Hom_{\mathcal{F}}(P, S) \subseteq Hom_G(P, S)$ (taking restrictions of the original $\widetilde{f}$ if necessary). Let then $h \in N_G(P, S)$ be such that $\widetilde{f} = c_h$, and note that $(c_h)_{|Q} = \widetilde{f}_{|Q} = f = c_g$. Hence, $h = gx$, for some $x \in C_G(Q) = Z(Q)$, and $x \in P$, $c_x \in Aut_{\mathcal{F}}(P)$. Thus, $f = c_g \in Hom_{\mathcal{F}}(P, S)$, and $f(P) = gPg^{-1} \leq P'$.

Finally, we check property (iii) in the statement. Since G is artinian and locally finite, and has Sylow p-subgroups, we can apply Theorem 8.7 in [BLO07] to see that G induces a p-local compact group $(S, \mathcal{F}', \mathcal{L}')$. But in fact we have already proved that $\mathcal{F}' = \mathcal{F}_S(G)$, and by Proposition 2.1.2, there is (up to isomorphism) a unique centric linking system $\mathcal{L}$ associated to $\mathcal{F}_S(G)$. Hence, $\mathcal{L} \cong \mathcal{L}_S(G) \cong \mathcal{L}'$.

Let $Q \leq S$ be $\mathcal{F}$-centric and $\mathcal{F}$-normal. We want to prove that $G \cong Aut_{\mathcal{L}}(Q)$. Let $Q' = O_p(G)$, and note that $C_G(Q') = Z(Q')$ since G is p-constrained (there exists some discrete p-toral subgroup $P \lhd G$ which is centric, and hence $Q' \geq P$ by definition of Q'). Since Q is normal in $\mathcal{F}$, each $c_g \in Aut_G(Q')$ extends to some $c_h \in Aut_G(QQ')$. It follows that $g^{-1}h \in C_G(Q') = Z(Q')$, $h \in N_G(QQ')$, and so $g \in N_G(QQ')$. This shows that QQ' is a normal discrete p-toral subgroup of G, and hence, by definition of Q', $Q \leq Q'$. Hence, for any $g \in G$, $c_g \in Aut_G(Q')$ restricts to $c_g \in Aut_{\mathcal{F}}(Q)$, and $Q \lhd G$. In particular,

$$Aut_{\mathcal{L}}(Q) = N_G(Q)/O^p(C_G(Q)) = G/\{1\} = G.$$

□

2.2 Robinson groups realizing fusion systems

Now we can extend the results from [Rob07] to the compact case. This models for fusion systems are amalgams of locally finite, and this make them a rather *wild* model, sometimes difficult to deal with. For instance, while such a group will have S as a Sylow p-subgroup by construction, it is not at all clear that an arbitrary subgroup $G' \leq G$ has Sylow p-subgroups at all.

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and choose representatives $P_1 = S, P_2, \ldots, P_r$ of the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups, such that P_j is fully $\mathcal{F}$-normalized for all j. From 1.3.2 (i) and 1.3.4 we already know that there are finitely many $\mathcal{F}$-conjugacy classes of such subgroups.

Now, for each P_j in the list above, we have proved in 1.6.2 that the fusion system $\mathcal{N}_j \overset{def}{=} N_{\mathcal{F}}(P_j)$ is saturated, since we have chosen P_j to be fully $\mathcal{F}$-normalized. Furthermore, since P_j is $\mathcal{F}$-centric, it is also $\mathcal{N}_j$-centric, and it is normal in $\mathcal{N}_j$ just by definition of the normalizer fusion subsystem. Thus, $\mathcal{N}_j$ is a constrained saturated fusion system, and we may apply Proposition 2.1.3 to it: $\mathcal{N}_j$ is realized by a certain group L_j such that $N_S(P_j) \in Syl_p(L_j)$ by 2.1.3 (i).

Definition 2.2.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. A* ***fusion-controlling set*** *is a set $\mathcal{P} = \{P_1 = S, P_2, \ldots, P_r\}$ of representatives of the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric ($\mathcal{F}$-radical) subgroups such that, for each j, P_j is fully $\mathcal{F}$-normalized.*

Note that the group L_j realizing the normalizer fusion subsystem $N_{\mathcal{F}}(P_j)$ is determined up to isomorphism by P_j and $\mathcal{F}$, and hence there is no need of fixing L_j too.

Since $Ob(\mathcal{F}^{\bullet})$ contains finitely many $\mathcal{F}$-conjugacy classes, it follows that so does such a set $\mathcal{P}$. Furthermore, as long as $\mathcal{P}$ contains $\mathcal{F}$-centric fully $\mathcal{F}$-normalized subgroups, the following hold for all $P \in \mathcal{P}$:

(i) the normalizer fusion subsystem $N_{\mathcal{F}}(P)$ is a saturated a saturated fusion system over $N_S(P)$, and

(ii) P is a $N_{\mathcal{F}}(P)$-centric subgroup which is normal in $N_{\mathcal{F}}(P)$.

In particular, $N_{\mathcal{F}}(P)$ is constrained and Theorem 2.1.3 applies to it.

Lemma 2.2.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $\mathcal{P}$ be a fusion-controlling set for $\mathcal{F}$. For each $P \in \mathcal{P}$, let L be the group realizing the fusion system $N_{\mathcal{F}}(P)$. Then, the following holds:*

(i) $P = O_p(L)$;

(ii) $L/P = Out_{\mathcal{F}}(P)$;

(iii) $N_S(P) \in Syl_p(L)$.

Proof. It is immediate by definition. ∎

The main result of this section is stated below.

Theorem 2.2.3. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $\mathcal{P} = \{P_1 = S, P_2, \ldots, P_r\}$ be a fusion-controlling set for $\mathcal{F}$. Furthermore, for each $P_j \in \mathcal{P}$, let $N_j = N_S(P_j)$ and L_j be the group obtained in 2.1.3.*

Then, the group

$$G = L_1 *_{N_2} L_2 *_{N_3} \ldots *_{N_r} L_r$$

satisfies the following properties:

(i) $P_1 = S$ *is a Sylow p-subgroup of G;*

(ii) G realizes the fusion system $\mathcal{F}$*, that is* $\mathcal{F} = \mathcal{F}_S(G)$*.*

We will call the group G in the statement a **Robinson group** realizing the fusion system $\mathcal{F}$. Notice that G depends on the fusion-controlling set $\mathcal{P}$ that we have previously fixed, but different choices of fusion-controlling sets give rise to isomorphic groups.

Before proving the theorem, we study the case where $r = 2$, that is,

$$G = L_1 *_{N_2} L_2.$$

The proof for Theorem 2.2.3 is reduced then to show the following result.

Theorem 2.2.4. *Let H and K be groups which have Sylow p-subgroups and such that such Sylow p-subgroups are discrete p-toral groups, and let $S \in Syl_p(H)$, $P \in Syl_p(K)$ be such that $P \leq S$. Then, the group $G = H *_P K$ has Sylow p-subgroup S and the G-fusion system on S is precisely that generated by the H-fusion system on S and the K-fusion system on P.*

A lemma is needed before giving a proof for the theorem.

Lemma 2.2.5. *(Lemma 1 [Rob07]). Let $G = A *_C B$, where C is a common subgroup of A and B. Let also $X \leq A$ and $g \in G \setminus A$ be such that $X^g \leq A$ or $X^g \leq B$. Write $g = a_0b_1a_1 \ldots b_sa_sb_\infty$, where $a_i \in A \setminus C$ for $i = 0, \ldots, s$ and $b_i \in B \setminus C$ for $i = 1, \ldots, s, \infty$. Set $X_0 = X^{a_0}$, $Y_1 = X_0^{b_1}$, $X_1 = Y_1^{a_1}$, etc. Then,*

$$\langle X_0, Y_i, X_i | 1 \leq i \leq s\rangle \leq C.$$

This lemma is proved in [Rob07] without finiteness restrictions on A, B, C or X. Thus the proof is the same here.

Proof. (of Theorem 2.2.4). By (the corollary of) Theorem 4.3.8 in [Ser03], and using the Sylow hypothesis on $S \leq H$ and $P \leq K$, it follows that every finite p-subgroup of G is subconjugate to S. Since every discrete p-toral subgroup of G, R, can be expressed as a union of finite p-subgroups, $R = \cup R_j$, it follows then that every discrete p-toral subgroup of G is subconjugate to S, and hence S is a Sylow p-subgroup of G.

It is easy to see then that the fusion system over S induced by G contains the fusion system over S induced by H and the fusion system over P induced by K, $\mathcal{F} \subseteq \mathcal{F}_S(G)$, and we have to prove the equality.

By Lemma 2.2.5, if $R \leq H$, $h_1, \ldots, h_{r+1} \in K \setminus P$ and $g_1, \ldots, g_r \in H \setminus S$ are such that

$$R' = R^{h_1g_1 \ldots h_rg_rh_{r+1}} \leq H,$$

then all the intermediate subgroups $R, R^{h_1}, R^{h_1g_1}, \ldots, R'$ are subgroups of P.

Let now $R \leq S$ be a non-trivial subgroup, and suppose that $\langle R, R^g\rangle \leq S$ for some $g \in G$, and we have to show that conjugation by g is a morphism in $\mathcal{F}$. If $g \in H$, there is nothing to show, so we can assume that this is not the case. Thus, there exist (possibly identity) elements $g_{-\infty}, g_\infty \in H$, and elements $h_1, \ldots, h_r, h_{r+1} \in H \setminus S$ and $g_1, \ldots, g_r \in K \setminus P$ such that

$$g = g_{-\infty} h_1 g_1 \ldots h_r g_r h_{r+1} g_\infty.$$

If we set $R_0 = R^{g_{-\infty}}$ and $R_1 = (R_0)^{h_1 g_1 \ldots h_r g_r h_{r+1}}$, then $\langle R_0, R_1\rangle \leq H$, and hence by Lemma 2.2.5, all the intermediate subgroups

$$R_0, (R_0)^{h_1}, (R_0)^{h_1 g_1}, \ldots, (R_0)^{h_1 g_1 \ldots h_{r+1}}$$

are contained in P.

Now, conjugation from R to R_0 is a morphism in $\mathcal{F}_S(H)$ (that is, it corresponds to conjugation by $g_{-\infty} \in H$). Conjugation from R_0 to $(R_0)^{h_1}$ is a morphism in $\mathcal{F}_P(K)$ (conjugation by $h_1 \in K$). In general, conjugation from $(R_0)^{h_1 \ldots h_j}$ to $(R_0)^{h_1 \ldots h_j g_j}$ is a morphism in $\mathcal{F}_S(H)$ (for $j = 1, \ldots, r$), while conjugation from $(R_0)^{h_1 \cdots g_j}$ to $(R_0)^{h_1 \ldots h_{j+1}}$ is a morphism in $\mathcal{F}_P(K)$ (for $j = 1, \ldots, r$). Finally, conjugation from $(R_0)^{h_1 \ldots h_{r+1}}$ to $(R_0)^{h_1 \ldots h_{r+1} g_\infty}$ is a morphism in $\mathcal{F}_S(H)$.

This shows that every morphism induced by conjugation by an element of G on subgroups of S can be expressed as a sequence of conjugations in $\mathcal{F}_S(H)$ and $\mathcal{F}_P(K)$, and hence $\mathcal{F} = \mathcal{F}_S(G)$.

□

Proof. (of Theorem 2.2.3). This is just an iterated application of Theorem 2.2.4.

□

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, let $\mathcal{P}$ be an (enlarged) fusion-controlling set for $\mathcal{F}$, and let G be the corresponding Robinson group realizing $\mathcal{F}$. Let also $P \leq S$. Then, for each $Q \in \mathcal{P}$ such that $P \leq Q$, there is an obvious inclusion

$$\iota_{P,Q} : P \xrightarrow{incl} Q \xrightarrow{\delta} L_Q \xrightarrow{incl} G, \tag{2.3}$$

where L_Q is the group realizing the normalizer fusion subsystem of Q.

Chapter 3

Connected p-local compact groups

The notion of p-local compact groups was introduced as a unifying homotopical setting for (p-completed classifying spaces of) compact Lie groups and p-compact groups. As such, one would like then to have classification of p-local compact groups, and, if possible, as close as possible to the classification of compact Lie groups and p-compact groups ([AGMV08] and [AG09]).

If this is to be the case, then a notion of connectivity on p-compact groups is needed in order to do a first reduction from general p-local compact groups to connected p-local compact groups. This is in fact a major gap in the theory, closely related to the fact that Bousfield-Kan p-completion ([BK72]) does not preserve fibrations in general.

We introduce in this chapter the notion of connected p-local compact groups, and exploit it in the case of rank 1 p-local compact groups. In this sense, it is not difficult to give a list of all connected p-local compact groups of rank 1, but it is not at all clear yet how one can reduce to consider only connected p-local compact groups.

This chapter is then organized as follows. We start by giving a notion of connectivity for p-local compact groups and showing some of its properties. The second section is devoted to study connectivity on p-local compact groups of rank 1. In this chapter we prove that given a rank 1 p-local compact group $\mathcal{G}$, a unique connected p-local compact subgroup is determined by $\mathcal{G}$, $\mathcal{G}_0$, which we consider as the connected component of $\mathcal{G}$, and furthermore, it can be equipped with an inclusion of p-local compact groups. Finally, the third section discusses briefly the existence of such connected components in the general rank case.

The author wants to thank C. Broto, R. Levi and B. Oliver for their useful notes on connectivity for p-local compact groups, as well as many talks on the subject. Some of the notions in this chapter were "suspected" to the author, but got the final shape thanks to them.

3.1 A notion of connectivity

As a first attempt to define connectivity on p-local compact groups we could say that $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is *(topologically) connected* if $B\mathcal{G} = |\mathcal{L}|^{\wedge}_p$ is simply connected. This turns out to be quite a confusing definition, since some p-local finite groups can be connected in this sense. Thus, a stronger notion is required, and since the "only" difference between saturated fusion systems over finite p-groups and discrete p-toral groups is the existence of a maximal torus of positive rank, this notion should take into account this maximal torus.

Lemma 3.1.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let T be its maximal torus. Then, there exists a (unique) minimal strongly $\mathcal{F}$-closed subgroup $S_0 \leq S$ containing T.*

The subgroup S_0 will be called the **connected component** of S with respect to $\mathcal{F}$. Note that if S is finite, then $S_0 = \{1\}$.

Proof. The existence of S_0 can be proved by taking the intersection of all strongly $\mathcal{F}$-closed subgroups of S containing T, since the intersection of two strongly $\mathcal{F}$-closed subgroups is again a strongly $\mathcal{F}$-closed subgroup.

□

An alternative characterization of the subgroup S_0 can be given as follows. Let P_1 be the subgroup of S generated by all the elements of S which are $\mathcal{F}$-subconjugate to T, and let in general P_{n+1} be the subgroup of S generated by all the elements in S which are $\mathcal{F}$-subconjugate to P_n. Let also $S_0' = \cup P_n$.

Lemma 3.1.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let S_0 be the connected component of S with respect to $\mathcal{F}$ and S_0' be defined as above. Then, $S_0' = S_0$.*

Definition 3.1.3. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let S_0 be the connected component of S with respect to $\mathcal{F}$. The* ***group of components*** *of $\mathcal{F}$ is defined then as the saturated fusion system*

$$\pi_0(\mathcal{F}) \stackrel{def}{=} \mathcal{F}/S_0$$

over the finite p-group S/S_0.

*Similarly, if $\mathcal{G}$ is a p-local compact group, and S_0 is the connected component of S with respect to $\mathcal{F}$, then the p-**local finite group of components** of $\mathcal{G}$ is defined as the p-local finite group*

$$\pi_0(\mathcal{G}) \stackrel{def}{=} (S/S_0, \pi_0(\mathcal{F}), \pi_0(\mathcal{L}) \stackrel{def}{=} (\mathcal{L}/S_0)^c).$$

The fact that $\pi_0(\mathcal{F})$ is saturated follows by Proposition A.1.1, since S_0 is strongly $\mathcal{F}$-closed. Similarly, by Proposition A.3.3, together with Proposition A.2.6, $\pi_0(\mathcal{G})$ is indeed a p-local finite group.

For a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, let $O^{p'}(\mathcal{F})$ over S be the minimal saturated fusion subsystem of $\mathcal{F}$ of index prime to p, whose existence has been proved in Theorem B.4.3. By Theorem B.4.4, an associated centric linking system $\mathcal{L}$ on $\mathcal{F}$ induces a centric linking system $O^{p'}(\mathcal{L})$ associated to $O^{p'}(\mathcal{F})$.

Definition 3.1.4. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let S_0 be the minimal strongly $\mathcal{F}$-closed subgroup of S containing T. Then, we say that $\mathcal{F}$ is* ***connected*** *if each $x \in S$ is $\mathcal{F}$-subconjugate to T and $\mathcal{F} = O^{p'}(\mathcal{F})$.*

Let $\mathcal{G}$ be a p-local compact group. Then, we say that $\mathcal{G}$ is ***connected*** *if the corresponding fusion system $\mathcal{F}$ is connected in the sense above.*

Note that if $\mathcal{F}$ is connected, then in particular $S = S_0$. The above condition on $O^{p'}(\mathcal{F})$ appears in order to avoid "unnatural" situations, like the p-compact groups known as the Sullivan spheres, whose induced p-local compact groups satisfy that all $x \in S$ is $\mathcal{F}$-subconjugate to T, but $O^{p'}(\mathcal{F}) \subsetneq \mathcal{F}$, and can in fact be seen as extensions of finite groups by a discrete p-torus of rank 1. Note also that, if S is finite and $\mathcal{F}$ is a connected saturated fusion system over S, then $S = \{1\}$ and $\mathcal{F}$ is the trivial fusion system.

Lemma 3.1.5. *If $\mathcal{G}$ is a connected p-local compact group, then $B\mathcal{G}$ is topologically connected.*

Proof. By Proposition 4.4 [BLO07], there is a natural epimorphism $S \twoheadrightarrow \pi_1(B\mathcal{G})$, where $\pi_1(B\mathcal{G})$ is in fact a finite p-group. Thus, if we set K for the kernel of this epimorhphism, then K contains the maximal torus T of S.

We next claim that K is a strongly $\mathcal{F}$-closed subgroup. Indeed, let $P \in Ob(\mathcal{L})$, $\varphi \in Aut_{\mathcal{L}}(P)$, $f = \rho(\varphi) \in Aut_{\mathcal{F}}(P)$ and $g \in P \cap K$. Then, by axiom (C) on linking systems,

$$\delta_P(x) = \varphi \circ \delta_P(g) \circ \varphi^{-1},$$

where $x = f(g)$. Let also $\omega : Mor(\mathcal{L}) \to \pi_1(|\mathcal{L}|)$ the map from (the proof of) 4.4 [BLO07], and set for simplicity $\omega : Mor(\mathcal{L}) \to \pi_1(B\mathcal{G})$ the composition of the former ω with the p-completion. In particular, this map sends compositions to products.

It follows then that $\omega(\delta_P(g)) = [1] \in \pi_1(B\mathcal{G})$, since $g \in K$, and hence

$$\omega(\delta_P(x)) = \omega(\varphi) \cdot \omega(\delta_P(g)) \cdot \omega(\varphi^{-1}) = [1],$$

which means that $x \in K$. Alperin's fusion theorem then implies that K is strongly $\mathcal{F}$-closed.

Now, by hypothesis, S is its own connected component with respect to $\mathcal{F}$, and in particular, if S_0 denotes the minimal strongly $\mathcal{F}$-closed subgroup of S containing T, then $S_0 = S$. Since K is strongly $\mathcal{F}$-closed and contains T, it follows that $K = S$.

□

Before finishing this section, we prove some technical lemmas that will be useful later on in this chapter.

Lemma 3.1.6. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $P \leq S$ be $\mathcal{F}$-subconjugate to T. Then,*

$$C_P(T) \overset{def}{=} C_S(T) \cap P = T \cap P.$$

The last lemma can be understood as follows. If $x \in S$ is $\mathcal{F}$-conjugate to an element of the torus T, then either x is already an element in T or x acts nontrivially on T.

Proof. Let $f : P \to T$ be a morphism in $\mathcal{F}$. We can assume that $P = \langle x \rangle$ (otherwise we can restrict f to all the cyclic subgroups of P), and furthermore we can suppose that $P' = f(P)$ is a fully $\mathcal{F}$-centralized, since it is a subgroup of T. Hence, we can apply axiom (II) of saturated fusion system to f: f extends to a morphism $\widetilde{f} \in Hom_{\mathcal{F}}(N_f, S)$, where

$$N_f = \{g \in N_S(P) | f c_g f^{-1} \in Aut_S(P')\}.$$

In particular, note that $C_S(P) \cdot P \leq N_f$.

Now, suppose that P acts trivially on T, that is, $T \leq C_S(P)$, and in particular f extends to

$$PT \xrightarrow{\widetilde{f}} S,$$

such that $\widetilde{f}(P) = f(P) \leq T$ and $\widetilde{f}(T) \leq T$ since T is the maximal infinitely p-divisible subgroup of S. Thus, $\widetilde{f}(PT) = T$, which means that $PT = T$, since f is a monomorphism, and $P \leq T$.

□

A discrete p-toral group S with maximal torus T fits by definition in an extension

$$T \longrightarrow S \longrightarrow S/T,$$

where S/T is a finite p-group. This extension, in turn, comes equipped with a morphism

$$\omega_S : S/T \longrightarrow Aut(T), \tag{3.1}$$

defined by $\omega_S(xT)(t) = xtx^{-1}$ for all $t \in T$, where $x \in S$ is a representative of the class xT.

Lemma 3.1.7. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let S_0 be the connected component of S with respect to $\mathcal{F}$. Then, the restriction of ω_S to $S_0/T \leq S/T$ is a monomorphism. In particular, S_0/T acts faithfully on T.*

Proof. Let $xT \in S_0/T$, and suppose that $\omega_S(xT) = id_T$. Then, for any $x \in S_0$ representing xT, x acts trivially on T, and, by definition of S_0, is $\mathcal{F}$-subconjugate to T. Thus, by Lemma 3.1.6, $x \in T$ and $xT = 1T$.

□

Lemma 3.1.8. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let T be the maximal torus. If T is central in S, that is, if $S = C_S(T)$, then T is $\mathcal{F}$-normal, and $\mathcal{F}$ is an extension of a saturated fusion system (over the finite p-group S/T) by T.*

If in addition $\mathcal{L}$ is an associated centric linking system for $\mathcal{F}$, then $\mathcal{L}$ is an admissible extension of $\mathcal{L}/T$ by T (in the sense of definition A.5.4), and there is a fibration

$$BT \longrightarrow |\mathcal{L}| \longrightarrow |\mathcal{L}/T|.$$

Note that $\mathcal{L}/T$ is not, in general, a linking system associated to $\mathcal{F}/T$, but a transporter system (in the sense of definition A.2.1) such that the set $Ob(\mathcal{L}/T)$ contains all $\mathcal{F}/T$-centric $\mathcal{F}/T$-radical subgroups.

Let for instance $S = T$ (a discrete p-torus of any rank), and let $K \leq Aut(T)$ be a finite p' group (that is, p does not divide $|K|$). Let also $\mathcal{F}$ be the fusion system over S spanned by $Aut_{\mathcal{F}}(S) = K$, that is, all morphisms in $\mathcal{F}$ are restrictions of automorphisms in $Aut_{\mathcal{F}}(S)$, and let $\mathcal{L}$ be the category with $Ob(\mathcal{L}) = \{S\}$ and $Aut_{\mathcal{L}}(S) = T \rtimes K$, where K acts on T by automorphisms. Then, $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is clearly a p-local compact group, and T satisfies the conditions in the above lemma.

In this case, the quotient fusion system $\mathcal{F}/T$ is just the trivial fusion system over the trivial p-group $\{1\}$, and the quotient $\mathcal{L}/T$ is clearly not a centric linking system associated to $\mathcal{F}/T$, since axiom (A) is not satisfied. It is, however, a transporter system associated to the quotient fusion system $\mathcal{F}/T$.

No matter how nice this lemma may seem, if we change in the statement the conditions on T by the same conditions on a subtorus $T' \lneqq T$ of strictly lower rank, then it fails to be true in general.

Proof. Let $f : P \to P'$ be a morphism in $\mathcal{F}$, and we have to check that it extends to some $\gamma : PT \to P'T$ such that $\gamma_{|T} \in Aut_{\mathcal{F}}(T)$.

Suppose first that P' is fully $\mathcal{F}$-centralized. We may then apply axiom (II) to f to see that this morphism extends to some $\gamma : P \cdot C_S(P) \to P' \cdot C_S(P')$. In particular, since $T \leq C_S(P), C_S(P')$, γ restricts to a morphism

$$PT \xrightarrow{\gamma} P'T.$$

Furthermore, since T is the maximal infinitely p-divisible subgroup of S, it follows that $\gamma(T) \leq T$, that is, $\gamma_{|T} \in Aut_{\mathcal{F}}(T)$.

If P' is not fully $\mathcal{F}$-centralized, we may consider $P'' \in \langle P' \rangle_{\mathcal{F}}$ such that P'' is fully $\mathcal{F}$-centralized, and morphisms $f' : P \to P''$ and $f'' : P' \to P''$ in $\mathcal{F}$ such that the following diagram is commutative

$$\begin{tikzcd} & P'' & \\ P \arrow[ur, "f'"] \arrow[rr, "f"'] & & P' \arrow[ul, "f''"'] \end{tikzcd}$$

By applying the above arguments on f' and f'', we see that these morphisms extend respectively to $\gamma' : PT \to P''T$ and $\gamma'' : P'T \to P''T$, and hence the composition

$$\gamma = (\gamma'')^{-1} \circ \gamma' : PT \to P'T$$

extends the original f. Again, since T is the maximal infinitely p-divisible subgroup of S, it follows that the restriction of γ to T is an automorphism (in $\mathcal{F}$) of T.

Thus T is $\mathcal{F}$-normal, and the extension theory developed for p-local compact groups in appendix §A applies: Proposition A.1.1 says that $\mathcal{F}/T$ is a saturated fusion system over S/T.

The second part of the statement also follows from the extension theory developed in appendix §A. In particular, the extension is easily seen to be admissible since T is central in S.

□

Since we intend to study connectivity of p-local compact groups, we expect the "connected components" that we might find to be somehow invariant in the original p-local compact group. We introduce for this purpose the following notion of invariance of fusion subsystems (definition 3.1 in [Lin06]). Actually, what we call here *invariant subsystems* are called *normal subsystems* in [Lin06], but, by the time we started this work, this notion had already been renamed as *invariant subsystems* in [Asc08], where a stronger notion of invariance is also introduced. We do not mention this last (stronger) condition here since it is not clear that connected components will satisfy it in general.

Definition 3.1.9. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $S' \leq S$ and $\mathcal{F}' \subseteq \mathcal{F}$ be a fusion subsystem over S'. Then, $\mathcal{F}'$ is* ***invariant*** *in $\mathcal{F}$ if*

(i) *S' is strongly $\mathcal{F}$-closed;*

(ii) *for each $P \leq Q \leq S'$ and each $\gamma \in Hom_{\mathcal{F}}(Q, S)$, the map*

$$\begin{tikzcd} Hom_{\mathcal{F}'}(P,Q) \arrow[r, "\gamma^*"] & Hom_{\mathcal{F}'}(\gamma(P),\gamma(Q)) \\ f \arrow[r, mapsto] & \gamma \circ f \circ \gamma^{-1} \end{tikzcd}$$

is a bijection.

For instance, let G be a finite group, let $S \in Syl_p(G)$, and let N be a normal subgroup of G. Let also $\mathcal{F}_S(G)$ be the (saturated) fusion system over S induced by G. Then, the fusion subsystem $\mathcal{F}_{S\cap N}(N)$ (i.e., the fusion system over $S \cap N$ induced by N) is invariant in $\mathcal{F}_S(G)$.

Note that, if $\mathcal{F}'$ in the definition above is saturated, then it is enough to check condition (ii) only for $\mathcal{F}'$-centric $\mathcal{F}'$-radical, by Alperin's fusion Theorem 1.3.5.

3.2 Rank 1 p-local compact groups and their connected components

In this section we study the connected component of a p-local compact group of rank 1. Even in such a concrete situation, the connectivity problem becomes a rather difficult issue, and in general all proofs follow *ad hoc* arguments. We provide in this section a list of all connected p-local compact groups of rank 1, and show how, given a rank 1 p-local compact group $\mathcal{G}$, it always contains a unique connected p-local compact subgroup $\mathcal{G}_0$ over the connected component of S with respect to $\mathcal{F}$. Furthermore, we are able to define inclusion functors $\mathcal{L}_0 \to \mathcal{L}$, which we consider as the inclusion of connected components, in all cases. However, we have not managed to express any $\mathcal{G}$ in terms of its connected component together with the p-local finite group of components. We list below the main results of this section.

Theorem 3.2.1. *Let $\mathcal{F}$ be a connected saturated fusion system over a discrete p-toral group S of rank 1. Then, $\mathcal{F}$ has a unique associated centric linking system $\mathcal{L}$.*

Furthermore, the p-local compac group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is induced by one (and only one) of the connected compact Lie groups in the list below:

(i) *$G_1 = S^1$,*

(ii) *$G_2 = SO(3)$, or*

(iii) *$G_3 = S^3$,*

where cases (ii) and (iii) appear only for $p = 2$. In particular, if $\mathcal{G}$ is induced by G_j, then

$$B\mathcal{G} \simeq (BG_j)^\wedge_p.$$

Note that there is nothing "unexpected" in the list above, in the sense that all connected rank 1 p-local compact groups are induced by connected compact Lie groups, and this list coincides with the classification of rank 1 connected compact Lie groups, and also with the classification of rank 1 connected p-compact groups. The following corollary was also an expected result.

Corollary 3.2.2. *Let $\mathcal{G}_2$ and $\mathcal{G}_3$ be the connected p-local compact groups induced by the compact Lie groups G_2 and G_3 in the list above. Let also T be the maximal torus of $\mathcal{G}_3$, and let $\mathbb{Z}/2 \cong T_1 \leq T$ be the order 2 subgroup of T.*

Then, T_1 is a central subgroup of $\mathcal{G}_3$, and $\mathcal{G}_3$ is an admissible extension (in the sense of definition A.5.4) of $\mathcal{G}_2$ by T_1. In particular, there is a fibration

$$B\mathbb{Z}/2 \simeq BT_1 \longrightarrow B\mathcal{G}_3 \longrightarrow B\mathcal{G}_2.$$

Another important feature of rank 1 p-local compact groups is that one can assign a connected p-local compact group $\mathcal{G}_0$ (from the list in Theorem 3.2.1) to each p-local compact group $\mathcal{G}$. In fact, this $\mathcal{G}_0$ is uniquely determined by the connected component of S with respect to $\mathcal{F}$. In this result we use the notion of invariant fusion subsystem, 3.1.9, which corresponds to definition 3.1 [Lin06].

Theorem 3.2.3. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S whose maximal torus has rank 1, and let S_0 be the connected component of S with respect to $\mathcal{F}$. Then, $\mathcal{F}$ uniquely determines a fusion subsystem $\mathcal{F}_0$ over S_0 such that the following holds:*

(i) $\mathcal{F}_0$ is a connected saturated fusion system,

(ii) $\mathcal{F}_0 \subseteq \mathcal{F}$ is invariant in $\mathcal{F}$.

We say then that $\mathcal{F}_0$ is the ***connected component of*** *$\mathcal{F}$.*

Similarly, let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a rank 1 p-local compact group, and let S_0 be the connected component of S with respect to $\mathcal{F}$. Then, $\mathcal{G}$ uniquely determines a p-local compact group $\mathcal{G}_0 = (S_0, \mathcal{F}_0, \mathcal{L}_0)$ such that the following holds:

(i) $\mathcal{F}_0$ is the connected component of $\mathcal{F}$, and

(ii) $\mathcal{G}_0$ is isomorphic as a p-local compact group to one of the p-local compact groups listed in Theorem 3.2.1.

We say then that $\mathcal{G}_0$ is the ***connected component of $\mathcal{G}$****.*

In fact, 2-local compact groups whose connected component is induced by S^3 depend completely on 2-local compact groups whose connected component is induced by $SO(3)$. Corollary 3.2.2 above is a particular case of the following result.

Corollary 3.2.4. *Let $\mathcal{G}$ be a 2-local compact group of rank 1 whose connected component is the 2-local compact group $\mathcal{G}_3$ induced by the compact Lie group $G_3 = S^3$ (in Theorem 3.2.1), let T be the maximal torus of $\mathcal{G}$, and let $\mathbb{Z}/2 \cong T_1 \leq T$ be the order 2 subgroup of T.*

Then, T_1 is a central subgroup of $\mathcal{G}$, and $\mathcal{G}$ is an (admissible) central extension (in the sense of A.5.1) of a 2-local compact group $\mathcal{G}'$ by T_1, where the connected component of $\mathcal{G}'$ is the 2-local compact group $\mathcal{G}_2$ induced by the compact Lie group $G_2 = SO(3)$ (in Theorem 3.2.1). In particular, there is a fibration

$$BT_1 \longrightarrow B\mathcal{G} \longrightarrow B\mathcal{G}'.$$

The p-local compact group $\mathcal{G}_0$ in Theorem 3.2.3 can be considered as the connected component of $\mathcal{G}$ since it is connected and uniquely determined by $\mathcal{G}$. However, in general, the centric linking system $\mathcal{L}_0$ is not a subcategory of $\mathcal{L}$, since in general $\mathcal{F}_0$-centric subgroups will not be $\mathcal{F}$-centric subgroups. Instead, given the p-local compact group $\mathcal{G}$, together with its connected component, we will construct a functor $\iota_0 : \mathcal{L}_0 \to \mathcal{L}$ which will play the role of the inclusion of the connected component.

Corollary 3.2.5. *Let $\mathcal{G}$ be a rank 1 p-local compact group, and let $\mathcal{G}_0$ and $\pi_0(\mathcal{G})$ be the corresponding connected component and the p-local finite group of components. For each $P_0 \in \mathcal{L}_0^{cr}$, let $P = P_0 \cdot C_S(P_0)$. Then,*

(i) P is $\mathcal{F}$-centric, and

(ii) $Aut_{\mathcal{L}}(P)$ contains a subgroup isomorphic to $Aut_{\mathcal{L}_0}(P_0)$.

In particular, a faithful functor $\iota^{cr} : \mathcal{L}_0 \to \mathcal{L}$ can be defined by sending each P_0 to the corresponding P, and such that

(iii) the composition

$$|\mathcal{L}_0| \simeq |\mathcal{L}_0^{cr}| \longrightarrow |\mathcal{L}| \longrightarrow |\pi_0(\mathcal{L})|$$

is nullhomotopic, and

(iv) there is a natural transformation from the inclusion functor $incl : \mathcal{F}_0^{cr} \to \mathcal{F}^c$ to the functor $i_0 : \mathcal{F}_0^{cr} \to \mathcal{F}^c$ induced by ι_0.

All the above results will be proved in the following subsections by describing explicitly each possible situation. In particular, we will treat separately the case $p > 2$ from the case $p = 2$. This is motivated by Lemma 3.2.6 below, where (as a corollary) we prove that in the case p is an odd prime, the maximal torus T is always $\mathcal{F}$-normal. Thus, the p odd version of the results above is proved almost automatically in Proposition 3.2.7 (except from the construction of the faithful functor, which will be treated in Proposition 3.2.26).

The case $p = 2$ is significantly longer to explain. In this case, we will start by studying which discrete 2-toral groups S_0 can appear as the connected component of a discrete 2-toral group S with respect to a saturated fusion system $\mathcal{F}$ on S. Thus, Theorem 3.2.1 is proved as Proposition 3.2.19, Theorem 3.2.3 is then a consequence of Proposition 3.2.18, and Corollary 3.2.4 follows by Lemma 3.2.17. The construction and properties of the faithful functors (Corollary 3.2.5) for $p = 2$ will be studied in Proposition 3.2.26, 3.2.30 and 3.2.32.

We start by stating a well-known fact about the automorphism group of a discrete p-torus of rank 1. and in this case it is not difficult to give a complete description of the whole group. In particular, our interest lies in the finite subgroups of $Aut(\mathbb{Z}/p^\infty)$, since for a saturated fusion system $\mathcal{F}$, the group $Aut_{\mathcal{F}}(T)$ has to be finite.

Lemma 3.2.6. *Let $T \cong \mathbb{Z}/p^\infty$. Then,*

$$Aut(T) \cong \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}_2^\wedge, & p = 2, \\ \mathbb{Z}/(p-1) \times \mathbb{Z}_p^\wedge, & p > 2. \end{cases}$$

Proof. Let $f \in Aut(T)$. Recall that

$$T \cong \mathbb{Z}/p^\infty = \bigcup_{n=1}^{\infty} \mathbb{Z}/p^n = \varinjlim \mathbb{Z}/p^n.$$

Thus, we see that the restriction of f to every cyclic subgroup $T_n = \mathbb{Z}/p^n$ gives an automorphism of this subgroup, f_n, such that the following squares are commutative for any n:

$$\begin{tikzcd} T_n \arrow[r, "f_n"] \arrow[d, "incl_{T_n}^{T_{n+1}}"'] & T_n \arrow[d, "incl_{T_n}^{T_{n+1}}"] \\ T_{n+1} \arrow[r, "f_{n+1}"'] & T_{n+1}. \end{tikzcd}$$

Thus, it is clear that we can identify the automorphism group $Aut(T)$ with the inverse limit of the inverse system $\{Aut(T_n)\}$. Finally, by Theorem 4.4.7 [RZ00], there is an isomorphism

$$\varprojlim Aut(T_n) \cong \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}_2^\wedge, \; p = 2, \\ \mathbb{Z}/(p-1) \times \mathbb{Z}_p^\wedge, \; p > 2. \end{cases}$$

□

As a consequence, for $p > 2$, the maximal torus T will always be central in S, and hence, by Lemma 3.1.8, if $\mathcal{F}$ is a saturated fusion system over S, then T is $\mathcal{F}$-normal. We restate and prove Theorems 3.2.1 and 3.2.3 for p odd as the following proposition.

Proposition 3.2.7. *Let $\mathcal{F}_0$ be a connected saturated fusion system over a discrete p-toral group S_0, with p odd. Then, the following holds:*

(i) $S_0 = T$, and

(ii) $\mathcal{F}_0$ is isomorphic to the saturated fusion system induced by the compact Lie group S^1,

$$\mathcal{F}_0 \cong \mathcal{F}_T(S^1).$$

In particular, $\mathcal{F}_0$ has a unique associated linking system $\mathcal{L}_0 \cong \mathcal{L}_T(S^1)$, and

$$|\mathcal{L}_0|_p^\wedge \simeq (BS^1)_p^\wedge.$$

Let p be an odd prime, and let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a rank 1 p-local compact group, with maximal torus T. Then,

(i) T is a $\mathcal{F}$-normal subgroup, and in particular it is strongly $\mathcal{F}$-closed. Hence, the connected component of S with respect to $\mathcal{F}$ is T itself, and

(ii) the connected component of $\mathcal{F}$ is (isomorphic to) the fusion system $\mathcal{F}_T(S^1)$.

In particular, $\mathcal{G}$ is an admissible extension of $\mathcal{G}/T$ by T, and there is a fibration

$$BT \longrightarrow |\mathcal{L}| \longrightarrow |\mathcal{L}/T|.$$

Proof. Let first $\mathcal{F}$ be any saturated fusion system over a discrete p-toral group S of rank 1, with $p > 2$. Since for p odd the maximal torus $T \leq S$ is central in S, it follows by Lemma 3.1.8 that T is a $\mathcal{F}$-normal subgroup, and in particular it is strongly $\mathcal{F}$-closed.

Thus, if $\mathcal{F}_0$ is connected and saturated over S_0, then $S_0 = T$ by definition of the connected component of S_0 with respect to $\mathcal{F}_0$. Furthermore, since T is abelian, the saturated fusion system $\mathcal{F}_0$ over T is completely determined by $Aut_{\mathcal{F}_0}(T)$, which is a finite subgroup of $Aut(T)$. Thus, $Aut_{\mathcal{F}_0}(T) \leq \mathbb{Z}/(p-1)$, but, if $Aut_{\mathcal{F}_0}(T)$ is not the trivial subgroup of $Aut(T)$, then $\mathcal{F}_0 \supsetneq O^{p'}(\mathcal{F}_0)$, and hence cannot be connected. This determines completely the list of connected saturated fusion systems over discrete p-toral groups of rank 1, when $p > 2$.

It is now an easy calculation to show that the obstructions to the existence and uniqueness of an associated centric linking system associated to $\mathcal{F}_0$ both vanish, since the orbit category $\mathcal{O}(\mathcal{F}_0^c)$ contains a single object with trivial automorphism group. Since S^1 already induces a centric linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$, it follows that this has to be the unique centric linking system associated to $\mathcal{F}_0$. By Theorem 1.4.5, it follows that $|\mathcal{L}|_p^\wedge \simeq (BS^1)_p^\wedge$.

Let now $\mathcal{G}$ be a p-local compact group of rank 1, with p odd. Again, T is $\mathcal{F}$-normal, and in particular strongly $\mathcal{F}$-closed. It is also clear that $\mathcal{F}$ contains $\mathcal{F}_0$ above as a subsystem, since $Aut_{\mathcal{F}_0}(T) = \{id\}$. It remains then to check the invariance of $\mathcal{F}_0$ in $\mathcal{F}$, and in particular we only have to check that condition (ii) in definition 3.1.9 holds, but this is obvious since for any two subgroups $P \leq Q$ of T, $Hom_{\mathcal{F}_0}(P, Q) = \{incl_P^Q\}$.

The last part of the statement follows easily since $T \leq Z(S)$, and hence the extension is admissible.

□

3.2.1 Some rank 1 discrete 2-toral groups

We are thus left to study the case $p = 2$. The strategy followed here is rather exhaustive. We will first study some properties of rank 1 discrete 2-toral groups and see which ones can appear as the connected component S_0 of a bigger discrete 2-toral group S with respect to a saturated fusion system $\mathcal{F}$. This will give rise to a short list of possibilities for S_0, and for each one we will check in later subsections that there only exists a unique saturated, connected fusion system $\mathcal{F}_0$ over S_0, and furthermore that for each such $\mathcal{F}_0$ there exists a unique associated centric linking system $\mathcal{L}_0$.

Let then S be a discrete 2-toral group of rank 1 with maximal torus T. In this section we want to study which discrete 2-toral groups can appear as the connected component of a bigger discrete 2-toral group with respect to some saturated fusion system.

In this case, Lemma 3.2.6 says that there only exists a nontrivial automorphism of $\mathbb{Z}/2^\infty$ of (finite) order 2, which corresponds to the automorphism

$$\begin{array}{rcl} T & \xrightarrow{\tau} & T \\ t & \longmapsto & t^{-1}. \end{array} \tag{3.2}$$

This is obviously an order 2 automorphism, and we can now deduce the following result.

Lemma 3.2.8. *Let S be a discrete 2-toral group of rank 1, and let $T = \cup T_n$ be the maximal 2-torus, where $T_n \cong \mathbb{Z}/2^n$ for all n. Then, either $T \cap Z(S) = T$ or $T \cap Z(S) = T_1$.*

Furthermore, if $T \cap Z(S) = T_1$ and $x \in S \setminus C_S(T)$, then all the elements in the coset xT are S-conjugate to x.

Proof. The condition $T \cap Z(S) = T$ means that $C_S(T) = S$. Thus, suppose that there exists some $x \in S \setminus C_S(T)$. Since S is locally finite, the automorphism c_x defined by $c_x(y) = xyx^{-1}$ has order a finite power of 2. In particular, when restricted to T, it has to be a nontrivial finite automorphism, and hence $(c_x)_{|T} = \tau$ in (3.2).

Now, there is only an order 2 element in T, namely t_1, and hence $\tau(t_1) = t_1$, that is, $T \cap Z(S) \geq T_1$. On the other hand, if $t \in T$ is of higher order, then clearly $\tau(t) \neq t$, and hence the first part of the statement is proved.

Let $x \in S$ be as above. Then, for each $t \in T$, we have $xtx^{-1} = t^{-1}$, and hence

$$t^{-1}xt = xt^2,$$

which proves the second part of the statement since all the elements in T are infinitely 2-divisible (that is, each element in T has at least one square root).

□

Lemma 3.2.9. *Let S be a rank 1 discrete 2-toral group, with maximal torus $T \cong \mathbb{Z}/2^\infty$, and let $\mathcal{F}$ be a saturated fusion system over S. Then, the connected component of S with respect to $\mathcal{F}$ is isomorphic to one (and only one) of the following discrete 2-toral groups:*

(i) T,

(ii) $D_{2^\infty} = \cup D_{2^n} = \langle x, T | x^2 = 1, xtx^{-1} = t^{-1}$ *for all* $t \in T\rangle = T \rtimes \langle x\rangle$, *or*

(iii) $Q_{2^\infty} = \cup Q_{2^n} = \langle y, T | y^4 = 1, y^2 = t_1, yty^{-1} = t^{-1}$ *for all* $t \in T\rangle$,

where t_1 is the single order 2 element in T.

Proof. Let S be such a discrete 2-toral group. By Lemma 3.1.7, if $S_0 \leq S$ is to be the connected component of S with respect to some fusion system over S, then S_0/T has to act faithfully on T, and hence by Lemma 3.2.6 S_0/T has to be isomorphic to a subgroup of $\mathbb{Z}/2$. If $S_0/T = \{1\}$, then $S_0 = T$.

Suppose otherwise that $S_0/T \cong \mathbb{Z}/2$. Then, by II.3.8 in [AM04], the isomorphism type of S_0 is determined by the group

$$H^2(\mathbb{Z}/2; T^\tau) \cong \mathbb{Z}/2,$$

where the superindex means that the coefficients are twisted by the automorphism τ (3.2). In particular this means that there are (up to isomorphism) two possible discrete 2-toral groups S_0 of rank 1 with the desired action on T and such that $S_0/T \cong \mathbb{Z}/2$.

Consider the following two families of 2-groups:

$$\begin{array}{c} \{D_{2^n} = \langle t_n, x_n | t_n^{2^{n-1}} = 1 = x_n^2, t_n x_n = x_n t_n^{2^{n-1}-1}\rangle\}_{n\geq 2}, \\ \{Q_{2^n} = \langle t_n, x_n | t_n^{2^{n-2}} = x_n^2, t_n^{2^{n-1}} = 1, t_n x_n = x_n t_n^{2^{n-1}-1}\rangle\}_{n\geq 3}, \end{array} \tag{3.3}$$

There are obvious injections $D_{2^n} \to D_{2^{n+1}}$ and $Q_{2^n} \to Q_{2^{n+1}}$ by sending (in both cases) t_n to t_{n+1}^2 and x_n to x_{n+1}, which allow us to consider

$$D_{2^\infty} = \bigcup_{n=3}^\infty D_{2^n} \quad \text{and} \quad Q_{2^\infty} = \bigcup_{n=2}^\infty Q_{2^n},$$

which are two nonisomorphic discrete 2-toral groups of rank 1 with the desired properties.

□

We have already studied the automorphism group T in Lemma 3.2.6, and now we seek a better understanding of the automorphism groups of D_{2^∞} and Q_{2^∞}.

Lemma 3.2.10. *Let S_0 be isomorphic to D_{2^∞} or Q_{2^∞}. Then, the group $Aut(S_0)$ fits in an extension*

$$Inn(S_0) \longrightarrow Aut(S_0) \longrightarrow \mathbb{Z}_2^\wedge.$$

Hence, $Inn(S_0)$ is a locally finite artinian subgroup of $Aut(S_0)$, and is maximal among all subgroups of $Aut(S_0)$ with these properties.

Proof. Let S_0 be as in the statement, with maximal torus T. Then, $S_0/T \cong \mathbb{Z}/2$ and by 2.8.7 in [Suz82] there is an exact sequence

$$0 \to H^1(\mathbb{Z}/2; T^\tau) \longrightarrow Aut(S_0)/Aut_T(S_0) \xrightarrow{\Phi} Aut(T) \times Aut(\mathbb{Z}/2),$$

where the superindex τ on T means that the coefficients are twisted by the automorphism τ (3.2) and $Aut_T(S_0) = \{c_t \in Aut(S_0) | t \in T\}$. Now, $Aut(\mathbb{Z}/2) = \{id\}$, and, by II.3.8 in [AM04], $H^1(\mathbb{Z}/2; T^\tau) = 0$. Now, we know that the factor $\mathbb{Z}/2 \leq Aut(T)$ is in the image of Φ, since it corresponds to the automorphism τ, that is, conjugation (in S_0) by an element $x \in S_0 \setminus T$. In fact, it is not difficult to see that any automorphism $f \in \mathbb{Z}_2^\wedge \leq Aut(T)$ can be extended to an automorphism of S_0, and hence Φ is epi and $Aut(S_0)$ fits in an extension

$$Aut_T(S_0) \longrightarrow Aut(S_0) \longrightarrow Aut(T).$$

Now, by definition, $Aut_T(S_0) \leq Inn(S_0)$, and in fact the pull-back of $Aut(S_0) \to Aut(T) \leftarrow \langle \tau \rangle$ is $Inn(S_0)$, and hence we deduce an extension

$$Inn(S_0) \longrightarrow Aut(S_0) \longrightarrow \mathbb{Z}_2^\wedge \cong Aut(T)/\langle \tau \rangle.$$

Since $\mathbb{Z}_2^\wedge$ has no finite subgroups, it follows that $Inn(S_0)$ is the greatest artinian locally finite subgroup of $Aut(S_0)$.

□

Lemma 3.2.11. *Let $S_0 \cong Q_{2^\infty}$, T be its maximal torus, and $T_1 \leq T$ be the subgroup of T of order 2. Then, $Z(S_0) = T_1$ and $S_0/T_1 \cong D_{2^\infty}$.*

Lemma 3.2.12. *Let $S_0 \cong D_{2^\infty}$, with maximal torus T. Then, $\forall x \in S_0 \setminus T$, $x^2 = 1$.*

Let $S_0 \cong Q_{2^\infty}$, with maximal torus T. Then, $\forall y \in S_0 \setminus T$, $y^4 = 1$ and $y^2 = t_1$.

Proof. Let first $S_0 \cong D_{2^\infty}$. In this case, $S_0 = T \rtimes \langle x \rangle$ for some $x \in S_0 \setminus T$. Furthermore, there is a section $s : S_0/T \to \langle x \rangle \leq S_0$ which is a group homomorphism, since $D_{2^{n+1}} = T_n \rtimes \langle x \rangle$ for all n. Since all the elements in the coset xT are S_0-conjugate as shown in Lemma 3.2.8, the first part of the statement follows.

Let now $S_0 \cong Q_{2^\infty}$. Then again, all the elements $y \in S_0 \setminus T$ are S_0-conjugate, by Lemma 3.2.8. Also, by definition of Q_{2^∞}, there is at least one $y \in S_0 \setminus T$ such that $y^2 = t_1 \in T$ and such that $y^4 = 1$, where t_1 is the order 2 element in T. Thus the second part of the statement is proved.

□

3.2.2 Connected saturated fusion systems over discrete 2-toral groups of rank 1

So far we know which discrete 2-toral groups of rank 1 can appear as the Sylow 2-subgroup of a connected saturated fusion system, we still do not know whether for each S_0 in 3.2.9 we can find connected saturated fusion systems over S_0 or not (note that if one drops the connectivity condition, then one can always consider the trivial saturated fusion system over S_0, which will not be connected in general). In this subsection we prove the following result.

Proposition 3.2.13. *Let S_0 be any of the discrete 2-toral groups in Lemma 3.2.9. Then, there is a connected saturated fusion system $\mathcal{F}_0$ over S_0 which is unique up to isomorphism.*

To prove this result we will study each case in 3.2.9 separately. The first one is actually quite easy to check.

Lemma 3.2.14. *Let $S_0 = T$. Then, there is a unique connected saturated fusion system $\mathcal{F}_0$ over S_0, namely, $\mathcal{F}_0 = \mathcal{F}_T(S^1)$.*

Proof. It is clear that $\mathcal{F}_T(S^1)$ is both connected and saturated over $S_0 = T$. Thus we have to check that this is the only choice.

Let $\mathcal{F}_0$ be a saturated fusion system over T. Since T is abelian, the whole fusion system is determined by $Aut_{\mathcal{F}_0}(T)$ by axiom (II), which has to be a finite subgroup of $Aut(T)$ and furthermore has to satisfy axiom (I) for fusion systems. Thus, $Aut_{\mathcal{F}_0}(T) = \{id\}$ and hence $\mathcal{F}_0 \cong \mathcal{F}_T(S^1)$.

□

The main part of this section is devoted to prove the existence of a unique connected saturated fusion system over D_{2^∞}, which in turn will imply the existence of a unique connected saturated fusion system over Q_{2^∞}. Actually, these fusion systems do not come out of nowhere since they are induced by the compact connected Lie groups $SO(3)$ (in the case $S_0 \cong D_{2^\infty}$) and S^3 (in the case $S_0 \cong Q_{2^\infty}$), but we will give a full descriptive construction of them, since we want to prove uniqueness too.

Lemma 3.2.15. *Let S_0 be a discrete 2-toral group with the isomorphism type of either D_{2^∞} or Q_{2^∞}, with maximal torus T, and suppose $\mathcal{F}$ is a saturated fusion system over S_0. Then,*

$$Aut_{\mathcal{F}}(T) = Aut_{S_0}(T) \cong \mathbb{Z}/2.$$

Proof. By hypothesis, there already exists some $x \in S_0$ such that $(c_x)_{|T} = \tau$ in (3.2), and hence $Aut_{\mathcal{F}}(T)$ cannot be the trivial group. Since then $Aut_{S_0}(T)$ is the greatest finite subgroup of $Aut(T)$ and we are assuming $\mathcal{F}$ to be saturated, it follows that $Aut_{\mathcal{F}}(T) = Aut_{S_0}(T)$.

□

Lemma 3.2.16. *Let $S_0 \cong D_{2^\infty}$. Then, there exists a unique connected saturated fusion system $\mathcal{F}_0$ over S_0, which is induced by the compact Lie group $SO(3)$.*

Proof. Start assuming the existence of such a fusion system $\mathcal{F}$. We will find some restrictions for it to actually exist, which will lead to both existence and uniqueness.

Recall that $S_0 = T \rtimes \langle x \rangle$, where x is an order 2 element acting nontrivially on T. Since we know that $Aut_{\mathcal{F}}(T) = \langle c_x \rangle$, we may apply the functor $(\_)^{\bullet}$ from 1.3.1 to $\mathcal{F}$. If $\mathcal{F}$ is saturated, then $Ob(\mathcal{F}^{\bullet})$ contains all the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S_0.

It is a rather easy exercise to obtain a list of representatives of the S_0-conjugacy classes of subgroups in $\mathcal{F}^{\bullet}$, and we outline the details below.

Set $T = \cup T_n$, where $T_n \cong \mathbb{Z}/2^n$ for each n. Let also $A \leq T$ be any subgroup, and set as usual

$$I(A) = \{t \in T | \omega(t) = t \text{ for all } \omega \in Aut_{\mathcal{F}_0}(T) \text{ such that } \omega_{|A} = id_A\}.$$

Then, since $Aut_{\mathcal{F}_0}(T) = \langle c_x \rangle$, we easily deduce that

- if $A \leq T_1$, then $I(A) = T_1$ and $I(A)_0 = \{1\}$, since the element x acts trivially on T_1;
- if $A \geq T_2$, then $I(A) = T = I(A)_0$, since c_x does not restrict to the identity on T_2.

Now, in the notation of the definition of the functor $(\_)^{\bullet}$, 1.3.1, $e = 1$, since $S_0/T \cong \mathbb{Z}/2$, and thus for any $P \leq S_0$, the following holds:

- if $P^{[1]} \leq T_1$, then $P^{\bullet} = P$;
- if $P^{[1]} \geq T_2$, then $P^{\bullet} = P \cdot T$.

Now it is easy to deduce the following list of representatives of the S_0-conjugacy classes of subgroups in $\mathcal{F}^{\bullet}$. We set for simplicity $C = \langle x \rangle$.

$$\{\{1\}, T_1, T_2, T, C, T_1 \times C, T_2 \rtimes C, T \rtimes C = S_0\}. \tag{3.4}$$

We can discard many of the elements in the list below by easy arguments:

- the subgroups $\{1\}$, T_1, T_2 and C cannot be $\mathcal{F}$-centric;
- the subgroup T cannot be $\mathcal{F}$-radical since its outer automorphism group (in $\mathcal{F}$) is a 2-groups by Lemma 3.2.15;
- the subgroup $T_2 \rtimes C$ cannot be $\mathcal{F}$-radical because, as we prove below, its outer automorphism group in $\mathcal{F}$ is a 2-group.

  Indeed, $T_2 \rtimes C \cong D_8$, whose automorphism group is isomorphic to D_8. Its inner automorphism group is

  $$Inn(T_2 \rtimes C) = (T_2 \rtimes C)/(T_1) \cong \mathbb{Z}/2 \times \mathbb{Z}/2,$$

  and hence $Out(T_2 \rtimes C) \cong Z/2$. Now, it is easy to check that the element $t_3 \in T_3$ induces by conjugation a nontrivial automorphism of $T_2 \rtimes C$ which is clearly not in $Inn(T_2 \rtimes C)$. Hence,

  $$\mathbb{Z}/2 \cong Out_{S_0}(T_2 \rtimes C) \leq Out_{\mathcal{F}}(T_2 \rtimes C) \leq Out(T_2 \rtimes C) \cong \mathbb{Z}/2,$$

  and $Out_{\mathcal{F}}(T_2 \rtimes C) \cong \mathbb{Z}/2$ is a 2-group.

This leaves us with only two S_0-conjugacy classes to study: $\langle S_0\rangle_{S_0} = \{S_0\}$ and $\langle T_1 \times C\rangle_{S_0}$. Set for simplicity $P = T_1 \times C$.

Now, from Lemma 3.2.10 we deduce that $Aut_{\mathcal{F}}(S_0) = Inn(S_0)$, so we really do not have choice here.

Consider now P. It is an elementary abelian 2-group of rank 2 (i.e., isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$), and in fact the set $\langle P\rangle_{S_0}$ already contains all the elementary abelian subgroups of S_0 of rank 2. Hence $\langle P\rangle_{\mathcal{F}} = \langle P\rangle_{S_0}$.

On the one hand, the automorphism group of P is then isomorphic to the symmetric group on three letters,

$$Aut(P) \cong \Sigma_3.$$

On the other hand, using the group relation $xtx = t^{-1}$ (for all $t \in T$), is follows easily that $C_{S_0}(P) = P$ and $N_{S_0}(P) = T_2 \rtimes C$. Since $\langle P\rangle_{\mathcal{F}} = \langle P\rangle_S$, it follows that P is fully $\mathcal{F}$-normalized, and hence

$$\mathbb{Z}/2 \cong (T_2 \rtimes C)/P = N_{S_0}(P)/C_{S_0}(P) \leq Aut_{\mathcal{F}}(P) \leq Aut(P) \cong \Sigma_3.$$

Thus, either $Aut_{\mathcal{F}}(P) \cong \mathbb{Z}/2$ or $Aut_{\mathcal{F}}(P) \cong \Sigma_3$. Note, however, that if $Aut_{\mathcal{F}}(P) \cong \mathbb{Z}/2$, then $\mathcal{F}$ is not connected.

On the other hand, if $\mathcal{F}_0$ is the fusion system over S_0 determined by $Aut_{\mathcal{F}_0}(S_0) = Inn(S_0)$ and $Aut_{\mathcal{F}_0}(P) \cong \Sigma_3$, then it is easy to check that $\mathcal{F}_0$ is both connected and saturated. The uniqueness now follows since we have already seen that, with the connectivity condition, there is no other choice to build up a saturated fusion system over S_0.

Now, the compact Lie group $SO(3)$ induces a connected saturated fusion system over S_0, and hence $\mathcal{F}_0 = \mathcal{F}_{S_0}(SO(3))$.

□

Finally, we deal with the case $S_0 \cong Q_{2^\infty}$. This case is also easy to handle, since it can be shown to completely depend on the case $S_0 \cong D_{2^\infty}$.

Lemma 3.2.17. *Let $S_0 \cong Q_{2^\infty}$. Then, there is a unique connected saturated fusion system $\mathcal{F}_0$ over S_0, which is induced by the compact Lie group S^3.*

Furthermore, if we set $T_1 \leq T$ for the order 2 subgroup, then $\mathcal{F}_0$ satisfies the following properties:

(i) T_1 is central in $\mathcal{F}_0$,

(ii) there is an isomorphism of saturated fusion systems $\mathcal{F}_0/T_1 \cong \mathcal{F}_{D_{2^\infty}}(SO(3))$,

(iii) the map $\theta : Ob(\mathcal{F}_0) \to Ob(\mathcal{F}_0/T_1)$ defined by $\theta(P) = P/T_1$ gives a bijective correspondence between the set of $\mathcal{F}_0$-centric subgroups and the set of $\mathcal{F}_0/T_1$-centric subgroups which restricts in turn to a correspondence between $\mathcal{F}_0$-centric $\mathcal{F}_0$-radical subgroups and $\mathcal{F}_0/T_1$-centric $\mathcal{F}_0/T_1$-radical subgroups.

Proof. The uniqueness of such $\mathcal{F}_0$ is proved by explicitly constructing it as we did in (the proof of) Lemma 3.2.16. To show property (i), recall that in Lemma 3.2.12 we have seen that all the elements $y \in S_0 \setminus T$ have order 4. Thus, S_0 contains a single order 2 element, the generator of T_1, which means that T_1 is strongly $\mathcal{F}_0$-closed for any saturated fusion system over $\mathcal{F}_0$. Since in addition $T_1 = Z(S_0)$, the condition of being $\mathcal{F}_0$-central is easily checked using axiom (II) on $\mathcal{F}_0$. Property (ii) is immediate, and property (iii) follows easily by inspection of the subgroups in Q_{2^∞} and D_{2^∞}.

□

We have then given a list of all possible connected saturated fusion systems of rank 1, and the next question to solve is whether, given a general saturated fusion system, $\mathcal{F}$, it does have a uniquely determined connected component $\mathcal{F}_0$, and if so, which is the relation between $\mathcal{F}_0$ and $\mathcal{F}$.

With all the information we have, we can state the following proposition.

Proposition 3.2.18. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S of rank* 1*. Then, there is a unique connected saturated fusion subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$, which is invariant in $\mathcal{F}$ in the sense of definition 3.1.9.*

Proof. For p odd, this has been shown in Proposition 3.2.7. For $p = 2$, it is not difficult to see that

($\star$) determining S_0, the connected component of S with respect to $\mathcal{F}$, also determines the fusion subsystem $\mathcal{F}_0$, in particular as a fusion subsystem of $\mathcal{F}$.

Indeed, let $\mathcal{F}$ be such a saturated fusion system over S. The statement in ($\star$) is clear if the connected component of S is T. Suppose now that the connected component of S is $S_0 = T \rtimes \langle x \rangle \cong D_{2^\infty}$, and let $P = T_1 \times \langle x \rangle \leq S_0$. It follows then that, for the element x to be in S_0, a morphism $f : \langle x \rangle \to T$ in $\mathcal{F}$ has to exists (since otherwise, the connected component of S with respect to $\mathcal{F}$ would be T). Using axiom (II) we can then see that $Aut_{\mathcal{F}}(P) \cong \Sigma_3$, and hence the full fusion system $\mathcal{F}_0$ describe in (the proof of) Lemma 3.2.16 is contained in $\mathcal{F}$. Since $\mathcal{F}_0$ has been proved to be the only connected saturated fusion system over S_0, ($\star$) is clear in this case. The statement ($\star$) when $S_0 \cong Q_{2^\infty}$ follows also by the same arguments.

The invariance issue can be easily checked by inspection of each case. We check it below in the case $S_0 \cong D_{2^\infty}$. Note that we only need to prove the invariance condition on the $\mathcal{F}_0$-centric $\mathcal{F}_0$-radical subgroups of S_0 since $\mathcal{F}_0$ is saturated. Set, as usual, $P = T_1 \times \langle x \rangle \leq S_0$. Then, by Lemma 3.2.16, S_0 and P are representatives of the S_0-conjugacy classes of $\mathcal{F}_0$-centric $\mathcal{F}_0$-radical subgroups. The invariance conditions is now easily seen to hold for both conjugacy classes since, again by Lemma 3.2.16,

$$Aut_{\mathcal{F}_0}(S_0) = Aut_{\mathcal{F}}(S_0) = Inn(S_0),$$

and

$$Aut_{\mathcal{F}_0}(P) = Aut_{\mathcal{F}}(P) = Aut(P).$$

□

3.2.3 Centric linking systems associated to connected saturated fusion systems in rank 1

In this section we study the existence and uniqueness of centric linking systems associated to the saturated fusion systems in Proposition 3.2.13. We study each case separately, and describe explicitly the (unique) linking system in the case $S_0 \cong D_{2^\infty}$.

The following proposition is a restatement of Theorem 3.2.1 for $p = 2$, and provides a list of all 2-connected compact groups of rank 1.

Proposition 3.2.19. *Let $S_1 = T$, $S_2 = D_{2^\infty}$ and $S_3 = Q_{2^\infty}$ be the discrete 2-toral groups listed in Lemma 3.2.9, and for $j = 1, 2, 3$ let $\mathcal{F}_j$ be the unique connected saturated fusion system over S_j from Proposition 3.2.13. Consider also the (connected) compact Lie groups $G_1 = S^1$, $G_2 = SO(3)$ and $G_3 = S^3$.*

Then, for each j, there is a unique centric linking system $\mathcal{L}_j$ associated to $\mathcal{F}_j$. Furthermore, this linking system satisfies $\mathcal{L}_j \cong \mathcal{L}_{S_j}(G_j)$, and

$$|\mathcal{L}_j|^\wedge_2 \simeq (BG_j)^\wedge_2.$$

Thus, for each j, the resulting 2-local compact group $\mathcal{G}_j$ is connected, and these are the only connected 2-local compact groups of rank 1.

Again, we prove this result by steps, considering each S_j separately. The case $j = 1$ is as usual easy to check.

Lemma 3.2.20. *Let $S_0 = T$, and let $\mathcal{F}_0 = \mathcal{F}_{S_0}(S_0)$. Then there is a unique centric linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$, whith classifying space*

$$|\mathcal{L}_0|^\wedge_2 \simeq (BS^1)^\wedge_2.$$

That is, $\mathcal{G}_0 = (S_0, \mathcal{F}_0, \mathcal{L}_0)$ is the 2-local compact group induced by S^1.

Proof. As happened in the proof of Proposition 3.2.7, showing that the obstructions to the existence and uniqueness of a centric linking system associated to $\mathcal{F}_0$ vanish is immediate, since $\mathcal{O}(\mathcal{F}_0^c)$ contains a single object with trivial automorphism group, and this centric linking system has to be the one induced by S^1.

Alternatively, since the only $\mathcal{F}_0$-centric subgroup is S_0 itself, a centric linking system associated to $\mathcal{F}_0$ will have a single object, namely S_0. Furthermore, $Aut_{\mathcal{L}_0}(S_0)$ is completely determined by the extension

$$S_0 \xrightarrow{\delta_{S_0}} Aut_{\mathcal{L}_0}(S_0) \longrightarrow Aut_{\mathcal{F}_0}(S_0) = \{id\}.$$

Thus, $Aut_{\mathcal{L}_0}(S_0) = S_0$, and there is no choice there. Finally, the classifying space of such a linking system with a single object has homotopy type

$$|\mathcal{L}_0|^\wedge_2 \simeq (BAut_{\mathcal{L}_0}(S_0))^\wedge_2 \simeq (BS^1)^\wedge_2.$$

□

Lemma 3.2.21. *Let $S_0 \cong D_{2^\infty}$, and let $\mathcal{F}_0$ be its corresponding connected saturated fusion system. Then, there is a unique centric linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$, whose classifying space satisfies*

$$|\mathcal{L}_0|^\wedge_2 \simeq (BSO(3))^\wedge_2.$$

That is, $\mathcal{G}_0 = (S_0, \mathcal{F}_0, \mathcal{L}_0)$ is the 2-local compact group induced by $SO(3)$.

Proof. Recall that the obstructions for the existence and uniqueness of associated linking systems lie in the groups $\varprojlim^j_{\mathcal{O}(\mathcal{F}_0)}(\mathcal{Z}_{\mathcal{F}_0})$, for $j = 2, 3$ respectively (by Proposition 1.5.6). In this particular case, using that for all j there is an isomorphism

$$\varprojlim{}^j_{\mathcal{O}(\mathcal{F}_0)}(\mathcal{Z}_{\mathcal{F}_0}) \cong \varprojlim{}^j_{\mathcal{O}(\mathcal{F}_0^{cr})}(\mathcal{Z}_{\mathcal{F}_0}),$$

it follows easily by Proposition B.1 [BM07] that these groups are trivial in particular for $j = 2, 3$, and hence there exists a unique centric linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$.

More concretely, in the terminology of Appendix B [BM07], the calculations that we want to perform are equivalent to the calculation of $\varprojlim^j_{\mathbb{I}(1)}(\mathbf{M})$, for $j = 2, 3$, where $G = \Sigma_3$, $H_1 = \mathbb{Z}/2 \leq G$, $\mathbb{I}(1)$ is the category with objects $\{0, 1\}$ and morphism sets

$$\begin{array}{ll} Aut_{\mathbb{I}(1)}(0) = G & Aut_{\mathbb{I}(1)}(1) = \{id\} \\ Hom_{\mathbb{I}(1)}(0, 1) = \emptyset & Hom_{\mathbb{I}(1)}(1, 0) = H_1 \setminus G, \end{array}$$

and $\mathbf{M}$ is the diagram

$$G \circlearrowright (\mathbb{Z}/2 \times \mathbb{Z}/2) \Lleftarrow \mathbb{Z}/2.$$

Proposition B.1 [BM07] is then easily seen to apply in this case.

Nevertheless, we will give an explicit description of such $\mathcal{L}_0$ and its construction, since it will help in later sections. Recall the notation in the proof of Lemma 3.2.16, and recall that $\mathcal{F}_0$ is determined by $Aut_{\mathcal{F}_0}(S_0) = Inn(S_0)$ and $Aut_{\mathcal{F}_0}(P) \cong \Sigma_3$.

Suppose $\mathcal{L}_0$ is a centric linking system associated to $\mathcal{F}_0$. Then, $\mathcal{L}_0$ is determined by the groups $Aut_{\mathcal{L}_0}(S_0)$ and $Aut_{\mathcal{L}_0}(P)$. This is, for instance, a consequence of Proposition 4.2.2, the version of Alperin's fusion theorem for linking systems developed as Proposition 4.8 [Jun09].

Consider first the subgroup S_0. Then, $Aut_{\mathcal{L}_0}(S_0)$ fits in an extension

$$S_0 \xrightarrow{\delta_{S_0}} Aut_{\mathcal{L}_0}(S_0) \longrightarrow Out_{\mathcal{F}_0}(S_0) = \{1\}.$$

Thus, $Aut_{\mathcal{L}_0}(S_0) \cong S_0$, and there is no choice here.

Consider now the subgroup P. The group $Aut_{\mathcal{L}_0}(P)$ fits in an extension

$$P = Z(P) \xrightarrow{\delta_P} Aut_{\mathcal{L}_0}(P) \longrightarrow Aut_{\mathcal{F}_0}(P) = Out_{\mathcal{F}_0}(P) \cong \Sigma_3.$$

Furthermore, $D_8 \cong N_{S_0}(P) \in Syl_p(Aut_{\mathcal{L}_0}(P))$. Thus, using for instance the list of groups of order 24, we see that the only choice is $Aut_{\mathcal{L}_0}(P) \cong \Sigma_4$, the symmetric group on 4 letters.

Now it is easy to check that the category $\mathcal{L}_0$ with object set $Ob(\mathcal{L}_0) = \{R \in \mathcal{F}_0^c\}$ and morphisms spanned by $Aut_{\mathcal{L}_0}(S_0) = S_0$ and $Aut_{\mathcal{L}_0}(P) \cong \Sigma_4$ satisfies the axioms of a linking system.

Finally, it remains to check that the compact Lie group $SO(3)$ induces the 2-local compact group $(S_0, \mathcal{F}_0, \mathcal{L}_0)$, which is an easy exercise.

□

Lemma 3.2.22. *Let $S_0 \cong Q_{2^\infty}$, and let $\mathcal{F}_0$ be its corresponding connected saturated fusion system. Then, there exists a unique centric linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$, with classifying space*

$$|\mathcal{L}_0|^\wedge_2 \simeq (BS^3)^\wedge_2.$$

That is, $\mathcal{G}_0 = (S_0, \mathcal{F}_0, \mathcal{L}_0)$ is the 2-local compact group induced by S^3.

Proof. In this case, it would be easy to apply again Proposition B.1 [BM07] to show that the obstructions to the existence and uniqueness of $\mathcal{L}_0$ vanish, as we did in the proof for Lemma 3.2.21.

We give also an alternative proof. By Lemma 3.2.17, there is a correspondence between the sets of $\mathcal{F}_0$-centric subgroups and $\mathcal{F}_0/T_1$-centric subgroups, which in fact restricts to a correspondence between $\mathcal{F}_0$-centric $\mathcal{F}_0$-radical subgroups and $\mathcal{F}_0/T_1$-centric $\mathcal{F}_0/T_1$-radical subgroups.

Also by Lemma 3.2.17, any linking system $\mathcal{L}$ associated to $\mathcal{F}_0$ has to be an extension of the linking system induced by $SO(3)$, $\mathcal{L}_{D_{2^\infty}}(SO(3))$, by T_1, and these extensions are determined by the group

$$H^2((BSO(3))^\wedge_2; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Thus, there exists (up to isomorphism) only one nontrivial extension of $\mathcal{L}_{D_{2^\infty}}(SO(3))$ by $\mathbb{Z}/2$, and the group S^3 induces one. This proves the statement.

□

This finishes the proofs of Theorems 3.2.1 and 3.2.3, since we have completed the list of connected 2-local compact groups, and we also have checked that, given a saturated fusion system $\mathcal{F}$ over a discrete 2-toral group S, the connected component of S with respect to $\mathcal{F}$, S_0, is completely determined by $\mathcal{F}$. The proof for Corollary 3.2.4 holds easily now.

Lemma 3.2.23. *Let $\mathcal{G}$ be a 2-local compact group whose connected component is the 2-local compact group induced by S^3. Let also $T_1 \leq T$ be the order 2 subgroup of T. Then, T_1 is a $\mathcal{F}$-central subgroup, and $\mathcal{G}/T_1$ is a 2-local compact group whose connected component is the 2-local compact group induced by $SO(3)$.*

In particular, $\mathcal{G}$ is an admissible extension of $\mathcal{G}/T_1$ by T_1.

Proof. It follows clearly from Lemma 3.2.11. The extension is admissible because T_1 is central in $\mathcal{G}$.

□

The following notion is then well-defined for rank 1 p-local compact groups.

Definition 3.2.24. *Let $\mathcal{G}$ be a p-local compact group of rank 1, and let $\mathcal{G}_0$ be the unique rank1 connected p-local compact group determined by $\mathcal{G}$ and whose Sylow subgroup S_0 is strongly $\mathcal{F}$-closed. We call then $\mathcal{G}_0$ the* ***connected component*** *of $\mathcal{G}$.*

3.2.4 Inclusions of connected components

We have then assigned to each p-local compact group $\mathcal{G}$ a connected p-local compact group $\mathcal{G}_0$, uniquely determined by $\mathcal{G}$, and now we define inclusions (of p-local compact groups) of $\mathcal{G}_0$ into $\mathcal{G}$. More concretely, we will define a functor $\iota_0 : \mathcal{L}_0^{cr} \to \mathcal{L}$ which will induce the inclusion map $B\mathcal{G}_0 \to B\mathcal{G}$. As usual in this section, we will deal with each case separately (depending on the connected component of $\mathcal{G}$).

Lemma 3.2.25. *Let $\mathcal{G}$ be a rank 1 p-local compact group, and let $\mathcal{G}_0$ be its connected component. Then, for each $P_0 \in \mathcal{L}_0$, P_0 is fully $\mathcal{F}$-centralized, and hence $P = P_0 \cdot C_S(P_0)$ is $\mathcal{F}$-centric.*

Proof. The first part of the statement follows since $\langle P_0 \rangle_{\mathcal{F}} = \langle P_0 \rangle_S$ in all cases, and the second part follows by Proposition 1.2.6.

□

We first consider p-local compact groups whose connected component is the p-local compact group induced by S^1. This is, as usual, the easiest case to deal with. Let $\mathcal{G}_0$ be the connected component of $\mathcal{G}$. Then, $\mathcal{L}_0$ has a single object, T, with automorphism group

$$Aut_{\mathcal{L}_0}(T) = \delta_T(T) \cong T.$$

In particular, $\mathcal{L}_0^{cr} = \mathcal{L}_0$. One can then define the functor ι_0 as follows.

$$(3.5)\qquad \begin{array}{rcl} \mathcal{L}_0 & \xrightarrow{\quad\iota_0\quad} & \mathcal{L} \\ T & \longmapsto & \iota_0(T) = T \cdot C_S(T) \\ Aut_{\mathcal{L}_0}(T) & \longmapsto & \iota_0(Aut_{\mathcal{L}_0}(T)) = \iota_S(T) \leq \iota_S(S) \leq Aut_{\mathcal{L}}(T). \end{array}$$

Proposition 3.2.26. *Let $\mathcal{G}$ be a rank 1 p-local compact group whose connected component $\mathcal{G}_0$ is the p-local compact group induced by S^1. Let also $\pi_0(\mathcal{G})$ be its p-local finite group of components.*

Then, T is normal in $\mathcal{F}$, $\mathcal{G}$ is an extension (in the sense of A.5.1) of $\mathcal{G}/T$ by T, and there is a fibration

$$|\mathcal{L}_0| \longrightarrow |\mathcal{L}| \longrightarrow |\pi_0(\mathcal{G})|,$$

where the left arrow is (homotopically equivalent to) the map induced by the functor ι_0, and the right arrow is the map induced by the functor $\mathcal{L} \to \mathcal{L}/S_0$ (between transporter systems).

Proof. In this case, either using Lemma 3.1.8 or Lemma B.2.5, one can prove that T is normal in $\mathcal{F}$. Hence by Proposition A.3.3, there is indeed a fibration like in the statement. The rest of the statement is also clear by the extension theory developed in appendix §A.

□

We now turn to the most difficult case to deal with. That is, that of 2-local compact groups whose connected component is the 2-local compact group induced by $SO(3)$. As in the previous discussion, we start by defining and describing the functor $\iota_0 : \mathcal{L}_0^{cr} \to \mathcal{L}$.

We recall some notation from the previous sections. The torus T can be seen as the union of all cyclic groups of order 2^n, $T_n = \langle t_n \rangle$. Also, we fix an order 2 element $x \in S_0 \leq S$ such that $xtx^{-1} = t^{-1}$ for all $t \in T$. Recall also that in this case, $Ob(\mathcal{L}_0^{cr})$ contains only two S_0-conjugacy classes (which in fact correspond to S-conjugacy classes):

$$\langle S_0 \rangle_{\mathcal{F}_0} = \langle S_0 \rangle_{S_0} = \langle S_0 \rangle_S = \langle S_0 \rangle_{\mathcal{F}} = \{S_0\}$$
$$\langle P_0 \rangle_{\mathcal{F}_0} = \langle P_0 \rangle_{S_0} = \langle P_0 \rangle_S = \langle P_0 \rangle_{\mathcal{F}},$$

where $P_0 = T_1 \times \langle x \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Fix then the representatives S_0 and P_0 for each conjugacy class above. Since for each representative its conjugacy class both in $\mathcal{F}$ and in $\mathcal{F}_0$ is completely determined by S_0, it will be enough to define ι_0 on these two objects. We already have candidates for $\iota_0(S_0)$ and $\iota_0(P_0)$ by Lemma 3.2.25. Namely, $S' = S_0 \cdot C_S(S_0)$ and $P = P_0 \cdot C_S(P_0)$ respectively.

Lemma 3.2.27. *The subgroup $S' \leq S$ is, in fact, S itself.*

Proof. Recall that $Aut_{\mathcal{F}}(S_0) = Aut_S(S_0) = Inn(S_0)$. Hence, since S_0 is clearly fully $\mathcal{F}$-normalized (and $\mathcal{F}$ is saturated by hypothesis), there is a commutative diagram

$$\begin{tikzcd}
Z(S_0) \arrow[r] \arrow[d] & C_S(S_0) \arrow[r] \arrow[d] & C_S(S_0)/Z(S_0) \arrow[d, equal] \\
S_0 \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & S/S_0 \\
Inn(S_0) \arrow[r, equal] & Aut_S(S_0). &
\end{tikzcd}$$

The statement clearly follows from this diagram.

□

Thus, the functor ι_0 could be already defined on objects by $\iota_0(S_0) = S$ and $\iota(P_0) = P$. However, it is not clear *a priori* that such an injective functor can be defined on the level of morphisms. In fact, since $Aut_{\mathcal{L}_0}(S_0) = \delta_{S_0}(S_0)$, there is an obvious inclusion of $Aut_{\mathcal{L}_0}(S_0)$ into $Aut_{\mathcal{L}}(S)$, induced by the inclusion $S_0 \leq S$:

$$\delta_{S_0}(S_0) = Aut_{\mathcal{L}_0}(S_0) \hookrightarrow \delta_S(S_0) \leq \delta_S(S) \leq Aut_{\mathcal{L}}(S), \tag{3.6}$$

and the problem lies in showing a similar statement for P_0. Recall that there are isomorphisms $Aut_{\mathcal{F}_0}(P_0) = Aut_{\mathcal{F}}(P_0) \cong \Sigma_3$ and $Aut_{\mathcal{L}_0}(P_0) \cong \Sigma_4$.

Lemma 3.2.28. *There is a subgroup $B \leq Aut_{\mathcal{L}}(P)$ which is isomorphic to the automorhism group $Aut_{\mathcal{L}_0}(P_0)$, and which satisfies the following properties:*

(i) for each $\varphi \in B$, the morphism $\rho(\varphi) \in Mor(\mathcal{F})$ restricts to an automorphism of P_0, and

(ii) the subgroup B contains the subgroup $\delta_P(N_{S_0}(P_0))$.

Proof. We first check that there is an inclusion $Aut_{\mathcal{F}_0}(P_0) = Aut_{\mathcal{F}}(P_0) \leq Aut_{\mathcal{F}}(P)$. Note that, via the axiom (II) for saturated fusion systems, we can in fact embed $Aut_{\mathcal{F}_0}(P_0)$ into $Aut_{\mathcal{F}}(P)$, but only as sets.

By 2.8.7 in [Suz82], there is an exact sequence

$$0 \to H^1(P/P_0; P_0) \longrightarrow Aut_{\mathcal{F}}(P)/Aut_{P_0}(P) \xrightarrow{\Phi_P} Aut_{\mathcal{F}}(P_0) \times Aut(P/P_0), \tag{3.7}$$

where, in fact, $Aut_{P_0}(P) = \{c_y \in Aut_{\mathcal{F}}(P) | y \in P_0\} = \{id\}$ since $P = P_0 \cdot C_S(P_0)$ and P_0 is abelian. Note also that by definition of P, the natural action (by conjugation) of P/P_0 on P_0 is trivial.

Let then $f_0 \in Aut_{\mathcal{F}_0}(P_0)$ be an order 3 automorphism, and let $f_1 \in Aut_{\mathcal{F}}(P)$ be an extension of f_0 (in the sense of axiom (II)). Then,

$$\Phi_P(f_1) = (f_0, \bar{f})$$

for some $\bar{f} \in Aut(P/P_0)$. Let also $\omega_2 = c_{t_2}$, and note that $\Phi_P(\omega_2) = (\omega_2, id)$, since $P/P_0 \leq S/S_0$ and S/S_0 acts trivially on S_0. Thus, $f_2 = \omega_2 \circ f_1 \circ \omega_2^{-1}$ satisfies

$$\Phi_P(f_2) = (\omega_2 \circ f_0 \circ \omega_2^{-1}, \bar{f}) = (f_0^{-1}, \bar{f}),$$

and hence $f = f_2 \circ f_1^{-1} = [\omega_2, f_1]$ (the commutator of ω_2 and f_1) is such that

$$\Phi_P(f) = (f_0^{-2}, id) = (f_0, id).$$

Also, $\Phi_P(f^3) = (id, id)$, that is, $f^3 \in H^1(P/P_0; P_0)$, which is clearly a (finite) 2-group. It follows that the subgroup $\langle f \rangle \leq Aut_{\mathcal{F}}(P)$ has a subgroup of order 3. We can assume that f itself has order 3 without loss of generality.

We now check that the subgroup $A = \langle \omega_2, f \rangle$ is isomorphic to $Aut_{\mathcal{F}_0}(P_0)$. Mainly, we have to show that

$$\omega_2 \circ f \circ \omega_2^{-1} = f^{-1}.$$

By definition, $f = [\omega_2, f_1] = \omega_2 f_1 \omega_2^{-1} f_1^{-1}$. Thus, by replacing f by this expression in $\omega_2 \circ f \circ \omega_2^{-1}$, (and using that ω_2 has order 2), we obtain

$$\omega_2 \circ f \circ \omega_2^{-1} = \omega_2(\omega_2 f_1 \omega_2^{-1} f_1^{-1})\omega_2^{-1} = f_1 \omega_2 f_1^{-1} \omega_2^{-1} = f_1 f_2^{-1} = f^{-1}$$

and A has the desired isomorphism type. Note also that all the elements in A are extensions in the sense of axiom (II) of the automorphisms in $Aut_{\mathcal{F}_0}(P_0)$.

Next, we show the existence of the subgroup $B \leq Aut_{\mathcal{L}}(P)$. This subgroup will be in fact a lifting in $\mathcal{L}$ of the subgroup A above, and hence property (i) will follow. Let $\widetilde{A} \leq Aut_{\mathcal{L}}(A)$ be the pull-back of $Aut_{\mathcal{L}}(P) \xrightarrow{\rho} Aut_{\mathcal{F}}(P) \leftarrow A$. There is a commutative diagram

$$\begin{tikzcd}
Z(P) \arrow[r, "\delta_P"] \arrow[d, equal] & Aut_{\mathcal{L}}(P) \arrow[r, "\rho"] & Aut_{\mathcal{F}}(P) \\
Z(P) \arrow[r, "\delta_P"'] & \widetilde{A} \arrow[r] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

and it follows from the Sylow theorems that $\widetilde{A}$ contains an order 3 element, namely φ, which is a lifting in $\mathcal{L}$ of the morphism $f \in A$.

Let also $B = \langle \delta_P(N_{S_0}(P_0)), \varphi \rangle \leq \widetilde{A}$. We claim that B is isomorphic to the group $Aut_{\mathcal{L}_0}(P_0)$. If we check that B has order 24, then, the fact that there is an extension $\delta_P(P_0) \to B \to A$, together with the list of all groups of order 24 will imply that B has the desired isomorphism type. Note that $\delta_P(P_0) \triangleleft Aut_{\mathcal{L}}(P)$, since S_0 is strongly $\mathcal{F}$-closed (use axiom (C) to check it).

Thus, we only have to check what happens with the composition $\delta(t_2) \circ \varphi \circ \delta(t_2^{-1})$. In particular, we want to see that this conjugation is a composition of a morphism in $\delta_P(N_{S_0}(P_0))$ followed by a power of φ. Since $\rho(\delta(t_2)\varphi\delta(t_2^{-1})) = \omega_2 f \omega_2^{-1} = f^{-1}$, it follows by axiom (A) that

$$\delta(t_2) \circ \varphi \circ \delta(t_2^{-1}) = \delta(z) \circ \varphi^{-1},$$

for some $z \in Z(P)$. Furtheremore, since $\varphi^3 = \delta(1)$, the identity element in $Aut_{\mathcal{L}}(P)$, one has (using axiom (C)):

$$\delta(1) = \delta(zf(z)f^2(z)).$$

This is the same as saying that $z \in C^*(\langle f \rangle; Z(P))$ is a cocicle (see II.3.8 in [AM04]). However, since f has order 3 and $Z(P)$ is a finite 2-group, it follows that the reduced cohomology $\widetilde{H}^*(\langle f \rangle; Z(P)) = 0$, and hence z is also a coboundary. That is, there is some $y \in Z(P)$ such that

$$z = y^{-1} f(y).$$

It follows now that the projection of z in P/P_0 is the trivial element, since, by assumption, f induces the identity on P/P_0. Hence, $z \in P_0$, and B has the desired isomorphism type. Properties (i) and (ii) in the statement now hold by construction.

$\square$

In fact, Lemma 3.2.28 is a particular case of the following result.

Lemma 3.2.29. *Let $R \leq S$ be any subgroup, and let $R_0 = R \cap S_0$ as usual. Let also $\Phi_R : Aut_{\mathcal{F}}(R) \to Aut_{\mathcal{F}}(R_0) \times Aut(R/R_0)$ the natural map which sends an automorphism f to the restriction to R_0 on the first factor and to the induced automorphism on R/R_0 on the second factor.*

Then, $Im(\Phi_R)$ is a direct product of a subgroup H of $Aut_{\mathcal{F}}(R_0)$ by a subgroup of $Aut(R/R_0)$, and there is a section $s : H \hookrightarrow Aut_{\mathcal{F}}(R)$.

Proof. If R_0 is not in the $\mathcal{F}$-conjugacy class of $P_0 = T_1 \times \langle x \rangle$, then $Aut_{\mathcal{F}}(R_0) = Aut_{S_0}(R_0)$, and the statement is clear since $N_{S_0}(R_0) \leq N_S(R)$. If R_0 is in the $\mathcal{F}$-conjugacy class of P_0, then the same arguments used to prove Lemma 3.2.28 above apply here.

$\square$

Once the subgroup $B \leq Aut_{\mathcal{L}}(P)$ is fixed, we can identify $Aut_{\mathcal{L}_0}(P_0)$ with B in a way that the "distinguished" Sylow 2-subgroup $\delta_{P_0}(N_{S_0}(P_0)) \leq Aut_{\mathcal{L}_0}(P_0)$ is sent to the subgroup $\delta_P(N_{S_0}(P_0)) \leq B$. The functor $\iota_0 : \mathcal{L}_0^{cr} \to \mathcal{L}$ is then spanned by the following

(3.8)
$$\begin{array}{rcl}
\mathcal{L}_0^{cr} & \xrightarrow{\quad\iota_0\quad} & \mathcal{L} \\
S_0 & \longmapsto & \iota_0(S_0) = S \\
P_0 & \longmapsto & \iota_0(P_0) = P \\
Aut_{\mathcal{L}_0}(S_0) & \longmapsto & \iota_0(Aut_{\mathcal{L}_0}(S_0)) = \delta_S(S_0) \\
Aut_{\mathcal{L}_0}(P_0) & \longmapsto & \iota_0(Aut_{\mathcal{L}_0}(P_0)) = B,
\end{array}$$

and it follows that it is a functor because the conjugacy classes of S_0 and P_0 in $\mathcal{F}$ satisfy $\langle S_0 \rangle_{\mathcal{F}} = \{S_0\}$ and $\langle P_0 \rangle_{\mathcal{F}} = \langle P_0 \rangle_{S_0}$.

We cannot prove such a strong statement as 3.2.26 in this case, since it is clear that S_0 is not an $\mathcal{F}$-normal subgroup of S. It is in fact an a future project to study further consequences of the existence of this inclusion functor, and to see if one can define some kind of "action" of $\mathcal{L}$ on $\mathcal{L}_0$ through ι_0. It is not clear at all to what extend we have the right to refer to $\mathcal{L}_0$ as the connected component of $\mathcal{L}$.

Proposition 3.2.30. *Let $\mathcal{G}$ be a rank 1 2-local compact group whose connected component is the 2-local compact group induced by $SO(3)$. Let also $\pi_0(\mathcal{G})$ be the corresponding 2-local finite group of components.*

Then, the composition $|\mathcal{L}_0| \simeq |\mathcal{L}_0^{cr}| \longrightarrow |\mathcal{L}| \longrightarrow |\pi_0(\mathcal{G})|$ (where the arrows are induced by ι_0 and the projection $\mathcal{L} \to \mathcal{L}/S_0$ respect.) is nullhomotopic.

Proof. It follows by construction of ι_0 and by definition of $\pi_0(\mathcal{G}) = \mathcal{G}/S_0$.

$\square$

Let F be the fiber of the projection $|\mathcal{L}| \to |\pi_0(\mathcal{L})|$. Then, the above result implies the existence of the dotted arrow in the diagram below.

$$\begin{tikzcd} & {|\mathcal{L}_0|} \arrow[dl, dotted] \arrow[d] \arrow[dr, "\simeq *"] & \\ F \arrow[r] & {|\mathcal{L}|} \arrow[r] & {|\pi_0(\mathcal{L})|.} \end{tikzcd}$$

However, it does not seem in general that the above map $|\mathcal{L}_0| \to F$ will be a (mod p) homotopy equivalence.

Even if $|\mathcal{L}_0|$ is far from being the fiber of the projection $|\mathcal{L}| \to |\pi_0(\mathcal{L})|$, having an inclusion functor as ι_0 is an important property of $\mathcal{L}_0$. However, there are still some details to check.

Let $\mathcal{G}$ be a rank 1 p-local compact group, and let $\mathcal{G}_0$ be its connected component, together with the inclusion functor $\iota_0 : \mathcal{L}_0^{cr} \to \mathcal{L}$. It is easy to see that the following square is not commutative in general:

$$\begin{tikzcd} \mathcal{L}_0^{cr} \arrow[r, "\iota_0"] \arrow[d, "\rho_0"'] \arrow[dr, phantom, "\mathrm{X}"] & \mathcal{L} \arrow[d, "\rho"] \\ \mathcal{F}_0^{cr} \arrow[r, "incl."'] & \mathcal{F}. \end{tikzcd}$$

Instead, let $i_0 : \mathcal{F}_0^{cr} \to \mathcal{F}$ be the functor induced by projecting ι_0 on $\mathcal{F}$ (through the projection functor ρ). More explicitly, on objects, i_0 is defined by sending an object $Q_0 \in Ob(\mathcal{F}_0^{cr})$ to $i_0(Q_0) = \iota_0(Q_0) = Q_0 \cdot C_S(Q_0)$. On morphisms, let $f \in Hom_{\mathcal{F}_0^{cr}}(Q_0, Q_0')$ and let $\varphi \in Mor_{\mathcal{L}_0^{cr}}(Q_0, Q_0')$ be a lifting of f, and define $i_0(f) = \rho(\iota_0)(\varphi)$.

Corollary 3.2.31. *The following hold for the functor i_0 defined above:*

(i) For each $f \in Mor(\mathcal{F}_0^{cr})$, $i_0(f)$ is an extension of f in the sense of axiom (II).

(ii) There is a natural transformation between the functors incl and i_0.

In particular, the square above commutes up to homotopy after realization.

Proof. Point (i) is immediate.

To prove point (ii), define $\theta : incl. \to i_0$ by

$$\theta(Q_0) = [Q_0 = incl(Q_0) \hookrightarrow i_0(Q_0) = Q_0 \cdot C_S(Q_0)]$$

on objects and by

$$\begin{tikzcd} Q_0 \arrow[r, "incl"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\theta(f_0)"] & Q_0 \cdot C_S(Q_0) \arrow[d, "f"] \\ Q_0' \arrow[r, "incl"'] & Q_0' \cdot C_S(Q_0') \end{tikzcd}$$

for each morphism $f_0 \in Mor(\mathcal{F}_0^{cr})$, where $f = \rho(\iota(\varphi_0))$, for some lifting φ_0 of f_0 in $\mathcal{L}_0^{cr}$.

To prove that θ is a natural transformation we have then to check that the above square is commutative. Now, by point (i), $f = i_0(f_0)$ is an extension of f_0 in the sense of axiom (II), and hence the square is indeed commutative. Thus, after realization, θ induces a homotopy equivalence.

□

The following question arises naturally after the above discussion. Several choices are made in order to define the functor ι_0 (3.8), mainly regarding the subgroup $B \leq Aut_{\mathcal{F}}(P_0 \cdot C_S(P_0))$, and different sets of choices may give rise to different functors ι_0', which in turn give rise to homotopy commutative squares

$$\begin{tikzcd} |\mathcal{L}_0^{cr}| \arrow[r, "|\iota_0'|"] \arrow[d, "|\rho_0|"'] \arrow[dr, phantom, "h"] & |\mathcal{L}| \arrow[d, "|\rho|"] \\ |\mathcal{F}_0^{cr}| \arrow[r, "|incl.|"'] & |\mathcal{F}^c|, \end{tikzcd}$$

where all arrows are independent of the choices except for (maybe) $|\iota_0'|$. What is then the relation (if any) among all possible inclusion functors ι_0? We suspect that this is related with some notion (still to be made clear) of action of $\mathcal{L}$ on $\mathcal{L}_0$, but it seems that developing this in detail and with the enough generality would take too long in time and space, and hence we leave it as an open question.

Finally, let $\mathcal{G}$ be a 2-local compact group whose connected component $\mathcal{G}_0$ is induced by S^3. Then, by Corollary 3.2.4, $\mathcal{G}$ is an extension of a 2-local compact group $\mathcal{G}'$ by $\mathbb{Z}/2$, where the connected component of $\mathcal{G}'$, $\mathcal{G}_0'$ is the 2-local compact group induced by $SO(3)$. Furthermore, the projection $\mathcal{G} \to \mathcal{G}'$ restricts to the projection $\mathcal{G}_0 \to \mathcal{G}_0'$, which in turn induces a bijective correspondence between $\mathcal{F}_0$-centric $\mathcal{F}_0$-radical subgroups and $\mathcal{F}_0'$-centric $\mathcal{F}_0'$-radical subgroups.

Then, we can define a functor $\iota : \mathcal{L}_0^{cr} \to \mathcal{L}$ by pulling back the functor $\iota_0' : \mathcal{L}_0' \to \mathcal{L}'$. Thus, Proposition 3.2.30 and Corollary 3.2.31 can be extended as below, and the corresponding proofs are given by the commutative diagram

$$\begin{tikzcd} B\mathbb{Z}/2 \arrow[r] \arrow[d, equal] & |\mathcal{L}_0| \arrow[r] \arrow[d] & |\mathcal{L}_0'| \arrow[d] \\ B\mathbb{Z}/2 \arrow[r] & |\mathcal{L}| \arrow[r] \arrow[d] & |\mathcal{L}'| \arrow[d] \\ & |\pi_0(\mathcal{L})| \arrow[r, equal] & |\pi_0(\mathcal{L}')|. \end{tikzcd}$$

Proposition 3.2.32. *Let $\mathcal{G}$ be a rank 1 2-local compact group whose connected component is the 2-local compact group induced by S^3. Let also $\pi_0(\mathcal{G})$ be the corresponding 2-local finite group of components.*

Then, the composition $|\mathcal{L}_0| \simeq |\mathcal{L}_0^{cr}| \longrightarrow |\mathcal{L}| \longrightarrow |\pi_0(\mathcal{G})|$ (where the arrows are induced by ι_0 and the projection $\mathcal{L} \to \mathcal{L}/S_0$ respect.) is nullhomotopic.

Corollary 3.2.33. *The following hold for the functor $i_0 : \mathcal{L}_0^{cr} \to \mathcal{L}$:*

(i) For each $f \in Mor(\mathcal{F}_0^{cr})$, $i_0(f)$ is an extension of f in the sense of axiom (II).

(ii) The induced maps $|incl.|$ and $|i_0|$ are homotopy equivalent.

3.3 Connectivity on p-local compact groups of general rank

It is clear that the exhaustive description of rank 1 p-local compact groups to study connectivity is out of range if we think of doing the same in the general case. We want, however, to discuss superficially the issue of connectivity in the general rank case. In this sense, we will describe some rather natural construction that could lead to the existence of connected components for all p-local compact groups. Our construction is, however, quite primitive.

We first show some easy proposition about a class of p-local compact groups which admit connected components. We are thinking of constrained p-local compact groups.

Proposition 3.3.1. *Let $\mathcal{F}$ be a constrained saturated fusion system over a discrete p-toral group S, with maximal torus T of rank r, and let $\mathcal{L}$ be the unique centric linking system associated to $\mathcal{F}$, and $\mathcal{G}$ the corresponding p-local compact group.*

Then, the p-local compact group induced by the connected compact Lie group $(S^1)^r$, $\mathcal{G}_0 = (T, \mathcal{F}_0, \mathcal{L}_0)$, is the (unique) connected component of $\mathcal{G}$. Furthermore, the following holds:

(i) T is normal in $\mathcal{F}$,

(ii) $\mathcal{F}_T((S^1)^r)$ is invariant in $\mathcal{F}$ in the sense of [Lin06], and

(iii) if $P \leq S$ is $\mathcal{F}$-centric and $\mathcal{F}$-normal, then $T \leq P$ and there is a commutative diagram

$$\begin{tikzcd}
BT \arrow[r] \arrow[d, equal] & BP \arrow[r] \arrow[d] & BP/T \arrow[d] \\
BT \arrow[r] & B\mathcal{G} \arrow[r] \arrow[d] & B\mathcal{G}/T \arrow[d] \\
 & B\mathcal{G}/P \arrow[r, equal] & B\mathcal{G}/P,
\end{tikzcd}$$

where $\mathcal{G}/T$, $\mathcal{G}/P$ are p-local finite groups.

In particular, a functor $\iota_0 : \mathcal{L}_T((S^1)^r) \to \mathcal{L}$ can be defined by

$$\begin{array}{ccc}
\mathcal{L}_0 & \xrightarrow{\quad\iota_0\quad} & \mathcal{L} \\
T & \longmapsto & \iota_0(T) = R \overset{def}{=} T \cdot C_S(T) \\
Aut_{\mathcal{L}_0}(T) & \longmapsto & \iota_0(Aut_{\mathcal{L}_0}(T)) = \delta_R(T) \leq \delta_R(R) \leq Aut_{\mathcal{L}}(R),
\end{array} \tag{3.9}$$

such that, if $\pi_0(\mathcal{G})$ is the p-local finite group of components, then there is a fibration

$$BT \simeq |\mathcal{L}_0| \xrightarrow{|\iota_0|} |\mathcal{L}| \longrightarrow |\pi_0(\mathcal{L})|.$$

Proof. Let $P \leq S$ be $\mathcal{F}$-centric and $\mathcal{F}$-normal. Then, $rk(S) = rk(NS(P)) = rk(P)$, since $Out_S(P) = N_S(P)/P$ is a finite p-group. Thus, $T \leq P$, and since P is $\mathcal{F}$-normal and T is the maximal infinitely p-divisible subgroup of S, it follows that T is also normal in $\mathcal{F}$, and in particular it is strongly $\mathcal{F}$-closed.

Hence, the connected component of S with respect to $\mathcal{F}$ is T, and it is clear that there is a unique connected saturated fusion system over T, $\mathcal{F}_0 = \mathcal{F}_T((S^1)^r)$, as well as a unique associated centric liniking system $\mathcal{L}_0 = \mathcal{L}_T((S^1)^r)$ on $\mathcal{F}_0$.

The invariance of $\mathcal{F}_0$ in $\mathcal{F}$ is obvious since $Aut_{\mathcal{F}_0}(T) = \{id\}$, and T is the only $\mathcal{F}_0$-centric object in $\mathcal{F}_0$. The existence of the commutative diagram is also clear since both

T and P are $\mathcal{F}$-normal, and the existence of the fibration follows by the extension theory from appendix §A.

□

Let $\mathcal{G}$ be a p-local compact group (of rank greater than 1), and let S_0 be the connected component of S with respect to $\mathcal{F}$. The following questions remain open

- Is there a connected saturated fusion subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ over S_0? maybe more than one?
- Suppose such a connected saturated fusion subsystem exists. Is it invariant in $\mathcal{F}$ (in the sense of definition 3.1.9)?
- Suppose such a connected saturated fusion subsystem exists. Is there an associated linking system $\mathcal{L}_0$?
- Furthermore, suppose such a linking system exists. Can we construct an inclusion functor $\iota_0 : \mathcal{L}_0^{cr} \to \mathcal{L}$ as in Corollary 3.2.5?

As a matter of fact, we can algorithmically construct a fusion system over S_0. However, the saturation issue of our candidate remains unsolved. We describe this construction below, after the following interesting property about the functor $(\_)^\bullet$.

Lemma 3.3.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $(S_0, \mathcal{F}_0)$ be a saturated fusion subsystem of $\mathcal{F}$ over a strongly $\mathcal{F}$-closed subgroup $S_0 \leq S$ such that $T \leq S_0$. Let $(\_)^\bullet$ and $(\_)^\bullet_0$ be the "bullet" functors defined in 1.3.1 for $\mathcal{F}$ and $\mathcal{F}_0$. Then,*

$$Ob(\mathcal{F}_0^\bullet) \subseteq Ob(\mathcal{F}^\bullet).$$

Proof. It follows easily from the definition of the functor $(\_)^\bullet$ in 1.3.1, together with the fact that $S_0/T \leq S/T$ and $Aut_{\mathcal{F}_0}(T) \leq Aut_{\mathcal{F}}(T)$.

□

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let S_0 be the connected component of S with respect to $\mathcal{F}$. We want to construct a fusion subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ over S_0, and the first step to take is reduce the S_0-conjugacy classes of subgroups of S_0 to consider to a finite number of them, just as is done via the functor $(\_)^\bullet$ in $\mathcal{F}$. Since, to define $(\_)^\bullet$, the only information needed is $Aut_{\mathcal{F}}(T)$, we first need to determine who should be the automorphism group in $\mathcal{F}_0$ of T.

Let $\widehat{\mathcal{K}}$ be the set of subgroups of $Aut_{\mathcal{F}}(T)$ such that $Aut_{S_0}(T) \in Syl_p(H)$ for each $H \in \widehat{\mathcal{K}}$. There is a partial ordering in $\widehat{\mathcal{K}}$ given by inclusion of subgroups, and we may then consider the subset $\mathcal{K}$ of maximal subgroups in $\widehat{\mathcal{K}}$ under this order relation. Let

$$Aut_{\mathcal{F}_0}(T) \stackrel{def}{=} \cap_{H \in \mathcal{K}} H \leq Aut_{\mathcal{F}}(T) \tag{3.10}$$

Lemma 3.3.3. *The subgroup $Aut_{\mathcal{F}_0}(T)$ has $Aut_{S_0}(T)$ as a Sylow p-subgroup.*

Proof. It is obvious since each $H \in \mathcal{K}$ has $Aut_{S_0}(T)$ as a Sylow p-subgroup.

□

We would like connected components of fusion systems to be also invariant in the sense of definition 3.1.9 (3.1 in [Lin06]). Note that in particular this would mean that $Aut_{\mathcal{F}_0}(T)$ ought to be normal in $Aut_{\mathcal{F}}(T)$, which is not at all clear to be the case.

Now, we can define an operation $(\_)_0^\bullet$ on the set of subgroups of S_0 following the steps in definition 1.3.1. Let $\mathcal{H}_0^\bullet = \{(P)_0^\bullet | P \leq S\}$. Lemma 1.3.2 (i) applies also in this case to show that $\mathcal{H}_0^\bullet$ contains finitely many S_0-conjugacy classes, and by Lemma 3.3.2, $\mathcal{H}_0^\bullet \subseteq Ob(\mathcal{F}^\bullet)$.

Since the set $\mathcal{H}_0^\bullet$ comes equipped with a partial order relation given by inclusion of subgroups, one could now start an inductive process (starting on S_0) to define $Aut_{\mathcal{F}_0}(P)$ for each $P \in \mathcal{H}_0^\bullet$ out of $Aut_{\mathcal{F}}(P)$ whenever P can be fully $\mathcal{F}_0$-normalized (this too can be done inductively).

This defines a fusion subsystem $\mathcal{F}_0$ of $\mathcal{F}$ over S_0 which satisfies axiom (I) (and probably axiom (III) too) by construction of $\mathcal{F}_0$. However, checking axiom (II) is a difficult problem, as well as checking invariance of $\mathcal{F}_0$.

It would be also interesting studying the following situation. Let G be a compact Lie group with connected component G_0, and let $\mathcal{G}, \mathcal{G}_0$ be the corresponding induced p-local compact groups. One should the check first whether $\mathcal{G}_0$ is connected in the sense of definition 3.1.4, and, if this is the case, then try constructing an inclusion functor $\iota_0 : \mathcal{L}_0 \to \mathcal{L}$ as in Corollary 3.2.5.

Chapter 4

Unstable Adams operations on p-local compact groups and Robinson groups

We introduce in this chapter a powerful tool for p-local compact groups: unstable Adams operations. These operations play an important role in the theory of compact Lie groups, for instance when describing the space of self-maps of classifying spaces (see Theorem 1 [JMO90]), in K-theory (see [AC77]), and many others, and we expect the analogous operations on p-local compact groups to be as central in the theory as they are for compact Lie groups.

Unstable Adams operations for p-local compact groups were prove to exist in [Jun09], where a explicit construction of operations can be found. We will recall in this chapter the constructions from the original source, avoiding proofs when they can be checked in [Jun09]. As an improvement of these results, we construct as well unstable Adams operations on Robinson groups realizing saturated fusion systems.

First, we define unstable Adams operations, both for p-local compact groups and for Robinson groups realizing fusion systems.

Definition 4.0.4. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group, and let q be a prime different from p, and $m \in \mathbb{N}$. We define an **unstable Adams operation** Ψ **of degree** q^m on $\mathcal{G}$ as a triple $(\psi_S, \psi_{\mathcal{F}}, \psi_{\mathcal{L}})$, where*

(i) $\psi_S : S \to S$ is a fusion preserving automorphism such that, for all $t \in T$, $\psi(t) = t^{q^m}$;

(ii) $\psi_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is the natural functor induced by ψ_S;

(iii) $\psi_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ is a functor such that $\psi_{\mathcal{L}}(P) = \psi_S(P)$ and such that

$$\rho \circ \psi_{\mathcal{L}} = \psi_{\mathcal{F}} \circ \rho,$$

where $\rho : \mathcal{L} \to \mathcal{F}$ is the usual projection functor.

*Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, G be a Robinson group associated to $\mathcal{F}$, and let q be a prime different from p, and $m \in \mathbb{N}$. Let also $\iota_T : T \to G$ be the canonical monomorphism from Lemma 2.3. An **unstable Adams operation** of degree q^m on G is a group automorphism $\Psi_G : G \to G$ such that*

(i) Ψ_G induces a fusion preserving automorphism on S;

(ii) the restriction of Ψ_G to T is the q^m-th power map.

Note that, in the definition of unstable Adams operations for p-local compact groups, since $\psi_{\mathcal{L}}(S) = \psi_S(S)$, in particular it follows that for each $P \in Ob(\mathcal{L})$ and each $t \in T \cap P$,

$$\Psi_{\mathcal{L}}(\delta_P(t)) = \delta_{\psi_S(P)}(\psi_S(t)).$$

4.1 Unstable Adams operations on saturated fusion systems

We start by constructing unstable Adams operations of saturated fusion systems. This section does not contain any new result, but describes explicitly how to construct unstable Adams operations on fusion systems. In fact, all we have to do is define fusion preserving automorphisms of the Sylow S, restricting to a certain automorphism of the maximal torus. Mainly, the aim of this section is showing the following result.

Theorem 4.1.1. *(3.2 [Jun09]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S with maximal torus T, and let q be a prime different from p.Then, there exists $m \in \mathbb{N}$ and $\psi_m : S \to S$ such that ψ_m is a fusion preserving automorphism and such that, for each $t \in T$, $\psi_m(t) = t^{q^m}$.*

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $T \leq S$ be its maximal torus. Then, there is an extension

$$T \longrightarrow S \longrightarrow S/T,$$

where S/T is a finite p-group. Thus, we can write $S/T = \{a_1, \ldots, a_l\}$. Furthermore, if we fix a (finite) set $\chi \subseteq S$ of representatives of the a_j, then we may write $S = \langle T, \chi\rangle$, and every $y \in S$ can be uniquely writen as

$$y = t_y x_j,$$

for some $t_y \in T$ and some $x_j \in \chi$. From now on, consider the set χ fixed.

We can now define, for each $m \in \mathbb{N}$, a map from S to S itself as follows:

$$\begin{array}{ccc} S & \xrightarrow{\psi_m} & S \\ y = t_y x_j & \longmapsto & \psi_m(y) = t_y^{q^m} x_j. \end{array} \tag{4.1}$$

This, of course, may not even be a group homomorphism. However, by increasing m, this map can be greatly improved. Mainly all the results in this sections are based in the following lemma.

Lemma 4.1.2. *Let p, q be different prime numbers, and let $n, m \in \mathbb{N}$ be such that $q^m - 1$ is congruent with 0 modulo p^n but not modulo p^{n+1}. Then, $q^{pm} - 1$ is congruent with 0 modulo p^{n+1}.*

Proof. We may write $q^m - 1 = p^n k$, for some $k \in \mathbb{N}$ not congruent with 0 modulo p, $k \leq p - 1$. We distinguish two different cases.

Suppose first that $p > 2$. Consider $(q^m - 1)^p$:

$$p^{pn}k^p = (q^m - 1)^p = \sum_{j=0}^{p} \binom{p}{j} (-1)^j (q^m)^{(p-j)} = A + B,$$

where

$$A = q^{pm} - 1 \quad \text{and} \quad B = \textstyle\sum_{j=1}^{p-1} \binom{p}{j} (-1)^j (q^m)^{p-j}.$$

Obviously, $p^{pn}k^p$ is congruent with 0 modulo p^{n+1}, since $n + 1 < pn$. Hence, we have to check that $B \equiv 0$ modulo p^{n+1}.

Seen as a polynomial on q^m, B has an even number of terms, and, since $j = 1, \ldots, p-1$, it follows that p divides all the coefficients in B. Furthermore, since $\binom{a}{b} = \binom{a}{a-b}$, it follows that 1 is a root of this polynomial, that is

$$B = p(q^m - 1)B',$$

where B' is a polynomial on q^m of lower degree. Since $q^m - 1 = p^n k$ by hypothesis, it follows now that p^{n+1} divides B, and hence the statement is true for p odd.

Suppose now that $p = 2$. This case is easier: if $q^m - 1 = 2^n$, then

$$2^{2n} = (q^m - 1)^2 = q^{2m} - 2q^m + 1 = (q^{2m} - 1) - (2q^m - 2) = (q^{2m} - 1) - 2(q^m - 1),$$

and $q^{2m} - 1$ is congruent with 0 modulo 2^{n+1}.

□

Lemma 4.1.3. *(3.3 [Jun09]). Fix $P \leq S$ and $f \in Aut_{\mathcal{F}}(P)$. Then the following holds:*

(i) there exists some m_1 such that, for all $i \geq m_1$, $\psi_i : S \to S$ is a group automorphism;

(ii) there exists some m_2 such that, for all $i \geq m_2$, the automorphism ψ_i satisfies $\psi_i(P) = P$;

(iii) there exists some m_3 such that, for all $i \geq m_3$, the automorphism ψ_i satisfies $\psi_i(P) = P$ and $\psi_i f = f \psi_i$. In particular, $\psi_i f \psi_i^{-1} = f \in Aut_{\mathcal{F}}(P)$.

Actually, since the lemma above only depends on certain finiteness conditions, we can deduce the following.

Corollary 4.1.4. *(3.4 [Jun09]). Let $\mathcal{H} = \{P_1, \ldots, P_n\}$ a finite set of subgroups of S, and for each j let $\mathcal{M}_j \subseteq Aut_{\mathcal{F}}(P_j)$ be a finite subset of automorphisms. Then, there exists $m \in \mathbb{N}$ such that, for all $i \geq m$, for each $P_j \in \mathcal{H}$ and each $f \in \mathcal{M}_j$,*

$$\psi_i(P_j) = P_j \quad \textit{and} \quad \psi_i f \psi_i^{-1} = f.$$

Next step towards Theorem 4.1.1 is the following lemma.

Lemma 4.1.5. *(3.5 [Jun09]). Let $P \leq S$, $f \in Aut_{\mathcal{F}}(P)$ and m be such that, for all $i \geq m$, ψ_i is a group automorphism, $\psi_i(P) = P$ and $\psi_i f \psi_i^{-1} \in Aut_{\mathcal{F}}(P)$. Then, for each $g \in P$,*

$$\psi_i \circ (c_g \circ f) \circ \psi_i^{-1} \in Aut_{\mathcal{F}}(P).$$

Since, for each $P \leq S$ the group $Out_{\mathcal{F}}(P)$ is finite, if we fix P and a finite set of representatives in $Aut_{\mathcal{F}}(P)$ of the elements in $Out_{\mathcal{F}}(P)$, then there exists m such that, for all $i \geq m$, $\psi_i(P) = P$ and

$$\psi_i \circ Aut_{\mathcal{F}}(P) \circ \psi_i^{-1} = Aut_{\mathcal{F}}(P).$$

Proposition 4.1.6. *(3.7 [Jun09]). Let $P \leq S$ and m be such that, for all $i \geq m$, $\psi_i(P) = P$ and $\psi_i Aut_{\mathcal{F}}(P)\psi_i^{-1} = Aut_{\mathcal{F}}(P)$. Then, for each $g \in S$,*

$$\psi_i \circ Aut_{\mathcal{F}}(P^g) \circ \psi_i^{-1} = Aut_{\mathcal{F}}(P^{\psi_i(g)}).$$

Now, to prove Theorem 4.1.1 we just need to fix a set $\mathcal{H}$ of representatives of each of the S-conjugacy classes of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S, and for each $P \in \mathcal{H}$, also fix a set of representatives $\mathcal{M}_P$ in $Aut_{\mathcal{F}}(P)$ of the elements in $Out_{\mathcal{F}}(P)$, and increase m until all the previous results hold for each $P \in \mathcal{C}$. Note that, by Lemma 1.3.2 (i) and Corollary 1.3.4, the set $\mathcal{H}$ is finite. Alperin's fusion theorem (Theorem 1.3.5) finishes the proof.

Theorem 4.1.7. *(3.9 [Jun09]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let m be as in Theorem 4.1.1. Then, there exist infinitely many functors $\Psi_{im} : \mathcal{F} \to \mathcal{F}$ such that, for each i, when restricted to the maximal torus $T \leq S$, Ψ_{im} is the q^{im}-th power map.*

Proof. Since we have already shown the existence of at least one functor Ψ_m, we can now compose Ψ_m with itself, giving rise to a new such functor of degree $2m$. Iterating the process proves the result.

□

Note too that the construction of ψ_m is done by "making choices" of representatives, the main choice made when fixing the set $\chi \subseteq S$ of representatives of the elements in S/T. Thus, a different set of choices may lead to a different unstable Adams operation on the same fusion system.

Proposition 4.1.8. *(3.10 [Jun09]). Let $\chi_1, \chi_2 \subseteq S$ be different sets of choices, and let m_1, m_2 such that χ_j gives rise to an unstable Adams operation Ψ_{m_j} on the same $\mathcal{F}$, for $j = 1, 2$. Then, there exists M such that, for all $i \geq M$, as functors on $\mathcal{F}$,*

$$\Psi_{im_1} = \Psi_{im_2}.$$

One can already see the similarities between the construction of Robinson groups realizing saturated fusion systems and the construction of unstable Adams operations, since it mainly depends on a choice of a fusion-controlling set for $\mathcal{F}$.

4.2 Unstable Adams operations on linking systems

In this section we finish introducing the work from [Jun09] with the construction of unstable Adams operations on linking systems associated to saturated fusion systems over discrete p-toral groups. Again, nothing new is proved in this section.

Definition 4.2.1. *Let $\mathcal{G}$ be a p-local compact group, and let $\Psi_{\mathcal{F}}$, $\Psi_{\mathcal{L}}$ be functors defined on $\mathcal{F}$ and $\mathcal{L}$ respectively. We say that they are **compatible functors** if*

$$\rho \circ \Psi_{\mathcal{L}} = \Psi_{\mathcal{F}} \circ \rho,$$

where $\rho : \mathcal{L} \to \mathcal{F}$ is the usual projection functor.

We start by giving an analog of Alperin's fusion theorem for linking systems.

Proposition 4.2.2. *(4.8 [Jun09]). Let $\mathcal{L}$ be a centric linking system associated to a saturated fusion systems $\mathcal{F}$ over a discrete p-toral group S. Then, for each $\varphi \in Iso_{\mathcal{L}}(P, P')$ there exist sequences of objects in $\mathcal{L}$,*

$$P = P_0, P_1, \ldots, P_k = P' \quad \text{and} \quad Q_1, \ldots, Q_k,$$

and morphisms $\phi_j \in Aut_{\mathcal{L}}(Q_j)$ such that

(i) Q_j is $\mathcal{F}$-centric $\mathcal{F}$-radical for each $j = 1, \ldots, k$;

(ii) $P_{j-1}, P_j \leq Q_j$, and $\rho(\phi_j)(P_{j-1}) = P_j$ for each $j = 1, \ldots, k$; and

(iii) $\iota_{P',Q_k} \circ \varphi = \phi_k \circ \phi_{k-1} \circ \ldots \circ \phi_1 \circ \iota_{P,Q_1}$.

The proposition above is actually all we need in order to prove the following result. Let χ be a set of representatives in S of the elements in S/T, $\mathcal{H}$ be a set of representatives of the S-conjugacy classes of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups, $\mathcal{M}_P$ be a set of representatives in $Aut_{\mathcal{F}}(P)$ of the elements in $Out_{\mathcal{F}}(P)$, for each $P \in \mathcal{H}$, and m be such that ψ_m is a fusion preserving automorphism of S as in Theorem 4.1.1 (constructed with respect to all these choices we have just done). Let also $\psi_{\mathcal{F}}$ be the self-functor on $\mathcal{F}$ defined by ψ_m.

Recall, by Theorem 4.1.7, that in fact we have a whole infinite family $\{\psi_{im}\}_i$ of fusion preserving automorphisms defined on S.

Now, for each $P \in \mathcal{H}$, let $\widehat{\mathcal{M}}_P$ be a set of liftings in $Aut_{\mathcal{L}}(P)$ of the elements in $\mathcal{M}_P$. Then, each $\phi \in Aut_{\mathcal{L}}(P)$ can be uniquely writen as $\phi = \delta_P(g) \circ \varphi$ for some $g \in P$ and some $\varphi \in \widehat{\mathcal{M}}_P$, and we can define maps

$$\begin{array}{ccc} Aut_{\mathcal{L}}(P) & \xrightarrow{\Psi_{P,i}} & Aut_{\mathcal{L}}(P) \\ \phi = \delta_P(g) \circ \varphi & \longmapsto & \delta_P(\psi_{im}(g)) \circ \varphi. \end{array}$$

Lemma 4.2.3. *(4.10 [Jun09]). For each $P \in \mathcal{H}$ and each $\varphi \in \widehat{\mathcal{M}}_P$, there exists some m_P such that, for all $i \geq m_P$, the map $\Psi_{P,i}$ is a group automorphism.*

Thus, once we find M such that, for all $i \geq M$, the above lemma holds for all $P \in \mathcal{H}$, the same arguments used to prove Theorem 4.1.1 apply to show the following result.

Theorem 4.2.4. *(4.12 [Jun09]). Let $\mathcal{G}$ be a p-local compact group, and let q be a prime different from p. Then, there exists $m \in \mathbb{N}$ and an automorphism $\Psi_m = (\psi_S, \psi_{\mathcal{F}}, \psi_{\mathcal{L}})$ such that*

(i) $\psi_{\mathcal{L}}$ and $\psi_{\mathcal{F}}$ are compatible functors,

(ii) $\psi_{\mathcal{L}}(\delta_S(S)) = \delta_S(\psi_S(S)) \leq Aut_{\mathcal{L}}(S)$, *and*

(iii) the restriction to the maximal torus T *is the* q^m*-th power map.*

Again, once such an automorphism on $\mathcal{G}$ has been constructed, its iterated compositions give rise to a whole family of automorphisms of increasingly greater degrees.

Corollary 4.2.5. *Let* $\mathcal{G}$ *be a p-local compact group, and let q be a prime different from p. Let also m and* Ψ_m *be as in Theorem 4.2.4. Then, there is a whole family of unstable Adams operations,* $\{\Psi_{im}\}$*, defined on* $\mathcal{G}$*. For each i,* Ψ_{im} *has degree* q^{im}*.*

By definition, an unstable Adams operation Ψ on $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is a triple $\Psi = (\psi_S, \psi_{\mathcal{F}}, \psi_{\mathcal{L}})$. It happens, however, that for the unstable Adams operations constructed in [Jun09], the whole triple Ψ can be recovered from the functor $\psi_{\mathcal{L}}$. This does not mean that this is the case for all unstable Adams operations.

4.3 Unstable Adams operations on groups realizing fusion systems

Finally, in this section, we construct unstable Adams operations for Robinson groups realizing saturated fusion systems. This construction is almost immediate since both Robinson groups and the unstable Adams operations from [Jun09] share some points in their constructions. Let then $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Whether $\mathcal{F}$ comes together with an associated centric linking system or not is of no importance here. Let also q be a prime different from p.

As in the previous section, let χ be a set of representatives in S of the elements in S/T, $\mathcal{H}$ be a set of representatives of the S-conjugacy classes of $\mathcal{F}$-centric ($\mathcal{F}$-radical) subgroups, $\mathcal{M}_P$ be a set of representatives in $Aut_{\mathcal{F}}(P)$ of the elements in $Out_{\mathcal{F}}(P)$, for each $P \in \mathcal{H}$, and m be such that ψ_m is a fusion preserving automorphism of S as in Theorem 4.1.1 (constructed with respect to all these choices we have just done), together with the self-functor on $\mathcal{F}$, $\psi_{\mathcal{F}}$, defined by ψ_m. Let also $\{\psi_{im}\}_i$ be the whole family of fusion preserving automorphisms on S induced by iterated compositions of ψ_m with itself.

Let $\mathcal{P} \subseteq \mathcal{H}$ be the subset of fully $\mathcal{F}$-normalized subgroups (since we have fixed in $\mathcal{H}$ representatives of all the S-conjugacy classes of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups, we have, in particular, fixed fully $\mathcal{F}$-normalized representatives of the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric ($\mathcal{F}$-radical) subgroups). Then, by Alperin's fusion theorem, they form an (enlarged) fusion-controlling set for $\mathcal{F}$, and we may apply Theorem 2.2.3 to obtain a group G realizing $\mathcal{F}$.

Fix $P \in \mathcal{P}$, and let $N_P = N_S(P)$. Since P is fully $\mathcal{F}$-normalized, the normalizer fusion subsystem $N_{\mathcal{F}}(P)$ is saturated and constrained, and we may apply Proposition 2.1.3 to it: there is a unique centric linking system $\mathcal{L}_P$ associated to $N_{\mathcal{F}}(P)$, which is induced by the group $L_P = Aut_{\mathcal{L}_P}(P)$. Furthermore, $N_P \in Syl_p(L_P)$.

Now, by definition, there is a commutative diagram

$$\begin{tikzcd}
Z(P) \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\delta_P"] & Inn(P) \arrow[d] \\
Z(P) \arrow[r, "\delta_P"'] & L_P \arrow[r] \arrow[d] & Aut_{\mathcal{F}}(P) \arrow[d] \\
 & Out_{\mathcal{F}}(P) \arrow[r, equal] & Out_{\mathcal{F}}(P),
\end{tikzcd}$$

so we can fix a set of liftings $\widehat{\mathcal{M}_P}$ in L_P of the elements in $\mathcal{M}_P \subseteq Aut_{\mathcal{F}}(P)$. Once this is done, we can construct a family of unstable Adams operations, $\{\Psi_{P,i}\}$ on $\mathcal{L}_P$ as done in Theorem 4.2.4. By construction, the automorphism $\Psi_{P,i}$ sends $\delta_P(N_P)$ to $\delta_P(N_P)$ for all i. Let $\psi_{N_P,i}$ be the automorphism of N_P induced this way.

Lemma 4.3.1. *For each $P \in \mathcal{P}$, let $N_P = N_S(P)$, let $\mathcal{L}_P$ be the normalizer linking system and let $\{\Psi_{P,i}\}$ the family of operations on $\mathcal{L}_P$ constructed above. Then, there exists some M such that, for all $i \geq M$, for each pair $P, P' \in \mathcal{P}$ and all $x \in N_P \cap N_{P'}$,*

$$\psi_{N_P,i}(x) = \psi_{N_{P'},i}(x).$$

Proof. We prove the above lemma for a pair $P, P' \in \mathcal{P}$, the general statement being an easy consequence. Let $T_P \leq N_P$, $T_{P'} \leq N_{P'}$ be the corresponding maximal tori. Then, by definition, the automorphisms $\psi_{N_P,i}$ and $\psi_{N_{P'},i}$ restrict to the q^{im}-th power map on T_P and $T_{P'}$, and hence the statement is true for all $x \in T_P \cap T_{P'}$.

To finish the proof, note that $(N_P \cap N_{P'})/(T_P \cap T_{P'})$ is a finite p-group ($T_P \cap T_{P'}$ being the maximal torus of the discrete p-toral group $N_P \cap N_{P'}$). If we fix representatives $x_1, \ldots, x_k$ of the elements of this quotient in $N_P \cap N_{P'}$, then, it is implicit in Lemma 4.1.3 that there exists M such that, for all $i \geq M$ and each of these representatives x_j,

$$\psi_{N_P,i}(x_j) = x_j = \psi_{N_{P'},i}(x_j),$$

and hence the statement holds for the pair P, P'.

□

Proposition 4.3.2. *Let G be the Robinson group realizing $\mathcal{F}$ constructed from the (enlarged) fusion-controlling set $\mathcal{P}$. Then, there exists some M such that, for all $i \geq M$, the automorphisms $\{\Psi_{P,i}\}_{P \in \mathcal{H}'}$ induce an automorphism*

$$\Psi_{G,i} : G \longrightarrow G.$$

Furthermore, if T is the maximal torus of S, then, $\Psi_{G,i}$ restricts to the q^{im}-th power map on T.

Proof. Again, the proof reduces to showing the case $\mathcal{H}'$ contains only two elements, P, P'. That is, suppose $G = L_1 *_{N_2} L_2$, where $N_j \in Syl_p(L_j)$, $j = 1, 2$, and $N_2 \leq N_1$.

By Lemma 4.3.1 above, there is some M such that, for all $i \geq M$, the automorphisms $\psi_{N_1,i}$ and $\psi_{N_2,i}$ agree on the intersection $N_1 \cap N_2 = N_2$. The universal property of amalgams (push-outs) implies now that $\Psi_{1,i}$ together with $\Psi_{2,i}$ induce an automorphism

$\Psi_{G,i} : G \to G$:

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & L_1 \arrow[d, "i_1"] \arrow[ddr, bend left, "i_1\circ\Psi_{1,i}"] & \\
L_2 \arrow[r, "i_2"'] \arrow[drr, bend right, "i_2\circ\Psi_{2,i}"'] & G \arrow[dr, dotted, "\Psi_{G,i}"] & \\
& & G,
\end{tikzcd}$$

where $I = N_1 \cap N_2$.

The last part of the statement follows by definition of the automorphisms $\Psi_{1,i}$ and $\Psi_{2,i}$.

□

Theorem 4.3.3. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group, $\mathcal{H}$ be a set of representatives of the S-conjugacy classes of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups, $\mathcal{P}$ be the (enlarged) fusion-controlling set inside $\mathcal{H}$, and let G be the Robinson group realizing $\mathcal{F}$ built up from $\mathcal{P}$.*

Let also $\psi_{\mathcal{F},i}$ be a family of Adams operations defined on $\mathcal{F}$ as in Theorem 4.1.7, and $\{\Psi_{G,i}\}$ a family of Adams operations defined on G as in Proposition 4.3.2. Then, there exists some M such that, for all $i \geq M$, the functor $\psi_{\mathcal{F},i}$ is naturally induced by $\Psi_{G,i}$.

Proof. This is immediate, since the functor $\psi_{\mathcal{F},i}$ is induces by the fusion preserving automorphism $\psi_{S,i}$, which in turn is implicit in $\Psi_{G,i}$ by definition.

□

Chapter 5

Fixed points of p-local compact groups under the action of unstable Adams operations

Let G be a group, and let $f : BG \to BG$ be a self-map. Then, one can consider the subspace of homotopy fixed points of BG under f, BG^{hf}, defined as the following homotopy pull-back:

$$\begin{tikzcd} BG^{hf} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & BG \arrow[d, "\Delta"] \\ BG \arrow[r, "id \times f"'] & BG \times BG, \end{tikzcd}$$

and in some particular cases it can be shown that BG^{hf} is homotopy equivalent to the classifying space of another group H, $BG^{hf} \simeq BH$. For instance, this happens when $f = B\alpha$, with $\alpha \in Aut(G)$ an actual automorphism of G. However, this is in general a difficult problem to solve, and usually one looks for such a homotopy equivalence *after* completing the space BG^{hf} on some suitable prime p.

An interesting example of this situation is deduced from the following result by E. M. Friedlander and G. Mislin.

Theorem 5.0.4. *(1.4 [FM84]). Let G be a reductive complex Lie group G, let q be a prime, and let $\bar{\mathbb{F}}_q$ be the algebraic closure of the field $\mathbb{F}_q$. Then, for each prime p different from q, there is a map*

$$BG(\bar{\mathbb{F}}_q) \longrightarrow BG,$$

inducing isomorphisms in mod p cohomology, where $G(\bar{\mathbb{F}}_q)$ is the discrete group of $\bar{\mathbb{F}}_q$-rational points of a Chevalley integral group scheme associated to G.

We can then easily prove the following corollary.

Corollary 5.0.5. *Let G and q be as in Theorem 5.0.4. Let also p be a prime different from q, and let Ψ be an unstable Adams operation of degree a p-adic unit acting on BG. Then, there is a homotopy equivalence*

$$(BG(\mathbb{F}_{q^n}))^\wedge_p \simeq (BG^{h\Psi})^\wedge_p$$

for certain n.

This result is proved by considering the Frobenius map α on $G(\bar{\mathbb{F}}_q)$ with same degree as Ψ. In this case, the space of homotopy fixed points of $BG(\bar{\mathbb{F}}_q)$ under $B\alpha$ is simply the classifying space of the fixed-point subgroup of $G(\bar{\mathbb{F}}_q)$ by α, $G(\mathbb{F}_{q^n})$ for a certain n, which is the same as in the statement of the corollary above.

This is of even greater importance since, up to homotopy, all self-maps of compact connected simple Lie groups are compositions of an actual automorphism of the group followed by an unstable Adams operation, as proved by S. Jackowski, J. McClure and B. Oliver in [JMO92a].

Theorem 5.0.6. *Let G be a compact connected simple Lie group with maximal torus T and Weyl group W. Then there is a bijection*

$$Rep(G,G) \wedge \{k \geq 0 | k = 0 \text{ or } (k, |W|) = 1\} \xrightarrow{\cong} [BG, BG]$$

of monoids with zero element, which sends the pair (α, k) to the composition $\Psi^k \circ B\alpha$.

Here, $Rep(G,G) \cong \{0\} \amalg Out(G)$, and Ψ^k is the unique (up to homotopy) unstable Adams operation of degree k on BG.

A similar situation was studied by C. Broto and J. Møller in [BM07], where the authors replace the compact connected Lie group G by a p-compact group (X, BX, e) such that BX is 1-connected.

Theorem 5.0.7. *(Theorem A [BM07]). Let p be an odd prime, and let (X, BX, e) be a 1-connected p-compact group. Let also q be a prime power, prime to p, and let τ be an automorphism of the p-compact group of finite order prime to p.*

Then, the space of homotopy fixed points of BX by the action of $\tau \circ \Psi^q$, denoted by $BX^{h\tau\Psi^q}$, is the classifying space of a p-local finite group.

The results from [BM07] suggest that a more general setting is needed if we intend to unify all these results in a single theorem about spaces of homotopy fixed points, and p-local compact groups seem to be an appropriate candidate.

Let then $\mathcal{G}$ be a p-local compact group, let Ψ be an unstable Adams operation, and let X be the space of homotopy fixed points of $B\mathcal{G}$ under Ψ. There are then (at least) two main different ways of studying the space X, namely from the topological point of view and from the combinatorial point of view.

The topological approach to this problem can be sketched as follows. Consider the natural map $BS \to B\mathcal{G}$. This maps induces in turn a map $f : B(S^{\Psi}) \to X$, where S^{Ψ} stands for the subgroup of all $x \in S$ such that $\Psi(x) = x$, and then one can reproduce the construction from section §7 in [BLO03b] to obtain a triple $(S^{\Psi}, \mathcal{F}_{S^{\Psi},f}(X), \mathcal{L}^c_{S^{\Psi},f}(X))$. Then, one "only" has to check that all conditions in Theorem 7.5 [BLO03b] hold for this triple to see that X is the classifying space of a p-local finite group.

The combinatorial approach starts by studying the invariants in $\mathcal{G}$ as a triple, that is by checking if one obtains a p-local finite group $(S^{\Psi}, \mathcal{F}^{\Psi}, \mathcal{L}^{\Psi})$ by means of the combinatorial description of Ψ as a triple $(\psi_S, \psi_{\mathcal{F}}, \psi_{\mathcal{L}})$. Then, one "only" has to compare the homotopy type of the classifying space of this p-local finite group with the X.

In this work we have focused on the combinatorial approach of the problem, and, more concretely, on the problem of obtaining p-local finite groups from unstable Adams operations on $\mathcal{G}$, and the issue of comparing the homotopy types is left as a future project.

We consider then a fixed p-local compact group and a family of unstable Adams operations as constructed in [Jun09]. The advantage of considering a whole family instead of a single operation is clear, since some properties hold only after considering operations of suitably higher degrees.

We obtain a family of triples $\{\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i)\}$ such that, for all i, $\mathcal{G}_i$ looks "almost" like a p-local finite group, in the sense that $\mathcal{F}_i$ is an $\mathcal{H}_i$-generated, $\mathcal{H}_i$-saturated fusion system over S_i (for a certain subset $\mathcal{H}_i \subseteq Ob(\mathcal{F}_i)$ which we will properly describe), and $\mathcal{L}_i$ satisfies the axiom of a linking system associated to $\mathcal{F}_i$, while the question of the saturation of $\mathcal{F}_i$ remains, in general, unsolved.

One particular situation that we have managed to solve (in the positive case) is that of p-local compact groups of rank 1. The detailed knowledge of such p-local compact groups that we have acquired in chapter §3 clearly helps, specially the existence of connected components in this case and the explicit description of them. Thus, one may ask whether we can develop a similar strategy for the general case or not, the first problem being that yet we have not proved the existence and uniqueness of connected components in the general case.

The first section of this chapter contains some technical results on invariance in linking systems (under the action of unstable Adams operations). This results are then exploited in the second section, in the case where a whole family of operations acts on a p-local compact group. The definition and properties of the family $\{\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i)\}$ are described in section §5.2.2. After introducing these triples, we exploit further their properties, specially in the case where eventually the $\mathcal{G}_i$ become p-local finite groups. In this sense, we prove in section §5.3 an analog of the Stable Elements theorem for p-local compact groups. Examples are then studied in section §5.4. The assumption of the existence of a centric linking system $\mathcal{L}$ is of great importance in our construction, but a section is devoted at then end to discuss the situation where no linking system is assumed to exist.

We state below the main results of this section. The first result is a compendium of the results from section §5.2. The second result is the Stable Elements theorem (proved later on as Theorem 5.3.8). The third result states that all rank 1 p-local compact groups can be approximated by p-local finite groups via families of unstable Adams operations (and corresponds to Theorem 5.4.1).

Theorem 5.0.8. *Let $\mathcal{G}$ be a p-local compact group, and let $\{\Psi_i\}$ be a family of unstable Adams operations defined on $\mathcal{G}$, and such that, for all i, $\Psi_{i+1} = (\Psi_i)^p$.*

Then, there exists some M such that, for all $i \geq M$, there exists a set $\{\mathcal{H}_i\}$ of subgroups of S_i, a triple $\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i)$ (defined in terms of the action of Ψ_i on $\mathcal{L}$) and a faithful functor $\Theta_i : \mathcal{L}_i \to \mathcal{L}_{i+1}$ such that the following holds:

(i) the functor $(\_)^{\bullet} : \mathcal{F} \to \mathcal{F}$ gives an inclusion of sets $\mathcal{H}_i \hookrightarrow \mathcal{H}_{i+1}$,

(ii) $\mathcal{F}_i$ is an $\mathcal{H}_i$-generated $\mathcal{H}_i$-saturated fusion system,

(iii) $\mathcal{L}_i$ satisfies the axioms of a linking system with respect to the full subcategory of $\mathcal{F}_i$ with object set $\mathcal{H}_i$, and

(iv) there is a homotopy equivalence $(\mathrm{hocolim}\ B\mathcal{G}_i)^{\wedge}_p \simeq B\mathcal{G}$, where, by analogy with p-local finite groups, $B\mathcal{G}_i$ stands for the p-completion of the realization of the nerve of $\mathcal{L}_i$.

Theorem 5.0.9. *Let $\mathcal{G}$ be a p-local compact group such that admits an approximation by p-local finite groups (in the sense of definition 5.3.5). Then, there is an isomorphism*

$$H^*(|\mathcal{L}|;\mathbb{F}_p) \xrightarrow{\cong} H^*(\mathcal{F}) \stackrel{def}{=} \varprojlim_{O(\mathcal{F}^c)} H^*(\_;\mathbb{F}_p) \subseteq H^*(BS;\mathbb{F}_p),$$

and in particular $H^*(B\mathcal{G};\mathbb{F}_p) = H^*(|\mathcal{L}|;\mathbb{F}_p)$ is noetherian.

Theorem 5.0.10. *Let $\mathcal{G}$ be a rank 1 p-local compact group, and let $\{\Psi_i\}$ be a family of unstable Adams operations defined on $\mathcal{G}$. Then, $\{\Psi_i|\}$ induces an approximation of $\mathcal{G}$ by p-local finite groups (in the sense of definition 5.3.5).*

It is also worth mentioning that, in the cases where the $\mathcal{G}_i$ are p-local finite groups, we obtain also inclusions of p-local finite groups, that is, functors between the linking systems. These functors are of a certain interest since they are not the identity on objects.

5.1 Unstable Adams operations acting on centric linking systems

Fix $\mathcal{G}$ a p-local compact group and q a prime different from p, and let Ψ be an unstable Adams operation on $\mathcal{G}$ (whose existence has been shown in Theorem 4.2.4). In this section we study invariance in $\mathcal{G}$ with respect to Ψ.

We will in fact restrict ourselves to consider only unstable Adams operations as constructed in chapter §4. Such an Adams operation $\Psi = (\psi_S, \psi_{\mathcal{F}}, \psi_{\mathcal{L}})$ is completely determined by the functor $\psi_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$, as noted at the end of section §4.2, and this is already a good reason to restrict in turn our study to the action of $\psi_{\mathcal{L}}$.

In fact, restricting to $\psi_{\mathcal{L}}$ is also justified by the following. Let $S^{\Psi} \leq S$ be the subgroup of fixed elements of S under ψ_S, and let $P \leq S^{\Psi}$ be any (nontrivial) subgroup. Then, it follows by construction of $\psi_{\mathcal{F}}$ that all the elements $f \in Aut_{\mathcal{F}}(P)$ remain invariant under $\psi_{\mathcal{F}}$. On the other hand, in general, the set $N_S(P) \setminus N_{S^{\Psi}}(P)$ will not be empty, and it stands to reason that an automorphism of P induced by conjugation by an element in this set should not be considered as invariant.

Definition 5.1.1. *We say that $R \in \mathcal{L}$ is **Ψ-invariant** if $\Psi(R) = R$. Similarly, a morphism $\varphi \in Mor_{\mathcal{L}}(R, R')$ is **Ψ-invariant** if $\Psi(\varphi) = \varphi$.*

In this chapter, we will work with a family of unstable Adams operations defined by iterations of an original operation Ψ (of the kind constructed in chapter §4), and the choices made to define such an operation will be rather relevant. Thus, it is worth compiling here in a few lines the list of choices to take in order to construct Ψ. Let $\mathcal{G}$ be a fixed p-local compact group and q be a fixed prime different from p.

Basic Ingredients - 1. Fix the following list of elements, objects and morphisms in $\mathcal{G}$:

(i) a set χ of representatives in S of the elements in S/T;

(ii) a set $\mathcal{H}$ of representatives of the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric subgroups in the set $Ob(\mathcal{F}^{\bullet})$;

(iii) for each $P \in \mathcal{H}$, a set $\mathcal{H}_P$ of representatives of the S-conjugacy classes in $\langle P \rangle_{\mathcal{F}}$ (and such that P itself is in $\mathcal{H}_P$ representing its own S-conjugacy class);

(iv) for each $P \in \mathcal{H}$ and each pair $R, R' \in \mathcal{H}_P$, a set $\mathcal{M}_{R,R'}$ of representatives in $Iso_{\mathcal{F}}(R, R')$ of the elements in $Rep_{\mathcal{F}}(R, R')$;

(v) for each $P \in \mathcal{H}$, and for each pair $R, R' \in \mathcal{H}_P$, a set $\widehat{\mathcal{M}}_{R,R'}$ of liftings in $Iso_{\mathcal{L}}(R, R')$ of the elements in $\mathcal{M}_{R,R'}$.

This list will remain fixed for the rest of the chapter. We think of it as the "basic ingredients" needed to construct Ψ, and will refer to it as BI-1. The reason for the numbering is that at some point we will want to "enlarge" it by adding to it some more objects or morphisms, although always a finite number of them.

Note that we are including in BI-1 more elements than are really needed in order to define an Adams operation on $\mathcal{G}$, but this is not a problem, as has been shown in chapter §4 ([Jun09]), since all the properties there hold for a finite number of subgroups and morphisms.

Finally, let $m \in \mathbb{N}$ be such that there is a well-defined Adams operation $\Psi_m = (\psi_S, \psi_{\mathcal{F}}, \psi_{\mathcal{L}})$ on $\mathcal{G}$, of degree q^m as in Theorem 4.2.4. To (slightly) simplify notation, we will refer to the functor $\psi_{\mathcal{L}}$ as Ψ, since there will not be place for confusion.

5.1.1 Detecting Ψ-invariants in a linking system

Note that, by construction of Ψ, all the objects and morphisms in BI-1 are Ψ-invariant, and now we want to "detect" which other objects Q and morphisms φ' in $\mathcal{L}$ are Ψ-invariant. The only way to do so is by comparing any other object or morphism with some object or morphism which we know, *a priori*, to be Ψ-invariant, i.e., by comparing Q and φ' with the elements in BI-1.

Consider the subgroup

$$S^{\Psi} = \{x \in S \mid \Psi(x) = x\} \leq S, \tag{5.1}$$

and note that for any pair Q, Q' of Ψ-invariant objects in $\mathcal{L}$, all the morphisms in the subset $\delta_{Q,Q'}(N_S(Q, Q') \cap S^{\Psi}) \subseteq Mor_{\mathcal{L}}(Q, Q')$ are Ψ-invariant too.

Lemma 5.1.2. *Let R, R' be subgroups fixed in BI-1, together with $\widehat{\mathcal{M}}_{R,R'}$. Then, an isomorphism $\gamma \in Iso_{\mathcal{L}}(R, R')$ is Ψ-invariant if and only if $\gamma = \delta_{R'}(g) \circ \varphi$ for some $g \in R' \cap S^{\Psi}$ and some $\varphi \in \widehat{\mathcal{M}}_{R,R'}$.*

Proof. By definition of $\widehat{\mathcal{M}}_{R,R'}$, any isomorphism in $Iso_{\mathcal{L}}(R, R')$ can be written uniquely as $\gamma = \delta_{R'}(g) \circ \varphi$ for some $g \in R'$ and some $\varphi \in \widehat{\mathcal{M}}_{R,R'}$. Thus, the statement is obvious. □

Lemma 5.1.3. *Let R be fixed in BI-1, and let $Q \in \langle R \rangle_S$. Then, Q is Ψ-invariant if and only if, for all $x \in N_S(R, Q)$, $x^{-1}\Psi(x) \in N_S(R)$.*

Proof. First, assume that Q is Ψ-invariant, and write $Q = xRx^{-1}$ for some $x \in N_S(R, Q)$. Thus, if we apply Ψ to this equality, we get

$$x \cdot R \cdot x^{-1} = Q = \Psi(Q) = \Psi(x) \cdot R \cdot \Psi(x)^{-1},$$

and hence $x^{-1}\Psi(x) \in N_S(R)$.

Now, suppose that, for any $x \in N_S(R,Q)$, $x^{-1}\Psi(x) \in N_S(R)$. Fix some $x \in N_S(R,Q)$, and note that the condition above is equivalent to saying that

$$\Psi(x)x^{-1} = x(x^{-1}\Psi(x))x^{-1} \in N_S(Q).$$

Thus, we may write $R = x^{-1} \cdot Q \cdot x$, and, by applying Ψ, we get $R = \Psi(x)^{-1} \cdot \Psi(Q) \cdot \Psi(x)$, and hence

$$\Psi(Q) = \Psi(x) \cdot R \cdot \Psi(x)^{-1} = \Psi(X)x^{-1} \cdot Q \cdot x\Psi(x)^{-1} = Q.$$

□

Lemma 5.1.4. *Let R, R' fixed in BI-1, and let $Q \in \langle R\rangle_S$, $Q' \in \langle R'\rangle_S$. Then, a morphism $\varphi' \in Iso_{\mathcal{L}}(Q,Q')$ is Ψ-invariant if and only if there exist $a \in N_S(R,Q)$, $b \in N_S(R',Q')$ and $\varphi \in \widehat{\mathcal{M}}_{R,R'}$ such that*

(i) $\varphi' = \delta(b) \circ \varphi \circ \delta(a^{-1})$, *and*

(ii) $\delta(b^{-1}\Psi(b)) \circ \varphi = \varphi \circ \delta(a^{-1}\Psi(a))$.

Proof. Note that condition (ii) above is equivalent to the following condition

(ii′) $\delta(\Psi(b)b^{-1}) \circ \varphi' = \varphi' \circ \delta(\Psi(a)a^{-1})$.

Suppose first that φ' is Ψ-invariant. Choose any $x \in N_S(R,Q)$ and $y \in N_S(R',Q')$, and let $\phi = \delta(y^{-1}) \circ \varphi' \circ \delta(x)$. Then, there exists $f \in \mathcal{M}_{R,R'}$ (a morphism in $\mathcal{F}$) such that

$$[f] = [\rho(\phi)] \in Rep_{\mathcal{F}}(R,R').$$

Furthermore, by definition of $Rep_{\mathcal{F}}(R,R')$, and by Lemma 4.3 (a) in [BLO07], it follows that there exists a unique $z \in R'$ such that

$$\varphi = \delta(z) \circ \phi,$$

where $\varphi \in \widehat{\mathcal{M}}_{R,R'}$ is the fixed lifting of f in $\mathcal{L}$.

Thus, we have a commutative diagram in $\mathcal{L}$

$$\begin{tikzcd} Q \arrow[r, "\varphi'"] \arrow[d, "\delta(x)"'] & Q' \arrow[d, "\delta(zy^{-1})"] \\ R \arrow[r, "\varphi"'] & R', \end{tikzcd}$$

where the horizontal arrows are Ψ-invariant morphisms.

If we set now $a = x \in N_S(R,Q)$ and $b = yz^{-1} \in N_S(R',Q')$, then condition (i) is already satisfied, and we have to check that condition (ii) also holds. Since both φ and φ' are Ψ-invariant, we may apply Ψ to (i) to get the following equality

$$\delta(b) \circ \varphi \circ \delta(a^{-1}) = \varphi' = \Psi(\varphi') = \delta(\Psi(b)) \circ \varphi \circ \delta(\Psi(a)^{-1}),$$

which is clearly equivalent to condition (ii), since morphisms in $\mathcal{L}$ are epimorphisms in the categorical sense by Lemma A.2.2 (iv).

Suppose now that conditions (i) and (ii) are satisfied for certain a, b and φ. Write $\varphi = \delta(b^{-1}) \circ \varphi' \circ \delta(a)$, and apply Ψ to this equality. Since φ is Ψ-invariant, we get

$$\delta(\Psi(b)^{-1}) \circ \Psi(\varphi') \circ \delta(\Psi(a)) = \delta(b^{-1}) \circ \varphi' \circ \delta(a).$$

Thus, after reordering the terms in this equality and using condition (ii′) above, we obtain

$$\Psi(\varphi') \circ \delta(\Psi(a)a^{-1}) = \varphi' \circ \delta(\Psi(a)a^{-1}),$$

which in turn implies that $\Psi(\varphi') = \varphi'$ since morphisms in $\mathcal{L}$ are epimorphisms in the categorical sense by Lemma A.2.2 (iv).

□

Note that, for any $y \in S$,

$$x^{-1}\Psi(x) \in T,$$

since $x = ty$ for some $t \in T$ and $y \in \chi$, and all the elements in χ are Ψ-invariant. This already justifies the feeling that this problem is closely related to the existence and properties of a hypothetical connected component of $\mathcal{G}$ (or at least of $\mathcal{F}_0$).

5.2 Families of unstable Adams operations acting on a linking system

Expecting that the invariants of $\mathcal{L}$ under the action of Ψ will give rise to a p-local finite group would be excesively optimistic, and probably false in general too, but we can improve the situation quite a lot if instead of considering a single unstable Adams operation on $\mathcal{G}$ we consider a whole family of operations on $\mathcal{G}$ of increasingly higher degrees.

As we have seen in chapter §4, given an unstable Adams operation Ψ on $\mathcal{G}$, different iterations of it give rise to new, different unstable Adams operations Ψ' on $\mathcal{G}$. Fix then the p-local compact group $\mathcal{G}$ and fix also such an operation Ψ of degree q^m, together with a list like BI-1. Set then $\Psi_0 = \Psi$, and

$$\Psi_{i+1} = (\Psi_i)^p,$$

that is, the operation Ψ_i iterated p times. Thus, since Ψ_0 has degree q^m for certain m, for each i the operation Ψ_i has degree $q^{p^i m}$. We fix the family

$$\{\Psi_i\}_{i \in \mathbb{N}} \tag{5.2}$$

for the rest of this section. Note that, if an object or a morphism in $\mathcal{L}$ is Ψ_j-invariant for some j, then it is Ψ_i-invariant for all $i \geq j$ by definition of the family above.

Before getting into further details, we want to fix some notations that we will use from now on. For each i, set

$$S_i \stackrel{def}{=} \{x \in S \mid \Psi_i(x) = x\}, \tag{5.3}$$

and more generally we will use the following notation

$$R_i \stackrel{def}{=} R \cap S_i$$

for the subgroup of fixed elements of R under Ψ_i.

Also, from time to time we will need to discard the first M operations (for some finite M) from the family (5.2) in order for a certain property to hold. When this happens, we will re-label the family following the formula "$\Psi_i \Rightarrow \Psi_{i-M}$". This way, we can keep using expressions as "for all i" instead of constantly pointing out which is the lower bound of operations to consider in (5.2).

The following results are consequences of Lemma 4.1.2.

Lemma 5.2.1. *For each i, $T_i \lneq T_{i+1}$, and hence*

$$T = \cup_{i \in \mathbb{N}} T_i.$$

Proof. Fix i, and suppose $T_i \cong (\mathbb{Z}/p^n)^r$ for some n, where $r = rk(T)$. Since for each $t \in T$, $\Psi_i(t) = t^{q^{p^i m}}$, this is the same as saying that $q^{p^i m} - 1$ is congruent with 0 modulo p^n, but not modulo p^{n+1}. By Lemma 4.1.2, it follows that $q^{p^{i+1} m} - 1$ is congruent with 0 modulo p^{n+1} (at least), and hence T_{i+1} strictly contains T_i.

□

Corollary 5.2.2. *For each $x \in S$ there exists some finite M_x such that, for all $i \geq M_x$, x is Ψ_i-invariant.*

Proof. In BI-1 we have fixed a set χ of representatives of the elements in S/T, and it is clear that every $x \in S$ can be uniquely written as $x = yt$ for certain $y \in \chi$ and $t \in T$. Now, the elements in χ are Ψ_i-invariant for all i, by construction, and it is clear then that x is Ψ_i-invariant if and only if t is Ψ_i-invariant.

□

Finally, we can prove the following result.

Proposition 5.2.3. *The following hold in $\mathcal{L}$:*

(i) Let $Q \in Ob(\mathcal{L})$. Then, there exists some M_Q such that, for all $i \geq M_Q$, Q is Ψ_i-invariant.

(ii) Let φ be a morphism in $\mathcal{L}$. Then, there exists some M_φ such that, for all $i \geq M_\varphi$, φ is Ψ_i-invariant.

Proof. (i) Let $Q \in \mathcal{L}$, and let T_Q be its maximal torus. Fix also representatives in Q of the elements in Q/T_Q. Since this is a finite set, and $\Psi_i(T_Q) = T_Q$ for all i, the result follows from Corollary 5.2.2.

(ii) Let now $\varphi : Q \to Q'$ be a morphism in $\mathcal{L}$. By (i), there exists some M' such that, for all $i \geq M'$, both Q and Q' are Ψ_i-invariant.

We can suppose that φ is an isomorphism. Furthermore, we can assume that φ is not any of the morphisms fixed in BI-1, since otherwise we are finished.

Now, $Q \in \langle R \rangle_S$, $Q' \in \langle R' \rangle_S$ for certain R, R' fixed in BI-1, and we can write

$$\varphi = \delta(b) \circ \varphi' \circ \delta(a^{-1})$$

for some $\varphi' \in \widehat{\mathcal{M}}_{R,R'}$, $a \in N_S(R, Q)$ and $b \in N_S(R', Q')$. Thus, it is enough to see that there exists some M_φ such that $\Psi_i(a) = a$, $\Psi_i(b) = b$ for all $i \geq M_\varphi$, and this has been shown in Corollary 5.2.2

□

5.2.1 A stronger invariance condition for linking systems

Once we have fixed the family of operations which will act on $\mathcal{G}$, the next step is deciding which invariants we will consider, keeping in mind that we want to provide them with some structure. More concretely, we would like to define triples $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ such that the following eventually holds:

(i) for all i, each of these triples is a p-local finite group , and

(ii) for each i some relation exists between the triples $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ and $(S_{i+1}, \mathcal{F}_{i+1}, \mathcal{L}_{i+1})$.

As a first attempt, one could then define a fusion system over S_i with morphism sets spanned by compositions of restrictions of all the Ψ_i-invariant morphisms in $\mathcal{L}$, and unfortunately this is addressed to fail in being a saturated fusion system, mainly because first, given $R \in Ob(\mathcal{L})$, the subgroup $R_i = R \cap S_i$ need not be $\mathcal{F}_i$-centric in general, and second, given a morphism $f_i : R_i \to S_i$ in $\mathcal{F}_i$ (which will be in general the restriction of some $f \in Mor(\mathcal{F})$) on which one might apply axiom (II), there is no way to relate the subgroup N_{f_i} (in $\mathcal{F}_i$) with the subgroup N_f (in $\mathcal{F}$), hence one does not have any way to make sure that the extensions of axiom (II) holds in $\mathcal{F}_i$. The key point to avoid these problems lies in the functor $(\_)^{\bullet}$ defined in 1.3.1.

Definition 5.2.4. *Let K be a subgroup of S. We say that the subgroup $R \in Ob(\mathcal{F}^{\bullet})$ is K-**determined** if*

$$(R \cap K)^{\bullet} = R.$$

*For a K-determined subgroup R we call the subgroup $R \cap K$ the K-**root** of R.*

It is clear that our interest lies in the case $K = S_i$ (5.3). We first prove the existence of such groups.

Lemma 5.2.5. *Let $R \in Ob(\mathcal{F}^{\bullet})$. Then, there exists some M_R such that, for all $i \geq M_R$, R is S_i-determined.*

Proof. Let T_R be the maximal torus of R, and note that R can be expressed as $R = \cup R_i$. Thus, there exists some M such that, for all $i \geq M$, R_i contains a set of representatives of the elements in R/T_R.

Now recall the definition of the functor $(\_)^{\bullet}$: for $Q \leq S$, $Q^{\bullet} = Q \cdot I(Q^{[e]})_0$, where e is such that $S/T = p^e$. Thus, since $R = R^{\bullet}$ and $Aut_{\mathcal{F}}(T)$ is finite, it is easy to check that there exists some $M' \geq M$ such that, for all $i \geq M'$,

$$I(R_i^{[e]}) = I(R^{[e]}),$$

which finishes the proof. $\square$

Implicit in the lemma above there is the fact that, if R is S_j-determined, then it is S_i-determined for all $i \geq j$, since

$$R = (R \cap S_i)^{\bullet} \leq (R \cap S_j)^{\bullet} \leq R^{\bullet} = R.$$

As an immediate consequence, it follows that all the subgroups fixed in BI-1 eventually become S_i-determined.

Lemma 5.2.6. *There exists some finite M_1 such that, for all $i \geq M_1$, all the subgroups fixed in BI-1 are S_i-determined.*

Proof. Since we have only fixed finitely many subgroups in BI-1, the statement follows by Lemma 5.2.5. □

We now list some properties of S_i-determined subgroups.

Lemma 5.2.7. *Let $R \in Ob(\mathcal{F}^\bullet)$ be an S_i-determined subgroup, for some i. Then,*

(i) $R = R_i \cdot T_R$, where T_R is the maximal torus of R;

(ii) R_i contains a set of representatives of all the elements in R/T_R; and

(iii) R_i contains a set of representatives of all the elements in $R/(R \cap T)$, where T is the maximal torus of S.

Proof. (i) Since R is S_i-determined, we have

$$R = (R_i)^\bullet = R_i \cdot I(R_i^{[e]})_0.$$

Furthermore, since R_i is finite, it follows that $I(R_i^{[e]})_0$ has the same rank as R, and thus there is an equality $I(R_i^{[e]})_0 = T_R$.

(ii) It has already been pointed out before, since $R/T_R = (R_i \cdot T_R)/T_R$.

(iii) It follows from (ii) and the commutative diagram

$$\begin{tikzcd}
T_R \arrow[r] \arrow[d, equal] & R \cap T \arrow[r] \arrow[d] & (R \cap T)/T_R \arrow[d] \\
T_R \arrow[r] & R \arrow[r] \arrow[d] & R/T_R \arrow[d] \\
 & R/(R \cap T) \arrow[r, equal] & R/(R \cap T).
\end{tikzcd}$$

□

Lemma 5.2.8. *The following holds in $\mathcal{L}$ for all i:*

(i) if R is S_i-determined, then it is Ψ_i-invariant;

(ii) if R is S_i-determined and $\varphi \in Iso_{\mathcal{L}}(R, R')$ is Ψ_i-invariant, then R' is also S_i-determined.

Proof. (i) is immediate after the previous lemma. Indeed,

$$\Psi_i(R) = \Psi_i(R_i \cdot T_R) \leq R,$$

since $\Psi_i(x) = x$ for all $x \in R_i$, and $\Psi_i(T_R) = T_R$ by the very definition of Ψ_i.

(ii) Now, let R be S_i-determined and $\varphi \in Iso_{\mathcal{L}}(R, R')$ be a Ψ_i-invariant morphism. Then, for each $x \in R_i$ we can apply axiom (C) of $\mathcal{L}$ to get a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d, "\delta(x)"'] & R' \arrow[d, "\delta(y)"] \\
R \arrow[r, "\varphi"'] & R',
\end{tikzcd}$$

where $y = \rho(\varphi)(x)$. Since three of the four arrows are Ψ_i-invariant by hypothesis, it follows that so is $\delta(y)$, i.e., $y \in R'_i$.

The proof is now finished by applying the functor $(\_)^\bullet$ and its properties to $f : R_i \to R'_i$, together with the hypothesis of R being S_i-determined.

□

Caution!. *Let R, R' be S_i-determined. If R_i and R'_i are $\mathcal{F}$-conjugate, then by Proposition 1.3.3, it follows that R and R' are also $\mathcal{F}$-conjugate. However, the opposite may not be true.*

The following results justify working with S_i-determined subgroups.

Proposition 5.2.9. *There exists some M_2 such that, for all $i \geq M_2$, if R is S_i-determined, then*

$$C_S(R_i) = C_S(R).$$

Proof. We will prove the statement for a single S-conjugacy class in $Ob(\mathcal{F}^\bullet)$, since by Lemma 1.3.2 (i) there are only finitely many such conjugacy classes.

Let then $R \in Ob(\mathcal{F}^\bullet)$, consider the S-conjugacy class $\langle R \rangle_S$, and let

$$\mathbb{T}_R = \{T_Q | Q \in \langle R \rangle_S\}$$

be the set of maximal tori of subgroups in the S-conjugacy class of R. First we show that this is a finite set. Indeed, for any two $Q, Q' \in \langle R \rangle_S$ and any $f \in Iso_{\mathcal{F}}(Q, Q')$, the infinite p-divisibility property on T_Q and $T_{Q'}$ implies that $f(T_Q) = T_{Q'}$. Now, by Lemma 1.2.3, $f_{|T_Q}$ is the restriction of an automorphism in $Aut_{\mathcal{F}}(T)$, and this automorphism group is finite because T is abelian.

For each $T_Q \in \mathbb{T}_R$, write $T_Q = \cup (T_Q)_i$, where $(T_Q)_i$ is the subgroup of Ψ_i-invariant elements of T_Q. Thus, for each i we have inclusions

$$(T_Q)_i \leq (T_Q)_{i+1} \leq T_Q,$$

and we may take their centralizers in S. This reverses the inclusions, and hence the artinian condition of S implies that there exists M_{T_Q} such that, for all $i \geq M_{T_Q}$,

$$C_S(T_Q) = C_S((T_Q)_i).$$

Since $\mathbb{T}_R$ is finite, we may finally consider $M_R = max\{M_{T_Q} | T_Q \in \mathbb{T}_R\}$.

Now, let $i \geq M_R$, and let $Q \in \langle R \rangle_S$ be S_i-determined. Then, by Lemma 5.2.7 (ii), Q_i contains a set $\bar{Q}$ of representatives of the elements of Q/T_Q, and the subgroup $H_Q = \langle \bar{Q} \rangle$ is finite because S is locally finite. Furthermore, we can write

$$Q = H_Q \cdot T_Q \quad \text{and} \quad Q_i = H_Q \cdot (T_Q)_i.$$

Thus, by taking centralizers on both equalities (and since $i \geq M_R$ as above), we get

$$C_S(Q) = C_S(H_Q) \cap C_S(T_Q) = C_S(H_Q) \cap C_S((T_Q)_i) = C_S(Q_i).$$

Finally, take M_2 to be the maximum of the M_R, for a set of representatives of the S-conjugacy classes in $Ob(\mathcal{F}^\bullet)$.

□

As an easy consequence, we have the following.

Corollary 5.2.10. *For all $i \geq M_2$ and all R which is S_i-determined, if $C_S(R) = Z(R)$ then $C_{S_i}(R_i) = Z(R_i)$.*

Proof. If $C_S(R) = Z(R)$, then

$$C_{S_i}(R_i) = S_i \cap C_S(R_i) = S_i \cap C_S(R) = S_i \cap Z(R) \leq R_i.$$

□

Also, the S_i-determined condition allows us to prove the following.

Proposition 5.2.11. *There exists some M_3 such that, for all $i \geq M_3$, if $R \in Ob(\mathcal{F}^\bullet)$ is an S_i-determined subgroup, and R_i is its S_i-root, then, $N_S(R_i) \leq N_S(R)$.*

Proof. Fix $R \in Ob(\mathcal{F}^\bullet)$, and let $\mathbb{T}_R = \{T_Q | Q \in \langle R \rangle_S\}$. Then, since this set is finite (and because $T \leq C_S(T_Q)$ and $T \leq C_S((T_Q)_i)$), it is clear that there exists some M_R such that, for all $i \geq M_R$ and all $Q \in \langle R \rangle_S$, if $g \in N_S((T_Q)_i)$ then $g \in N_S(T_Q)$.

Let then $i \geq M_R$, $Q \in \langle R \rangle_S$ such that Q is S_i-determined, and $g \in N_S(Q_i)$. Then, in particular, $g \in N_S((T_Q)_i) \leq N_S(T_Q)$, and hence $g \in N_S(Q)$.

□

As a corollary, the following holds.

Corollary 5.2.12. *Let $i \geq M_3$, and let R, R' S_i-determined subgroups, and R_i, R'_i be the corresponding S_i-roots. Let also $f : R \to R'$ be a morphism in $\mathcal{F}$ which restricts to a morphism f_i between the S_i-roots, and set*

$$N_f = \{g \in N_S(R) | f c_g f^{-1} \in Aut_S(R')\},$$
$$N'_{f_i} = \{h \in N_{S_i}(R_i) | f_i c_h f_i^{-1} \in Aut_{S_i}(R'_i)\}.$$

Then, there is an inclusion $N'_{f_i} \leq N_f$.

5.2.2 A family of "quasi"-p-local finite groups

Using the notion of S_i-determined subgroups we can now define a family of triples $(S_i, \mathcal{F}_i, \mathcal{L}_i)$, which, as we will show, behave almost as p-local finite groups, in a sense to be made precise below.

First, we recall some notation and a result from [BCG+05] which will be used in this chapter.

Definition 5.2.13. *Let $\mathcal{F}$ be a fusion system over a finite p-group S, and let $\mathcal{H} \subseteq Ob(\mathcal{F})$ be a subset of objects. Then, we say that $\mathcal{F}$ is **$\mathcal{H}$-generated** if every morphism in $\mathcal{F}$ is a composite of restrictions of morphisms in $\mathcal{F}$ between subgroups in $\mathcal{H}$, and we say that $\mathcal{F}$ is **$\mathcal{H}$-saturated** if the saturation axioms in definition 1.2.2 hold for all subgroups in the set $\mathcal{H}$.*

The following result is somehow the key for our constructions to work. Given a set $\mathcal{H}$ of subgroups in a fusion system over a finite p-group, this theorem provides a tool to determine whether the fusion system is saturated.

Theorem 5.2.14. *(Theorem A [BCG+05]). Let $\mathcal{F}$ be a fusion system over a finite p-group S, and let $\mathcal{H}$ be a subset of objects of $\mathcal{F}$ closed under $\mathcal{F}$-conjugacy and such that $\mathcal{F}$ is $\mathcal{H}$-generated and $\mathcal{H}$-saturated. Suppose further that, for each $\mathcal{F}$-centric subgroup $P \notin \mathcal{H}$, P is $\mathcal{F}$-conjugate to some P' such that*

$$Out_S(P') \cap O_p(Out_{\mathcal{F}}(P')) \neq \{1\}. \tag{5.4}$$

Then, $\mathcal{F}$ is saturated.

Note that, if $\mathcal{F}$ turned out to be saturated, it would mean that $\mathcal{H}$ contains all $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S.

To be more specific, then, we will define triples $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ for all i, and we will then show that for i big enough there exists a set $\mathcal{H}_i$ of subgroups of S_i such that $\mathcal{F}_i$ is an $\mathcal{H}_i$-generated $\mathcal{H}_i$-saturated fusion system and that $\mathcal{L}_i$ satisfies the axioms of a linking system with respect to $\mathcal{F}_i$.

As a final step, we would like to apply Theorem 5.2.14 on $\mathcal{F}_i$ to prove saturation, although there appear certain difficulties with the technical condition (5.4), which we will discuss later on in this chapter.

For each i, consider the sets

$$\begin{gathered} \mathcal{H}_i^\bullet = \{R \leq S | R \text{ is } \mathcal{F}\text{-centric and } S_i\text{-determined}\}, \\ \mathcal{H}_i = \{R_i = R \cap S_i | R \in \mathcal{H}_i^\bullet\}. \end{gathered} \tag{5.5}$$

Note that the functor $(\_)^\bullet$ gives a one-to-one correspondence between these two sets. Also, for each pair $R, R' \in \mathcal{H}_i^\bullet$, consider the sets

$$\begin{gathered} A_{\mathcal{L}}(R, R')_i = \{\varphi \in Iso_{\mathcal{L}}(R, R') | \varphi \text{ is } \Psi_i\text{-invariant}\}, \\ A_{\mathcal{F}}(R, R')_i = \{f = \rho(\varphi) | \varphi \in A_{\mathcal{L}}(R, R')_i\}. \end{gathered}$$

Lemma 5.2.15. *Let $R, R' \in \mathcal{H}_i^\bullet$, and let $\varphi \in A_{\mathcal{L}}(R, R')_i$. Then, $f = \rho(\varphi)$ restricts to an isomorphism $f_i : R_i \to R'_i$.*

Proof. Let $x \in R_i \leq R$. Then, by applying axiom (C) of linking systems, we get a commutative diagram in $\mathcal{L}$

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ {\scriptstyle \delta(x)}\downarrow & & \downarrow{\scriptstyle \delta(y)} \\ R & \xrightarrow[\varphi]{} & R', \end{array}$$

where $y = f(x)$. Since both φ and $\delta(x)$ are Ψ_i-invariant morphisms in $\mathcal{L}$, then so is $\delta(y)$, i.e., $y \in S_i \cap R' = R'_i$. □

We may consider then the sets

$$A(R_i, R'_i) = \{f_i = res^R_{R_i}(f) | f \in A_{\mathcal{F}}(R, R')_i\} \subseteq Iso_{\mathcal{F}}(R_i, R'_i). \tag{5.6}$$

Note that this set can be identified with $A_{\mathcal{F}}(R, R')_i$ via the functor $(\_)^\bullet$. We can now define the following categories.

Definition 5.2.16. *For each i, define the* ***fixed-point fusion system*** $\mathcal{F}_i$ *to be the fusion system over S_i with morphism sets generated by compositions of restrictions of morphisms in the sets $A(R_i, R'_i)$, for $R_i, R'_i \in \mathcal{H}_i$.*

Also, for each i, define the ***fixed-point linking system*** $\mathcal{L}_i$ *be the category with object set $\mathcal{H}_i$ and with morphism sets (formally) spanned by the sets $A_{\mathcal{L}}(R, R')$, after identifying $\mathcal{H}_i$ with $\mathcal{H}_i^{\bullet}$ via the functor $(\_)^{\bullet}$.*

Finally, set $\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i)$.

We can think of $\mathcal{L}_i$ also as a subcategory of $\mathcal{L}$, and this identification will be rather helpful at some points, but the proper definition makes more sense since we expect it to be a linking system associated to $\mathcal{F}_i$ (as will be the case in some important examples that we show later on).

One can also extend $\mathcal{L}_i$ to a category $\widetilde{\mathcal{L}}_i$ which is closed under overgroups in the sense that, if $R_i \in \mathcal{H}_i$ and $R_i \leq Q_i \leq S_i$, then $Q_i \in Ob(\widetilde{\mathcal{L}}_i)$, and where the morphism sets in this bigger category are compositions of restrictions of morphisms in $\mathcal{L}_i$, but this is not of much interest for us.

The projection functor $\rho : \mathcal{L} \to \mathcal{F}$ naturally induces now functors

$$\rho_i : \mathcal{L}_i \longrightarrow \mathcal{F}_i \tag{5.7}$$

which are the identity on objects and $\rho_i(\varphi) = res^R_{R_i}(\rho(\varphi))$, for all $R_i, R'_i \in \mathcal{H}_i$ and all $\varphi \in A_{\mathcal{L}}(R, R')_i$. Also, the "distinguished monomorphisms" $\delta : R \to Aut_{\mathcal{L}}(R)$ naturally induce "distinguished monomorphisms"

$$\delta_i : R_i \longrightarrow Aut_{\mathcal{L}_i}(R_i). \tag{5.8}$$

The following lemma is obvious.

Lemma 5.2.17. *For all $R_i \in \mathcal{H}_i$ and all $x \in R_i$, the functor ρ_i sends $\delta_i(x) \in Aut_{\mathcal{L}_i}(R_i)$ to $c_x \in Aut_{\mathcal{F}_i}(R_i)$.*

Proposition 5.2.18. *For all i, the fusion system $\mathcal{F}_i$ is $\mathcal{H}_i$-generated. Furthermore, for all $R_i, R'_i \in \mathcal{H}_i$, there are equalities of sets*

$$A(R_i, R'_i) = Hom_{\mathcal{F}_i}(R_i, R'_i) \quad \text{and} \quad A_{\mathcal{L}}(R, R')_i = Mor_{\mathcal{L}_i}(R_i, R'_i).$$

Proof. The $\mathcal{H}_i$-generation of $\mathcal{F}_i$ is clear by definition of $\mathcal{F}_i$.

To show the second part of the statement, it is enough to check only the equality of morphisms sets in $\mathcal{L}_i$. Now, this isomorphism holds since $A_{\mathcal{L}}(R, R')_i$ is the set of all Ψ_i-invariant isomorphisms in $\mathcal{L}$, and any other morphism that could appear in $Mor_{\mathcal{L}_i}(R_i, R'_i)$ would be a composition of restrictions of Ψ_i-invariant automorphisms, thus Ψ_i-invariant too.

□

We now face the question of the $\mathcal{H}_i$-saturation of $\mathcal{F}_i$. This in fact will require adding some more items to the list BI-1, in order to make sure that the axiom (II) holds. First, we make a brief discussion on the Ψ_i-invariance of extensions of morphisms, which will motivate the list BI-2 below.

Let $R \xrightarrow{\varphi} R'$ be a morphism in $\mathcal{L}$. An extension of φ in $\mathcal{L}$ is a morphism $\widetilde{\varphi} : \widetilde{R} \to \widetilde{R}'$, where $R \leq \widetilde{R}$, $R' \leq \widetilde{R}'$, and

$$\delta(1) \circ \varphi = \widetilde{\varphi} \circ \delta(1).$$

In general, the condition of φ being Ψ_i-invariant will not imply that the extension $\widetilde{\varphi}$ is also Ψ_i-invariant. This problem becomes easier if we compare extensions to morphisms which we know to be Ψ_i-invariant *a priori*.

Suppose then that both φ and $\widetilde{\varphi}$ are Ψ_i-invariant. Let also $Q \in \langle R \rangle_S$, $Q' \in \langle R' \rangle_S$, $a \in N_S(R, Q)$ and $b \in N_S(R', Q')$ be such that the morphism

$$\varphi' = \delta(b) \circ \varphi \circ \delta(a^{-1}) : Q \longrightarrow Q'$$

satisfies the equality

$$\delta(b^{-1}\Psi_i(b)) \circ \varphi = \varphi \circ \delta(a^{-1}\Psi_i(a)). \tag{5.9}$$

Note that, in particular, it follows by Lemma 5.1.4 that φ' is Ψ_i-invariant.

In this situation, we can deduce an extension for φ' from the extension $\widetilde{\varphi}$ of φ. Indeed, if we set $\widetilde{\varphi}' = \delta(b) \circ \widetilde{\varphi} \circ \delta(a^{-1})$, then it follows easily that

$$\delta(1) \circ \varphi' = \widetilde{\varphi}' \circ \delta(1).$$

The next lemma states that, under the assumption of $\widetilde{\varphi}$ being Ψ_i-invariant, we can give conditions for $\widetilde{\varphi}'$ to be also Ψ_i-invariant.

Lemma 5.2.19. *In the situation above, the morphism $\widetilde{\varphi}'$ is Ψ_i-invariant if $a^{-1}\Psi_i(a) \in \widetilde{R}$.*

Note that the condition $a^{-1}\Psi_i(a) \in \widetilde{R}$ is equivalent to the condition $b^{-1}\Psi_i(b) \in \widetilde{R}'$.

Proof. By Lemma 5.1.4, we only have to show that

$$\delta(b^{-1}\Psi_i(b)) \circ \widetilde{\varphi} = \widetilde{\varphi} \circ \delta(a^{-1}\Psi_i(a)).$$

Assume then that $a^{-1}\Psi_i(a) \in \widetilde{R}$. Then, axiom (C) yields a commutative square

$$\begin{tikzcd} \widetilde{R} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\delta(a^{-1}\Psi_i(a))"'] & \widetilde{R}' \arrow[d, "\delta(y)"] \\ \widetilde{R} \arrow[r, "\widetilde{\varphi}"'] & \widetilde{R}', \end{tikzcd}$$

where $y = \rho(\widetilde{\varphi})(a^{-1}\Psi_i(a))$. On the other hand, the equality (5.9) above is a restriction of this square, and thus $y = b^{-1}\Psi_i(b)$ and the statement follows by Lemma 5.1.4.

□

Now, in list BI-1 we fixed a finite set of morphisms, but we did not fixed extensions for the morphisms in that set. This is why now we extend BI-1 by adding some more objects and morphisms. For a morphism $f : R \to R'$ in $\mathcal{F}$, set, as usual,

$$N_f = \{g \in N_S(R) | f \circ c_g \circ f^{-1} \in Aut_S(R')\}.$$

Basic Ingredients - 2. In addition to the elements in the lists BI-1, we fix the following elements:

(vi) for each pair R, R' of subgroups in BI-1, and for each $f \in \mathcal{M}_{R,R'}$, an extension $\widetilde{f} : N_f \to S$ in the sense of axiom (II) applied to f in $\mathcal{F}$;

(vii) for each $\widetilde{f}$ as above, the induced morphism $\widetilde{f^\bullet} : N_f^\bullet \to S$, after applying Proposition 1.3.3;

(viii) for each $\widetilde{f^\bullet}$ as above, a lifting $\widetilde{\varphi} : N_f^\bullet \to S$ in $\mathcal{L}$. We choose these liftings such that the following diagram commutes in $\mathcal{L}$:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\delta(1)"'] & R' \arrow[d, "\delta(1)"] \\ N_f^\bullet \arrow[r, "\widetilde{\varphi}"'] & S, \end{tikzcd}$$

where $\varphi \in \widehat{\mathcal{M}}_{R,R'}$ is the lifting of f in $\mathcal{L}$ fixed in BI-1.

This part of the list needs some justification:

- Since the subgroups fixed in BI-1 are all $\mathcal{F}$-centric, there is no problem in applying axiom (II) to the morphisms fixed in that list.
- Furthermore, since the R, R' are $\mathcal{F}$-centric, it follows by Proposition 1.2.6 that so are N_f and $N_f^\bullet$. In particular, $N_f^\bullet \in Ob(\mathcal{L}^\bullet)$.

Note that this still involves only a finite number of items. Note also that the subgroup $N_f^\bullet$ may not coincide with the representative of its S-conjugacy class fixed previously in BI-1, but this is not a problem, since it is clear that there exists a finite M_4 such that, for all $i \geq M_4$, the following holds:

(i) all the subgroups fixed above are S_i-determined,

(ii) all the morphisms fixed above are Ψ_i-invariant, and

(iii) each $N_f^\bullet$ is S_i-conjugate to the corresponding representative of its S-conjugacy class fixed in BI-1.

We can now prove the $\mathcal{H}_i$-saturation of $\mathcal{F}_i$. We will use the equivalent axioms from Proposition 1.7.1. For the sake of a better reading of the proof, we will recall the statement of each axiom before proving it.

Proposition 5.2.20. *For all i, the fusion system $\mathcal{F}_i$ is $\mathcal{H}_i$-saturated.*

Proof. (I') The subgroup $Inn(S_i)$ is a Sylow p-subgroup of $Aut_{\mathcal{F}_i}(S_i)$.

By Proposition 5.2.18, $Aut_{\mathcal{F}_i}(S_i)$ is isomorphic to $A_{\mathcal{F}}(S)_i \leq Aut_{\mathcal{F}}(S)$. Since in BI-1 we have fixed a set $\mathcal{M}_S$ of representatives of all the elements in $Out_{\mathcal{F}}(S)$, it follows that $A_{\mathcal{F}}(S)_i$ fits in an extension

$$Inn(S_i) \longrightarrow A_{\mathcal{F}}(S)_i \longrightarrow Out_{\mathcal{F}}(S).$$

Since $\mathcal{F}$ is saturated, $\{1\} \in Syl_p(Out_{\mathcal{F}}(S))$, and hence

$$Inn(S_i) \in Syl_p(Aut_{\mathcal{F}_i}(S_i)).$$

(II′) Let $f_i \in Hom_{\mathcal{F}_i}(R_i, S_i)$ be such that $R_i' = f_i(R_i)$ is fully $\mathcal{F}_i$-normalized. Then, there exists a morphism $\widetilde{f_i} \in Hom_{\mathcal{F}_i}(\bar{N}_{f_i}, S_i)$ extending f_i, where

$$\bar{N}_{f_i} = \{g \in N_{S_i}(R_i) | f_i \circ c_g \circ f_i^{-1} \in Aut_{S_i}(R_i')\}.$$

This part of the proof will be done by steps.

Let $f_i : R_i \to R_i'$ be such a morphism in $\mathcal{F}_i$, and let $f = (f_i)^\bullet \in A_{\mathcal{F}}(R, R')_i$ and $\varphi \in A_{\mathcal{L}}(R, R')_i$ such that $\rho(\varphi) = f$. Set also $\bar{N} = \bar{N}_{f_i}$ for simplicity, and note that, by Lemma 5.2.12, there is an inclusion $\bar{N} \leq N_f$.

- Case 1. R and R' are subgroups fixed in BI-1.

We claim that there exist $f' \in \mathcal{M}_{R,R'}$, $\varphi' \in \widehat{\mathcal{M}}_{R,R'}$ and $x \in R_i'$ such that

(i) $[f] = [f'] \in Rep_{\mathcal{F}}(R, R')$,

(ii) $N_f = N_{f'}$,

(iii) $\rho(\varphi') = f'$, and

(iv) $\varphi = \delta(x) \circ \varphi'$ in $\mathcal{L}$.

The existence of f' and φ' satisfying (i) and (iii) is clear from BI-1. Thus, by definition of $Rep_{\mathcal{F}}(R, R')$, there exists $y \in R'$ such that

$$f = c_y \circ f'.$$

Furthermore, $N_f = N_{f'}$ since f and f' only differ by conjugation by an element of S. Now, we only have to apply Lemma 4.3 (a) in [BLO07] to see that there exists $x \in R'$ such that

$$\varphi = \delta(x) \circ \varphi'.$$

Since both φ and φ' are Ψ_i-invariant, $x \in R_i'$.

Let now $\widetilde{\varphi}'$ be the extension of φ' fixed in BI-2, and let

$$\widetilde{\varphi} = \delta(x) \circ \widetilde{\varphi}' : N_{f'}^\bullet \longrightarrow S.$$

Again, since $\widetilde{\varphi}'$ and $\delta(x)$ are both Ψ_i-invariant, then so is $\widetilde{\varphi}$. Furthermore, since $N_{f'}^\bullet$ is S_i-determined by BI-2, it follows that so is $Im(\rho(\widetilde{\varphi}))$, $\widetilde{\varphi}$ is a morphism in $\mathcal{L}_i$, and $\rho_i(\widetilde{\varphi})$ is an extension of f_i in the sense of axiom (II).

- Case 2. One (and possibly both) of the subgroups R, R' is not in BI-1.

Since φ is Ψ_i-invariant, by Lemma 5.1.4 it follows that there exist H, H' and $\varphi' \in \widehat{\mathcal{M}}_{H,H'}$ in BI-1, and $a \in N_S(H, R)$, $b \in N_S(H', R')$ such that

(i) $\varphi = \delta(b) \circ \varphi' \circ \delta(a^{-1})$, and

(ii) $\delta(b^{-1}\Psi_i(b)) \circ \varphi' = \varphi' \circ \delta(a^{-1}\Psi_i(a))$.

Let $f' = \rho(\varphi')$, and let $\widetilde{\varphi}' \in Mor_{\mathcal{L}}(N_{f'}^{\bullet}, S)$ be the extension of φ' fixed in BI-2. Then, by Lemma 5.2.19, the morphism

$$\widetilde{\varphi} = \delta(b) \circ \widetilde{\varphi}' \circ \delta(a^{-1}) : a(N_{f'}^{\bullet})a^{-1} \longrightarrow S$$

is also Ψ_i-invariant. This holds since, by (ii) above, $a^{-1}\Psi_i(a) \in N_{f'} \leq N_{f'}^{\bullet}$.

Note that, in particular, $\widetilde{f} = \rho(\widetilde{\varphi})$ is an extension of f in the sense of axiom (II). However, this is not enough to finish the proof, since it does not imply that $\widetilde{\varphi}$ is a morphism in $\mathcal{L}_i$. To finish the proof, we have to show that either $N_f^{\bullet} = aN_{f'}^{\bullet}a^{-1}$ is S_i-determined, or, more generally, that there exists a subgroup $Q \leq N_f^{\bullet}$ such that

(i) $\bar{N} \leq Q$,

(ii) Q is S_i-determined, and

(iii) Q is $\mathcal{F}$-centric.

Note first that $\bar{N}^{\bullet} \in \mathcal{L}$, since $R_i \leq \bar{N}$ and $R = R_i^{\bullet}$ is $\mathcal{F}$-centric. Recall also that by Lemma 5.2.12 there this an inclusion $\bar{N} \leq N_f$, and hence

$$\bar{N}^{\bullet} \leq N_f^{\bullet}.$$

Thus, if $\bar{N}^{\bullet}$ is S_i-determined, we take $Q = \bar{N}^{\bullet}$. Suppose then that $\bar{N}^{\bullet}$ is not S_i-determined, and let $Q_i = \bar{N}^{\bullet} \cap S_i$ and $Q = Q_i^{\bullet}$.

The subgroup Q_i satisfies that $\bar{N} \leq Q_i \leq \bar{N}^{\bullet}$, since $\bar{N} \leq N_f \cap S_i$ by definition, and this proves (i). It follows then by 1.3.2 (ii) and 1.3.3 (iii) that $Q = \bar{N}^{\bullet}$, and thus

$$Q \cap S_i = \bar{N}^{\bullet} \cap S_i = Q_i.$$

Applying now the functor $(\_)^{\bullet}$ above proves (ii). Finally, to see that Q is $\mathcal{F}$-centric, just note that $R_i \leq \bar{N} \leq Q_i$. Hence, since $R = R_i$ is $\mathcal{F}$-centric, so is Q.

This case is then solved by taking the restriction of $\widetilde{\varphi}$ to Q, which is a morphism in $\mathcal{L}_i$.

□

Finally, we check that $\mathcal{L}_i$ satisfies the axioms of a linking system.

Proposition 5.2.21. *For all i, the category $\mathcal{L}_i$ is a linking system associated to $\mathcal{F}_i$.*

Proof. Recall the definition of the functor $\rho_i : \mathcal{L}_i \to \mathcal{F}_i$ in (5.7) and the "distinguished monomorphisms" $\delta_i : R_i \to Aut_{\mathcal{L}_i}(R_i)$ from (5.8). Thus, we have to prove that the axioms in definition 1.4.1 hold for $\mathcal{L}_i$.

(A) The functor ρ_i is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $R_i, R_i' \in \mathcal{L}_i$, $Z(R_i)$ acts freely on $Mor_{\mathcal{L}_i}(R_i, R_i')$ by composition (upon identifying $Z(R_i)$ with $\delta_i(Z(R_i)) \leq Aut_{\mathcal{L}_i}(R_i)$), and ρ_i induces a bijection

$$Mor_{\mathcal{L}_i}(R_i, R_i')/Z(R_i) \xrightarrow{\cong} Hom_{\mathcal{F}_i}(R_i, R_i').$$

By Proposition 5.2.9 (and its corollary), for all $R_i \in \mathcal{L}_i$,

$$Z(R_i) = Z(R) \cap S_i,$$

and thus the free action of $Z(R_i)$ on $Mor_{\mathcal{L}_i}(R_i, R_i')$ follows from the free action of $Z(R)$ on $Mor_{\mathcal{L}}(R, R')$. The isomorphism follows simply by definition of $\mathcal{F}_i$ and $\mathcal{L}_i$.

(B) For each $R_i \in \mathcal{L}_i$ and each $g \in R_i$, ρ_i sends $\delta_i(g) \in Aut_{\mathcal{L}_i}(R_i)$ to $c_g \in Aut_{\mathcal{F}_i}(R_i)$. This is Lemma 5.2.17, proved above.

(C) For each $\varphi \in Mor_{\mathcal{L}_i}(R_i, R_i')$ and each $g \in R_i$, the following square commutes in $\mathcal{L}_i$:

$$\begin{tikzcd} R_i \arrow[r, "\varphi"] \arrow[d, "\delta_i(g)"'] & R_i' \arrow[d, "\delta_i(\rho_i(\varphi)(g))"] \\ R_i \arrow[r, "\varphi"'] & R_i'. \end{tikzcd}$$

This holds by definition of $\mathcal{L}_i$ (as a subcategory of $\mathcal{L}$) and because axiom (C) already holds in $\mathcal{L}$.

□

We would like to prove the saturation of $\mathcal{F}_i$ via Theorem 5.2.14 because that would mean that all $\mathcal{F}_i$-centric $\mathcal{F}_i$-radical subgroups are in the set $\mathcal{H}_i$, on which we have full control. To do so, we still have to show that, for any $\mathcal{F}_i$-centric subgroup $H_i \leq S_i$ which is not in $\mathcal{H}_i$, there is some $H_i' \in \langle H_i \rangle_{\mathcal{F}_i}$ such that

$$Out_{S_i}(H_i') \cap O_p(Out_{\mathcal{F}_i}(H_i')) \neq \{1\}.$$

Let $H_i \leq S_i$ be such an $\mathcal{F}_i$-centric subgroup not in $\mathcal{H}_i$. Then there are two different situations to distinguish:

(a) H_i is not an S_i-root, that is, $(H_i)^\bullet \cap S_i \gneqq H_i$, or

(b) H_i is an S_i-root, that is, $(H_i)^\bullet \cap S_i = H_i$ but $(H_i)^\bullet$ is not $\mathcal{F}$-centric.

Here, the difficult case to treat is (b), but we can "get rid of" (a) quite easily, as we show below.

Proposition 5.2.22. *Let $H_i \leq S_i$ be a $\mathcal{F}_i$-centric subgroup not in $\mathcal{H}_i$ and such that H_i is not an S_i-root. Then, condition (5.4) holds.*

Proof. Let $H = (H_i)^\bullet$. Since H_i is not an S_i-root, if follows that

$$H_i \lneqq H \cap S_i \overset{def}{=} H_i',$$

and there is a natural inclusion

$$Aut_{\mathcal{F}_i}(H_i) \leq Aut_{\mathcal{F}_i}(H_i'),$$

induced by the functor $(\_)^\bullet$ as follows. For each $f_i \in Aut_{\mathcal{F}_i}(H_i)$, let $f = (f_i)^\bullet$ be the unique induced automorphism in $Aut_{\mathcal{F}}(H)$, and let $f_i' = res^H_{H_i'}(f) \in Aut_{\mathcal{F}_i}(H_i')$.

Consider the following subgroup of $Aut_{\mathcal{F}_i}(H_i)$,

$$A = \{c_x \in Aut_{\mathcal{F}_i}(H_i) | x \in N_{H_i'}(H_i)\}.$$

Via the previous inclusion, we can see A as

$$A = Aut_{\mathcal{F}_i}(H_i) \cap Inn(H_i').$$

Furthermore, since $H_i \lneqq H_i'$, it follows that $H_i \lneqq N_{H_i'}(H_i)$, and hence $Inn(H_i) \lneqq A$.

Now, $Aut_{\mathcal{F}_i}(H_i)$, seen as a subgroup of $Aut_{\mathcal{F}_i}(H_i')$, normalizes $Inn(H_i')$. Thus, $A \lhd Aut_{\mathcal{F}_i}(H_i')$, and

$$\{1\} \neq A/Inn(H_i) \leq O_p(Out_{\mathcal{F}_i}(H_i)).$$

Finally, by definition of A, we have $A \leq Aut_{S_i}(H_i)$. Thus, $A/Inn(S_i) \leq Out_{S_i}(H_i)$, and condition (5.4) holds.

□

Proving that condition (5.4) holds for subgroups of S_i which are S_i-roots of non-$\mathcal{F}$-centric subgroups is a considerably more difficult issue. One might consider then to prove saturation of the fusion systems $\mathcal{F}_i$ by other means.

This is, however, not reasonable, since then one would not know whether $\mathcal{H}_i$ contains or not all $\mathcal{F}_i$-centric $\mathcal{F}_i$-radical subgroups. This poses two main problems, the first being that the category $\mathcal{L}_i$ may not contain enough objects to span a whole centric linking system associated to $\mathcal{F}_i$, and the second being that there could not be a natural way to define inclusions $\mathcal{L}_i \hookrightarrow \mathcal{L}_{i+1}$, while there actually is one when $\mathcal{H}_i$ contains all $\mathcal{F}_i$-centric $\mathcal{F}_i$-radical subgroups, as we discuss in the following section.

Somehow it seems then that we are forced to prove saturation of $\mathcal{F}_i$ by applying Theorem 5.2.14. A better understanding of the properties of the objects in $Ob(\mathcal{F}^\bullet) \subseteq Ob(\mathcal{F})$ which are not $\mathcal{F}$-centric is then needed if we are to prove saturation through Theorem 5.2.14.

Definition 5.2.23. *Let $\mathcal{G}$ be a p-local compact group, and let $\{\Psi_i\}$ be a family of unstable Adams operations defined on $\mathcal{G}$. We say that, for some i, $\mathcal{F}_i$ is **5A-saturated** if Theorem A [BCG+05] applies to prove the saturation of $\mathcal{F}_i$ with respect to the set $\mathcal{H}_i$.*

We finish this section by studying the behaviour of families of unstable Adams operations and extensions of p-local compact groups in the sense of A.5.1. Given such an extension, $A \to \mathcal{G} \to \mathcal{G}/A$, a family of unstable Adams operations on the quotient p-local compact group induces a family of unstable Adams operations on the extension and viceversa. This is a rather helpful fact, since, roughly speaking, $\mathcal{G}_i$ will be a p-local finite group if and only if $(\mathcal{G}/A)_i$ is a p-local finite group.

Proposition 5.2.24. *Let $\mathcal{G}$ be a p-local compact group, $A \leq T$ be a $\mathcal{F}$-normal subgroup, and $\mathcal{G}/A$ be the quotient p-local compact group. Then, the following holds:*

(i) A family $\{\Psi_i\}$ of unstable Adams operations on $\mathcal{G}$ induces a family $\{\Psi_i/A\}$ of unstable Adams operations on $\mathcal{G}/A$ such that, for each i, both Ψ_i and Ψ_i/A have the same degree.

(ii) A family $\{\bar{\Psi}_i\}$ of unstable Adams operations on $\mathcal{G}/A$ induces a family $\{\Psi_i\}$ of unstable Adams operations on $\mathcal{G}$ such that, for each i, both $\bar{\Psi}_i$ and Ψ_i have the same degree.

(iii) For all i, there is an extension of transporter systems $A_i \to \mathcal{L}_i \to (\mathcal{L}/A)_i$.

(iv) If for some i $\mathcal{F}_i$ is saturated, then so is $(\mathcal{F}/A)_i$, and

(v) If, for some i, the extension $A_i \to \mathcal{L}_i \to (\mathcal{L}/A)_i$ is admissible and $(\mathcal{F}/A)_i$ is 5A-saturated, then $\mathcal{F}_i$ is 5A-saturated.

Proof. Points (i) and (ii) are a consequence of the extension theory developed in appendix §A, toghether with the fact that the maximal torus of S/A, $\bar{T}$, is the quotient of the maximal torus of S by A, and that any unstable Adams operation respects the maximal torus (in the sense that any rank 1 subtorus is sent to itself by the Adams operation).

Point (iii) follows by construction. Note that, in particular, there is an equality $\mathcal{F}_i/A_i = (\mathcal{F}/A)_i$, where the fusion system on the left part of the equality is the quotient of the fusion system $\mathcal{F}_i$ by A_i, and the fusion system on the right part of the equality is the fusion system defined in 5.2.16.

By Proposition A.1.1, if $\mathcal{F}_i$ is saturated, then so is $\mathcal{F}_i/A_i = (\mathcal{F}/A)_i$, and this proves point (iv). Finally, if the extension in point (iii) is admissible, then by Theorem A.5.5 it follows that $\mathcal{F}_i$ is 5A-saturated.

□

5.3 Some interesting consequences

Let $\mathcal{G}$ be a p-local compact group, and let $\{\Psi_i\}$ be a family of unstable Adams operations defined on $\mathcal{G}$. Some of the properties that we study in this section can be deduced without proving the saturation of the fusion systems $\mathcal{F}_i$, while others will require further assumptions.

Our interest in the family $\{\mathcal{G}_i\}$ goes beyond the individual structure of each $\mathcal{G}_i$: we want also inclusions $\mathcal{G}_i \hookrightarrow \mathcal{G}_{i+1}$ (of p-local finite groups when possible). In this section we will see first that such inclusions always exist, independently from the $\mathcal{G}_i$ being p-local finite groups. Once this will be proved, we will restrict our study to p-local compact groups and families of unstable Adams operations giving rise to a family of p-local finite groups, and further properties will be deduced in this case.

Let us start by describing the inclusions $\mathcal{G}_i \hookrightarrow \mathcal{G}_{i+1}$, regardless of further hypothesis. Define, for all i,

(5.10)
$$\begin{array}{rcl} \mathcal{L}_i & \xrightarrow{\Theta_i} & \mathcal{L}_{i+1} \\ R_i & \longmapsto & (R_i)^\bullet \cap S_{i+1} = R_{i+1} \\ \varphi & \longmapsto & \varphi. \end{array}$$

It is easy then to check that this is a well-defined functor: since $R_i \in \mathcal{H}_i$ is the (unique) S_i-root of an S_i-determined subgroup $R \in Ob(\mathcal{L})$, it follows by construction that $R_{i+1} \in \mathcal{H}_{i+1}$. Furthermore, by definition of $\mathcal{L}_i$, there is an inclusion of sets

$$Mor(\mathcal{L}_i) \subseteq Mor(\mathcal{L}_{i+1}).$$

Furthermore, it is obvious that this functor is faithful.

Note that these inclusion functors do not induce in general commutative squares

$$\begin{tikzcd} \mathcal{L}_i \arrow[r, "\Theta_i"] \arrow[d, "\rho_i"'] \arrow[dr, phantom, "\mathrm{X}"] & \mathcal{L}_{i+1} \arrow[d, "\rho_{i+1}"] \\ \mathcal{F}_i^c \arrow[r, "incl."'] & \mathcal{F}_{i+1}, \end{tikzcd}$$

since, in general, for any $R \in \mathcal{H}_i^\bullet \cong Ob(\mathcal{L}_i)$,

- $\rho_{i+1}(\Theta_i(R)) = R \cap S_{i+1}$, while
- $incl(\rho_i(R)) = R \cap S_i$.

This is, in fact, not much of a problem, since it can be "fixed" as we "fixed" a similar problem which appeared when defining inclusions of connected components of rank 1 p-local compact groups (see Corollary 3.2.31). Let $\theta_i : \mathcal{F}_i^c \to \mathcal{F}_{i+1}$ be the functor induced by Θ_i.

Proposition 5.3.1. *For all i, there is a natural transformation τ_i between the functors $incl : \mathcal{F}_i^c \to \mathcal{F}_{i+1}$ and $\theta_i : \mathcal{F}_i^c \to \mathcal{F}_{i+1}$.*

Proof. The natural transformation is induced by the functor $(\_)^\bullet$ on $\mathcal{F}$. Indeed, define $\tau_i : incl \to \theta_i$ by

$$\tau_i(R_i) = [incl(R_i) = R_i \hookrightarrow R_{i+1} = (R_i)^\bullet \cap S_{i+1}]$$

on objects and by

$$\begin{tikzcd} R_i \arrow[r, "incl"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\tau_i(f_i)"] & R_{i+1} \arrow[d, "f_{i+1}"] \\ R'_i \arrow[r, "incl"'] & R'_{i+1}, \end{tikzcd}$$

for each morphism $f_i : R_i \to R'_i$ in $\mathcal{F}_i^c$, where $f_{i+1} = res^R_{R_{i+1}}((f_i)^\bullet)$.

This is well defined since R_i is the (unique) S_i-root of the S_i-determined subgroup $R \leq S$, which means that

$$(R_i)^\bullet = R.$$

Thus, to check that this is indeed a natural transformation we only have to prove that the previous square is commutative, as it actually is because, by Proposition 1.3.3, the morphism f_i extends to a unique $f = (f_i)^\bullet : (R_i)^\bullet = R \to R' = (R'_i)^\bullet$, which in turn restricts to a unique f_{i+1} between the corresponding S_{i+1}-roots.

□

Thus, we can consider the triple $(incl_{S_i}^{S_{i+1}}, incl_{\mathcal{F}_i}^{\mathcal{F}_{i+1}}, \Theta_i)$ as an inclusion of the triple $\mathcal{G}_i$ into $\mathcal{G}_{i+1}$. We will refer to the whole triple as Θ_i for simplicity. As an easy consequence of the existence of such inclusions, we deduce the following result.

Theorem 5.3.2. *Let $\mathcal{G}$ be a p-local compact group, $\{\Psi_i\}$ be a family of unstable Adams operations acting on $\mathcal{G}$, and $\{\mathcal{G}_i\}$ the family of triples obtained from $\{\Psi_i\}$. Then,*

$$(hocolim\ (|\mathcal{L}_i|)^\wedge_p \xrightarrow{\simeq} B\mathcal{G}.$$

Proof. The statement follows since, as categories, $\mathcal{L} = \cup \mathcal{L}_i$.

□

Next, we study an important property of the subgroups in the set $\mathcal{H}_i$ when seen as objects in $\mathcal{F}_j$, for some $j \geq i$.

Proposition 5.3.3. *Let $R_i \in \mathcal{H}_i$. Then, the following holds:*

(i) R_i is $\mathcal{F}_i$-centric,

(ii) R_i *is* $\mathcal{F}$*-quasicentric, and*

(iii) R_i *is* $\mathcal{F}_j$*-quasicentric for all* $j \geq i$*.*

In fact, the above result is just a particular case of the following more general proposition.

Proposition 5.3.4. *Let* $R \leq S$ *be* $\mathcal{F}$*-quasicentric and* S_i*-determined for some* i*. Then,*

(i) R_i *is* $\mathcal{F}$*-quasicentric, and*

(ii) R_i *is* $\mathcal{F}_j$*-quasicentric for all* $j \geq i$*.*

Proof. First, we claim that, for all $Q_i \in \langle R_i \rangle_{\mathcal{F}}$, there are equalities

$$C_S(Q_i) = C_S(Q),$$

where $Q = Q_i^{\bullet}$. Indeed, for R_i this holds directly by Proposition 5.2.9, since by hypothesis R is S_i-determined. Also, since R is S_i-determined, it follows by Lemma 5.2.7 (ii) that R_i contains a set of representatives of the elements in R/T_R, namely $\bar{R}$, where T_R is the maximal torus of R. Thus, if we set $H_R = \langle \bar{R} \rangle$, we can write

$$R = H_R \cdot T_R \quad \text{and} \quad R_i = H_R \cdot (T_R)_i.$$

Let then $Q_i \in \langle R_i \rangle_{\mathcal{F}}$, let $Q = Q_i^{\bullet}$, and let $f \in Iso_{\mathcal{F}}(R_i, Q_i)$. Then, using Proposition 1.3.3 and the infinitely p-divisibility of T_R,

$$Q = f(H_R) \cdot T_Q \quad \text{and} \quad Q_i = f(H_R) \cdot (T_Q)_i,$$

from where it follows that

$$C_S(Q_i) = C_S(f(H_R)) \cap C_S((T_Q)_i) = C_S(f(H_R)) \cap C_S(T_Q),$$

since the set $\{T_Q \mid Q \in \langle R \rangle_{\mathcal{F}}\}$ is finite.

Next, we claim that both $C_{\mathcal{F}_i}(R_i)$ and $C_{\mathcal{F}}(R_i)$ can be indentified with certain subcategories of $C_{\mathcal{F}}(R)$ (and the same happens for each $Q_i \in \langle R_i \rangle_{\mathcal{F}_i}$, with respect to $Q = (Q_i)^{\bullet}$).

Indeed, let C be either $C_{\mathcal{F}}(R_i)$ or $C_{\mathcal{F}_i}(R_i)$ for simplicity, and note that, since $C_S(R_i) = C_S(R)$, the Sylow p-subgroup of C will certainly be a subgroup of the Sylow p-subgroup of $C_{\mathcal{F}}(R)$. Let then $f : H \to H'$ a morphism in C. Then, by definition of the centralizer fusion subsystem, there is a morphism $\widetilde{f} : H \cdot R_i \to H' \cdot R_i$ in C and a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{f} & H' \\ \downarrow & & \downarrow \\ H \cdot R_i & \xrightarrow{\widetilde{f}} & H' \cdot R_i \\ \uparrow & & \uparrow \\ R_i & = & R_i, \end{array}$$

where the vertical arrows are inclusions. By applying the functor $(\_)^\bullet$ to this diagram, one then gets a new commutative diagram

$$\begin{tikzcd} H^\bullet \arrow[r, "f"] \arrow[d] & (H')^\bullet \arrow[d] \\ (H \cdot R_i)^\bullet \arrow[r, "\widetilde{f}"] & (H' \cdot R_i)^\bullet \\ R \arrow[u] \arrow[r, equal] & R, \arrow[u] \end{tikzcd}$$

from where it follows that $f : H \to H'$ is also a morphism in $C_{\mathcal{F}}(R)$, since $H \cdot R \leq (H \cdot R_i)^\bullet$ and $H' \cdot R \leq (H' \cdot R_i)^\bullet$. Note also that all these arguments are still valid if we change R_i by any other $Q_i \in \langle R_i \rangle_{\mathcal{F}}$.

This, together with the natural inclusion of categories $C_{\mathcal{F}}(R) \subseteq C_{\mathcal{F}}(R_i)$ implies that

$$C_{C_S(R)}(C_S(R)) = C_{\mathcal{F}}(R) = C_{\mathcal{F}}(R_i),$$

where the first equality holds since, by hypothesis, R is quasi-centric. Since the same holds for any $Q_i \in \langle R_i \rangle_{\mathcal{F}}$, this proves point (i). Point (ii) for $j = i$ also follows easily from

$$C_{\mathcal{F}_i}(R_i) \subseteq C_{\mathcal{F}}(R) = C_{\mathcal{F}}(R_i) = C_{C_S(R)}(C_S(R)).$$

Point (ii) in general now follows since we can deduce equalities

$$C_{\mathcal{F}_j}(R_i) = C_{\mathcal{F}_j}(R_j) = C_{C_{S_j}(R_j)}(C_{S_j}(R_j))$$

using the same arguments as above, and the same holds for any $Q_i \in \langle R_i \rangle_{\mathcal{F}_j}$.

□

Definition 5.3.5. *We say that the family $\{\Psi_i\}$* ***induces an approximation of $\mathcal{G}$ by p-local finite groups*** *if, for all i,*

(i) $\mathcal{G}_i$ is a p-local finite group, and

(ii) each $R_i \in \mathcal{F}_i^{cr}$ is $\mathcal{F}_{i+1}$-quasicentric.

Clearly, by Proposition 5.3.3, if $\mathcal{G}_i$ is 5A-saturated for all i, then $\{\Psi_i\}$ induces an approximation of $\mathcal{G}$ by p-local finite groups in the sense of definition above. Condition (ii) above is the lightest condition that we can ask for in order to have inclusions $\mathcal{G}_i \to \mathcal{G}_{i+1}$ for all i. Indeed, by Theorem B in [BCG$^+$07], if $\mathcal{G}_{i+1}$ is a p-local finite group, then there exists a unique quasicentric linking system $\mathcal{L}_{i+1}^q$ containing $\mathcal{L}_{i+1}$ as a full subcategory and such that the inclusion of categories induces

$$|\mathcal{L}_{i+1}|_p^\wedge \simeq |\mathcal{L}_{i+1}^q|_p^\wedge.$$

This, way, there are faithful functors

$$\mathcal{L}_i^{cr} \longrightarrow \mathcal{L}_{i+1}^q, \tag{5.11}$$

which can be considered as inclusions of p-local finite groups. Note that this inclusion functors do induce commutative diagrams

$$\begin{tikzcd} \mathcal{L}_i^{cr} \arrow[r, "\Xi_i"] \arrow[d, "\rho_i"'] & \mathcal{L}_{i+1}^q \arrow[d, "\rho_{i+1}"] \\ \mathcal{F}_i^{cr} \arrow[r, "incl"'] & \mathcal{F}_{i+1}^q. \end{tikzcd}$$

The following result is obvious.

Theorem 5.3.6. *Let $\mathcal{G}$ be a p-local compact group, $\{\Psi_i\}$ be a family of unstable Adams operations inducing an approximation of $\mathcal{G}$ by p-local finite groups, then there is a homotopy equivalence*

$$(hocolim\ (|\mathcal{L}_i|, |\Xi_i|))^\wedge_p \xrightarrow{\simeq} B\mathcal{G}.$$

The notation $(|\mathcal{L}_i|, |\Xi_i|)$ is used here to mark the difference with the homotopy colimit from Theorem 5.3.2, where the spaces are the same, but the maps are not. In fact, as we next discuss, there is no need of this difference in the notation.

Let $\mathcal{G}$ be a p-local compact group, and let $\{\Psi_i\}$ be a family of unstable Adams operations inducing an approximation of $\mathcal{G}$ by p-local finite groups. Consider also, for all i, the diagram

(5.12)
$$\begin{tikzcd} \mathcal{L}_i^{cr} \arrow[r, "\Xi_i"] \arrow[dr, "\Theta_i"'] & \mathcal{L}_{i+1}^q \\ & \mathcal{L}_{i+1}, \arrow[u] \end{tikzcd}$$

where Ξ_i is the natural inclusion from (5.11), the vertical arrow is the inclusion from Theorem B [BCG+05], and Θ_i is the inclusion functor defined in (5.10). Let also $\widehat{\Theta}_i$ be the composition

$$\widehat{\Theta}_i : \mathcal{L}_i^{cr} \xrightarrow{\Theta_i} \mathcal{L}_{i+1} \hookrightarrow \mathcal{L}_{i+1}^q.$$

The diagram (5.12) need not be commutative in general, but we can easily prove the following.

Proposition 5.3.7. *For all i, there is a homotopy equivalence $|\widehat{\Theta}_i| \simeq |\Xi_i|$.*

Proof. We will see that there is a natural transformation Υ_i between Ξ_i and $\widehat{\Theta}_i$, thus proving the statement. Indeed, define

$$\Upsilon_i(R_i) = [R_i = incl(R_i) \overset{\delta(1)}{\hookrightarrow} (\widehat{\Theta})_i(R_i) = R_{i+1}]$$

on objects and

$$\begin{tikzcd} R_i \arrow[r, "\delta(1)"] \arrow[d, "\varphi_i"'] \arrow[dr, phantom, "\Upsilon_i(\varphi_i)"] & R_{i+1} \arrow[d, "\varphi_{i+1}"] \\ R_i' \arrow[r, "\delta(1)"'] & R_{i+1}', \end{tikzcd}$$

for each morphism $\varphi_i \in Mor(\mathcal{L}_i)$, where, in fact, $\varphi_{i+1} = \varphi_i$ as morphisms in $\mathcal{L}$ (the reader may think of φ_i and φ_{i+1} as the "restriction" of φ to R_i and R_{i+1}, although neither R_i nor R_{i+1} need to be objects in $\mathcal{L}$). Thus, we have to show that the above square is commutative, which is, in fact, obvious by definition of $\mathcal{L}_{i+1}$.

□

An interesting consequence of having an approximation of a p-local compact group $\mathcal{G}$ by p-local finite groups is the analog of the Stable Elements theorem (Theorem 5.8 in [BLO03b]) for $\mathcal{G}$, which determines the mod p cohomology of $\mathcal{G}$ in terms of the cohomology of the fusion system $\mathcal{F}$ as a subring of the mod p cohomology of BS.

Such an statement cannot be proved in the compact case following the same arguments as in the finite case, but there is a rather easy proof in the case $\mathcal{G}$ admits an approximation by p-local finite groups. We state below the Stable Elements theorem for p-local compact groups.

Theorem 5.3.8. *Let $\mathcal{G}$ be a p-local compact group. Then, the natural map*

$$H^*(|\mathcal{L}|;\mathbb{F}_p) \xrightarrow{\cong} H^*(\mathcal{F}) \overset{def}{=} \varprojlim_{O(\mathcal{F}^c)} H^*(\_;\mathbb{F}_p) \subseteq H^*(BS;\mathbb{F}_p)$$

is an isomorphism, and $H^*(B\mathcal{G};\mathbb{F}_p) = H^*(|\mathcal{L}|;\mathbb{F}_p)$ is noetherian.

Let then $\mathcal{G}$ be a p-local compact group, and let $\{\mathcal{G}_i\}$ be an approximation of $\mathcal{G}$ by p-local finite groups, induced by a family of unstable Adams operations $\{\Psi_i\}$. Let also $\Theta_i : \mathcal{L}_i \hookrightarrow \mathcal{L}_{i+1}$ be the inclusion functors defined in (5.10). We start by proving that, in this particular case, the functor $H^*(\_;\mathbb{F}_p)$ commutes with the homotopy colimit from Theorem 5.3.2 (or Theorem 5.3.6).

Proposition 5.3.9. *The functors Θ_i induce natural isomorphisms*

$$H^*(BS;\mathbb{F}_p) \cong \varprojlim H^*(BS_i;\mathbb{F}_p) \quad \textit{and} \quad H^*(B\mathcal{G};\mathbb{F}_p) \cong \varprojlim H^*(B\mathcal{G}_i;\mathbb{F}_p).$$

Proof. Let X be $B\mathcal{G}$ (respectively BS), and for each i let also X_i be $B\mathcal{G}_i$ (respectively BS_i). Consider also the homotopy colimit spectral sequence for cohomology (which, for coefficients in $\mathbb{F}_p$, is dual to the Bousfield-Kan spectral sequence defined in XII.5.7 in [BK72]):

$$E_2^{r,s} = \varprojlim{}^r H^s(X_i;\mathbb{F}_p) \Longrightarrow H^{r+s}(X;\mathbb{F}_p).$$

We will see that, for $r \geq 1$, $E_2^{r,s} = \{0\}$, which, in particular, will imply the statement.

For each s, let $H_i^s = H^s(X_i;\mathbb{F}_p)$, and let F_i be the induced morphism in cohomology (in degree s) by the map $|\Theta_i|$ (respectively $Bincl : BS_i \to BS_{i+1}$).

Now, $H^*(X_i;\mathbb{F}_p)$ is noetherian by Theorem 5.8 [BLO03b], and it follows in particular that H_i^s is an $\mathbb{F}_p$-vector space of finite dimension. It is clear then that the inverse system $\{H_i^s; F_i\}$ satisfies the Mittag-Leffler condition (see 3.5.6 in [Wei94]), and as a consequence the higher limits $\varprojlim^r H_i^s$ all vanish for $r \geq 1$. This in turn implies that all the differentials in the above spectral sequence are trivial, and the spectral sequence collapses.

□

Proof. (of Theorem 5.3.8). We omit the coefficients $\mathbb{F}_p$ in all cohomology rings for simplicity. Since, by hypothesis, $\mathcal{G}_i$ is a p-local finite group for each i, we can apply the Stable Elements theorem (Theorem 5.8 in [BLO03b]) to it: there is a natural isomorphism

$$H^*(|\mathcal{L}_i|;\mathbb{F}_p) \xrightarrow{\cong} H^*(\mathcal{F}_i) \stackrel{def}{=} \varprojlim_{\mathcal{O}(\mathcal{F}_i^c)} H^*(\_;\mathbb{F}_p) \subseteq H^*(BS_i;\mathbb{F}_p)$$

In particular, for each i, $H^*(\mathcal{F}_i) \subseteq H^*(BS_i)$. Thus, by the above isomorphisms, together with Proposition 5.3.9, it follows that

$$H^*(B\mathcal{G}) \cong \varprojlim H^*(B\mathcal{G}_i) \cong \varprojlim H^*(\mathcal{F}_i) \subseteq \varprojlim H^*(BS_i) \cong H^*(BS).$$

Furthermore, there are natural inclusions $\mathcal{O}(\mathcal{F}_i^c) \subseteq \mathcal{O}(\mathcal{F}_{i+1}^c)$ (induced by the functor $(\_)^\bullet$ in the same way as it induces the inclusion functors Θ_i in (5.10)) such that, as categories,

$$\mathcal{O}(\mathcal{F}) \cong \varinjlim \mathcal{O}(\mathcal{F}_i^c).$$

Thus, it follows that

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)} H^*(\_;\mathbb{F}_p) \cong \varprojlim_i \varprojlim_{\mathcal{O}(\mathcal{F}_i^c)} H^*(\_;\mathbb{F}_p).$$

Finally, since $H^*(BS)$ is noetherian by Proposition 12.1 [DW98], it follows then that so is $H^*(B\mathcal{G};\mathbb{F}_p)$.

□

5.4 Examples

In this section we study several examples of p-local compact groups for whom families of unstable Adams operations give rise to approximations by p-local finite groups. Since the ideas used to study each example differ significantly from the ohers, we treat each example in separate sections.

The first example we will study is probably the most important one: we will see that all rank 1 p-local compact groups can be approximated by p-local finite groups. The second example treats the p-local compact groups induced by the linear torsion groups $GL_n(\bar{\mathbb{F}}_q)$. The third example studies the 3-local compact groups which one obtains as limits of families of 3-local finite groups in [DRV07].

The examples in this section are also to be understood as a list of ideas to prove the existence of approximations of p-local compact groups by p-local finite groups through unstable Adams operations. One open question then is whether we can find a unifying argument to explain all the examples shown here.

5.4.1 Rank 1 p-local compact groups

To study these examples, we will make strong use of Theorem 3.2.1, and the explicit descriptions of the connected components of p-local compact groups of rank 1 that we have given in chapter §3. The main result of this example is the following.

Theorem 5.4.1. *Let $\mathcal{G}$ be a p-local compact group of rank* 1*, and let* $\{\Psi_i\}$ *be a family of unstable Adams operations on $\mathcal{G}$. Then, this family induces an approximation of $\mathcal{G}$ by p-local finite groups.*

This theorem will be proved as Propositions 5.4.4, 5.4.16 and 5.4.17, by checking that for i big enough, the fusion systems $\mathcal{F}_i$ are 5A-saturated, in a case-by-case argument, depending on the connected component of $\mathcal{G}$.

First, we study the problem for connected rank 1 p-local compact groups.

Proposition 5.4.2. *Let $\mathcal{G}$ be a connected p-local compact group of rank* 1*. Then, for any family of unstable Adams operation* $\{\Psi_i\}$*, there exists some M such that, for all $i \geq M$, $\mathcal{F}_i$ is 5A-saturated and $\mathcal{G}_i$ is a p-local finite group.*

Proof. For each connected p-local compact group $\mathcal{G}$ (in the list of Theorem 3.2.1) and each family of Adams operations $\{\Psi_i\}$ defined on $\mathcal{G}$, we have to prove that there exists some M such that, for all $i \geq M$, the finite fusion system $\mathcal{F}_i$ is 5A-saturated. Recall the statement of Theorem 3.2.1: there are only three cases to check, that is

(i) $\mathcal{G} = (T, \mathcal{F}_T(S^1), \mathcal{L}_T(S^1))$,

(ii) $\mathcal{G} = (D_{2^\infty}, \mathcal{F}_S(SO(3)), \mathcal{L}_S(SO(3)))$, and

(iii) $\mathcal{G} = (Q_{2^\infty}, \mathcal{F}_S(S^3), \mathcal{L}_S(S^3))$,

where the last two cases only happen for $p = 2$.

(i) In this case it is obvious that $\mathcal{G}_i$ is a p-local finite group, since $\mathcal{G}_i = (T_i, \mathcal{F}_{T_i}(T_i), \mathcal{L}_{T_i}(T_i))$, and $\mathcal{H}_i = \{T_i\}$ contains all $\mathcal{F}_i$-centric subgroups.

(ii) To simplify notation, set $S = D_{2^\infty}$, $\mathcal{F} = \mathcal{F}_S(SO(3))$ and $\mathcal{L} = \mathcal{L}_S(SO(3))$. Also, remember the set of representatives of the S-conjugacy classes of subgroups in $Ob(\mathcal{F}^\bullet)$ which we listed in (3.4):

$$\{\{1\}, T_1, T_2, T, C, T_1 \times C, T_2 \rtimes C, T \rtimes C = S\},$$

where only T, $T_1 \times C$, $T_2 \rtimes C$ and S represent S-conjugacy classes of $\mathcal{F}$-centric subgroups. Thus, the following is easily seen to hold. If $R \in Ob(\mathcal{F}^\bullet)$ is S-conjugate to a subgroup in the above list which is not $\mathcal{F}$-centric, then

$$C_T(R) \setminus (Z(R) \cap T) \neq \emptyset.$$

Now, by applying Lemma 5.4.3 below, it follows that there exists some M such that, for all $i \geq M$, if $R \in Ob(\mathcal{F}^\bullet)$ is S_i-determined but not $\mathcal{F}$-centric, then R_i is not $\mathcal{F}_i$-centric. Hence, Theorem 5.2.14 (Theorem A in [BCG+05]), together with Proposition 5.2.22, applies to show that $\mathcal{F}_i$ is saturated.

(iii) This case holds because of Proposition 5.2.24 and because in this case $\mathcal{G}$ is an admissible extension of the 2-local compact group in (ii) by $\mathbb{Z}/2$ by Corollary 3.2.2. □

Lemma 5.4.3. *Suppose $R \in Ob(\mathcal{F}^\bullet)$ is such that $(Z(R) \cap T) \lneqq C_T(R)$. Then, there exists M such that, for all $i \geq M$, if $Q \in \langle R \rangle_S$ is S_i-determined then*

$$(Z(Q_i) \cap T_i) \lneqq C_{T_i}(Q_i).$$

Proof. For such a subgroup R, consider the set

$$\mathcal{C}_T = \{C_T(Q) | Q \in \langle R \rangle_S\}.$$

Since $Aut_{\mathcal{F}}(T)$ is finite, so is this set. The lemma is proved then by taking M such that, for all $i \geq M$, $C_T(Q) \in \mathcal{C}_T$ contains a set of representatives of the elements in $C_T(Q)/T'$, where T' is the maximal torus of $C_T(Q)$. □

We now study the general case. Let $\mathcal{G}$ be a rank 1 p-local compact group, and let $\mathcal{G}_0$ be its connected component. The proof will depend on the homotopy type of $\mathcal{G}_0$. The case where $\mathcal{G}_0$ is the p-local compact group induced by S^1 is, as usual, easy to solve.

Proposition 5.4.4. *Let $\mathcal{G}$ be a rank 1 p-local compact group whose connected component has the homotopy type of $(BS^1)^\wedge_p$. Then, there exists some M such that, for all $i \geq M$, $\mathcal{F}_i$ is 5A-saturated.*

Proof. Suppose first that $T \leq Z(S)$, and recall that this is always the case for $p > 2$, by Lemma 3.2.6. Furthermore, in this case the extension $T \to \mathcal{G} \to \mathcal{G}/T$ is admissible by Proposition 3.2.7 (this proposition was stated for p odd, but it is clear that, under the assumption of $T \leq Z(S)$, the statement was also valid for $p = 2$). We can now apply Proposition 5.2.24 to this extension: by 5.2.24 (i), the family $\{\Psi_i\}$ of operations defined on $\mathcal{G}$ induces a family $\{\Psi_i/T\}$ of operations on $\mathcal{G}/T$, and since $\mathcal{G}/T$ is a p-local finite group, it follows that there exists some m_0 such that, for all $i \geq m_0$, Ψ_i/T is the identity on $\mathcal{G}/T$. In particular, for $i \geq m_0$, $(\mathcal{G}/T)_i = \mathcal{G}/T$ is a p-local finite groups whose fusion system is 5A-saturated. Thus, by 5.2.24 (iii) and (v), for all $i \geq m_0$ the fusion system $\mathcal{F}_i$ is 5A-saturated since the extension $T_i \to \mathcal{L}_i \to \mathcal{L}/T$ is admissible.

Suppose now that T is not central in S (this means, in particular, that $p = 2$), and recall that, by Lemma 3.2.6 together with (3.2), there is some $x \in S$ such that $xtx^{-1} = t^{-1}$, for all $t \in T$.

Even if this is the case, we know that T is normal in $\mathcal{F}$ by Corollary 3.2.4, and hence all $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups have rank 1. Now, since S has rank 1, it follows that there are finitely many subgroups of S of rank 1.

Note that this means that there exists some m_1 such that, for all $i \geq m_1$, all the subgroups $R \in Ob(\mathcal{F}^\bullet)$ of rank 1 are S_i-determined, and, furthermore, R is $\mathcal{F}$-centric if and only if R_i is $\mathcal{F}_i$-centric.

Thus, we only have to deal with the finite subgroups of S in $Ob(\mathcal{F}^\bullet)$. Now, since this set contains finitely many S-conjugacy classes, there is some m_2 such that, for all $i \geq m_2$ and all $R \in Ob(\mathcal{F}^\bullet)$ which has rank 0 and is S_i-determined, $R_i \cap T_i \lneqq T_i$. In fact, note that if R is finite and S_i-determined, then $R = R_i$.

We may assume that m_2 is also big enough such that, for all $i \geq m_2$, Lemma 5.4.3 applies. Let then H be a finite S_i-determined subgroup. If there is no $y \in H$ acting nontrivially on T, then $C_T(H) = H$, and hence by Lemma 5.4.3 $C_{T_i}(H) \gneqq H$, and H is not $\mathcal{F}_i$-centric.

Assume otherwise that there is some $y \in H$ acting nontrivially on T, and let $T_n = H \cap T$. It is clear that if $n = 0$, then H is not $\mathcal{F}_i$-centric, since $T_1 \leq C_{T_i}(H)$. Thus, suppose $n \geq 1$, and note that, since y acts by sending $t \in T$ to t^{-1}, it follows that $T_{n+1} \leq N_{S_i}(H)$.

We claim that in this case H satisfies condition (5.4). Indeed, let

$$A \stackrel{def}{=} \langle Inn(H), Aut_{T_i}(H) \rangle,$$

where $Aut_{T_i}(R_i) \leq Aut_{\mathcal{F}_i}(R_i)$ is the subgroup of conjugations by elements of T_i which send R_i to itself. It is clear that $Inn(H) \lneqq A$, since conjugation by t_{n+1} is not an automorphism in $Inn(H)$, and we prove below that $A \lhd Aut_{\mathcal{F}_i}(H)$.

The group $Aut_{\mathcal{F}_i}(H)$ is by definition spanned by compositions of restrictions of subgroups R_i such that $R = (R_i)^\bullet$ has rank 1, and the groups $Aut_{\mathcal{F}_i}(R_i)$ which span it are all projections (through the functor ρ_i) of the groups $Aut_{\mathcal{L}_i}(R_i)$, which can be considered as subgroups of $Aut_{\mathcal{L}}(R)$.

Now, $T \leq R$ means that T can be seen also as a subgroup of $Aut_{\mathcal{L}}(R)$ above, and since T is $\mathcal{F}$-normal, it means that, for all $\varphi \in Aut_{\mathcal{L}}(R)$, conjugation by φ in $Aut_{\mathcal{L}}(R)$ sends T to itself. Hence, $A \lhd Aut_{\mathcal{F}_i}(H)$, and thus condition (5.4) holds on H.

We have then proved that for all $i \geq M = m_2$, $\mathcal{F}_i$ is 5A-saturated.

□

Let now $\mathcal{G}$ be a 2-local compact group whose connected component $\mathcal{G}_0$ is the 2-local compact group induced by $SO(3)$. This is, as usual, the most difficult to deal with. In this case, we will show that there exists some M such that, for all $i \geq M$, if $R_i \leq S_i$ is an S_i-root but $R = (R_i)^\bullet$ is not $\mathcal{F}$-centric, then either condition (5.4) holds for R_i or R_i cannot be $\mathcal{F}_i$-centric. This, together with Proposition 5.2.22, will imply the 5A-saturation of $\mathcal{F}_i$.

We start by proving several technical results that will be used later on. First, we show an interesting property which will allow us to "detect" S_i-determined subgroups. In fact, this property holds for all rank 1 p-local compact groups, regardless of their connected component, but we will used it in this particular case and hence it seemed more convenient to place it here. After this general property we will prove next a property of the p-group S/T when $S_0 \cong D_{2^\infty}$ which will make calculations easier later on.

Lemma 5.4.5. *Let $R \in Ob(\mathcal{F}^\bullet)$ be an S_i-determined subgroup. There exists some M_R such that, for all $i \geq M_R$, if $Q \in \langle R \rangle_S$ is S_i-determined, then $Q_i \in \langle R_i \rangle_S$.*

Proof. Let R be such a subgroup. Then, since S has rank 1, it follows that either R contains the whole T or R is finite. In the first case, the conjugacy class of R in $\mathcal{F}$ contains finitely many subgroups, and hence the statement follows easily.

Suppose then that R is finite. Then, for any $Q \in \langle R \rangle_S$, Q is S_i-determined if and only if $Q = Q_i$, and again the statement is easily seen to hold.

□

As a consequence, we can now "detect" S_i-determined subgroups. The following result can be regarded as a particular case of Lemma 5.1.3.

Lemma 5.4.6. *Let $R \in Ob(\mathcal{F}^\bullet)$, and let M_R be as in Lemma 5.4.5. Then, $Q \in \langle R \rangle_S$ is S_i-determined if and only if for all $y \in N_S(R_i, Q_i)$,*

$$y^{-1}\Psi_i(y) \in C_T(R).$$

Note that the condition above is equivalent to $\Psi_i(y)y^{-1} \in C_T(Q)$.

Proof. Suppose first that Q is S_i-determined. Then, by Lemma 5.4.5, $Q_i \in \langle R_i \rangle_S$. Let $y \in N_S(R_i, Q_i)$, and note that, by Proposition 5.2.11, $a \in N_S(R, Q)$. Now, for each $g \in R_i$, let $h = ygy^{-1} \in Q_i$. Since both g and h are Ψ_i-invariant, we have $h = \Psi_i(y)g\Psi_i(y^{-1})$, and hence

$$g(y^{-1}\Psi_i(y)) = (y^{-1}\Psi_i(y))g.$$

Thus, $y^{-1}\Psi_i(y) \in C_T(R_i) = C_T(R)$ by Proposition 5.2.9.

Assume now that, for all $y \in N_S(R_i, Q_i)$, $y^{-1}\Psi_i(y) \in C_T(R)$, and fix such an element y. For each $g \in R_i$, let $h = ygy^{-1}$, and write $g = y^{-1}hy$. Since g is Ψ_i-invariant, and $\Psi_i(y)y^{-1} \in C_T(Q)$, it is easy to check now (by applying Ψ_i on the latter equality) that h is also Ψ_i-invariant. Proposition 1.3.3 finishes the proof.

□

The following result is a property of 2-local compact groups whose connected component is the 2-local compact group induced by $SO(3)$.

Lemma 5.4.7. *Let $\mathcal{G}$ be a 2-local compact group whose connected component $\mathcal{G}_0$ is the 2-local compact group induced by $SO(3)$, let S and S_0 be the corresponding Sylow subgroups, and let T be the maximal torus of S. Consider also the group extension*

$$1 \to T \longrightarrow S \longrightarrow S/T \to 1,$$

and let $\omega : S/T \to Aut(T)$ be the morphism which realizes the action of S/T on T, and let $K = Ker(\omega)$. Then,

$$S/T = S_0/T \times K.$$

Proof. Since $S_0 \cong D_{2^\infty}$, we know that S_0 already contains a nontrivial element x which acts nontrivially on T, and hence $|S/T| = |S_0/T| \cdot |K|$. Furthermore, by definition of S_0/T and K, it is clear that

$$S_0/T \cap K = \{1\}.$$

Finally, the elements in K commute with the elements in S_0/T since $S_0/T \cong \mathbb{Z}/2$ is a normal subgroup of S/T.

□

Consider then the group extension $1 \to S_0 \to S \to K \to 1$. Let R be any subgroup of S, and let i be big enough such that $S_i/T_i = S/T$. Let also $R_i = R \cap S_i$ as usual, and let $(R_0)_i = R_i \cap S_0$. Then, R_i fits in an extension

$$1 \to (R_0)_i \longrightarrow R_i \longrightarrow R_i/(R_0)_i = R/R_0 \to 1, \tag{5.13}$$

where $R/R_0 \le K$ and hence acts trivially on T.

Lemma 5.4.8. *There exists some k such that, if $T_k \le R$, then $T \le R^\bullet$. As a consequence, there exists m_1 such that, for all $i \ge m_1$, the following holds:*

(i) $T_k \le S_i$, and

(ii) if R is S_i-determined and $T_k \le R_i$, then $T \le R$.

Furthermore, we can choose m_1 such that, for all $i \ge m_1$, R is $\mathcal{F}$-centric if and only if R_i is $\mathcal{F}_i$-centric.

Proof. The existence of such k is obvious, since $T = \cup T_i$ and $Ob(\mathcal{F}^\bullet)$ contains finitely many S-conjugacy classes of objects: we just have to take k such that

$$(T_k)^\bullet = T.$$

Thus, the existence of m_0 satisfying (i) and (ii) is also clear since $S = \cup S_i$ and, if R is S_i-determined, then

$$T = (T_k)^\bullet \leq (R_i)^\bullet = R.$$

The last part of the statement is easy to prove as well, since S contains finitely many subgroups R of rank 1, and hence we can consider m_1 to be big enough so that any property that we want holds on R.

□

This lemma provides a first reduction. Indeed, we can now consider only subgroups R which are S_i-determined and such that $R_i \cap T_i \gneqq T_k$, for k as in Lemma 5.4.8. Note that, such a subgroups R is S_i-determined if and only if $R = R_i$.

Let then R be such an S_i-determined subgroup, let also $(R_0)_i = R_i \cap S_0$. Then, $(R_0)_i$ is in the S_0-conjugacy class on one of the subgroups in the following list

$$\{\{1\}, T_n, \langle x\rangle, T_n \rtimes \langle x\rangle\}, \tag{5.14}$$

where $n < k$ (this can be easily seen from the study that we have already made on rank 1 2-local compact groups in chapter 3). Depending on the S_0-conjugacy class of $(R_0)_i$ (within the above list), we will now show that there exists some M such that, for all $i \geq M$, if R_i is an S_i-root but $R = (R_i)^\bullet$ is not $\mathcal{F}$-centric, then either R_i is not $\mathcal{F}_i$-centric or it satisfies condition (5.4).

Lemma 5.4.9. *There exists some m_2 such that, for all $i \geq m_2$, if $(R_0)_i$ is in the S-conjugacy class of $\{1\}$, T_n or $\langle x\rangle$, then R_i is not $\mathcal{F}_i$-centric.*

Proof. Since $n < k$ and $T_k \leq S_i$, in all three cases it follows that

$$C_T(R) \setminus (Z(R) \cap T) \neq \emptyset,$$

and the statement follows then by Lemma 5.4.3.

□

Let $\mathcal{F}_0$ be the saturated fusion system over S_0 induced by $SO(3)$, and note that the subgroups $\{1\}$, T_n and $\langle x\rangle$ are the non-$\mathcal{F}_0$-centric subgroups within the list (5.14). To deal with the $\mathcal{F}_0$-centric subgroups within that list, we need the following result.

Lemma 5.4.10. *If $(R_0)_i$ is $\mathcal{F}_0$-centric, then*

$$N_{S_0}((R_0)_i) \leq N_{S_i}(R_i).$$

Proof. If $(R_0)_i$ is $\mathcal{F}_0$-centric, then $(R_0)_i \in \langle T_n \rtimes \langle x\rangle\rangle_{S_0}$ for some (finite) n. Assume for simplicity that $(R_0)_i = T_n \rtimes \langle x\rangle$. Thus, by (5.13), it follows that

$$T_{n+1} \rtimes \langle x\rangle = N_{S_0}((R_0)_i) \leq N_{S_i}(R_i).$$

□

Let $R_i \leq S_i$ be any subgroup, and set $H = (R_0)_i$ for simplicity. Then, there is an exact sequence (2.8.7 in [Suz82])

$$(5.15) \qquad 0 \to H^1(R_i/H; Z(H)) \to Aut_{\mathcal{F}_i}(R_i)/Aut_H(R_i) \stackrel{\Phi_{R_i}}{\to} Aut_{\mathcal{F}_i}(H) \times Aut(R_i/H).$$

We will use this exact sequence repeatedly to prove several technical lemmas.

Lemma 5.4.11. *There exists some m_3 such that, for all $i \geq m_3$, if $(R_0)_i$ is in the S_0-conjugacy class of $T_n \rtimes \langle x \rangle$, for some $n \geq 2$, and such that $(R_0)_i \lneqq (S_0)_i$, then R_i satisfies condition (5.4).*

Proof. Let $t \in T_{n+1}$ be a generator. Then, since $(R_0)_i \lneqq (S_0)_i$, it follows that $t \in T_i$, and hence $Aut_{\mathcal{F}_i}((R_0)_i) = Aut_{\mathcal{F}}((R_0)_i) = Aut_{S_0}((R_0)_i)$. Consider the exact sequence (5.15) applied in this case. By Lemma 3.2.29, the image of Φ_{R_i} is a direct product $K \times K'$, for certain subgroups $K \leq Aut_{\mathcal{F}_i}((R_0)_i)$ and $K' \leq Aut(R_i/(R_0)_i)$, and there is a partial section $s : K \hookrightarrow Aut_{\mathcal{F}_i}(R_i)$. In fact, in this case ($n \geq 2$), it is easy to see that

$$K = Aut_{(S_0)_i}((R_0)_i) = Aut_{\mathcal{F}_i}((R_0)_i),$$

and $\omega = c_t \in Aut_{\mathcal{F}_i}(R_i)$ gives rise to a nontrivial element in $Out_{\mathcal{F}_i}(R_i)$.

Let $L \leq Aut_{\mathcal{F}_i}(R_i)$ be the subgroup such that

$$L/Aut_{(R_0)_i}(R_i) = H^1(R_i/(R_0)_i; Z((R_0)_i)),$$

which is clearly a 2-group, and let $L' = \langle L, s(K)\rangle$, which is again a 2-subgroup since L' fits in an extension

$$L \longrightarrow L' \longrightarrow K.$$

We want to see that L' is normal in $Aut_{\mathcal{F}_i}(R_i)$. Note that, by definition, for all $\gamma \in L'$,

$$\Phi_{R_i}(\gamma) = (\gamma_0, id)$$

in the extension (5.15), for some $\gamma_0 \in Aut_{\mathcal{F}_i}((R_0)_i)$. In fact, L' is the subgroup of $Aut_{\mathcal{F}_i}(R_i)$ of automorphisms satisfying this condition, since $K = Aut_{\mathcal{F}_i}((R_0)_i)$.

Let $f \in Aut_{\mathcal{F}_i}(R_i)$ and $\gamma \in L'$. Then,

$$\Phi_{R_i}(f\gamma f^{-1}) = (\gamma', id),$$

which implies that $f\gamma f^{-1} \in L'$. Since, as noted above, $\omega = c_t$ induces a nontrivial element in $Out_{S_i}(R_i)$, and clearly $\omega \in L'$, it follows that

$$Out_{S_i}(R_i) \cap O_p(Out_{\mathcal{F}_i}(R_i)) \neq \{1\}$$

as desired.

□

We are then left to study the case where R satisfies that $(R_0)_i$ is in the S_0-conjugacy class of $T_1 \times \langle x \rangle$. This case requires a longer discussion and also involves another enlargement of the list BI-1+2. Note that, for such subgroup R,

$$(5.16) \qquad C_T(R) = C_T(R_i) \leq C_T((R_0)_i) = T_1,$$

where the first equality holds by Proposition 5.2.9.

Basic Ingredients - 3. Fix the following list of objects and morphisms, in addition to those already fixed in lists BI-1 and BI-2:

(ix) a set $\mathcal{H}'$ of representatives of the $\mathcal{F}$-conjugacy classes of non-$\mathcal{F}$-centric subgroups $Q \in Ob(\mathcal{F}^\bullet)$ such that $Q_0 \in \langle T_1 \times \langle x \rangle \rangle_{S_0}$;

(x) for each $P \in \mathcal{H}'$, a set $\mathcal{H}'_P$ of representatives of the S-conjugacy classes in $\langle P \rangle_{\mathcal{F}}$ (and such that P itself is in $\mathcal{H}'_P$ representing its own S-conjugacy class);

(xi) for each $P \in \mathcal{H}'$ and each $R \in \mathcal{H}'_P$ such that $C_S(R) = Z(R)$, a morphism $f_R : R \to R'$, where $R' \in \mathcal{H}'_P$ is such that $C_S(R') \gneqq Z(R')$.

(xii) for each such a morphism $f_R : R \to R'$, an "Alperin-like" decomposition

(5.17)
$$\begin{tikzcd}[column sep=small] & L_1 \arrow[r, "\gamma_1"] & L_1 & & L_2 \arrow[r, "\gamma_2"] & L_2 & & & & L_k \arrow[r, "\gamma_k"] & L_k & \\ R_0 \arrow[ur] \arrow[rrr, "f_1"'] & & & R_1 \arrow[ul] \arrow[ur] \arrow[rrr, "f_2"'] & & & R_2 \arrow[ul] \arrow[r] & \cdots \arrow[r] & R_{k-1} \arrow[ur] \arrow[rrr, "f_k"'] & & & R_k, \arrow[ul] \end{tikzcd}$$

where $R_0 = R$, $R_k = R'$, L_j is $\mathcal{F}$-centric $\mathcal{F}$-radical and fully $\mathcal{F}$-normalized for $j = 1, \ldots, k$, and
$$f_R = f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1.$$

(xiii) for each γ_j above, a lifting φ_j in $\mathcal{L}$.

Let also M_6 be such that, for all $i \geq M_6$, all the subgroups fixed in the list above are S_i-determined, all the morphisms φ_j are Ψ_i-invariant (that is, φ_j is a morphism in $\mathcal{L}_i$), and such that for each $P \in \mathcal{H}'$, if $R \in \mathcal{H}'_P$ is such that $C_S(R) \gneqq Z(R)$, then $C_{S_i}(R_i) \gneqq Z(R_i)$.

In fact,points (xii) and (xiii) in this list can be improved using the following technical lemma.

Lemma 5.4.12. *Let $Q \leq S$ be a $\mathcal{F}$-centric $\mathcal{F}$-radical fully $\mathcal{F}$-normalized subgroup such that $Q_0 \in \langle T_n \rtimes \langle x \rangle \rangle_{S_0}$ for some $n \geq 2$. Then, every automorphism $\gamma \in Aut_{\mathcal{F}}(Q)$ extends to some $\widetilde{\gamma} \in Aut_{\mathcal{F}}(Q \cdot S_0)$, and similarly every automorphism $\varphi \in Aut_{\mathcal{L}}(Q)$ is the restriction of an automorphism $\widetilde{\varphi} \in Aut_{\mathcal{L}}(Q \cdot S_0)$.*

Proof. Suppose for simplicity that $Q_0 = T_n \rtimes \langle x \rangle$. By (5.13), and since $N_{S_0}(Q_0) = T_{n+1} \rtimes \langle x \rangle$, it follows that
$$N_{S_0}(Q_0) \leq N_S(Q).$$
If we prove that any $\gamma \in Aut_{\mathcal{F}}(Q)$ extends to $\gamma' \in Aut_{\mathcal{F}}(Q \cdot T_{n+1})$, then the statement will be proved by iteration of this, together with axiom (III) for saturated fusion system.

Consider the exact sequence (5.15) above. Then, since $Aut_{\mathcal{F}}(Q_0) = Aut_{S_0}(Q_0)$, the (partial) section of the morphism Φ_Q in this exact sequence given by Lemma 3.2.29 can be taken such that $s(Aut_{\mathcal{F}}(Q_0)) \leq Aut_S(Q)$. Let $L \leq Aut_{\mathcal{F}}(Q)$ be the subgroup such that $L/Aut_{Q_0}(Q) = H^1(Q/Q_0; Z(Q_0))$. Then, L is clearly a normal p-subgroup of $Aut_{\mathcal{F}}(Q)$, and since Q is fully $\mathcal{F}$-normalized, it follows by axiom (I) of saturated fusion systems that $L \leq Aut_S(Q)$. Furthermore, the subgroup $L' \leq Aut_{\mathcal{F}}(Q)$ generated by L together with $s(Aut_{\mathcal{F}}(Q_0))$ is also a subgroup of $Aut_S(Q)$. Note that L' is also the subgroup of $Aut_{\mathcal{F}}(Q)$ of automorphisms γ such that $\Phi_Q(\gamma) = (?, id)$.

Now, let $t \in T_{n+1}$ be a generator, $\omega = c_t \in Aut_S(Q)$, and $\gamma \in Aut_{\mathcal{F}}(Q)$. Then, $\omega' = \gamma\omega\gamma^{-1}$ satisfies

$$\Phi_Q(\omega') = (?, id),$$

and hence $\omega' \in L' \leq Aut_S(Q)$. By axiom (II) for saturated fusion system, γ extends to some $\gamma' : Q \cdot T_{n+1} \to S$. It follows also that $Im(\gamma') = Q \cdot T_{n+1}$, since γ' extends γ and S_0 is strongly $\mathcal{F}$-closed.

Iterating this process, we obtain a family $\gamma_j \in Aut_{\mathcal{F}}(Q \cdot T_{n+j})$ of automorphisms extending the original γ, which, in the limit, yield an automorphism $\widetilde{\gamma} \in Aut_{\mathcal{F}}(Q \cdot S_0)$.

The same arguments now apply in $\mathcal{L}$ to prove the second part of the statement. One may use the axioms of transporter systems on $\mathcal{L}$ to prove it (see definition A.2.1).

□

Lemma 5.4.13. *We can choose each morphisms φ_j in the list BI-3 (xiii) to be such that,*

$$\varphi_j \in N_{Aut_{\mathcal{L}}(L_j)}(\delta_{L_j}(N_T(L_j))).$$

Proof. It is enough to check the statement for a single $\varphi \in Aut_{\mathcal{L}}(L)$ (we suppress here the subindex in order to simplify notation). Also, we can assume that $R_0 = T_1 \times \langle x \rangle$ for simplicity (after point (ix) in the BI-3), and hence $L_0 = T_n \rtimes \langle x \rangle$ for some $n \geq 1$.

Suppose first that $n \geq 2$. Then, by Lemma 5.4.12, φ is the restriction of some $\widetilde{\varphi} \in Aut_{\mathcal{L}}(L \cdot S_0)$. Since $T \leq L \cdot S_0$, and T is the maximal infinitely 2-divisible subgroup of S, if we apply axiom (C) for linking systems to $\widetilde{\varphi}$ and $\delta(t)$, it follows that

$$\widetilde{\varphi} \circ \delta(t) = \delta(t') \circ \widetilde{\varphi}$$

for some $t' \in T$, and hence, by taking restriction of the above, the statement follows.

Suppose now that $n = 1$. Note that in this case $L_0 = R_0$. Suppose also we are given any $\gamma \in Aut_{\mathcal{F}}(L)$, and let $f_0 = res^L_{L_0}(\gamma)$ and $\omega = s(f_0)$, where s is the (partial) section to morphism Φ_L in the exact sequence (5.15) given by Lemma 3.2.29. Then, $\gamma' = \omega^{-1} \circ \gamma$ satisfies

$$\Phi_L(\gamma') = (id, \bar{\gamma}),$$

where $\bar{\gamma} : L/L_0 \to L/L_0$ is the automorphism induced by γ. Equivalently,

$$\Phi_L(\gamma'\gamma^{-1}) = (id, id).$$

Thus, if $H \leq Aut_{\mathcal{F}}(L)$ is the subgroup such that $H/Aut_{L_0}(L) = H^1(L/L_0; L_0)$, then, $H \leq Aut_S(L)$ because H is a normal p-subgroup of $Aut_{\mathcal{F}}(L)$ and L is fully $\mathcal{F}$-normalized, and $\gamma' \circ \gamma^{-1} \in H$. In particular, this means that $\gamma'(R)$ and $\gamma(R)$ are S-conjugated.

To finish the proof, lift γ' to some $\varphi' \in Aut_{\mathcal{L}}(L)$. In this case, we have to check that

$$\varphi' \circ \delta(t_2) \circ (\varphi')^{-1} \in \delta(T_2),$$

because $N_T(L) = T_2$. Since $res^L_{L_0}(\gamma') = id$, we can now apply the same arguments as in Lemma 5.4.12 to check that φ' extends to some $\widetilde{\varphi}' \in Aut_{\mathcal{L}}(L \cdot S_0)$. The statement follows now because S_0 is strongly $\mathcal{F}$-closed.

□

As a consequence of Lemmas 5.4.5 and 5.4.6, we can now deduce the following.

Lemma 5.4.14. *There exists some M_7 such that, for all $i \geq M_7$ and for all $Q \in Ob(\mathcal{F}^\bullet)$ which is S_i-determined, if $C_S(Q) \gneqq Z(Q)$, then $C_{S_i}(Q_i) \gneqq Z(Q_i)$.*

Proof. Clearly, the statement holds for the representatives fixed in list BI-3. Furthermore, since we have fixed only finitely many representatives, it is enough to prove the lemma for the S-conjugacy class of one of them, namely $\langle R \rangle_S$.

By Proposition 5.2.9, $C_T(R_i) = C_T(R)$. Set then

$$C_T(R_i) = T'_R \times F_R,$$

for some subtorus $T'_R \leq T$ and some finite subgroup $F_R \leq T$. Let T_R be the maximal torus of R, and note that if $rk(T_R) \lneqq rk(T'_R)$, then the statement follows immediately by Lemma 5.4.3, since the same will hold for all $Q \in \langle R \rangle_S$. Also, since $Aut_{\mathcal{F}}(T)$ is finite and F_R is finite, there exists some M such that, for all $i \geq M$ and all $Q \in \langle R \rangle_S$, $F_Q \leq S_i$ (here we apply the same arguments as those used to show Proposition 5.2.9).

Hence, we can assume without loss of generality that $rk(T_R) = rk(T'_R)$. Let now $Q \in \langle R \rangle_S$ be an S_i-determined subgroup, and let Q_i be its S_i-root. By Lemma 5.4.5, $Q_i \in \langle R_i \rangle_S$. Fix also some $y \in N_S(R_i, Q_i)$.

Suppose first that F_R is not contained in R_i, and let $z \in F_R$ be such that $z \notin R_i$. Set also $z' = yzy^{-1}$. Since $z \in T_i$ and T_i is normal in S, it follows that $z' \in T_i$, and hence (by Lemma 5.4.6), $z' \in C_{S_i}(Q_i) \setminus Z(Q_i)$.

Suppose otherwise that $F_R \leq R_i$. This means that $C_T(R_i) = C_T(R) \leq Z(R_i)$, and hence, for any $z \in C_{S_i}(R_i) \setminus Z(R_i)$ (and by Lemma 5.4.6),

$$z(y^{-1}\Psi_i(y)) = (y^{-1}\Psi_i(y))z,$$

and hence $z' = yzy^{-1} \in C_{S_i}(Q_i) \setminus Z(Q_i)$.

□

Finally, we check that, for i big enough, if $Q \in Ob(\mathcal{F}^\bullet)$ is S_i-determined, non-$\mathcal{F}$-centric, and such that $(Q_0)_i \in \langle T_1 \times \langle x \rangle \rangle_{S_0}$, then Q_i is not $\mathcal{F}_i$-centric.

Lemma 5.4.15. *There exists some m_4 such that, for all $i \geq m_4$, if $(Q_0)_i$ is in the S_0-conjugacy class of $T_1 \times \langle x \rangle$, then Q_i is not $\mathcal{F}_i$-centric.*

Proof. The statement holds for the representatives fixed in BI-3, by assumption. Let R be any of these representatives. We will prove the statement for $\langle R \rangle_S$, and the general statement will follow easily.

If $C_S(R) \gneqq Z(R)$, then by Lemma 5.4.14 the statement holds for $\langle R \rangle_S$. Thus, suppose $C_S(R) = Z(R)$. Let $Q \in \langle R \rangle_S$ and consider too the morphism $f_R : R \to R'$ and its Alperin-like decomposition fixed in points (xi) and (xii) of BI-3.

Consider the first commutative square in this decomposition of f_R:

$$\begin{tikzcd} L_1 \arrow[r, "\gamma_1"] & L_1 \\ R \arrow[u, "incl."] \arrow[r, "f_1"'] & R_1. \arrow[u, "incl."'] \end{tikzcd}$$

Since $C_T(R) = C_T(R_i) = T_1 \cong \mathbb{Z}/2$ (as noted in (5.16)), for all $Q \in \langle R \rangle_S$ which is S_i-determined and all $y \in N_S(R_i, Q_i)$, either $y^{-1}\Psi_i(y) = 1$ or $y^{-1}\Psi_i(y) = t_1$, the generator of T_1. In the first case, Q_i is S_i-conjugate to R_i, and thus there is nothing to show. Hence suppose otherwise.

Fix some $y \in N_S(R_i, Q_i)$, and let $L_1' = yL_1y^{-1}$. Since $y^{-1}\Psi_i(y) = t_1$ and $T_1 \leq Z(S)$, it follows that L_1' is also S_i-determined by Lemma 5.4.6. Let now $\varphi_1 \in Aut_{\mathcal{L}}(L_1)$ be the lifting of γ_1 fixed in point (xiii) of list BI-3. Since, by Lemma 5.4.13, the restriction of γ_1 to (R_0) is the identity, it follows now by Lemma 5.1.4 that $\varphi' = \delta(y)\varphi_1\delta(y^{-1}) \in Aut_{\mathcal{L}}(L_1')$ is also Ψ_i-invariant, and hence a morphism in $\mathcal{L}_i$.

This way, we obtain from (5.17) a chain of morphisms in $\mathcal{F}_i$ (by taking restrictions) from Q_i to some $Q_i' \in \langle R_i' \rangle_S$. Since $C_{S_i}(R_i') \gneqq Z(R_i')$ by hypothesis, it follows by Lemma 5.4.14 that $C_{S_i}(Q_i') \gneqq Z(Q_i')$, and hence Q_i is not $\mathcal{F}_i$-centric.

□

Proposition 5.4.16. *Let $\mathcal{G}$ be a rank 1 p-local compact group whose connected component has the homotopy type of $(BSO(3))^\wedge_2$. Then, there exists some M such that, for all $i \geq M$, $\mathcal{F}_i$ is 5A-saturated.*

Proof. We have proved in Lemmas 5.4.8, 5.4.9, 5.4.11 and 5.4.15 that there exists some M such that, for all $i \geq M$, if $R_i \leq S_i$ is an S_i-root such that $R_i \notin \mathcal{H}_i$, then, depending on the subgroup $(R_0)_i = R_i \cap S_0$, either R_i is not $\mathcal{F}_i$-centric, or it is $\mathcal{F}_i$-centric and satisfies condition (5.4). This, together with Propositions 5.2.18, 5.2.20 and 5.2.22, and Theorem A in [BCG+05] (5.2.14) implies that $\mathcal{F}_i$ is 5A-saturated.

□

Finally, we deal with 2-local compact groups whose connected component is the 2-local compact group induced by S^3. As usual, the central group extension $\mathbb{Z}/2 \to S^3 \to SO(3)$ helps in simplifying the proof.

Proposition 5.4.17. *Let $\mathcal{G}$ be a rank 1 p-local compact group whose connected component has the homotopy type of $(BS^3)^\wedge_2$. Then, there exists some M such that, for all $i \geq M$, $\mathcal{F}_i$ is 5A-saturated.*

Proof. In this case, by Corollary 3.2.4, $\mathcal{G}$ is an admissible extension of $\mathbb{Z}/2$ by a 2-local compact group $\mathcal{G}'$ whose connected component has the homotopy type of $(BSO(3))^\wedge_2$. Hence, the statement follows from Lemma 5.4.16 together with Proposition 5.2.24.

□

5.4.2 General linear groups

Another example of p-local compact groups which can be approximated by p-local finite groups are those induced by the groups $GL_n(\bar{\mathbb{F}}_q)$, where $\bar{\mathbb{F}}_q$ is the algebraic closure of the field of q elements. The arguments used here are slightly different from those used to study rank 1 p-local compact groups.

Let q be a prime. The algebraic closure of the field $\mathbb{F}_q$ can be seen as the union

$$\bar{\mathbb{F}}_q = \cup_m \mathbb{F}_{q^m}$$

of the finite fields of q^m. Thus, $\bar{\mathbb{F}}_q$ is a locally finite field, and $G = GL_n(\bar{\mathbb{F}}_q)$ is a linear torsion group. Theorem 8.10 in [BLO07] (1.4.5) applies now to assure that, for any prime p different from q, G induces a p-local compact group $\mathcal{G}$.

The main result in this section is the following.

Theorem 5.4.18. *Let $\mathcal{G}$ be the p-local compact group induced by $G = GL_n(\bar{\mathbb{F}}_q)$, and let $\{\Psi_i\}$ be a family of unstable Adams operations on $\mathcal{G}$. Then, this family induces an approximation of $\mathcal{G}$ by p-local finite groups.*

Again, we prove that, for i big enough, the fusion systems $\mathcal{F}_i$ are 5A-saturated, and hence the statement above holds.

We describe in some detail the choice of the Sylow p-subgroup of $\mathcal{G}$, since we will need some understading of it. Let $\widehat{T} \leq G$ be the subgroup of diagonal matrices. That is, the matrices in $\widehat{T}$ take coeffients in $(\bar{\mathbb{F}}_q)^{\times} \overset{def}{=} (\bar{\mathbb{F}}_q) \setminus \{0\}$. Let also $N = N_G(\widehat{T})$. Then, there is an extension

$$\widehat{T} \longrightarrow N \longrightarrow \Sigma_n,$$

where Σ_n is the symmetric group on n letters, acting on $\widehat{T}$ by permutations of the diagonal entries. And, in fact, this extension is easily seen to split.

On the other hand, $(\bar{\mathbb{F}}_q)^{\times}$ contains a copy of $\mathbb{Z}/p^{\infty}$, and hence $\widehat{T}$ contains a subgroup $T \cong (\mathbb{Z}/p^{\infty})^n$, which in fact is normal in N since $(\bar{\mathbb{F}}_q)^{\times}$ is the direct product of all $\mathbb{Z}/l^{\infty}$, varying l. Thus, if $\pi \in Syl_p(\Sigma_n)$, then

$$S = T \rtimes \pi \tag{5.18}$$

is a Sylow p-subgroup for $\mathcal{G}$.

With such a description of S it is easy now to deduce the following result.

Lemma 5.4.19. *Let $P \leq S$. Then, $C_T(P)$ is connected in the sense of definition 1.1.1, i.e.,*

$$C_T(P)/T_{C_T(P)} = \{1\}.$$

Proof. Since the action of π on T is by permutation of the entries in the diagonal of an element in T, it follows that, for each rank 1 subtorus $T' \leq T$ and each $x \in S$, either $xtx^{-1} = t$ for all $t \in T'$ or $xtx^{-1} \neq t$ for all $t \in T'$. The statement follows easily from this. $\square$

Now, let $\{\Psi_i\}$ be a family of unstable Adams operations defined on $\mathcal{G}$, and fix a list like BI-1+2. As we have already pointed out several times in this chapter, even if R and Q are S-conjugate and S_i-determined it does not mean that the corresponding S_i-roots are S-conjugate. However, since S has a special isomorphism type (being a semidirect product $T \rtimes \pi$), one can actually deduce so. This is rather interesting, since this will allow us then to apply Lemma 5.4.6 to this example.

Lemma 5.4.20. *Let $R \in Ob(\mathcal{F}^{\bullet})$ be any subgroup. There is some M_R such that, for all $i \geq M_R$, if $Q, Q' \in \langle R \rangle_S$ are S_i-determined, then Q_i and Q'_i are S-conjugate.*

Proof. Write $R = H_R \cdot T'_R$, where $T'_R = R \cap T$ and H_R is a finite subgroup which acts on T'_R by permutations. It is clear then that $R = T'_R \rtimes H_R$. Note that $T'_R = T_R \times F_R$, where T_R is the maximal torus of R and F_R is some finite subgroup of T. Then, since $Aut_{\mathcal{F}}(T)$ is finite, it follows (by the same arguments used to prove Proposition 5.2.9) that the S-conjugacy class of F_R is finite, and hence there exists some m_1 such that, for all $i \geq m_1$ and all $Q \in \langle R \rangle_S$, $F_Q \leq S_i$. We can also assume that, for all $i \geq m_1$, R is S_i-determined.

Thus, $R_i = ((T_R)_i \times F_R) \rtimes H_R$, since, by Lemma 5.2.7, R_i contains a set of representatives of R/T_R and of $R/(R \cap T)$. It is easy to see now that, for $i \geq m_1$, if $Q \in \langle R \rangle_S$ is S_i-determined then Q_i has the same isomorphism type of R_i, that is, we can write

$$Q_i = ((T_Q)_i \times F_Q) \rtimes H_Q$$

where H_Q is a subgroup acting by permutations on T'_Q, which in fact is isomorphic to H_R (although it may not be S-conjugate to H_R).

Consider the sets $\mathbb{T}'_R = \{T'_Q | Q \in \langle R \rangle_S\}$ and $\mathbb{H}_R = \{H_Q | Q \in \langle R \rangle_S\}$. It follows then that the first set is finite because $Aut_{\mathcal{F}}(T)$ is a finite group, and the second set contains finitely many S-conjugacy classes, since Σ_n contains finitely many subgroups and S contains finitely many S-conjugacy classes of subgroups of the isomorphism type of H_R by Lemma 1.1.3. Thus, there exist m_2 and representatives $R_1, \ldots, R_l \in \langle R \rangle_S$ such that, for each $i \geq m_2$,

(i) R_k is S_i-determined for each $k = 1, \ldots, l$, and,

(ii) if $Q \in \langle R \rangle_S$ is any other S_i-determined subgroup, then there exists $x \in S$ and R_j is this list such that

$$T'_Q = x(T'_{R_j})x^{-1} \quad \text{and} \quad H_Q = x(H_{R_j})x^{-1}.$$

In particular, the S_i-root of Q is then S-conjugate to the S_i-root of R_j.

Finally, since we only have to fix finitely many representatives $R_1, \ldots, R_l$, it is clear that there exists some m_3 such that, for all $i \geq m_3$, the S_i-root of R_j is S_i-conjugate to the S_i-root of R, for $j = 1, \ldots, l$. Taking $M_R = m_3$ finishes the proof.

□

Proposition 5.4.21. *There is some M such that, for all $i \geq M$, if R and Q are S-conjugate and S_i-determined, then their S_i-roots are S_i-conjugate.*

Proof. Let $R \in Ob(\mathcal{F}^\bullet)$, and let M_R be as in Lemma 5.4.20. Let also $Q \in \langle R \rangle_S$ be S_i-determined. Then, by Lemma 5.4.6, for all $y \in N_S(R_i, Q_i)$,

$$y^{-1}\Psi_i(y) \in C_T(R_i).$$

Furthermore, since, by point (i) in BI-1, S_i contains a set of representatives χ of the elements in S/T, we can assume that $y \in T$.

Consider now, for each i, the map

$$\begin{array}{ccc} T & \xrightarrow{\Psi_i^*} & T \\ t & \longmapsto & t^{-1}\Psi_i(t). \end{array}$$

Since T is abelian, this turns out to be a group homomorphism, and since T is infinitely p-divisible, this is in fact an epimorphism, whose kernel is the subgroup of fixed points T_i. Furthermore, for any subtorus $T' \leq T$, Ψ_i^* restricts to an epimorphism $T' \twoheadrightarrow T'$, just by definition of Ψ_i.

Since, by Lemma 5.4.19, $C_T(R_i)$ is some subtorus T' of T, it follows now that $y = t_1 t_2$, where $t_1 \in T_i$ and $t_2 \in T' = C_T(R_i)$, and hence Q_i is S_i-conjugate to R_i.

□

We can now prove Theorem 5.4.18 above.

Proof. (of Theorem 5.4.18). We want to apply Theorem A in [BCG+05] (5.2.14) to prove that the fusion system $\mathcal{F}_i$ is saturated for all i big enough, and by Propositions 5.2.18, 5.2.20 and 5.2.22 it will be enough to prove that there is some M such that, for all $i \geq M$, if $Q \in Ob(\mathcal{F}^\bullet)$ is S_i-determined but not $\mathcal{F}$-centric, then Q_i is not $\mathcal{F}_i$-centric.

Fix representatives in $Ob(\mathcal{F}^\bullet)$ of the S-conjugacy classes of non-$\mathcal{F}$-centric subgroups. By Lemma 1.3.2, there is only a finite number of them. For each such representatives R, let M_R be as in Lemma 5.4.20, and let $M'_R \geq M_R$ be such that, for all $i \geq M'_R$, R_i is not $\mathcal{F}_i$-centric. Then, by Proposition 5.4.21 above, for any other $Q \in \langle R \rangle_S$ which is S_i-determined, the S_i-roots R_i and Q_i are S_i-conjugate, and hence neither Q_i is $\mathcal{F}_i$-centric.

Since we have fixed only finitely many representatives in $Ob(\mathcal{F}^\bullet)$ of S-conjugacy classes of non-$\mathcal{F}$-centric subgroups, the statement follows.

□

5.4.3 3-local compact groups from families in [DRV07]

We study now some examples of 3-local compact group which appear as limits of certain families of 3-local finite groups from [DRV07]. In particular, we study the families $\mathcal{F}(3^{2k+1}, j)$, for $j = 2, 3$, in (table 6 of) Theorem 5.10 [DRV07].

In fact, before starting the study of such families, we would like to justify why we restrict to these two families. First, we wanted to study families of (finite) saturated fusion systems not realized by finite groups, hence we discarded all those realized by actual finite groups. We also discarded all fusion systems from tables 5 and 6 in 5.10 [DRV07] which are extensions of other fusion systems also listed in these tables. This left us with the families $\{\mathcal{F}(3^{2k}, j)\}_{j=1,2}$ and $\{\mathcal{F}(3^{2k+1}, j)\}_{j=1,\ldots,4}$.

The families $\{\mathcal{F}(3^{2k+1}, j)\}_{j=1,4}$ were discarded simply because, for any k, there is no inclusion $\mathcal{F}(3^{2k+1}, j) \nsubseteq \mathcal{F}(3^{2k+3}, j)$, and hence we cannot even expect to obtain a p-local compact group from them, and the same happens with the family $\{\mathcal{F}(3^{2k}, 2)\}$. Now, one could indeed obtain a p-local compact group, $\mathcal{G}$, from the family $\{\mathcal{F}(3^{2k}, 1)\}$ by the same arguments that we will use later on in this section. However, it happens that for any k, $\mathcal{F}(3^{2k}, 1)$ cannot appear as the fixed-points p-local finite subgroup of the action of any unstable Adams operation Ψ acting on $\mathcal{G}$, since the maximal abelian normal subgroup of $\mathcal{F}(3^{2k}, 1)$, R_k, is isomorphic to $\mathbb{Z}/3^k \times \mathbb{Z}/3^{k-1}$ by Lemma A.11 [DRV07] (this subgroup should correspond to the fixed points of the maximal torus $T \leq \mathcal{G}$ under Ψ, and hence should be isomorphic either to $(\mathbb{Z}/3^k)^2$ or to $(\mathbb{Z}/3^{k-1})^2$).

Thus, we focus on the fusion systems of the type $\mathcal{F}(3^{2k+1}, j)$, $k \geq 2$ and $j = 2, 3$. We will recall first the notation from [DRV07], and check that indeed by taking limits over k (after fixing j), we obtain p-local compact groups $\mathcal{G}(j)$. We will then consider a family of unstable Adams operations $\{\Psi_i\}$ acting on the resulting p-local compact group, and compare the triples $\mathcal{G}_i = (S_i, \mathcal{F}'_i, \mathcal{L}'_i)$ induced by the Adams operations on $\mathcal{G}(j)$ with the families $\{\mathcal{F}(3^{2k+1}, j)\}$ to see that indeed each $\mathcal{F}_i$ corresponds to a fusion system in these families.

Note that each of the fusion systems listed in 5.10 [DRV07] has a unique associated linking system by Corollary 3.5 [BLO03b]. We may then not mention the existence of this linking system, but consider it implicit.

We start by giving a brief description of the families we want to study. For each of these families, let $B_k \cong B(3, 2k+1; 0, 0, 0)$ be the 3-group described in Theorem A.2 [DRV07]: B_k is the 3-group of order 3^{2k+1} generated by elements $\{s, s_1, \ldots, s_{2k}\}$ satifying the following list of relations:

- $s_j = [s_{j-1}, s]$, for $j \in \{2, 3, \ldots, 2k\}$,
- $[s_1, s_j] = 1$, for $j \in \{2, 3, \ldots, 2k\}$,
- $s^3 = 1$,
- $s_j^3 s_{j+1}^3 s_{j+2} = 1$, for $j \in \{1, 2, \ldots, 2k\}$, with $s_{2k+1} = s_{2k+2} = 1$.

Using Lemma A.11 [DRV07] amb combining these relations, we can now deduce the following presentation for B_k:

$$B_k = \langle s, s_1, s_2 | s^3 = s_1^{3^k} = s_2^{3^k} = 1, [s_1, s_2] = 1, [s, s_1] = s_2, [s, s_2] = (s_1 s_2)^{-3} \rangle. \tag{5.19}$$

Thus, if we set now $\gamma_k = \langle s_1, s_2 \rangle \leq B_k$, then γ_k is a normal subgroup of B_k which is isomorphic to $\mathbb{Z}/3^k \times \mathbb{Z}/3^k$, and there is an extension

$$\gamma_k \longrightarrow B_k \longrightarrow \pi = \langle \bar{s} \rangle \cong \mathbb{Z}/3,$$

which, by Proposition A.9 [DRV07], is split. Furthermore, since the action of π on γ_k is not trivial, it follows that

$$Z(B_k) = (\gamma_k)^\pi = \langle s_2^{3^{k-1}} \rangle \cong \mathbb{Z}/3. \tag{5.20}$$

For simplicity, set $z_k = s_2^{3^{k-1}}$. Set also $z'_k = s_1^{3^{k-1}}$.

Consider now B_k and B_{k+1}, with generator sets (with respect to the presentation (5.19)) $\{s, s_1, s_2\}$ and $\{t, t_1, t_2\}$ respectively. Then, one can define monomorphisms $B_k \hookrightarrow B_{k+1}$ naturally by

$$\begin{array}{rcl} B_k & \xrightarrow{i_k} & B_{k+1} \\ s & \longmapsto & t \\ s_1 & \longmapsto & t_1^3 \\ s_2 & \longmapsto & t_2^3. \end{array} \tag{5.21}$$

It is then natural to consider the limit of this family of inclusions, $S = \cup B_k$, which is a discrete p-toral group with maximal torus $T = \cup_k \gamma_k$ of rank 2. In fact, since the splittings of the B_k are natural with respect to the inclusions $B_k \leq B_{k+1}$, it follows that $S = T \rtimes \langle s \rangle$.

Consider the following subgroups of S_k:

- $V_l = \langle z_k, s s_1^l \rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/3$, for $l \in \{-1, 0, 1\}$, and
- $E_l = \langle z_k, z'_k, s s_1^l \rangle \cong 3^{1+2}_+$, the extraspecial group of order 3^3 and exponent 3, for $l \in \{-1, 0, 1\}$.

The fusion systems $\mathcal{F}(3^{2k+1}, j)$, for $j = 2, 3$, are determined then by the following table (a summary of table 6 in 5.10 [DRV07] for the families of our interest).

<table>
<tr><th>B_k</th><th>V_0</th><th>$V_{\pm 1}$</th><th>E_0</th><th>$E_{\pm 1}$</th><th>γ_k</th><th>3-lfg</th></tr>
<tr><td rowspan="2">$\mathbb{Z}/2 \times \mathbb{Z}/2$</td><td>-</td><td>-</td><td>$GL_2(3)$</td><td>-</td><td rowspan="2">$GL_2(3)$</td><td>$\mathcal{F}(3^{2k+1}, 2)$</td></tr>
<tr><td>$GL_2(3)$</td><td>-</td><td>-</td><td>-</td><td>$\mathcal{F}(3^{2k+1}, 3)$</td></tr>
</table>

where the column under B_k is the isomorphism type of $Out_{\mathcal{F}_k(j)}(B_k)$, and the rest of columns give the isomorphism type of $Aut_{\mathcal{F}_k(j)}(P)$ for the corresponding subgroup P, when it is $\mathcal{F}_k(j)$-centric $\mathcal{F}_k(j)$-radical, or a dash when P is not $\mathcal{F}_k(j)$-centric $\mathcal{F}_k(j)$-radical.

Set for simplicity $\mathcal{F}_k(j) = \mathcal{F}(3^{2k+1}, j)$, for some j, and let $\mathcal{L}_k(j)$ be the unique associated centric linking system. Consider again B_k, B_{k+1} with generators s, s_1, s_2 and t, t_1, t_2 respectively. Then, the inclusion i_k (5.21) sends z_k to z_{k+1}, z'_k to z'_{k+1}, and hence i_k can be extended to an inclusion functor $i_k : \mathcal{F}_k(j) \to \mathcal{F}_{k+1}(j)$ of saturated fusion systems.

Let then $\mathcal{F}(j)$ be the fusion system over S obtained by taking the limit of these inclusion functors i_k and spanning morphism sets from that. We want to check then that $\mathcal{F}(j)$ is saturated. Note that the subgroups $V_l, E_l \leq B_k$, for $l \in \{-1, 0, 1\}$, which are not $\mathcal{F}_k(j)$-conjugate, become $\mathcal{F}_{k+1}(j)$-conjugate after taking their images through i_k (since, with the notation above, $t_2 t t_2^{-1} = t t_1^3$ and $t_2^{-1} t t_2 = t t_1^{-3}$).

Lemma 5.4.22. *Let $P \leq S$ be a $\mathcal{F}(j)$-centric $\mathcal{F}(j)$-radical subgroup. Then, P is in the $\mathcal{F}(j)$-conjugacy class of one of the following subgroups: S, V_0, E_0 or T.*

Proof. For $P \leq S$ a subgroup in the list above, the automorphism group is

$$Aut_{\mathcal{F}(j)}(P) = \varinjlim Aut_{\mathcal{F}_k(j)}(P).$$

Thus, it is clear that the subgroups listed in the statement are $\mathcal{F}(j)$-centric $\mathcal{F}(j)$-radical by definition of $\mathcal{F}(j)$ (depending on the automorphism groups in the table above). Let then $P \leq S$ be any other $\mathcal{F}(j)$-centric $\mathcal{F}(j)$-radical subgroup, and suppose it is not $\mathcal{F}(j)$-conjugate to any of the subgroups in the list above.

Clearly, P cannot be finite, since if this was the case, then there would exist some k such that $P \leq B_k$ and such that $Aut_{\mathcal{F}_k(j)}(P) = Aut_{\mathcal{F}(j)}(P)$. Set $T = T_1 \times T_2$, where T_m, $m = 1, 2$, is the union of all the subgroups $\langle s_m \rangle$ (in the notation of (5.19)), and let $\Delta \leq T$ be the diagonal rank 1 subtorus. Thus, since $S = T \rtimes \langle s \rangle$, and $\langle s \rangle \cong \mathbb{Z}/3$, we claim that P satisfies one (and only one) of the properties below:

(a) $P \leq T$,

(b) $P = S$, or

(c) $T_P = \Delta$ and P contains an element x which acts nontrivially on T.

We are assuming P is not a finite subgroup, that is, that $1 \leq rk(P) \leq 2$. Suppose then that P is not a subgroup of T. This means then that P contains some element $x = ts$ (for some $t \in T$) which acts nontrivially on T. If $rk(P) = 2$, then $P = S$, so suppose that $rk(P) = 1$. Then, the maximal torus of P has to be Δ, since the only rank 1 subtorus of T which is sent by conjugation by x to itself is Δ.

Now, we are assuming that P is neither S nor T, so we have to assume that P has rank 1, in which case the automorphism group of P, $Aut_{\mathcal{F}(j)}(P)$, is generated by the automorphism groups of S by definition of $\mathcal{F}(j)$. Since $P \lneqq S$, it follows that $P \lneqq N_S(P)$, and (since all automorphisms of P are restrictions of automorphisms of S),

$$Out_S(P) \cap O_p(Out_{\mathcal{F}(j)}(P)) \neq \{1\}.$$

□

Proposition 5.4.23. *The fusion system $\mathcal{F}(j)$ is saturated.*

Proof. Let $\mathcal{H}$ be the set formed by the $\mathcal{F}(j)$-conjugacy classes of the subgroups listed in Lemma 5.4.22. Then it is clear that $\mathcal{F}(j)$ is $\mathcal{H}$-generated, and it is not difficult to prove that it is also $\mathcal{H}$-saturated. Since, in addition, $\mathcal{H}$ contains the list of all $\mathcal{F}(j)$-centric $\mathcal{F}(j)$-radical subgroups, it follows by Theorem A in [BCG+05] that $\mathcal{F}(j)$ is saturated.

□

The following list describes the two different saturated fusion systems which are obtained as the limits of the families that we are considering.

S	V_0	E_0	T	3-lcg
$\mathbb{Z}/2 \times \mathbb{Z}/2$	-	$GL_2(3)$	$GL_2(3)$	$\mathcal{F}(2)$
	$GL_2(3)$	-		$\mathcal{F}(3)$

Next we check that we can naturally associate a linking system to $\mathcal{F}(j)$. For each P in the list of $\mathcal{F}(j)$-centric $\mathcal{F}(j)$-radical subgroups of Lemma 5.4.22, define

$$Aut_{\mathcal{L}(j)}(P) \overset{def}{=} \varinjlim Aut_{\mathcal{L}_k(j)}(P)$$

(it is not difficult to check that this groups are well-defined), and let $\mathcal{L}^{cr}(j)$ be the category with object set the $\mathcal{F}(j)$-conjugacy classes of the subgroups listed in 5.4.22 and with morphism sets spanned by the automorphism groups above.

Proposition 5.4.24. *The category $\mathcal{L}^{cr}(j)$ spans a centric linking system associated to the fusion system $\mathcal{F}(j)$.*

Proof. The functor ρ and the "distinguished monomorphisms" δ_P for $P \in Ob(\mathcal{L}^{cr}(j))$ are defined as the limits of the corresponding functors ρ_k and "distinguished monomorphisms" in $\mathcal{L}_k(j)$. The axioms then follow easily since $\mathcal{L}^{cr}(j)$ has been defined as some direct limit of linking systems. Since $Ob(\mathcal{L}^{cr}(j))$ already contains all $\mathcal{F}(j)$-centric $\mathcal{F}(j)$-radical subgroups, it follows that a full centric linking system $\mathcal{L}(j)$ with object set $Ob(\mathcal{L}(j)) = Ob(\mathcal{F}^c(j))$ can be spanned from $\mathcal{L}^{cr}(j)$.

□

Let now $\{\Psi_i\}$ be a family of unstable Adams operations defined on $\mathcal{G}(j)$, and for each i let $\mathcal{G}(j)_i = (S_i, \mathcal{F}(j)'_i, \mathcal{L}(j)'_i)$ be the triple induced by Ψ_i as in definition 5.2.16. It is not difficult to prove now the main result of this section.

Proposition 5.4.25. *There is some M such that, for all $i \geq M$, there is some k_i such that*

$$\mathcal{F}'_i(j) = \mathcal{F}_{k_i}(j),$$

where $\mathcal{F}_{k_i}(j)$ is one of the (saturated) fusion systems listed in [DRV07]. In particular, $\mathcal{F}'_i(j)$ is 5A-saturated.

Proof. Fix lists like BI-1 and BI-2 for $\mathcal{G}(j)$. In particular, fix V_0 and E_0 in the tables above as the representatives of their S-conjugacy classes. For each i, let k_i be such that the subgroup of fixed points in S under the action of Ψ_i, S_i, equals the group B_{k_i} (5.19).

Note that, since $\langle S\rangle_{\mathcal{F}(j)} = \{S\}$ and $\langle T\rangle_{\mathcal{F}(j)} = \{T\}$, it is clear that there is some m_1 such that, for all $i \geq m_1$, the automorphism groups of S_i and T_i in $\mathcal{F}_i'(j)$ equal the automorphism groups of B_{k_i} and γ_{k_i} in $\mathcal{F}_{k_i}(j)$ respectively, and we only have to deal with the $\mathcal{F}(j)$-conjugacy classes of V_0 and E_0. Note that, for any i, the subgroups E_0, E_1 and E_{-1} (repectively V_0, V_1 and V_{-1}, see Lemma 5.2 in [DRV07]) are no longer S_i-conjugate, since the elements that conjugate them are not Ψ_i-invariant.

Now, it is easy to see that the subgroups V_0 and E_0 are $\mathcal{F}(j)$-centric by Lemma 5.2 [DRV07], since by definition of $\mathcal{F}(j)$ there is no morphism in this fusion system conjugating V_0 with $P_1 = \langle z, z'\rangle$, where z and z' represent in T the elements z_k and z_k' defined in (5.20). Furthermore, by Lemma A.15 [DRV07], any subgroup of S of the isomorphism type of $\mathbb{Z}/3 \times \mathbb{Z}/3$ (except from P_1) is $\mathcal{F}(j)$-conjugate to V_0. Hence, the only non-$\mathcal{F}(j)$-centric subgroups of S are P_1 and all subgroups of the isomorphism type of $\mathbb{Z}/3$.

Let $P \leq S$ be a non-$\mathcal{F}(j)$-centric subgroup. We claim then that

$$C_T(P) \gneqq (Z(P) \cap T).$$

Suppose first that $P \in \langle P_1\rangle_{\mathcal{F}}(j)$. Then, in fact $P = P_1 \leq T$, and the statement follows immeadiately. Suppose now that P is isomorphic to $\mathbb{Z}/3$. Then, either $P \leq T$ and the statement follows, or $P = \langle ts\rangle$, for some $t \in T$, in which case P does not contain $Z(S) \leq T$, and the statement holds for P.

Thus, by Lemma 5.4.3, there is some m_1 such that, for all $i \geq m_1$, if $R \in Ob(\mathcal{F}(j)^\bullet)$ is S_i-determined but not $\mathcal{F}(j)$-centric, then R_i is not $\mathcal{F}(j)_i'$-centric. This, together with Proposition 5.2.22, implies that condition (5.4) holds for all $i \geq m_1$, and hence $\mathcal{F}(j)_i'$ is 5A-saturated (and in particular saturated).

Now, since both V_0 and E_0 are finite subgroups of S, it is easy to see that there is some m_2 such that, for all $i \geq m_2$, if k_i is such that $|B_{k_i}| = |S_i|$, then

$$Aut_{\mathcal{F}(j)_i'}(V_0) = Aut_{\mathcal{F}(j)}(V_0) = Aut_{\mathcal{F}_{k_i}(j)}(V_0)$$
$$Aut_{\mathcal{F}(j)_i'}(E_0) = Aut_{\mathcal{F}(j)}(E_0) = Aut_{\mathcal{F}_{k_i}(j)}(E_0)$$

and hence, by comparison with table 6 in Theorem 5.10 [DRV07], it follows that

$$\mathcal{F}(j)_i' = \mathcal{F}_{k_i}(j).$$

□

The following result is then obvious.

Theorem 5.4.26. *Let $\mathcal{G}(j)$, for $j = 2, 3$, be one of the 3-local compact groups defined by Propositions 5.4.23 and 5.4.24, and let $\{\Psi_i\}$ be a family of unstable Adams operations acting on $\mathcal{G}$.*

Then, the family $\{\Psi_i\}$ induces an approximation of $\mathcal{G}(j)$ by p-local finite groups.

5.5 Saturated fusion systems without associated linking system

Up until now, we have always assumed in this chapter that, given a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, there was always an associated linking system $\mathcal{L}$ (forgetting about uniqueness of such linking system), but it is also worth considering the situation where no associated linking system is assumed to exist.

When this is the case, Robinson groups realizing (saturated) fusion systems (see Theorem 2.2.3) and the unstable Adams operations that we have defined for them (Proposition 4.3.2) provide a setting where we might try to reproduce the constructions done previously in this chapter. However, working with these infinite groups is more complicated than it could seem on first sight.

For instance, while we have not proved an analogue of property (ii) in Theorem 2 [Rob07], which says that the centralizer of a centric subgroup has a normal free subgroup of finite index, it is fair to expect something similar to happen to the models we use in this work. Furthermore, the subgroups $N_G(P)$, for $P \leq S$, need not be neither artinian nor locally finite. This means that one has to be careful when thinking of Robinson groups as substitutes of centric linking systems.

Let then $\mathcal{F}$ be a saturated fusion system, and let G be a Robinson group realizing $\mathcal{F}$. Let also $\{\Psi_i\}$ be a family of unstable Adams operations acting on G. Since, in particular, the Ψ_i are group automorphisms, we may then consider for each i the subgroup

$$G_i = \{g \in G | \Psi_i(x) = x\}.$$

Naturally, the subgroup of fixed elements of S under Ψ_i, S_i, is a subgroup of G_i for all i, and here is appears the first question. Is S_i a Sylow p-subgroup of G_i (in the sense that any other finite p-subgroup of G_i is subconjugate to S_i) or not?

What is not difficult to prove is the following lemma, which does not seem to be of much interest if we cannot progress in this problem.

Lemma 5.5.1. *For each $g \in G$ there exists some M_g such that, for all $i \geq M_g$, $\Psi_i(g) = g$. In particular, G is the union of all the subgroups G_i.*

Proof. Let $g \in G$ be any element, and write it as $g = g_1 * g_2 * \ldots * g_m$, where each g_n is in L_j for some $P_j \in \mathcal{P}$. Since Ψ is constructed from unstable Adams operations on each L_j, it follows by Proposition 5.2.3 that for each g_m there exists some M_{g_m} such that the statement holds. We just have to take M_g to be the maximum of all the M_{g_m}'s to finish the proof.

□

Appendix A

Extensions of p-local compact groups with discrete p-toral kernel

While morphisms between any arbitrary two p-local compact groups are not defined in general, there are some situations where morphisms can be considered. This is even more interesting when such morphisms allow us to transfer information from one p-local compact group to another, and this is the case of extensions. Extensions of p-local finite groups were first studied in [BCG+07], and then in [OV07], where the authors classified extensions of p-local finite groups with p-group kernel.

In this chapter, we extend the results from [OV07] to extensions of p-local compact groups with discrete p-toral group kernel. In the original paper, such extensions were used to study the construction of new exotic p-local finite groups. Leaving apart the fact that it is not at all clear what an *exotic* p-local compact groups ought to be, we are not interested in extensions of p-local compact groups from this point of view, and hence the part of [OV07] regarding exoticity has been skipped here.

This chapter is then organized following the structure of the original source. In the first section we define quotients of saturated fusion systems by weakly closed subgroups, and prove that one obtains then a saturated fusion system. In the second section we introduce the notion of transporter system, which generalizes that of linking system, and prove some of their properties, such as the fact that a linking system is always a transporter system, and that given a transporter system we can obtain a linking system under certain reasonable hypothesis. The third section proves extends the results on quotients of fusion systems to quotients of transporter systems. In the fourth section we deal with higher limits over orbit categories and show some properties about the homotopy type of classifying spaces of transporter systems, and finally in section five we classify a certain class of extensions of transporter systems called admissible, extending thus the classification of admissible extensions of finite transporter systems from [OV07].

A.1 Quotients of fusion systems

In 1.2.8 we have introduced the notion of weakly closed subgroups of fusion systems, as well as stronger conditions such as strongly closed and normal subgroups. Given a (saturated) fusion system $\mathcal{F}$ over S and a weakly closed subgroup A, the quotient fusion system $\mathcal{F}/P$ over S/A is defined naturally as the fusion system where morphisms are induced by morphisms in $\mathcal{F}$. In this section, we prove that saturation on $\mathcal{F}$ implies saturation on $\mathcal{F}/A$.

Let then $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $A \leq S$ be a subgroup. Whenever A is weakly $\mathcal{F}$-closed, we define the **quotient fusion system** $\mathcal{F}/A$ over S/A as the fusion system with morphism sets

$$Hom_{\mathcal{F}/A}(P/A, P'/A) = \{f/A | f \in Hom_{\mathcal{F}}(P, P')\}. \tag{A.1}$$

Note that S/A can be again a discrete p-toral group.

Proposition A.1.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $A \leq S$ be a weakly $\mathcal{F}$-closed subgroup. Then, $\mathcal{F}/A$ is a saturated fusion system over S/A.*

One lemma is needed before giving a proof for this result.

Lemma A.1.2. *In the situation above, a subgroup $P \leq S$ containing A is fully $\mathcal{F}$-normalized if and only if $P/A \leq S/A$ is fully $\mathcal{F}/A$-normalized.*

Proof. This holds since for each $P \leq S$ such that $A \leq P$,

$$N_{S/A}(P/A) = N_S(P)/A,$$

and since the $\mathcal{F}/A$-conjugacy class of P/A corresponds to the $\mathcal{F}$-conjugacy class of P. □

Proof. (of Proposition A.1.1). We can skip checking axiom (III) in 1.2.2 for $\mathcal{F}/A$ since it follows from axiom (III) on $\mathcal{F}$. We will use the equivalent saturation axioms from [KS08], proved in 1.7.1 for the compact case.

(I′) $Inn(S/A) \in Syl_p(Aut_{\mathcal{F}/A}(S/A))$.

This is equivalent to showing that $\{1\} \in Syl_p(Out_{\mathcal{F}/A}(S/A))$, and this follows immediately from the epimorphism of finite groups

$$Out_{\mathcal{F}}(S) \twoheadrightarrow Out_{\mathcal{F}/A}(S/A)$$

and the fact that $\mathcal{F}$ is saturated.

(II′) Let $f/A : P/A \to S/A$ be a morphism in $\mathcal{F}/A$ such that $P'/A = f/A(P/A)$ is fully $\mathcal{F}/A$-normalized, then f/A extends to a morphisms $\widetilde{f/A} \in Hom_{\mathcal{F}}(N_{f/A}, S/A)$, where

$$N_{f/A} = \{gA \in N_{S/A}(P/A) | f/A \circ c_{gA} \circ (f/A)^{-1} \in Aut_{S/A}(P'/A)\}.$$

Let $f \in Hom_{\mathcal{F}}(P, S)$ be a representative of f/A in $\mathcal{F}$. By Lemma A.1.2 above, $P'/A = f/A(P/A)$ is fully $\mathcal{F}/A$-normalized if and only if $P' = f(P)$ is fully $\mathcal{F}$-normalized. In particular, since $\mathcal{F}$ is saturated, P' is also fully $\mathcal{F}$-centralized, and we may apply axiom (II) in $\mathcal{F}$ to the morphism f.

Axiom (II′) now follows since there is an epimorphism

$$N_f \twoheadrightarrow N_{f/A},$$

and hence an extension of f in $\mathcal{F}$ gives rise to the desired extension of f/A in $\mathcal{F}/A$. □

A.2 Transporter systems associated to fusion systems

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $A \leq S$ be weakly $\mathcal{F}$-closed. Suppose in addition that there exists a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$. As shown in section 2 of [OV07], even if we may define a quotient linking system $\mathcal{L}/A$ associated to $\mathcal{F}/A$, this need not be a centric linking system, since $\mathcal{F}$-centric subgroups need not correspond to $\mathcal{F}/A$-centric subgroups. This is the reason why we introduce the wider class of transporter systems.

Let G be an artinian locally finite group such that has Sylow p-subgroups, and let $S \in Syl_p(G)$. We define $\mathcal{T}_S(G)$ as the category whose object set is $Ob(\mathcal{T}_S(G)) = \{P \leq S\}$, and such that

$$Mor_{\mathcal{T}_S(G)}(P, P') = N_G(P, P') = \{g \in G | gPg^{-1} \leq P'\}.$$

For a subset $\mathcal{H} \subseteq Ob(\mathcal{T}_S(G))$, $\mathcal{T}_{\mathcal{H}}(G)$ denotes the full subcategory of $\mathcal{T}_S(G)$ with object set $\mathcal{H}$.

Definition A.2.1. *Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. A* ***transporter system*** *associated to $\mathcal{F}$ is a category $\mathcal{T}$ such that*

(i) $Ob(\mathcal{T}) \subseteq Ob(\mathcal{F})$;

(ii) for all $P \in Ob(\mathcal{T})$, $Aut_{\mathcal{T}}(P)$ is an artinian locally finite group;

together with a couple of functors

$$\mathcal{T}_{Ob(\mathcal{T})}(S) \xrightarrow{\varepsilon} \mathcal{T} \xrightarrow{\rho} \mathcal{F},$$

satisfying the following axioms:

(A1) $Ob(\mathcal{T})$ is closed under $\mathcal{F}$-conjugacy and overgroups. Also, ε is the identity on objects and ρ is inclusion on objects.

(A2) For each $P \in Ob(\mathcal{T})$, let

$$E(P) = Ker(Aut_{\mathcal{T}}(P) \to Aut_{\mathcal{F}}(P)).$$

Then, for each $P, P' \in Ob(\mathcal{T})$, $E(P)$ acts freely on $Mor_{\mathcal{T}}(P, P')$ by right composition, and $\rho_{P,P'}$ is the orbit map for this action. Also, $E(P')$ acts freely on $Mor_{\mathcal{T}}(P, P')$ by left composition.

(B) For each $P, P' \in Ob(\mathcal{T})$, $\varepsilon_{P,P'} : N_S(P, P') \to Mor_{\mathcal{T}}(P, P')$ is injective, and the composite $\rho_{P,P'} \circ \varepsilon_{P,P'}$ sends $g \in N_S(P, P')$ to $c_g \in Hom_{\mathcal{F}}(P, P')$.

(C) For all $\varphi \in Mor_{\mathcal{T}}(P, P')$ and all $g \in P$, the following diagram commutes in $\mathcal{T}$:

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\varepsilon_{P,P}(g)"'] & P' \arrow[d, "\varepsilon_{P',P'}(\rho(\varphi)(g))"] \\ P \arrow[r, "\varphi"'] & P'. \end{tikzcd}$$

(I) $\varepsilon_{S,S}(S) \in Syl_p(Aut_{\mathcal{T}}(S))$.

(II) Let $\varphi \in Iso_{\mathcal{T}}(P, P')$, and $P \lhd R \leq S$, $P' \lhd R' \leq S$ such that

$$\varphi \circ \varepsilon_{P,P}(R) \circ \varphi^{-1} \leq \varepsilon_{P',P'}(R').$$

Then, there is some $\widetilde{\varphi} \in Mor_{\mathcal{T}}(R, R')$ such that $\widetilde{\varphi} \circ \varepsilon_{P,R}(1) = \varepsilon_{P',R'}(1) \circ \varphi$, that is, the following diagram is commutative in $\mathcal{T}$:

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\varepsilon_{P,R}(1)"'] & P' \arrow[d, "\varepsilon_{P',R'}(1)"] \\ R \arrow[r, "\widetilde{\varepsilon}"'] & R'. \end{tikzcd}$$

(III) Let $P_1 \leq P_2 \leq \ldots$ be an increasing sequence of subgroups in $Ob(\mathcal{T})$, and $P = \bigcup_{n=1}^{\infty} P_n$. Suppose in addition that there exists $\psi_n \in Mor_{\mathcal{T}}(P_n, S)$ such that

$$\psi_n = \psi_{n+1} \circ \varepsilon_{P_n,P_{n+1}}(1)$$

for all n. Then, there exists $\psi \in Mor_{\mathcal{T}}(P, S)$ such that $\psi_n = \psi \circ \varepsilon_{P_n,P}(1)$ for all n.

Given a transporter system $\mathcal{T}$, we will refer to the p-completion of the realization of the nerve of $\mathcal{T}$, $|\mathcal{T}|^{\wedge}_p$, as the ***classifying space*** *of the transporter system.*

Note that, in axiom (III), P is an object in $Ob(\mathcal{T})$, since $Ob(\mathcal{T})$ is a set of subgroups of S closed under $\mathcal{F}$-conjugacy and overgroups. As in [OV07], the axioms are labelled to show their relation with the axioms for linking and fusion systems respectively.

The following lemma is an analog of Lemmas 3.2 in [OV07] and 4.3 in [BLO07].

Lemma A.2.2. *Let $\mathcal{T}$ be a transporter system associated to a fusion system $\mathcal{F}$ (over a discrete p-toral group), and let $\rho : \mathcal{T} \to \mathcal{F}$ be the projection functor.*

(i) Fix morphisms $f \in Hom_{\mathcal{F}}(P, Q)$ and $f' \in Hom_{\mathcal{F}}(Q, R)$, where $P, Q, R \in \mathcal{T}$. Then, for any pair of liftings $\varphi' \in \rho^{-1}_{Q,R}$ and $\omega \in \rho^{-1}_{P,R}(f'f)$, there is a unique lifting $\varphi \in \rho^{-1}_{P,Q}(f)$ such that $\varphi' \circ \varphi = \omega$.

(ii) All morphisms in $\mathcal{T}$ are monomorphisms in the categorical sense. That is, for all $P, Q, R \in \mathcal{T}$ and all $\varphi_1, \varphi_2 \in Mor_{\mathcal{T}}(P, Q)$, $\psi \in Mor_{\mathcal{T}}(Q, R)$, if $\psi \circ \varphi_1 = \psi \circ \varphi_2$ then $\varphi_1 = \varphi_2$.

(iii) For every morphism $\varphi \in Mor_{\mathcal{T}}(P, Q)$ and every $P_0, Q_0 \in \mathcal{T}$ such that $P_0 \leq P$, $Q_0 \leq Q$ and $\rho(\varphi)(P_0) \leq Q_0$, there is a unique morphism $\varphi_0 \in Mor_{\mathcal{T}}(P_0, Q_0)$ such that $\varphi \circ \iota_{P_0,P} = \iota_{Q_0,Q} \circ \varphi_0$. In particular, every morphism in $\mathcal{T}$ is a composite of an isomorphism followed by an inclusion.

(iv) All morphisms in $\mathcal{T}$ are epimorphisms in the categorical sense. In other words, for all $P, Q, R \in \mathcal{T}$ and all $\varphi \in Mor_{\mathcal{T}}(P, Q)$ and $\psi_1, \psi_2 \in Mor_{\mathcal{T}}(Q, R)$, if $\psi_1 \circ \varphi = \psi_2 \circ \varphi$ then $\psi_1 = \psi_2$.

Proof. Let $\mathcal{F}_{\mathcal{T}}$ be the full subcategory of $\mathcal{F}$ with object set $Ob(\mathcal{F}_{\mathcal{T}}) = Ob(\mathcal{T})$. Since the functor $\rho : \mathcal{T} \to \mathcal{F}_{\mathcal{T}}$ is both source regular and target regular (because of axiom (A2) of transporter systems, see definition A.5 in [OV07]), the proof for Lemma 3.2 in [OV07] applies as well in this case.

□

The rest of this section is devoted to prove first that linking systems are transporter systems, and second that from a transporter system (satisfying some mild conditions) we can always obtain a linking system.

Let $\mathcal{G}$ be a p-local compact group, and, for each $P \in \mathcal{L}$ fix a lifting of $incl_P^S : P \to S$ in $\mathcal{L}$, $\iota_{P,S} \in Mor_{\mathcal{L}}(P, S)$. Then, by Lemma 1.4.3, we may complete this to a family of inclusions $\{\iota_{P,P'}\}$ in a unique way and such that $\iota_{P,S} = \iota_{P',S} \circ \iota_{P,S}$ whenever it makes sense.

Lemma A.2.3. *Fix such a family of inclusions $\{\iota_{P,P'}\}$ in $\mathcal{L}$. Then, for each $P, P' \in \mathcal{L}$, there are unique injections*

$$\delta_{P,P'} : N_S(P, P') \longrightarrow Mor_{\mathcal{L}}(P, P')$$

such that

(i) $\iota_{P',S} \circ \delta_{P,P'}(g) = \delta_S(g) \circ \iota_{P,S}$, for all $g \in N_S(P, P')$, and

(ii) δ_P is the restriction to P of $\delta_{P,P}$.

These injections thus form a functor from $\mathcal{T}_{Ob(\mathcal{L})}(S)$ to $\mathcal{L}$.

Proof. Note that, for $P = P' = S$, the injection δ_S is already defined by the definition of linking system. Let then $P, P' \in Ob(\mathcal{L})$, and consider the following commutative diagram in $\mathcal{F}$:

$$\begin{tikzcd} P \arrow[r, "c_g"] \arrow[d, "incl_P^S"'] & P' \arrow[d, "incl_{P'}^S"] \\ S \arrow[r, "c_g"'] & S, \end{tikzcd}$$

where $g \in N_S(P, P')$. Since $g \in S$ is a lifting in $\mathcal{L}$ of c_g, and we have fixed liftings for $incl_P^S$ and $incl_{P'}^S$ respectively, Lemma 4.3 (a) in [BLO07] applies, and it follows that there exists a unique morphism $\delta_{P,P'}(g) \in Mor_{\mathcal{L}}(P, P')$ making the following diagram commute:

$$\begin{tikzcd} P \arrow[r, "\delta_{P,P'}(g)"] \arrow[d, "\iota_{P,S}"'] & P' \arrow[d, "\iota_{P',S}"] \\ S \arrow[r, "\delta_S(g)"'] & S, \end{tikzcd}$$

This gives a map $\delta_{P,P'} : N_S(P, P') \to Mor_{\mathcal{L}}(P, P')$, and conditions (i) and (ii) follow by 4.3 (a) in [BLO07]. Hence, it remains to check the injectivity of $\delta_{P,P'}$.

Let $g, h \in N_S(P, P')$ be such that $\delta_{P,P'}(g) = \delta_{P,P'}(h)$. This would imply that

$$c_g = \pi(\delta_{P,P'}(g)) = \pi(\delta_{P,P'}(h)) = c_h$$

in $\mathcal{F}$ and hence, that $hg^{-1} \in C_S(P) = Z(P)$ (since P is centric). By construction of $\delta_{P,P'}$ and by Lemma 4.3 (b) in [BLO07], it would follow then that there exists $z \in Z(S)$, a nontrivial element, such that

$$\delta_S(h) = \delta_S(z) \circ \delta_S(g)$$

and hence that $\delta_{P,P'}(h) = \delta_P(z) \circ \delta_{P,P'}(g)$. Hence, $g = h$ and $\delta_{P,P'}$ is injective.

□

Proposition A.2.4. *Let $\mathcal{T}$ be a transporter system associated to a fusion system $\mathcal{F}$ over a discrete p-toral group S. Then, for any $P \in \mathcal{T}$,*

(i) P is fully $\mathcal{F}$-normalized if and only if $\varepsilon_{P,P}(N_S(P)) \in Syl_p(Aut_{\mathcal{T}}(P))$.

(ii) P is fully $\mathcal{F}$-centralized if and only if $\varepsilon_{P,P}(C_S(P)) \in Syl_p(E(P))$.

Proof. For both (i) and (ii) the "if" implication is immediate, since they only depend on the $\mathcal{F}$-conjugation class of P. Also, for S the "only if" part in (i) and (ii) also hold by axiom (I) in the definition of transporter systems.

(i) "only if":

Suppose P is fully $\mathcal{F}$-normalized but $\varepsilon_{P,P}(N_S(P))$ is not a Sylow p-subgroup of $Aut_{\mathcal{T}}(P)$. Suppose further that P is of maximal order satisfying that (i.e, no over-group of P satisfies those two conditions).

Let now $H \in Syl_p(Aut_{\mathcal{T}}(P))$ such that $\varepsilon_{P,P}(N_S(P)) \leq H$. In fact, this inclusion is by hypothesis strict. Being both H and $\varepsilon_{P,P}(N_S(P))$ discrete p-toral groups, the inclusion

$$\varepsilon_{P,P}(N_S(P)) \leq H_0 = N_H(\varepsilon_{P,P}(N_S(P)))$$

is again strict. We have in fact a sequence of strict inclusions

$$\varepsilon_{P,P}(P) \lneqq \varepsilon_{P,P}(N_S(P)) \lneqq H_0$$

since $P \lneqq S$ implies $P \lneqq N_S(P)$.

Let $\varphi \in H_0 \setminus \varepsilon_{P,P}(N_S(P))$. Then,

$$\varphi^{-1} \circ \varepsilon_{P,P}(x) \circ \varphi \in \varepsilon_{P,P}(N_S(P))$$

for all $x \in N_S(P)$ since $\varepsilon_{P,P}(N_S(P)) \lhd H_0$. It follows by axiom (II) for transporter systems that there is an extension of φ, namely $\widetilde{\varphi} \in Aut_{\mathcal{T}}(H_S(P))$, making the following diagram commute:

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\iota_{P,N_S(P)}"'] & P \arrow[d, "\iota_{P,N_S(P)}"] \\ N_S(P) \arrow[r, "\widetilde{\varphi}"'] & N_S(P). \end{tikzcd}$$

Since $Aut_{\mathcal{T}}(N_S(P))$ is locally finite, $|\widetilde{\varphi}| = p^k m$, where $p \nmid m$. Thus, for r such that $r \equiv 0 \pmod m$, $r \equiv 1 \pmod{p^k}$, $(\widetilde{\varphi})^r$ is an automorphism of $N_S(P)$ of order p^k which is again an extension of φ. Thus, we can assume directly that $\widetilde{\varphi}$ has order a power of p.

Choose R fully $\mathcal{F}$-normalized and $\mathcal{F}$-conjugate to $N_S(P)$, and let $\gamma \in Iso_{\mathcal{T}}(N_S(P), R)$. Since we have taken P maximal with respect to the above assumptions, it follows that

$$\varepsilon_{R,R}(N_S(R)) \in Syl_p(Aut_{\mathcal{T}}(R)),$$

and hence $\gamma \circ \widetilde{\varphi} \circ \gamma^{-1}$ is conjugate to $\varepsilon_{R,R}(y)$ for some $y \in N_S(R)$. We can suppose without loss of generality that

$$\gamma \circ \widetilde{\varphi} \circ \gamma^{-1} = \varepsilon_{R,R}(y).$$

By construction, $\widetilde{\varphi} \in Aut_{\mathcal{T}}(R)$ restricts to $\varphi \in Aut_{\mathcal{T}}(P)$. Sigui $P' = \rho(\gamma)(P) \leq R$ i $\gamma_0 = \gamma_{|P,P'} \in Iso_{\mathcal{T}}(P, P')$. There is a commutative diagram

$$\begin{tikzcd} P' \arrow[r, "\gamma_0^{-1}"] \arrow[d, "\iota_{P',R}"'] & P \arrow[r, "\varphi"] & P \arrow[r, "\gamma_0"] & P' \arrow[d, "\iota_{P',R}"] \\ R \arrow[rrr, "\varepsilon_{R,R}(y)=\gamma\widetilde{\varphi}\gamma^{-1}"'] & & & R, \end{tikzcd}$$

from where we deduce that $\gamma_0 \circ \varphi \circ \gamma_0^{-1} \in Aut_{\mathcal{T}}(P')$ is a restriction of $\varepsilon_{R,R}(y)$, i.e., $\rho(\varepsilon_{R,R}(y)) = c_y \in Aut_{\mathcal{F}}(R)$ restricts to an automorphism of P' (in $\mathcal{F}$), which means that $y \in N_S(P')$ and $\gamma_0 \circ \varphi \circ \gamma_0^{-1} = \varepsilon_{P',P'}(y)$.

On the other hand. since P is fully $\mathcal{F}$-normalized, it follows that

$$\rho(\gamma)(N_S(P)) = N_S(P'),$$

and $y = \rho(\gamma)(z)$ for some $z \in N_S(P)$. We thus have the following two commutative squares in $\mathcal{T}$, where all arrows are isomorphism:

$$\begin{tikzcd} P \arrow[r, "\gamma_0"] \arrow[d, "\varphi"'] & P' \arrow[d, "\varepsilon_{P',P'}(y)"] \\ P \arrow[r, "\gamma_0"'] & P' \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\gamma_0"] \arrow[d, "\varepsilon_{P,P}(z)"'] & P' \arrow[d, "\varepsilon_{P',P'}(y)"] \\ P \arrow[r, "\gamma_0"'] & P'. \end{tikzcd}$$

Comparing the two squares, it follows that $\varphi = \varepsilon_{P,P}(z)$ (since all morphisms in $\mathcal{T}$ are monomorphisms in the categorical sense, by Lemma A.2.2 (b)), in contradiction with the assumption that $\varphi \notin \varepsilon_{P,P}(N_S(P))$.

(ii) "only if":

Let $P \in Ob(\mathcal{T})$, and let P' be $\mathcal{F}$-conjugate to P and fully $\mathcal{F}$-normalized. Then

$$\varepsilon_{P',P'}(N_S(P')) \in Syl_p(Aut_{\mathcal{T}}(P'))$$

by (i), and hence

$$\varepsilon_{P',P'}(C_S(P')) = \varepsilon_{P',P'}(N_S(P')) \cap E(P')$$

is a Sylow p-subgroup of $E(P')$. Also, $E(P) \cong E(P')$, so P is fully $\mathcal{F}$-centralized if and only if $C_S(P) \cong C_S(P')$. Equivalently,

$$\varepsilon_{P',P'}(C_S(P')) \in Syl_p(E(P)).$$

□

Using this proposition we can give an alternative statement for axiom (I) in definition A.2.1:

(I') If P is fully $\mathcal{F}$-normalized, then $\varepsilon_{P,P}(N_S(P)) \in Syl_p(Aut_{\mathcal{T}}(P))$.

This will simplify the proof that a linking system is a transporter system.

Proposition A.2.5. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then, $\mathcal{L}$ is a transporter system associated to $\mathcal{F}$.*

Proof. The usual projection functor $\rho : \mathcal{L} \to \mathcal{F}$ in the definition of a linking system plays also the role of the projection functor in the definition of transporter system. Also, in Lemma A.2.3 we have defined a functor $\varepsilon : \mathcal{T}_{Ob(\mathcal{L})}(S) \to \mathcal{L}$. It remains to check that $\mathcal{L}$ satisfies the axioms in definition A.2.1.

(A1) This follows from axiom (A) on $\mathcal{L}$.

(A2) By axiom (A) on $\mathcal{L}$, we know that, for all $P, P' \in \mathcal{L}$, $E(P) = Z(P)$ acts freely on $Mor_{\mathcal{L}}(P, P')$ and that $\rho_{P,P'}$ is the orbit map of this action.

Thus, we have to check that $E(P') = Z(P')$ acts freely on $Mor_{\mathcal{L}}(P, P')$. Suppose $\varphi \in Mor_{\mathcal{L}}(P, P')$ and $x \in E(P')$ are such that

$$\varepsilon_{P'}(x) \circ \varphi = \varphi.$$

Then, x centralizes $\rho(\varphi)(P)$, so $x = \rho(\varphi)(y)$ for some $y \in Z(P)$, since P is $\mathcal{F}$-centric. Hence,

$$\varphi = \delta_{P'}(x) \circ \varphi = \varphi \circ \delta_P(y)$$

by axiom (C) for linking systems, and thus by axiom (A) we deduce that $y = 1$, $x = 1$ and the action is free.

(B) By construction of the functor ε, we know that

$$\varepsilon_{P,P'} : N_S(P, P') \longrightarrow Mor_{\mathcal{L}}(P, P')$$

is injective for all $P, P' \in \mathcal{L}$.

Thus, we have to check that the composite $\rho_{P,P'} \circ \varepsilon_{P,P'}$ sends $g \in N_S(P, P')$ to $c_g \in Hom_S(P, P')$. Note that the following holds for any $P, P' \in \mathcal{L}$ and any $x \in N_S(P, P')$:

$$\iota_{P'} \circ \varepsilon_{P,P'}(x) = \varepsilon_{P',S}(1) \circ \varepsilon_{P,P'}(x) = \varepsilon_S(x) \circ \varepsilon_{P,S}(1) = \delta_S(x) \circ \iota_P$$

in $\mathcal{L}$ and hence so does the following:

$$incl_{P'}^S \circ \rho_{P,P'}(\varepsilon_{P,P'}(x)) = \rho_{P,S}(\iota_{P'} \circ \varepsilon_{P,P'}(x)) = \rho_{P,S}(\delta_S(x) \circ \iota_P) = c_x.$$

in $\mathcal{F}$.

(C) This follows from axiom (C) for linking systems.

(I′) Let $P \in \mathcal{L}$ be fully $\mathcal{F}$-normalized. We want to check that

$$\varepsilon_{P,P}(N_S(P)) \in Syl_p(Aut_{\mathcal{L}}(P)).$$

Now, if P is fully $\mathcal{F}$-normalized (and $\mathcal{F}$-centric, since it is an object in $\mathcal{L}$), then $C_S(P) = Z(P)$ and $N_S(P)/Z(P) = Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$. Since $E(P) = Z(P)$, axiom (I′) follows.

(II) Let $\varphi \in Iso_{\mathcal{L}}(P, P')$, $P \lhd R$, $P' \lhd R'$ be such that

$$\varphi \circ \varepsilon_{P,P}(R) \circ \varphi^{-1} \leq \varepsilon_{P',P'}(R').$$

We want to see that there exists $\widetilde{\varphi} \in Mor_{\mathcal{L}}(R, R')$ such that

$$\widetilde{\varphi} \circ \varepsilon_{P,R}(1) = \varepsilon_{P',R'}(1) \circ \varphi.$$

Since P' is $\mathcal{F}$-centric, it is fully $\mathcal{F}$-centralized. Then, we may apply axiom (II) for fusion systems to the morphism $f = \rho(\varphi)$, that is, f extends to some $\widetilde{f} \in Hom_{\mathcal{F}}(N_f, S)$, where

$$N_f = \{g \in N_S(P) | f c_g f^{-1} \in Aut_S(P')\},$$

and clearly $R \leq N_f$. Hence, $\widetilde{f}$ restricts to a morphism in $Hom_{\mathcal{F}}(R, S)$. Furthermore, $\widetilde{f}(R) \leq R'$ since f conjugates $Aut_R(P)$ into $Aut_{R'}(P')$.

Now, $(\iota_{P',R'} \circ \varphi) \in Mor_{\mathcal{L}}(P, R')$ is a lifting in $\mathcal{L}$ for $incl_{P'}^{R'} \circ f \in Hom_{\mathcal{F}}(P, R')$, and we can fix a lifting $\psi \in Mor_{\mathcal{L}}(R, R')$ for $\widetilde{f}$. Thus, by Lemma 1.4.3 (i) there exists a unique $\widetilde{\iota} \in Mor_{\mathcal{L}}(P, R)$, a lifting of $incl_P^R$, such that

$$\iota_{P',R'} \circ \varphi = \psi \circ \widetilde{\iota}.$$

Since $\rho(\widetilde{\iota}) = incl_P^R = \rho(\iota_{P,R})$, by axiom (A) it follows that there exists some $z \in Z(P)$ such that

$$\widetilde{\iota} = \iota_{P,R} \circ \delta_P(z) = \delta_R(\rho(\iota_{P,R})(z)) \circ \iota_{P,R},$$

where the second equality holds by axiom (C). Hence

$$\iota_{P',R'} \circ \varphi = (\psi \circ \delta_R(\rho(\iota_{P,R})(z))) \circ \iota_{P,R}.$$

(III) Let $P_1 \leq P_2 \leq \ldots$ be an increasing sequence of objects in $\mathcal{L}$, $P = \cup P_n$, and $\varphi_n \in Mor_{\mathcal{L}}(P_n, S)$ satisfying $\varphi_n = \varphi_{n+1} \circ \iota_{P_n,P_{n+1}}$ for all n. We want to see that there exists some $\varphi \in Mor_{\mathcal{L}}(P, S)$ such that $\varphi_n = \varphi \circ \iota_{P_n,P}$ for all n.

Set $f_n = \rho(\varphi_n)$ for all n. Then, by hypothesis, $f_n = f_{n+1} \circ incl_{P_n}^{P_{n+1}}$ for all n. Now, it is clear that $\{f_n\}$ forms a nonempty inverse system, and there exists $f \in Hom_{\mathcal{F}}(P, S)$ such that $f_n = f_{|P_n}$ for all n (the existence follows from Proposition 1.1.4 in [RZ00], and the fact that f is a morphism in $\mathcal{F}$ follows from axiom (III) for fusion systems).

Consider now the following commutative diagram (in $\mathcal{F}$):

$$\begin{tikzcd} & P \arrow[dd, "f"] \\ P_1 \arrow[ur, "incl"] \arrow[dr, "f_1"'] & \\ & S. \end{tikzcd}$$

The same arguments used to prove that axiom (II) for transporter systems holds on $\mathcal{L}$ above apply now to show that there exists a unique $\varphi \in Mor_{\mathcal{L}}(P, S)$ such that $\varphi_1 = \varphi \circ \iota_{P_1,P}$. Combining this equality with $\varphi_1 = \varphi_2 \circ \iota_{P_1,P_2}$ and Lemma A.2.2 (iv) (morphisms in $\mathcal{L}$ are epimorphisms in the categorical sense), it follows that

$$\varphi_2 = \varphi \circ \iota_{P_2,P}.$$

Proceeding by induction it now follows that φ satisfies the desired condition.

□

Finally, we show that, given a transporter system $\mathcal{T}$, we can always obtain a linking system $\mathcal{L}$. However, this linking system may not be a centric linking system, in the sense that $Ob(\mathcal{L})$ may not contain all $\mathcal{F}$-centric subgroups of may contain subgroups which are not $\mathcal{F}$-centric.

Proposition A.2.6. *Let $\mathcal{T}$ be a transporter system associated to a fusion system $\mathcal{F}$ over a discrete p-toral group S. Then, for every $P \in \mathcal{T}$ which is $\mathcal{F}$-centric,*

$$E(P) = Z(P) \times E_0(P),$$

where all the elements in $E_0(P)$ are of order prime to p.

Hence, the category $\mathcal{L}$ defined by $Ob(\mathcal{L}) = \{P \in \mathcal{T} | P \text{ is } \mathcal{F} - \text{centric}\}$ and by

$$Mor_{\mathcal{L}}(P, P') = Mor_{\mathcal{T}}(P, P')/E_0(P)$$

is a linking system associated to $\mathcal{F}$.

Proof. Recall that, for each $P \in \mathcal{T}$, $E(P) = Ker(Aut_{\mathcal{T}}(P) \twoheadrightarrow Aut_{\mathcal{F}}(P))$. Thus, by axiom (C) for transporter systems it follows that $E(P)$ commutes with $\varepsilon_P(P)$: the following diagrama is commutative,

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\varepsilon_P(g)"'] & P \arrow[d, "\varepsilon_P(g)"] \\ P \arrow[r, "\varphi"'] & P, \end{tikzcd}$$

for all $g \in P$ and all $\varphi \in E(P)$. That is,

$$\varepsilon_P(P) \leq C_{Aut_{\mathcal{T}}(P)}(E(P)),$$

and hence $\varepsilon_P(Z(P))$ is central in $E(P)$ (this in fact holds for all $P \in \mathcal{T}$).

Now, suppose $P \in \mathcal{T}$ is $\mathcal{F}$-centric. Then, P is fully $\mathcal{F}$-centralized, and hence by Proposition A.2.4 (ii),

$$\varepsilon_P(C_S(P)) = \varepsilon_P(Z(P)) \in Syl_p(E(P)),$$

and $E(P) = \varepsilon_P(Z(P)) \times E_0(P)$, where each element in $E_0(P)$ has order prime to p.

Finally, let $\mathcal{T}^c \subseteq \mathcal{T}$ be the full subcategory of $\mathcal{T}$ with object set the $\mathcal{F}$-centric subgroups in $\mathcal{T}$. Then, $\mathcal{T}^c$ is again a transporter system, and $\mathcal{L}$ defined as in the statement is a quotient category of $\mathcal{T}^c$, which means that composition is well-defined in $\mathcal{L}$. The axioms for linking systems on $\mathcal{L}$ now hold because of axioms (A1), (A2), (B) and (C) on $\mathcal{T}^c$.

□

A.3 Quotients of transporter systems

Quotients of (saturated) fusion systems have been introduced and studied in section A.1, and now we want to extend those definitions and properties to transporter systems. More specifically, we have proved in Proposition A.1.1 that the quotient of a saturated fusion system by a weakly $\mathcal{F}$-closed subgroup is again a saturated fusion system, and now we prove that the quotient of a transporter system by a weakly closed subgroup is again a transporter system.

Definition A.3.1. *Let $\mathcal{T}$ be a transporter system associated to a fusion system $\mathcal{F}$ over a discrete p-toral groups S, and fix $A \leq S$, not necessarily an object in $\mathcal{T}$. Then,*

- *we say that A is **weakly $\mathcal{T}$-closed** if for any $P, P' \in \mathcal{T}$ containing A as a subgroup and any $\varphi \in Mor_{\mathcal{T}}(P, P')$, the morphism $f = \rho(\varphi) \in Hom_{\mathcal{F}}(P, P')$ satisfies $f(A) \leq A$;*
- *we say that A is **stronly $\mathcal{T}$-closed** if for any pair of objects $P, P' \in \mathcal{T}$ and any $\varphi \in Mor_{\mathcal{T}}(P, P')$, the morphism $f = \rho(\varphi)$ satisfies $f(P \cap A) \leq P' \cap A$;*

- *we say that A is $\mathcal{T}$***-normal** *if for any pair of objects $P, P' \in \mathcal{T}$ and any $\varphi \in Mor_{\mathcal{T}}(P, P')$, there exists $\psi \in Mor_{\mathcal{T}}(PA, P'A)$ such that $\varepsilon_{P',P'A}(1) \circ \varphi = \psi \circ \varepsilon_{P,PA}(1)$ and $\rho(\psi)(A) \leq A$.*

Lemma A.3.2. *If A is $\mathcal{T}$-normal, then it is strongly $\mathcal{T}$-closed. If A is strongly $\mathcal{T}$-closed,then it is weakly $\mathcal{T}$-closed.*

Let $\mathcal{T}$ be a transporter system associated to a fusion system $\mathcal{F}$ over a discrete p-toral group S, and let $A \leq S$ be a weakly $\mathcal{T}$-closed subgroup. We define the **quotient transporter system** of $\mathcal{T}$ by A, $\mathcal{T}/A$, as the category with object set $Ob(\mathcal{T}/A) = \{P/A | P \in \mathcal{T}, A \leq P\}$, and with morphism sets

(A.2) $$Mor_{\mathcal{T}/A}(P/A, P'/A) = Mor_{\mathcal{T}}(P, P')/\varepsilon_P(A) = \varepsilon_{P'}(A) \setminus Mor_{\mathcal{T}}(P, P').$$

As happened in Proposition A.1.1 with quotient fusion systems, the structure on $\mathcal{T}$ induces some structure on $\mathcal{T}/A$.

Proposition A.3.3. *Let $\mathcal{T}$ be a transporter system associated to a fusion system $\mathcal{F}$ over a discrete p-toral group S, and let $A \leq S$ be a weakly $\mathcal{T}$-closed subgroup. Then, $\mathcal{T}/A$ is a transporter system associated to the fusion system $\mathcal{F}/A$ over S/A.*

Proof. The functor $\rho/A : \mathcal{T}/A \to \mathcal{F}/A$ is induced by ρ, and the functor $\varepsilon/A : \mathcal{T}_{Ob(\mathcal{T}/A)}(S/A) \to \mathcal{T}/A$ is defined by

$$(\varepsilon/A)_{P/A,P'/A}(gA) = [\varepsilon_{P,P'}(g)] \in Mor_{\mathcal{T}}(P, P')/\varepsilon_P(A).$$

Axioms (A1) and (B) hold for $\mathcal{T}/A$ because they already hold for $\mathcal{T}$.

Axioms (A2), (C), (I) and (II) hold using the same arguments in the proof for Proposition 3.10 in [OV07].

Thus, we just have to show that axiom (III) also holds for $\mathcal{T}/A$. Suppose that we are given an ascending chain of subgroups in S/A, $P_1/A \leq P_2/A \leq \ldots$. Set then $P/A = \cup P_n/A$, and for each n let $\varphi_n/A \in Mor_{\mathcal{T}/A}(P_n/A, S/A)$ be such that

$$\varphi_n/A = \varphi_{n+1}/A \circ \varepsilon/A_{P_n/A,P_{n+1}/A}(1).$$

We want to see that there exists $\varphi/A \in Mor_{\mathcal{T}}(P/A, S/A)$ such that for each n φ_n/A is the corresponding restriction of φ/A.

We start by chosing liftings in $\mathcal{T}$ of the morphisms φ_n/A so that we can apply axiom (III) in $\mathcal{T}$. Start by chosing a lifting $\varphi_1 \in Mor_{\mathcal{T}}(P_1, S)$ of φ_1/A, and now suppose we have already chosen liftings $\varphi_1, \ldots, \varphi_n$ such that, for each $i = 1, \ldots, n-1$,

$$\varphi_i = \varphi_{i+1} \circ \varepsilon_{P_i,P_{i+1}}(1),$$

and choose a lifting $\varphi'_{n+1} \in Mor_{\mathcal{T}}(P_{n+1}, S)$ of φ_{n+1}/A. This lifting may not satisfy that $\varphi_n = \varphi'_{n+1} \circ \varepsilon_{P_n,P_{n+1}}(1)$, but by definition of $\mathcal{T}/A$ there exists some $a \in A$ such that

$$\varphi_n = \varphi'_{n+1} \circ \varepsilon_{P_n,P_{n+1}}(1) \circ \varepsilon_{P_n}(a) = (\varphi'_{n+1} \circ \varepsilon_{P_{n+1}}(a)) \circ \varphi_{P_n,P_{n+1}}(1),$$

where the second equality holds by axiom (C) for transporter systems applied on $\mathcal{T}$. Thus, $\varphi_{n+1} = \varphi'_{n+1} \circ \varepsilon_{P_{n+1}}(a)$ satisfies de desired condition. Inductively, we obtain liftings for all φ_n/A such that each lifting is the restriction of the next one.

Now, we can apply axiom (III) for transporter systems on $\mathcal{T}$ for the family $\{\varphi_n\}$: there exists some $\varphi \in Mor_{\mathcal{T}}(P, S)$ such that $\varphi_n = \varphi \circ \varepsilon_{P_n,P}(1)$ for all n, and the induced morphism $\varphi/A \in Mor_{\mathcal{T}}(P/A, S/A)$ is the morphism in $\mathcal{T}/A$ we were looking for.

□

A.4 Homotopy properties of transporter systems

We have seen in Proposition A.2.6 that, given a transporter system, we can obtain a linking system by taking suitable quotients of the automorphism groups in the transporter system. We will now prove that actually in this situation, the classifying spaces of the transporter system and of the linking system are homotopy equivalent. Some calculations on higher limits over orbit categories will be needed in order to prove that.

Thus, let $\mathcal{T}$ be a transporter system. We first introduce the orbit category of $\mathcal{T}$, which is analogous to the orbit category of a fusion system as defined in 1.5.1.

Definition A.4.1. *The* ***orbit category*** *of $\mathcal{T}$ is the category $\mathcal{O}(\mathcal{T})$ with object set $Ob(\mathcal{O}(\mathcal{T})) = Ob(\mathcal{T})$ and with morphism sets*

$$Mor_{\mathcal{O}(\mathcal{T})}(P, Q) = \varepsilon_{Q,Q}(Q) \setminus Mor_{\mathcal{T}}(P, Q).$$

Lemma A.4.2. *Fix a transporter system $\mathcal{T}$ associated to a fusion system $\mathcal{F}$ over a discrete p-toral group S, and let*

$$\Phi : \mathcal{O}(\mathcal{T})^{op} \longrightarrow \mathbb{Z}_{(p)} - mod$$

be any functor which vanishes except on the $\mathcal{F}$-conjugacy class of one subgroup $Q \in \mathcal{O}(\mathcal{T})$. Then,

$$\varprojlim{}^{*}_{\mathcal{O}(\mathcal{T})}(\Phi) \cong \Lambda^{*}(Aut_{\mathcal{O}(\mathcal{T})}(P); \Phi(P)).$$

Proof. This lemma is proved by the same arguments used to prove Proposition 5.4 in [BLO07] (an analog for transporter systems of Proposition 2.8 in [BLO07] can easily be deduced precisely using this result). □

Corollary A.4.3. *Fix a transporter system $\mathcal{T}$ associated to a fusion system over a discrete p-toral group S.*

(i) Assume the functor $\Phi : \mathcal{O}(\mathcal{T})^{op} \to \mathbb{Z}_{(p)} - mod$ has the property that for all $P \in \mathcal{T}$ such that $\Phi(P) \neq 0$, there is an element of order p in $Aut_{\mathcal{O}(\mathcal{T})}(P)$ which acts trivially on $\Phi(P)$. Then,

$$\varprojlim{}^{*}_{\mathcal{O}(\mathcal{T})}(\Phi) = 0.$$

(ii) Assume the functor $\Psi : \mathcal{T}^{op} \to \mathbb{Z}_{(p)} - mod$ has the property that for all $P \in \mathcal{T}$ such that $\Psi(P) \neq 0$, there is $g \in C_S(P) \setminus P$ such that $\varepsilon_{P,P}(g)$ acts trivially on $\Psi(g)$. Then,

$$\varprojlim{}^{*}_{\mathcal{T}}(\Psi) = 0.$$

Proof. (i) By Lemma 5.12 in [BLO07] together with the hypothesis on (i), it follows that $\Lambda^{*}(Aut_{\mathcal{O}(\mathcal{T})}(P); \Phi(P)) = 0$ for all $P \in \mathcal{T}$. By Lemma A.4.2 and an appropriate filtration of Φ and the long exact sequences in higher limits for extensions of functors on $\mathcal{O}(\mathcal{T})$ finishes the proof of (i).

(ii) It follows from point (i) together with Proposition A.11 in [OV07]. □

Proposition A.4.4. *Let $\mathcal{T}$ be a transporter system associated to a $\mathcal{F}$ system, and suppose that $\mathcal{T}$ contains all the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups as objects. Let $\mathcal{T}^c$ be the full subcategory of $\mathcal{T}$ with object set the $\mathcal{F}$-centric objects of $\mathcal{T}$, and let $\mathcal{L}$ be the centric linking system obtained from $\mathcal{T}^c$ (Proposition A.2.6).*

Then, the inclusion $\mathcal{T}^c \hookrightarrow \mathcal{T}$ and the projection $\mathcal{T}^c \twoheadrightarrow \mathcal{L}$ induce homotopy equivalences

$$|\mathcal{T}|_p^\wedge \simeq |\mathcal{T}^c|_p^\wedge \simeq |\mathcal{L}|_p^\wedge.$$

Proof. Lemma 1.3 in [BLO03a] holds also if we ask $K(c)$ to be locally finite instead of finite. Thus we can apply it in these case to see that the projection $\mathcal{T}^c \twoheadrightarrow \mathcal{L}$ induces a homotopy equivalence

$$|\mathcal{T}^c|_p^\wedge \simeq |\mathcal{L}|_p^\wedge.$$

We have to prove then that the inclusion $\mathcal{T}^c \hookrightarrow \mathcal{T}$ induces too a homotopy equivalence. Consider the functor $\Phi : \mathcal{T}^{op} \to Ab$ which sends all objects to $\mathbb{F}_p$ and all morphisms to the identity, and let $\Phi_0 \subseteq \Phi$ be the subfunctor

$$\Phi_0(P) = \begin{cases} \Phi(P) & \text{if } P \notin \mathcal{T}^c, \\ 0 & \text{if } P \in \mathcal{T}^c. \end{cases}$$

Then, there are isomorphisms

$$H^*(|\mathcal{T}|; \mathbb{F}_p) \cong \varprojlim{}^*{}_{\mathcal{T}}(\Phi) \quad \text{and} \quad H^*(|\mathcal{T}|, |\mathcal{T}^c|; \mathbb{F}_p) \cong \varprojlim{}^*{}_{\mathcal{T}}(\Phi_0).$$

Now, for each $P \notin \mathcal{T}^c$ which is fully centralized and each $g \in C_S(P) \setminus P$, $\varepsilon_{P,P}(g)$ acts trivially on $\Phi(P)$, and by Lemma A.4.3 (ii) it follows then that $\varprojlim{}^*{}_{\mathcal{T}}(\Phi_0) = 0$, and hence the inclusion of $|\mathcal{T}^c|$ in $|\mathcal{T}|$ is a mod p homology isomorphisms and induces a homotopy equivalence of p-completions.

□

A.5 Extensions of transporter systems by discrete p-toral groups

Finally, we define extensions of transporter systems with discrete p-toral group kernel, and introduce also the notion of admissible extensions. We will then classify all admissible extensions, extending thus the results from [OV07].

Definition A.5.1. *Let $\mathcal{T}$ be a transporter system. An* ***extension*** *of $\mathcal{T}$ by a discrete p-toral group is a category $\widetilde{\mathcal{T}}$, together with a functor $\tau : \widetilde{\mathcal{T}} \to \mathcal{T}$ which is the identity on objects, and such that, for all $\bar{P}, \bar{Q} \in Ob(\widetilde{\mathcal{T}})$, the following hold:*

(i) $K_{\bar{P}} \overset{def}{=} Ker[Aut_{\widetilde{\mathcal{T}}}(\bar{P}) \to Aut_{\mathcal{T}}(P)]$ *is a discrete p-toral group;*

(ii) $K_{\bar{P}}$ *acts freely on* $Mor_{\widetilde{\mathcal{T}}}(\bar{P}, \bar{Q})$ *by right composition and* τ *is the orbit map of this action; and*

(iii) $K_{\bar{Q}}$ *acts freely on* $Mor_{\widetilde{\mathcal{T}}}(\bar{P}, \bar{Q})$ *by left composition and* τ *is the orbit map of this action.*

By definition, the functor τ is source and target regular functor in the sense of definition A.5 in [OV07]. In particular, Lemma A.7 in [OV07] says then that, for all $\bar{P}, \bar{Q} \in \widetilde{\mathcal{T}}$,

$$K_{\bar{P}} = K_{\bar{Q}} = A,$$

for a certain discrete p-toral group A. Thus, we can talk about extensions of $\mathcal{T}$ by the discrete p-toral group A.

First we will check that such an extension is again a transporter system. Let then $\widetilde{S}$ be the pull-back of $\tau_{\bar{S}}$ and ε_S:

(A.3)
$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & Aut_{\widetilde{\mathcal{T}}}(\bar{S}) \arrow[d, "\tau_{\bar{S}}"] \\ S \arrow[r, "\varepsilon_S"'] & Aut_{\mathcal{T}}(S). \end{tikzcd}$$

Let $q : \widetilde{S} \to S$ and $\widetilde{\varepsilon}_{\widetilde{S}} : \widetilde{S} \to Aut_{\widetilde{\mathcal{T}}}(\bar{S})$ be the structure maps of the pull-back. Then, q is surjective and $\widetilde{\varepsilon}_{\widetilde{S}}$ is injective.

Set also $A = Ker(q)$, and $\widetilde{P} = q^{-1}(P)$ for each $P \in \mathcal{T}$. Then, for each $P \in \mathcal{T}$, the subgroup $\widetilde{P}$ fits in an extension

$$A \longrightarrow \widetilde{P} \xrightarrow{q_{|P}} P,$$

and hence we can identify the set of objects in $\widetilde{\mathcal{T}}$ with the set of subgroups $\widetilde{P} \leq \widetilde{S}$ defined as above for each $P \in \mathcal{T}$. Note that $\widetilde{S}$ is a discrete p-toral group since both S and A are.

A fusion system over $\widetilde{S}$ can also be defined as follows. Define a functor $\widetilde{\rho} : \widetilde{\mathcal{T}} \to Gps$ to be the identity on objects and sending a morphism $\widetilde{\varphi}$ in $\widetilde{\mathcal{T}}$ to the unique group homomorphism defined in Lemma 5.5 in [OV07]. Define $\widetilde{\mathcal{F}}$ as the image of $\widetilde{\rho}$. As shown in [OV07], it turns out to be a fusion system over $\widetilde{S}$.

Proposition A.5.2. *The category $\widetilde{\mathcal{T}}$ defines a transporter system associated to the fusion system $\widetilde{\mathcal{F}}$. In particular, $\widetilde{\mathcal{F}}$ is $Ob(\widetilde{\mathcal{T}})$-saturated. Furthermore, A is $\widetilde{\mathcal{T}}$-normal, and*

$$(S, \mathcal{F}, \mathcal{T}) \cong (\widetilde{S}/A, \widetilde{\mathcal{F}}/A, \widetilde{\mathcal{T}}/A).$$

Proof. Since Lemmas 5.2, 5.3, 5.4, 5.5 in [OV07] hold also in this case, the proof of Proposition 5.6 in [OV07] also applies here, and we only have to show that $\widetilde{\mathcal{T}}$ satisfies axiom (III) for transporter systems.

Let then $\widetilde{P}_1 \leq \widetilde{P}_2 \leq \ldots$ be an increasing sequence of subgroups in $Ob(\widetilde{\mathcal{T}})$, and let $\widetilde{P} = \cup \widetilde{P}$. Suppose also that for all n there exists $\widetilde{\varphi}_n \in Mor_{\widetilde{\mathcal{T}}}(\widetilde{P}_n, \widetilde{S})$ such that

$$\widetilde{\varphi}_n = \widetilde{\varphi}_{n+1} \circ \widetilde{\varepsilon}_{\widetilde{P}_n, \widetilde{P}_{n+1}}(1).$$

We have to prove then that there exists $\widetilde{\varphi} \in Mor_{\widetilde{\mathcal{T}}}(\widetilde{P}, \widetilde{S})$ such that

$$\widetilde{\varphi}_n = \widetilde{\varphi} \circ \widetilde{\varepsilon}_{\widetilde{P}_n, \widetilde{P}}(1).$$

By projecting all the $\widetilde{P}_n$ and the $\widetilde{\varphi}_n$ to $\mathcal{T}$, we get a family of subgroups $\{P_n\}$ and morphisms $\{\varphi_n\}$ like the above, and we can apply axiom (III) on $\mathcal{T}$ to see that there exists $\varphi \in Mor_{\mathcal{T}}(P, S)$ such that $\varphi_n = \varphi \circ \varepsilon_{P_n,P}(1)$ for all n.

Let $\widetilde{\varphi}' \in Mor_{\widetilde{\mathcal{T}}}(\widetilde{P}, \widetilde{S})$ be a lifting in $\widetilde{\mathcal{T}}$ of φ. Since, by construction, the projections of $\widetilde{\varphi}_1$ and of $\widetilde{\varphi}' \circ \widetilde{\varepsilon}_{\widetilde{P}_1,\widetilde{P}}(1)$ on $\mathcal{T}$ are equal, it follows that, in $\widetilde{\mathcal{T}}$, they differ by a morphism in $A = Ker(q)$. This means that there exists some $a \in A$ such that

$$\widetilde{\varphi} = \widetilde{\varphi}' \circ \widetilde{\varepsilon}(a)$$

restricts to $\widetilde{\varphi}_1$ and is still a lifting of φ.

Using that $\widetilde{\varphi}_1 = \widetilde{\varphi}_2 \circ \widetilde{\varepsilon}_{\widetilde{P}_1,\widetilde{P}_2}(1)$ and that $\widetilde{\varepsilon}_{\widetilde{P}_1,\widetilde{P}}(1) = \widetilde{\varepsilon}_{\widetilde{P}_2,\widetilde{P}}(1) \circ \widetilde{\varepsilon}_{\widetilde{P}_1,\widetilde{P}_2}(1)$, we have the following equalities

$$\widetilde{\varphi} \circ \widetilde{\varepsilon}_{\widetilde{P}_2,\widetilde{P}}(1) \circ \widetilde{\varepsilon}_{\widetilde{P}_1,\widetilde{P}_2}(1) = \widetilde{\varphi}_1 = \widetilde{\varphi}_2 \circ \widetilde{\varepsilon}_{\widetilde{P}_1,\widetilde{P}_2}(1).$$

Since the natural projection functor $\pi : \widetilde{\mathcal{T}} \to \mathcal{T}$ is, in particular, target regular by definition, it follows by Lemma A.8 in [OV07] that morphisms in $\widetilde{\mathcal{T}}$ are epimorphisms in the categorical sense, and hence, from the above equalities we deduce that the restriction of $\widetilde{\varphi}$ to $\widetilde{P}_2$ is $\widetilde{\varphi}_2$ as desired. Repeating this proces we see that axiom (III) holds in $\widetilde{\mathcal{T}}$.

□

Finally, we classify extensions of a fixed transporter system $\mathcal{T}$ by a fixed discrete p-toral group A. Here, by an action of the transporter systems $\mathcal{T}$ (or of its fundamental group $\pi_1(|\mathcal{T}|)$) on A we mean the natural action described in Lemmas 5.7 and A.7 in [OV07].

Proposition A.5.3. *For a given transporter system $\mathcal{T}$, the extensions of $\mathcal{T}$ by a given discrete p-toral group A are in one-to-one correspondence with actions Φ of $\pi_1(|\mathcal{T}|)$ on A, together with elements of $H^2(|\mathcal{T}|; A)$ (with coefficients twisted by Φ).*

Proof. The same proof as for Proposition 5.8 in [OV07] applies here.

□

Definition A.5.4. *Let $\mathcal{T}$ be a transporter system associated to a fusion system $\mathcal{F}$ over a discrete p-toral group S.*

Let $\Phi : \pi_1(|\mathcal{T}|) \to \Gamma$ be a group homomorphism, and let $S_1 = Ker(\Phi \circ \varepsilon_{S,S})$. The morphism Φ is called ***admissible*** *if, for all fully $\mathcal{F}$-centralized $P \leq S$ such that $C_{S_1}(P) \leq P$, $P \in Ob(\mathcal{T})$.*

The action of a transporter system $\mathcal{T}$ on A, $\Phi : \pi_1(|\mathcal{T}|) \to Out(A)$, is called ***admissible*** *if the morphism Φ is admissible.*

An extension $A \to \widetilde{\mathcal{T}} \to \mathcal{T}$ is ***admissible*** *if the natural action of $\mathcal{T}$ on A is admissible.*

Theorem A.5.5. *Let $\mathcal{T}$ be a transporter system, A be a discrete p-toral group, and $A \to \widetilde{\mathcal{T}} \to \mathcal{T}$ be an admissible extension of transporter systems. Then, the following hold:*

(i) $Ob(\widetilde{\mathcal{T}})$ contains all $\widetilde{\mathcal{F}}$-centric $\widetilde{\mathcal{F}}$-radical subgroups. Moreover, if $\widetilde{P} \leq \widetilde{S}$ is a $\widetilde{\mathcal{F}}$-centric not in $\widetilde{\mathcal{T}}$, then it is $\widetilde{\mathcal{F}}$-conjugate to some $\widetilde{P}'$ such that

$$Out_{\widetilde{S}}(\widetilde{P}') \cap O_p(Out_{\widetilde{\mathcal{F}}}(\widetilde{P}')) \neq \{1\}.$$

(ii) $\widetilde{\mathcal{F}}$ is a saturated fusion system.

Appendix B

Fusion subsystems of p-power index and index prime to p

Extensions of p-local finite groups have been exhaustively studied in [BCG+07] and [OV07], and, while the results from the later paper have been proved to fully extend to p-local compact groups (see appendix §A), this is not the case for the former paper, the reason being that quasicentric subgroups play a central role in the arguments used in [BCG+07].

Nevertheless, some results can be extended to the compact case, such as the Hyperfocal subgroup theorem, or some results about detection of saturated fusion subsystems, which are of interest in this work. In fact, the main interest for us in extending the work in [BCG+07] to p-local compact groups is in order to give a well-defined notion of connectivity of p-local compact groups, and, somehow surprinsingly, this part of [BCG+07] extends without problem to the compact case.

This chapter is then organized as follows. The first section extends the Hyperfocal subgroup theorem to p-local compact groups. Far from being an independent section, the results from section §1 will be used all along the rest of this chapter. The second section develops then some criteria to decide whether a certain subsystem of a saturated fusion system is saturated. Sections three and four are then devoted to detect all saturated fusion subsystems of a given saturated fusion system of p-power index and index prime to p respectively.

B.1 The Hyperfocal subgroup theorem for p-local compact groups

For a finite group G, the subgroup $O^p(G)$ is defined as the minimal normal subgroup of p-power index, or, equivalently, as the subgroup of G generated by all elements of order prime to p. In section §1.8, we have generalized this notion to a (possibly infinite) group G as the subgroup generated by all infinitely p-divisible elements.

Definition B.1.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and define the **hyperfocal subgroup** of $\mathcal{F}$ as the subgroup of S*

$$O^p_{\mathcal{F}}(S) = \langle \{g^{-1}\alpha(g) | g \in P \leq S, \alpha \in O^p(Aut_{\mathcal{F}}(P))\}, T\rangle.$$

Note that, since $T \leq O^p_{\mathcal{F}}(S)$, its index in S is a finite power of p.

We want to prove that for a p-local compact group $\mathcal{G}$, $\pi_1(B\mathcal{G}) \cong S/O^p_{\mathcal{F}}(S)$. Since we follow the steps in [BCG+07], we first need a version of Puig's hyperfocal theorem for artinian locally finite groups (Lemma 2.2 in [BCG+07]).

Let then G be an artinian locally finite group such that has Sylow p-subgroups, fix $S \in Syl_p(G)$, and define

$$\begin{aligned} O^p_G(S) \stackrel{def}{=} O^p_{\mathcal{F}_S(G)}(S) &= \langle \{g^{-1}\alpha(g) | g \in P \leq S, \alpha \in O^p(Aut_G(P))\}, T\rangle \\ &= \langle \{[g,x] | g \in P \leq S, x \in N_G(P) \text{ of } p' \text{ order} p\}, T\rangle. \end{aligned}$$

Lemma B.1.2. *Let G be an artinian locally finite group such that has Sylow p-subgroups, and let $S \in Syl_p(G)$. Then,*

$$O^p_G(S) = S \cap O^p(G).$$

Proof. Let $\underline{O}^p(G)$ be the subgroup of G generated by all elements of order prime to p in G, which is a subgroup of $O^p(G)$. We prove the following result:

$$\underline{O}^p_G(S) \stackrel{def}{=} \langle \{g^{-1}\alpha(g) | g \in P \leq S, \alpha \in O^p(Aut_G(P))\}\rangle = S \cap \underline{O}^p(G),$$

which is equivalent to the statement in the proposition.

The inclusion $\underline{O}^p_G(S) \leq S \cap \underline{O}^p(G)$ holds by the same arguments as in Lemma 2.2 [BCG+07], and we want to see the converse inclusion. Let $\{G_i\}$ be a family of finite subgroups of G such that $G = \cup G_i$, and set $S_i = S \cap G_i$. It follows then that $S = \cup S_i$. We can choose the subgroups G_i such that $S_i \in Syl_p(G_i)$ for all i.

We first check that $\underline{O}^p_G(S) = \cup \underline{O}^p_{G_i}(S_i)$. This is in fact clear since, for each $g^{-1}\alpha(g) \in O^p_G(S)$, we can find M_1 such that, for all $i \geq M_1$, $g \in S_i$ and α is conjugation by an element in G_i, and hence $g^{-1}\alpha(g) \in \underline{O}^p_{G_i}(S_i)$.

Next we show that $S \cap \underline{O}^p(G) = \cup(S_i \cap \underline{O}^p(G_i))$. Note that, since $S = \cup S_i$, we have

$$S \cap \underline{O}^p(G) = \cup(S_i \cap \underline{O}^p(G)).$$

Since, for each i, $\underline{O}^p(G_i)$ is generated by all elements in G_i of order prime to p, it follows that for all i we have inclusions

$$\begin{aligned} S_i \cap \underline{O}^p(G_i) &\leq S_i \cap \underline{O}^p(G), \\ S_i \cap \underline{O}^p(G_i) &\leq S_{i+1} \cap \underline{O}^p(G_{i+1}). \end{aligned}$$

Let $y \in S_i \cap \underline{O}^p(G)$, in particular, $y \in \underline{O}^p(G)$, and hence $y \in \underline{O}^p(G_j)$ for all $j \geq i$ big enough. Since $S_i \leq S_j$ for $j \geq i$, it follows that, for each i, there exists some j such that

$$S_i \cap \underline{O}^p(G_i) \leq S_i \cap \underline{O}^p(G) \leq S_j \cap \underline{O}^p(G_j).$$

Hence, $S \cap \underline{O}^p(G) = \cup(S_i \cap \underline{O}^p(G_i))$, and by the hyperfocal subgroup theorem in the finite case, Lemma 2.2 in [BCG+07],

$$\underline{O}^p_G(S) = \cup \underline{O}^p(G_i) = \cup(S_i \cap \underline{O}^p(G_i) = S \cap \underline{O}^p(G).$$

□

Let $\mathcal{G}$ be a p-local compact group, with fundamental group

$$\pi_1(B\mathcal{G}) = \pi_1(|\mathcal{L}|^{\wedge}_p).$$

Assume that a compatible set of inclusions $\{\iota_{P,Q}\}$ has been fixed in $\mathcal{L}$.

Consider the space $|\mathcal{L}|$, and fix $S \in \mathcal{L}$ as the basepoint of the realization of the category $\mathcal{L}$. For each morphism $\varphi \in Mor_{\mathcal{L}}(P,Q)$, let $J(\varphi) \in \pi_1(|\mathcal{L}|)$ be the homotopy class of the loop $\iota_Q \circ \varphi \circ \iota_P^{-1}$ in $|\mathcal{L}|$. Then, this defines a functor

$$J : \mathcal{L} \longrightarrow \mathcal{B}(\pi_1(|\mathcal{L}|)),$$

where all objects in $\mathcal{L}$ are sent to the unique object in $\mathcal{B}(\pi_1(|\mathcal{L}|))$, and where all inclusion morphisms are sent to the identity automorphism. Let $j : S \to \mathcal{B}(\pi_1(|\mathcal{L}|))$ be the composition of J with the distinguished monomorphism $\delta_S : S \hookrightarrow Aut_{\mathcal{L}}(S)$.

In fact, the same can be considered for the full subcategory $\mathcal{L}^\bullet \subseteq \mathcal{L}$, whose object set is $Ob(\mathcal{L}^\bullet) = \{P \in Ob(\mathcal{L}) | P = (P)^\bullet\}$, since the functor $(\_)^\bullet : \mathcal{L} \to \mathcal{L}^\bullet$ is left adjoint to the inclusion $\mathcal{L}^\bullet \subseteq \mathcal{L}$, and hence $|\mathcal{L}| \simeq |\mathcal{L}^\bullet|$ by Corollary 1 in [Qui73]. In particular, this means that there is a natural isomorphism

$$\pi_1(|\mathcal{L}|) \cong \pi_1(|\mathcal{L}^\bullet|).$$

Furthermore, by 1.3.4, $Ob(\mathcal{L}^\bullet)$ contains finitely many $\mathcal{F}$-conjugacy classes of objects, among which are contained all $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups. We will thus in general work on $\mathcal{L}^\bullet$ rather than on $\mathcal{L}$. Nevertheless, we keep the above notation for simplicity.

Proposition B.1.3. *Let $\mathcal{G}$ be a p-local compact group. Then the following hold:*

(i) For any group Γ and any functor $\lambda : \mathcal{L}^\bullet \to \mathcal{B}(\Gamma)$ which sends inclusions to the identity, there is a unique homomorphism $\bar{\lambda} : \pi_1(|\mathcal{L}^\bullet|) \to \Gamma$ such that $\lambda = \mathcal{B}(\bar{\lambda}) \circ J$.

(ii) For $g \in P \in \mathcal{L}$, $J(\delta_P(g)) = J(\delta_S(g))$. In particular, $J(\delta_P(g)) = 1$ in $\pi_1(|\mathcal{L}^\bullet|)$ if and only if $\delta_P(g)$ is nulhomotopic as a loop based at the vertex P of $|\mathcal{L}^\bullet|$.

(iii) If $\alpha \in Mor_{\mathcal{L}^\bullet}(P,Q)$, and $\rho(\alpha)(x) = y$, then

$$j(y) = J(\alpha) \cdot j(x) \cdot J(\alpha)^{-1}.$$

(iv) If any x and y are $\mathcal{F}$-conjugate elements of S, then $j(x)$ and $j(y)$ are conjugate in $\pi_1(|\mathcal{L}^\bullet|)$.

As a consequence, the above properties hold also for the functor $\lambda \circ (\_)^\bullet$.

Proof. It is clear that any functor $\lambda : \mathcal{L}^\bullet \to \mathcal{B}(\Gamma)$ induces, after geometric realizations, a homomorphism $\bar{\lambda} : \pi_1(|\mathcal{L}^\bullet|) \to \Gamma$. If, in addition, the functor λ sends inclusion morphisms to the identity morphism in $\mathcal{B}(\Gamma)$, then $\lambda = \mathcal{B}(\bar{\lambda}) \circ J$ by definition of J. The rest of the properties follow from axiom (C) for linking systems.

□

Thus, we want to construct functors from $\mathcal{L}^\bullet$ to $\mathcal{B}(\Gamma)$ sending inclusions to the identity automorphism, so that we can obtain some information on the fundamental group $\pi_1(|\mathcal{L}^\bullet|)$. The following lemma is the key result to build up such functors inductively. The properties of the functor $(\_)^\bullet$ will be implicitely used in all the proofs (see chapter §4 in [Jun09]).

Lemma B.1.4. *Let $\mathcal{G}$ be a p-local compact group, together with a fixed compatible set of inclusions $\{\iota_{P,Q}\}$. Let $\mathcal{H}_0$ be a subset of objects in $\mathcal{L}^\bullet$ which is closed under $\mathcal{F}$-conjugacy and overgroups (in $\mathcal{L}^\bullet$). Let also $\mathcal{P}$ be an $\mathcal{F}$-conjugacy class of $\mathcal{F}$-centric subgroups maximal among those not contained in $\mathcal{H}_0$, set $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{P}$, and let $\mathcal{L}_{\mathcal{H}_0} \subseteq \mathcal{L}_{\mathcal{H}} \subseteq \mathcal{L}^\bullet$ be the full subcategories with these object sets. Assume, for some Γ, that*

$$\lambda_0 : \mathcal{L}_{\mathcal{H}_0} \longrightarrow \mathcal{B}(\Gamma)$$

is a functor which sends inclusions to the indetity. Fix $P \in \mathcal{P}$ which is fully $\mathcal{F}$-normalized, and fix a homomorphism $\lambda_P : Aut_{\mathcal{L}^\bullet}(P) \to \Gamma$. Assume also that

(∗) *for all $P \lneqq Q \leq N_S(P)$ such that Q is fully normalized in $N_{\mathcal{F}}(P)$ and for all $\alpha \in Aut_{\mathcal{L}^\bullet}(P)$ and $\beta \in Aut_{\mathcal{L}}(Q)$ such that $\alpha = \beta_{|P}$, $\lambda_P(\alpha) = \lambda_0(\beta^\beta)$.*

Then, there exists a unique extension of λ_0 to a functor $\lambda : \mathcal{L}_{\mathcal{H}} \to \mathcal{B}(\Gamma)$ which sends inclusions to the identity, and such that $\lambda(\alpha) = \lambda_P(\alpha)$ for all $\alpha \in Aut_{\mathcal{L}^\bullet}(P)$.

Note that, in condition (∗), Q may not be an object in $Ob(\mathcal{L}^\bullet)$, but, by Proposition 1.3.3, for all $\beta \in Aut_{\mathcal{L}}(Q)$, $\beta^\bullet \in Aut_{\mathcal{L}^\bullet}(Q^\bullet)$.

Proof. The uniqueness of the extension is an immediate consequence of Alperin's fusion Theorem (1.3.5). We have to prove then the existence of λ. We prove first that (∗) implies the following statement:

(∗∗) For all $Q, Q' \in Ob(\mathcal{L}^\bullet)$ which strictly contain P, and for all $\beta \in Mor_{\mathcal{L}^\bullet}(Q, Q')$ and $\alpha \in Aut_{\mathcal{L}^\bullet}(P)$ such that $\alpha = \beta_{|P}$, $\lambda_P(\alpha) = \lambda_0(\beta)$.

In fact, as we next justify, it is enough to consider the case where $P \leq N = (N_Q(P))^\bullet$, $N' = (N_{Q'}(P))^\bullet$. Indeed, since $\rho(\beta)(P) = \rho(\alpha)(P) = P$, then $\rho(\beta)((N_Q(P)) \leq N_{Q'}(P)$. Also, by the properties of $(\_)^\bullet$, there are chains of inclusions

$$P \leq N_Q(P) \leq N \leq Q$$
$$P \leq N_{Q'}(P) \leq N' \leq Q'$$

and $\rho(\beta)(N) \leq N'$, which in turn implies that the morphism β restricts to some $\bar{\beta} \in Mor_{\mathcal{L}}(N, N')$ (since P is $\mathcal{F}$-centric, the so are $N_Q(P)$ and $N_{Q'}(P)$), and since P is strictly contained in Q and Q', then $P \lneqq N, N'$, and by the induction hypothesis, $\lambda_0(\beta) = \lambda_0(\bar{\beta})$.

Now, by the Alperin's fusion Theorem for linking systems, Proposition 4.2.2, β is a composition of restrictions of automorphisms of subgroups in $\mathcal{L}^\bullet$ which, as in proof of Lemma 2.3 in [BCG+07], can be chosen such that P is contained in all of them,

$$\beta = \beta_k \circ \beta_{k-1} \circ \ldots \circ \beta_0,$$

and hence it follows that

$$\begin{aligned} \lambda_0(\beta) &= \lambda_0(\beta_k)\lambda_0(\beta_{k-1}) \ldots \lambda_0(\beta_0) = \\ &= \lambda_P((\beta_k)_{|P})\lambda_P((\beta_{k-1})_{|P}) \ldots \lambda_P((\beta_0)_{|P}) = \lambda_P(\beta_{|P}) = \lambda_P(\alpha), \end{aligned}$$

where the second equality holds by (∗).

Now we can extend λ_0 to a functor defined on all morphisms in $\mathcal{L}_{\mathcal{H}}$ not in $\mathcal{L}_{\mathcal{H}_0}$. Let $\varphi \in Mor_{\mathcal{L}}(P_1, Q)$ be such a morphism, and let $P_2 = \rho(\varphi)(P_1)$. Then, $P_1, P_2 \in \mathcal{P}$, and there is a unique $\varphi' \in Iso_{\mathcal{L}}(P_1, P_2)$ such that $\varphi = \iota_{P_2,Q} \circ \varphi'$. Since P is fully $\mathcal{F}$-normalized by hypothesis, it follows by Lemma 1.3 in [BCG+07] that there exist isomorphisms $\widetilde{\varphi}_i \in Iso_{\mathcal{L}}((N_S(P_i))^\bullet, N_i)$, $i = 1, 2$, for certain $N_i \leq (N_S(P))^\bullet$, such that their restrictions to P_1, P_2 are isomorphisms $\varphi_i \in Iso_{\mathcal{L}}(P_i, P)$. Set then $\psi = \varphi_2 \circ \varphi' \circ \varphi_1^{-1} \in Aut_{\mathcal{L}^\bullet}(P)$, and define

$$\lambda(\varphi) = \lambda(\varphi') = \lambda_0(\widetilde{\varphi}_2^{-1})\lambda_P(\psi)\lambda_0(\widetilde{\varphi}_1).$$

We have to make sure that the above definition does not depend on the decomposition $\varphi' = \varphi_2^{-1}\psi\varphi_1$. Let then $\varphi' = (\varphi_2')^{-1}\psi'(\varphi_1')$ be another decomposition, and consider the commutative diagram

$$\begin{tikzcd} P \arrow[d, "\psi"'] & P_1 \arrow[l, "\varphi_1"'] \arrow[r, "\varphi_1'"] \arrow[d, "\varphi'"] & P \arrow[d, "\psi'"] \\ P & P_2 \arrow[l, "\varphi_2"] \arrow[r, "\varphi_2'"'] & P, \end{tikzcd}$$

where, for $i = 1, 2$ the isomorphisms φ_i, respect. φ_i', are restrictions of isomorphisms $\widetilde{\varphi}_i$, respect. $\widetilde{\varphi}_i'$, defined on $(N_S(P_i))^\bullet$. Then, to show that $\lambda(\varphi')$ is well defined, we have to show that

$$\lambda_P(\psi') \cdot \lambda_0(\widetilde{\varphi}_1' \circ \widetilde{\varphi}_1^{-1}) = \lambda_0(\widetilde{\varphi}_2' \circ \widetilde{\varphi}_2^{-1}) \circ \lambda_P(\psi),$$

and this holds since, by (∗∗),

$$\lambda_0(\widetilde{\varphi}_i' \circ \widetilde{\varphi}_i^{-1}) = \lambda_P(\varphi_i' \circ \varphi_i^{-1}).$$

Furthermore, the functor λ sends, by definition, inclusion morphisms to the indentity, and compositions to products.

□

Proposition B.1.5. *Let $\mathcal{G}$ be a p-local compact group, and fix a compatible set of inclusions $\{\iota_{P,Q}\}$ in $\mathcal{L}$. Then, there is a unique functor*

$$\widehat{\lambda} : \mathcal{L} \longrightarrow \mathcal{B}(S/O^p_{\mathcal{F}}(S))$$

which sends inclusions to the identity, and such that $\lambda(\delta_S(g)) = g$ for all $g \in S$.

Proof. We construct a functor $\lambda : \mathcal{L}^\bullet \to \mathcal{B} \overset{def}{=} \mathcal{B}(S/O^p_{\mathcal{F}}(S))$ inductively, using Lemma B.1.4. The functor $\widehat{\lambda} : \mathcal{L} \to \mathcal{B}$ will be then defined as the composition $\lambda \circ \bullet$, and its uniqueness follows from the uniqueness of λ.

Thus, let $\mathcal{H}_0 \subseteq Ob(\mathcal{L}^\bullet)$ be a subset which is closed under $\mathcal{F}$-conjugacy and overgroups. Note that this set may be empty. Let also $\mathcal{P}$ be an $\mathcal{F}$-conjugacy class of subgroups in $\mathcal{L}^\bullet$ maximal among those not in $\mathcal{H}_0$, set $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{P}$, and let $\mathcal{L}_{\mathcal{H}_0} \subseteq \mathcal{L}_{\mathcal{H}} \subseteq \mathcal{L}^\bullet$ be the corresponding full subcategories.

Suppose also that a functor

$$\lambda_0 : \mathcal{L}_{\mathcal{H}_0} \longrightarrow \mathcal{B}$$

has already been defined, satisfying that

- $\lambda_0(\delta_S(g)) = g$ for all $g \in S$ (if $S \in \mathcal{H}_0$), and

- λ_0 sends inclusions to the identity.

Fix also $P \in \mathcal{P}$ such that it is fully $\mathcal{F}$-normalized, and let $\delta_P : N_S(P) \to Aut_{\mathcal{L}^\bullet}(P)$ the monomorphism of Lemma A.2.3. Then, by Proposition A.2.4, $Im(\delta_P) \in Syl_p(Aut_{\mathcal{L}^\bullet}(P)$. To simplify the notation, we will refer to $Im(\delta_P)$ as $N_S(P)$. Then,

$$Aut_{\mathcal{L}^\bullet}(P)/O^p(Aut_{\mathcal{L}^\bullet}(P)) \cong N_S(P)/(N_S(P) \cap O^p(Aut_{\mathcal{L}^\bullet}(P))) = N_S(P)/N_0,$$

where, by Lemma B.1.2, N_0 is the subgroup generated by all commutators $[g, x]$ for $g \in Q \leq N_S(P)$ and $x \in N_{Aut_{\mathcal{L}^\bullet}(P)}(Q)$, together with the maximal torus of P (since P being $\mathcal{F}$-centric implies that $rk(P) = rk(NS_(P)))$.

Thus, conjugation by x lies in the automorphism group $Aut_{\mathcal{F}}(Q)$ by 1.13 (d) in [BCG+07] (since we are not using quasicentric linking systems, we may apply this result), and hence $[g, x] = gc_x(g)^{-1} \in O^p_{\mathcal{F}}(S)$, $N_0 \leq O^p_{\mathcal{F}}(S)$, and the inclusion $N_S(P) \leq S$ extends to a homomorphism

$$\lambda_P : Aut_{\mathcal{L}^\bullet}(P) \longrightarrow S/O^p_{\mathcal{F}}(S).$$

Next, we check that λ_0 and λ_P satisfy condition $(*)$ in Lemma B.1.4. Let $N = (N_S(P))^\bullet$, and let $P \lneqq Q \leq N$ be fully $\mathcal{F}$-normalized in $N_{\mathcal{F}}(P)$, $Q \in \mathcal{L}^\bullet$, and let $\alpha \in Aut_{\mathcal{L}^\bullet}(P)$, $\beta \in Aut_{\mathcal{L}^\bullet}(Q)$ be such that $\alpha = \beta_{|P}$. We have to see that $\lambda_P(\alpha) = \lambda_0(\beta)$.

First note that, after taking the k-th power of both α and β, for some suitable $k \equiv 1$ (mod p), we can suppose that both automorphisms have order a (finite) power of p. Then, since Q is fully $\mathcal{F}$-normalized, it follows that

$$Aut_{N_S(P)}(Q) \in Syl_p(Aut_{N_{\mathcal{F}}(P)}(Q),$$

and thus, there is an automorphism $\widetilde{\gamma} \in O^p(Aut_{N_{\mathcal{L}}(P)}(Q))$ such that $\widetilde{\gamma}\beta\widetilde{\gamma}^{-1} = \delta_Q(g)$ for some $g \in N_S(Q) \cap N_S(P)$. Note that, in particular, λ_0 sends $\widetilde{\gamma}$ to the identity since it is a composite of automorphisms of infinitely p-divisible order. If we now set $\gamma = \widetilde{\gamma}_{|P}$, it follows that $\gamma \in O^p(Aut_{\mathcal{L}}(P))$ and hence $\lambda_P(\gamma) = 1$, and by axiom (C),

$$\lambda_0(\beta) = \lambda_0(\delta_Q(g)) = g = \lambda_P(\delta_P(g)) = \lambda(\alpha).$$

This implies that we can now apply Lemma B.1.4 to extend λ_0 to a functor defined on $\mathcal{L}_{\mathcal{H}}$.

□

Theorem B.1.6. *(The Hyperfocal subgroup theorem for p-local compact groups). Let $\mathcal{G}$ be a p-local compact group. Then*

$$\pi_1(B\mathcal{G}) \cong S/O^p_{\mathcal{F}}(S).$$

More precisely, the natural map $\tau : S \to \pi_1(B\mathcal{G})$ is surjective with kernel $Ker(\tau) = O^p_{\mathcal{F}}(S)$.

Proof. Let $\widehat{\lambda} : \mathcal{L} \to \mathcal{B}(S/O^p_{\mathcal{F}}(S))$ be the functor of Proposition B.1.5, and let $|\lambda|$ be the induced map between geometric realizations. Since $S/O^p_{\mathcal{F}}(S)$ is a finite p-group, $|\mathcal{B}(S/O^p_{\mathcal{F}}(S))|$ is p-complete, and $|\lambda|$ factors through the p-completion $B\mathcal{G} = |\mathcal{L}|^\wedge_p$. Consider the following commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "j"] \arrow[dr, "\tau"'] & \pi_1(|\mathcal{L}|) \arrow[d, "(_)^\wedge_p"] \arrow[dr, "\pi_1(|\lambda|)"] & \\
& \pi_1(B\mathcal{G}) \arrow[r, "\pi_1(|\lambda|^\wedge_p)"'] & S/O^p_{\mathcal{F}}(S).
\end{tikzcd}$$

The morphism τ is surjective by Proposition 1.4.4. Also, by construction, the composite $\pi_1(|\lambda|_p^\wedge) \circ \tau = \pi_1(|\lambda|) \circ j$ is the natural projection. Hence $Ker(\tau) \leq O^p_{\mathcal{F}}(S)$. The converse inclusion holds by the same arguments as those used to prove the hyperfocal subgroup theorem for p-local finite groups (Theorem 2.5 in [BCG+07]).

□

The next result is again a generalization to p-local compact groups of a result on p-local finite groups. In this case, it corresponds to Proposition 2.6 in [BCG+07], and needs no prove.

Proposition B.1.7. *Let $\mathcal{G}$ be a p-local compact group. Then, the induced map*

$$\pi_1(|\mathcal{L}|) \longrightarrow \pi_1(|\mathcal{F}^c|)$$

is surjective, and its kernel is generated by elements of p-power order.

B.2 Finding saturated fusion subsystems

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. In this section we study two specific situations in which we can obtain a saturated fusion subsystem $\mathcal{F}'$ over a subgroup $S' \leq S$ from $\mathcal{F}$. One of this situations can be in fact extended to a result on p-local compact groups, while the same cannot (yet) be done for the second one since quasicentric linking systems are involved in the proofs in [BCG+07].

As usual, for an artinian locally finite group G, $O^p(G)$ is the subgroup generated by all infinitely p-divisible elements in G, and $O^{p'}(G)$ is the subgroup generated by all elements of p-power order.

Definition B.2.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $(S', \mathcal{F}') \subseteq (S, \mathcal{F})$ be a saturated fusion subsystem, that is, $S' \leq S$ is a subgroup, and $\mathcal{F}'$ is a subcategory of $\mathcal{F}$ which is saturated as a fusion system over the subgroup S'.*

*(i) We say that $(S', \mathcal{F}')$ is of p-**power index** in $(S, \mathcal{F})$ if*

$$S' \geq O^p_{\mathcal{F}}(S) \quad \textit{and} \quad Aut_{\mathcal{F}'}(P) \geq O^p(Aut_{\mathcal{F}}(P))$$

for all $P \leq S'$.

*(ii) We say that $(S', \mathcal{F}')$ is of **index prime to** p in $(S, \mathcal{F})$ if*

$$S' = S \quad \textit{and} \quad Aut_{\mathcal{F}'}(P) \geq O^{p'}(Aut_{\mathcal{F}}(P))$$

for all $P \leq S$.

We will also use the following notation.

Definition B.2.2. *Fix a discrete p-toral group S. A **restrictive category over** S is a category $\mathcal{F}$ such that $Ob(\mathcal{F})$ is the set of all subgroups of S, such that all morphisms in $\mathcal{F}$ are group monomorphisms between the subgroups, and with the following additional properties:*

(i) for each $P' \leq P \leq S$ and $Q' \leq Q \leq S$, and for each $f \in Hom_{\mathcal{F}}(P, Q)$ such that $f(P') \leq Q'$, $f_{|P'} \in Hom_{\mathcal{F}}(P', Q')$,

(ii) for each $P \leq S$, $Aut_{\mathcal{F}}(P)$ is artinian and locally finite, and

(iii) for each pair $P, Q \leq S$ of finite subgroups, the set $Hom_{\mathcal{F}}(P, Q)$ is finite.

*A restrictive category $\mathcal{F}$ over S is **normalized** by an automorphism $\gamma \in Aut_{\mathcal{F}}(S)$ if for each $P, Q \leq S$, and each $f \in Hom_{\mathcal{F}}(P, Q)$,*

$$\gamma f \gamma^{-1} \in Hom_{\mathcal{F}}(\gamma(P), \gamma(Q)).$$

For any restrictive category $\mathcal{F}$ over S and any (artinian locally finite) subgroup $A \leq Aut(S)$, $\langle \mathcal{F}, A\rangle$ is the smallest restrictive category over S which contains $\mathcal{F}$ together with all automorphisms in A and their restrictions.

Definition B.2.3. *Let $\mathcal{F}$ be any fusion system over a discrete p-toral group S.*

(i) $O^p_*(\mathcal{F}) \subseteq \mathcal{F}$ denotes the smallest restrictive subcategory of $\mathcal{F}$ whose morphism set contains $O^p(Aut_{\mathcal{F}}(P))$ for all $P \leq S$.

(ii) $O^{p'}_*(\mathcal{F}) \subseteq \mathcal{F}$ denotes the smallest restrictive subcategory of $\mathcal{F}$ whose morphism set contains $O^{p'}(Aut_{\mathcal{F}}(P))$ for all $P \leq S$.

The subcategory $O^p_*(\mathcal{F})$ is not in general a fusion system, and this is the reason to use restrictive categories. The subcategory $O^{p'}_*(\mathcal{F})$ is always a fusion system (since $Aut_S(P) \leq O^{p'}(Aut_{\mathcal{F}}(P))$ for all $P \leq S$), but in general fails to be saturated.

Lemma B.2.4. *The following holds for any fusion system $\mathcal{F}$ over a discrete p-toral group S:*

(i) $O^p_*(\mathcal{F})$ and $O^{p'}_*(\mathcal{F})$ are normalized by $Aut_{\mathcal{F}}(S)$.

(ii) If $\mathcal{F}$ is saturated, then $\mathcal{F} = \langle O^p_*(\mathcal{F}), Aut_{\mathcal{F}}(S)\rangle = \langle O^{p'}_*(\mathcal{F}), Aut_{\mathcal{F}}(S)\rangle$.

(iii) If $\mathcal{F}' \subseteq \mathcal{F}$ is any restrictive subcategory normalized by $Aut_{\mathcal{F}}(S)$ and such that $\mathcal{F} = \langle \mathcal{F}', Aut_{\mathcal{F}}(S)\rangle$, then for each $P, Q \leq S$ and $f \in Hom_{\mathcal{F}}(P, Q)$, there is an automorphism $\gamma \in Aut_{\mathcal{F}}(S)$, and morphisms $f' \in Hom_{\mathcal{F}'}(\gamma(P), Q)$ and $f'' \in Hom_{\mathcal{F}'}(P, \gamma^{-1}(Q))$ such that $f = f' \circ \gamma_{|P} = \gamma \circ f''$.

Proof. The same proof as for Lemma 3.4 in [BCG+07] applies here, since all the properties needed there have an analogous for saturated fusion systems over discrete p-toral groups.

□

Lemma B.2.5. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Fix a normal subgroup $S_0 \lhd S$ which is strongly $\mathcal{F}$-closed. Let $(S_0, \mathcal{F}_0)$ be a saturated fusion subsystem of $(S, \mathcal{F})$. Then, for any $P \leq S$ which is $\mathcal{F}$-centric $\mathcal{F}$-radical, $P \cap S_0$ is $\mathcal{F}_0$-centric.*

Proof. Let $P \leq S$ be a $\mathcal{F}$-central $\mathcal{F}$-radical subgroup, and let $P_0 = P \cap S_0$. Choose then $P_0' \in \langle P_0\rangle_{\mathcal{F}}$ such that it is fully $\mathcal{F}$-normalized. It follows then that P_0' is fully $\mathcal{F}$-centralized, and there is some $f \in Hom_{\mathcal{F}}(N_S(P_0), N_S(P_0'))$ such that $f(P_0) = P_0'$. Furthermore, since S_0 is strongly $\mathcal{F}$-closed, $P \leq N_S(P_0)$. Let then $P' = f(P)$, and note that $P_0' = P' \cap S_0$.

For any other $P_0'' \in \langle P_0\rangle_{\mathcal{F}_0}$, there is some $\gamma \in Hom_{\mathcal{F}}(P_0'' \cdot C_S(P_0''), P_0' \cdot C_S(P_0'))$ such that $\gamma(P_0'') = P_0'$ by axiom (II), and thus $\gamma(C_{S_0}(P_0'')) \leq C_{S_0}(P_0')$. So, if $C_{S_0}(P_0') = Z(P_0')$, then $C_{S_0}(P_0'') = Z(P_0'')$ for all $P_0'' \in \langle P_0'\rangle_{\mathcal{F}_0}$, and P_0 is $\mathcal{F}_0$-centric.

Without loss of generality, we can assume that $P' = P$ and $P_0' = P_0$. Since S_0 is strongly $\mathcal{F}$-closed, it follows that, for every $\alpha \in Aut_{\mathcal{F}}(P)$, $\alpha(P_0) = P_0$. Let then $A^0 \leq Aut_{\mathcal{F}}(P)$ be the subgroup of elements which restricts to the identity on P_0 and on P/P_0. This turns out to be a normal discrete p-toral subgroup of $Aut_{\mathcal{F}}(P)$ since, by the exact sequence (2.8.7 in [Suz82])

$$0 \to H^1(P/P_0; Z(P_0)) \longrightarrow Aut_{\mathcal{F}}(P)/Aut_{P_0}(P) \xrightarrow{\Phi_P} Aut_{\mathcal{F}}(P_0) \times Aut(P/P_0),$$

there is an isomorphism $A^0/Aut_{P_0}(P) \cong H^1(P/P_0; Z(P_0))$. Thus, A^0 is a subgroup of $O_p(Aut_{\mathcal{F}}(P))$, and since P is $\mathcal{F}$-radical it follows that $A^0 \leq Inn(P)$.

Let now $x \in C_{S_0}(P_0)$, and assume that the coset $x \cdot Z(P_0) \in C_{S_0}(P_0)/Z(P_0)$ is fixed by the conjugation action of P. Hence, $x \in S_0$, $[x, P_0] = 1$ and $[x, P] \leq Z(P_0)$, which implies that $c_x \in A^0$ and $xg \in C_S(P)$ for some $g \in P$. Since P is $\mathcal{F}$-centric, this means that $xg \in P$, and thus that $x \in C_{S_0}(P_0) \cap P = Z(P_0)$. That is, $[C_{S_0}(P_0)/Z(P_0)]^P = 1$, so $C_{S_0}(P_0)/Z(P_0) = 1$, and P_0 is $\mathcal{F}_0$-centric.

□

Let $\mathcal{G}$ be a p-local compact group and let Γ be a group. We may think of a functor $\mathcal{L} \to \mathcal{B}(\Gamma)$ which sends inclusions to the identity as a function $\bar{\Theta} : Mor(\mathcal{L}) \to \Gamma$ which sends composites to products and inclusions to the identity. Given such a function and a subgroup $H \leq \Gamma$, we may consider then $\mathcal{L}_H \subseteq \mathcal{L}$, the subcategory of $\mathcal{L}$ with the same object set and with $Mor(\mathcal{L}_H) = \bar{\Theta}^{-1}(H)$. This in turn induces a fusion subsystem $\mathcal{F}_H \leq \mathcal{F}$ via the projection functor $\rho : \mathcal{L} \to \mathcal{F}$.

Now, assume there is no associated linking system to the fusion system $\mathcal{F}$. We want to reproduce somehow the previous constructions in this new setting.

Let $\mathfrak{Sub}(\Gamma)$ denote the set of nonempty subsets of Γ. Given a function $\bar{\Theta}$ as above, there is an obvious associated function $\Theta : Mor(\mathcal{F}^c) \to \mathfrak{Sub}(\Gamma)$, which sends a morphism $f \in Mor(\mathcal{F}^c)$ to $\bar{\Theta}(\rho^{-1}(f))$. Furthermore, the function $\bar{\Theta}$ also induces a homomorphism $\theta = \bar{\Theta} \circ \delta_S$ from S to Γ.

Definition B.2.6. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be any full subcategory such that $Ob(\mathcal{F}_0)$ is closed under $\mathcal{F}$-conjugacy. A **fusion mapping triple** for $\mathcal{F}_0$ consists of a triple (Γ, θ, Θ), where Γ is a group, $\theta : S \to \Gamma$ is a homomorphism, and*

$$\Theta : Mor(\mathcal{F}_0) \longrightarrow \mathfrak{Sub}(\Gamma)$$

is a map which satisfies the following conditions for all subgroups $P, Q, R \leq S$ which lie in $\mathcal{F}_0$:

(i) For all $P \xrightarrow{f} Q \xrightarrow{f'} R$ in $\mathcal{F}_0$ and all $x \in \Theta(f')$, $\Theta(f' \circ f) = x \cdot \Theta(f)$.

(ii) If P is fully $\mathcal{F}$-centralized, then $\Theta(Id_P) = \theta(C_S(P))$.

(iii) If $f = c_g \in Hom_{\mathcal{F}}(P, Q)$, where $g \in N_S(P, Q)$, then $\theta(g) \in \Theta(f)$.

(iv) For all $f \in Hom_{\mathcal{F}}(P, Q)$, all $x \in \Theta(f)$ and all $g \in P$, $x\theta(g)x^{-1} = \theta(f(g))$.

(v) If $P \lneqq (P)^\bullet$, then for all $f \in Aut_{\mathcal{F}}(P)$, $\Theta(f) = \Theta(f^\bullet)$.

For any fusion mapping triple (Γ, θ, Θ) and any $H \leq \Gamma$, we let $\mathcal{F}_H^* \subseteq \mathcal{F}$ the smallest restrictive subcategory which contains all $f \in Mor(\mathcal{F}^c)$ such that $\Theta(f) \cap H \neq \emptyset$. Let also $\mathcal{F}_H \subseteq \mathcal{F}_H^*$ be the full subcategory whose objects are the subgroups of $\theta^{-1}(H)$.

The following lemma states further properties of fusion mapping triples.

Lemma B.2.7. *Fix a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, let $\mathcal{F}_0$ be a full subcategory such that $Ob(\mathcal{F}_0)$ is closed under $\mathcal{F}$-conjugacy, and let (Γ, θ, Θ) be a fusion mapping triple for $\mathcal{F}_0$. Then the following hold for all $P, Q, R \in Ob(\mathcal{F}_0)$:*

(vi) *$\Theta(Id_P)$ is a subgroup of Γ, and Θ restricts to a homomorphism*

$$\Theta_P : Aut_{\mathcal{F}}(P) \longrightarrow N_\Gamma(\Theta(Id_P))/\Theta(Id_P).$$

Thus $\Theta_P(f) = \Theta(f)$ (as a coset of $\Theta(Id_P)$) for all $f \in Aut_{\mathcal{F}}(P)$.

(vii) *For all $P \xrightarrow{f} Q \xrightarrow{f'} R$ in $\mathcal{F}_0$ and all $x \in \Theta(f)$, $\Theta(f' \circ f) \supseteq \Theta(f') \cdot x$, with equality if $f(P) = Q$. In particular, if $P \leq Q$ then $\Theta(f'_{|P}) \supseteq \Theta(f')$.*

(viii) *Assume $S \in Ob(\mathcal{F}_0)$. Then for any $f \in Hom_{\mathcal{F}}(P, Q)$, any $\gamma \in Aut_{\mathcal{F}}(S)$ and any $x \in \Theta(\gamma)$, $\Theta(\gamma f \gamma^{-1}) = x\Theta(f)x^{-1}$, where*

$$\gamma \circ f \circ \gamma^{-1} in Hom_{\mathcal{F}}(\gamma(P), \gamma(Q)).$$

Proof. (vi) By (i), for any $\alpha, \beta \in Aut_{\mathcal{F}}(P)$ and any $x \in \Theta(\alpha)$, there is an equality $\Theta(\alpha\beta) = x \cdot \Theta(\beta)$. When applied with $\alpha = \beta = Id_P$, this implies that indeed $\Theta(Id_P) \leq \Gamma$ (and note that $\Theta(Id_P)$ is not the empty set by definition of $\mathfrak{Sub}(\Gamma)$). When applied with $\beta = \alpha^{-1}$, then $x^{-1} \in \Theta(\alpha^{-1})$ if $x \in \Theta(\alpha)$.

Thus, $\Theta(\alpha) = x \cdot \Theta(Id_P)$ implies that

$$\Theta(\alpha^{-1}) = \Theta(Id_P)x^{-1} \quad \text{and} \quad \Theta(\alpha^{-1}) = x^{-1}\Theta(Id_P),$$

which in turn implies that $\Theta(\alpha)$ is both a left and right coset. Thus $\Theta(\alpha) \subseteq N_\Gamma(\Theta(Id_P))$ for all $\alpha \in Aut_{\mathcal{F}}(P)$, and the induced map Θ_P is then a homomorphism.

(vii) By (i), $\Theta(\alpha\beta) \supseteq \Theta(\alpha) \cdot \Theta(\beta)$ for any pair of composable arrows in $\mathcal{F}_0$, and in particular

$$\Theta(\alpha\beta) \supseteq \Theta(\alpha) \cdot x$$

if $x \in \Theta(\beta)$. If β is an isomorphism, then $1 \in \Theta(Id_P) = x \cdot \Theta(\beta^{-1})$ by (vi) and (i), and hence $x^{-1} \in \Theta(\beta^{-1})$. This yields inclusions

$$\Theta(\alpha) = \Theta(\alpha\beta\beta^{-1}) \supseteq \Theta(\alpha\beta)x^{-1} \supseteq \Theta(\alpha)xx^{-1},$$

and hence they are all equalities. The last part of (vii) is the special case where $P \leq Q$ and $\beta = incl_{P,Q}$ ($1 \in \Theta(incl_{P,Q})$ by point (iii)).

(viii) For $x \in \Theta(\alpha)$, $\Theta(\alpha\beta) = x \cdot \Theta(\beta) = \Theta(\alpha\beta\alpha^{-1}) \cdot x$ by (i) and (vii).

□

Recall that, given a group Γ, where a functor $\mathcal{L} \to \mathcal{B}(\Gamma)$ induces a functor $\mathcal{L}^\bullet \to \mathcal{B}(\Gamma)$ and viceversa. We now prove a similar statement regarding fusion systems and fusion mapping triples. This will again allow us to reduce problems involving infinitely many $\mathcal{F}$-conjugacy classes to finitely many.

Lemma B.2.8. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a full subcategory, closed under $\mathcal{F}$-conjugacy, and let $\mathcal{F}_0^\bullet$ be the full subcategory of $\mathcal{F}_0$ with object set $\{P \in Ob(\mathcal{F}_0) | P = (P)^\bullet\}$. Let also (Γ, θ, Θ) be a fusion mapping triple for $\mathcal{F}_0$. Then, $(\Gamma, \theta, \Theta^\bullet)$ is fusion mapping triple for $\mathcal{F}_0^\bullet$, where $\Theta^\bullet = \Theta \circ incl$.*

Reciprocally, let $\mathcal{F}_0^\bullet \subseteq \mathcal{F}$ be a full subcategory which is closed under $\mathcal{F}$-conjugacy and such that each $P \in \mathcal{F}_0^\bullet$ satiesfies $P = (P)^\bullet$, and let $\mathcal{F}_0$ be the greatest subcategory of $\mathcal{F}$ such that for all $P \in \mathcal{F}_0$, $(P)^\bullet \in \mathcal{F}_0^\bullet$ and such that

$$\{f^\bullet | f \in Mor(\mathcal{F}_0)\} = Mor(\mathcal{F}_0^\bullet).$$

Let also (Γ, θ, Θ) be a fusion mapping triple for $\mathcal{F}_0^\bullet$. Then, $(\Gamma, \theta, \Theta^\circ)$ is a fusion mapping triple for $\mathcal{F}_0$, where $\Theta^\circ = \Theta \circ (\_)^\bullet$.

Proof. In each case, we have to check that the new maps, $\Theta^\bullet$ and Θ° respectively, satisfy the conditions in definition B.2.6, but this is immediate by condition (v) in this definition.

□

For reasons that will be made clear in later sections, we need to work with $\mathcal{F}$-quasicentric subgroups. For a (saturated) fusion system $\mathcal{F}$ over a discrete p-toral group S, let $\mathcal{F}^q \subseteq \mathcal{F}$ be the full subcategory of $\mathcal{F}$ with object set all the $\mathcal{F}$-quasicentric subgroups.

Lemma B.2.9. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $P \leq S$ be an $\mathcal{F}$-quasicentric subgroup. Then, $P^\bullet$ is also $\mathcal{F}$-quasicentric.*

Proof. It is a consequence of Proposition 1.3.3, together with the fact that, since $P \leq P^\bullet$, then $C_S(P) \geq C_S(P^\bullet)$.

□

The following result is the induction step that we need in order to construct fusion mapping triples for fusion subsystems. It can be thought of as an equivalent of Lemma B.1.4 for fusion systems.

Lemma B.2.10. *Fix a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S. Let $\mathcal{H}_0$ be a set of $\mathcal{F}$-quasicentric subgroups of S which is closed under $\mathcal{F}$-conjugacy and overgroups, and such that for all $P \in \mathcal{H}_0$, $P = (P)^\bullet$. Let $\mathcal{P}$ be an $\mathcal{F}$-conjugacy class of $\mathcal{F}$-quasicentric subgroups maximal among those not in $\mathcal{H}_0$, and such that for all $P \in \mathcal{P}$, $P = (P)^\bullet$, and set $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{P}$. Let also $\mathcal{F}_{\mathcal{H}_0} \subseteq \mathcal{F}_{\mathcal{H}} \subseteq \mathcal{F}^q$ be the full subcategories with these object sets. Fix a group Γ and a homomorphism $\theta : S \to \Gamma$, and let*

$$\Theta : Mor(\mathcal{F}_{\mathcal{H}_0}) \longrightarrow \mathfrak{Sub}(\Gamma)$$

be such that (Γ, θ, Θ) is a fusion mapping triple for $\mathcal{F}_{\mathcal{H}_0}$. Let $P \in \mathcal{P}$ be fully normalized in $\mathcal{F}$, and fix a homomorphism

$$\Theta_P : Aut_{\mathcal{F}}(P) \longrightarrow N_\Gamma(\theta(C_S(P)))/\theta(C_S(P))$$

such that the following conditions hold:

(a) $x\theta(g)x^{-1} = \theta(f(g))$ *for all* $g \in P$, $f \in Aut_{\mathcal{F}}(P)$ *and* $x \in \Theta_P(f)$.

(b) For all $P \lneqq Q \leq S$ such that $P \triangleleft Q$ and Q is fully normalized in $N_{\mathcal{F}}(P)$, and for all $f \in Aut_{\mathcal{F}}(P)$ and $f' \in Aut_{\mathcal{F}}(Q)$ such that $f = f'_{|P}$, $\Theta_P(f) \supseteq \Theta((f')^{\bullet})$.

Then, there is a unique extension of Θ to a fusion mapping triple $(\Gamma, \theta, \widetilde{\Theta})$ on $\mathcal{F}_{\mathcal{H}}$ such that $\widetilde{\Theta}(f) = \Theta_P(f)$ for all $f \in Aut_{\mathcal{F}}(P)$.

Proof. Note that the above conditions (a) and (b) are necessary if such an extension of Θ is to be constructed, since they correspond to points (vi) and (vii) in Lemma B.2.7. Now, the uniqueness of the extension follows from Alperin's fusion theorem, 1.3.5, and the proof for the existence is identical to the proof of Lemma B.1.4.

□

We now present a criterion to detect saturated fusion subsystems when a fusion mapping triple is provided.

Proposition B.2.11. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let (Γ, θ, Θ) be any fusion mapping triple on $\mathcal{F}^q$, where Γ is either a p-group or a p'-group.*
Then, the following hold for any subgroup $H \leq \Gamma$, where we set $S_H = \theta^{-1}(H)$:

(i) $\mathcal{F}_H$ is a saturated fusion system.

(ii) If γ is a p-group, then a subgroup $P \leq S_H$ is $\mathcal{F}_H$-quasicentric if and only if it is $\mathcal{F}$-quasicentric. Also, $\mathcal{F}_H^* \supseteq O_*^p(\mathcal{F})$.

(iii) If Γ is a p'-group, then $S_H = S$. A subgroup $P \leq S$ is $\mathcal{F}_H$-quasicentric (fully $\mathcal{F}_H$-centralized, fully $\mathcal{F}_H$-normalized) if and only if it is $\mathcal{F}$-centric (fully $\mathcal{F}$-centralized, fully $\mathcal{F}$-normalized). Also, $\mathcal{F}_H^* \supseteq O_*^{p'}(\mathcal{F})$.

Proof. Points (ii) and (iii) correspond to points (b) and (c) in Proposition 3.8 in [BCG+07], and hold by the same arguments.

About the saturation of $\mathcal{F}_H$, axioms (I) and (II) hold by the same arguments used to prove point (a) in 3.8 [BCG+07]. Thus, we only have to prove that axiom (III) also holds for $\mathcal{F}_H$. Let $P_1 \leq P_2 \leq \ldots$ be an ascending family of subgroups of S_H, $P = \cup P_n$, and let $f \in Hom(P, S_H)$ be such that $f_n = f_{|P_n} \in Hom_{\mathcal{F}_H}(P_n, S_H)$ for all n. We have to show that $f \in Hom_{\mathcal{F}_H}(P, S_H)$.

Since $\mathcal{F}$ is saturated, it follows that $f \in Hom_{\mathcal{F}}(P, S_H)$. Using point (vi) in Lemma B.2.7 above, we see that, for all n, $\Theta(f_{n+1}) \subseteq \Theta(f_n)$. Now, since Γ is finite by assumption, and since by hypothesis f_n is a morphism in $\mathcal{F}_H$ for all n, it follows that there exists some M such that, for all $n \geq M$,

$$\Theta(f_n) \cap H = \Theta(f_{n+1}) \cap H \neq \emptyset.$$

By point (v) in definition of fusion mapping triples, it implies that $\Theta(f) \neq \emptyset$, and hence that

$$\Theta(f) \cap H \neq \emptyset.$$

□

Proposition B.2.11 can be extended to a result on p-local compact groups when we restrict to p'-groups Γ, since in the analogous situation for p-local finite groups, quasicentric linking systems are not involved and hence we can reproduce the proofs given there.

For a p-local compact group $\mathcal{G}$, recall the homomorphism $j : S \to \pi_1(|\mathcal{L}|)$ induced by the distinguished monomorphism $\delta_S : S \to Aut_{\mathcal{L}}(S)$, and the functor $J : \mathcal{L} \to \mathcal{B}(\pi_1(|\mathcal{L}|))$ which sends morphisms to (homotopy classes of) loops. Let $\widehat{\theta}$ be a homomorphism from $\pi_1(|\mathcal{L}|)$ to a p'-group Γ, and set

$$\theta = \widehat{\theta} \circ j \in Hom(S, \Gamma) \quad \text{and} \quad \widehat{\Theta} = \mathcal{B}(\widehat{\theta}) \circ J : \mathcal{L} \to \mathcal{B}(\Gamma).$$

Since J depends on a choice of a compatible set of inclusions $\{\iota_{P,Q}\}$, so do these definitions above. For any $H \leq \Gamma$, let $\mathcal{L}_H \subseteq \mathcal{L}$ be the subcategory with the same object set and morphism set $\widehat{\Theta}^{-1}(H)$. Let also $\mathcal{F}_H$ be the fusion subsystem over S_H generated by $\rho(\mathcal{L}_H) \subseteq \mathcal{F}^c$.

Theorem B.2.12. *Let $\mathcal{G}$ be a p-local compact group, together with a fixed compatible set of inclusions $\{\iota_{P,Q}\}$ in $\mathcal{L}$. Fix a p'-group Γ and a surjective homomorphism $\widehat{\theta} : \pi_1(|\mathcal{L}|) \twoheadrightarrow \Gamma$. Then, for each $H \leq \Gamma$, $\mathcal{G}_H = (S_H, \mathcal{F}_H, \mathcal{L}_H)$ is a p-local compact group whose classifying space is homotopy equivalent to the covering space of $|\mathcal{L}|$ with fundamental group $\widehat{\theta}^{-1}(H)$.*

Proof. Define $\Theta : Mor(\mathcal{F}^{\bullet c}) \to \mathfrak{Sub}$ by setting $\Theta(\alpha) = \widehat{\Theta}(\rho^{-1}(\alpha))$, where $\widehat{\Theta} = \mathcal{B}(\widehat{\theta}) \circ J$ as above. Then, θ and Θ satisfy conditions (i)-(v) in B.2.6: (i) and (ii) follow from axiom (A) for linking systems, (iii) follows from Proposition 1.13 [BCG+07], (vi) follows from axiom (C) for linking systems, and (v) follows from the properties of the functor $(\_)^{\bullet}$ (in particular, because given $\varphi \in Mor(\mathcal{L})$ and $f = \rho(\varphi)$, there is an equality $f^{\bullet} = (\rho(\varphi))^{\bullet} = \rho(\varphi^{\bullet})$). Hence, (Γ, θ, Θ) is a fusion mapping triple on $\mathcal{F}^{\bullet c}$, which in turn induces a fusion mapping triple $(\Gamma, \theta, \Theta^{\circ})$ on $\mathcal{F}^c$ by Lemma B.2.8. Also, by Proposition B.2.11, for all $H \leq \Gamma$, $\mathcal{F}_H$ is a saturated fusion subsystem of $\mathcal{F}$ over S_H.

Let $O^{p'}_*(\mathcal{F})$ be the category defined in B.2.3, and let $O^{p'}_*(\mathcal{F})^c$ be the full subcategory whose objects are the $\mathcal{F}$-centric subgroups of S, and let $\mathcal{L}^*_H = \widehat{\Theta}^{-1}(1)$. By (iii) B.2.11, $\rho(\mathcal{L}^*_H)$ contains $O^{p'}_*(\mathcal{F})^c$, and hence by (ii) B.2.4, all morphisms in $\mathcal{L}$ are compositions of morphisms in $\mathcal{L}^*_H$ and restrictions in $Aut_{\mathcal{L}}(S)$. Thus, by definition of $\mathcal{L}^*_H$, and since $\widehat{\Theta}(\alpha) = \widehat{\Theta}(\beta)$ whenever α is a restriction of β, it follows that $\widehat{\Theta}$ restricts to a surjection of $Aut_{\mathcal{L}}(S)$ onto Γ. In particular,

(1) for all $P \in Ob(\mathcal{L})$ and all $g \in \Gamma$, there exists some $P' \leq S$ and $\alpha \in Iso_{\mathcal{L}}(P, P')$ such that $\widehat{\Theta}(\alpha) = g$,

where α can be chosen to be the restriction of an automorphism of S.

We now prove that $\mathcal{L}_H$ is a (centric) linking system associated to $\mathcal{F}_H$. Since Γ is a p'-group, it follows by (iii) B.2.11 that $S_H = S$ for all $H \leq \Gamma$. Let then $P \leq S$ be a $\mathcal{F}$-centric subgroup, and let $g \in P$. Then, by construction, $\widehat{\Theta}(\delta_S(g)) = \theta(g)$ and $\widehat{\Theta}(\iota_P) = 1$, which in particular implies that the inclusion morphisms are in $\mathcal{L}_H$.

Also, since $\iota_P \circ \delta_P(g) = \delta_S(g) \circ \iota_P$, it follows that $\widehat{\Theta}(\delta_P(g)) = \theta(g)$, and in particular $\delta_P(g) \in Aut_{\mathcal{L}_H}(P)$ if and only if $g \in S$. From this we see then that the distinguished monomorphism δ_P restricts to a distinguished monomorphism

$$P \longrightarrow Aut_{\mathcal{L}_H}(P).$$

It also implies that the axioms (A), (B) and (C) for $\mathcal{L}_H$ hold because they already hold on $\mathcal{L}$. Thus, $(S, \mathcal{F}_H, \mathcal{L}_H)$ is a p-local compact group.

Finally, we prove that $|\mathcal{L}_H|$ is indeed homotopy equivalent to a certain covering space of $|\mathcal{L}|$. Note that $Mor(\mathcal{L}_H) = Mor(\mathcal{L}) \cap \widehat{\Theta}^{-1}(H)$. Let $\mathcal{E}_\Gamma(\gamma/H)$ the category with object set Γ/H, and with a morphism g from the coset aH to the coset gaH for each $g \in \Gamma$ and each $aH \in \Gamma/H$. Thus,

$$Aut_{\mathcal{E}_\Gamma}(1 \cdot H) \cong H \quad \text{and} \quad |\mathcal{E}_\Gamma(\Gamma/H)1 = EG/H \simeq BH.$$

Let then $\widetilde{\mathcal{L}}$ be the pullback category in the following square:

$$\begin{tikzcd} \widetilde{\mathcal{L}} \arrow[r] \arrow[d] & \mathcal{E}_\Gamma(\Gamma/H) \arrow[d] \\ \mathcal{L} \arrow[r] & \mathcal{B}(\Gamma). \end{tikzcd}$$

Thus, $Ob(\widetilde{\mathcal{L}}) = Ob(\mathcal{L}) \times \Gamma/H$ and $Mor(\widetilde{\mathcal{L}})$ is the set of pairs of morphisms in $\mathcal{L}$ and $\mathcal{E}_\Gamma(\Gamma/H)$ which are sent to the same morphism in $\mathcal{B}(\Gamma)$. The linking system $\mathcal{L}_H$ can then be identified with the full subcategory of $\widetilde{\mathcal{L}}$ whose objects are the pairs $(P, 1 \cdot H)$, for $P \in Ob(\mathcal{L})$. It follows by (1) that this inclusion $\mathcal{L}_H \hookrightarrow \widetilde{\mathcal{L}}$ is an isomorphism of categories, and thus $|\mathcal{L}_H| \simeq |\widetilde{\mathcal{L}}|$. On the other hand, by construction, $|\widetilde{\mathcal{L}}|$ is the covering space of $|\mathcal{L}|$ with fundamental group $\theta^{-1}(H)$.

□

B.3 Fusion subsystems of p-power index

In this section, we classify all saturated fusion subsystems of (finite) p-power index in a given saturated fusion system $\mathcal{F}$, and in particular show the existence of a unique minimal subsystem $O^p(\mathcal{F})$ of this type. This will be done by applying the results from the previous section. Since quasicentric linking systems are deeply involved in the analogous situation for p-local finite groups, we will not be able to extend the results from section §4 in [BCG+07] to p-local compact groups.

Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, let $O^p_{\mathcal{F}}(S)$ be its hyperfocal subgroup, and define $\Gamma_p(\mathcal{F}) = S/O^p_{\mathcal{F}}(S)$, which is a finite p-group since $T \leq O^p_{\mathcal{F}}(S)$. We will show that there is a bijective correspondence between subgroups of $\Gamma_p(\mathcal{F})$ and saturated fusion subsystems of p-power index in $\mathcal{F}$.

We start by constructing a fusion mapping triple for $\mathcal{F}^q$, so that we can apply Proposition B.2.11.

Lemma B.3.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let*

$$\theta : S \longrightarrow \Gamma_p(\mathcal{F})$$

be the natural projection. Then there is a fusion mapping triple $(\Gamma_p(\mathcal{F}), \theta, \Theta)$ on $\mathcal{F}^q$.

Proof. For simplicity, set $\Gamma = \Gamma_p(\mathcal{F})$. We will construct a fusion mapping triple (Γ, θ, Θ) for $\mathcal{F}^{\bullet q}$ and then extend it to a fusion mapping triple for $\mathcal{F}^q$ by Lemma B.2.8. The map Θ will be constructed inductively using Lemma B.2.10.

Let $\mathcal{H}_0 \subseteq Ob(\mathcal{F}^{\bullet q})$ be a (possibly empty) subset which is closed under $\mathcal{F}$-conjugacy and overgroups. Let also $\mathcal{P}$ be an $\mathcal{F}$-conjugacy class in $Ob(\mathcal{F}^{\bullet q})$ maximal among those not in $\mathcal{H}_0$, set $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{P}$, and let $\mathcal{F}_{\mathcal{H}_0} \subseteq \mathcal{F}_{\mathcal{H}} \subseteq \mathcal{F}^{\bullet q}$ be the corresponding full subcategories. Assume also that a fusion mapping triple $(\Gamma, \theta, \Theta_0)$ has been already constructed for $\mathcal{F}_{\mathcal{H}_0}$.

Recall also that, for an artinian locally finite group G which has Sylow p-subgroups,

$$O^p_G(S) = \langle \{[g,x] | g \in P \leq S, x \in N_G(P) \text{ of order prime to } p\}, T \rangle,$$

where T is the maximal p-discrete torus of S. By Lemma B.1.2, $O^p_G(S) = S \cap O^p(G)$, and thus

$$G/O^p(G) \cong S/O^p_G(S).$$

Fix $P \in \mathcal{P}$ such that it is fully $\mathcal{F}$-normalized, and let N_0 be the subgroup generated by commutators $[g,x]$ for $g \in N_S(P)$ and $x \in N_{Aut_{\mathcal{F}}(P)}(N_S(P))$ of order prime to p, together with the maximal torus of $N_S(P)$. In this situation, $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$ because P is fully $\mathcal{F}$-normalized, and $Aut_{N_0}(P) = O^p_{Aut_{\mathcal{F}}(P)}(Aut_S(P))$, and by Lemma B.1.2,

$$Aut_{\mathcal{F}}(P)/O^p(Aut_{\mathcal{F}}(P)) \cong Aut_S(P)/Aut_{N_0}(P) \cong N_S(P)/\langle N_0, C_S(P) \rangle.$$

Also, $N_0 \leq O^p_{\mathcal{F}}(S)$, and so the inclusion of $N_S(P)$ in S induces a homomorphism

$$\Theta_P : Aut_{\mathcal{F}}(P) \twoheadrightarrow N_S(P)/\langle N_0, C_S(P) \rangle \to N_S(C_S(P) \cdot S_0)/C_S(P) \cdot S_0,$$

where $S_0 = O^p_{\mathcal{F}}(S)$ for short. Point (i) in Lemma B.2.10 holds by construction of Θ_P.

Thus, we have to prove that point (ii) in B.2.10 also holds. Let then $P \lneqq Q \leq N_S(P)$ be a subgroup which is fully normalized in $N_{\mathcal{F}}(P)$, and let $\alpha \in Aut_{\mathcal{F}}(P)$ and $\beta \in Aut_{\mathcal{F}}(Q)$ be such that $\alpha = \beta_{|P}$. We have to check that $\Theta_P(\alpha) \supseteq \Theta_0(\beta^\bullet)$.

Taking the k-th power of both α and β ($\beta^\bullet$) for some appropriate k congruent with 1 modulo p, we can assume that both morphisms have order a power of p. Now, since Q is fully normalized in $N_{\mathcal{F}}(P)$ (and this is a saturated fusion system), it follows that $Aut_{N_S(P)}(Q) \in Syl_p(Aut_{N_{\mathcal{F}}(P)}(Q))$, and thus there are automorphisms $f \in Aut_{\mathcal{F}}(Q)$, $f' = f_{|P} \in Aut_{\mathcal{F}}(P)$ such that $f \beta f^{-1} = (c_g)_{|Q}$ for some $g \in N_S(Q) \cap N_S(P)$. Thus, $(f')\alpha(f')^{-1} = (c_g)_{|P}$. Furthermore, by the properties of the functor $(\_)^\bullet$, $f^\bullet \in Aut_{\mathcal{F}}(Q^\bullet)$, and $f^\bullet \beta^\bullet (f^\bullet)^{-1} = (c_g)_{|Q^\bullet}$. It follows then that

$$\Theta_0(\beta^\bullet) = \Theta((c_g)_{|Q^\bullet}) = g \cdot \theta(C_S(Q^\bullet)) \subseteq g \cdot \theta(C_S(P)) = \Theta_P((c_g)_{|P}) = \Theta_P(\alpha).$$

Thus, using Lemma B.2.10, we can extend Θ_0 to a fusion mapping triple defined on $\mathcal{F}_{\mathcal{H}}$. Since $\mathcal{F}^{\bullet q}$ contains finitely many $\mathcal{F}$-conjugacy classes, we obtain then a fusion mapping triple for $\mathcal{F}^{\bullet q}$, and a fusion mapping triple for $\mathcal{F}^q$.

□

We can now classify all fusion subsystems of p-power index.

Theorem B.3.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, let $O^p_{\mathcal{F}}(S)$ be the hyperfocal subgroup of $\mathcal{F}$, and let $\Gamma_p(\mathcal{F}) = S/O^p_{\mathcal{F}}(S)$. Then, for each $R \leq S$ containing $O^p_{\mathcal{F}}(S)$ there is a unique saturated fusion subsystem $\mathcal{F}_R \subseteq \mathcal{F}$ over R with p-power index in $\mathcal{F}$. Furthermore, $\mathcal{F}_R$ satisfies:*

(i) a subgroup $P \leq R$ is $\mathcal{F}_R$-quasicentric if and only if it is $\mathcal{F}$-quasicentric, and

(ii) for each pair $P, Q \leq R$ of $\mathcal{F}_R$-quasicentric subgroups,

$$Hom_{\mathcal{F}_R}(P,Q) = \{f \in Hom_{\mathcal{F}}(P,Q) | \Theta(f) \cap (R/O^p_{\mathcal{F}}(S)) \neq \emptyset\}.$$

Here, Θ is the map in the fusion mapping triple constructed in Lemma B.3.1.

Proof. Let $\mathcal{F}_R \subseteq \mathcal{F}$ be the fusion system over R defined on $\mathcal{F}$-quasicentric subgroups by the formula given in (i), and then extended to arbitrary subgroups by taking compositions of restrictions of these morphisms. Note that this fusion system corresponds to the fusion system $\mathcal{F}_{R/O^p_{\mathcal{F}}(S)}$ from Proposition B.2.11, but we adopt this notation for the sake of simplicity.

By Proposition B.2.11 (i) and (ii), $\mathcal{F}_R$ is a saturated fusion system over R, a subgroup $P \leq R$ is $\mathcal{F}_R$-quasicentric if and only if P is $\mathcal{F}$-quasicentric, and $Aut_{\mathcal{F}_R}(P) \geq O^p(Aut_{\mathcal{F}}(P))$ for all $P \leq R$.

Let now $\mathcal{F}'_R \subseteq \mathcal{F}$ another saturated fusion subsystem over R which has p-power index. We have to prove that $\mathcal{F}'_R = \mathcal{F}_R$. By hypothesis, for all $P \leq R$, both automorphism groups $Aut_{\mathcal{F}_R}(P)$ and $Aut_{\mathcal{F}'_R}(P)$ contain $O^p(Aut_{\mathcal{F}}(P))$, and hence each of them is generated by $O^p(Aut_{\mathcal{F}}(P)$ together with a Sylow p-subgroup. Thus if P is fully $\mathcal{F}$-normalized in both $\mathcal{F}_R$ and $\mathcal{F}'_R$, then

$$Aut_{\mathcal{F}_R}(P) = \langle O^p(Aut_{\mathcal{F}}(P)), Aut_R(P)\rangle = Aut_{\mathcal{F}'_R}(P).$$

Let $T_R \leq R$ be its maximal torus. Since T_R has finite index in R, it is easy to see (inductively) that, for all $T_R \leq P \leq R$, $Aut_{\mathcal{F}_R}(P) = Aut_{\mathcal{F}'_R}(P)$. We can now define functors $(\_)^{\bullet}_i$, $i = 1, 2$ on $\mathcal{F}_R$ and $\mathcal{F}'_R$ respectively, and, since $Aut_{\mathcal{F}_R}(T_R) = Aut_{\mathcal{F}'_R}(T_R)$, satisfy

$$\mathcal{H}^1 \overset{def}{=} \{(P)^{\bullet}_1 | P \leq R\} = \{(P)^{\bullet}_2 | P \leq R\} \overset{def}{=} \mathcal{H}_2.$$

Furthermore, the first set contains all $\mathcal{F}_R$-centrc $\mathcal{F}_R$-radical subgroups and finitely many $\mathcal{F}_R$-conjugacy classes, and the same happens to the second set with respect to $\mathcal{F}'_R$. It is not difficult to see, inductively on the order of the subgroups in the above sets, and using Alperin's fusion theorem, that for each $P \in \mathcal{H}_1, \mathcal{H}_2$, P is fully $\mathcal{F}_R$-normalized if and only if it is $\mathcal{F}'_R$-normalized, and that

$$Aut_{\mathcal{F}_R}(P) = Aut_{\mathcal{F}'_R}(P).$$

Hence, $\mathcal{F}_R = \mathcal{F}'_R$, and hence the uniqueness of $\mathcal{F}_R$ holds.

□

We can now define $O^p(\mathcal{F})$ as the minimal saturated fusion subsystem of $\mathcal{F}$ of p-power index, which is a saturated fusion (sub)system over the hyperfocal subgroup $O^p_{\mathcal{F}}(S)$.

B.4 Fusion subsystems of index prime to p

In this section, we classify all saturated fusion subsystems of (finite) index prime to p in a given saturated fusion system $\mathcal{F}$, and in particular show that there exists a unique minimal subsystem $O^{p'}(\mathcal{F})$ of this type. We will apply the results from the section §B.2.

Let G be a (possibly infinite) group, and let $O^{p'}(G)$ be the intersection of all normal subgroups in G of finite index prime to p. Given an epimorphism $f : G \twoheadrightarrow H$ such that $Ker(f) \leq O^{p'}(G)$, it follows that f induces an isomorphism $G/O^{p'}(G) \cong H/O^{p'}(H)$. Thus, given a p-local compact group $\mathcal{G}$, Proposition B.1.7 says that the projection $\rho : \mathcal{L} \to \mathcal{F}^c$ induces an isomorphism

$$\pi_1(|\mathcal{L}|)/O^{p'}(\pi_1(|\mathcal{L}|)) \cong \pi_1(|\mathcal{F}^c|)/O^{p'}(\pi_1(|\mathcal{F}^c|)).$$

Fix a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, and define

$$\Gamma_{p'}(\mathcal{F}) = \pi_1(|\mathcal{F}^c|)/O^{p'}(\pi_1(|\mathcal{F}^c|)).$$

We will show that the natural functor

$$\varepsilon_{\mathcal{F}^c} : \mathcal{F}^c \longrightarrow \mathcal{B}(\Gamma_{p'}(\mathcal{F})),$$

induces a bijective correspondence between subgroups of $\Gamma_{p'}(\mathcal{F})$ and fusion subsystems of $\mathcal{F}$ of index prime to p.

Recall (definition B.2.3) the subcategory $O^{p'}_*(\mathcal{F}) \subseteq \mathcal{F}$, the smallest fusion subsystem which contains $O^{p'}(Aut_{\mathcal{F}}(P))$ for all $P \leq S$. Define

$$Out^0_{\mathcal{F}}(S) = \langle [f] \in Out_{\mathcal{F}}(S) | f_{|P} \in Mor_{O^{p'}_*(\mathcal{F})}(P, S), \text{ some } \mathcal{F}\text{-centric} P \leq S\rangle.$$

Then $Out^0_{\mathcal{F}}(S) \lhd Out_{\mathcal{F}}(S)$, since $O^{p'}_*(\mathcal{F})$ is normalized by $Aut_{\mathcal{F}}(S)$ by Lemma B.2.4 (i).

Proposition B.4.1. *There is a unique functor*

$$\widehat{\theta} : \mathcal{F}^c \longrightarrow \mathcal{B}(Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S))$$

with the following properties:

(i) $\widehat{\theta}(f) = [f]$ *for all* $f \in Aut_{\mathcal{F}}(S)$.

(ii) $\widehat{\theta}(f) = [1]$ *if* $f \in Mor(O^{p'}_*(\mathcal{F})^c)$. *In particular,* $\widehat{\theta}$ *sends inclusion morphisms to the identity.*

Furthermore, there is an isomorphism

$$\bar{\theta} : \Gamma_{p'}(\mathcal{F}) \xrightarrow{\cong} Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S)$$

such that $\widehat{\theta} = \mathcal{B}(\bar{\theta}) \circ \varepsilon_{\mathcal{F}^c}$.

Proof. By Lemma B.2.4 (iii), each morphism $f \in Hom_{\mathcal{F}^c}(P, Q)$ factors as a composite of the restriction of some $\alpha \in Aut_{\mathcal{F}}(S)$ followed by a morphism $f' \in Hom_{O^{p'}_*(\mathcal{F})^c}(\alpha(P), Q)$, $f = f' \circ \alpha_{|P}$. Thus, if we have two such decompositions

$$f = f'_1 \circ (\alpha_1)_{|P} = f'_2 \circ (\alpha_2)_{|P},$$

then (after factoring out inclusions), we have

$$(\alpha_2 \circ \alpha_1^{-1})_{|P} = (f'_2)^{-1} \circ f'_1 \in Iso_{O^{p'}_*(\mathcal{F})^c}(\alpha_1(P), \alpha_2(P)).$$

This implies then that $\alpha_2 \circ \alpha_1^{-1} \in Out^0_{\mathcal{F}}(S)$, and hence we can define

$$\widehat{\theta}(f) = [\alpha_1] = [\alpha_2] \in Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S).$$

This prove that $\widehat{\theta}$ is well defined on morphisms, and sends all objects in $\mathcal{F}^c$ to the unique object of $\mathcal{B}(Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S))$. By Lemma B.2.4 (iii) again, this functor preserves compositions, and thus is a well defined functor. Furthermore, it satisfies conditions (i) and (ii) above by construction. The uniqueness of $\widehat{\theta}$ is clear.

It remains then to prove the isomorphism at the end of the statement. Since $Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S)$ is a finite p'-group, $\pi_1(|\widehat{\theta}|)$ factors through a homomorphism

$$\bar{\theta} : \pi_1(|\mathcal{F}^c|)/O^{p'}(\pi_1(|\mathcal{F}^c|)) \longrightarrow Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S),$$

and the inclusion of $BAut_{\mathcal{F}}(S)$ into $|\mathcal{F}^c|$ (as a subcomplex with one vertes S) induces then a homomorphism

$$\tau : Out_{\mathcal{F}}(S) \longrightarrow \pi_1(|\mathcal{F}^c|)/O^{p'}(\pi_1(|\mathcal{F}^c|)).$$

Furthermore, τ is an epimorphism since $\mathcal{F} = \langle O^{p'}_*(\mathcal{F}), Aut_{\mathcal{F}}(S)\rangle$ (by Lemma B.2.4 (ii)), and because any automorphism in $O^{p'}_*(\mathcal{F})$ is a composite of restrictions of automorphisms of p-power order.

By (i), and since θ restricted to $Aut_{\mathcal{F}}(S)$ is the projection onto $Out_{\mathcal{F}}(S)$, the composite $\bar{\theta} \circ \tau$ is the projection of $Out_{\mathcal{F}}(S)$ onto the quotient group $Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S)$. Finally, $Out^0_{\mathcal{F}}(S) \leq Ker(\tau)$ by definition of $Out^0_{\mathcal{F}}(S)$, and hence $\bar{\theta}$ is an isomorphism. □

In order to apply Proposition B.2.11 to prove Theorem B.4.3, we need to prove that fusion mapping triples for $\mathcal{F}^c$ can be extended to fusion mapping triples on $\mathcal{F}^q$.

Lemma B.4.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let (Γ, θ, Θ) be a fusion mapping triple on $\mathcal{F}^c$. Then there is a unique extension*

$$\widetilde{\Theta} : Mor(\mathcal{F}^q) \longrightarrow \mathfrak{Sub}(\Gamma)$$

of Θ such that $(\Gamma, \theta, \widetilde{\Theta})$ is a fusion mapping triple on $\mathcal{F}^q$.

Proof. We have seen in Lemma B.2.8 that a fusion mapping triple for $\mathcal{F}^c$ induces a fusion mapping triple for $\mathcal{F}^{\bullet c}$. Let (Γ, θ, Θ) be also this induced fusion mapping triple for simplicity. We will then extend Θ to a fusion mapping triple for $\mathcal{F}^{\bullet q}$, and then apply Lemma B.2.8 again to obtain a fusion mappin triple for $\mathcal{F}^q$.

Let then $\mathcal{H}_0 \subseteq \mathcal{F}^{\bullet q}$ be a set closed under $\mathcal{F}$-conjugacy and overgroups (in $\mathcal{F}^{\bullet q}$), and such that it contains $Ob(\mathcal{F}^{\bullet c})$, and let $\mathcal{P}$ be a conjugacy class in $\mathcal{F}^{\bullet q}$ maximal among those not in $\mathcal{H}_0$. We want to extend Θ to $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{P}$.

Let $P \in \mathcal{P}$ be fully $\mathcal{F}$-normalized. For each $\alpha \in Aut_{\mathcal{F}}(P)$, there is an extension $\beta \in Aut_{\mathcal{F}}(R)$, where $R = P \cdot C_S(P)$, which in turn induces a unique $\beta^\bullet \in Aut_{\mathcal{F}}(R^\bullet)$. Furthermore, by Proposition 1.2.6, both R and $R^\bullet$ are $\mathcal{F}$-centric (because P is fully $\mathcal{F}$-normalized), and in particular $R^\bullet \in \mathcal{H}_0$. We can define then a map

$$\Theta_P : Aut_{\mathcal{F}}(P) \longrightarrow \mathfrak{Sub}(N_\Gamma(\theta(C_S(P))))$$

by $\Theta_P(\alpha) = \Theta(\beta^\bullet) \cdot \theta(C_S(P))$. By (i) and (ii) in the definition of fusion mapping triples, $\Theta(\beta^\bullet)$ is a left coset of $\theta(C_S(R))$ (because, by Lemma 1.3.2 (iv), $Z(R) = Z(R^\bullet)$), and by (iv) it is also a right coset (where the left and right coset representatives can be chosen to be the same). Hence, $\Theta_P(\alpha)$ is a left and right coset of $\theta(C_S(P))$ (again with the same coset representative on both sides).

If $\beta' \in Aut_{\mathcal{F}}(R)$ is any other extension of α, then by Lemma 3.8 in [BCG+05] (which applies as well in this case), there is some $g \in C_S(P)$ such that $\beta' = c_g \circ \beta$, and then (again by definition B.2.6) $\Theta((\beta')^\bullet) = \Theta(c_g\beta^\bullet) = \theta(g)\Theta(\beta^\bullet)$, and

$$\begin{aligned}\Theta((\beta')^\bullet) \cdot \theta(C_S(P)) &= \theta(g)\Theta(\beta^\bullet) \cdot \theta(C_S(P)) = \\ &= \Theta(\beta^\bullet)\theta(\beta^\bullet(g)) \cdot \theta(C_S(P)) = \Theta(\beta^\bullet) \cdot \theta(C_S(P))\end{aligned}$$

and so the definition of $\Theta_P(\alpha)$ is independent of the choice of the extension of β. This shows that Θ_P is well defined.

Note also that Θ_P respects compositions and, since $\Theta_P(\alpha) = x\cdot\theta(C_S(P)) = \theta(C_S(P))\cdot x$ for some $x \in \Gamma$, we conclude that $x \in N_\Gamma(\theta(C_S(P)))$. Thus, Θ_P induces a homomorphism

$$\Theta_P : Aut_{\mathcal{F}}(P) \longrightarrow N_\Gamma(\theta(C_S(P)))/\theta(C_S(P)).$$

We can now apply Lemma B.2.10 to extend Θ to $\mathcal{H}$.

If $\alpha \in Aut_{\mathcal{F}}(P)$ and $x \in \Theta_P(\alpha)$, then $x = y \cdot \theta(h)$ for some $h \in C_S(P)$ and $y \in \Theta(\beta^\bullet)$, where $\bullet$ is an extension of α to $R = PC_S(P)$. Hence, for any $g \in P$,

$$x\theta(g)x^{-1} = y \cdot (hgh^{-1})y^{-1} = y\theta(g)y^{-1} = \theta(\beta^\bullet(g)) = \theta(\alpha(g)).$$

This shows that point (i) in Lemma B.2.10 holds.

Assume now that $P \lneqq Q \leq N_S(P)$, and let $\alpha \in Aut_{\mathcal{F}}(P)$, $\beta \in Aut_{\mathcal{F}}(Q)$ be such that $\alpha = \beta_{|P}$. Then, in the notation of axiom (II) for saturated fusion systems, $Q\cdot C_S(P) \leq N_\alpha$, and hence α extends to some other $\gamma \in Aut_{\mathcal{F}}(Q \cdot C_S(P))$, and

$$\Theta_P(\alpha) = \Theta(\gamma^\bullet) \cdot \theta(C_S(P))$$

by definition of Θ_P. By Lemma 3.8 [BCG+05] again, $\gamma_{|Q} = c_g \circ \beta$ for some $g \in C_S(P)$, and hence by definition B.2.6, $\Theta(\gamma^\bullet) = \Theta(c_g \circ \beta^\bullet) = \theta(g) \cdot \Theta(\beta^\bullet)$, and so

$$\begin{aligned}\Theta_P(\alpha) &= \theta(g) \cdot \Theta(\beta^\bullet) \cdot \theta(C_S(P)) = \\ &= \Theta(\beta^\bullet)\theta(\beta^\bullet(g)) \cdot \theta(C_S(P)) = \Theta(\beta^\bullet) \cdot \theta(C_S(P)).\end{aligned}$$

In particular, $\Theta_P(\alpha) \supseteq \Theta(\beta^\bullet)$, and point (ii) in Lemma B.2.10 also holds. Thus we can extend Θ to $\mathcal{H}$.

□

Theorem B.4.3. *For any saturated fusion system $\mathcal{F}$ over a discrete p-toral group S, there is a bijective correspondence between subgroups*

$$H \leq \Gamma_{p'}(\mathcal{F}) = Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S)$$

and saturated fusion subsystems $\mathcal{F}_H$ of $\mathcal{F}$ over S of index prime to p in $\mathcal{F}$. The correspondence is given by associating to H the fusion system generated by $\widehat{\theta}^{-1}(\mathcal{B}(H))$, where $\widehat{\theta}$ is the functor of Proposition B.4.1.

Proof. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be any saturated fusion subsystem over S which contains $O^{p'}_*(\mathcal{F})$. Then $Out^0_{\mathcal{F}}(S) \lhd Out_{\mathcal{F}_0}(S)$, and one can set $H = Out_{\mathcal{F}_0}(S)/Out^0_{\mathcal{F}}(S)$. We first show that a morphism $f \in Mor(\mathcal{F}^c)$ is in $\mathcal{F}_0$ if and only if $\widehat{\theta}(f) \in H$, which in turn implies that

$$\mathcal{F}_0 = \widehat{\theta}^{-1}(H).$$

Clearly it is enough to prove this for isomorphisms in $\mathcal{F}^c$.

Let $P, Q \leq S$ be $\mathcal{F}$-centric, $\mathcal{F}$-conjugate subgroups, and fix an isomorphism $f \in Iso_{\mathcal{F}}(P,Q)$. By Lemma B.2.4, we can write $f = f' \circ \alpha_{|P}$, where $\alpha \in Aut_{\mathcal{F}}(S)$ and $f' \in Iso_{O^{p'}_*(\mathcal{F})}(\alpha(P), Q)$. Then, f is in $\mathcal{F}_0$ if and only if $\alpha_{|P}$ is in $\mathcal{F}_0$. Also, by definition of $\widehat{\theta}$ (and of H), $\widehat{\theta}(f) \in H$ if and only if $\alpha \in Aut_{\mathcal{F}_0}(S)$. Thus we have to prove that $\alpha_{|P} \in Mor(\mathcal{F}_0)$ if and only if $\alpha \in Aut_{\mathcal{F}_0}(S)$.

The "if" part is obvious, and we have to check the "only if" part. The same argument used to prove Proposition B.2.11 (iii) shows here that $\alpha(P)$ is $\mathcal{F}_0$-centric, and hence fully $\mathcal{F}$-centralized in $\mathcal{F}_0$. Since $\alpha_{|P}$ extends to an (abstract) automorphism of S, axiom (II) implies that it extends to some $\alpha_1 \in Hom_{\mathcal{F}_0}(N_S(P), S)$. By Proposition 2.8 [BLO07], $\alpha_1 = (\alpha_{|NS(P)}) \circ c_g$ for some $g \in Z(P)$, and hence $\alpha_{|N_S(P)} \in Hom_{\mathcal{F}_0}(N_S(P), S)$. Furthermore, $P \subsetneq N_S(P)$ since, by hypothesis, $P \subsetneq S$. Applying this process repeatedly (and using the functor $(\_)^\bullet$ in $\mathcal{F}_0$), it follows that $\alpha \in Aut_{\mathcal{F}_0}(S)$.

Now, fix a subgroup $H \leq Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S)$, and let $\mathcal{F}_H$ be the smallest fusion system over S which contains $\widehat{\theta}^{-1}(\mathcal{B}(H))$. We show then that $\mathcal{F}_H$ is a saturated fusion system over S of index prime to p in $\mathcal{F}$. For $\mathcal{F}$-centric subgroups $P, Q \leq S$, $Hom_{\mathcal{F}_H}(P,Q)$ is the set of all morphisms $f \in Hom_{\mathcal{F}}(P,Q)$ such that $\widehat{\theta}(f) \in H$. Thus, in particular, $\mathcal{F}_H \supseteq O^{p'}_*(\mathcal{F})$ because all morphisms in $O^{p'}_*(\mathcal{F})$ are sent by $\widehat{\theta}$ to the identity.

Define a map $\Theta : Mor(\mathcal{F}^c) \to \mathfrak{Sub}(\Gamma_{p'}(\mathcal{F}))$ by setting $\Theta(f) = \{\widehat{\theta}(f)\}$, that is, each image is a subset with one element. Let also $\theta \in Hom(S, \Gamma_{p'}(\mathcal{F}))$ be the trivial (and unique) homomorphism. Then, it follows that $(\Gamma_{p'}(\mathcal{F}), \theta, \Theta)$ is a fusion mapping triple of $\mathcal{F}^c$, which, by Lemma B.4.2, can be extended to a fusion mapping triple for $\mathcal{F}^q$, and thus $\mathcal{F}_H$ is saturated by Proposition B.2.11.

By Alperin's fusion Theorem (1.3.5), $\mathcal{F}_H$ is the unique saturated fusion subsystem of $\mathcal{F}$ with the property that a morphism $f \in Hom_{\mathcal{F}}(P,Q)$ between $\mathcal{F}$-centric subgroups of S lies in $\mathcal{F}_H$ if and only if $\widehat{\theta}(f) \in H$. This shows that the correspondence is bijective. □

Finally, we extend this result to a theorem on p-local compact groups.

Theorem B.4.4. *Fix a p-local compact group $\mathcal{G}$. Then, for each $H \leq Out_{\mathcal{F}}(S)$ which contains $Out^0_{\mathcal{F}}(S)$, there is a unique p-local compact subgroup $\mathcal{G}_H = (S, \mathcal{F}_H, \mathcal{L}_H)$ such that*

(i) *$\mathcal{F}_H$ has index prime to p in $\mathcal{F}$,*

(ii) *$Out_{\mathcal{F}_H}(S) = H$, and*

(iii) *$\mathcal{L}_H = \rho^{-1}(\mathcal{F}_H)$.*

Furthermore, $|\mathcal{L}_H|$ is homotopy equivalent, via its inclusion into $|\mathcal{L}|$, to the covering space of $|\mathcal{L}|$ with fundamental group $\widetilde{H}$, where $\widetilde{H}$ is the subgroup of $\pi_1(|\mathcal{L}|)$ such that $\bar{\theta}(\widetilde{H}/O^{p'}(\pi_1(|\mathcal{L}|)))$ corresponds to $H/Out^0_{\mathcal{F}}(S)$ under the isomorphism $\bar{\theta}$ from Proposition B.4.1.

Proof. The statement is proved by Theorem B.2.12, applied to the composite functor

$$\mathcal{L} \xrightarrow{\rho} \mathcal{F}^c \xrightarrow{\widehat{\theta}} \mathcal{B}(Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S)).$$

Indeed, this result says then that $(S, \mathcal{F}_H, \mathcal{L}_H)$ is a p-local compact groups, and that $|\mathcal{L}_H|$ is homotopy equivalent to the covering space of $\mathcal{L}$ with fundamental group $\widetilde{H}$. Uniqueness follows from Theorem B.4.3.

□

In particular, if we take $H = Out^0_{\mathcal{F}}(S)$, it follows that there exists a unique minimal saturated fusion subsystem (resp. p-local compact subgroup) $O^{p'}(\mathcal{F})$ (resp. $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$) of index prime to p in $\mathcal{F}$. Furthermore, a centric linking system associated to $\mathcal{F}$ induces a (unique) centric linking system $O^{p'}(\mathcal{L})$ associated to $O^{p'}(\mathcal{F})$.

Bibliography

[AC77] J. F. Adams and F. W. Clarke. Stable operations on complex K-theory. *Illinois J. Math.*, 21(4):826–829, 1977.

[AG09] Kasper K. S. Andersen and Jesper Grodal. The classification of 2-compact groups. *J. Amer. Math. Soc.*, 22(2):387–436, 2009.

[AGMV08] K. K. S. Andersen, J. Grodal, J. M. Møller, and A. Viruel. The classification of p-compact groups for p odd. *Ann. of Math. (2)*, 167(1):95–210, 2008.

[AM04] Alejandro Adem and R. James Milgram. *Cohomology of finite groups*, volume 309 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.

[Asc08] Michael Aschbacher. Normal subsystems of fusion systems. *Proc. Lond. Math. Soc. (3)*, 97(1):239–271, 2008.

[BCG$^+$05] Carles Broto, Natàlia Castellana, Jesper Grodal, Ran Levi, and Bob Oliver. Subgroup families controlling p-local finite groups. *Proc. London Math. Soc. (3)*, 91(2):325–354, 2005.

[BCG$^+$07] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver. Extensions of p-local finite groups. *Trans. Amer. Math. Soc.*, 359(8):3791–3858 (electronic), 2007.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[BLO03a] Carles Broto, Ran Levi, and Bob Oliver. Homotopy equivalences of p-completed classifying spaces of finite groups. *Invent. Math.*, 151(3):611–664, 2003.

[BLO03b] Carles Broto, Ran Levi, and Bob Oliver. The homotopy theory of fusion systems. *J. Amer. Math. Soc.*, 16(4):779–856 (electronic), 2003.

[BLO07] Carles Broto, Ran Levi, and Bob Oliver. Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups. *Geom. Topol.*, 11:315–427, 2007.

[BM07] Carles Broto and Jesper M. Møller. Chevalley p-local finite groups. *Algebr. Geom. Topol.*, 7:1809–1919, 2007.

[DRV07] Antonio Díaz, Albert Ruiz, and Antonio Viruel. All p-local finite groups of rank two for odd prime p. *Trans. Amer. Math. Soc.*, 359(4):1725–1764 (electronic), 2007.

[DW94] W. G. Dwyer and C. W. Wilkerson. Homotopy fixed-point methods for Lie groups and finite loop spaces. *Ann. of Math. (2)*, 139(2):395–442, 1994.

[DW98] W. G. Dwyer and C. W. Wilkerson. The elementary geometric structure of compact Lie groups. *Bull. London Math. Soc.*, 30(4):337–364, 1998.

[FM84] Eric M. Friedlander and Guido Mislin. Cohomology of classifying spaces of complex Lie groups and related discrete groups. *Comment. Math. Helv.*, 59(3):347–361, 1984.

[JMO90] Stefan Jackowski, James E. McClure, and Bob Oliver. Self-maps of classifying spaces of compact simple Lie groups. *Bull. Amer. Math. Soc. (N.S.)*, 22(1):65–72, 1990.

[JMO92a] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy classification of self-maps of BG via G-actions. I. *Ann. of Math. (2)*, 135(1):183–226, 1992.

[JMO92b] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy classification of self-maps of BG via G-actions. II. *Ann. of Math. (2)*, 135(2):227–270, 1992.

[Jun09] Fabien Junod. Unstable adams operations on p-local compact groups. *PhD Thesis*, pages iv+31, 2009.

[KS08] Radha Kessar and Radu Stancu. A reduction theorem for fusion systems of blocks. *J. Algebra*, 319(2):806–823, 2008.

[Lin06] Markus Linckelmann. Simple fusion systems and the Solomon 2-local groups. *J. Algebra*, 296(2):385–401, 2006.

[LS07] Ian J. Leary and Radu Stancu. Realising fusion systems. *Algebra Number Theory*, 1(1):17–34, 2007.

[Møl07] Jesper M. Møller. N-determined 2-compact groups. II. *Fund. Math.*, 196(1):1–90, 2007.

[OV07] Bob Oliver and Joana Ventura. Extensions of linking systems with p-group kernel. *Math. Ann.*, 338(4):983–1043, 2007.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Rob07] Geoffrey R. Robinson. Amalgams, blocks, weights, fusion systems and finite simple groups. *J. Algebra*, 314(2):912–923, 2007.

[RZ00] Luis Ribes and Pavel Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[Ser03] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[Suz82] Michio Suzuki. *Group theory. I*, volume 247 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1982. Translated from the Japanese by the author.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

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