UNED Escuela Internacional de Doctorado EIDUNED

TESIS DOCTORAL
2017

On the existence of abelian groups of automorphisms of Klein surfaces

Jesús Rodríguez Martín de los Santos

Programa de Doctorado en Ciencias

Director: D. Francisco Javier Cirre Torres
Codirector: D. Emilio Bujalance García

Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED.

Programa de Doctorado en Ciencias.

Tesis Doctoral: On the existence of abelian groups of automorphisms
of Klein surfaces.

Año: 2017.

Autor: Jesús Rodríguez Martín de los Santos.

Director: D. Francisco Javier Cirre Torres.

Codirector: D. Emilio Bujalance García.

Agradecimientos

Todo este trabajo que aquí expongo está precedido del realizado en el Máster en Matemáticas Avanzadas de la UNED. Tras concluirlo, enseguida comencé las investigaciones que finalmente cristalizan en esta memoria. Emilio Bujalance y Francisco Javier Cirre me dieron esa oportunidad. Desde entonces, muchas veces me detengo a recordar cómo han transcurrido estos años de trabajo intenso y en esos recuerdos siempre encuentro el apoyo, el ánimo, la cordialidad, la amabilidad que Emilio y Javier me han ofrecido. Me siento privilegiado por haber podido contar con ellos, guiándome con erudición y sensatez, y les estoy enormemente agradecido. También quisiera extender este agradecimiento a todos los miembros del Departamento de Matemáticas Fundamentales de la UNED, que siempre se han mostrado afables y atentos.

A mis padres, Juan y Consuelo, y mis hermanos, gracias por haber mostrado vuestro interés y haberme dado ánimos.

Clothilde, sin tu paciencia y todo el apoyo que me has dado no me hubiera sido posible recorrer este camino. Entretanto, Heliodoro ha llegado. A vosotros os dedico esta memoria.

Contents

Resumen

Esta memoria está dedicada al estudio de grupos de automorfismos de superficies de Klein. Más concretamente, a la búsqueda de condiciones bajo las que un grupo abeliano finito es isomorfo a un grupo de automorfismos de alguna superficie de Klein compacta de cierto género algebraico, diferenciando entre superficies orientables sin borde, superficies no orientables sin borde y superficies con borde. Además, esos resultados nos permitirán responder a los llamados problemas de *género mínimo* y *orden máximo* de forma diferente y, ciertamente, más concisa que las conocidas hasta ahora.

Los aspectos teóricos en los que se basan los desarrollos propuestos se presentan en el primer capítulo. Si bien Klein propuso por primera vez la utilización de superficies de Klein (asociando una superficie posiblemente no orientable o con borde a cada curva algebraica compleja), no fue hasta finales de los años 60 del pasado siglo cuando Alling y Greenleaf realizan diversos trabajos, los recopilan en [1] y comienzan el tratamiento moderno de tales superficies.

Una superficie de Klein está dotada de una estructura *dianalítica*, ampliación de la estructura analítica de superficies de Riemann que permite funciones de transición *antianalíticas*, i.e., tales que su conjugada compleja es analítica. Además, el dominio de las funciones de transición entre cartas puede ser un abierto de $\mathbb{C}^+$, el semiplano superior cerrado del plano complejo. De esa forma se generaliza el concepto de superficie de Riemann, que consta de superficies orientables y sin borde, al de superficie de Klein, en el que se incluyen, además, tanto superficies no orientables como con borde. El *género algebraico* de una superficie de Klein es el número entero $\eta g + k - 1$, donde $\eta = 2$ o 1, dependiendo de si la superficie es orientable o no, g es el género topológico de la superficie y k el número de componentes conexas de su borde.

La composición de *morfismos* entre superficies de Klein posee una serie de

propiedades que hacen que, cuando se trata de morfismos biyectivos de una superficie de Klein en sí misma, i.e., de *automorfismos* de una superficie de Klein, en conjunto dispongan de una estructura de grupo. Entre los distintos grupos de automorfismos de los que puede disponer una superficie de Klein, los que actúan *propiamente discontinuamente* tienen especial importancia. Esencialmente, ningún automorfismo de un grupo tal consigue acercar tanto como queramos a dos puntos que pertenecen a órbitas diferentes, y hay un número finito de automorfismos que a un punto dado lo llevan arbitrariamente cerca de sí mismo. Cabe reseñar que el espacio de órbitas de un grupo de automorfismos que actúa propiamente discontinuamente admite una única estructura dianalítica para la que la proyección canónica es un morfismo.

Entre los grupos que actúan propiamente discontinuamente, destacamos los subgrupos *discretos* del grupo de automorfismos $\mathrm{Aut}(\mathcal{H})$ del semiplano superior complejo $\mathcal{H}$. Son llamados discretos en tanto que se trata de subespacios topológicos discretos de $\mathrm{Aut}(\mathcal{H})$, ya que este es isomorfo al grupo topológico $PGL(2,\mathbb{R})$.

Si el espacio cociente $\mathcal{H}/\Lambda$ de un subgrupo discreto Λ de $\mathrm{Aut}(\mathcal{H})$ es compacto, decimos que Λ es un *grupo cristalográfico no euclídeo*, o, de forma abreviada por sus siglas en inglés, un *grupo NEC*. En particular, si $\mathcal{H}/\Lambda$ es una superficie de Riemann, Λ es un grupo *Fuchsiano*, lo cual equivale a que todos los elementos de Λ actúen manteniendo la orientación en $\mathcal{H}$. En los demás casos, cuando $\mathcal{H}/\Lambda$ es no orientable o tiene borde, el grupo NEC es *propio*. Se dice que un grupo NEC es *de superficie* si ninguno de sus elementos de orden finito mantiene la orientación. En todo caso, el cociente $\mathcal{H}/\Lambda$ admite una única estructura dianalítica tal que la proyección canónica $\mathcal{H} \to \mathcal{H}/\Lambda$ es un morfismo.

Un punto clave es el hecho de que los grupos NEC *uniformizan* a las superficies de Klein compactas de género algebraico mayor que uno, es decir, toda superficie de Klein tal puede ser representada por el cociente $\mathcal{H}/\Gamma$ para cierto grupo NEC de superficie Γ. Este resultado fue establecido por Preston [31] y generaliza el concepto de uniformización de superficies de Riemann propuesto anteriormente por Poincaré y Klein como fruto de un apasionante intercambio epistolar entre 1880 y 1882.

Macbeath [23] y Wilkie [38] asociaron una colección de números enteros y símbolos a cada grupo NEC que permite diferenciarlo de otros grupos NEC. Tal colección es llamada *signatura*, que, en general, es de la siguiente forma:

$$(g;\, \pm\,;\, [m_1, \ldots, m_r];\, \{(n_{i1}, \ldots, n_{is_i}),\, i = 1, \ldots, k\}).$$

Para un grupo Fuchsiano se utiliza $(g;\, m_1, \ldots, m_r)$ de forma abreviada. Un grupo NEC es de superficie si su signatura es de la forma $(g;\, \pm\,;\, [-];\, \{(-), \overset{k}{\ldots}, (-)\})$. La signatura de un grupo NEC Λ determina la estructura algebraica y topológica del espacio cociente $\mathcal{H}/\Lambda$. El área de cualquier región fundamental de $\mathcal{H}/\Lambda$ es $2\pi\mu(\Lambda)$, donde

$$\mu(\Lambda) = \eta g + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right),$$

donde $\eta = 1$ si la signatura tiene signo '$-$' y es 2 si el signo es '$+$'. Para cualquier subgrupo Λ' de Λ de índice finito se verifica $[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda)$, que es la fórmula de Riemann-Hurwitz asociada al recubrimiento $\mathcal{H}/\Lambda' \to \mathcal{H}/\Lambda$. La signatura proporciona asimismo la presentación *canónica* del grupo NEC.

Una consecuencia directa de la fórmula de Riemann-Hurwitz y del hecho de que $\mathrm{Aut}(\mathcal{H}/\Gamma)$ es el cociente $N(\Gamma)/\Gamma$, donde Γ es un grupo de superficie y $N(\Gamma)$ es el normalizador de Γ en $\mathrm{Aut}(\mathcal{H})$, es que todo grupo de automorfismos de una superficie de Klein compacta de género algebraico mayor que uno tiene orden finito. Así mismo, los grupos de automorfismos de $\mathcal{H}/\Gamma$ están caracterizados como cocientes Λ/Γ para cierto grupo NEC Λ del que Γ es subgrupo normal de índice finito. De esta forma, un grupo G será de automorfismos de $\mathcal{H}/\Gamma$ si y solo si existe un epimorfismo $\Lambda \to G$ cuyo núcleo sea el grupo NEC de superficie Γ. A dicho tipo de epimorfismos se les denomina epimorfismos *con núcleo de superficie*. De forma equivalente, un epimorfismo con núcleo de superficie se caracteriza por mantener el orden de todo elemento de Λ de orden finito que conserva la orientación.

Para algunas familias de grupos finitos actuando sobre superficies de determinada clase (de Riemann, de Riemann no orientables, o con borde; en todo caso, compactas y de género algebraico mayor que uno) ha sido posible encontrar condiciones sobre los parámetros de la signatura de Λ y de la estructura algebraica de G para que existan epimorfismos con núcleo de superficie $\Lambda \to G$. Este tipo de condiciones, junto con la fórmula de Riemann-Hurwitz, permiten establecer de forma precisa cuándo un grupo de la familia correspondiente actúa sobre alguna superficie compacta de un género algebraico dado (mayor que uno). El primer resultado en este sentido fue el de Harvey [19], en el que se detallan dichas condiciones para grupos cíclicos de automorfismos de superficies de Riemann, resultado que ampliaron Bujalance, Etayo, Gamboa y Gromadzki [11] a superficies de Klein con borde y Breuer [3] a acciones de grupos abelianos finitos en superficies de Riemann. Las otras dos situaciones para las que se han obtenido dichas condiciones son para grupos cíclicos actuando sobre superficies de Riemann no orientables, establecido por Bujalance [4], y para grupos diédricos actuando sobre superficies de Riemann, por Bujalance, Cirre, Gamboa y Gromadzki [8]. Para el caso de p-grupos el problema ha sido estudiado por Kulkarni y Maclachlan [21].

Es conocido que todo grupo finito actúa como grupo de automorfismos de alguna superficie de Riemann [20], de alguna superficie de Riemann no orientable, así como de alguna superficie de Klein con borde [5] (en todos los casos se entiende que las superficies son compactas y de género algebraico mayor que uno). Un grupo finito dado puede actuar en superficies de distinto género. El llamado *problema de género mínimo* de un grupo finito consiste en encontrar el menor de los géneros algebraicos de las superficies sobre las que puede actuar. Dependiendo del tipo de superficie a la que nos refiramos, se han dado diferentes denominaciones a dicho género mínimo: *género simétrico* si se trata de superficies de Riemann y *género simétrico fuerte* si, además, se exige que los automorfismos conserven la orientación, *número cross-cap simétrico* para superficies de Riemann no orientables, y *género real* para superficies de Klein con borde.

Por otra parte, sobre superficies compactas de un mismo género algebraico dado pueden actuar diversos grupos. Cuando el género algebraico es mayor que uno, el número de tales grupos es finito. Llamamos *problema del máximo orden* al cálculo del mayor orden entre los grupos de una familia de grupos que actúan sobre superficies de un género algebraico determinado (se distinguen entre diferentes tipos de superficies al igual que en el problema del género mínimo).

El Capítulo 1 finaliza con una serie de consideraciones sobre la factorización de epimorfismos con núcleo de superficie a través de la abelianización del grupo NEC. Las condiciones para la existencia de epimorfismos entre grupos abelianos expuestas por Breuer en [3] serán aplicadas en nuestra situación en capítulos posteriores y aportarán información importante para el estudio de la existencia de epimorfismos con núcleo de superficie.

En el Capítulo 2 se inician las aportaciones originales de esta memoria. Se centra en automorfismos de superficies de Riemann. Breuer [3] amplió para grupos abelianos las condiciones de Harvey para grupos cíclicos de automorfismos [19]. Las condiciones de Breuer establecen ciertas relaciones entre los factores invariantes de un grupo abeliano A y la signatura de un grupo Fuchsiano Λ para la existencia de epimorfismos de superficie $\Lambda \to A$. Una de esas condiciones requiere la existencia de un epimorfismo $\Lambda \to A$. En esta memoria proponemos una modificación de dicha condición, planteándola, como el resto de condiciones, como una relación entre los citados parámetros del grupo abeliano y el grupo Fuchsiano. Para ello, utilizamos las condiciones, también expuestas por Breuer, para la existencia de epimorfismos $\Lambda_{ab} \to A$ de la abelianización de Λ sobre A. Aprovecharemos, en los capítulos posteriores, dichos aspectos sobre la abelianización de un grupo NEC para estudiar los epimorfismos de superficie de un grupo NEC sobre un grupo abeliano.

Las condiciones de existencia de epimorfismos con núcleo de superficie nos permitirán demostrar, de forma más breve, la expresión de Maclachlan [24] para el género simétrico fuerte $\sigma^o(A)$ de un grupo abeliano A. Dicha expresión está indicada como el valor mínimo dentro un conjunto de valores candidatos a género mínimo del grupo abeliano. Es posible concretar ese valor mínimo para diferentes tipos de grupos abelianos, o, al menos, reducir ese conjuto de valores candidatos con una simple inspección de los cocientes entre los sucesivos factores invariantes del grupo abeliano.

Nos permitirán, así mismo, encontrar el menor de los géneros simétricos fuertes entre los de los grupos abelianos con igual orden. Un aplicación inmediata de esto último es una nueva prueba del máximo orden que puede tener un grupo abeliano que actúa sobre una superficie de Riemann de determinado género.

Los resultados presentados en el Capítulo 2 han sido publicados por el autor en el artículo "*Some results on abelian groups of automorphisms of compact Riemann surfaces*" [32].

En el Capítulo 3 estudiamos los grupos de automorfismos abelianos en superficies de Klein compactas con borde. También ha sido posible en este caso obtener las condiciones sobre la signatura de un grupo NEC Λ y los factores invariantes de un grupo abeliano A para la existencia de epimorfismos $\theta : \Lambda \to A$ con núcleo de superficie.

El hecho de que la superficie de Klein posea borde implica ciertas restricciones en la signatura del grupo NEC: debe tener algún ciclo-período (de forma que el cociente $\mathcal{H}/\ker\theta$ puede tener borde) y sus períodos de enlace deben tener valor dos. Las condiciones se obtienen, *grosso modo*, buscando los grupos NEC con generadores suficientes y asegurándonos de que el epimorfismo sea, en efecto, sobreyectivo y que mantenga las relaciones del grupo NEC.

Al igual que en el Capítulo 2, la factorización del epimorfismo $\theta : \Lambda \to A$ a través de la abelianización del grupo NEC nos permitirá obtener una condición que asegure un número suficiente de generadores en el grupo NEC. Como ya se ha comentado, se trata de trasladar a nuestra situación las condiciones de Breuer para que exista algún epimorfismo entre grupos abelianos.

Una vez obtenidas las condiciones de existencia de epimorfismos con núcleo de superficie, las utilizamos para calcular el género real $\sigma(A)$ de un grupo abeliano A, ya sea cíclico o no, anteriormente establecido por Bujalance, Etayo, Gamboa y Martens en [12] y por McCullough en [29], respectivamente. Al igual que para superficies de Riemann, esto nos permite calcular el menor de los géneros reales entre los de los grupos abelianos del mismo orden y, con ello, abordar de forma sencilla el problema del máximo orden, ya resuelto antes por Bujalance, Etayo, Gamboa y Gromadzki [11], para grupos abelianos actuando sobre superficies de Klein compactas con borde de género algebraico dado (mayor que uno).

Los resultados presentados en el Capítulo 3 han sido publicados por el autor en el artículo "*Abelian actions on compact bordered Klein surfaces*" [33].

El Capítulo 4 lo dedicamos a superficies de Riemann no orientables. Como en los capítulos precedentes, hemos estudiado las condiciones necesarias y suficientes para que un grupo abeliano sea un grupo de automorfismos de alguna superficie de Riemann no orientable de género topológico mayor que dos. Pero, en este caso, solo ha sido posible obtenerlas para ciertos tipos de grupos abelianos, los de orden impar y aquellos cuyo 2-subgrupo de Sylow es cíclico. Para el resto de grupos abelianos de orden par no ha sido posible obtenerlas, como se comenta en la Sección 4.3.

En el caso de grupos abelianos de orden impar, la signatura del grupo NEC no puede tener ciclo-períodos si queremos que un epimorfismo $\theta : \Lambda \to A$ tenga núcleo de superficie tal que $\mathcal{H}/\ker\theta$ sea una superficie de Riemann no orientable. En tal caso, práctimente es suficiente con la condición de Breuer sobre la existencia de epimorfismos de la abelianización de Λ sobre el grupo abeliano A.

Cuando consideramos, a continuación, grupos abelianos cuyo 2-subgrupo de Sylow es cíclico la signatura del grupo NEC puede tener ciclo-períodos, pero deben ser vacíos. Aquí la existencia de epimorfismos con núcleo de superficie requiere, además, alguna otra condición si la signatura del grupo NEC no tiene ciclo-períodos o si solo tiene uno.

Finalmente, las condiciones para que un grupo abeliano de esos tipos actúe sobre una superficie de Riemann no orientable de nuevo nos permiten constatar, de forma más sencilla que las hasta ahora conocidas, la solución al problema del género mínimo correspondiente, i.e., el cálculo del número *cross-cap* simétrico $\widetilde{\sigma}(A)$ de un grupo abeliano A de los tipos indicados, establecido por Etayo [14] y Gromadzki [18].

Introduction

Computing groups of automorphisms of Riemann and Klein surfaces is a classical problem initiated by Schwartz, Hurwitz, Klein and Wiman, among others, at the end of the 19th century. Surfaces with a nontrivial finite group of automorphisms are of particular importance, since they correspond to the singular locus of the moduli space of such surfaces. By the uniformization theorem, compact Riemann and Klein surfaces of algebraic genus greater than one can be seen as the quotient of the hyperbolic plane under the action of a discrete subgroup of its isometries (a non-Euclidean crystallographic group, in general, or a Fuchsian group if it only contains orientation-preserving isometries). This approach gave rise to the use of combinatorial methods, which have proven the most fruitful in computing groups of automorphisms.

Thus far, research has focused on low genus surfaces or on surfaces with a certain group of automorphisms endowing the surface with significant properties (for instance, hyperelliptic, elliptic-hyperelliptic, Wiman, Accola-Maclachlan and Kulkarni surfaces).

Not surprisingly, cyclic groups were tackled firstly [39]. Combinatorial methods were first applied by Harvey [19]. He found necessary and sufficient conditions for a cyclic group to act on a Riemann surface. Such conditions are expressed in terms of the algebraic structure of the Fuchsian group associated to the action. Harvey's Theorem has been widely used since. Similar results have only been found for dihedral [8] and abelian groups [3, Theorem 9.1]. For p-groups, the problem has been studied by Kulkarni and Maclachlan [21].

For cyclic actions on Klein surfaces with boundary, the result corresponding to Harvey's Theorem was proven in [11, §3.1]. A similar theorem for abelian actions remained unknown, although some meaningfull, partial results were well-known, such as the answer to the minimum genus problem for cyclic [12] and noncyclic abelian groups [29].

Minimum genus and maximum order problems have been studied for a number of families of groups using diverse techniques. Some thorough surveys on these topics can be found in [9, 6, 7]. One of these techniques takes advantage of previously established conditions for the existence of surface-kernel epimorphisms onto a group of the family. This approach usually provides a shorter proof to the solution to the minimum genus and maximum order problems, as we will see in subsequent chapters.

In this thesis, we obtain the following results:

Chapter 2. We establish a refinement of Breuer's conditions [3, Theorem 9.1] for the existence of abelian actions on compact Riemann surfaces of genus greater than one. In this new form, every condition is entirely expressed in terms of the invariant factors of the abelian group and the signature of the Fuchsian group. As a consequence, we obtain a new, shorter proof of Maclachlan's solution to the minimum genus problem and, in many cases, an explicit expression using some results concerning the invariant factors of the abelian group. We find the least strong symmetric genus for the family of abelian groups, cyclic or not, of the same given order, as well as the unique abelian group attaining such minimum genus, which leads to a new proof of the maximum order problem for the family of abelian groups acting on Riemann surfaces of a given genus greater than one. These results were published in [32].

Chapter 3. We state conditions for an abelian group to act on some compact bordered Klein surfaces of algebraic genus greater than one, expressing such conditions in terms of the algebraic structure of the NEC group associated to that action. We then deduce by new, more concise methods the real genus of an abelian group and solve the related maximum order problem. We also find the expression for the least real genus of abelian groups of the same given order. The results in this chapter are already published in [33].

Chapter 4. We find conditions of existence of actions of abelian groups of odd order or with cyclic Sylow 2-subgroup on compact nonorientable Riemann surfaces of topological genus greater than two. That makes it easier to obtain the known expression of the symmetric cross-cap number of such groups.

1 Preliminaries

We devote this chapter to look over the underlying matters that are referred to in this thesis. Uniformization of Klein surfaces makes it possible to address the study of actions of finite groups on Klein surfaces by means of combinatorial methods of NEC groups. We also gather a number of results concerning abelian groups and the abelianization of NEC groups.

1.1 Klein surfaces

Klein surfaces constitute a generalization of Riemann surfaces that include bordered and nonorientable surfaces. They broaden the scope of Riemann surfaces by allowing transition functions that may include complex conjugation besides analytic functions and domains in the closed upper half-plane $\mathbb{C}^+$. This makes up what is called a dianalytic structure [1]. The topological genus g, the number k of boundary components and the orientability are known as the *topological type* of a Klein surface, and the integer $p = \eta g + k - 1$ as its *algebraic genus*, where $\eta = 2$ if the surface is orientable and $\eta = 1$ otherwise.

The equivalence between categories of compact Riemann surfaces and complex projective and smooth algebraic curves was extended to categories of Klein surfaces and real algebraic curves by Alling and Greenleaf [1]. By means of this equivalence, any result on automorphisms of compact Klein (Riemann) surfaces turns into a corresponding result on birational transformations of real (complex) algebraic curves.

We now introduce the main results concerning Klein surfaces which will be used herein (a thorough account on this topic can be found in [11] chapters 0 and 1).

Definition 1.1.1. A *surface* is a connected Hausdorff topological space X together with a *topological atlas*, i.e., a family $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ of *charts* such that $\{U_i : i \in I\}$ is an open covering of X and each map $\varphi_i : U_i \to \varphi_i(U_i)$ is a homeomorphism onto an open subset of $\mathbb{C}$ or $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant 0\}$ —the closure of the open upper half-plane $\mathcal{H}$. The homeomorphisms

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are called *transition functions* of $\mathcal{A}$. Assuming the identification of $\mathbb{C}$ with $\mathbb{R}^2$, the *orientability* of X is defined as for a real 2-manifold. The *boundary* of X is

$$\partial X = \{x \in X : \exists i \in I \text{ such that } x \in U_i,\ \varphi_i(x) \in \mathbb{R} \text{ and } \varphi_i(U_i) \subseteq \mathbb{C}^+\}.$$

A nonorientable surface or a surface with nonempty boundary do not admit an analytic structure. However, a small generalization of the notion of analytic map will enable us to define a proper structure on such surfaces.

Definition 1.1.2. Let U be an open set of $\mathbb{C}$. A map $f : U \to \mathbb{C}$ is *antianalytic in U* if its complex conjugate $\overline{f}$ is analytic in U, and f is *dianalytic in U* if it is analytic or antianalytic on each connected component of U.

If U is connected and f is both analytic and antianalytic on U, then f is constant. An analytic map is orientation-preserving, while an antianalytic map reverses the orientation. If f and g are both analytic or both antianalytic, then $g \circ f$ is analytic; if one is analytic and the other is antianalytic, then $g \circ f$ is antianalytic.

In order to deal with surfaces with boundary, it will be also necessary to consider maps having an open subset of $\mathbb{C}^+$ as domain.

Definition 1.1.3. Let A be an open set of $\mathbb{C}^+$ that is not open in $\mathbb{C}$. A map $f : A \to \mathbb{C}$ is *dianalytic in A* if it is the restriction of a dianalytic map whose domain is an open set of $\mathbb{C}$ containing A.

Definition 1.1.4. A topological atlas $\mathcal{A}$ is *dianalytic* if its transition functions are dianalytic. Two atlases $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if $\mathcal{A} \cup \mathcal{B}$ is dianalytic. A *dianalytic structure* on X is the equivalence class of a dianalytic atlas of X. A pair consisting of a surface X and a dianalytic structure on X will be called a *Klein surface*.

Morphisms between Klein surfaces can be orientation-reversing and can generate boundary. The proper definition of such morphisms is achieved by means of the following map:

Definition 1.1.5. The *folding map* is the continuous and open map

$$\Phi : \mathbb{C} \to \mathbb{C}^+ : a + b\sqrt{-1} \mapsto a + |b|\sqrt{-1}.$$

Definition 1.1.6. A *morphism* between Klein surfaces X and Y is a continuous map $f : X \to Y$ such that $f(\partial X) \subseteq \partial Y$ and for all $x \in X$ there exist dianalytic charts (U, φ) and (V, ψ) of X and Y, respectively, with $x \in U$ and $f(x) \in V$, and an analytic function $F : \varphi(U) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[d, "\varphi"'] & & V \arrow[d, "\psi"] \\ \varphi(U) \arrow[r, "F"] & \mathbb{C} \arrow[r, "\Phi"] & \mathbb{C}^+ \end{tikzcd}$$

The composition of morphisms is ruled by the following result [2]:

Proposition 1.1.7. *Let X, X' and X'' be Klein surfaces and $f : X \to X'$ and $g : X' \to X''$ be nonconstant continuous maps such that $f(\partial X) \subseteq \partial X'$ and $g(\partial X') \subseteq \partial X''$. Consider the following assertions:*

(1) *f is a morphism;*
(2) *g is a morphism;*
(3) *$g \circ f$ is a morphism.*

Then,

a) (1) *and* (2) *imply* (3)*;*
b) *if f is onto,* (1) *and* (3) *imply* (2)*;*
c) *if f is open,* (2) *and* (3) *imply* (1).

Definition 1.1.8. An *automorphism* of a Klein surface X is an isomorphism $X \to X$ in the category of Klein surfaces.

It follows from Proposition 1.1.7 that the set $\mathrm{Aut}(X)$ of all automorphisms of X is a group under the operation of composition of morphisms. The group $\mathrm{Aut}(X)$ is called the *full group of automorphisms* of X

Definition 1.1.9. A *group of automorphisms* of a Klein surface X is a subgroup of $\mathrm{Aut}(X)$.

When a group G is isomorphic to a group of automorphisms of a Klein surface we say that G *acts on* that surface; if G acts on some surface of algebraic genus p, the group G *acts on genus* p.

Let G be a group of automorphisms of a Klein surface X. The stabilizer of $x \in X$ is the subgroup $G_x = \{f \in G : f(x) = x\}$. Given two subsets U and V of X, we also define $G(U,V) = \{f \in G : U \cap f(V) \neq \varnothing\}$ and denote $G(U,U)$ by G_U (so that $G_x = G_{\{x\}}$).

Definition 1.1.10. A group G of automorphisms of a Klein surface X acts *properly discontinuously* if the following conditions hold:

i) Each $x \in X$ has a neighborhood U such that G_U is finite.

ii) If $x, y \in X$ and $x \notin O_y$, then there exist a neighborhood U of x and a neighborhood V of y such that $G(U,V) = \varnothing$.

iii) If $x \in X$, (U, φ) is a chart with $x \in U$, $f \in G_x$ is not the identity and the map $\varphi \circ f \circ \varphi^{-1}$ (suitable restricted) is analytic, then x is isolated in the set of fixed points of f.

Groups of automorphisms of a Klein surface X acting properly discontinuously hold some important features. As stated in [1] Theorem 1.8.4, the quotient of X under the action of such a group can be endowed with a unique dianalytic structure.

Theorem 1.1.11. *If a group G of automorphisms of a Klein surface X acts properly discontinuously on X, then the quotient X/G admits a unique dianalytic structure such that the canonical projection $X \to X/G$ is a morphism.*

1.2 Non-Euclidean crystallographic groups and uniformization of Klein surfaces

The group $\mathrm{Aut}(\mathcal{H})$ of automorphisms of the upper complex half-plane is isomorphic to $PGL(2,\mathbb{R})$. Indeed, recall that $\mathrm{Aut}(\mathcal{H})$ is the set of all transformations $z \mapsto \frac{az+b}{cz+d}$ and $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ for real numbers a, b, c and d such that $ad - bc > 0$ and

$ad - bc < 0$, respectively, and the group epimorphism

$$\begin{aligned} \mathrm{GL}(2,\mathbb{R}) &\to \mathrm{Aut}(\mathcal{H}) \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto f_A : z \mapsto \begin{cases} \frac{az+b}{cz+d} & \text{if } \det A > 0, \\ \frac{a\bar{z}+b}{c\bar{z}+d} & \text{if } \det A < 0 \end{cases} \end{aligned}$$

has kernel $\{\lambda I_2 : \lambda \in \mathbb{R} - \{0\}\}$, where I_2 is the identity matrix, and the quotient $GL(2,\mathbb{R})/\{\lambda I_2\}$ is just $PGL(2,\mathbb{R})$. As a topological space, $\mathrm{Aut}(\mathcal{H})$ contains subgroups made up of isolated elements:

Definition 1.2.1. A subgroup of $\mathrm{Aut}(\mathcal{H})$ is *discrete* if it is discrete as a topological subspace of $\mathrm{Aut}(\mathcal{H})$.

Proposition 1.2.2. *Every discrete subgroup of $Aut(\mathcal{H})$ acts properly discontinuously on $\mathcal{H}$.*

The following result is an immediate consequence of Theorem 1.1.11.

Corollary 1.2.3. *The orbit space $\mathcal{H}/G$ of $\mathcal{H}$ under the action of a discrete subgroup G of $Aut(\mathcal{H})$ admits a unique dianalytic structure such that the canonical projection $\mathcal{H} \to \mathcal{H}/G$ is a morphism.*

Definition 1.2.4. A *non-Euclidean crystallographic* (NEC) group Λ is a discrete subgroup of $\mathrm{Aut}(\mathcal{H})$ for which $\mathcal{H}/\Lambda$ is compact.

An NEC group is a *Fuchsian* group if it contains only orientation preserving automorphisms; otherwise, it is said to be a *proper* NEC group. An NEC group with no orientation preserving elements of finite order is called *surface* NEC group.

By an important result stated by Preston [31], surface NEC groups *uniformize* compact Klein surfaces:

Theorem 1.2.5. *If X is a compact Klein surface of algebraic genus $p \geqslant 2$, then there exists a surface NEC group Γ such that X and $\mathcal{H}/\Gamma$ are isomorphic as Klein surfaces.*

Theorem 1.2.6. *Let Γ and Γ' be surface NEC groups. The compact Klein surfaces $\mathcal{H}/\Gamma$ and $\mathcal{H}/\Gamma'$ of algebraic genus greater than or equal to 2 are isomorphic if and only if Γ and Γ' are conjugate subgroups in $\mathrm{Aut}(\mathcal{H})$.*

1.3 Signature and canonical presentation of an NEC group

Nonisomorphic NEC groups differ from one another in the *signature*. It was introduced by Macbeath [23] and Wilkie [38] and is as follows:

$$(g;\, \pm\,;\, [m_1, \ldots, m_r];\, \{(n_{i1}, \ldots, n_{is_i}),\, i = 1, \ldots, k\}).$$

The signature of a Fuchsian group is usually denoted by $(g;\, m_1, \ldots, m_r)$. For a surface NEC group, it is of the form $(g;\, \pm\,;\, [-];\, \{(-), \overset{k}{\ldots}, (-)\})$ and we say that the surface group is *unbordered* if $k = 0$ and *bordered* otherwise. The signature of an NEC group Λ determines both its algebraic structure and the topological structure of the orbit space $\mathcal{H}/\Lambda$.

The integers $m_i \geqslant 2$ are called *proper periods*, $n_{ij} \geqslant 2$ are the *link periods*, $(n_{i1}, \ldots,\, n_{is_i})$ are the *period cycles* and g is the *orbit genus*. The orbit space $\mathcal{H}/\Lambda$ has topological genus g, k boundary components and is orientable if the sign of the signature is '+' and nonorientable otherwise. The covering map $\mathcal{H} \to \mathcal{H}/\Lambda$ ramifies over r interior points with ramification indices m_i and, on each boundary component, over s_i points with ramification indices n_{ij}. The integer $\eta g + k - 1$ is the algebraic genus of $\mathcal{H}/\Lambda$, where $\eta = 2$ if the sign of the signature is '+' and $\eta = 1$ otherwise. An arbitrary set of such numbers and symbols defines the signature of an NEC group if and only if

$$\eta g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right) > 0. \tag{1.1}$$

The expression in the left side is denoted by $\mu(\Lambda)$. The hyperbolic area of any fundamental region of $\mathcal{H}/\Lambda$ is $2\pi\mu(\Lambda)$. We will call $\mu(\Lambda)$ the *reduced area* of Λ. Also, if Λ' is a subgroup of Λ of finite index, then Λ' is an NEC group and

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda), \tag{1.2}$$

which is the Riemann-Hurwitz formula associated to the covering $\mathcal{H}/\Lambda' \to \mathcal{H}/\Lambda$.

The signature provides a *canonical* presentation of Λ with the following *canon-*

ical generators and relations depending on the sign of the signature:

$x_1, \dots, x_r$		(elliptic elements),
$c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k}$		(hyperbolic reflections),
$e_1, \dots, e_k$		(hyperbolic or elliptic elements),
$a_1, b_1, \dots, a_g, b_g$	if the sign is '+'	(hyperbolic translations),
$d_1, \dots, d_g$	if the sign is '−'	(glide reflections),

$$x_i^{m_i} = 1, \qquad c_{ij}^2 = 1, \qquad (c_{ij-1}c_{ij})^{n_{ij}} = 1, \qquad e_i^{-1}c_{i0}e_ic_{is_i} = 1,$$

$$x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \qquad \text{if the sign is '+' and}$$

$$x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1 \qquad \text{if the sign is '−'.}$$

The last one is called the *long relation.* An abstract group with such a presentation is an NEC group with signature as above if and only if (1.1) is fulfilled.

For further purposes, the following should be considered. Hereinafter, we assume factorizations $m_i = p_1^{\mu_i(p_1)} \cdots p_s^{\mu_i(p_s)}$ with prime numbers $p_1 < \cdots < p_s$ and integers $\mu_i(p_j) \geqslant 0$ such that $\mu_1(p_j) + \cdots + \mu_r(p_j) > 0$. For each prime p_j, we rearrange the integers $\mu_1(p_j), \dots, \mu_r(p_j)$ to obtain an increasing sequence of integers $\widehat{\mu}_1(p_j) \leqslant \widehat{\mu}_2(p_j) \leqslant \cdots \leqslant \widehat{\mu}_r(p_j)$ and define $\widehat{m}_i = p_1^{\widehat{\mu}_i(p_1)} \cdots p_s^{\widehat{\mu}_i(p_s)}$. Then, $\widehat{m}_i | \widehat{m}_{i+1}$ and there is an integer $\widehat{r}$ such that $\widehat{m}_i = 1$ for $i = 1, \dots, r - \widehat{r}$ and $\widehat{m}_i > 1$ for the $\widehat{r}$ integers $i = r - \widehat{r} + 1, \dots, r$. Also, we can check that

$$\sum_{i=1}^{r} \left(1 - \frac{1}{\widehat{m}_i}\right) \leqslant \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right). \tag{1.3}$$

For, consider the following matrices with the factors of periods $m_i, \widehat{m}_i$ in their rows:

$$\mathcal{M} = \begin{pmatrix} p_1^{\mu_1(p_1)} & \cdots & p_s^{\mu_1(p_s)} \\ \vdots & & \vdots \\ p_1^{\mu_r(p_1)} & \cdots & p_s^{\mu_r(p_s)} \end{pmatrix} \qquad \widehat{\mathcal{M}} = \begin{pmatrix} p_1^{\widehat{\mu}_1(p_1)} & \cdots & p_s^{\widehat{\mu}_1(p_s)} \\ \vdots & & \vdots \\ p_1^{\widehat{\mu}_r(p_1)} & \cdots & p_s^{\widehat{\mu}_r(p_s)} \end{pmatrix}.$$

We can get $\widehat{\mathcal{M}}$ from $\mathcal{M}$ by an interchange of entries in pairs of consecutive rows of $\mathcal{M}$. Let $(i, j) = p_j^{\mu_i(p_j)}$ be the entry of $\mathcal{M}$ in row i and column j. First, take the first and second rows and, for each $j \in \{1, \dots, s\}$, interchange $(1, j)$ and $(2, j)$ if $(2, j) < (1, j)$. Then we proceed with the second and third rows, and so on for the remaining rows on the produced matrices. We obtain $\widehat{\mathcal{M}}$ repeating the whole process if necessary. It is enough to consider only one of these steps, say, for two successive rows

$$\begin{pmatrix} p_1^{a_1} & \cdots & p_{j_1}^{b_{j_1}} & \cdots & p_{j_q}^{b_{j_q}} & \cdots & p_s^{a_s} \\ p_1^{b_1} & \cdots & p_{j_1}^{a_{j_1}} & \cdots & p_{j_q}^{a_{j_q}} & \cdots & p_s^{b_s} \end{pmatrix},$$

with integers a_j, b_j such that $0 \leqslant a_j \leqslant b_j$; we take consecutive unordered columns for readability. If we let

$$\begin{aligned} m_1 &= p_1^{a_1} \cdots p_{j_1}^{b_{j_1}} \cdots p_{j_q}^{b_{j_q}} \cdots p_s^{a_s} \qquad & m_1' &= p_1^{a_1} \cdots p_s^{a_s} \\ m_2 &= p_1^{b_1} \cdots p_{j_1}^{a_{j_1}} \cdots p_{j_q}^{a_{j_q}} \cdots p_s^{b_s} \qquad & m_2' &= p_1^{b_1} \cdots p_s^{b_s}, \end{aligned}$$

then $m_1 m_2 = m_1' m_2'$ and thus

$$\frac{1}{m_1} + \frac{1}{m_2} - \left(\frac{1}{m_1'} + \frac{1}{m_2'}\right) = \frac{1}{m_1'}\left(\frac{m_1}{m_2'} - 1\right)\left(1 - \frac{m_1'}{m_1}\right) \leqslant 0,$$

since $m_1' \leqslant m_1 \leqslant m_2'$. It follows that $-\frac{1}{m_1'} - \frac{1}{m_2'} \leqslant -\frac{1}{m_1} - \frac{1}{m_2}$.

1.4 Surface-kernel epimorphisms

In this section, we introduce some of the results on which matters considered in subsequent chapters are directly based.

Theorem 1.4.1. [26] *Let Γ be a surface NEC group. Then,*

i) *the normalizer $N(\Gamma)$ of Γ in $\mathrm{Aut}(\mathcal{H})$ is an NEC group and*
ii) $\mathrm{Aut}(\mathcal{H}/\Gamma) \approx N(\Gamma)/\Gamma$.

As a consequence, when $X = \mathcal{H}/\Gamma$ is a Klein surface of algebraic genus $p \geqslant 2$, the order of $\mathrm{Aut}(X)$ equals the index $[N(\Gamma) : \Gamma]$, which is finite by the Riemann-Hutwitz formula (1.2). We highlight this important fact:

Theorem 1.4.2. *The group $\mathrm{Aut}(X)$ of automorphisms of a compact Klein surface X of algebraic genus $p \geqslant 2$ is finite.*

The following result is well known for Riemann surfaces [22], and it was established in [34, §2] and [26, Proposition 3] for nonorientable Riemann surfaces and bordered Klein surfaces, respectively.

Theorem 1.4.3. *Let Γ be a surface NEC group. A group G is a group of automorphisms of $\mathcal{H}/\Gamma$ if and only if G is isomorphic to the factor group Λ/Γ for some NEC group Λ containing Γ as a normal subgroup.*

Therefore, the action of G on $\mathcal{H}/\Gamma$ can then be given by an epimorphism $\theta : \Lambda \to G$ whose kernel is Γ.

Definition 1.4.4. An epimorphism $\theta : \Lambda \to G$ from an NEC group Λ onto a group G whose kernel is a surface NEC group is called a *surface-kernel* epimorphism. We say that θ is a *orientable*, *nonorientable*, *unbordered* or *bordered* surface-kernel epimorphism if $\mathcal{H}/\ker\theta$ is orientable, nonorientable, has empty boundary or has nonempty boundary, respectively.

Since the order of $\theta(x)$ divides the order of an element $x \in G$ of finite order, the next result follows easily:

Lemma 1.4.5. *A homomorphism $\theta : \Lambda \to G$ from an NEC group Λ onto a group G is a surface-kernel epimorphism if and only if the order of $\theta(x)$ equals the order of x for every orientation-preserving element $x \in \Lambda$ of finite order.*

One of the main goals of this thesis is to find, for a given integer $p \geqslant 2$, conditions on a finite abelian group A to be a group of automorphisms of some compact Klein surface of algebraic genus p. As we have noted above, this is equivalent to finding conditions for the existence of an epimorphism $\Lambda \to A$ whose kernel is a surface NEC group with signature $(g; \pm;\ [-]; \{(-), \overset{k}{\ldots}, (-)\})$ such that $p = \eta g + k - 1$. Such conditions will be established in terms of the defining parameters of the NEC group Λ and of the abelian group A.

1.5 Minimum genus and maximum order problems

We mentioned in Theorem 1.4.2 that every group of automorphisms of a compact Klein surface of algebraic genus $p \geqslant 2$ is finite. It is also well-known that every group of finite order acts on some compact Klein surface of algebraic genus greater than one. More precisely, every group of finite order acts on some compact Riemann surface [20], on some compact nonorientable Riemann surface and on some compact bordered Klein surface [5] (in the latter case, we can also distinguish between orientable and nonorientable surfaces) of algebraic genus greater than one.

A finite group may act on Klein surfaces of different genera. Given a finite group, the *minimum genus problem* consists in finding the least genus on which a group acts. In this respect, Riemann surfaces, nonorientable Riemann surfaces and bordered Klein surfaces are considered separately. The following terminology was introduced in [36, 37, 27, 28].

Definition 1.5.1. The *strong symmetric genus* $\sigma^o(G)$ of a finite group G is the minimum topological genus of the compact Riemann surfaces of genus greater than one on which G acts preserving orientation.

The *symmetric genus* $\sigma(G)$ of a finite group G is the minimum topological genus of the compact Riemann surfaces of genus greater than one on which G acts, either preserving or reversing orientation.

The *symmetric cross-cap number* $\tilde{\sigma}(G)$ of a finite group G is the minimum topological genus of the compact nonorientable Riemann surfaces of topological genus greater than two on which G acts.

The *real genus* $\rho(G)$ of a finite group G is the minimum algebraic genus of the compact bordered Klein surfaces of algebraic genus greater than one on which G acts. ■

Remark 1.5.2. Some authors allow values 0 and 1 in the definition of real genus. Cyclic groups and $\mathbb{Z}_2^2 \approx D_2$ are the only abelian groups that act on genus 0 —the closed disk is the unique bordered surface of algebraic genus 0—, and $\mathbb{Z}_2^3 \approx \mathbb{Z}_2 \times D_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_{2u}$ $(u > 1)$ are the only noncyclic abelian groups that act on genus 1 —the closed annulus and the Möbius strip are the unique bordered surfaces of algebraic genus 1—, see [27] theorems 3 and 4. ■

The *maximum order problem* is closely related to the minimum genus problem. Several groups may act on Klein surfaces of a given algebraic genus. When the algebraic genus is greater than one, there are only finitely many such groups. Computing the largest group order in a family of groups which act on a given algebraic genus is what we call the *maximum order problem* for that family. We will also distinguish between actions on Riemann surfaces, nonorientable Riemann surfaces and bordered Klein surfaces.

1.6 A brief remainder on abelian groups

According to the previous comments, we are interested in examining a specific type of homomorphisms onto finite groups, namely, surface-kernel epimorphisms, and, in particular, we will focus on surface-kernel epimorphisms onto finite abelian groups. In this respect, it is worth mentioning some elementary features of abelian groups.

When A is a finitely generated abelian group, its *invariant factor decomposition* is $A \approx \mathbb{Z}^n \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ for integers $n > 0$, called *torsion-free rank* of A, and $v_i > 1$, called *invariant factors* of A, with v_i dividing v_{i+1}, and *primary decomposition* $A \approx \mathbb{Z}^n \oplus A_{q_1} \oplus \cdots \oplus A_{q_\lambda}$, where $q_1 < \cdots < q_\lambda$ are the prime numbers dividing the order of A and $A_q = \{x \in A | q^n x = 0 \text{ for some } n \geqslant 0\}$ is the *q-primary component* of A —the q-Sylow subgroup $Syl_q(A)$. We also assume $v_i = q_1^{\alpha_i(q_1)} \cdots q_\lambda^{\alpha_i(q_\lambda)}$ for $i = 1, \ldots, t$, so $0 \leqslant \alpha_1(q) \leqslant \cdots \leqslant \alpha_t(q)$ and $A_q \approx \mathbb{Z}_{q^{\alpha_1(q)}} \oplus \cdots \oplus \mathbb{Z}_{q^{\alpha_t(q)}}$. The integers $q_j^{\alpha_i(q_j)}$ are the *elementary divisors* of A.

Below it will be helpful to express a finite abelian group as follows:

$$A \approx \mathbb{Z}_2^n \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t},$$

(for readability, $\mathbb{Z}_v \oplus \overset{n}{\cdots} \oplus \mathbb{Z}_v$ will be denoted by $\mathbb{Z}_v^n$) where $v_i > 2$ and v_i divides v_{i+1}, so that there exists some integer $m \leqslant t$ such that $v_1, \ldots, v_{t-m}$ are odd and the m integers $v_{t-m+1}, \ldots, v_t$ are multiple of 4; note that, though unique, this expression may not coincide with the invariant factor decomposition of A.

Also, an element of the finite abelian group $\mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ will be denoted by $(a_1, \ldots, a_t)$ where the integer a_i is to be understood as its residue class modulo v_i.

1.7 Abelianization of NEC groups

We are mainly concerned with conditions of existence of epimorphisms $\phi : \Lambda \to A$ from an NEC group onto a finite abelian group. In this context, the abelianization Λ_{ab} of Λ provides significant information.

Recall that the *derived subgroup* G' of a group G is the subgroup generated by the set of all commutators of G (elements of the form $[x,y] = xyx^{-1}y^{-1}$ with $x, y \in G$). It is easily seen that $G' \trianglelefteq G$. The *abelianization* of G is the quotient $G_{ab} = G/G'$, which is obviously an abelian group. Moreover, if A is an abelian group and $\phi : G \to A$ is a group homomorphism, its kernel contains G' and so ϕ factors through G_{ab}, i.e., there exists a (unique) group homomorphism $\overline{\phi} : G_{ab} \to A$ such that $\phi = \overline{\phi} \circ \pi$, where $\pi : G \to G_{ab}$ is the canonical projection. Note also that, π being surjective, the composition $\phi = \overline{\phi} \circ \pi$ is onto for every epimorphism $\overline{\phi} : G_{ab} \to A$.

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & A \\ & G_{ab} \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

Therefore, it is worth considering epimorphisms between abelian groups. Breuer stated conditions for the existence of such epimorphisms as a set of inequations on the rank and the number of cyclic factors in the primary decomposition of the abelian groups:

Lemma 1.7.1. [3, lemmas A.1 and A.2] *Let q be a prime number and $R, N_1, \ldots, N_s, r, n_1, \ldots, n_s$ be non-negative integers. There is an epimorphism*

$$\mathbb{Z}^R \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{q^i}^{N_i} \quad \to \quad \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{q^i}^{n_i}$$

if and only if

$$R \geqslant r \quad \text{and} \quad R + \sum_{i=j}^{s} N_i \geqslant r + \sum_{i=j}^{s} n_i \qquad \text{for} \quad j = 1, \ldots, s. \tag{1.4}$$

For arbitrary finite abelian groups A and B, there is an epimorphism $\mathbb{Z}^R \oplus A \to \mathbb{Z}^r \oplus B$ if and only if there is an epimorphism $\mathbb{Z}^R \oplus A_q \to \mathbb{Z}^r \oplus B_q$ for each prime q dividing the order of B.

In order to study these conditions for epimorphisms $\Lambda_{ab} \to A$, we need to know the structure of Λ_{ab} in terms of the signature of Λ. Here a distinction is made between Fuchsian and proper NEC groups. For a Fuchsian group Λ, we find its abelianization in [3, Lemma A.3]; with the notation of Section 1.3 it reads as follows:

Lemma 1.7.2. *The abelianization of a Fuchsian group Λ with signature $(g;\, m_1, \ldots, m_r)$ is isomorphic to $\mathbb{Z}^{2g}$ if $r = 0$ or 1 and*

$$\Lambda_{ab} \approx \mathbb{Z}^{2g} \oplus \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_{r-1}}$$

otherwise.

Now, we compute the abelianization of proper NEC groups. When the signature has some period cycle, the abelianization is obtained by some considerations on the canonical presentation of the proper NEC group. Otherwise, if it has no period cycle (hence the sign of the signature is '$-$'), we will compute the *Smith normal* form of the *relation matrix* of the canonical presentation of Λ.

Let $\langle X \mid R, [X, X]\rangle$ be a presentation of a finitely generated abelian group A, where $X = \{x_1, \dots, x_n\}$ is a set of generators, $[X, X]$ stands for the commutation relations between all pairs of generators and R is the set of all other (noncommutation) relations. The elements of a row of the *relation matrix* $\mathbf{R}$ of this presentation are the exponents of the generators of a relation (multiplicatively written). If the cardinal of R is m, then $\mathbf{R}$ is an $m \times n$ integer matrix. We can apply elementary row and column operations,

a) interchange two rows (columns),
b) multiply by -1 a row (column),
c) add an integer multiple of one row (column) to another,

in order to transform $\mathbf{R}$ into another integer matrix (ϵ_{ij}) of the same dimensions, called the *Smith normal form* of $\mathbf{R}$, such that $\epsilon_{ij} = 0$ if $i \neq j$ and ϵ_{ii} divides $\epsilon_{i+1,i+1}$. Smith proved its existence in [35]. Let q be the number of non-null integers ϵ_{ii}. It is well-known that the non-null integers ϵ_{ii} are the invariant factors of A and $n - q$ is the torsion-free rank (see, for instance, [25, Section 3.3] or [30, Chapter 2]). Applications of elementary row and column operations to the relation matrix correspond to Tietze transformations of the group presentation, and leave the associated abelian group unchanged.

The integers ϵ_{ii} can be computed as follows: $\epsilon_{11} = \rho_1$, $\epsilon_{22} = \rho_2/\rho_1$, $\epsilon_{33} = \rho_3/\rho_2$, etc., where ρ_i is the greatest common divisor of the determinants of the submatrices of $\mathbf{R}$ of order i. This method for computing the Smith normal form was firstly stated by Smith [35]; see also, for instance, [25, Section 3.3] or [30, Chapter 2].

Lemma 1.7.3. *The abelianization of a proper NEC group Λ with signature $(g; \pm; [m_1, \dots, m_r]; \{(n_{i1}, \dots, n_{is_i}),\ i = 1, \dots, k\})$ is*

$$\Lambda_{ab} \approx \mathbb{Z}^{\eta g+k-1} \oplus \mathcal{T}(\Lambda_{ab}), \tag{1.5}$$

where

$$\mathcal{T}(\Lambda_{ab}) \approx \begin{cases} \mathbb{Z}_2 & \text{if } k = r = 0, \\ \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_{r-1}} \oplus \mathbb{Z}_{2\widehat{m}_r} & \text{if } k = 0,\ r > 0, \\ \mathbb{Z}_2^S \oplus \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_r} & \text{otherwise} \end{cases}$$

is the torsion subgroup of Λ_{ab}*,* η *equals* 2 *if the sign of the signature is '+' and* 1 *otherwise, and*

$$S = \#\{\textit{period cycles with no even link periods}\} + \#\{\textit{even link periods}\}.$$

Proof. When $k > 0$, we remove one generator e_i by the long relation in the abelianized presentation, only remaining relations containing generators of finite order. The remaining canonical generators e_i, a_i, b_i and d_i provide the factor $\mathbb{Z}^{\eta g+k-1}$, while the elliptic generators and their relations turn into $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r} \approx \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_r}$.

The factor $\mathbb{Z}_2^S$ originates from the generators c_{ij}. There are $k + s_1 + \cdots + s_k$ of such generators; we remove k of them (each relation $e_i^{-1}c_{i0}e_ic_{is_i} = 1$ lets us remove c_{i0} or c_{is_i} when abelianized) and also those generators c_{ij} for which n_{ij} is odd (when abelianized, the relation $(c_{ij-1}c_{ij})^{n_{ij}} = 1$ becomes either $c_{ij-1}c_{ij} = 1$ or trivial for odd and even values of n_{ij}, respectively).

If $k = 0$ (hence the sign of the signature is '$-$' since Λ is proper), the presentation of Λ_{ab} is

$$\langle x_1, \ldots, x_r, d_1, \ldots, d_g \mid x_i^{m_i},\ x_1 \cdots x_r d_1^2 \cdots d_g^2, [x_i, x_j], [x_i, d_j], [d_i, d_j]\rangle. \tag{1.6}$$

The relation matrix of this presentation is

$$\mathbf{R} = \begin{pmatrix} m_1 & & & & & & \\ & m_2 & & & & 0 & \\ & & \ddots & & & & \\ & 0 & & m_r & & & \\ 1 & \cdots & 1 & 1 & 2 & \overset{g}{\cdots} & 2 \end{pmatrix}$$

and its Smith normal form is the integer matrix

$$\begin{pmatrix} \epsilon_1 & & & & & 0 \\ & \epsilon_2 & & & & \\ & & \ddots & & & \\ 0 & & & \epsilon_{r+1} & 0 & \overset{g-1}{\cdots} & 0 \end{pmatrix}$$

where ϵ_i divides ϵ_{i+1}. The integers ϵ_i are the invariant factors of an abelian group with presentation (1.6) and, as we noted above, they can be computed as follows: $\epsilon_1 = \rho_1$, $\epsilon_2 = \rho_2/\rho_1, \dots, \epsilon_{r+1} = \rho_{r+1}/\rho_r$, where ρ_i is the greatest common divisor of the determinants of the submatrices of $\mathbf{R}$ of order i.

Clearly, $\rho_1 = 1$. Non-null determinants of 2×2 submatrices of $\mathbf{R}$ are $m_i, 2m_i$ or $m_{i_1}m_{i_2}$ and thus $\rho_2 = \gcd(m_1, \dots, m_r)$. Likewise, non-null 3×3 determinants take values $m_{i_1}m_{i_2}$, $2m_{i_1}m_{i_2}$ or $m_{i_1}m_{i_2}m_{i_3}$ so that $\rho_3 = \gcd(m_1m_2, \dots, m_1m_r, m_2m_3, \dots, m_{r-1}m_r)$. In general, we can easily check that

$$\begin{aligned} \rho_k &= \gcd\{m_{i_1}\cdots m_{i_{k-1}}\}_{1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant r} \quad \text{for } k = 2,\dots,r-1 \text{ and} \\ \rho_{r+1} &= 2m_1\cdots m_r. \end{aligned}$$

Obviously, if p is a prime number dividing some m_i, then ρ_k contains as factors the $k-1$ smallest powers of p in the factor decomposition of $m_1, \dots, m_r$:

$$\begin{aligned} \rho_1 &= 1, \\ \rho_2 &= p_1^{\widehat{\mu}_{11}}\cdots p_s^{\widehat{\mu}_{1s}} = \widehat{m}_1, \\ \rho_3 &= p_1^{\widehat{\mu}_{11}+\widehat{\mu}_{21}}\cdots p_s^{\widehat{\mu}_{1s}+\widehat{\mu}_{2s}} = \widehat{m}_1\widehat{m}_2, \\ &\ \vdots \\ \rho_r &= p_1^{\widehat{\mu}_{11}+\cdots+\widehat{\mu}_{r-1,1}}\cdots p_s^{\widehat{\mu}_{1s}+\cdots+\widehat{\mu}_{r-1,s}} = \widehat{m}_1\cdots\widehat{m}_{r-1}, \\ \rho_{r+1} &= 2p_1^{\widehat{\mu}_{11}+\cdots+\widehat{\mu}_{r1}}\cdots p_s^{\widehat{\mu}_{1s}+\cdots+\widehat{\mu}_{rs}} = 2\widehat{m}_1\cdots\widehat{m}_r, \end{aligned}$$

where, in the notation introduced in Section 1.3, $\widehat{\mu}_{ij} = \widehat{\mu}_i(p_j)$, and thus

$$\epsilon_1 = 1,\ \epsilon_2 = \widehat{m}_1,\ \epsilon_3 = \widehat{m}_2,\ \dots,\ \epsilon_r = \widehat{m}_{r-1},\ \epsilon_{r+1} = 2\widehat{m}_r.$$

Therefore, $\Lambda_{ab} \approx \mathbb{Z}^{g-1} \oplus \mathbb{Z}_{\widehat{m}_{r-\widehat{r}+1}} \oplus \cdots \oplus \mathbb{Z}_{\widehat{m}_{r-1}} \oplus \mathbb{Z}_{2\widehat{m}_r}$ when $k = 0$. ■

It follows that $Syl_q(\mathcal{T}(\Lambda_{ab}))$ is trivial if $r \leqslant 1$ and isomorphic to $\mathbb{Z}_{q^{\widehat{\mu}_1(q)}} \oplus \cdots \oplus \mathbb{Z}_{q^{\widehat{\mu}_{r-1}(q)}}$ otherwise when Λ is a Fuchsian group and, if Λ is a proper NEC group,

$$Syl_2(\mathcal{T}(\Lambda_{ab})) \approx \begin{cases} \mathbb{Z}_2 & \text{if } k = r = 0, \\ \mathbb{Z}_{2^{\widehat{\mu}_1(2)}} \oplus \cdots \oplus \mathbb{Z}_{2^{\widehat{\mu}_{r-1}(2)}} \oplus \mathbb{Z}_{2^{\widehat{\mu}_r(2)+1}} & \text{if } k = 0 \text{ and } r > 0, \\ \mathbb{Z}_2^S \oplus \mathbb{Z}_{2^{\widehat{\mu}_1(2)}} \oplus \cdots \oplus \mathbb{Z}_{2^{\widehat{\mu}_r(2)}} & \text{if } k > 0, \end{cases}$$

$$Syl_q(\mathcal{T}(\Lambda_{ab})) \approx \quad \mathbb{Z}_{q^{\widehat{\mu}_1(q)}} \oplus \cdots \oplus \mathbb{Z}_{q^{\widehat{\mu}_r(q)}} \qquad \text{for } q > 2.$$

By Lemma 1.7.1, we obtain necessary and sufficient conditions for an epimorphism $\Lambda_{ab} \to A$ to exist. Indeed, these conditions are expressed in terms of the number of cyclic factors of $Syl_q(\mathcal{T}(\Lambda_{ab}))$ and A_q. Let $N_q(i)$ and $n_q(i)$ be the number of cyclic factors of $Syl_q(\mathcal{T}(\Lambda_{ab}))$ and A_q, respectively, of order greater than or equal to q^i. As a consequence of Lemma 1.7.1, the existence of an epimorphism $\Lambda \to A$ is thus equivalent to the fulfillment of the following inequalities:

$$\eta g + k - 1 + N_q(i) \geqslant n_q(i) \tag{1.7}$$

for each prime q dividing $|A|$ and every integer $i > 0$

($2g+N_q(i)$ in the left term if Λ is a Fuchsian group). Note that $n_q(1)$ is the number of nontrivial cyclic factors of A_q and, if $q > 2$ and Λ is a proper NEC group, $N_q(1)$ is the number of proper periods divisible by q. Also, $N_q(i) = 0$ if $q \nmid m_{i'}$ for all $i' \in \{1, \ldots, r\}$.

It can be helpful to represent graphically the inequalities (1.7) overlapping the graphs of the number of factors of $Syl_q(\mathcal{T}(\Lambda_{ab}))$ and A_q for a given q. Conditions are fulfilled if and only if the second line never places above the first one.

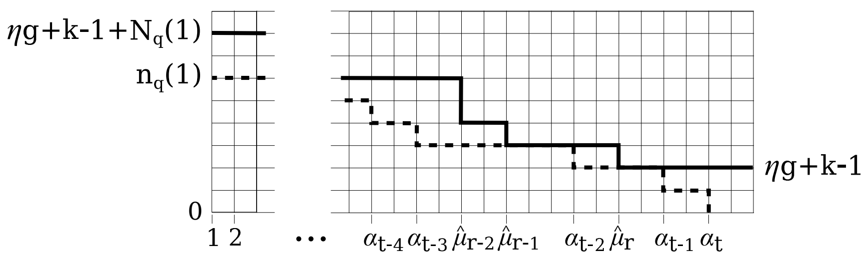


Figure 1.1: Example fulfilling conditions (1.7) for a prime q dividing $|A|$. We show the values $\alpha_1(q), \ldots, \alpha_t(q)$ and $\widehat{\mu}_1(q), \ldots, \widehat{\mu}_{r-1}(q)$ on the horizontal axis. As we move from right to left along each integer value x on the horizontal axis, the dotted line cumulatively adds up the number of factors of order q^x in A_q. Likewise, the solid line adds up the factors of $Syl_q(\mathcal{T}(\Lambda_{ab}))$ starting from the right with value $\eta g + k - 1$. In this example, Λ is a proper NEC group, $q > 2$, $\eta g + k - 1 = 2$ and $\widehat{\mu}_{r-3} = \widehat{\mu}_{r-2}$.

2 Abelian actions on Riemann surfaces

In this chapter, Breuer conditions for orientation-preserving abelian actions on compact Riemann surfaces of genus $g > 1$ are refined so that every condition is entirely expressed in terms of the invariant factors of the abelian group and the signature of the Fuchsian group. This alternative statement results in a more concise proof of Maclachlan's solution to the minimum genus problem; in many cases, we can fix an explicit expression using some results about the invariant factors of the abelian group. Finally, we find an explicit solution to the minimum genus problem for the family of abelian groups, cyclic or not, of the same given order, as well as the unique abelian group attaining the minimum genus.

2.1 Surface-kernel epimorphisms onto an abelian group

Conditions for the existence of a surface-kernel epimorphism from a Fuchsian group Λ onto a finite abelian group A were first stated by Breuer [3, Theorem. 9.1]. Unlike for other families of groups, one of these conditions simply claims the existence of an epimorphism $\Lambda \to A$, relying this issue upon lemmas A.1 and A.2 in [3], collected here in the inequalities (1.7) for each prime q dividing the order of A. In Theorem 2.1.2, we embed conditions (1.7) for the existence of an epimorphism $\Lambda_{ab} \to A$ into Breuer's theorem. This way we achieve conditions only in terms of the signature of the Fuchsian group and the invariant factors defining the abelian group.

Theorem 2.1.1. (Breuer) *Let A be a finite abelian group, Λ a Fuchsian group with signature $(g; m_1, \dots, m_r)$ and $M = \operatorname{lcm}(m_1, \dots, m_r)$. There exists a surface-kernel epimorphism $\psi : \Lambda \to A$ if and only if the following conditions are satisfied:*

(o) *There exists an epimorphism $\Lambda \to A$.*

(i) $\operatorname{lcm}(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_r) = M$ *for all i.*

(ii) $M \big| \exp A$; *if* $g = 0$, *then* $M = \exp A$.
(iii) $r \neq 1$; *if* $g = 0$, *then* $r \geqslant 3$.
(iv) *If* M *is even and only one of the elementary divisors of* A *is divisible by the maximum power of* 2 *dividing* M, *then the number of periods* m_i *divisible by such power of* 2 *is even.*

Now, we make use of conditions (1.7) for the existence of an epimorphism $\Lambda_{ab} \to A$ to replace condition (o) in Breuer's theorem:

Theorem 2.1.2. *Let* Λ *be a Fuchsian group with signature* $(g; m_1, \dots, m_r)$, $M = \operatorname{lcm}(m_1, \dots, m_r)$ *and integers* $t \geqslant 1$ *and* $v_1, \dots, v_t$ *with* $v_i > 1$ *and* $v_1 | \cdots | v_t$. *There exists a surface-kernel epimorphism* $\Lambda \to \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ *if and only if the following conditions are satisfied:*

(i) $\operatorname{lcm}(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_r) = M$ *for all* i.
(ii) $M \big| v_t$; *if* $g = 0$, *then* $M = v_t$.
(iii) $r \neq 1$; *if* $g = 0$, *then* $r \geqslant 3$.
(iv) *If* M *is even and only one of the elementary divisors of* $\mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ *is divisible by the maximum power of* 2 *dividing* M, *then the number of periods* m_i *divisible by such power of* 2 *is even.*
(v) *If* $t > 2g$, *then* $r \geqslant t - 2g + 1$ *and every elementary divisor of* $\mathbb{Z}_{v_k}$ *divides, at least,* $t - 2g - k + 2$ *periods* m_i *for* $k = 1, \dots, t - 2g$.

Proof. Condition (v) replaces condition (o) in Theorem 2.1.1. Below we prove that both conditions are equivalent. Hence, theorems 2.1.1 and 2.1.2 are equivalent.

If there exists a surface-kernel epimorphism $\Lambda \to A$, then we know, by Theorem 2.1.1, that conditions (i)-(iv) are satisfied. Conditions (i) and (ii) imply that $\widehat{\mu}_{r-1}(q) = \widehat{\mu}_r(q) \leqslant \alpha_t(q)$ for each prime q dividing the order of A, and so dividing v_t. (For readability, we will write α_i and $\widehat{\mu}_i$ instead of $\alpha_i(q)$ and $\widehat{\mu}_i(q)$ in what follows.)

If $t \leqslant 2g$, the inequalities (1.7) are always fulfilled for every prime q dividing v_t, since $t \geqslant n_q(\alpha_1) \geqslant \cdots \geqslant n_q(\alpha_t)$. However, if $t > 2g$, that need not necessarily be the case. Now, we show that, when $t > 2g$, conditions (1.7) hold if and only if

$$r \geqslant t - 2g + 1 \quad \text{and} \quad \alpha_k \leqslant \widehat{\mu}_{r-1-t+2g+k} \quad \text{for} \quad k = 1, \dots, t - 2g, \tag{2.1}$$

or, explicitly,

$$\begin{aligned} \alpha_{t-2g} &\leqslant \widehat{\mu}_{r-1}, \\ \alpha_{t-2g-1} &\leqslant \widehat{\mu}_{r-2}, \\ &\vdots \\ \alpha_1 &\leqslant \widehat{\mu}_{r-t+2g}, \\ \text{and } r &\geqslant t-2g+1. \end{aligned}$$

If (2.1) holds, let $k \in \{1,\ldots,t-2g\}$ and l be the smallest integer in $\{1,\ldots,k\}$ such that $\alpha_l = \alpha_k$. Then $n_q(\alpha_k) = n_q(\alpha_l) = t-l+1$ and $N_q(\alpha_k) = N_q(\alpha_l)$. By (2.1), $\alpha_l \leqslant \widehat{\mu}_{r-1-t+2g+l}$, so

$$N_q(\alpha_l) \geqslant r-1-(r-1-t+2g+l)+1 = t-2g-l+1.$$

Hence, $2g + N_q(\alpha_k) \geqslant n_q(\alpha_k)$. This proves that (2.1) implies (1.7).

Now, assume that conditions (1.7) hold. Since $v_1 \neq 1$, there exists some prime q dividing v_t for which $0 < \alpha_1 \leqslant \cdots \leqslant \alpha_t$ and $n_q(\alpha_1) = t$. The inequality $2g + N_q(\alpha_1) \geqslant n_q(\alpha_1) = t$ in (1.7) implies $N_q(\alpha_1) \geqslant t-2g > 0$, so that $r > 1$ (by Lemma 1.7.2, $N_q(i) = 0$ if $r = 0$ or 1), hence $r-1 \geqslant N_q(\alpha_1)$ and thus $r \geqslant t-2g+1$.

The last $2g$ values $\alpha_{t-2g+1},\ldots,\alpha_t$ can be smaller than (except α_t, by condition (ii)), equal to or greater than $\widehat{\mu}_{r-1}$. However, it is always $\alpha_{t-2g} \leqslant \widehat{\mu}_{r-1}$: otherwise, the last $2g+1$ values $\alpha_{t-2g},\ldots,\alpha_t$, at least, would be greater than $\widehat{\mu}_{r-1}$, thus $N_q(\alpha_{t-2g}) = 0$ and $n_q(_{t-2g}) \geqslant 2g+1$, so the inequality $2g + N_q(\alpha_{t-2g}) \geqslant n_{t-2g}$ would not be fulfilled and an epimorphism would not exist.

Also, the next value α_{t-2g-1} smaller than or equal to α_{t-2g} must satisfy $\alpha_{t-2g-1} \leqslant \widehat{\mu}_{r-2}$; otherwise, some inequality in (1.7) would not be fulfilled. Likewise, it must be $\alpha_{t-2g-2} \leqslant \widehat{\mu}_{r-3}$, $\alpha_{t-2g-3} \leqslant \widehat{\mu}_{r-4}$, and so on. Hence, (1.7) implies (2.1).

Now, since $\widehat{\mu}_i \leqslant \widehat{\mu}_{i+1}$, it is clear that conditions (2.1) hold if and only if $\alpha_k \leqslant \widehat{\mu}_i$ for each $k \in \{1,\ldots,t-2g\}$ and $i = r-1-t+2g+k,\ldots,r-1$. Since $\widehat{\mu}_r = \max\{\mu_i\}_{i=1,\ldots,r}$, this means that q^{α_k} divides $q^{\widehat{\mu}_i}$ for $i = r-1-t+2g+k,\ldots,r$ and thus also divides, at least, $t-2g-k+2$ periods m_i. This is condition (v). ■

2.2 Explicit expression for the strong symmetric genus

Conditions of Theorem 2.1.2 allow us to obtain a new shorter proof —see Theorem 2.2.1— of Maclachlan's solution of the minimum genus problem for a finite noncyclic abelian group [24, Theorem. 4]. Furthermore, upon closer study of this solution, we find an explicit expression for the minimum genus in many cases, as is shown in Remarks 2.2.3, 2.2.4, 2.2.5 and subsequent comments.

In particular, condition 2.1.2.(v) makes it possible to use the invariant factors $v_1, \ldots, v_{t-2g}$ as periods of a Fuchsian group, and this group is a candidate to minimize $\mu(\Lambda) = 2(g-1) + \sum_{i=1}^{r}(1 - 1/m_i)$: elementary divisors of $\mathbb{Z}_{v_k}$ always divide certain periods m_i of any Fuchsian group satisfying conditions of Theorem 2.1.2, and this suggests to compose a signature with $v_1, \ldots, v_{t-2g}$ as periods.

Theorem 2.2.1. (Maclachlan) *Let $A \approx \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$, with $v_i > 1$ and $v_1 | \cdots | v_t$, be a noncyclic abelian group of order $|A| > 9$. The strong symmetric genus $\sigma^o(A)$ of A satisfies*

$$\frac{2(\sigma^o(A) - 1)}{|A|} = \min_{0 \leqslant 2g < t'} \left\{ 2(g-1) + \sum_{i=1}^{t-2g} \left(1 - \frac{1}{v_i}\right) + 1 - \frac{1}{v_{t-2g}} \right\}$$

where $t' = t$ if $t = 2$ or odd, and $t' = t+1$ if $t > 2$ is even (interpreting v_0 as 1).

Remark 2.2.2. [24, p. 711] The strong symmetric genera of noncyclic abelian groups of order smaller than or equal to 9 are:

$$\begin{aligned} \sigma^o(\mathbb{Z}_2 \oplus \mathbb{Z}_2) &= 2, \\ \sigma^o(\mathbb{Z}_2 \oplus \mathbb{Z}_4) &= 3, \\ \sigma^o(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) &= 3, \\ \sigma^o(\mathbb{Z}_3 \oplus \mathbb{Z}_3) &= 4. \end{aligned}$$

Proof of Theorem 2.2.1. Let $\mathcal{F}$ be the family of Fuchsian groups for which there exists a surface-kernel epimorphism onto A, and let Λ_g be any Fuchsian group in $\mathcal{F}$ with orbit genus g, say, with signature $(g;\ m_1, \ldots, m_r)$.

First, we note that, for any integer g such that $0 \leqslant 2g < t$, the signature $(g\,;\, v_1\,,\, \ldots\,,\, v_{t-2g-1}\,,\, v_{t-2g}\,,\, v_{t-2g})$ defines a Fuchsian group if $|A| = v_1 \cdots v_t > 9$ —recall (1.1). For, let

$$\mu = 2g - 2 + \sum_{i=1}^{t-2g} \left(1 - \frac{1}{v_i}\right) + 1 - \frac{1}{v_{t-2g}} = t - 1 - \frac{1}{v_1} - \cdots - \frac{1}{v_{t-2g-1}} - \frac{2}{v_{t-2g}}.$$

Since $v_i \geqslant 2$ and $0 \leqslant 2g < t$, it follows that $\mu \geqslant \frac{t-3}{2}$. Clearly, $\mu > 0$ if $t \geqslant 4$, and also if $t = 3$ (in this case, $\mu = 0$ if $g = 0$ and $v_1 = v_2 = v_3 = 2$, but then $v_1v_2v_3 = 8 < 9$; otherwise, $\mu > 0$ since $v_1v_2v_3 > 9$ and thus $v_i > 2$ for some i). If $t = 2$, then $g = 0$ and $\mu = 1 - 1/v_1 - 2/v_2$, so that $\mu \leqslant 0$ if $v_1 = v_2 = 2$, $2v_1 = v_2 = 4$ or $v_1 = v_2 = 3$ (but $|A| < 9$ is these cases) and $\mu > 0$ otherwise.

We denote this Fuchsian group by $\widetilde{\Lambda}_g$. The signature fulfills conditions of Theorem 2.1.2, so $\widetilde{\Lambda}_g \in \mathcal{F}$. We notice that, by condition (v) of Theorem 2.1.2 and since $2g < t$, it must be $r \geqslant 2$. By conditions (i) and (v) of Theorem 2.1.2, we have $v_{t-2g}|\widehat{m}_r$, $v_{t-2g-1}|\widehat{m}_{r-1}$, $v_{t-2g-2}|\widehat{m}_{r-2}$, …, $v_1|\widehat{m}_{r-t+2g}$. It follows that

$$\left(1-\frac{1}{v_1}\right)+\cdots+\left(1-\frac{1}{v_{t-2g}}\right)+\left(1-\frac{1}{v_{t-2g}}\right)\leqslant\left(1-\frac{1}{\widehat{m}_1}\right)+\cdots+\left(1-\frac{1}{\widehat{m}_r}\right),$$

so the signature $(g;\ \widehat{m}_{r-\hat{r}+1},\ldots,\widehat{m}_r)$ defines a Fuchsian group, $\widehat{\Lambda}_g$, since $\mu(\widetilde{\Lambda}_g) > 0$, and $\mu(\widetilde{\Lambda}_g) \leqslant \mu(\widehat{\Lambda}_g)$ (recall $\hat{r}$ is such that $\widehat{m}_{r-\hat{r}} = 1$ and $\widehat{m}_{r-\hat{r}+1} > 1$). Then, by (1.3), $\mu(\widetilde{\Lambda}_g) \leqslant \mu(\widehat{\Lambda}_g) \leqslant \mu(\Lambda_g)$.

For any g with $2g \geqslant t > 1$, the signature $(g; —)$ defines a Fuchsian group Γ_g and fulfills conditions of Theorem 2.1.2, so $\Gamma_g \in \mathcal{F}$. Then $\mu(\Gamma_g) = 2(g-1) \leqslant \mu(\Lambda_g)$ for any $\Lambda_g \in \mathcal{F}$. Also, $\mu(\Gamma_{g'}) < \mu(\Gamma_g)$ if $g' < g$. Let $\overline{g} = \min\{g \in \mathbb{Z} \mid 2g \geqslant t, g > 1\}$. Then, $\overline{g} = 2$ if $t = 2$, $\overline{g} = (t+1)/2$ if t is odd, and $\overline{g} = t/2$ if $t > 2$ is even; the corresponding values of $\mu(\Gamma_{\overline{g}})$ are 2, $t-1$ and $t-2$, respectively.

Therefore, we have to compare $\mu(\widetilde{\Lambda}_g)$ and $\mu(\Gamma_{\overline{g}})$ for any g satisfying $0 \leqslant 2g < t$. If $g = 0$, then $\mu(\widetilde{\Lambda}_g) < \mu(\Gamma_{\overline{g}})$ for $t = 2$ or t odd (recall that $\mu(\widetilde{\Lambda}_g) > 0$ when $|A| > 9$ and $0 \leqslant g < t$). But, if t is even, it could be $\mu(\widetilde{\Lambda}_g) > \mu(\Gamma_{\overline{g}})$ for every g such that $2g < t$. To take this possibility into account, we define $v_0 = 1$, $t' = t$ if $t = 2$ or odd, and $t' = t + 1$ if $t > 2$ is even. ■

Now we get some insight into the expression for the strong symmetric genus in Theorem 2.2.1. In many cases, we can avoid calculating the minimum, since it is possible to find out in advance which g satisfying $0 \leqslant g < t$ provides it, by simple inspection of the invariant factors of the abelian group. In the remainder of this section we assume $t \geqslant 3$ since the case $t = 2$ was solved in [24, Theorem 4], see example 2.2.6.

We can write $\sigma^o(A) = 1 + \frac{|A|}{2}\ \min\{\mu_0, \mu_1, \ldots, \mu_{g'}\}$, where

$$\mu_g = 2(g-1) + \sum_{i=1}^{t-2g}\left(1-\frac{1}{v_i}\right)+\left(1-\frac{1}{v_{t-2g}}\right)$$

and $g' = \lfloor t/2 \rfloor$ is the integer part of $t/2$. Let also

$$q_i = \frac{v_{i+1}}{v_i} \quad \text{for } i = 1, \dots, t-1.$$

If t is odd, we arrange the integers $\{q_1, \dots, q_{t-1}\}$ into pairs, reversing the order of subindices:

$$(q_{t-1}, q_{t-2}), \dots, (q_4, q_3), (q_2, q_1).$$

If t is even, we can consider A as an abelian group with $t+1$ invariant factors, $A \approx \{0\} \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$, without changing the expressions for μ_g and $\sigma^o(A)$; now the $(t+1)-1 = t$ quotients of two consecutive invariant factors become $v_1/1 = v_1$, $v_2/v_1 = q_1$, …, $v_t/v_{t-1} = q_{t-1}$, resulting in the sequence of pairs

$$(q_{t-1}, q_{t-2}), \dots, (q_3, q_2), (q_1, v_1).$$

The expression of μ_{g+1} in terms of the foregoing μ_g is

$$\mu_{g+1} = \mu_g + \frac{2}{v_{t-2g}} + \frac{1}{v_{t-2g-1}} - \frac{1}{v_{t-2g-2}} = \mu_g + \frac{2-(q_{t-2g-2}-1)q_{t-2g-1}}{v_{t-2g}}$$

for $g = 0, \dots, g'-1$ (interpreting $v_0 = 1$ and $q_0 = v_1$ if t is even). Writing them explicitly,

$$\begin{aligned} \mu_1 &= \mu_0 + \frac{2-(q_{t-2}-1)q_{t-1}}{v_t}, \\ \mu_2 &= \mu_1 + \frac{2-(q_{t-4}-1)q_{t-3}}{v_{t-2}}, \quad \text{etc.,} \end{aligned}$$

we observe that the difference $\mu_1 - \mu_0$ depends on q_{t-1}, q_{t-2}, v_t, which are the largest subindices for q and v. These subindices reduce by 2 each step as g increases. The last difference $\mu_{g'} - \mu_{g'-1}$ depends on q_2, q_1, v_3 (or on $q_0 = v_1$ and q_1 if t is even). It follows that:

$$\begin{aligned}
&a)\quad \mu_0 < \mu_1 \iff \begin{cases} q_{t-2} = 1 \\ \quad \text{or} \\ (q_{t-1}, q_{t-2}) = (1,2) \end{cases} \iff \begin{cases} v_{t-2} = v_{t-1} \\ \quad \text{or} \\ 2v_{t-2} = v_{t-1} = v_t \end{cases} \\
&b)\quad \mu_0 = \mu_1 \iff (q_{t-1}, q_{t-2}) = \begin{cases} (2,2) \\ \quad \text{or} \\ (1,3) \end{cases} \iff \begin{cases} 4v_{t-2} = 2v_{t-1} = v_t \\ \quad \text{or} \\ 3v_{t-2} = v_{t-1} = v_t \end{cases} \qquad (2.2) \\
&c)\quad \mu_0 > \mu_1 \quad \text{otherwise.}
\end{aligned}$$

The same happens for each pair μ_g, μ_{g+1} and (q_{t-2g-1}, q_{t-2g-2}). So, in general, μ_g is smaller as g increases, and $\sigma^o(A)$ is mostly given by $\mu_{g'}$:

Remark 2.2.3. If $v_1 \neq v_2 \neq \cdots \neq v_t$, then

$$\sigma^o(A) = 1 + \frac{|A|}{2}\,\mu_{g'}$$

since, in this case, $q_i > 1$ for all i, so $\mu_g \geqslant \mu_{g+1}$ for all $g \in \{0, \ldots, g'-1\}$. ■

On the other hand, we can have $\mu_g \leqslant \mu_{g+1}$ for all g (the following two remarks fix the inaccurate results noted in [32] remarks 4.5 and 4.6):

Remark 2.2.4. Assume that either $v_{i+1} = v_i$ or $v_{i+1} = 2v_i$ for each $i \in \{1, \ldots, t-1\}$. If t is odd or $t = 2$, then

$$\sigma^o(A) = 1 + \frac{|A|}{2}\,\mu_0.$$

When $t > 2$ is even, $\mu_{t/2}$ may be smaller than μ_0. Indeed, this occurs very often since

$$\mu_{t/2} = \mu_0 + \frac{1}{v_1} + \cdots + \frac{1}{v_{t-1}} + \frac{2}{v_t} - 1$$

and thus

$$\sigma^o(A) = \begin{cases} 1 + \dfrac{|A|}{2}\,\mu_{t/2} & \text{if } \frac{1}{v_1} + \cdots + \frac{1}{v_{t-1}} + \frac{2}{v_t} \leqslant 1, \text{ and} \\ 1 + \dfrac{|A|}{2}\,\mu_0 & \text{otherwise.} \end{cases}$$

■

In both remarks 2.2.3 and 2.2.4, we have only made use of the case $(q_{t-1}, q_{t-2}) = (2, 2)$ in (2.2.b). It is straightforward to include the other case $(q_{t-2g-1}, q_{t-2g-2}) = (1, 3)$, for values of g in $\{1, \ldots, g'\}$, to enlarge the set of cases with the same result $\sigma^o(A) = 1 + |A|\mu_0/2$ or $1 + |A|\mu_{g'}/2$.

Also, we can get $\sigma^o(A) = 1 + |A|\mu_g/2$ for each $g \in \{0, \ldots, g'\}$, at least when t is odd, by combining conditions in both remarks to get $\mu_0 \geqslant \mu_1 \geqslant \ldots \geqslant \mu_{g-1} \geqslant \mu_g \leqslant \mu_{g+1} \leqslant \ldots \leqslant \mu_{g'}$:

Remark 2.2.5. Let t be odd and $g \in \{0, \ldots, g'\}$. If $v_{t-2g-1} \neq v_{t-2g} \neq \cdots \neq v_t$ and $v_{i+1} = v_i$ or $v_{i+1} = 2v_i$ for all $i \in \{1, \ldots, t-2g-2\}$, then

$$\sigma^o(A) = 1 + \frac{|A|}{2}\,\mu_g.$$

■

Conditions (2.2) show that it is difficult to study which μ_g provides $\sigma^o(A)$ in general. But we can still give some hints that ease the situation by reducing the number of candidates among $\{\mu_0, \dots, \mu_{g'}\}$ to obtain $\sigma^o(A)$.

It can help to assign a symbol to each pair (q_{t-2g-1}, q_{t-2g-2}) for $g = 0, \dots, g'-1$ as follows:

$$
\begin{array}{ll}
{}_g\nearrow^{g+1} & \text{if } (q_{t-2g-1}, q_{t-2g-2}) = (..., 1) \text{ or } (1,2), \\
{}_g\rightarrow_{g+1} & \text{if } (q_{t-2g-1}, q_{t-2g-2}) = (2,2) \text{ or } (1,3), \\
{}^g\searrow_{g+1} & \text{otherwise.}
\end{array}
$$

So, for example, the sequence of pairs $(q_{t-1}, q_{t-2}), \dots, (q_4, q_3), (q_2, q_1)$ —ending with (q_1, v_1) if t is even— is represented by

$$
{}^0\searrow_1\nearrow^2\searrow_3\searrow_4\longrightarrow_5 \;\cdots\; {}^{g'-2}\searrow^{g'-1}\searrow_{g'}
$$

if and only if

$$
\mu_0 > \mu_1 < \mu_2 > \mu_3 > \mu_4 = \mu_5 \;\cdots\; \mu_{g'-2} > \mu_{g'-1} > \mu_{g'}.
$$

Obviously, the case ${}^{g-1}\searrow_g\nearrow^{g+1}$ indicates that μ_g is a candidate to obtain $\sigma^o(A)$, but not μ_{g-1} or μ_{g+1}. Also, μ_0 or $\mu_{g'}$ are candidates if ${}_0\nearrow$ or $\searrow_{g'}$ appear, respectively, but not if we have ${}^0\searrow$ or $\nearrow^{g'}$.

When ${}_g\nearrow^{g+1}$ occurs, then $\mu_{g+1} = \mu_g + 1/v_{t-2g}$ or $\mu_{g+1} = \mu_g + 2/v_{t-2g}$ corresponding to each possibility in (2.2.a).

It is easy to see that, in general, $\mu_{g+1} > \mu_g \geqslant \mu_{g+2}$ in the case ${}_g\nearrow^{g+1}\searrow_{g+2}$, with the only exception when $(q_{t-2g-1}, q_{t-2g-2}) = (1,1)$ and $(q_{t-2g-3}, q_{t-2g-4}) = (3,2)$, in which case $\mu_{g+1} > \mu_{g+2} > \mu_g$, since $\mu_{g+1} = \mu_g + 2/v_{t-2g}$ and $\mu_{g+2} = \mu_g + 1/v_{t-2g}$. We introduce another symbol for this last situation:

$$
\begin{aligned}
{}_g\nearrow\!\!\searrow g+2 &\iff \mu_g < \mu_{g+2} < \mu_{g+1} \\
&\iff (q_{t-2g-1}, q_{t-2g-2}) = (1,1) \text{ and } (q_{t-2g-3}, q_{t-2g-4}) = (3,2),
\end{aligned}
$$

so μ_g is a candidate to obtain $\sigma^o(A)$, but not μ_{g+1} or μ_{g+2}. In the following examples, we assume $\mu_{g+2} < \mu_g < \mu_{g+1}$ whenever we write ${}_g\nearrow^{g+1}\searrow_{g+2}$ (the case $\nearrow\!\!\searrow$ is excluded from that sequence and we split it apart as a separate case).

Example 2.2.6. [24, Theorem 4] If $t = 2$ and $|A| > 9$, then $\sigma^o(A) = \frac{1}{2}(v_1 v_2 - v_2) - v_1 + 1$.

Example 2.2.7. $t = 3$. Here $g = 0, 1$, $\mu_0 = 2 - \frac{1}{v_1} - \frac{1}{v_2} - \frac{2}{v_3}$, $\mu_1 = 2 - \frac{2}{v_1}$ and, considering the pair (q_2, q_1), there are only three cases (in each one, we point out which μ_g provides the strong symmetric genus):

$$^{0}\searrow_{1} \quad \mu_1 \qquad\qquad {}_0\!\rightarrow_1 \quad \mu_1 = \mu_0 \qquad\qquad {}_0\nearrow^{1} \quad \mu_0.$$

Example 2.2.8. $t = 4$. We have $g = 0, 1, 2$ and

$$\mu_0 = 3 - \frac{1}{v_1} - \frac{1}{v_2} - \frac{1}{v_3} - \frac{2}{v_4} \qquad \mu_1 = 3 - \frac{1}{v_1} - \frac{2}{v_2} \qquad \mu_2 = 2.$$

We consider the sequence $(q_3, q_2), (q_1, v_1)$; there are ten cases:

$^{0}\searrow^{1}\searrow_{2}$	μ_2	${}_0\nearrow^{1}\nearrow^{2}$	μ_0	$^{0}\rightarrow^{1}\searrow_{2}$	μ_2
$^{0}\searrow_{1}\rightarrow_{2}$	$\mu_2 = \mu_1$	${}_0\nearrow^{1}\rightarrow^{2}$	μ_0	${}_0\rightarrow_1\rightarrow_2$	$\mu_2 = \mu_1 = \mu_0$
$^{0}\searrow_{1}\nearrow^{2}$	μ_1	${}_0\nearrow^{1}\searrow_{2}$	μ_2	${}_0\rightarrow_1\nearrow^{2}$	$\mu_1 = \mu_0$
		${}_0\nearrow\!\!\searrow{}^{2}$	μ_0		

Example 2.2.9. $t = 5$. Here, the sequence is $(q_4, q_3), (q_2, q_1)$, $g = 0, 1, 2$ and

$$\mu_0 = 4 - \frac{1}{v_1} - \frac{1}{v_2} - \frac{1}{v_3} - \frac{1}{v_4} - \frac{2}{v_5} \qquad \mu_1 = 4 - \frac{1}{v_1} - \frac{1}{v_2} - \frac{2}{v_3} \qquad \mu_2 = 4 - \frac{2}{v_1}.$$

The cases are the same as in the previous example.

Example 2.2.10. The strong symmetric genus of $A \approx \mathbb{Z}_7 \oplus \mathbb{Z}_{14} \oplus \mathbb{Z}_{14} \oplus \mathbb{Z}_{84} \oplus \mathbb{Z}_{336}$ is $\sigma^o(A) = 1 + |A|\, \mu_2/2 = 1 + |A|\, (t - 1 - 2/v_1)/2 = 71914753$, since the corresponding sequence is $(4, 6), (1, 2)$, so the diagram is ${}_0\nearrow^{1}\searrow_{2}$.

Example 2.2.11. Consider the groups

$$\begin{aligned}
&A \approx \mathbb{Z}_{15} \oplus \mathbb{Z}_{90} \oplus \mathbb{Z}_{450} \oplus \mathbb{Z}_{900} \oplus \mathbb{Z}_{900}, && \text{with sequence } (1, 2), (5, 6),\\
&B \approx \mathbb{Z}_{3} \oplus \mathbb{Z}_{450} \oplus \mathbb{Z}_{450} \oplus \mathbb{Z}_{900} \oplus \mathbb{Z}_{900}, && \text{with sequence } (1, 2), (1, 150), \text{ and}\\
&C \approx \mathbb{Z}_{3} \oplus \mathbb{Z}_{450} \oplus \mathbb{Z}_{450} \oplus \mathbb{Z}_{810000}, && \text{with sequence } (1800, 1), (150, 3).
\end{aligned}$$

The three groups have order $N = 492075000000$ and diagram ${}_0\nearrow^1\searrow_2$. Hence

$$\sigma^o(A) = 1 + N\mu_2/2 = 1 + N(t-1-2/v_1)/2 = 951345000001,$$
$$\sigma^o(B) = 1 + N\mu_2/2 = 1 + N(t-1-2/v_1)/2 = 820125000001,$$
$$\sigma^o(C) = 1 + N\mu_2/2 = 1 + N(t-2)/2 = 492075000001,$$

for the corresponding value of μ_2, t and v_1 in each case. We notice that $\sigma^o(C) < \sigma^o(B) < \sigma^o(A)$. The following section deals with this issue.

2.3 Least strong symmetric genus of abelian groups of the same order

Harvey solved the minimum genus problem for cyclic groups [19, Theorem 6]. Given an integer $N > 1$, we now focus on abelian groups of order N. Each of these groups has a strong symmetric genus; in this thesis, the lowest of these genera will be called the *least strong symmetric genus* of abelian groups of order N, and will be denoted by $\sigma^o(N)$. In this section, we obtain $\sigma^o(N)$ by means of Theorem 2.2.1. Taking advantage of this result, we also obtain a new proof of the solution to the maximum order problem for abelian groups stated by Breuer [3, Corollary 9.6].

We first consider, in Theorem 2.3.1, abelian groups having order N and a fixed number t of nontrivial invariant factors. Applying this result, we obtain, in Theorem 2.3.2, the abelian group of order N that provides the least strong symmetric genus by comparing the resulting genera for admissible values of t.

We observe that, if $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ is the prime factorization of N and $\mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ is an abelian group of order $v_1 \cdots v_t = N$, with $v_i > 1$ and $v_1 | \cdots | v_t$, then $t \leqslant \max_{j=1,\ldots,s}\{\alpha_j\}$.

Theorem 2.3.1. *The least strong symmetric genus of all abelian groups of order $N > 1$ with $t > 1$ nontrivial invariant factors is*

$$\sigma^o(N,t) = \begin{cases} (t-1)(p-1)\frac{N}{2p} - p^{t-1} + 1 & \textit{if } t = 2,\ t \textit{ is odd or } p - t + 1 \leqslant \frac{2p^t}{N}, \textit{ and} \\ (t-2)\frac{N}{2} + 1 & \textit{otherwise,} \end{cases}$$

where p is the smallest prime such that $p^t | N$. Moreover, the least strong symmetric genus is attained by $\mathbb{Z}_p \oplus \overset{t-1}{\cdots} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{N/p^{t-1}}$.

Proof. Let $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the prime factorization of N and $A \approx \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ with $t > 1$, $|A| = v_1 \cdots v_t = N$, $v_i > 1$ and $v_1 | \cdots | v_t$. Hence, there must be, at least, a prime $p_j \in \{p_1, \ldots, p_s\}$ such that $p_j | v_i$ for all $i \in \{1, \ldots, t\}$, so $p_j^t | N$. Let p be the smallest such prime,

$$p = \min_{j=1,\ldots,s} \left\{ p_j \mid p_j^t \text{ divides } N \right\}.$$

By Theorem 2.2.1, we obtain the strong symmetric genus of A from the smallest μ_g for the admissible values $g = 0, \ldots, g'$, with $g' = 0$ if $t = 2$ and $g' = \lfloor t/2 \rfloor$ otherwise. If we let A vary with t and its order fixed, then the invariant factors

$$v_1 = \cdots = v_{t-1} = p \qquad v_t = \frac{N}{p^{t-1}} \tag{2.3}$$

give the smallest μ_g for each $g = 0, \ldots, g'$: this is straightforward to check when $g > 0$, since $1/v_i \leqslant 1/p$, $i = 1, \ldots, t-1$, for any other invariant factors such that $v_1 \cdots v_t = N$. For $g = 0$, we observe that $0 \leqslant (\frac{m}{n} - 2)(s - 1)$, which is equivalent to

$$-\frac{1}{n} - \frac{2}{sm} \leqslant -\frac{1}{ns} - \frac{2}{m}, \tag{2.4}$$

for any integers $s \geqslant 2$, $m \geqslant 2$ and $n \geqslant 1$ such that $ns | m$. Let $v_1, \ldots, v_t$ be invariant factors such that $v_1 \cdots v_t = N$. Applying (2.4) repeatedly, it follows that

$$-\frac{1}{q} - \overset{t-1}{\cdots} - \frac{1}{q} - \frac{2q^{t-1}}{N} \leqslant -\frac{1}{v_1} - \cdots - \frac{1}{v_{t-1}} - \frac{2}{v_t}$$

for any prime q dividing v_1. Since $q^t | N$, we have $p \leqslant q$. In case that $q \neq p$, $p^2 q^t$ divides N and thus $q^{t-1} < N/2pq$ since $p \geqslant 2$. Hence,

$$(q-p)(t-1)\frac{N}{2pq} + p^{t-1} - q^{t-1} \geqslant \frac{N}{2pq} + p^{t-1} - q^{t-1} > 0$$

and thus

$$-\frac{t-1}{p} - \frac{2p^{t-1}}{N} \leqslant -\frac{t-1}{q} - \frac{2q^{t-1}}{N}.$$

Therefore, $\mathbb{Z}_p \oplus \overset{t-1}{\cdots} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{N/p^{t-1}}$ attains the least strong symmetric genus $\sigma^o(N, t)$.

For the invariant factors (2.3), we have $\mu_0 = (t-1)(p-1)/p - 2p^{t-1}/N$, and $\sigma^o(A) = 1 + \frac{N}{2}\mu_0$ if $t = 2$. When $t > 2$, and following Section 2.2, the corresponding sequences $(N/p^t, 1), (1, 1), \ldots, (1, 1)$ if t is odd, and $(N/p^t, 1), (1, 1), \ldots, (1, 1), (1, p)$

if t is even, lead to the cases

$$\begin{array}{ll} {}_0\nearrow^1\nearrow^2\cdots\nearrow^{(t-1)/2} & \text{if } t \text{ is odd,} \\ {}_0\nearrow^1\nearrow^2\cdots\nearrow^{t/2} & \text{if } t \text{ is even and } p=2, \\ {}_0\nearrow^1\nearrow^2\cdots\nearrow^{t/2-1}\rightarrow^{t/2} & \text{if } t \text{ is even and } p=3, \\ {}_0\nearrow^1\nearrow^2\cdots\nearrow^{t/2-1}\searrow_{t/2} & \text{if } t \text{ is even and } p\geqslant 5. \end{array}$$

Hence, $\mu_0 < \mu_i$ when $i > 0$, so that $\sigma^o(A) = 1+\frac{N}{2}\mu_0$ unless $t > 2$ is even and $p \geqslant 5$, in which case $\sigma^o(A) = 1 + \frac{N}{2}\min\{\mu_0, \mu_{t/2}\}$. If $t > 2$ is even, then $\mu_{t/2} = t - 2$, so $\mu_0 \leqslant \mu_{t/2}$ if and only if $p - t + 1 \leqslant 2p^t/N$ (this includes $p \in \{2, 3\}$). ■

Now, we consider different values of $t \in \{2, \ldots, \max_{j=1,\ldots,s}\{\alpha_j\}\}$. By comparing the values $\sigma^o(N,t)$ in Theorem 2.3.1 with the strong symmetric genus $\sigma^o(\mathbb{Z}_N)$ of the cyclic group $\mathbb{Z}_N$, we obtain the following theorem, that can be seen as a generalization of Harvey's solution of the minimum genus problem for cyclic groups.

Theorem 2.3.2. *The minimum genus of a compact Riemann surface of genus greater than one that admits an abelian group of automorphisms of order N is*

$$\sigma^o(N) = \begin{cases} 2 & \text{if } N = 2,3,4,5,6 \text{ or } 8, \\ 3 & \text{if } N = 7 \text{ or } 9, \\ \dfrac{1}{2}(N-1) & \text{if } N > 9 \text{ is prime}, \\ \dfrac{1}{2}(p-1)\left(\dfrac{N}{p}-1\right) & \text{if } N > 9 \text{ is not prime and } p^2 \nmid N, \\ (p-1)\left(\dfrac{N}{2p}-1\right) & \text{otherwise}, \end{cases}$$

where p is the smallest prime divisor of N. The abelian group

i) $\mathbb{Z}_N$ if $N \leqslant 9$ or $p^2 \nmid N$,

ii) $\mathbb{Z}_p \oplus \mathbb{Z}_{N/p}$ otherwise,

attains the minimum genus.

Proof. Let $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the prime factorization of N. If $N \leqslant 9$, by Theorem 6 in [19], Theorem 2.2.1 and Remark 2.2.2, $\sigma^o(N)$ is given by the cyclic group $\mathbb{Z}_N$. For $N > 9$, we distinguish two cases: a) $\alpha_1 = 1$; b) $\alpha_1 > 1$. Let $p(t)$ be the smallest prime q such that $q^t | N$, $\tau = \max_{j=1,\dots,s}\{\alpha_j\}$ and

$$\mu_0(t) = (t-1)\left(1 - \frac{1}{p(t)}\right) - \frac{2}{N}\, p(t)^{t-1}. \tag{2.5}$$

a) $\alpha_1 = 1$. If $N = p_1 \cdots p_s$, there is only one abelian group of order N, the cyclic group $\mathbb{Z}_N$. Otherwise, and also for noncyclic groups to take place, we consider $N = p_1 p(t)^2 m(t)$ for primes $p_1 < p(t)$, and $m(t) \geqslant 1$ an integer such that $p_1 < q$ if q is a prime dividing $m(t)$ —$p(t)$ and $m(t)$ may be different for each $t \in \{2, \dots, \tau\}$.

By Theorem 6 in [19],

$$\sigma^o(\mathbb{Z}_N) = \frac{1}{2}(p_1 - 1)\left(\frac{N}{p_1} - 1\right) = \underbrace{\frac{1}{2}(p_1-1)\frac{N}{p_1}}_{A} \ \underbrace{-\frac{p_1}{2}}_{B} \ + \frac{1}{2},$$

and, by Theorem 2.3.1, either $\sigma^o(N,t) = 1 + \frac{N}{2}\mu_0(t)$ or $\sigma^o(N,t) = (t-2)N/2 + 1$. In the first case,

$$\begin{aligned} \sigma^o(N,t) &= 1 + \frac{1}{2}(t-1)(p(t)-1)\frac{N}{p(t)} - p(t)^{t-1} \\ &= \underbrace{\frac{1}{2}(p(t)-1)\frac{N}{p(t)}}_{A'} + \underbrace{\frac{1}{2}(t-2)(p(t)-1)\frac{N}{p(t)} - p(t)^{t-1}}_{C} + 1. \end{aligned}$$

We notice that $\sigma^o(\mathbb{Z}_N) < \sigma^o(N,t)$ since

$$A < A', \quad B < 0, \quad C = -p(t) \text{ if } t = 2 \quad \text{ and } C \geqslant 0 \text{ if } t > 2.$$

The last inequality holds since $p(t)^t | N$ and, therefore, $N/p(t) \geqslant p(t)^{t-1}$. In case that $\sigma^o(N,t) = (t-2)N/2+1$, then $\sigma^o(\mathbb{Z}_N) < \sigma^o(N,t)$ as well, since $(p_1-1)/p_1 < 1 < t-2$ for $t > 2$ even in this case. Hence, $\sigma^o(\mathbb{Z}_N) < \sigma^o(N,t)$ if $t \in \{3, \dots, \tau\}$.

We address the case $t = 2$ as follows: if $N = 2 \cdot 3^2$, then $\sigma^o(\mathbb{Z}_N) = \sigma^o(N,2) = 4$. Any other noncyclic abelian group with $\alpha_1 = 1$ has order $N > 18$. In this case,

(a) $p_1 > 2$, or

(b) $p_1 = 2$ and $p(2) > 3$, or

(c) $p_1 = 2$, $p(2) = 3$ and $m > 1$.

It follows that

$$(p(2) - p_1)\, N \;\geqslant\; 2p_1 p(2)^2,$$

since $p(2) - p_1 \geqslant 2$ and $N \geqslant p_1 p(2)^2$ in (a) and (b), and, in (c), $p(2) - p_1 = 1$ and $N > 2p_1 p(2)^2$. Hence,

$$(p(2) - p_1)\, N - 2p_1 p(2)^2 + p_1 p(2)(1 + p_1) \;>\; (p(2) - p_1)\, N - 2p_1 p(2)^2 \;\geqslant\; 0,$$

thus

$$p(2)N + p_1^2 p(2) \;>\; p_1 N + 2p_1 p(2)^2 - p_1 p(2).$$

Dividing by $-2p_1 p(2)$ and adding $N/2 + 1/2$ to both sides, we get $\sigma^o(\mathbb{Z}_N) < \sigma^o(N, 2)$.

Then, $\sigma^o(\mathbb{Z}_N) \leqslant \sigma^o(N, t)$ for any $t \in \{2, \dots, \tau\}$ —equality holds only when $N = 18$— and the cyclic group $\mathbb{Z}_N$ attains the minimum genus when $\alpha_1 = 1$.

b) If $\alpha_1 > 1$ then, by Theorem 6 in [19],

$$\sigma^o(\mathbb{Z}_N) = \frac{1}{2}(p_1 - 1)\frac{N}{p_1}.$$

For noncyclic groups of order N and $t \in \{2, \dots, \tau\}$ invariant factors, either

$$\sigma^o(N, t) = 1 + \frac{N}{2}\mu_0(t) = 1 + (t - 1)\left(1 - \frac{1}{p(t)}\right) - \frac{2}{N}\, p(t)^{t-1}$$

or $\sigma^o(N, t) = (t - 2)N/2 + 1$ by Theorem 2.3.1. In particular, if $t = 2$,

$$\sigma^o(N, 2) = \frac{1}{2}(p(2) - 1)\frac{N}{p(2)} - (p(2) - 1).$$

Since $1 < p_1 \leqslant p(2)$, it follows that $\sigma^o(N, 2) < \sigma^o(\mathbb{Z}_N)$. Also $\sigma^o(N, 2) < \sigma^o(N, t)$ if $\sigma^o(N, t) = (t - 2)N/2 + 1$, since $(p(2) - 1)/p(2) < 1 < t - 2$ for $t > 2$ even.

Now it remains to check that $\sigma^o(N, 2) < \sigma^o(N, t)$ for all $t \in \{3, \dots, \tau\}$ whenever $\sigma^o(N, t) = 1 + \frac{N}{2}\mu_0(t)$. For, we first change slightly the notation and define

$$\mu_0(p, t) = (t - 1)\left(1 - \frac{1}{p}\right) - \frac{2}{N}\, p^{t-1},$$

If we prove that

i) $\mu_0(p, t-1) < \mu_0(p,t)$ for all $t \in \{3, \dots, \alpha\}$ if $N = p^\alpha m$ with p prime and integers $\alpha \geqslant 3$ and $m \geqslant 1$ such that $p \nmid m$, and

ii) $\mu_0(p,t) < \mu_0(q,t)$ for an integer $t \geqslant 2$ and $p < q$ primes such that $p^t | N$ and $q^t | N$,

then it follows that $\mu_0(p_1, 2)$ minimizes $\mu_0(p,t)$ for $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ with primes $p_j < p_{j+1}$, $p \in \{p_1, \dots, p_s\}$ and $t \in \{2, \dots, \tau\}$, thus $\sigma^o(N, 2) \leqslant \sigma^o(N,t)$ for all $t \in \{2, \dots, \tau\}$.

i) We notice that

$$\mu_0(p,t) - \mu_0(p,t-1) = 1 - \frac{p^{\alpha-1}m - 2(p^{t-1} - p^{t-2})}{N} > 0$$

for all $t \in \{3, \dots, \alpha\}$, since $p^{\alpha-1}m - 2(p^{t-1} - p^{t-2}) < N$ if $t \geqslant 3$, $p > 1$, $\alpha \geqslant 3$ and $m \geqslant 1$.

ii) Since $p < q$, we can write $N = p^{\alpha_1} q^{\alpha_2} m$, where $\alpha_1 > 1$, $\alpha_2 > 1$ and $m \geqslant 1$ is an integer such that $p \nmid m$ and $q \nmid m$. If $t \in \{2, \dots, \min\{\alpha_1, \alpha_2\}\}$, then

$$\begin{aligned}
\mu_0(q,t) - \mu_0(p,t) &= (t-1)\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{2q^{t-1} - 2p^{t-1}}{N} \\
&= \frac{1}{pqm}\left((t-1)(q-p)m - A\right) > 0,
\end{aligned}$$

with

$$A = \frac{2}{p^{\alpha_1 - 1} q^{\alpha_2 - t}} - \frac{2}{p^{\alpha_1 - t} q^{\alpha_2 - 1}},$$

since $t \geqslant 2$, $q > p \geqslant 2$ and $m \geqslant 1$; therefore $(t-1)(q-p)m \geqslant 1$ and $0 < A < 1$. Then $\mu_0(p,t) < \mu_0(q,t)$ whenever $p < q$ and $t \in \{2, \dots, \min\{\alpha_1, \alpha_2\}\}$. ■

2.4 Maximum order problem

The group $\mathbb{Z}_2 \oplus \mathbb{Z}_{2g+2}$ has order $4g+4$ and acts as a group of automorphisms of a compact Riemann surface of genus g: for, consider the triangle group with signature $(0;\ 2, 2g+2, 2g+2)$ in Theorem 2.1.2 —see also [3, Example 9.9]. In fact, this is the maximum order for a finite abelian group acting on genus g, as Breuer [3, Corollary 9.6] proved from Maclachlan's result [24, Theorem 4] —see Theorem 2.2.1— for the minimum genus. This result follows easily from Theorem 2.3.2 as well.

Corollary 2.4.1. *The maximum order for an abelian group of automorphisms of a compact Riemann surface of genus $g > 1$ is $4g + 4$.*

Proof. Let A be any abelian group of automorphisms of a compact Riemann surface of genus g of order $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, with integers $s \geqslant 1$, $\alpha_j > 0$ and primes $p_j < p_{j+1}$. The genus g must be greater than or equal to the minimum genus $\sigma^o(N)$ provided by Theorem 2.3.2. Let $g^* = \sigma^o(N)$. It follows that $N \leqslant 4g + 4$ is trivially satisfied when $N \leqslant 9$: $N < 12 = 4g^* + 4 \leqslant 4g + 4$ if $N \in \{2, 3, 4, 5, 6, 8\}$, and $N < 16 = 4g^* + 4$ if $N \in \{7, 9\}$.

Now let $N > 9$. Then $N = 2g^* + 1 \leqslant 2g + 1 < 4g + 4$ if N is prime. If $N > 9$ is not prime and $\alpha_1 = 1$, let $N = p_1 q$ for an integer $q > 1$, $p_1 \nmid q$ and $p_1 < p$ for every prime $p|q$. Then $4g^* + 4 = N(2(p_1 - 1)/p_1 - 2/q) + 6$, so $4g^* + 4 = N + 2$ if $p_1 = 2$, $4g^* + 4 = N + q$ if $p_1 = 3$. Since $2(p_1 - 1)/p_1$ grows and $2/q$ decreases with increasing p_1, and $q \geqslant 7$ if $p_1 = 5$, then $4g^* + 4 \geqslant \frac{46}{35}N + 4$ if $p_1 \geqslant 5$. Hence, $4g + 4 \geqslant 4g^* + 4 > N$ provided that $N > 9$ is not prime and $\alpha_1 = 1$.

Finally, we prove that $4g^* + 4 \geqslant N$ if $N > 9$ and $\alpha_1 > 1$. Indeed, let us write $N = p_1^2 q$ for an integer $q \geqslant 3$ if $p_1 \in \{2, 3\}$ and $q = 1$ or $q \geqslant p_1$ otherwise. Then $4g^* + 4 = N(2(p_1 - 1)/p_1 - 4/p_1 q) + 8$, so $4g^* + 4 = N$ if $p_1 = 2$, and it is also straightforward to check that $4g^* + 4 \geqslant N$ for the cases $\{p_1 = 3, q = 3\}$ and $\{p_1 = 5, q = 1\}$. We have $4g^* + 4 \geqslant N$ also for greater values of p_1 and q, since $2(p_1 - 1)/p_1$ grows and $4/p_1 q$ decreases with increasing values of p_1 and q. ∎

3 Abelian actions on bordered Klein surfaces

In this chapter, we state necessary and sufficient conditions for a finite abelian group to act as a group of automorphisms of some compact bordered Klein surface of algebraic genus $p > 1$. This result provides a new method to obtain the real genus and to solve the maximum order problem of abelian groups. We also compute the least real genus of abelian groups of the same order.

3.1 Bordered surface-kernel epimorphisms

In this section, we establish necessary and sufficient conditions for a given finite abelian group to act as a group of automorphisms of some compact bordered Klein surface, i.e., we find conditions on the signature of a proper NEC group Λ so that an abelian group A is isomorphic to Λ/Γ for some bordered surface group Γ. In such a case, A is a group of automorphisms of the bordered Klein surface $\mathcal{H}/\Gamma$.

The signature of Λ must contain some period cycle —otherwise Λ would not have any normal bordered surface subgroup. Moreover, the following lemma shows how the period cycles of Λ look like. It was stated for nonorientable Riemann surfaces in [18, Corollary 2.3]. The proof included here for completeness is much the same though slight changes are needed so as to apply to bordered Klein surfaces.

Lemma 3.1.1. *Let A be a finite abelian group and Λ an NEC group. If there exists a surface-kernel epimorphism $\Lambda \to A$, then every link period equals 2 and no period cycle has only a single link period. If the order of A is odd, then every period cycle is empty.*

Proof. Let $\theta : \Lambda \to A$ be a surface-kernel epimorphism. As A is abelian, we have

$\theta((c_{ij-1}c_{ij})^2) = \theta(c_{ij-1}^2)\theta(c_{ij}^2) = 1$. Hence $n_{ij} = 2$; otherwise n_{ij} would be even and greater than 2 and thus $\ker\theta$ would contain the orientable element $(c_{ij-1}c_{ij})^2$ of finite order $n_{ij}/2$, so that $\ker\theta$ would not be a surface group.

Assume that $(n_{i1}) = (2)$ is a period cycle. Then $(c_{i0}c_{i1})^2 = 1$ and $\theta(c_{i0}c_{i1}) = \theta(e_i^{-1}c_{i0}e_ic_{i1}) = 1$. Therefore $c_{i0}c_{i1}$ would belong to $\ker\theta$ and would be an orientable element of finite order.

Finally, assume that the order of A is odd and the signature of Λ contains a nonempty period-cycle $(n_{i1},\dots) = (2,\dots)$. The order of $\theta(c_{ij})$ divides both $|A|$ and 2 (since $c_{ij}^2 = 1$), hence $\theta(c_{ij}) = 1$ for all j and thus $c_{ij-1}c_{ij} \in \ker\theta$ would be an orientable element of finite order 2. ■

Theorem 3.1.2. *Let Λ be an NEC group with signature $(g;\, \pm\,;\, [m_1,\dots,m_r];\, \{(-)^\varepsilon, (2,\overset{s_{\varepsilon+1}}{\dots},2),\dots,(2,\overset{s_k}{\dots},2)\})$, $k > 0$, $\varepsilon \geqslant 0$, $s_i \neq 1$, and a nontrivial abelian group $A \approx \mathbb{Z}_2^n \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$, where $t \geqslant 0$, $v_i > 2$, $v_i|v_{i+1}$, $v_1,\dots,v_{t-m}$ are odd and $v_{t-m+1},\dots,v_t$ are multiple of 4 for some integer $m \leqslant t$. Let also $w = \eta g + k - 1$, $S = \varepsilon + s_{\varepsilon+1} + \cdots + s_k$, $\eta = 2$ if '+' is the signature sign of Λ and $\eta = 1$ otherwise. Then, there exists a bordered surface-kernel epimorphism $\Lambda \to A$ if and only if*

(*i*) *$m_i = 2$ if $t = 0$, $m_i|v_t$ if $n = 0$ and $m_i \,|\, \operatorname{lcm}(2, v_t)$ otherwise for all i,*

(*ii*) *if $t > w$ and $i \in \{1,\dots,t-w\}$, then every elementary divisor of $\mathbb{Z}_{v_i}$ divides, at least, $t - w + 1 - i$ proper periods,*

(*iii*) *if $m + n > w + S - 1$, then at least $m + n - w - S + 1$ proper periods are even,*

(*iv*) *if $m + n = 0$, then $k = \varepsilon$,*

(*v*) *if $m + n = 1$, then s_i is even for all i.*

Proof. Let $\theta : \Lambda \to A$ be a bordered surface-kernel epimorphism.

(*i*) The order of $\theta(x_i)$ is m_i for all i (see Lemma 1.4.5). The order of every element of A divides

$$\exp A = \begin{cases} 2 & \text{if } t = 0,\\ v_t & \text{if } n = 0,\\ \operatorname{lcm}(2, v_t) & \text{otherwise.}\end{cases}$$

(*ii*) Suppose that $t > w$ and let q be a prime dividing v_t, $\widehat{\mu}_i = \widehat{\mu}_i(q)$ as defined in Section 1.3 and $\alpha_1, \dots, \alpha_t$ be integers such that $q^{\alpha_i} | v_i$ and $q^{\alpha_i+1} \nmid v_i$. By (1.7), the w integers $\alpha_{t-w+1}, \dots, \alpha_t$ may take any value, but the following $t-w$ inequalities must hold:

$$\alpha_1 \leqslant \widehat{\mu}_{r-t+w+1}, \quad \dots, \quad \alpha_{t-w-1} \leqslant \widehat{\mu}_{r-1}, \quad \alpha_{t-w} \leqslant \widehat{\mu}_r$$

(note that $\alpha_i \neq 1$ if $q = 2$ since either v_i is odd or $4 | v_i$). It follows that q^{α_1} divides, at least, $t-w$ proper periods, q^{α_2} divides, at least, $t-w-1$ proper periods, and so on —recall that $\widehat{\mu}_i \leqslant \widehat{\mu}_{i+1}$.

(*iii*) As $\theta : \Lambda \to A$ is bordered surface-kernel, then there is, at least, one reflection in Λ that belongs to $\ker\theta$. This reflection is conjugate in Λ to some canonical reflection, say c_{kl}, and thus $c_{kl} \in \ker\theta$ as well. Let $N = \langle c_{kl} \rangle^{\Lambda}$ be the normal subgroup generated by c_{kl}. A presentation of Λ/N is that of Λ with the additional relation $c_{kl} = 1$ and thus a presentation of $(\Lambda/N)_{ab}$ has generators $x_i, e_i, c_{ij}, a_i, b_i$ or x_i, e_i, c_{ij}, d_i and relations

$$x_i^{m_i} = 1, \; x_1 \cdots x_r e_1 \cdots e_k = 1, \; c_{ij}^2 = 1, \; c_{kl} = 1$$

$$\text{or} \quad x_i^{m_i} = 1, \; x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1, \; c_{ij}^2 = 1, \; c_{kl} = 1.$$

Computing the Smith normal form of its relation matrix gives $(\Lambda/N)_{ab} \approx \Lambda_{ab}/\mathbb{Z}_2$. This is the same abelianization as for Λ but

$$Syl_2(\mathcal{T}((\Lambda/N)_{ab})) \approx \mathbb{Z}_2^{S-1} \oplus \mathbb{Z}_{2^{\widehat{\mu}_1(2)}} \oplus \cdots \oplus \mathbb{Z}_{2^{\widehat{\mu}_r(2)}}.$$

Now, by the universal property of the quotient group, there exists a unique homomorphism $\phi : \Lambda/N \to A$ such that $\phi \circ \pi = \theta$, where $\pi : \Lambda \to \Lambda/N$ is the canonical epimorphism. Since θ and π are epimorphisms, ϕ is onto as well, so we can apply Breuer's conditions (1.7) —replacing Λ_{ab} by $(\Lambda/N)_{ab}$— to the epimorphism $\overline{\phi} : (\Lambda/N)_{ab} \to A$, where $\overline{\phi} \circ \pi' = \phi$ and $\pi' : \Lambda/N \to (\Lambda/N)_{ab}$ is the canonical epimorphism; in particular, for $q = 2$ and $i = 1$,

$$\eta g + k - 1 + N_2(1) \geqslant n_2(1) = m + n.$$

If we let r_2 be the number of even proper periods, then the number of nontrivial cyclic factors of $Syl_2(\mathcal{T}((\Lambda/N)_{ab}))$ is $N_2(1) = r_2 + S - 1$ and it follows that $\eta g + k - 1 + r_2 + S - 1 \geqslant m + n$, hence $r_2 \geqslant m + n - w - S + 1$.

(iv) The order of A is odd ($m+n=0$), so the claim follows from Lemma 3.1.1.

(v) Let $Syl_2(A) \approx \mathbb{Z}_{2^\alpha}$, $(n_{i1},\dots,n_{is_i}) = (2,2,\dots,2)$. Since c_{ij} has order two then $\theta(c_{ij})$ must belong to $\mathbb{Z}_{2^\alpha}$ and has order 1 or 2, so either $\theta(c_{ij}) = 0$ or $2^{\alpha-1}$ for all j. Also, $\theta(c_{i0}) = \theta(c_{is_i})$ by the relation $e_i^{-1}c_{i0}e_ic_{is_i} = 1$ and $\theta(c_{ij-1}) \neq \theta(c_{ij})$ for $j = 1,\dots,s_i$ (otherwise $\theta(c_{ij-1}c_{ij}) = 1$ and $c_{ij-1}c_{ij}$ would be an orientable element of order 2 in $\ker\theta$). This is possible only if s_i is even.

We prove the sufficiency of the conditions by defining epimorphisms $\theta_q : \Lambda \to A_q$ for each prime q in the set $\{q_1,\dots,q_\lambda\}$ of prime numbers dividing the order of A, and a surface-kernel epimorphism $\theta : \Lambda \to A$ as the direct product epimorphism

$$\theta : \Lambda \to A : g \mapsto \theta(g) = (\theta_{q_1}(g),\dots,\theta_{q_\lambda}(g)).$$

For readability, we let $\mu_i = \mu_i(q)$ —see Section 1.3— in the definition of each homomorphism θ_q. Also, we assume that $\mu_i \leqslant \mu_{i+1}$; otherwise, there is a permutation —in general, different for each value of q— such that $\widehat{\mu}_i = \mu_{\tau(i)}$, $\widehat{\mu}_i \leqslant \widehat{\mu}_{i+1}$ and we replace x_i by $x_{\tau(i)}$ and μ_i by $\widehat{\mu}_i$ in the definition of $\theta_q(x_i)$ below —so that the order of $\theta(x_i)$ is m_i.

Let $A_2 \approx \mathbb{Z}_{2^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{\alpha_{m+n}}}$ be the 2-Sylow subgroup of A ($\alpha_i = 1$ if $i \leqslant n$ and $\alpha_i > 1$ if $i > n$). If $m+n = 1$, we define $\theta(c_{10}) = 0$ if $k = \varepsilon$, $\theta(c_{10}) = 2^{\alpha_1-1}$ if $k > \varepsilon$, $\theta(c_{i0}) = 2^{\alpha_1-1}$ for $i = 2,\dots,\varepsilon$ and, for a nonempty period cycle, we assign 0 and 2^{α_1-1} alternatively. If $m+n > 1$, we consider the sequence

$$c_{20},\dots,c_{\varepsilon 0},c_{\varepsilon+1,0},\dots,c_{\varepsilon+1,s_{\varepsilon+1}-1},\dots,c_{k0},\dots,c_{k,s_k-1} \tag{3.1}$$

containing $S-1$ elements (we rule out the elements c_{10} and c_{is_i} for $i > \varepsilon$). We let $\theta_2(c_{10}) = (0,\dots,0)$ and assign $(2^{\alpha_1-1},0,\dots,0)$ to the first element in that sequence, $(0,2^{\alpha_2-1},0,\dots,0)$ to the second element, and so on until we assign $(0,\dots,0,2^{\alpha_{m+n}-1})$ to the $(m+n)$th element; then we assign again $(2^{\alpha_1-1},0,\dots,0)$ to the $(m+n+1)$th element, etc.

Finally, we define

$$\theta_2(c_{is_i}) = \theta_2(c_{i0}) \quad \text{for } i > \varepsilon.$$

It follows that θ_2 preserves the relations $c_{ij}^2 = 1$ and $e_i^{-1}c_{i0}e_ic_{is_i} = 1$, $\theta_2(c_{ij-1}c_{ij})$ has order 2 and $\theta_2(c_{ij})$ is trivial or has order 2.

We note that $c_{10} \in \ker\theta_2$ and that the images of the first $\min(n, S-1)$ elements in the sequence (3.1) generate the subgroup $\mathbb{Z}_2 \oplus \overset{\min(n,S-1)}{\cdots} \oplus \mathbb{Z}_2$ of A_2, since $\theta_2(c_{20}) = (1,0,\dots,0)$, $\theta_2(c_{30}) = (0,1,0,\dots,0)$, etc.

Let

$$\gamma_1 = e_1,\ \ldots,\ \gamma_{k-1} = e_{k-1},\ \gamma_k = a_1,\ \gamma_{k+1} = b_1,\ \ldots,\ \gamma_{w-1} = a_g,\ \gamma_w = b_g,$$

or

$$\gamma_1 = e_1, \ldots, \gamma_{k-1} = e_{k-1}, \gamma_k = d_1, \ldots, \gamma_w = d_g,$$

according to the sign of the signature of Λ, and

$$\delta = \begin{cases} -1 & \text{if } g = 0, \\ 0 & \text{if } g > 0 \text{ and sign}(\Lambda) \text{ is `}+\text{'}, \\ -2 & \text{if } g > 0 \text{ and sign}(\Lambda) \text{ is `}-\text{'}, \end{cases}$$

We define θ_2 on the canonical generators x_i, e_i, a_i, b_i or d_i as follows: if $S-1 \geqslant n$ and $w \geqslant m$,

$$\begin{aligned}
&\theta_2(x_i) = (0, \ldots, 0, 2^{\alpha_{m+n}-\mu_i}), \quad i = 1, \ldots, r, \\
&\theta_2(\gamma_i) = (0, \ldots, 0), \quad i = 1, \ldots, w-m, \\
&\theta_2(\gamma_i) = (0, \overset{m+n-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{w-i}{\ldots\ldots}, 0), \quad i = w-m+1, \ldots, w,
\end{aligned}$$

if $S-1 \geqslant n$ and $w < m$,

$$\begin{aligned}
\theta_2(x_i) &= (0, \ldots, 0, 2^{\alpha_{m+n}-\mu_i}), \quad i = 1, \ldots, r-m+w, \\
\theta_2(x_i) &= (0, \overset{m+n-r-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{r+w-i-1}{\ldots\ldots}, 0, 2^{\alpha_{m+n}-\mu_i}), \\
&\qquad\qquad i = r-m+w+1, \ldots, r, \\
\theta_2(\gamma_i) &= (0, \overset{m+n-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{w-i}{\ldots\ldots}, 0), \quad i = 1, \ldots, w,
\end{aligned}$$

if $S-1 < n$ and $m+n \leqslant w+S-1$ (hence $w \geqslant m$),

$$\begin{aligned}
&\theta_2(x_i) = (0, \ldots, 0, 2^{\alpha_{m+n}-\mu_i}), \quad i = 1, \ldots, r, \\
&\theta_2(\gamma_i) = (0, \ldots, 0), \quad i = 1, \ldots, w-m-n+S-1, \\
&\theta_2(\gamma_i) = (0, \overset{m+n-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{w-i}{\ldots\ldots}, 0), \quad i = w-m-n+S, \ldots, w,
\end{aligned}$$

if $S-1 < n$ and $m+n > w+S-1$,

$$\begin{aligned}
\theta_2(x_i) &= (0, \ldots, 0, 2^{\alpha_{m+n}-\mu_i}), \quad i = 1, \ldots, r-m-n+w+S-1, \\
\theta_2(x_i) &= (0, \overset{m+n-r-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{r+w-i-1}{\ldots\ldots}, 0, 2^{\alpha_{m+n}-\mu_i}), \\
&\qquad\qquad i = r-m-n+w+S, \ldots, r, \\
\theta_2(\gamma_i) &= (0, \overset{m+n-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{w-i}{\ldots\ldots}, 0), \quad i = 1, \ldots, w,
\end{aligned}$$

and

$$\theta_2(e_k) = \begin{cases} (0, \overset{\min(n,S-1)}{\ldots\ldots\ldots}, 0, -1, \ldots, -1, \delta, \overset{\eta g-1}{\ldots\ldots}, \delta, -u+\delta) & \\ & \text{if } m > \eta g > 0, \\ (0, \overset{\min(n,S-1)}{\ldots\ldots\ldots}, 0, \delta, \ldots, \delta, -u+\delta) & \text{if } m \leqslant \eta g \text{ or } g = 0, \end{cases}$$

where $u = \sum_{i=1}^{r} 2^{\alpha_{m+n}-\mu_i}$.

Now, let $q \neq 2$ be a prime number dividing $|A|$ and $A_q \approx \mathbb{Z}_{q^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{q^{\alpha_t}}$ be the q-Sylow subgroup of A —note that some factors of A_q may be trivial, i.e., $\alpha_1 = \cdots = \alpha_{t'} = 0$ for some $t' < t$. We define θ_q as follows —note that $r + w \geqslant t$ by condition (ii):

$$\theta_q(c_{i0}) = (0, \ldots, 0), \qquad i = 1, \ldots, k,$$

$$\theta_q(x_i) = (0, \ldots, 0, q^{\alpha_t-\mu_i}), \qquad i = \begin{cases} 1, \ldots, r-t+w & \text{if } t > w, \\ 1, \ldots, r & \text{if } t \leqslant w, \end{cases}$$

$$\theta_q(x_i) = (0, \overset{t-r-w+i-1}{\ldots\ldots\ldots}, 0, 1, 0, \overset{r+w-i-1}{\ldots\ldots\ldots}, 0, q^{\alpha_t-\mu_i}),$$
$$i = r-t+w+1, \ldots, r \quad \text{if } t > w,$$

$$\theta_q(\gamma_i) = (0, \ldots, 0), \qquad i = 1, \ldots, w-t \quad \text{if } t < w,$$

$$\theta_q(\gamma_i) = (0, \overset{t-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{w-i}{\ldots\ldots}, 0), \quad i = \begin{cases} 1, \ldots, w & \text{if } t \geqslant w, \\ w-t+1, \ldots, w & \text{if } t < w, \end{cases}$$

$$\theta_q(e_k) = \begin{cases} (-1, \ldots, -1, \delta, \overset{\eta g-1}{\ldots\ldots}, \delta, -u+\delta) & \text{if } t > \eta g > 0, \\ (\delta, \ldots, \delta, -u+\delta) & \text{if } t \leqslant \eta g \text{ or } g = 0, \end{cases}$$

where $u = \sum_{i=1}^{r} q^{\alpha_t-\mu_i}$. The long relation is clearly preserved by θ.

No element of finite order other than some reflections belongs to $\ker\theta$ since the elements $\theta(c_{ij-1}c_{ij})$ have order two and the order of $\theta(x_i)$ is m_i. Indeed, by condition (i), any prime number dividing m_i also divides $|A|$, hence $m_i = q_1^{\mu_i(q_1)} \cdots q_\lambda^{\mu_i(q_\lambda)}$ (recall that $\mu_i(q) = 0$ for a prime q not dividing m_i). Also, it follows that $|\theta_q(x_i)| = q^{\mu_i(q)}$ for all i (taking into account condition (ii) for $i = r-t+w+1, \ldots, r$ if $t > w$ and condition (iii) for $i = r-m-n+w+S, \ldots, r$ if $q = 2$ and $m + n > w + S - 1$). As $\theta_q(x_i)$ and $\theta_{q'}(x_i)$ belong to different primary components of A if $q \neq q'$, the order of $\theta(x_i) = (\theta_{q_1}(x_i), \ldots, \theta_{q_\lambda}(x_i))$ is $\operatorname{lcm}(q_1^{\mu_i(q_1)}, \ldots, q_\lambda^{\mu_i(q_\lambda)}) = q_1^{\mu_i(q_1)} \cdots q_\lambda^{\mu_i(q_\lambda)} = m_i$.

Therefore, θ is surface-kernel. We also notice that the surface group $\ker\theta$ is bordered since it contains, at least, one reflection (for instance c_{10}, or $c_{\varepsilon+1,0}$ if $m+n=1$ and $k>\varepsilon$).

Finally, θ_q is onto since, by conditions (*ii*) and (*iii*), A_q is generated by the images of the canonical generators x_i, e_i, c_{ij} and a_i, b_i or d_i. Therefore, θ is onto as well. For, consider an elementary divisor $q^{\alpha_i(q)}$ of A and the generator $h = (0,\ldots,0,1,0,\ldots,0)$ of some cyclic factor

$$H = \{0\} \oplus \cdots \oplus \{0\} \oplus \mathbb{Z}_{q^{\alpha_i(q)}} \oplus \{0\} \oplus \cdots \oplus \{0\}$$

of A_q. Then, $h = \theta_q(g)$ for some $g \in \Lambda$. Obviously, $\theta(g)$ may have nontrivial components in some other primary component $A_{q'}$ for a prime $q' \neq q$, but not the element $\frac{v_t}{q^{\alpha_t(q)}}\,\theta(g)$ since $\frac{v_t}{q^{\alpha_t(q)}}\,\theta_{q'}(g)$ is trivial whenever $q' \neq q$. Moreover, the element $\frac{v_t}{q^{\alpha_t(q)}}\,\theta(g)$ has order $q^{\alpha_i(q)}$ since $\gcd(q, v_t/q^{\alpha_t(q)}) = 1$. Hence, $\langle\theta(g^{v_t/q^{\alpha_t(q)}})\rangle = H$. ■

Remark 3.1.3. For the surface-kernel epimorphism θ defined in the proof of Theorem 3.1.2, the number k' of boundary components of the bordered Klein surface $\mathcal{H}/\ker\theta$ can be computed by means of [11, §2.3], namely, $k' = 2^n v_1 \cdots v_t/|\theta(e_1)|$ if $m+n \neq 1$ or $k=\varepsilon$, where $|\theta(e_1)| = 1, 2, v_{t-w+1}$ or $2v_{t-w+1}$ depending on the parameters t, m, n, w and S; if $m+n=1$ and $k>\varepsilon$, then $k' = 2^{n-1}v_1\cdots v_t(k-\varepsilon)$. Another epimorphism θ may provide a Klein surface $\mathcal{H}/\ker\theta$ of different topological type, and so with a different number of boundary components. ■

As a consequence of the Riemann-Hurwitz formula (1.2) and Theorem 3.1.2, we can find out whether an abelian group acts on genus $p>1$.

Corollary 3.1.4. *Let $A \approx \mathbb{Z}_2^n \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ be an abelian group, where $t \geqslant 0$, $v_i > 2$, $v_i|v_{i+1}$, $v_1,\ldots,v_{t-m}$ are odd and $v_{t-m+1},\ldots,v_t$ are multiple of 4 for some nonnegative integer $m \leqslant t$, and let $p>1$ be an integer. Then, A is a group of automorphisms of some compact bordered Klein surface of algebraic genus p if and only if there exist integers $\eta = 1$ or 2, $g, k, \epsilon, s_{\epsilon+1},\ldots,s_k, m_1,\ldots,m_r$ and $w = \eta g + k - 1$ such that*

- (*i*) *$m_i = 2$ if $t=0$, $m_i|v_t$ if $n=0$ and $m_i \,|\, \mathrm{lcm}(2,v_t)$ otherwise for all i,*
- (*ii*) *if $t>w$ and $i \in \{1,\ldots,t-w\}$, then every elementary divisor of $\mathbb{Z}_{v_i}$ divides, at least, $t-w+1-i$ proper periods,*
- (*iii*) *if $m+n > w+S-1$, then at least $m+n-w-S+1$ proper periods are even,*

(*iv*) *if* $m+n=0$, *then* $k=\varepsilon$,
(*v*) *if* $m+n=1$, *then* s_i *is even for all* i,
(*vi*) *and*

$$\frac{p-1}{2^n v_1\cdots v_t} = \eta g + k - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) + \frac{s_{\epsilon+1}+\cdots+s_k}{4}.$$

3.2 Real genus of an abelian group

The real genus of cyclic groups was first obtained by Bujalance, Etayo, Gamboa and Martens in [12]. Shortly after, McCullough stated the corresponding result for noncyclic abelian groups in [29, theorems 2.7 and 3.2]; it was given an explicit expression in [6, Theorem 4.1]. By means of rather different methods (that of the combinatorial theory of NEC groups used herein), both results also follow from Theorem 3.1.2, as we discuss hereunder.

Later, further results on the real genus of cyclic groups were established (for certain group orders) concerning cyclic groups acting on bordered Klein surfaces of fixed orientability and number of boundary components; see [11, Chapter 3] and [13, 15, 16]. However, those results are beyond the scope of Theorem 3.1.2, since we cannot fix the orientability and number of boundary components of the quotient $\mathcal{H}/\ker\theta$ in this theorem.

Theorem 3.2.1. [12] *The real genus of the cyclic group of order* N *is*

$$\rho(\mathbb{Z}_N) = \begin{cases} 2 & \text{if } N=2, \\ (q-1)\,(N/q-1) & \text{if } q^2\nmid N \text{ and } N/q>1, \\ (q-1)\,N/q & \text{otherwise,} \end{cases} \tag{3.2}$$

where q *is the smallest prime divisor of* N.

Proof. We obtain the expressions in (3.2) from NEC groups with signatures $(0;+;[2,2,2];\{(-)\})$, $(0;+;[q,N/q];\{(-)\})$ and $(0;+;[q,N];\{(-)\})$, respectively, by means of the Riemann-Hurwitz formula. These NEC groups fulfill conditions of Theorem 3.1.2. As we now prove, it follows from Theorem 3.1.2 that $1+N\mu(\Lambda)\geqslant\rho(\mathbb{Z}_N)$ for any other proper NEC group Λ with signature $(g;\pm;[m_1,\ldots,m_r];\{(-)^{k_o},(2,\overset{s_{k_o+1}}{\ldots},2),\ldots,(2,\overset{s_k}{\ldots},2)\})$, $k>0$, fulfilling conditions of Theorem 3.1.2 (hence $\sum_i s_i=0$ if N is odd and it is even if N is even).

Let $N = q^\alpha u$, with q not dividing u. We have to prove that

$$\mu^* \leqslant \mu(\Lambda) = w - 1 + \textstyle\sum_i (1 - 1/m_i) + \frac{s/2}{2},$$

where $w = \eta g + k - 1$, $s = \sum_i s_i$ and

$$\mu^* = \frac{\rho(\mathbb{Z}_N) - 1}{N} = \begin{cases} 1 - 1/q - 1/u & \text{if } \alpha = 1 \text{ and } u > 1, \\ 1 - 1/q - 1/N & \text{otherwise.} \end{cases}$$

This is obvious if $w > 1$. If $w = 1$, then $r \geqslant 1$ or $s > 0$ (otherwise, $\mu(\Lambda)$ would not be greater than 0); if $r \geqslant 1$, since $q \leqslant m_i$ by condition *(i)* of Theorem 3.1.2, it follows that $\mu^* < 1 - 1/q \leqslant 1 - 1/m_1 \leqslant \mu(\Lambda)$; if $s > 0$ then $q = 2$ and $\mu^* < 1/2 \leqslant \mu(\Lambda)$.

If $w = 0$, then $r \geqslant 1$ by condition *(i)* of Theorem 3.1.2. Also, $r + s/2 \geqslant 2$ since $\mu(\Lambda) > 0$. Clearly, $\mu^* < \mu(\Lambda)$ if $r + s/2 \geqslant 4$.

In case that $r + s/2 = 3$ we have $\mu(\Lambda) \geqslant -1 + 3/2 = 1/2$. If $q = 2$, then $\mu^* < 1/2 < \mu(\Lambda)$. If $q > 2$, then $s = 0$, $r = 3$ and $q \leqslant m_i$, hence $\mu^* < 1 - 1/q \leqslant 1 - 1/m_1 \leqslant \mu(\Lambda)$.

Otherwise, $r + s/2 = 2$. If $w = 0$, $r = 1$ and $s/2 = 1$, then $q = 2$ and $\mu(\Lambda) = 1/2 - 1/m_1$. Also, $\mu^* = 1/2 - 1/u$ if $\alpha = 1$ and $u > 1$ and $\mu^* = 1/2 - 1/N$ otherwise. But $\mu(\Lambda) = \mu^*$ in both cases: as a result of condition *(ii)* of Theorem 3.1.2, $m_1 = u$ if $\alpha = 1$ and $u > 1$ (since $v_1 = u$), and $m_1 = N$ otherwise ($v_1 = N$).

Finally, if $w = 0$, $r = 2$ and $s = 0$, we consider two cases:

i) $\gcd(m_1, m_2) > 1$. Then $m_1 m_2 = hN$, by conditions *(i)*, *(ii)* and *(iii)* of Theorem 3.1.2, for some integer h such that $h|N$ and $q \leqslant h \leqslant m_i \leqslant N$. Therefore, $1/m_1 + 1/m_2 \leqslant 1/h + 1/N$ and thus

$$\mu^* \leqslant 1 - \frac{1}{q} - \frac{1}{N} \leqslant 1 - \frac{1}{h} - \frac{1}{N} \leqslant 1 - \frac{1}{m_1} - \frac{1}{m_2} = \mu(\Lambda).$$

ii) $\gcd(m_1, m_2) = 1$. Then $m_1 m_2 = N$—hence N is not prime—by conditions *(i)*, *(ii)* and *(iii)* of Theorem 3.1.2 and $\mu(\Lambda) - \mu^* = f(m_1)$, where

$$f(x) = \frac{1}{q} + \frac{\epsilon}{N} - \frac{1}{x} - \frac{x}{N}, \qquad \epsilon = \begin{cases} q & \text{if } \alpha = 1 \text{ and } n > 1, \\ 1 & \text{otherwise.} \end{cases}$$

It suffices to show that $f(m_1) \geqslant 0$ for every admissible value of m_1. We first note that

$$f(q) = f\left(\tfrac{N}{q}\right) = \frac{\epsilon - q}{N}, \qquad f(q+1) = f\left(\tfrac{N}{q+1}\right) = \frac{1}{q(q+1)} + \frac{\epsilon - q - 1}{N}$$

and f is strictly convex upwards in $(0, +\infty)$ since $f''(x) = -2/x^3$. Since $m_1 m_2 = N$ and $\gcd(m_1, m_2) = 1$, the admissible values of m_1 and m_2 are in the interval $[q, \frac{N}{q}]$ if $\alpha = 1$ and in the interval $(q+1, \frac{N}{q+1})$ if $\alpha > 1$. In the first case, $u > 1$ (since N is not prime), $\epsilon = q$ and $f(q) = f\left(\frac{N}{q}\right) = 0$; in the second case, $f(q+1) = f\left(\frac{N}{q+1}\right) = \frac{1}{q(q+1)} - \frac{1}{q^{\alpha-1}u} > 0$ since $q \leqslant q^{\alpha-1}$ and $q < u$. In both cases, $f(m_1) \geqslant 0$ since is strictly convex upwards in the interval of admissible values of m_1. ■

Remark 3.2.2. The real genus of the groups $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_{2u}$ $(u > 1)$ was obtained in [29, Theorem 3.2]:

$$\rho(\mathbb{Z}_2^2) = 2, \quad \rho(\mathbb{Z}_2^3) = 3, \quad \rho(\mathbb{Z}_2 \oplus \mathbb{Z}_{2u}) = 2u - 1, \tag{3.3}$$

Theorem 3.1.2 allows us to obtain signatures of NEC groups attaining such algebraic genera: $(0; +; [-]; \{(2,2,2)\})$, $(0; +; [-]; \{(2,2,2,2)\})$ and $(0; +; [2, 2u]; \{(-)\})$, respectively, fulfill conditions of Theorem 3.1.2 and it can be proved that any other signature fulfilling such conditions leads to a greater or equal algebraic genus. ■

Theorem 3.2.3. [29, Theorem 3.2][6, Theorem 4.1] *The real genus of a noncyclic abelian group A different to $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_{2u}$ ($u \geqslant 2$) is $\rho(A) = 1 + |A|\,\mu^*$, where μ^* is, with the notation of Theorem* 3.1.2*,*

$$\begin{aligned}
&\text{a) } t - 1 - \frac{1}{v_1} - \cdots - \frac{1}{v_{t-n}} && \textit{if } n < m,\\
&\text{b) } t - 1 - \frac{1}{v_1} - \cdots - \frac{1}{v_{t-\epsilon}} + \frac{\delta}{2v_{t-\epsilon}} && \textit{if } 2m + \delta \leqslant m + n \leqslant 2t - \delta,\\
&\text{c) } t - 1 + \frac{m + n - 2t + 1}{4} && \textit{if } 2t < m + n,
\end{aligned} \tag{3.4}$$

$\epsilon = (m + n - \delta)/2$, $\delta = 1$ if $m + n$ is odd and $\delta = 0$ otherwise.

Proof. The real genus is attained by an NEC group Λ^* with signature

a) $(0; +; [v_1, \ldots, v_{t-n}]; \{(-)^{n+1}\})$

b) $(0; +; [v_1, \ldots, v_{t-\frac{m+n-\delta}{2}-1}, (\delta+1)v_{t-\frac{m+n-\delta}{2}}]; \{(-)^{\frac{m+n-\delta}{2}+1}\})$

c) $(0; +; [-]; \{(-)^t, (2, \overset{m+n-2t+1}{\ldots}, 2)\})$

respectively. This NEC group fulfills conditions of Theorem 3.1.2 and $\mu(\Lambda^*) = \mu^*$. Now, we prove that it follows from Theorem 3.1.2 that $\mu^* \leqslant \mu(\Lambda)$ for any other NEC group Λ fulfilling such conditions.

We can assume that Λ has signature

$$(0;\, +\, ;\, [m_1, \dots, m_r];\, \{(-)^{k-1}, (2, \overset{s}{\dots}, 2)\}), \tag{3.5}$$

where $m_i | m_{i+1}$, $s \neq 1$, s is even if $m+n=1$ and $s=0$ if $m+n=0$. The reasons for this are the following.

The signature

$$(g;\, \pm\, ;\, [\widehat{m}_{r-\widehat{r}+1}, \dots, \widehat{m}_r];\, \{(-)^{\varepsilon}, (2, \overset{s_{\varepsilon+1}}{\dots}, 2), \dots, (2, \overset{s_k}{\dots}, 2)\})$$

defines an NEC group $\widehat{\Lambda}$ that fulfills conditions of Theorem 3.1.2 and, by (1.3), $\mu(\widehat{\Lambda}) \leqslant \mu(\Lambda)$ if Λ is an NEC group with signature

$$(g;\, \pm\, ;\, [m_1, \dots, m_r];\, \{(-)^{\varepsilon}, (2, \overset{s_{\varepsilon+1}}{\dots}, 2), \dots, (2, \overset{s_k}{\dots}, 2)\})$$

also fulfilling such conditions. Therefore, we can assume that $m_1 | \cdots | m_r$.

In order to prove that $\mu(\widehat{\Lambda}) > 0$, we first note that $\widehat{\Lambda} \approx \Lambda$ if $r \leqslant 1$. So, in case that $k > 0$, this signature does not define an NEC group —i.e., $\mu(\widehat{\Lambda}) \leqslant 0 < \mu(\Lambda)$— if and only if $r > 1$, $g = 0$, $k = 1$, $s_1 = 0$ and $\widehat{m}_{r-1} = 1$ (note that, if $r = 2$, then $m_1 = m_2 = 2$ if and only if $\widehat{m}_1 = \widehat{m}_2 = 2$, so that also $\widehat{\Lambda} \approx \Lambda$ in this case).

Now, let $w = \eta g + k - 1$. Obviously, $\mu(\widehat{\Lambda}) > 0$ if $w > 0$. By condition *(ii)* of Theorem 3.1.2, $t > 1$ and $w = 0$ means that $\widehat{m}_{r-1} > 1$; if $t = 1$, then $m + n \geqslant 2$ since A is noncyclic, hence, if also $w = 0$, either $s_1 > 0$ or, by condition *(iii)*, there are two or more even proper periods and thus $\widehat{m}_{r-1} > 1$; finally, if $t = 0$, then $\widehat{m}_i = m_i = 2$ (and thus $\mu(\widehat{\Lambda}) = \mu(\Lambda)$). Therefore, $\mu(\widehat{\Lambda}) > 0$.

Finally, consider an NEC group Λ_o with signature

$$(0;\, +\, ;\, [m_1, \dots, m_r];\, \{(-)^{\eta g + k - 1}, (2, \overset{s}{\dots}, 2)\}),$$

where $s = \sum_{i=1}^{k} s_i$ and η, g and k are the parameters of Λ. It is straightforward to check that $\mu(\Lambda_o) = \mu(\Lambda)$ and Λ_o fulfills conditions of Theorem 3.1.2 if Λ does.

So assume that Λ has signature (3.5) and let

$$\mu = \mu(\Lambda) = w - 1 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{s}{4}$$

and $w^* = n$, $(m + n - \delta)/2$ or t for case a), b) and c), respectively, of (3.4).

If $t \leqslant w$, then $\mu^* < t-1 \leqslant w-1 \leqslant \mu$ for cases a) and b). For case c), if $m+n-2w-s+1 \leqslant 0$, then

$$\mu \geqslant w-1+\frac{s}{4} \geqslant w-1+\frac{m+n-2w+1}{4} = \mu^* + \frac{w-t}{2} \geqslant \mu^*,$$

and, if $m+n-2w-s+1 > 0$, then $\sum(1-1/m_i) \geqslant (m+n-2w-s+1)/2$ by condition *(iii)* and

$$\mu \geqslant w-1+\frac{m+n-2w-s+1}{2}+\frac{s}{4} = \mu^* + \frac{m+n-2t-s+1}{4} \geqslant \mu^*$$

since $m+n-2t-s+1 \geqslant m+n-2w-s+1 > 0$.

Otherwise, $t > w$. If $t > w > w^*$ —hence we factor out case c)—, then

$$\mu^* \; < \; w^*-1+\sum_{i=1}^{t-w}\left(1-\frac{1}{v_i}\right)+\sum_{i=t-w+1}^{t-w^*} 1 \; = \; w-1+\sum_{i=1}^{t-w}\left(1-\frac{1}{v_i}\right) \; \leqslant \; \mu$$

since, by condition *(ii)*, $v_1|m_{r-t+w+1}, \ldots, v_{t-w}|m_r$.

If $t > w = w^*$ —we also factor out case c)—, then, in case a) and case b) with $m+n$ even ($\delta = 0$),

$$\mu^* \; = \; w-1+\sum_{i=1}^{t-w}\left(1-\frac{1}{v_i}\right) \; \leqslant \; \mu$$

by condition *(ii)* as above. In case b) with $m+n$ odd ($\delta = 1$),

$$1-\frac{1}{2v_{t-w}} = 1-\frac{1}{v_{t-w}}+\frac{1}{2v_{t-w}} \leqslant 1-\frac{1}{m_r}+\frac{s}{4}$$

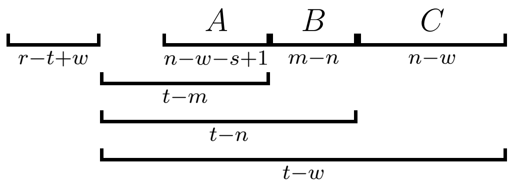
provided that $s \geqslant 2$, and, if $s = 0$, then there is, at least, $m+n-2w = m+n-(m+n-1) = 1$ even proper period by condition *(iii)* (note that $S-1 = w$ if $s = 0$) and thus $2v_{t-w}|m_r$ since v_{t-w} is odd (note that $t-w \leqslant t-m$ since, in case b), $w^* \geqslant m$). Therefore, $\mu^* \leqslant \mu$ either if $s > 0$ or $s = 0$.

Finally, if $t > w$ and $w^* > w$, we deal with cases a), b) and c) separately. For readability, we rename the first $t-w$ integers v_i by defining v_i' as follows:

$$\begin{array}{ccccccc} 1 & \cdots & 1 & v_1 & \cdots & v_{t-w} \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ v_1' & \cdots & v_{r-t+w}' & v_{r-t+w+1}' & \cdots & v_r' \end{array}$$

Hence, $v_i'|m_i$ for all i by condition *(ii)*, and thus $1-\frac{1}{v_i'} \leqslant 1-\frac{1}{m_i}$.

a) $t \geqslant m > n = w^* > w$. We consider the following partition of $\{1, \ldots, r\}$:



($\#A < t - m$ in the figure, but $\#A$ may be greater than $t - m$). Let $A = \varnothing$ if $n - w - s + 1 \leqslant 0$. In case that $s = 0$, let $\#A = n - w$.

Note that $4|v_i'$ if $i \in B \cup C$ and v_i' is odd otherwise, and $\mu^* = n - 1 + \sum_{i \notin C}(1 - \frac{1}{v_i'})$.

If $w + s > n$ (hence $s \geqslant 2$ since $w < n$), then

$$\sum_C \left(1 - \frac{1}{m_i}\right) + \frac{s}{4} > \frac{3(n-w)}{4} + \frac{n-w}{4} = n - w$$

since $4|m_i$ for $i \in C$, and thus $\mu > w - 1 + \sum_{i \notin C}(1 - \frac{1}{m_i}) + n - w \geqslant \mu$.

If $w + s - 1 < n$ and $s \geqslant 2$, then $m + n - 2w - s + 1 > 0$ since $m > n > w$, and, by condition *(iii)*, m_i is even if $i \in \{A \cup B \cup C\}$ and thus $r \geqslant m + n - 2w - s + 1 = \#\{A \cup B \cup C\}$. Let $C = C_1 \cup C_2$, with $C_1 = \{r - n + w + 1, \ldots, r - s + 1\}$ and $C_2 = \{r - s + 2, \ldots, r\}$, $\#C_1 = \#A$, $\#C_2 = s - 1$. For $i \in A$, v_i' is odd and m_i is even, hence $2v_i'|m_i$. Also, $4|m_j$ for $j \in C_1$, hence $4v_i'|m_j$ if $i \in A$. Then

$$\sum_A \left(1 - \frac{1}{m_i}\right) + \sum_{C_1} \left(1 - \frac{1}{m_i}\right) \geqslant \sum_A \left(1 - \frac{1}{2v_i'}\right) + \sum_A \left(1 - \frac{1}{4v_i'}\right)$$

$$= n - w - s + 1 + \sum_A \left(1 - \frac{1}{2v_i'} - \frac{1}{4v_i'}\right) > n - w - s + 1 + \sum_A \left(1 - \frac{1}{v_i'}\right)$$

As $4|m_i$ for $i \in C_2$ and $\#C_2 = s - 1$,

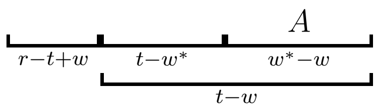
$$\sum_{C_2} \left(1 - \frac{1}{m_i}\right) + \frac{s}{4} \geqslant (s-1)\left(1 - \frac{1}{4}\right) + \frac{s}{4} = s - 1 + \frac{1}{4} \geqslant s - 1.$$

If $w + s - 1 < n$ and $s = 0$, then $C_2 = \varnothing$, $\#A = \#C = n - w$ and

$$\sum_{A \cup C} \left(1 - \frac{1}{m_i}\right) > n - w + \sum_A \left(1 - \frac{1}{v_i'}\right).$$

It follows that $\mu^* \leqslant \mu$ if either $s \geqslant 2$ or $s = 0$.

b) $t-\delta \geqslant \frac{m+n-\delta}{2} = w^* > w$. We partition $\{1,\dots,r\}$ as follows:



Note that $1-\frac{1}{(\delta+1)v_{t-w^*}} = 1-\frac{1}{v_{t-w^*}}+\frac{\delta}{2v_{t-w^*}}$ and thus $\mu^* = w^*-1+\sum_{i\notin A}(1-\frac{1}{v_i'})+\frac{\delta}{2v_{t-w^*}}$.

b.1) $s \geqslant 2(w^*-w)$. Hence $\sum_A(1-\frac{1}{m_i})+\frac{s}{4} \geqslant \frac{2}{3}(w^*-w)+\frac{1}{2}(w^*-w) = w^*-w+\frac{w^*-w}{6} \geqslant w^*-w+\frac{1}{6} \geqslant w^*-w+\frac{\delta}{2v_{t-w^*}}$ since $3 \leqslant v_1|m_i$ for $i\in A$ and $\#A = w^*-w$. Therefore, $\mu \geqslant w-1+\sum_{i\notin A}(1-\frac{1}{m_i})+w^*-w+\frac{\delta}{2v_{t-w^*}} \geqslant \mu$.

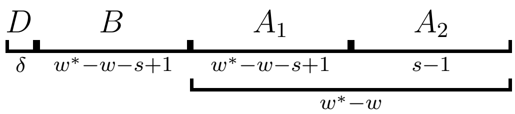
b.2) $w^*-w+1 < s \leqslant 2(w^*-w)$. If v_1 is even, then $4 \leqslant v_1|m_i$ for $i\in A$ and $t=m$, hence $m+n$ is even since $2m < m+n < 2t$ if $m+n$ is odd, and thus $w^*=n=m=t$ and $\mu^*=w^*-1$. Therefore, $\mu \geqslant w-1+\sum_A(1-\frac{1}{m_i})+\frac{s}{4} > w-1+\frac{3}{4}(w^*-w)+\frac{1}{4}(w^*-w) = w^*-1 = \mu^*$. Otherwise, $v_1 \geqslant 3$ is odd. By condition *(iii)*, there are, at least, $2w^*-2w-s+1>0$ even proper periods, hence $m_i \geqslant 6$ for these proper periods since also $v_1|m_i$ and $v_1 \geqslant 3$ is odd (note that $t-w > 2(w^*-w)-s+1$ since $t \geqslant w^*$ and $w^*-w-s+1<0$). Therefore,

$$\sum_A\left(1-\frac{1}{m_i}\right)+\frac{s}{4} > \frac{2}{3}(w+s-1-w^*)+\frac{5}{6}(2w^*-2w-s+1)+\frac{s}{4}$$

$$= w^*-w+\frac{s+2}{12} \geqslant w^*-w+\frac{1}{6} \geqslant w^*-w+\frac{\delta}{2v_{t-w^*}}$$

and $\mu > \mu^*$.

b.3) $s \leqslant w^*-w+1$. The number of even proper periods is, at least, $2w^*-2w-s+1+\delta \geqslant 0$ if $s \geqslant 2$ and $2w^*-2w+\delta > 0$ if $s=0$; we partition $\{r-2w^*+2w+s-\delta,\dots,r\}$ (but $\{r-2w^*+2w+1-\delta,\dots,r\}$ if $s=0$) as follows:



($\#A_1 = \#B = w^*-w$ and $A_2=\varnothing$ if $s=0$). As $w^* \geqslant m$, we have $w^*-w \geqslant m-w$ and thus v_i' is odd and $2v_i'|m_i$ for $i\in B$ or D. Therefore,

$$\sum_{D\cup B}\left(1-\frac{1}{m_i}\right)+\sum_{A_1}\left(1-\frac{1}{m_i}\right)$$

$$\geqslant \sum_{D\cup B}\left(1-\frac{1}{2v_i'}\right)+\sum_{B}\left(1-\frac{1}{2v_i'}\right)$$

$$= \sum_{D\cup B}\left(1-\frac{1}{v_i'}+\frac{1}{2v_i'}\right)+\sum_{B}\left(1-\frac{1}{2v_i'}\right)$$
$$= \sum_{D\cup B}\left(1-\frac{1}{v_i'}\right)+\sum_{D}\frac{1}{2v_i'}+\sum_{B}1$$
$$\geqslant \sum_{D\cup B}\left(1-\frac{1}{v_i'}\right)+\frac{\delta}{2v_{t-w^*}}+w^*-w-s+1$$

since $\#B = \#A_1$, $m_i \leqslant m_j$ if $i \in B$ and $j \in A_1$, and $v_i' \leqslant v'_{r-w^*+w} = v_{t-w^*}$ if $i \in D$. Also, if $s \geqslant 2$, then $m_i \geqslant 4$ for $i \in A_2$ since $2v_1|m_i$ (m_i is even, and v_1 is odd since $t > m$; recall that v_1 divides, at least, $t-w \geqslant w^*-w \geqslant s-1$ proper periods) and thus

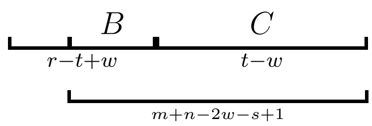
$$\sum_{A_2}\left(1-\frac{1}{m_i}\right)+\frac{s}{4} \geqslant \frac{3}{4}(s-1)+\frac{s}{4} = s-\frac{3}{4} > s-1$$

Therefore

$$\sum_{D\cup B\cup A_1\cup A_2}\left(1-\frac{1}{m_i}\right)+\frac{s}{4} > w^*-w+\sum_{D\cup B}\left(1-\frac{1}{v_i'}\right)+\frac{\delta}{2v_{t-w^*}}$$

and thus $\mu^* > \mu$.

c) $t = w^* > w$, $\mu^* = \frac{m+n+2t-3}{4}$. If $t-w < m+n-2w-s+1$ (but $t-w < m+n-2w$ when $s=0$), we partition $\{1,\ldots,r\}$ as follows:



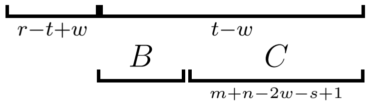
(with $\#\{B\cup C\} = m+n-2w$ when $s=0$). Then, $m_i \geqslant 4$ if $i \in C$ since m_i is even and $v_1 \geqslant 3$ divides m_i. Therefore,

$$\mu \geqslant w-1+\sum_{B\cup C}\left(1-\frac{1}{m_i}\right)+\frac{s}{4} \;\geqslant\; w-1+\frac{1}{2}(\#B)+\frac{3}{4}(\#C)+\frac{s}{4}$$
$$= w-1+\frac{1}{2}(m+n-t-w-s+1)+\frac{3}{4}(t-w)+\frac{s}{4}$$
$$= \mu^*+\frac{m+n-2w-s+1-(t-w)}{4} > \mu^*$$

if $s \geqslant 2$, and, if $s = 0$,

$$\mu \geqslant w-1+\frac{1}{2}(m+n-t-w)+\frac{3}{4}(t-w) = \mu^*+\frac{m+n-2w-1-(t-w)}{4} \geqslant \mu^*.$$

If $t-w \geqslant m+n-2w-s+1$ (hence $s \geqslant m+n-t-w+1 > 0$ since $2t < m+n$ and $t > w$), we partition $\{1,\dots,r\}$ as follows:



Then, $v_1 \geqslant 3$ divides m_i, hence $m_i \geqslant 3$, if $i \in B$ or C; in addition, m_i is even, hence $m_i \geqslant 4$, if $i \in C$. Therefore,

$$\mu \geqslant w-1+\sum_{B\cup C}\left(1-\frac{1}{m_i}\right)+\frac{s}{4} \geqslant w-1+\frac{2}{3}(\#B)+\frac{3}{4}(\#C)+\frac{s}{4}$$

$$=t-1+\frac{m+n-4t+2w+1+2s}{12}$$

and thus $\mu \geqslant \mu^*$ since $s \geqslant m+n-t-w+1$. ∎

The following examples highlight some cases that arise in the straightforward, routine proof of Theorem 3.2.3 by means of Theorem 3.1.2; as we noted, we can focus on NEC groups with signature $(0;+;[m_1,\dots,m_r];\{(-)^{k-1},(2,\overset{s}{\dots},2)\})$ and $m_i|m_{i+1}$.

Example 3.2.4. Let $q>2$ be a prime number.

a) Let $A \approx \mathbb{Z}_2\oplus\mathbb{Z}_q\oplus\mathbb{Z}_{4q}\oplus\mathbb{Z}_{4q}$. By (3.4.a), $\rho(A)=64q^3-40q^2+1$, attained by $(0;+;[q,4q];\{(-),(-)\})$. If Λ is an NEC group with signature $(0;+;[m_1,\dots,m_r];\{(-)\})$ fulfilling conditions of Theorem 3.1.2, then $r \geqslant 3$ and, at least, three proper periods are multiple of q, two proper periods are multiple of $4q$ and three proper periods are even. For the signature $(0;+;[2q,4q,4q];\{(-)\})$, we obtain $1+|A|\,\mu(\Lambda)=64q^3-32q^2+1>\rho(A)$.

b) Let $A \approx \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_q\oplus\mathbb{Z}_{4q}$. By (3.4.b), $\rho(A)=16q^2-8q+1$, attained by $(0;+;[2q];\{(-),(-)\})$. If we consider the signature $(0;+;[m_1,\dots,m_r];\{(-)\})$, then, by Theorem 3.1.2, $r \geqslant 3$ and, at least, two proper periods are multiple of q, one proper period is multiple of $4q$ and three proper periods are even. For instance, we obtain $1+|A|\,\mu(\Lambda)=24q^2-12q+1>\rho(A)$ for the signature $(0;+;[2,2q,4q];\{(-)\})$.

c) Let $A \approx \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_{4q}$. By (3.4.c), $\rho(A)=192q+1$, attained by $(0;+;[-];\{(-),(-),(2,2)\})$. For the signature $(0;+;[m_1,\dots,m_r];\{(-),(-)\})$, Theorem 3.1.2 gives $r \geqslant 3$ and, at least, one proper period is multiple of 4 and three proper periods are even. We obtain $1 + |A|\,\mu(\Lambda) = 224q + 1 > \rho(A)$ for the signature $(0; +; [2, 2, 4]; \{(-), (-)\})$. ■

3.3 Least real genus of abelian groups of the same order

We may take advantage of the results of the previous section: for a given integer $N > 1$, we find the least algebraic genus of any bordered Klein surface of algebraic genus $p > 1$ on which some abelian group of order N acts. For ease and by abuse of notation, we denote it by $\rho(N)$ (it is not the real genus of a group but the least real genus attained in a family of groups).

Theorem 3.3.1. *The least real genus of abelian groups of order $N > 1$ is*

$$\rho(N) = \begin{cases} 2 & \textit{if } N \leqslant 4, \\ 5 & \textit{if } N = 16, \\ N - 1 & \textit{if } N > 4 \textit{ is prime, and} \\ (q-1)\,(N/q - 1) & \textit{otherwise,} \end{cases}$$

where q is the smallest prime divisor of N.

Proof. In case that N is prime the result follows from (3.2), and the case $N = 16$ follows from computing the real genus of the five abelian groups of order 16 by means of (3.2), (3.3) and (3.4) —namely, $\rho(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) = 5$.

Now, assume that N is a composite odd number, say $N = q^\alpha u$, $q \nmid u$. For any noncyclic abelian group $A \approx \mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_2}$ of order N,

$$\begin{aligned} \rho(\mathbb{Z}_N) &\leqslant \rho(A) && \text{if } \alpha = 1 \text{ and} \\ \rho(\mathbb{Z}_q \oplus \mathbb{Z}_{N/q}) < \rho(\mathbb{Z}_N) &< \rho(A) && \text{if } \alpha > 1 \text{ and } A \not\approx \mathbb{Z}_q \oplus \mathbb{Z}_{N/q}. \end{aligned}$$

Therefore, $\rho(N) = (q-1)(N/q-1)$. For, we notice that

$$\frac{\rho(A) - 1}{N} = \frac{\rho(\mathbb{Z}_N) - 1}{N} + f(v_1),$$

where

$$f(x) = \frac{1}{q} + \frac{\epsilon}{N} - \frac{1}{x} - \frac{x}{N}, \qquad \epsilon = \begin{cases} q & \text{if } \alpha = 1 \text{ and } u > 1, \\ 1 & \text{otherwise.} \end{cases}$$

As $v_1v_2 = N$ and v_1 divides v_2, the admissible values of v_1 are in the interval $[q, \sqrt{N}]$. The function f is increasing in this interval since $f'(x) = 1/x^2 - 1/N$. Also, $v_1 \geqslant q$ and $f(q) = 0$ if $\alpha = 1$ (i.e., $\epsilon = q$), hence $f(v_1) \geqslant 0$ and $\rho(\mathbb{Z}_N) \leqslant \rho(A)$. If $\alpha > 1$ (i.e., $\epsilon = 1$), then $f(q) < 0$ and $f(q+1) > 0$ since $q \leqslant q^{\alpha-1}$ and $q < u$ (note that $v_1 = q$ if $\alpha = 2$ and $u = 1$, i.e., if $N = q^2$); hence $\rho(\mathbb{Z}_q \oplus \mathbb{Z}_{N/q}) < \rho(\mathbb{Z}_N) < \rho(A)$ for $A \not\approx \mathbb{Z}_q \oplus \mathbb{Z}_{N/q}$ in this case. Also, $\rho(\mathbb{Z}_N) < \rho(A)$ for any noncyclic abelian group $A \approx \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ of order N and $t > 2$. Indeed,

$$\frac{\rho(A) - 1}{N} = \frac{\rho(\mathbb{Z}_N) - 1}{N} + \frac{1}{q} + \frac{\epsilon}{N} - \frac{1}{v_1} - \frac{1}{v_2} + \sum_{i=3}^{t} \left(1 - \frac{1}{v_i}\right).$$

Therefore,

$$\frac{\rho(A) - 1}{N} > \frac{\rho(\mathbb{Z}_N) - 1}{N} + \frac{1}{q} + \frac{\epsilon}{N} - \frac{1}{v_1} - \frac{1}{v_2} + \frac{t-2}{2}$$

since $1 - 1/v_i > 1/2$. Hence $\rho(A) > \rho(\mathbb{Z}_N)$ if $t > 2$ since $1/v_i \leqslant 1/q$ and $(t-2)/2 > 1/v_i$.

Finally, we address even values of N different to 2 and 16. For $N = 4$, then $\rho(\mathbb{Z}_2^2) = \rho(\mathbb{Z}_4) = 2$ by (3.3) and (3.2). For $N = 8$, then $\rho(\mathbb{Z}_2^3) = \rho(\mathbb{Z}_2 \oplus \mathbb{Z}_4) = N/2 - 1 < \rho(\mathbb{Z}_8) = 4$ by (3.3) and (3.2). Otherwise, if $4 \nmid N$, then $\rho(\mathbb{Z}_N) = N/2 - 1$ and, if $4|N$, then $\rho(\mathbb{Z}_2 \oplus \mathbb{Z}_{N/2}) = N/2 - 1 < \rho(\mathbb{Z}_N) = N/2$ by (3.3) and (3.2), hence $\rho(N)$ is at most $N/2 - 1$ for such values of N. We now prove that $\mu^* \geqslant 1/2$, hence $\rho(A) = 1 + N\mu^* > N/2 - 1$, for any other noncyclic abelian group A of order N by examining the three cases in (3.4):

a) $n < m$. In this case, $\mu^* = n - 1 + \sum_{i=1}^{t-n}(1 - 1/v_i)$. As $1 - 1/v_i > 1/2$ and $n < t$, if follows that $\mu^* > 1/2$ (note that A is cyclic if $n = 0$ and $t = 1$).

b) $2m + \delta \leqslant m + n \leqslant 2t - \delta$. In this case,

$$\mu^* = \frac{m + n - \delta}{2} - 1 + \sum_{i=1}^{t - \frac{m+n-\delta}{2}} (1 - 1/v_i) + \delta/2v_{t - \frac{m+n-\delta}{2}}.$$

Clearly, $\mu^* \geqslant 1/2$ in the following cases: $m + n - \delta > 2$; $m + n - \delta = 2$ and $t > 1$; or $m + n - \delta = 0$ and $t > 2$ (note that m and $n - \delta$ have the same parity due to the definition of δ, so $m + n - \delta \neq 1$). Otherwise, either $\mu^* \geqslant 1/2$ or A is not noncyclic different to $\mathbb{Z}_2 \oplus \mathbb{Z}_{N/2}$, depending on the values of $m, n - \delta$ and t (note that $m \leqslant t$ and $m \leqslant n - \delta$):

- if $m = 1$, $n - \delta = 1$ and $t = 1$, then $n = 1$ or $n = 2$, but $n = 2$ is not possible since $m + n \leqslant 2t - \delta$, hence $n = 1$ and thus $A \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{N/2}$;

- if $m = 0$, $n - \delta = 2$ and $t = 0$, then $N = 4$ and $A \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{N/2}$;
- if $m = 0$, $n - \delta = 2$ and $t = 1$, then $N = 4u$ (u odd) and $A \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{N/2}$ ($n \neq 3$ since $m + n \leqslant 2t - \delta$);
- if $m = 0$, $n - \delta = 0$ and $t = 0$, then A is the trivial group;
- if $m = 0$, $n - \delta = 0$ and $t = 1$, then A is cyclic;
- if $m = 0$, $n - \delta = 0$ and $t = 2$, then $n = 1$ (N is odd if $n = 0$) and thus $\mu^* = 1 - 1/v_1 - 1/2v_2 \geqslant 1 - 1/3 - 1/6 = 1/2$ since v_1 and v_2 are odd (m=0) and $\delta = 1$.
- if $m = 1$, $n - \delta = -1$ and $t = 1$, then $n = 0$ and A is cyclic.
- if $m = 1$, $n - \delta = -1$ and $t = 2$, then $n = 0$ and $\mu^* = 1 - 1/v_1 - 1/v_2 \geqslant 1 - 1/3 - 1/12 > 1/2$ since v_1 is odd, $v_1|v_2$ and $4|v_2$.

c) $2t < m + n$. We have $\mu^* = t - 1 + (m + n - 2t + 1)/4$. Therefore, $\mu^* \geqslant 1/2$ if $t \geqslant 1$ since $2t < m + n$. If $t = 0$ (hence $m = 0$) and $n \geqslant 5$, then $\mu^* \geqslant 1/2$ as well. Otherwise, the abelian group ($\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$) has been addressed previously. ■

Remark 3.3.2. The abelian group of order N acting on genus $\rho(N)$ is unique ($\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $N = 16$ and either $\mathbb{Z}_N$ or $\mathbb{Z}_q \oplus \mathbb{Z}_{N/q}$ otherwise) unless $N = 4$ or 8. ■

3.4 Maximum order problem

The maximum order problem for abelian groups was solved in [3, Corollary 9.6] in case of Riemann surfaces of genus $g > 1$, and in [11, §4.5] for compact bordered Klein surfaces of algebraic genus $p > 1$. As we now prove, this last result follows easily from theorems 3.1.2, 3.3.1 and 3.2.3.

Corollary 3.4.1. *The largest order of an abelian group acting on a compact bordered Klein surface of algebraic genus $p \geqslant 2$ is 16 if $p = 5$ and $2p + 2$ otherwise.*

Proof. If $p = 5$, then the largest order is 16 since, by Theorem 3.3.1, $\rho(16) = 5$ and $\rho(N) \geqslant N/2 - 1 > 5$ for $N > 16$.

Otherwise, we notice that the abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_{p+1}$ acts on genus p since an NEC group with signature $(0; +; [2, p+1]; \{(-)\})$ fulfills conditions of Theorem 3.1.2, so the largest order is at least $2(p+1)$.

Consider an abelian group A of order N that acts on genus p, so $p \geqslant \rho(N)$. If $N \neq 16$, then $\rho(N) \geqslant N/2 - 1$ by Theorem 3.3.1 —note that $(q-1)(N/q-1) = N/2 - 1 + (q-2)(N-2q)/2q$— and thus $N \leqslant 2(p+1)$.

Now, suppose that $N = 16$, hence $p \geqslant \rho(16) = 5$. If $p \geqslant 7$, then $2(p+1) \geqslant 16$. Finally, $p \neq 6$ since $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not act on genus $p = 6$ (by Theorem 3.1.2 and the Riemann-Hurwitz formula) and $\rho(A) \geqslant 7$ for any other abelian group A of order 16 (by Remark 3.2.2 and Theorem 3.2.3). ■

Remark 3.4.2. There are only finitely many abelian groups acting on some compact bordered Klein surface of a given algebraic genus $p > 1$. For any abelian group A acting on such a surface $\mathcal{H}/\Gamma$, there is also a finite number of signatures of NEC groups satisfying conditions of Theorem 3.1.2 and the Riemann-Hurwitz formula.

Therefore, given an integer $p \geqslant 2$, Theorem 3.1.2 makes it possible to define an algorithm for computing the set of all abelian groups acting on some compact bordered Klein surface of algebraic genus p: for each abelian group A of order $|A| \leqslant 2p + 2$ (or $|A| \leqslant 16$ if $p = 5$), we can check whether there exists some NEC group Λ fulfilling conditions of Theorem 3.1.2 and such that $\mu(\Lambda) = (p-1)/|A|$.

For cyclic actions, Bujalance, Costa, Gamboa and Lafuente presented in [10] an effective algorithm to obtain the order and ramification indices of finite cyclic groups acting on some compact Klein surface of fixed topological type. Its design is based on known upper bounds to the order of the cyclic group and some results on cyclic actions, in the vein of Theorem 3.1.2, stated in [11, 17, 19]. ■

4 Abelian actions on nonorientable Riemann surfaces

With regard to nonorientable compact Riemann surfaces, we study in this chapter how to characterize the action of abelian groups on topological genus greater than two. The case of odd order abelian groups is easily addressed with techniques used above. However, even order abelian actions turn out to be more involved and we have settled only certain cases.

4.1 General results

In [4], Bujalance established conditions for the existence of surface-kernel epimorphisms onto cyclic groups, on the basis of the following result stated by Singerman [34, Theorem 1].

Theorem 4.1.1. *A finite group G is a group of automorphisms of a nonorientable unbordered Klein surface of topological genus $g > 2$ if and only if there exists a proper NEC group Λ and a homomorphism $\theta : \Lambda \to G$ such that $\ker\theta$ is a nonorientable surface group without period cycles and $\theta(\Lambda^+) = G$.*

This prompts to say that a homomorphism θ of a proper NEC group Λ into a finite group G is *nonorientable unbordered surface-kernel* if $\ker\theta$ is a nonorientable surface group without period cycles and $\theta(\Lambda^+) = G$. The condition on $\ker\theta$ can be translated into conditions on the orders of the image of elliptic generators x_i and reflections c_{ij}.

Theorem 4.1.2. [4, Prop. 3.2] *An epimorphism $\theta : \Lambda \to G$ of a proper NEC group with signature $(g; \pm; [m_1, \ldots, m_r]; \{(n_{i1}, \ldots, n_{is_i}), i = 1, \ldots, k\})$ onto a finite group G is nonorientable unbordered surface-kernel if and only if $\theta(x_i)$ has order m_i, $\theta(c_{ij})$ has order 2, $\theta(c_{ij-1} \cdot c_{ij})$ has order n_{ij} and $\theta(\Lambda^+) = G$.*

In order to apply theorems 4.1.1 and 4.1.2 in what follows, we observe that the existence of an element of Λ that belongs to both $\ker\theta$ and $\Lambda - \Lambda^+$ allows us to claim that $\theta(\Lambda^+) = G$ if θ is onto. Indeed, since Λ^+ is a subgroup of index two in Λ, if there exists an element $u \in \ker\theta \cap (\Lambda - \Lambda^+)$, then any $h \in \Lambda - \Lambda^+$ can be expressed as $h = h'u$ for some $h' \in \Lambda^+$, so that $\theta(h) = \theta(h')\theta(u) = \theta(h')$ and thus $\theta(\Lambda^+) = \theta(\Lambda)$. This fact will be used without further mention in the proofs below.

As a result of Theorem 4.1.2, the signature of the NEC group fulfills the following conditions when we consider abelian groups.

Lemma 4.1.3. [4, Corollary 3.3][18, Corollary 2.3] *Let A be a finite abelian group, Λ a proper NEC group and $\Lambda \to A$ a nonorientable unbordered surface-kernel epimorphism. Then every link period equals 2 and no period cycle has only a single link period. If the order of A is odd, then the signature of Λ has no period cycle.*

4.2 Odd order abelian actions

When $A \approx \Lambda/\Gamma$ is an odd order abelian group of automorphisms of the unbordered compact Klein surface $\mathcal{H}/\Gamma$, the signature of Λ has no period cycles by Lemma 4.1.3 and the signatures of Λ and Γ have the same sign. When this sign is '+', Λ is a Fuchsian group and the conditions for the existence of a surface-kernel epimorphism $\Lambda \to A$ were stated in [3] as we have seen in Chapter 2.

Now, we obtain the corresponding result for actions on nonorientable Riemann surfaces. The conditions of existence of a surface-kernel epimorphism are easily obtained from a basic property of groups and the factorization through the abelianization of the NEC group.

Theorem 4.2.1. *Let Λ be a proper NEC group with signature $(g; -; [m_1, \ldots, m_r]; \{-\})$ and $v_1, \ldots, v_t$ be odd integers such that $t \geqslant 1$, $v_i > 1$ and v_i divides v_{i+1}. Then, there exists a nonorientable unbordered surface-kernel epimorphism $\Lambda \to \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ if and only if*

(i) $m_i \mid v_t$ *for all i and,*

(ii) *if $t > g-1$, then every elementary divisor of $\mathbb{Z}_{v_i}$ divides, at least, $t-g-i+2$ proper periods for $i = 1, \ldots, t-g+1$.*

Proof. Let $\theta : \Lambda \to A \approx \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ be a nonorientable surface-kernel epimorphism. The order of $\theta(x_i)$ divides $\exp A = v_t$. By Theorem 4.1.2, $|\theta(x_i)| = m_i$ and thus m_i divides v_t.

Now, assume that $t > g-1$ and let q be any prime number dividing v_t. Let $\pi : \Lambda \to \Lambda_{ab}$ be the canonical projection. The epimorphism θ factors through Λ_{ab}, so there is a (unique) homomorphism $\overline{\theta} : \Lambda_{ab} \to A$ such that $\theta = \overline{\theta} \circ \pi$. Since θ is onto, $\overline{\theta}$ is also onto and thus (1.7) holds: $g - 1 + N_q(i) \geqslant n_q(i)$ for integers $i > 0$. These inequalities impose some restrictions on the values of $\alpha_1(q), \ldots, \alpha_{t-g+1}(q)$. If we suppose that $\alpha_1(q) > \widehat{\mu}_{r-t+g}(q)$, then, at most, $t-g$ of the integers $\widehat{\mu}_i(q)$ are greater than or equal to $\alpha_1(q)$ —recall that $\widehat{\mu}_i(q) \leqslant \widehat{\mu}_{i+1}(q)$— and thus $N_q(\alpha_1(q)) \leqslant t-g$. Also, $n_q(\alpha_1(q)) = t$ since $\alpha_1(q) > 0$, hence $g - 1 + N_q(\alpha_1(q)) \leqslant t - 1 < n_q(\alpha_1(q))$, which is not consistent with (1.7). Therefore, $\alpha_1(q) \leqslant \widehat{\mu}_{r-t+g}(q)$. Likewise we obtain

$$\alpha_2(q) \leqslant \widehat{\mu}_{r-t+g+1}(q), \quad \ldots, \quad \alpha_{t-g+1}(q) \leqslant \widehat{\mu}_r(q).$$

It follows from these $t-g+1$ inequalities that $q^{\alpha_1(q)}$ divides, at least, $t-g+1$ proper periods, $q^{\alpha_2(q)}$ divides, at least, $t-g$ proper periods, and so on. This proves condition *(ii)*.

Now, assuming that conditions *(i)* and *(ii)* hold, we build a nonorientable unbordered surface-kernel epimorphism as the direct product epimorphism $\theta : \Lambda \to A : g \mapsto \theta(g) = (\theta_{q_1}(g), \ldots, \theta_{q_\lambda}(g))$ of epimorphisms $\theta_q : \Lambda \to A_q$ for each $q \in \{q_1, \ldots, q_\lambda\}$ (the set of prime factors of the order of A).

For readability, let $\alpha_j = \alpha_j(q)$, $\mu_i = \mu_i(q)$ and assume that $\mu_i \leqslant \mu_{i+1}$ (otherwise, there is a permutation, in general, different for each value of q such that $\widehat{\mu}_i = \mu_{\tau(i)}$ and we replace x_i by $x_{\tau(i)}$ and μ_i by $\widehat{\mu}_i$ in the definition of $\theta_q(x_i)$ below so that the order of $\theta(x_i)$ is m_i). Let $n \equiv \sum_{i=1}^{r} q^{\alpha_t - \mu_i} \pmod{q^{\alpha_t}}$, $n \in \{0, 1, \ldots, q^{\alpha_t} - 1\}$, and define

$$u = \begin{cases} \frac{n}{2} & \text{if } g = 1 \text{ and } n \text{ is even,} \\ \frac{1}{2}(q^{\alpha_t} + n) & \text{if } g = 1 \text{ and } n \text{ is odd,} \\ 1 + \frac{n}{2} & \text{if } g > 1 \text{ and } n \text{ is even,} \\ 1 + \frac{1}{2}(q^{\alpha_t} + n) & \text{if } g > 1 \text{ and } n \text{ is odd} \end{cases}$$

and (note that condition *(ii)* gives $r \geqslant t-g+1$ if $t > g-1$, and thus $r+g-1 \geqslant t$

whether t is greater than $g-1$ or not)

$$\theta_q(x_i) = (0,0,0,\ldots,0,0,0,\ldots,0,q^{\alpha_t-\mu_i}), \quad i=1,\ldots,r-t+g-1,$$

$$\begin{aligned}
\theta_q(x_{r-t+g}) &= (1,0,0,\ldots,0,0,0,\ldots,0,q^{\alpha_t-\mu_{r-t+g}}),\\
\theta_q(x_{r-t+g+1}) &= (0,1,0,\ldots,0,0,0,\ldots,0,q^{\alpha_t-\mu_{r-t+g+1}}),\\
&\vdots\\
\theta_q(x_r) &= (0,0,0,\ldots,0,1,0,\ldots,0,q^{\alpha_t-\mu_r}),\\
\theta_q(d_1) &= (0,0,0,\ldots,0,0,1,\ldots,0,0),\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\theta_q(d_{g-1}) &= (0,0,0,\ldots,0,0,0,\ldots,0,1),\\
\theta_q(d_g) &= \Big(\frac{q^{\alpha_1}-1}{2},\ldots,\frac{q^{\alpha_{t-g+1}}-1}{2}, q^{\alpha_{t-g+2}}-1,\ldots,q^{\alpha_{t-1}}-1,-u\Big).
\end{aligned}$$

Observe that $\mu_r = \alpha_t$ by condition *(ii)*, hence $q^{\alpha_t-\mu_r}=1$, in case that $g=1$. The long relation is preserved by θ since $\theta_q(x_1\cdots x_r\, d_1^2\cdots d_g^2)=0$.

By condition *(i)*, we can write $m_i = q_1^{\mu_i(q_1)}\cdots q_\lambda^{\mu_i(q_\lambda)}$. It also follows that $|\theta_q(x_i)| = q^{\mu_i}$ for all i, taking into account condition *(ii)* for $i=r+g-t,\ldots,r$. Since $\theta_q(x_i)$ and $\theta_{q'}(x_i)$ belong to different primary components of A if $q\neq q'$, the order of $\theta(x_i) = (\theta_{q_1}(x_i),\ldots,\theta_{q_\lambda}(x_i))$ is

$$\operatorname{lcm}(q_1^{\mu_i(q_1)},\ldots,q_\lambda^{\mu_i(q_\lambda)}) = q_1^{\mu_i(q_1)}\cdots q_\lambda^{\mu_i(q_\lambda)} = m_i.$$

The homomorphism θ_q is onto since, by condition *(ii)*, A_q is generated by $\{\theta_q(x_{r+g-t}),\ldots,\theta_q(x_r),\theta_q(d_1),\ldots,\theta_q(d_{g-1})\}$. Therefore, θ is also onto. For, consider an elementary divisor $q^{\alpha_i(q)}$ of A and the generator $h=(0,\ldots,0,1,0,\ldots,0)$ of the cyclic factor

$$H = \{0\}\oplus\cdots\oplus\{0\}\oplus\mathbb{Z}_{q^{\alpha_i(q)}}\oplus\{0\}\oplus\cdots\oplus\{0\}$$

of A_q. Then, $h=\theta_q(g)$ for some $g\in\Lambda$. Obviously, $\theta(g)$ may have nontrivial components in some other primary component $A_{q'}$ for a prime $q'\neq q$, but not the element $\frac{v_t}{q^{\alpha_t(q)}}\,\theta(g)$ since $\frac{v_t}{q^{\alpha_t(q)}}\,\theta_{q'}(g)$ is trivial whenever $q'\neq q$. Moreover, the element $\frac{v_t}{q^{\alpha_t(q)}}\,\theta(g)$ has order $q^{\alpha_i(q)}$ since $\gcd(q, v_t/q^{\alpha_t(q)})=1$. Hence, $\langle\theta(g^{v_t/q^{\alpha_t(q)}})\rangle = H$.

Finally, $\theta(d_g^{v_t}) = 0$, so $d_g^{v_t} \in \ker\theta$, and, since v_t is odd, $d_g^{v_t} \in \Lambda - \Lambda^+$. Therefore, $\theta(\Lambda^+) = A$ and, by Theorem 4.1.2, θ is a nonorientable unbordered surface-kernel epimorphism. ■

Remark 4.2.2. If $g = 1$ and $k = 0$, then the signature of an NEC group Λ with sign '$-$' must have $r \geqslant 2$ proper periods; otherwise $\mu(\Lambda) < 0$ by the Riemann-Hurwitz formula (1.2). ■

Remark 4.2.3. If $t = 1$, then Theorem 4.2.1 becomes Theorem 3.7 in [4]. ■

Corollary 4.2.4. *Let $A \approx \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ be an abelian group of odd order, where v_i divides v_{i+1}, and let $g' > 2$ be an integer. Then, A is a group of automorphisms of some nonorientable compact Riemann surface of topological genus g' if and only if there exist integers $g, m_1, \ldots, m_r$ such that*

(i) $m_i \mid v_t$ *for all* i,

(ii) if $t > g-1$, then every elementary divisor of $\mathbb{Z}_{v_i}$ divides, at least, $t-g-i+2$ proper periods for $i = 1, \ldots, t-g+1$, and

(iii)

$$\frac{g'-2}{v_1 \cdots v_t} = g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).$$

Etayo's result [14] for the symmetric cross-cap number of a noncyclic abelian group of odd order (see also [18, Proposition 6.3]) follows easily from Theorem 4.2.1.

Corollary 4.2.5. *Let $A \approx \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ be a noncyclic abelian group of odd order, where v_i divides v_{i+1}. Then, the symmetric cross-cap number of A is*

$$\tilde{\sigma}(A) = 2 + v_1 \cdots v_t \left[-1 + \sum_{i=1}^{t} \left(1 - \frac{1}{v_i}\right)\right].$$

Proof. For, note that the signature $(1; -; [v_1, \ldots, v_t]; \{-\})$ defines an NEC group Λ^* and fulfills conditions of Theorem 4.2.1. Therefore, $A \approx \Lambda^*/\Gamma_{g^*}$ for some surface NEC group Γ_{g^*} with signature $(g^*; -; [-]; \{-\})$ and thus $\tilde{\sigma}(A) \leqslant g^*$ —the value of g^* is determined by the Riemann-Hurwitz formula (1.2),

$$\frac{g^*-2}{|A|} = -1 + \sum_{i=1}^{t} \left(1 - \frac{1}{v_i}\right). \tag{4.1}$$

Now we prove that, if Λ is another NEC group with signature $(g;\ -\,;\ [m_1,\ldots,m_r];\ \{-\})$ fulfilling conditions of Theorem 4.2.1 and $A \approx \Lambda/\Gamma_{g'}$, then $g^* \leqslant g'$ and thus $\widetilde{\sigma}(A) = g^*$. By the Riemann-Hurwitz formula (1.2),

$$\frac{g'-2}{|A|} = g - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

If $t \leqslant g-1$, then $-1+t \leqslant g-2$. By (4.1), $(g^*-2)/|A| < -1+t$ and thus

$$\frac{g^*-2}{|A|} < -1 + t + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) \leqslant g - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) = \frac{g'-2}{|A|}.$$

Therefore, let $t \geqslant g$. By (1.3), we may assume that $m_1|\cdots|m_r$, since $\mu(\widehat{\Lambda}) \leqslant \mu(\Lambda)$ if $\widehat{\Lambda}$ is an NEC group with signature $(g;\ -\,;[\widehat{m}_{r-\widehat{r}+1},\ldots,\widehat{m}_r];\ \{-\})$. By condition *(ii)* of Theorem 4.2.1, $v_1\,|\,m_{r+g-t}$, $v_2\,|\,m_{r+g-t+1}$, $\ldots$, $v_{t-g+1}\,|\,m_r$, hence

$$\sum_{i=1}^{t-g+1}\left(1-\frac{1}{v_i}\right) \leqslant \sum_{i=r-t+g}^{r}\left(1-\frac{1}{m_i}\right) \leqslant \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) \tag{4.2}$$

(recall that $r-t+g \geqslant 1$ by condition *(ii)* of Theorem 4.2.1). If $g=1$, then

$$\sum_{i=1}^{t}\left(1-\frac{1}{v_i}\right) \leqslant \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)$$

and

$$-1+\sum_{i=1}^{t}\left(1-\frac{1}{v_i}\right) \leqslant -1+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) = g-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right).$$

If $g>1$, then

$$\sum_{i=t-g+2}^{t}\left(1-\frac{1}{v_i}\right) < \sum_{i=t-g+2}^{t} 1 = g-1$$

and, adding up (4.2),

$$-1+\sum_{i=1}^{t}\left(1-\frac{1}{v_i}\right) < g-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right).$$

Therefore, $\frac{g^*-2}{|A|} \leqslant \frac{g'-2}{|A|}$ either if $g=1$ or $g>1$. ■

Remark 4.2.6. When comparing the strong symmetric genus $\sigma^o(A)$ in Theorem 2.2.1 and the symmetric cross-cap number $\widetilde{\sigma}(A)$ of a noncyclic abelian group A of odd order, we have

$$\sigma^o(A) < \frac{\widetilde{\sigma}(A)}{2}$$

(for, consider $g=0$ in McLachlan's expression in Theorem 2.2.1; for order 9, see Remark 2.2.2). ■

As a consequence, we can obtain the smallest topological genus of nonorientable Riemann surfaces on which some abelian group of given odd order acts.

Corollary 4.2.7. *Let $N > 1$ be an odd integer. The least symmetric cross-cap number of abelian groups of order N is*

$$\tilde{\sigma}(N) = \begin{cases} N & \text{if } N \text{ is prime, and} \\ (q-1)\,(N/q-1)+1 & \text{otherwise,} \end{cases}$$

where q is the smallest prime divisor of N. It is attained by

i) $\mathbb{Z}_N$ if N is a prime number or $q^2 \nmid N$, and
ii) $\mathbb{Z}_q \oplus \mathbb{Z}_{N/q}$ otherwise.

Proof. The symmetric cross-cap number of a cyclic group of odd order was obtained by Bujalance in [4]:

$$\tilde{\sigma}(\mathbb{Z}_N) = \begin{cases} N & \text{if } N \text{ is prime,} \\ (q-1)\,(N/q-1)+1 & \text{if } N \text{ is not prime and } q^2 \nmid N, \\ (q-1)\,(N/q-1)+q & \text{otherwise.} \end{cases}$$

Therefore, when N is an odd prime number, $\tilde{\sigma}(N) = N$. Otherwise, N is a composite odd number, say $N = q^\alpha u$, $q \nmid u$. For any noncyclic abelian group $A \approx \mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_2}$ of order N,

$$\begin{aligned} \tilde{\sigma}(\mathbb{Z}_N) \leqslant \tilde{\sigma}(A) \qquad & \text{if } \alpha = 1 \text{ and} \\ \tilde{\sigma}(\mathbb{Z}_q \oplus \mathbb{Z}_{N/q}) < \tilde{\sigma}(\mathbb{Z}_N) < \tilde{\sigma}(A) \qquad & \text{if } \alpha > 1 \text{ and } A \not\approx \mathbb{Z}_q \oplus \mathbb{Z}_{N/q}. \end{aligned}$$

Therefore, $\tilde{\sigma}(N) = (q-1)(N/q-1)+1$. For, we notice that

$$\frac{\tilde{\sigma}(A)-2}{N} = \frac{\tilde{\sigma}(\mathbb{Z}_N)-2}{N} + f(v_1),$$

where

$$f(x) = \frac{1}{q} + \frac{\epsilon}{N} - \frac{1}{x} - \frac{x}{N}, \qquad \epsilon = \begin{cases} q & \text{if } \alpha = 1 \text{ and } u > 1, \\ 1 & \text{otherwise.} \end{cases}$$

As $v_1 v_2 = N$ and v_1 divides v_2, the admissible values of v_1 are in the interval $[q, \sqrt{N}]$. The function f is increasing in this interval since $f'(x) = 1/x^2 - 1/N$. Also, $v_1 \geqslant q$ and $f(q) = 0$ if $\alpha = 1$ (i.e., $\epsilon = q$), hence $f(v_1) \geqslant 0$ and $\tilde{\sigma}(\mathbb{Z}_N) \leqslant \tilde{\sigma}(A)$. If $\alpha > 1$ (i.e., $\epsilon = 1$), then $f(q) < 0$ and $f(q+1) > 0$ since $q \leqslant q^{\alpha-1}$ and $q < u$ (note that $v_1 = q$ if $\alpha = 2$ and $u = 1$, i.e., if $N = q^2$); hence $\widetilde{\sigma}(\mathbb{Z}_q \oplus \mathbb{Z}_{N/q}) < \widetilde{\sigma}(\mathbb{Z}_N) < \widetilde{\sigma}(A)$ for $A \not\approx \mathbb{Z}_q \oplus \mathbb{Z}_{N/q}$ in this case. Also, $\widetilde{\sigma}(\mathbb{Z}_N) < \widetilde{\sigma}(A)$ for any noncyclic abelian group $A \approx \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ of order N and $t > 2$. Indeed,

$$\frac{\widetilde{\sigma}(A) - 2}{N} = \frac{\widetilde{\sigma}(\mathbb{Z}_N) - 2}{N} + \frac{1}{q} + \frac{\epsilon}{N} - \frac{1}{v_1} - \frac{1}{v_2} + \sum_{i=3}^{t} \left(1 - \frac{1}{v_i}\right).$$

Therefore,

$$\frac{\widetilde{\sigma}(A) - 2}{N} > \frac{\widetilde{\sigma}(\mathbb{Z}_N) - 2}{N} + \frac{1}{q} + \frac{\epsilon}{N} - \frac{1}{v_1} - \frac{1}{v_2} + \frac{t-2}{2}$$

since $1 - 1/v_i > 1/2$. Hence $\widetilde{\sigma}(A) > \widetilde{\sigma}(\mathbb{Z}_N)$ if $t > 2$ since $1/v_i \leqslant 1/q$ and $(t-2)/2 > 1/v_i$. ■

4.3 Even order abelian actions

In this section, we consider abelian groups whose Sylow 2-subgroup is cyclic. We find that it is necessary to add some conditions to that obtained for odd order abelian actions, namely concerning signatures without period cycles or with only one period cycle.

For other abelian groups of even order —those with noncyclic Sylow 2-subgroup—, we have not achieved a complete characterization of nonorientable unbordered surface-kernel epimorphisms $\theta : \Lambda \to A$ from a proper NEC group onto such an abelian group. Obviously, large enough values of the parameters of the signature of the NEC group allow relations of the presentation of the NEC group to be preserved and provide enough generators to ensure surjectivity and the existence of a nontrivial element in $\ker \theta \cap (\Lambda - \Lambda^+)$. Indeed, this is the usual situation. Challenges arise when the signature has no period cycles and some proper period is even, or when, having period cycles, the number of 'effective' canonical generators of Λ —more precisely, with the notation used herein, $w + S + r_2$, where r_2 is the number of even proper periods— equals the number of factors of the Sylow 2-subgroup of A. These types of signatures turn out to entail a complex task, so that the complete characterization of this kind of abelian actions on nonorientable Riemann surfaces of a given algebraic genus still remains as an open problem.

Consequently, we now focus on abelian groups with cyclic Sylow 2-subgroup. The following basic result sharpens the type of signatures we have to deal with.

Lemma 4.3.1. *Let A be a finite abelian group such that $Syl_2(A)$ is cyclic, Λ a proper NEC group and $\Lambda \to A$ a nonorientable unbordered surface-kernel epimorphism. Then, every period cycle in the signature of Λ is empty.*

Proof. If $Syl_2(A) \approx \mathbb{Z}_{2^\alpha}$ and $(n_{i1}, \dots, n_{is_i}) = (2, 2, \dots, 2)$ is a nonempty period cycle of Λ (recall Lemma 4.1.3), then $\theta(c_{ij-1}) = \theta(c_{ij}) = 2^{\alpha-1}$, since both $\theta(c_{ij-1})$ and $\theta(c_{ij})$ are elements of $Syl_2(A)$ of order 2. But then $\theta(c_{ij-1}c_{ij}) = 0$ and $\theta(c_{ij-1}c_{ij})$ cannot have order $n_{ij} = 2$, in contradiction with Theorem 4.1.2. ■

Theorem 4.3.2. *Let Λ be a proper NEC group with signature $(g; \pm; [m_1, \dots, m_r]; \{(-)^k\})$, and $A \approx \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ an abelian group, where $\alpha > 0$, $t \geqslant 0$, $v_i > 2$ is odd and v_i divides v_{i+1}. Let also $M = \mathrm{lcm}(m_1, \dots, m_r)$, $w = \eta g + k - 1$, $\eta = 2$ if the signature sign of Λ is '$+$' and $\eta = 1$ otherwise. Then, there exists a nonorientable unbordered surface-kernel epimorphism $\Lambda \to A$ if and only if the following conditions hold:*

(i) *m_i divides 2^α if $t = 0$, and m_i divides $2^\alpha v_t$ otherwise.*

(ii) *If $t > w$ and $i \in \{1, \dots, t-w\}$, then every elementary divisor of $\mathbb{Z}_{v_i}$ divides, at least, $t - w + 1 - i$ proper periods.*

(iii) *If $k = 0$ and $2^\alpha \nmid M$, then $g > 1$; if, in addition, $\alpha > 1$ and $2^{\alpha-1}$ divides an even number of proper periods, then $g > 2$.*

(iv) *If $k = 0$ and $2^\alpha \mid M$, then 2^α divides an even number of proper periods.*

(v) *If $g = 0$ and $k = 1$, then $2^\alpha \mid M$.*

Proof. Let $\theta : \Lambda \to A$ be a nonorientable unbordered surface-kernel epimorphism and $\theta_q = \pi_q \circ \theta$ for a prime number q, where $\pi_q : A \to Syl_q(A)$ is the canonical homomorphism. Conditions *(i)* and *(ii)* follow as in Theorem 4.2.1.

(iii) Suppose that $k = 0$, $2^\alpha \nmid M$ and $g = 1$. Hence, $\theta_2(x_i)$ is even for all i and $Syl_2(A) \approx \mathbb{Z}_{2^\alpha}$ must be generated by $\theta_2(d_1)$ and thus $\theta_2(d_1)$ is odd. Therefore, $\ker \theta \cap (\Lambda - \Lambda^+)$ is empty since any element in $\Lambda - \Lambda^+$ contains an odd number of canonical glide reflections.

Now, suppose that $k = 0$, $g = 2$, $\alpha > 1$, $2^\alpha \nmid M$ and $2^{\alpha-1}$ divides an even number of proper periods. If $2^{\alpha-1} \nmid m_i$, then $\theta_2(x_i)$ is multiple of four; otherwise, $\theta_2(x_i)$ is even but not multiple of four and, since there is an even number of such proper periods, $\theta_2(x_1\cdots x_r)$ is multiple of four. Also, $\theta_2(d_1)$ and $\theta_2(d_2)$ are of different parity, say, $\theta_2(d_1)$ is odd and $\theta_2(d_2)$ is even (otherwise, every element in $\ker\theta$ would contain an even number of glide reflections and would be orientation-preserving) and thus $\theta_2(d_1^2d_2^2)$ is even but not multiple of four. Therefore, the long relation would not be preserved, since $\theta_2(x_1\cdots x_rd_1^2d_2^2)$ would be even but not multiple of four.

(iv) Otherwise, $\theta_2(x_1\cdots x_rd_1^2\cdots d_g^2)$ is odd and the long relation would not be preserved.

(v) When $\alpha>1$, the claim follows from (1.7) since $2g+k-1=0$ and $N_2(\alpha)\geqslant n_2(\alpha)$ means that $\widehat{\mu}_r\geqslant\alpha$. If $\alpha=1$ and we suppose that M is odd, then $\theta_2(c_{10})=1$, $\theta_2(x_i)=0$ and, by the long relation, $\theta_2(e_1)=0$; hence, $\ker\theta\cap(\Lambda-\Lambda^+)=\varnothing$ since any element in $\Lambda-\Lambda^+$ contains c_{10} an odd number of times, and this is not consistent with Theorem 4.1.1.

We prove the sufficiency of the conditions by defining epimorphisms $\theta_q:\Lambda\to A_q$ for each prime q in the set $\{q_1,\dots,q_\lambda\}$ of prime numbers dividing the order of A, and a surface-kernel epimorphism $\theta:\Lambda\to A$ as the direct product epimorphism

$$\theta:\Lambda\to A: g\mapsto\theta(g)=(\theta_{q_1}(g),\dots,\theta_{q_\lambda}(g)).$$

For readability, we let $\mu_i=\mu_i(q)$ (see Section 1.3) in the definition of each homomorphism θ_q. Also, we assume that $\mu_i\leqslant\mu_{i+1}$; otherwise, there is a permutation (in general, different for each value of q) such that $\widehat{\mu}_i=\mu_{\tau(i)}$ and we replace x_i by $x_{\tau(i)}$ and μ_i by $\widehat{\mu}_i$ in the definition of $\theta_q(x_i)$ below —so that the order of $\theta(x_i)$ is m_i. Let

$$\gamma_1=e_1,\ \dots,\ \gamma_{k-1}=e_{k-1},\ \gamma_k=a_1,\ \gamma_{k+1}=b_1,\ \dots,\ \gamma_{w-1}=a_g,\ \gamma_w=b_g,$$

or

$$\gamma_1=e_1,\dots,\gamma_{k-1}=e_{k-1},\gamma_k=d_1,\dots,\gamma_w=d_g,$$

according to the sign of the signature of Λ, and

$$\delta=\begin{cases}-1 & \text{if } g=0,\\ 0 & \text{if } g>0 \text{ and sign}(\Lambda) \text{ is `+'},\\ -2 & \text{if } g>0 \text{ and sign}(\Lambda) \text{ is `}-\text{'}.\end{cases}$$

Let also $q\neq 2$ be a prime number dividing $|A|$ and $A_q\approx\mathbb{Z}_{q^{\alpha_1}}\oplus\cdots\oplus\mathbb{Z}_{q^{\alpha_t}}$ be the q-Sylow subgroup of A—note that some factors of A_q may be trivial, i.e.,

$\alpha_1 = \cdots = \alpha_{t'} = 0$ for some $t' < t$. We define θ_q as follows—note that $r + w \geqslant t$ by condition *(ii)*:

$$\theta_q(c_{i0}) = (0, \ldots, 0), \qquad i = 1, \ldots, k,$$

$$\theta_q(x_i) = (0, \ldots, 0, q^{\alpha_t - \mu_i}), \qquad i = \begin{cases} 1, \ldots, r - t + w & \text{if } t > w, \\ 1, \ldots, r & \text{if } t \leqslant w, \end{cases}$$

$$\theta_q(x_i) = (0, \overset{t-r-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{r+w-i-1}{\ldots\ldots}, 0, q^{\alpha_t - \mu_i}),$$
$$i = r - t + w + 1, \ldots, r \quad \text{if } t > w,$$

$$\theta_q(\gamma_i) = (0, \ldots, 0), \qquad i = 1, \ldots, w - t \quad \text{if } t < w,$$

$$\theta_q(\gamma_i) = (0, \overset{t-w+i-1}{\ldots\ldots}, 0, 1, 0, \overset{w-i}{\ldots\ldots}, 0), \quad i = \begin{cases} 1, \ldots, w & \text{if } t \geqslant w, \\ w - t + 1, \ldots, w & \text{if } t < w, \end{cases}$$

$$\theta_q(e_k) = \begin{cases} (-1, \ldots, -1, \delta, \overset{\eta g-1}{\ldots\ldots}, \delta, \delta - u) & \text{if } t > \eta g > 0, \\ (\delta, \ldots, \delta, \delta - u) & \text{if } t \leqslant \eta g \text{ or } g = 0, \end{cases}$$

where $u = \sum_{i=1}^{r} q^{\alpha_t - \mu_i}$.

Now, we define θ_2 considering the following cases.

a) $k = 0$, $2^\alpha \nmid M$ and $g > 2$.

$$\theta_2(x_i) = 2^{\alpha - \mu_i}, \qquad i = 1, \ldots, r,$$
$$\theta_2(d_1) = -1 - \sum_{i=1}^{r} 2^{\alpha - \mu_i - 1},$$
$$\theta_2(d_2) = 1,$$
$$\theta_2(d_i) = 0, \qquad i = 3, \ldots, g.$$

b) $k = 0$, $2^\alpha \nmid M$, $g = 2$ and $\alpha = 1$.

$$\theta_2(x_i) = 0, \qquad i = 1, \ldots, r,$$
$$\theta_2(d_1) = 1, \ \theta_2(d_2) = 0.$$

c) $k = 0$, $2^\alpha \nmid M$, $g = 2$ and $\alpha > 1$. By condition *(iii)*, $2^{\alpha-1}$ divides an odd number of proper periods and thus $\sum_{i=1}^{r} 2^{\alpha - \mu_i - 1}$ is odd.

$$\theta_2(x_i) = 2^{\alpha - \mu_i}, \qquad i = 1, \ldots, r,$$
$$\theta_2(d_1) = -\sum_{i=1}^{r} 2^{\alpha - \mu_i - 1},$$
$$\theta_2(d_2) = 0.$$

d) $k = 0$ and $2^\alpha \mid M$. Assume that $2^\alpha \mid m_r$. Then, $\sum_{i=1}^{r-1} 2^{\alpha-\mu_i}$ is odd by condition *(iv)*.

$$\begin{aligned}
\theta_2(x_i) &= 2^{\alpha-\mu_i}, && i = 1, \ldots, r-1, \\
\theta_2(x_r) &= -\sum_{i=1}^{r-1} 2^{\alpha-\mu_i}, && \\
\theta_2(d_i) &= 0, && i = 1, \ldots, g,
\end{aligned}$$

so that $\theta_2(x_r)$ generates $\mathbb{Z}_{2^\alpha}$.

e) $k = 1$, $g = 0$.

$$\begin{aligned}
\theta_2(x_i) &= 2^{\alpha-\mu_i}, \qquad i = 1, \ldots, r, \\
\theta_2(e_1) &= -\textstyle\sum_i 2^{\alpha-\mu_i}, \\
\theta_2(c_{10}) &= 2^{\alpha-1}.
\end{aligned}$$

By condition *(v)*, $\theta_2(x) = 1$ for some $x \in \{x_1, \ldots, x_r\}$ and thus $c_{10}x^{2^{\alpha-1}v_t} \in \ker\theta \cap (\Lambda - \Lambda^+)$.

f) Otherwise, we define

$$\begin{aligned}
\theta_2(x_i) &= 2^{\alpha-\mu_i}, \qquad i = 1, \ldots, r, \\
\theta_2(\gamma_i) &= 0, \qquad\qquad i = 1, \ldots, w-1, \\
\theta_2(\gamma_w) &= \begin{cases} 0 & \text{if } \alpha = 1 \text{ and } \operatorname{sign}(\Lambda) \text{ is `$-$'}, \\ 1 & \text{otherwise}, \end{cases} \\
\theta_2(e_k) &= \delta - \textstyle\sum_i 2^{\alpha-\mu_i}, \\
\theta_2(c_{i0}) &= 2^{\alpha-1}, \qquad i = 1, \ldots, k.
\end{aligned}$$

We observe that either $d_g^{v_t}$ $(\alpha = 1)$ or $c_{10}\gamma_w^{2^{\alpha-1}v_t}$ belongs to $\ker\theta \cap (\Lambda - \Lambda^+)$ and thus $\theta(\Lambda^+) = A$.

■

Corollary 4.3.3. *Let $A \approx \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ be an abelian group, where $\alpha > 0$, v_i is odd and v_i divides v_{i+1} for all i, and let $g' > 2$ be an integer. Then, A is a group of automorphisms of some nonorientable compact Riemann surface of topological genus g' if and only if there exist integers $\eta = 1$ or 2, $g, k, m_1, \ldots, m_r$ and $w = \eta g + k - 1$ such that* *(i)* m_i *divides* 2^α *if* $t = 0$*, and* m_i *divides* $2^\alpha v_t$ *otherwise;*

(ii) *if* $t > w$ *and* $i \in \{1, \ldots, t-w\}$*, then every elementary divisor of* $\mathbb{Z}_{v_i}$ *divides, at least,* $t - w + 1 - i$ *proper periods;*

(iii) *if* $k = 0$ *and* $2^\alpha \nmid M$*, then* $g > 1$*; if, in addition,* $\alpha > 1$ *and* $2^{\alpha-1}$ *divides an even number of proper periods, then* $g > 2$*;*

(iv) *if* $k = 0$ *and* $2^\alpha \mid M$*, then* 2^α *divides an even number of proper periods;*

(v) *if* $g = 0$ *and* $k = 1$*, then* $2^\alpha \mid M$*; and*

(vi)
$$\frac{g'-2}{2^\alpha v_1 \cdots v_t} = \eta g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

The symmetric cross-cap number of a noncyclic abelian group of even order with cyclic Sylow 2-subgroup, as stated in [18, Proposition 6.2] by Gromadzki, also follows from Theorem 4.3.2 (note the resemblance to the expression of Corollary 4.2.5 for abelian groups of odd order).

Corollary 4.3.4. *Let* $A \approx \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_t}$ *be a noncyclic abelian group of even order, where* v_i *is an odd integer and* v_i *divides* v_{i+1}*. Then, the symmetric cross-cap number of* A *is*

$$\tilde{\sigma}(A) = 2 + 2^\alpha v_1 \cdots v_t \left[\sum_{i=1}^{t-1}\left(1 - \frac{1}{v_i}\right) - \frac{1}{2^\alpha v_t}\right].$$

Proof. For, note that the signature $(0; +; [v_1, \ldots, v_{t-1}, 2^\alpha v_t]; \{(-)\})$ defines an NEC group Λ^* and fulfills conditions of Theorem 4.3.2. Therefore, $A \approx \Lambda^*/\Gamma_{g^*}$, where Γ_{g^*} is a surface NEC group with signature $(g^*; -; [-]; \{-\})$. By the Riemann-Hurwitz formula (1.2), $(g^* - 2)/|A| = \mu(\Lambda^*)$, where

$$\mu(\Lambda^*) = -1 + \sum_{i=1}^{t-1}\left(1 - \frac{1}{v_i}\right) + 1 - \frac{1}{2^\alpha v_t}.$$

Now we proof that, if Λ is another NEC group with signature $(g; \pm; [m_1, \ldots, m_r]; \{(-)^k\})$ fulfilling conditions of Theorem 4.3.2 and $A \approx \Lambda/\Gamma_{g'}$, then $g^* \leqslant g'$ and thus $\tilde{\sigma}(A) = g^*$. By the Riemann-Hurwitz formula (1.2), $(g' - 2)/|A| = \mu(\Lambda)$, where

$$\mu(\Lambda) = w - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right), \quad w = \eta g + k - 1.$$

If $t \leqslant w$, then $-1 + t \leqslant w - 1$ and

$$\mu(\Lambda^*) < -1 + t \leqslant -1 + t + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) \leqslant w - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) = \mu(\Lambda).$$

Now, we consider the case $t > w$. By (1.3), we may assume that $m_1|\cdots|m_r$, since $\mu(\widehat{\Lambda}) \leqslant \mu(\Lambda)$ if $\widehat{\Lambda}$ is an NEC group with signature $(g;\, \pm\,; [\widehat{m}_{r-\widehat{r}+1}, \ldots, \widehat{m}_r];\ \{(-)^k\})$. By condition *(ii)* of Theorem 4.3.2, $r-t+w+1 \geqslant 1$ and $v_1 \,|\, m_{r-t+w+1}, \ldots, v_{t-w} \,|\, m_r$. If $w > 0$, then

$$\sum_{i=1}^{t-w}\left(1-\frac{1}{v_i}\right) \leqslant \sum_{i=r-t+w+1}^{r}\left(1-\frac{1}{m_i}\right) \leqslant \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)$$

and

$$\sum_{i=t-w+1}^{t-1}\left(1-\frac{1}{v_i}\right) + 1 - \frac{1}{2^\alpha v_t} < \sum_{i=t-w+1}^{t} 1 = w,$$

hence, adding up both inequalities, $\mu(\Lambda^*) < \mu(\Lambda)$. If $w = 0$, then $2^\alpha \,|\, m_r$ by conditions *(iii)* and *(v)*, hence

$$\sum_{i=1}^{t-1}\left(1-\frac{1}{v_i}\right) + 1 - \frac{1}{2^\alpha v_t} \leqslant \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)$$

and

$$\mu(\Lambda^*) \leqslant -1 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) = \mu(\Lambda).$$

■

Conclusions and further developments

The results obtained in this thesis determine the algebraic genera of compact Klein surfaces on which a given abelian group acts as a group of automorphisms in terms of the invariant factors of the abelian group. More precisely, formerly known characterizations of abelian actions on Riemann surfaces are revisited and, moreover, they are extended to other families of Klein surfaces, namely nonorientable compact Riemann surfaces and compact bordered Klein surfaces. All abelian actions are addressed except those of even order abelian groups with noncyclic Sylow 2-subgroup on nonorientable compact Riemann surfaces. This latter case remains as an open problem.

Some of the techniques used in this thesis can be applied to other families of finite groups. Conditions for the existence of an epimorphism $\Lambda_{ab} \to G_{ab}$ from the abelianization of an NEC group Λ onto that of a finite group G may provide useful information in order to establish conditions for the existence of surface-kernel epimorphisms $\Lambda \to G$.

Other possible developments might extend these results to the study of abelian actions on Klein surfaces of given algebraic genus, number of boundary components and orientability, possibly distinguishing between orientation-preserving and orientation-reversing actions. When fixing the number of boundary components, a preliminary approach could be considering actions of abelian groups of specific type, say p-groups. Also, other specific types of surfaces could be considered (for instance, pseudo-real Riemann surfaces).

Notation

$\mathcal{H}$	Open upper half-plane in $\mathbb{C}$.
$\mathbb{C}^+$	Closed upper half-plane in $\mathbb{C}$.
$\approx$	Group Isomorphism.
$\|G\|, \|g\|$	Order of a group G, order of an element $g \in G$.
$\exp G$	Exponent of a group G.
$\mathcal{T}(G)$	Torsion set of a group G.
$Syl_p(G)$	Sylow p-subgroup of a group G.
G', $[G, G]$	Derived or commutator subgroup of a group G.
G_{ab}	Abelianization of a group G.
$\mathbb{Z}_n$	Cyclic group of order n.
$\mathrm{Aut}(X)$	Full group of dianalytic automorphisms of a Klein surface X.
$\mu(\Lambda)$	Reduced hyperbolic area of any fundamental region for an NEC group Λ.
$\sigma^o(G)$	Strong symmetric genus of a finite group G.
$\widetilde{\sigma}(G)$	Symmetric cross-cap number of a finite group G.
$\rho(G)$	Real genus of a finite group G.

References

[1] N. L. Alling and N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture Notes in Mathematics, Vol. 219, Springer-Verlag, Berlin-New York, 1971.

[2] C. Andreian Cazacu, *On the morphisms of Klein surfaces*, Rev. Roumaine Math. Pures Appl. **31** (1986), no. 6, 461–470.

[3] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series, vol. 280, Cambridge University Press, Cambridge, 2000.

[4] E. Bujalance, *Cyclic groups of automorphisms of compact nonorientable Klein surfaces without boundary*, Pacific J. Math. **109** (1983), no. 2, 279–289.

[5] ———, *A note on the group of automorphisms of a compact Klein surface*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **81** (1987), no. 3, 565–569. MR 950620

[6] E. Bujalance, F. J. Cirre, J. J. Etayo, G. Gromadzki, and E. Martínez, *A survey on the minimum genus and maximum order problems for bordered Klein surfaces*, Groups St Andrews 2009 in Bath. Volume 1, London Math. Soc. Lecture Note Ser., vol. 387, Cambridge Univ. Press, Cambridge, 2011, pp. 161–182.

[7] ———, *Automorphism groups of compact non-orientable Riemann surfaces*, Groups St Andrews 2013, London Math. Soc. Lecture Note Ser., vol. 422, Cambridge Univ. Press, Cambridge, 2015, pp. 183–193.

[8] E. Bujalance, F. J. Cirre, J. M. Gamboa, and G. Gromadzki, *On compact Riemann surfaces with dihedral groups of automorphisms*, Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 3, 465–477.

[9] E. Bujalance, F. J. Cirre, and G. Gromadzki, *A survey of research inspired by Harvey's theorem on cyclic groups of automorphisms*, Geometry of Riemann surfaces, London Math. Soc. Lecture Note Ser., vol. 368, Cambridge Univ. Press, Cambridge, 2010, pp. 15–37.

[10] E. Bujalance, A. F. Costa, J. M. Gamboa, and J. Lafuente, *An algorithm to compute orders and ramification indices of cyclic actions on compact surfaces. II*, Discrete Comput. Geom. **16** (1996), no. 1, 33–54.

[11] E. Bujalance, J. J. Etayo, J. M. Gamboa, and G. Gromadzki, *Automorphism groups of compact bordered Klein surfaces. A combinatorial approach*, Lecture Notes in Mathematics, vol. 1439, Springer-Verlag, Berlin, 1990.

[12] E. Bujalance, J. J. Etayo, J. M. Gamboa, and G. Martens, *Minimal genus of Klein surfaces admitting an automorphism of a given order*, Arch. Math. (Basel) **52** (1989), no. 2, 191–202.

[13] E. Bujalance, J. M. Gamboa, and C. Maclachlan, *Minimum topological genus of compact bordered Klein surfaces admitting a prime-power automorphism*, Glasgow Math. J. **37** (1995), no. 2, 221–232.

[14] J. J. Etayo, *Sobre grupos de automorfismos de superficies de Klein*, Ph.D. thesis, Universidad Complutense, 1983.

[15] J. J. Etayo and E. Martínez, *On the minimum genus problem on bordered Klein surfaces*, Mathematical contributions in honor of Juan Tarrés (Spanish), Univ. Complut. Madrid, Fac. Mat., Madrid, 2012, pp. 149–158.

[16] ———, *On the minimum genus problem on bordered Klein surfaces for automorphisms of even order*, Riemann and Klein surfaces, automorphisms, symmetries and moduli spaces, Contemp. Math., vol. 629, Amer. Math. Soc., Providence, RI, 2014, pp. 119–135.

[17] J. J. Etayo Gordejuela, *Nonorientable automorphisms of Riemann surfaces*, Arch. Math. (Basel) **45** (1985), no. 4, 374–384.

[18] G. Gromadzki, *Abelian groups of automorphisms of compact nonorientable Klein surfaces without boundary*, Comment. Math. Prace Mat. **28** (1989), no. 2, 197–217.

[19] W. J. Harvey, *Cyclic groups of automorphisms of a compact Riemann surface*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 86–97.

[20] A. Hurwitz, *Ueber algebraische Gebilde mit eindeutigen Transformationen in sich*, Math. Ann. **41** (1892), no. 3, 403–442.

[21] R. S. Kulkarni and C. Maclachlan, *Cyclic p-groups of symmetries of surfaces*, Glasgow Math. J. **33** (1991), no. 2, 213–221.

[22] A. M. Macbeath, *Discontinuous groups and birational transformations*, Proceedings of Dundee Summer School, Univ. of St. Andrews, 1961.

[23] ———, *The classification of non-euclidean plane crystallographic groups*, Canad. J. Math. **19** (1967), 1192–1205.

[24] C. Maclachlan, *Abelian groups of automorphisms of compact Riemann surfaces*, Proc. London Math. Soc. (3) **15** (1965), 699–712.

[25] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, Second revised ed., Dover Publications Inc., New York, 1976.

[26] C. L. May, *Large automorphism groups of compact Klein surfaces with boundary. I*, Glasgow Math. J. **18** (1977), no. 1, 1–10.

[27] ———, *Finite groups acting on bordered surfaces and the real genus of a group*, Rocky Mountain J. Math. **23** (1993), no. 2, 707–724.

[28] ———, *The symmetric crosscap number of a group*, Glasg. Math. J. **43** (2001), no. 3, 399–410.

[29] D. McCullough, *Minimal genus of abelian actions on Klein surfaces with boundary*, Math. Z. **205** (1990), no. 3, 421–436.

[30] M. Newman, *Integral matrices*, Academic Press, New York-London, 1972, Pure and Applied Mathematics, Vol. 45.

[31] R. R. Preston, Jr, *Fundamental domains and projective structures on compact klein surfaces*, Ph.D. thesis, University of Texas, 1983.

[32] J. Rodríguez, *Some results on abelian groups of automorphisms of compact Riemann surfaces*, Riemann and Klein surfaces, automorphisms, symmetries and moduli spaces, Contemp. Math., vol. 629, Amer. Math. Soc., Providence, RI, 2014, pp. 283–297.

[33] J. Rodríguez, *Abelian actions on compact bordered Klein surfaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. **111** (2017), no. 1, 189–204.

[34] D. Singerman, *Automorphisms of compact non-orientable Riemann surfaces*, Glasgow Math. J. **12** (1971), 50–59.

[35] H. J. S. Smith, *On systems of linear indeterminate equations and congruences*, Philos. Trans. R. Soc. Lond. **151** (1861), 293–326.

[36] T. W. Tucker, *Finite groups acting on surfaces and the genus of a group*, J. Combin. Theory Ser. B **34** (1983), no. 1, 82–98.

[37] ———, *Symmetric embeddings of Cayley graphs in nonorientable surfaces*, Graph theory, combinatorics, and applications. Vol. 2 (Kalamazoo, MI, 1988), Wiley-Intersci. Publ., Wiley, New York, 1991, pp. 1105–1120.

[38] H. C. Wilkie, *On non-Euclidean crystallographic groups*, Math. Z. **91** (1966), 87–102.

[39] A. Wiman, *Über die hyperelliptischen Curven und diejenigen vom Geschlechte* $p = 3$*, welche eindeutigen Transformationen in sich zulassen*, Bihang till Kongl. Svenska vetenskaps-akademiens handlingar, bd. 21, afd. 1, no. 1 (1895), 1–23.