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Ana Belén de Felipe Paramio

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Topología de espacios de valoraciones y geometría de las singularidades

Topologie des espaces de valuations et géométrie des singularités

Dirigida por: Evelia R. García Barroso / Bernard Teissier

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Institut de Mathématiques de Jussieu -
Paris Rive Gauche (IMJ-PRG)
UP7D - Campus des Grands Moulins
Bâtiment Sophie Germain
Case 7012
75205 Paris Cedex 13

Departamento de Matemáticas, Estadística e Investigación Operativa
Universidad de La Laguna - Facultad de Ciencias. Sección Matemáticas.
Avda. Astrofísico Francisco Sánchez s/n. Campus de Anchieta.
Apartado de Correos 456, CP 38200
La Laguna

Abstract

Topology of spaces of valuations and geometry of singularities

We study the fiber of the Riemann-Zariski space above a closed point x of an algebraic variety X defined over an algebraically closed field. We characterize its homeomorphism type for regular points and normal surface singularities. This is done by studying the relation between this space and the normalized non-Archimedean link of x in X. We prove that their behavior is the same.

Keywords: Riemann-Zariski space, normalized non-Archimedean link, valuative tree, dual graph.

Topología de espacios de valoraciones y geometría de las singularidades

Estudiamos la fibra del espacio de Riemann-Zariski sobre un punto cerrado x de una variedad algebraica X definida sobre un cuerpo algebraicamente cerrado. Caracterizamos su tipo de homeomorfismo para puntos regulares y singularidades normales de superficie. Hacemos esto estudiando la relación entre dicho espacio y el link no arquimediano normalizado de x en X. Demostramos que se comportan igual.

Palabras clave: espacio de Riemann-Zariski, link no arquimediano normalizado, árbol valorativo, grafo dual.

Topologie des espaces de valuations et géométrie des singularités

On étudie la fibre de l'espace de Riemann-Zariski au-dessus d'un point fermé x d'une variété algébrique X définie sur un corps algébriquement clos. On caractérise son type d'homéomorphisme pour des points réguliers et des singularités normales de surface. Cela est fait en étudiant le lien avec l'entrelac non Archimédien normalisé de x dans X. On démontre qu'ils ont le même comportement.

Mots-clefs: espace de Riemann-Zariski, entrelac non Archimédien normalisé, arbre valuative, graphe dual.

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Contents

Summary

Valuations are a fundamental tool in algebraic geometry. Historically they played an important role in [DW82] where valuations were introduced to construct algebraically the Riemann surface associated to a field of algebraic functions of one variable. Later Zariski generalized this idea and valuations became an essential ingredient in its approach to the problem of resolution of singularities of an algebraic variety X. In [Zar44], he endowed the set $\mathfrak{X}$ of all valuation rings of the function field K of X containing the base field with a topology and established its quasi-compactness. This was a key point in his program for resolution. It turns out to be also a key result in some recent attempts to solve this problem in positive characteristic following new strategies also using local uniformization (see [CP08, Tei14]).

In this work we suppose that X is an algebraic variety defined over an algebraically closed field k (i.e., an integral separated scheme of finite type over k) and we fix a closed point x in X. We initiate the study of the homeomorphism type of the space $\mathrm{RZ}(X,x)$ consisting of all valuations rings in $\mathfrak{X}$ dominating the local ring $\mathcal{O}_{X,x}$, endowed with the topology induced by the Zariski topology. We call $\mathrm{RZ}(X,x)$ the *Riemann-Zariski space* of X at x. Our goal is to clarify the relation between the topological properties of this space and the local geometry of X at x. Note that the one-dimensional case is well understood. If X is an algebraic curve then $\mathrm{RZ}(X,x)$ is in bijection with the local analytic branches of X at x. However, the situation is richer in higher dimension.

Similar preoccupations have appeared in the context of the theory of analytic spaces as developed by Berkovich and others after [Ber90]. Adopting this point of view one associates to X is analytification X^{an}. A point of X^{an} is an absolute value (giving rise to a rank one valuation by taking minus the logarithm) on the residue field of a point of X, extending the trivial absolute value of k. We may consider the subspace $\mathrm{L}(X,x)$ of all points in X^{an} which specialize to x excepting the trivial one. One nice feature of this space, established by Thuillier in [Thu07], is that it has the homotopy type of the dual complex associated to the exceptional divisor of a resolution of singularities of (X,x) whose exceptional divisor has simple normal crossings.

In fact, the space $\mathrm{RZ}(X,x)$ is closely related to the *normalized non-Archimedean link* $\mathrm{NL}(X,x)$ of x in X which is obtained from $\mathrm{L}(X,x)$ by identifying points defining equivalent valuations (see [Fan14a]). If X is reduced to x then $\mathrm{RZ}(X,x)$ is a singleton and $\mathrm{NL}(X,x)$ is empty. When X is a curve then $\mathrm{RZ}(X,x)$ and $\mathrm{NL}(X,x)$ are finite spaces of the same cardinality. Therefore we may assume that the algebraic varieties we consider have dimension at least two. Note that $\mathrm{NL}(X,x)$ is a compact space whereas $\mathrm{RZ}(X,x)$ is never Hausdorff. In general, there is a canonical continuous surjective map from $\mathrm{RZ}(X,x)$ to $\mathrm{NL}(X,x)$, and the latter appears to be the largest Hausdorff quotient of the former space

in the case of normal surfaces. This is no longer true in higher dimension. A detailed proof of these facts is given in Subsections 1.2.3 and 1.2.4 respectively.

We address first the regular case. We call d the dimension of X. Recall that the Krull dimension of a topological space is the supremum of the lengths of all chains of irreducible closed subspaces of the space. Observe that $\mathrm{NL}(X, x)$ has Krull dimension zero since it is Hausdorff. We look instead at its covering dimension as defined in [Pea75], and we show that $\mathrm{NL}(X, x)$ has covering dimension $d - 1$. If another regular closed point $y \in Y$ of an algebraic variety defined over the same field k is given, and $\mathrm{NL}(X, x)$ is homeomorphic to $\mathrm{NL}(Y, y)$, then this result on the dimension implies that X and Y have the same dimension. The converse also holds. Assume that X and Y have both dimension d. By Cohen's structure theorem there exists an isomorphism of k-algebras σ from the formal completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ to that of $\mathcal{O}_{Y,y}$ (they are isomorphic to the ring of formal power series in d variables with coefficients in k). This enables us to define a natural bijection between the semivaluations of $\widehat{\mathcal{O}}_{Y,y}$ and those of $\widehat{\mathcal{O}}_{X,x}$ by composition with σ. By a semivaluation we mean here a map from the ring to $[0, +\infty]$ verifying the standard axioms of valuations but which may send to infinity some non-zero element of the ring.

Observe that $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ do not need to be isomorphic as k-algebras, even if they have the same fraction field. Suppose that X is non singular and take two different closed points $x, x' \in X$. If there exists such an isomorphism between the local rings of X at these points, then we can find an isomorphism $U \to U'$ which sends x to x', where $U, U' \subset X$ are open neighborhoods of x and x' respectively. Hence we would have a birational map from X to X itself sending x to x'. If X is of general type and the ground field is algebraically closed of characteristic zero, the birational automorphism group of X is finite (see [HMX13] for a bound). In this case, there exists only a finite number of points x' for which $\mathcal{O}_{X,x}$ could be isomorphic as k-algebra to $\mathcal{O}_{X,x'}$.

Let us now go back to our problem. Note that it has been reduced to the study of the extensions of a semivaluation of $\mathcal{O}_{X,x}$ to its formal completion. In fact, one can show that a point ν in $\mathrm{NL}(X, x)$ defines in a canonical way a suitably normalized semivaluation $\widehat{\nu}$ of $\widehat{\mathcal{O}}_{X,x}$ whose restriction to k is trivial. It suffices to define $\widehat{\nu}(f) = \lim_{n \to +\infty} \nu(f_n)$, where $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{O}_{X,x}$ which converges to f. This result is true whenever the point x is analytically irreducible, that is, if $\widehat{\mathcal{O}}_{X,x}$ is an integral domain.

Our first main result is the following:

Theorem A. *Let X, Y be two algebraic varieties defined over the same algebraically closed field k. For all regular closed points $x \in X$, $y \in Y$, the spaces $\mathrm{NL}(X, x)$ and $\mathrm{NL}(Y, y)$ are homeomorphic if and only if X and Y have the same dimension.*

In dimension two, one can be more specific. A topological model for $\mathrm{NL}(\mathbf{A}^2_{\mathbf{C}}, 0)$ has already been proposed in [FJ04, Section 3.2.3]. In this monograph the normalized non-Archimedean link of the origin in the affine complex plane is referred to as the *valuative tree*. This space carries a canonical affine structure which allows one to perform convex analysis on it and which finds interesting applications in dynamics and complex analysis in [FJ07, FJ05]. More precisely it has a *rooted nonmetric* $\mathbf{R}$*-tree* structure (see [FJ04, Definition 3.1 and Theorem 3.14] and [Nov14]). Roughly speaking, this means that is a topological space where any two different points are joined by a unique real line interval. This structure was extended in [Gra07] to the case of a regular closed point of a surface. The homeomorphism type of an arbitrary Berkovich curve is also treated in [HLP14] under a countability assumption on the base field. Since $\mathrm{NL}(\mathbf{A}_k^2, 0)$ is homeomorphic to the closed unit ball over the discrete valued field $k(\!(t)\!)$, their result shows that $\mathrm{NL}(\mathbf{A}_k^2, 0)$ is a Ważewski universal dendrite when k is countable.

We now turn to the Riemann-Zariski setting. It is known that $\mathfrak{X}$ has the same Krull dimension as X. We show that $\mathrm{RZ}(X, x)$ has Krull dimension $d-1$. Again this fact implies that two varieties X and Y defined over k have the same dimension when $\mathrm{RZ}(X, x)$ and $\mathrm{RZ}(Y, y)$ are homeomorphic for some regular closed points $x \in X$, $y \in Y$. The converse also holds, however in this situation the proof is more involved. If one tries to reproduce the proof given in the case of the normalized non-Archimedean links, we are led to study the behavior of a valuation of K whose valuation ring dominates $\mathcal{O}_{X,x}$ under passage to the formal completion of the ring.

When x is an analytically normal point of a surface (i.e. the formal completion $\widehat{\mathcal{O}}_{X,x}$ is a normal domain), it is shown in [Spi90b] that a valuation of $\mathrm{RZ}(X, x)$ extends in a unique way to a valuation of the fraction field of $\widehat{\mathcal{O}}_{X,x}$ whose valuation ring dominates that ring. This result applies to our situation since any regular point is analytically normal. However this statement about extensions of valuations is no longer true in dimension greater than two. Such an extension always exists, but it is not difficult to construct some explicit examples where it is not unique. In fact, either it extends uniquely or it has infinitely many such extensions (see [HS91]).

Therefore one needs to follow a different strategy. Recall that the henselization $\widetilde{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ is the inductive limit of the system of equiresidual local étale $\mathcal{O}_{X,x}$-algebras. Let us take an open affine neighbourhood $U \subseteq X$ of x given by a finitely generated k-algebra $A = k[T_1, \dots, T_n]/I$ of dimension d. The Noether normalization Lemma states that there exist d elements $x_1, \dots, x_d$ in A algebraically independent over k such that A is a finitely generated $k[x_1, \dots, x_d]$-module and $\mathfrak{m} \cap k[x_1, \dots, x_d] = (x_1, \dots, x_d)$, where $\mathfrak{m}$ is the maximal ideal of A corresponding to the point x. Moreover, each x_i can be chosen to be the residue class of a generic k-linear combination $\sum_{1 \leq i \leq n} \alpha_{ij} T_j$ modulo I (see [Eis95, Theorem 13.3]). One can show that it is also possible to chose them in such a way that the tangent map $\mathrm{T}_{U,x} \to \mathrm{T}_{\mathbf{A}_k^d,0}$ is injective. Hence the finite morphism from U to $\mathbf{A}_k^d$ is étale in the point x. Then $\mathcal{O}_{U,x}$ is a local étale equiresidual $\mathcal{O}_{\mathbf{A}_k^d,0}$-algebra and therefore $\widetilde{\mathcal{O}}_{X,x}$ and the henselization of $k[x_1, \dots, x_d]_{(x_1,\dots,x_d)}$ coincide.

As a consequence, for any regular closed point $y \in Y$ of a d-dimensional algebraic variety defined over k, we can find an isomorphism of k-algebras σ from $\widetilde{\mathcal{O}}_{X,x}$ to the henselization $\widetilde{\mathcal{O}}_{Y,y}$ of $\mathcal{O}_{Y,y}$. Let us denote by $\widetilde{K}$ the fraction field of $\widetilde{\mathcal{O}}_{X,x}$ (note that $\widetilde{\mathcal{O}}_{X,x}$ is a subring of $\widehat{\mathcal{O}}_{X,x}$, which is an integral domain by hypothesis). The previous remark allows us to define a natural bijection between the valuations of $\widetilde{K}$ whose valuation ring dominates $\widetilde{\mathcal{O}}_{Y,y}$ and the ones whose valuation ring dominates $\widetilde{\mathcal{O}}_{X,x}$, by composition with σ. The question which naturally arises now is if a valuation of $\mathrm{RZ}(X, x)$ defines in a canonical way a valuation of $\widetilde{K}$ whose valuation ring dominates $\widetilde{\mathcal{O}}_{X,x}$. This turns out to be true. In order to show this fact we rely on a deep result of [HOST12] on the extension of valuations to a local étale ring extension, under the assumption of the excellence of the local domain.

Our second main result is the following:

Theorem A'. *Let X, Y be two algebraic varieties defined over the same algebraically closed field k. For all regular closed points $x \in X$, $y \in Y$, the spaces* $\mathrm{RZ}(X,x)$ *and* $\mathrm{RZ}(Y,y)$ *are homeomorphic if and only if X and Y have the same dimension.*

Two consequences of this statement are particularly noteworthy. On the one hand, this result shows that the homeomorphism type of $\mathrm{RZ}(X,x)$ and $\mathrm{NL}(X,x)$ depends only on the dimension of the variety X and the base field k. On the other hand, assuming that resolution of singularities holds, it reveals the *self-homeomorphic* structure of $\mathrm{RZ}(X,x)$ by considering a projective system of non-singular varieties. This property is also observed in the valuative tree and in the space of real places of $L(y)$ where $L = \mathbf{R}((t^{\mathbf{Q}}))$ (see [FJ04, Theorem 6.51] and [Kuh13, Corollary 21] respectively). We make this precise in Section 2.3.

Next, we consider a singular point x of a normal algebraic surface X. First of all let us point out that the spaces $\mathrm{RZ}(X,x)$ and $\mathrm{NL}(X,x)$ have more structure than just topology. Actually they are both locally ringed spaces. The second carries a natural analytic structure locally modeled on affinoid spaces over $k((t))$. These local $k((t))$-analytic structures are not canonical and cannot in general be glued to get a global one. This structure was studied in [Fan14a] and shown to determine the completion of the local ring $\mathcal{O}_{X,x}$ (see [Fan14a, Corollary 4.14]). However it is only the underlying topological spaces that concerns us here. We introduce the notions necessary to state our main result concerning the homeomorphism type of $\mathrm{RZ}(X,x)$ and $\mathrm{NL}(X,x)$ in this case.

By a *graph* we mean a finite connected graph with at least one vertex, without loops and without multiple edges. Recall that a graph Γ is a purely combinatorial object which can be seen as a finite one-dimensional CW-complex. To be precise, we endow the set of vertices V of Γ and its set of edges E with the discrete topology and the unit interval $[0,1]$ with the induced topology from the standard topology of the real line. The topological space $|\Gamma|$, which we call the *topological realization* of Γ, is the quotient space of the disjoint union $\mathrm{V} \sqcup (\mathrm{E} \times [0,1])$ under the natural identifications $v \sim (e,0)$ and $v' \sim (e,1)$ given by incidence of vertices and edges.

We say that a graph is a *tree* if its topological realization is simply connected. Following [Sta83] we associate to any graph its core (see also the definition of the skeleton of a quasipolyhedron given in [Ber90]). The *core of a graph* Γ which is not a tree is the subgraph of Γ obtained by repeatedly deleting a vertex of degree one and the edge incident to it, until no vertex of degree one remains. Recall that the degree of a vertex is the number of edges connected to it. We denote the core of Γ by $\mathrm{Core}\,(\Gamma)$.

Let Γ be a graph which is not a tree. Observe that if Γ has no vertex of degree one, then Γ is its own core. Note also that $|\Gamma|$ admits a deformation retraction to $|\mathrm{Core}\,(\Gamma)\,|$. The complement of $|\mathrm{Core}\,(\Gamma)\,|$ in $|\Gamma|$ is the set of points in $|\Gamma|$ which admit an open neighborhood whose closure is a tree and whose boundary is reduced to a vertex of Γ. We may thus think of Γ as its core with some disjoint trees attached to it.

We introduce an equivalence relation in the set of graphs on which the characterization given in our result relies. Two graphs Γ and Γ' are *equivalent* if either they are both trees or neither is a tree and $|\mathrm{Core}\,(\Gamma)\,|$ is homeomorphic to $|\mathrm{Core}\,(\Gamma')\,|$. Observe that this equivalence relation is stricter than the homotopy equivalence. The three graphs consisting of two triangles sharing a vertex, two triangles sharing a side, and a line segment with a triangle attached to each endpoint, have all homotopy equivalent topological realizations but are not pairwise equivalent.

Recall that a proper birational map $\pi_{X'} : X' \to X$ is a *good resolution* if X' is smooth and the exceptional locus $E_{X'} = \pi_{X'}^{-1}(x)_{\mathrm{red}}$ is a divisor with normal crossing singularities such that its irreducible components are smooth and the intersection of any two of them is at most a point. To any good resolution is associated its *dual graph* $\Gamma_{X'}$ whose vertices are in bijection with the irreducible components of $E_{X'}$ and where two vertices are adjacent if and only if the corresponding irreducible components of $E_{X'}$ intersect.

Our main result in this setting is the following:

Theorem B. *Let $x \in X$ and $y \in Y$ be singular points of normal algebraic surfaces defined over an algebraically closed field k and $\Gamma_{X'}$, $\Gamma_{Y'}$ the dual graphs associated to two good resolutions of (X, x) and (Y, y) respectively. The following statements are equivalent:*

1. *The spaces* $\mathrm{RZ}(X, x)$ *and* $\mathrm{RZ}(Y, y)$ *are homeomorphic.*

2. *The spaces* $\mathrm{NL}(X, x)$ *and* $\mathrm{NL}(Y, y)$ *are homeomorphic.*

3. *The graphs* $\Gamma_{X'}$ *and* $\Gamma_{Y'}$ *are equivalent.*

This statement implies that the spaces of valuations $\mathrm{RZ}(X, x)$ and $\mathrm{NL}(X, x)$ associated to *any* rational surface singularity (X, x) are homeomorphic to $\mathrm{RZ}(\mathbf{A}^2_k, 0)$ and $\mathrm{NL}(\mathbf{A}^2_k, 0)$ respectively. In order to obtain more precise information on the singularity (X, x) it will be necessary to explore finer structures of $\mathrm{RZ}(X, x)$.

As we stated before, $\mathrm{NL}(X, x)$ is the largest Hausdorff quotient of $\mathrm{RZ}(X, x)$. Therefore if *1* holds then $\mathrm{NL}(X, x)$ is homeomorphic to $\mathrm{NL}(Y, y)$. Let us now explain briefly our strategy to prove that *2* implies *3* and that *3* implies *1*.

The topological realization of any dual graph $\Gamma_{X'}$ can be embedded into $\mathrm{NL}(X, x)$ as a closed set and there exists a continuous *retraction map* $\mathrm{r}_{X'} : \mathrm{NL}(X, x) \to |\Gamma_{X'}|$ (see [Fav10]). The key observation is the following: any fiber $\mathrm{r}_{X'}^{-1}(\nu)$ under $\mathrm{r}_{X'}$ is a tree whose boundary is reduced to ν. A proof of this statement follows from [FJ04, Theorem 6.51]. In fact, it is shown in [Fan14a, Proposition 9.5 (i)] that the fiber $\mathrm{r}_{X'}^{-1}(\nu)$ is an analytic disk when endowed with its canonical analytic structure. We mean here by a *tree* a topological space which is homeomorphic to a rooted nonmetric $\mathbf{R}$-tree in the sense of [FJ04]. If a graph is a tree, then it is also a tree in this sense.

We define the *core* of $\mathrm{NL}(X, x)$ as the set of all points in $\mathrm{NL}(X, x)$ which do not admit an open neighborhood whose closure is a tree and whose boundary is reduced to a single point of $\mathrm{NL}(X, x)$ and we denote it by $\mathrm{Core}\,(\mathrm{NL}(X, x))$. In [Ber90] the core is referred to as the skeleton. Observe that by definition $\mathrm{Core}\,(\mathrm{NL}(X, x))$ is empty if and only if $\mathrm{NL}(X, x)$ is a tree. We show that, given a good resolution $\pi_{X'} : X' \to X$, the space $\mathrm{NL}(X, x)$ is a tree if and only if $\Gamma_{X'}$ is a tree. Furthermore, if neither is a tree then we have $\mathrm{Core}\,(\mathrm{NL}(X, x)) = |\mathrm{Core}\,(\Gamma_{X'})\,|$ as subspaces of $\mathrm{NL}(X, x)$. It is straightforward to deduce from this that *2* implies *3*.

Finally, let us present a sketch of the proof that *3* implies *1*, which is the most delicate part of the proof of Theorem B. We start with two good resolutions $\pi_{X'} : X' \to X$ and $\pi_{Y'} : Y' \to Y$, and suppose that their dual graphs are equivalent. Our goal is to construct an homeomorphism from $\mathrm{RZ}(X,x)$ to $\mathrm{RZ}(Y,y)$. We first construct two good resolutions $\pi_{X''} : X'' \to X$ and $\pi_{Y''} : Y'' \to Y$ which factor through $\pi_{X'}$ and $\pi_{Y'}$ respectively and such that $\Gamma_{X''}$ and $\Gamma_{Y''}$ are isomorphic graphs. This isomorphism determines a natural bijection between the irreducible components $\{E_i\}_{i=1}^m$ of $E_{X''}$ and those, say $\{D_i\}_{i=1}^m$, of $E_{Y''}$. We map the divisorial valuation in $\mathrm{RZ}(X,x)$ defined by E_i to the divisorial valuation in $\mathrm{RZ}(Y,y)$ defined by D_i. Thus, in order to define a bijection from $\mathrm{RZ}(X,x)$ to $\mathrm{RZ}(Y,y)$ it suffices to concentrate on the valuations having as center in X'' a closed point. To do so we choose a bijection σ between the set of closed points of $E_{X''}$ and $E_{Y''}$ such that $\sigma(E_i \cap E_j) = D_i \cap D_j$ and $\sigma(E_i) \subseteq D_i$. The idea is to apply Theorem A' to obtain an homeomorphism from $\mathrm{RZ}(X'',x'')$ to $\mathrm{RZ}(Y'',\sigma(x''))$. The construction of the bijection from $\mathrm{RZ}(X,x)$ to $\mathrm{RZ}(Y,y)$ using this idea requires a careful local study at the points of $E_{X''}$. The fact that it is an homeomorphism then follows by examination of the behaviors of sequences of centers and their images by σ.

We now outline the contents of the chapters:

In Chapter 1 we introduce the spaces of valuations we deal with in this work. Section 1.1 is devoted to Riemann-Zariski spaces. We start by recalling some basic facts about valuations. Then we present the main topological properties of the Riemann-Zariski space $\mathrm{Z}(K|R)$ associated to a field K and a subring R of K. The study of this space has received in the last few years some special attention (see [FFL13a, FFL13b, Olb15]). We explain in particular that it is a *spectral space* and we also consider $\mathrm{Z}(K|R)$ endowed with its *constructible topology*. Next we restrict ourselves to the case where K is the function field of X and R is the base field k. In this geometrical context the space $\mathrm{Z}(K|R)$ is denoted by $\mathfrak{X}$. We discuss briefly its locally ringed structure. Then we focus on the subspace of $\mathfrak{X}$ consisting of all valuations having a center in X, which is the projective limit of the projective system of all proper birational models of X, and the Riemann-Zariski space $\mathrm{RZ}(X,x)$ of X at a closed point x. We exhibit some topological properties of $\mathrm{RZ}(X,x)$, for instance its connectedness, and we compute its Krull dimension.

In Section 1.2 we turn to spaces of semivaluations. We consider the space $\mathcal{V}(A,\mathfrak{p})$ of *normalized semivaluations* associated to an integral domain A and an ideal $\mathfrak{p}$ of A and develop some of its topological features. The most significant example is the valuative tree. Then we concentrate on the normalized non-Archimedean link $\mathrm{NL}(X,x)$ of x in X. The section finishes with the study of the relationship between this space and $\mathrm{RZ}(X,x)$. This relation leads us to study the largest Hausdorff quotient of $\mathrm{RZ}(X,x)$.

The purpose of Chapter 2 is to give the proofs of Theorems A and A'. In Sections 2.1 and 2.3 we present detailed proofs of the statements which are needed to complete those proofs and which we have already mentioned. Since the henselization of a local ring appears as an essential tool, in Section 2.2 we include basic definitions and some results involving it useful in the sequel. Chapter 3 provides a short discussion about trees and graphs. More precisely, in Section 3.2 we define the equivalence relation on the set of graphs in two different ways and we show that they are equivalent. Finally, Chapter 4 contains the proof of Theorem B.

Resumen

Las valoraciones son una herramienta fundamental de la geometría algebraica. Históricamente jugaron un papel importante en [DW82] donde fueron introducidas para construir algebraicamente la superficie de Riemann asociada a un cuerpo de funciones algebraicas en una variable. Posteriormente Zariski generalizó esta idea y las valoraciones pasaron a ser un ingrediente esencial en su enfoque del problema de resolución de singularidades de una variedad algebraica X. En [Zar44], Zariski dota de una topología el conjunto $\mathfrak{X}$ de todos los anillos de valoración del cuerpo de funciones racionales K de X que contienen el cuerpo base y establece su casi-compacidad. Este es un punto clave en su estrategia para la resolución. Asimismo resulta ser un resultado importante en algunos intentos recientes de resolver este problema en característica positiva (siguiendo nuevas vías usando también uniformización local, ver [CP08, Tei14]).

En este trabajo suponemos que X es una variedad algebraica definida sobre un cuerpo algebraicamente cerrado k (es decir, un esquema separado íntegro de tipo finito sobre k) y fijamos un punto cerrado x de X. Iniciamos el estudio del tipo de homeomorfismo del espacio $\mathrm{RZ}(X, x)$ formado por todos los anillos de valoración de $\mathfrak{X}$ que dominan al anillo local $\mathcal{O}_{X,x}$, dotado de la topología introducida por Zariski. Llamamos a $\mathrm{RZ}(X, x)$ el *espacio de Riemann-Zariski* de X en x. Nuestro objetivo es clarificar la relación entre las propiedades topológicas de este espacio y la geometría local de X en x. Dicha relación es bien conocida en dimensión uno. Si X es una curva algebraica $\mathrm{RZ}(X, x)$ está en biyección con las ramas analíticas de X en x. Sin embargo la situación es más rica en dimensión superior.

Preocupaciones similares han aparecido en el marco de la teoría de espacios analíticos desarrollada por Berkovich y otros autores según [Ber90]. Adoptando este punto de vista asociamos a X su *analytification* X^{an}. Un punto de X^{an} es un valor absoluto definido en el cuerpo residual de un punto de X y que extiende el valor absoluto trivial de k (un valor absoluto da lugar a una valoración de rango uno tomando menos el logaritmo). Sea $\mathrm{L}(X, x)$ el subespacio de X^{an} formado por todos los puntos que se especializan en x, excepto el trivial. Una cualidad interesante de este espacio, establecida por Thuillier en [Thu07], es que tiene el mismo tipo de homotopía que el complejo dual asociado a una resolución de singularidades de (X, x) cuyo divisor excepcional es un divisor con cruzamientos normales simples.

De hecho, el espacio $\mathrm{RZ}(X, x)$ está íntimamente relacionado con el *link no arquimediano normalizado* $\mathrm{NL}(X, x)$ de x en X, obtenido a partir de $\mathrm{L}(X, x)$ identificando puntos que definen valoraciones equivalentes (ver [Fan14a]). Si X se reduce al punto x entonces el espacio de Riemann-Zariski $\mathrm{RZ}(X, x)$ es también un punto; por su parte, $\mathrm{NL}(X, x)$ es vacío. Cuando X es una curva, $\mathrm{RZ}(X, x)$ y $\mathrm{NL}(X, x)$ son espacios finitos del mismo

cardinal. Por tanto podemos suponer que las variedades algebraicas que consideramos son de dimensión al menos dos. Obsérvese que estos espacios disfrutan de propiedades topológicas diferentes: $\mathrm{NL}(X,x)$ es compacto mientras que $\mathrm{RZ}(X,x)$ no es Hausdorff. En general existe una aplicación canónica continua y sobreyectiva de $\mathrm{RZ}(X,x)$ en $\mathrm{NL}(X,x)$, y este último espacio es el mayor cociente Hausdorff del primero en el caso de superficies normales (esto no es cierto en dimensión superior). Demostraciones detalladas de estas afirmaciones pueden encontrarse en las Subseciones 1.2.3 y 1.2.4 respectivamente.

En primer lugar abordamos el caso regular. Sea d la dimensión de X. Recordemos que la dimensión de Krull de un espacio topológico es el supremo de las longitudes de todas las cadenas de subespacios cerrados irreducibles del mismo. Como $\mathrm{NL}(X,x)$ es Hausdorff, su dimension de Krull es cero. Miramos en su lugar la dimension de recubrimiento, definida como en [Pea75], y probamos que $\mathrm{NL}(X,x)$ tiene dimensión de recubrimiento $d-1$. Dado un segundo punto cerrado regular $y \in Y$ de una variedad algebraica definida sobre el mismo cuerpo k, si $\mathrm{NL}(X,x)$ es homeomorfo a $\mathrm{NL}(Y,y)$, entonces este resultado sobre la dimensión implica que las variedades X e Y son de la misma dimensión. El recíproco también se verifica. Supongamos que X e Y son ambas de dimension d. Por el Teorema de Cohen, existe un isomorfismo de k-álgebras σ del completado $\widehat{\mathcal{O}}_{X,x}$ del anillo local $\mathcal{O}_{X,x}$ en el completado $\widehat{\mathcal{O}}_{Y,y}$ de $\mathcal{O}_{Y,y}$ (ambos anillos son isomorfos al anillo de series formales en d variables con coeficientes en k). Esto nos permite definir una biyección entre las semivaloraciones de $\widehat{\mathcal{O}}_{Y,y}$ y las de $\widehat{\mathcal{O}}_{X,x}$ mediante composición con σ. Entendemos aquí por una semivaloración una aplicación del anillo en $[0,+\infty]$ que verifica los axiomas propios de una valoración, pero que puede enviar a infinito elementos diferentes del cero.

Obsérvese que $\mathcal{O}_{X,x}$ y $\mathcal{O}_{Y,y}$ no son necesariamente isomorfos como k-álgebras, ni siquiera cuando tienen el mismo cuerpo de fracciones. En efecto, supongamos que X es no singular y consideremos dos puntos cerrados distintos $x, x' \in X$. Si existe un isomorfismo entre los anillos locales de X en estos puntos, entonces podemos encontrar un isomorfismo $U \to U'$ que envía x en x', donde $U, U' \subset X$ son entornos abiertos de x y x' respectivamente. Tenemos por tanto una aplicación birracional de X en sí mismo enviando x en x'. Si X es de tipo general y el cuerpo base es algebraicamente cerrado de característica cero, el grupo de automorfismos birracionales de X es finito (ver [HMX13] para una cota). En este caso solo existe un número finito de puntos x' para los que $\mathcal{O}_{X,x}$ puede ser isomorfo como k-álgebra a $\mathcal{O}_{X,x'}$.

Volvamos a nuestro problema. El mismo se reduce ahora al estudio de las extensiones de una semivaloración de $\mathcal{O}_{X,x}$ a una semivaloración de su completado. Demostramos que un punto ν de $\mathrm{NL}(X,x)$ define de forma canónica una semivaloración convenientemente normalizada $\widehat{\nu}$ de $\widehat{\mathcal{O}}_{X,x}$ cuya restricción a k es trivial. Basta definir $\widehat{\nu}(f) = \lim_{n\to+\infty} \nu(f_n)$, donde $(f_n)_{n=1}^{\infty}$ es una sucesión de Cauchy en $\mathcal{O}_{X,x}$ que converge a f. Este resultado es cierto siempre que x es analíticamente irreducible, esto es, si $\widehat{\mathcal{O}}_{X,x}$ es un dominio de integridad.

Nuestro primer resultado principal es el siguiente:

Teorema A. *Sean X, Y dos variedades algebraicas definidas sobre el mismo cuerpo algebraicamente cerrado k. Para todo par de puntos cerrados regulares $x \in X$, $y \in Y$, los espacios $\mathrm{NL}(X,x)$ y $\mathrm{NL}(Y,y)$ son homeomorfos si y solo si X y Y son de la misma dimensión.*

En el caso de dimensión dos podemos ser más específicos. Un modelo topológico para $\mathrm{NL}(\mathbf{A}^2_{\mathbf{C}}, 0)$ ha sido propuesto en [FJ04, Section 3.2.3]. En esta monografía el link no arquimediano normalizado del origen en el plano complejo recibe el nombre de *árbol valorativo*. Este espacio cuenta con una estructura afín canónica que permite realizar análisis convexo y que encuentra aplicaciones interesantes en dinámica y análisis complejo en [FJ07, FJ05]. Más precisamente, posee estructura de *árbol real no métrico con raíz* (ver [FJ04, Definition 3.1 y Theorem 3.14] y [Nov14]). A grandes rasgos se trata de un espacio topológico donde cualesquiera dos puntos están unidos por un único intervalo real. Dicha estructura ha sido extendida en [Gra07] al caso de un punto cerrado regular de una superficie. Por su parte, el tipo de homeomorfismo de una curva de Berkovich arbitraria ha sido también tratado en [HLP14] bajo hipótesis de numerabilidad del cuerpo base. Puesto que $\mathrm{NL}(\mathbf{A}^2_k, 0)$ es homeomorfo a la bola unidad cerrada sobre el cuerpo $k(\!(t)\!)$ (equipado con su valoración discreta), este resultado muestra que $\mathrm{NL}(\mathbf{A}^2_k, 0)$ es una dendrita universal de Ważewski cuando k es numerable.

Tratemos ahora el problema en el marco Riemann-Zariski. Es sabido que $\mathfrak{X}$ tiene la misma dimensión de Krull que X. Demostramos que $\mathrm{RZ}(X, x)$ tiene dimensión de Krull $d-1$. De nuevo este resultado implica que dos variedades X e Y definidas sobre k son de la misma dimensión cuando podemos encontrar puntos cerrados regulares $x \in X$, $y \in Y$ tales que $\mathrm{RZ}(X, x)$ y $\mathrm{RZ}(Y, y)$ son homeomorfos. El recíproco también es cierto, aunque la demostración en este caso es más complicada. En un intento de reproducir la prueba dada para el link no arquimediano normalizado, nos interesamos primeramente por el comportamiento de una valoración de K cuyo anillo de valoración domina $\mathcal{O}_{X,x}$ cuando pasamos al completado de este último.

Suponiendo que x es un punto analíticamente normal de una superficie (esto es, el completado $\widehat{\mathcal{O}}_{X,x}$ es normal), en [Spi90b] se demuestra que una valoración de $\mathrm{RZ}(X, x)$ se extiende de forma única a una valoración del cuerpo de fracciones de $\widehat{\mathcal{O}}_{X,x}$ cuyo anillo de valoración domina dicho anillo. Este resultado se aplica en nuestra situación puesto que todo punto regular es analíticamente normal. Sin embargo este teorema sobre extensión de valoraciones no es cierto en dimensión mayor que dos. Dicha extensión siempre existe, pero no es difícil construir ejemplos explícitos donde no es única. Es más, o bien existe una única extensión, o bien hay infinitas (ver [HS91]).

Necesitamos por lo tanto adoptar una estrategia diferente. Recordemos que el henselizado $\widetilde{\mathcal{O}}_{X,x}$ de $\mathcal{O}_{X,x}$ es el límite inductivo del sistema de $\mathcal{O}_{X,x}$-álgebras locales étales equiresiduales. Sea $U \subseteq X$ un entorno afín abierto de x dado por una k-álgebra finitamente generada $A = k[T_1, \ldots, T_n]/I$ de dimensión d. El Lema de normalización de Noether afirma que existen d elementos $x_1, \ldots, x_d$ de A, algebraicamente independientes sobre k, tales que A es un $k[x_1, \ldots, x_d]$-módulo finitamente generado y $\mathfrak{m} \cap k[x_1, \ldots, x_d] = (x_1, \ldots, x_d)$, donde $\mathfrak{m}$ es el ideal maximal de A correspondiente al punto x. Cada x_i puede suponerse igual a la clase de una combinación k-lineal genérica $\sum_{1 \le i \le n} \alpha_{ij} T_j$ módulo I (ver [Eis95, Theorem 13.3]). Se puede probar que además es posible tomar dichos elementos de modo que la aplicación tangente $\mathrm{T}_{U,x} \to \mathrm{T}_{\mathbf{A}^d_k, 0}$ sea inyectiva. Así, el morfismo finito de U en $\mathbf{A}^d_k$ es étale en el punto x. Luego $\mathcal{O}_{U,x}$ es una $\mathcal{O}_{\mathbf{A}^d_k, 0}$-álgebra local étale equirresidual y $\widetilde{\mathcal{O}}_{X,x}$ y el henselizado de $k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$ coinciden.

Como consecuencia, para todo punto cerrado regular $y \in Y$ de una variedad algebraica de dimensión d definida sobre k, podemos encontrar un isomorfismo de k-álgebras σ de $\widetilde{\mathcal{O}}_{X,x}$

en el henselizado $\widetilde{\mathcal{O}}_{Y,y}$ de $\mathcal{O}_{Y,y}$. Sea $\widetilde{K}$ el cuerpo de fracciones de $\widetilde{\mathcal{O}}_{X,x}$ (obsérvese que $\widetilde{\mathcal{O}}_{X,x}$ es un subanillo de $\widehat{\mathcal{O}}_{X,x}$, que es un dominio de integridad por hipótesis). La observación anterior nos permite definir una biyección entre las valoraciones de $\widetilde{K}$ cuyo anillo de valoración domina $\widetilde{\mathcal{O}}_{Y,y}$ y aquellas cuyo anillo de valoración domina $\widetilde{\mathcal{O}}_{X,x}$, mediante composición con σ. Parece natural ahora preguntarse si una valoración de $\mathrm{RZ}(X,x)$ determina de forma canónica una valoración de $\widetilde{K}$ cuyo anillo de valoración domina $\widetilde{\mathcal{O}}_{X,x}$. La respuesta es afimativa. Para demostrarlo nos apoyamos en un resultado profundo de [HOST12] sobre extensión de valoraciones a una extensión local étale bajo la hipótesis de excelencia del anillo local.

Nuestro segundo resultado principal es el siguiente:

Teorema A'. *Sean X, Y dos variedades algebraicas definidas sobre el mismo cuerpo algebraicamente cerrado k. Para todo par de puntos cerrados regulares $x \in X$, $y \in Y$, los espacios $\mathrm{RZ}(X,x)$ y $\mathrm{RZ}(Y,y)$ son homeomorfos si y solo si X e Y son de la misma dimensión.*

De este teorema se derivan dos consecuencias particularmente notables. Por un lado demuestra que el tipo de homeomorfismo de $\mathrm{RZ}(X,x)$ y $\mathrm{NL}(X,x)$ depende únicamente de la dimensión de la variedad X y del cuerpo base k. Por otro lado, admitiendo la existencia de resolución de singularidades, pone de manifiesto la estructura *auto-homeomorfa* de $\mathrm{RZ}(X,x)$ si consideramos un sistema proyectivo de variedades no singulares. Esta propiedad se observa también en el árbol valorativo y en el espacio de lugares reales de $L(y)$ donde $L = \mathbf{R}(\!(t^{\mathbf{Q}})\!)$ (ver [FJ04, Theorem 6.51] y [Kuh13, Corollary 21] respectivamente). Precisamos esta noción en la Sección 2.3.

A continuación consideramos un punto singular x de una superficie algebraica normal X. Llegados a este punto queremos resaltar que los espacios $\mathrm{RZ}(X,x)$ y $\mathrm{NL}(X,x)$ tienen una estructura más rica que simplemente una topología. De hecho ambos son espacios localmente anillados. El segundo posee además una estructura analítica natural localmente modelada en espacios afinoides sobre $k(\!(t)\!)$. Estas estructuras $k(\!(t)\!)$-analíticas locales no son canónicas y en general no pueden pegarse para obtener una global. Dicha estructura ha sido estudiada en [Fan14a], donde se demuestra que determina el completado del anillo local $\mathcal{O}_{X,x}$ (ver [Fan14a, Corollary 4.14]). Sin embargo, son solo los espacios topológicos subyacentes los que nos ocupan aquí. Introducimos ahora las nociones necesarias para enunciar nuestro resultado principal sobre el tipo de homeomorfismo de $\mathrm{RZ}(X,x)$ y $\mathrm{NL}(X,x)$ en este caso.

Un *grafo* es un grafo finito, conexo, con al menos un vértice, sin bucles y sin aristas múltiples. Recordemos que un grafo Γ es un objeto puramente combinatorio que puede ser visto como un CW-complejo finito de dimensión uno. Más precisamente, dotamos el conjunto de vértices V y el conjunto de aristas E de Γ de la topología discreta, y el intervalo $[0,1]$ de la topología inducida por la topología usual de la recta real. El espacio topológico $|\Gamma|$, que llamamos la *realización topológica* de Γ, es el espacio cociente obtenido a partir de la unión disjunta $\mathrm{V} \sqcup (\mathrm{E} \times [0,1])$ mediante las identificaciones naturales $v \sim (e,0)$ y $v' \sim (e,1)$ dadas por las indicencias entre aristas y vértices.

Decimos que un grafo es un *árbol* si su realización topológica es simplemente conexa. De acuerdo con [Sta83], asociamos a todo grafo su núcleo (ver también la definición del esqueleto de un casipoliedro dada en [Ber90]). El *núcleo de un grafo* Γ que no es un árbol es el subgrafo de Γ obtenido a partir de Γ por eliminaciones sucesivas de un vértice de grado uno y la arista incidente a él, hasta que no queden vértices de grado uno. Entendemos por el grado de un vértice el número de aristas incidentes a él. Denotamos el núcleo de Γ como $\mathrm{Core}(\Gamma)$.

Sea Γ un grafo que no es un árbol. Obsérvese que si Γ no tiene vértices de grado uno entonces Γ es su propio núcleo. Obsérvese también que $|\Gamma|$ admite una retracción por deformación en $|\mathrm{Core}(\Gamma)|$. El complementario de $|\mathrm{Core}(\Gamma)|$ en $|\Gamma|$ es el conjunto de puntos de $|\Gamma|$ que admiten un entorno abierto cuya clausura es un árbol y cuya frontera se reduce a un vértice de Γ. Por tanto podemos imaginar Γ como su núcleo con algunos árboles disjuntos "colgando" de él.

Introducimos a continuación la relación de equivalencia en el conjunto de grafos en la que está basada la caracterización que damos en nuestro resultado. Dos grafos Γ y Γ' son *equivalentes* si ambos son árboles, o bien ninguno es un árbol y $|\mathrm{Core}(\Gamma)|$ es homeomorfo a $|\mathrm{Core}(\Gamma')|$. Esta relación es más estricta que la equivalencia de homotopía. Los tres grafos siguientes tienen realizaciones topológicas homotópicamente equivalentes, pero no son dos a dos equivalentes: dos triángulos compartiendo un vértice, dos triángulos compartiendo un lado, y un segmento con un triángulo pegado en cada vértice.

Recordemos que un morfismo propio birracional $\pi_{X'} : X' \to X$ es una *buena resolución* si X' es no singular y el lugar excepcional $E_{X'} = \pi_{X'}^{-1}(x)_{\mathrm{red}}$ es un divisor con cruzamientos normales tal que sus componentes irreducibles son lisas y la intersección de cualesquiera dos de ellas es a lo sumo un punto. A toda buena resolución se le asocia su *grafo dual* $\Gamma_{X'}$, cuyos vértices están en biyección con las componentes irreducibles de $E_{X'}$ y donde dos vértices son adyacentes si y solo si las correspondientes componentes irreducibles de $E_{X'}$ se intersectan.

Nuestro resultado principal en este contexto es el siguiente:

Teorema B. *Sean $x \in X$, $y \in Y$ puntos singulares de superficies algebraicas normales definidas sobre un cuerpo algebraicamente cerrado k, y $\Gamma_{X'}$, $\Gamma_{Y'}$ los grafos duales asociados a dos buenas resoluciones de (X,x) e (Y,y) respectivamente. Las siguientes afirmaciones son equivalentes:*

1. *Los espacios $\mathrm{RZ}(X,x)$ y $\mathrm{RZ}(Y,y)$ son homeomorfos.*
2. *Los espacios $\mathrm{NL}(X,x)$ y $\mathrm{NL}(Y,y)$ son homeomorfos.*
3. *Los grafos $\Gamma_{X'}$ y $\Gamma_{Y'}$ son equivalentes.*

Este teorema implica que los espacios de valoraciones $\mathrm{RZ}(X,x)$ y $\mathrm{NL}(X,x)$ asociados a *cualquier* singularidad racional de superficie (X,x) son homeomorfos a $\mathrm{RZ}(\mathbf{A}_k^2, 0)$ y $\mathrm{NL}(\mathbf{A}_k^2, 0)$ respectivamente. Para obtener información más precisa sobre la singularidad (X,x) será necesario explorar estructuras más finas de $\mathrm{RZ}(X,x)$.

Como dijimos anteriormente, $\mathrm{NL}(X,x)$ es el mayor cociente Hausdorff de $\mathrm{RZ}(X,x)$. Por lo tanto si la afirmación *1* del Teorema B se verifica, entonces $\mathrm{NL}(X,x)$ es homeomorfo a $\mathrm{NL}(Y,y)$. En las siguientes líneas explicamos brevemente nuestra estrategia para probar que *2* implica *3* y que *3* implica *1*.

La realización topológica de todo grafo dual $\Gamma_{X'}$ puede verse como un subespacio cerrado de $\mathrm{NL}(X,x)$ y existe una *aplicación retracción* continua $\mathrm{r}_{X'} : \mathrm{NL}(X,x) \to |\Gamma_{X'}|$ (ver [Fav10]). La idea clave de la demostración es ver que la fibra $\mathrm{r}_{X'}^{-1}(\nu)$ de $\mathrm{r}_{X'}$ es un árbol cuya frontera se reduce a ν, para toda $\nu \in \mathrm{NL}(X,x)$. Esta afimación es consecuencia de [FJ04, Theorem 6.51]. Es más, en [Fan14a, Proposition 9.5 (i)] se prueba que la fibra $\mathrm{r}_{X'}^{-1}(\nu)$ es un disco analítico cuando es dotada de su estructura analítica canónica. Entendemos aquí que un *árbol* es un espacio topológico que es homeomorfo a un árbol real no métrico con raíz en el sentido de [FJ04]. Si un grafo es un árbol entonces también lo es con esta definición.

Definimos el *núcleo* de $\mathrm{NL}(X,x)$ como el subconjunto de $\mathrm{NL}(X,x)$ formado por los puntos que no admiten un entorno abierto cuya clausura es un árbol y cuya frontera se reduce a un único punto de $\mathrm{NL}(X,x)$. Lo denotamos por $\mathrm{Core}\,(\mathrm{NL}(X,x))$ (en [Ber90] el núcleo es llamado el esqueleto). Obsérvese que por definición $\mathrm{Core}\,(\mathrm{NL}(X,x))$ es vacío si y solo si $\mathrm{NL}(X,x)$ es un árbol. Demostramos que, dada una buena resolución $\pi_{X'} : X' \to X$, el espacio $\mathrm{NL}(X,x)$ es un árbol si y solo si $\Gamma_{X'}$ es un árbol. Además, si ninguno es un árbol entonces tenemos $\mathrm{Core}\,(\mathrm{NL}(X,x)) = |\mathrm{Core}\,(\Gamma_{X'})\,|$ como subespacios de $\mathrm{NL}(X,x)$. De aquí se deduce directamente que *2* implica *3*.

Por último, presentamos un resumen de la demostración de que *3* implica *1*, que es la parte más delicada de la prueba del Teorema B. Comenzamos con dos buenas resoluciones $\pi_{X'} : X' \to X$ y $\pi_{Y'} : Y' \to Y$ cuyos grafos duales son equivalentes. Nuestro objetivo es construir un homeomorfismo de $\mathrm{RZ}(X,x)$ en $\mathrm{RZ}(Y,y)$. En primer lugar construimos dos buenas resoluciones $\pi_{X''} : X'' \to X$ y $\pi_{Y''} : Y'' \to Y$ que factorizan por $\pi_{X'}$ y $\pi_{Y'}$, respectivamente y tales que $\Gamma_{X''}$ y $\Gamma_{Y''}$ son grafos isomorfos. Este isomorfismo determina una biyección entre las componentes irreducibles $\{E_i\}_{i=1}^m$ de $E_{X''}$ y las de $E_{Y''}$, que llamamos $\{D_i\}_{i=1}^m$. Envíamos la valoración divisorial de $\mathrm{RZ}(X,x)$ definida por E_i a la valoración divisorial de $\mathrm{RZ}(Y,y)$ definida por D_i. Así, para definir una biyección de $\mathrm{RZ}(X,x)$ en $\mathrm{RZ}(Y,y)$ basta con que nos concentremos en las valoraciones que tienen como centro en X'' un punto cerrado. Con este fin, elegimos una biyección σ entre el conjunto de puntos cerrados de $E_{X''}$ y el de $E_{Y''}$ cumpliendo $\sigma(E_i \cap E_j) = D_i \cap D_j$ y $\sigma(E_i) \subseteq D_i$. Aplicando el Teorema A' obtenemos un homeomorfismo de $\mathrm{RZ}(X'',x'')$ en $\mathrm{RZ}(Y'',\sigma(x''))$. La construcción de la biyección de $\mathrm{RZ}(X,x)$ en $\mathrm{RZ}(Y,y)$ siguiendo esta idea require un estudio local minucioso en los puntos de $E_{X''}$. El hecho de que sea además un homemorfismo se demuestra tras examinar el comportamiento de sucesiones de centros y sus imágenes por σ.

Resumimos ahora el contenido de los diferentes capítulos:

En el Capítulo 1 introducimos los espacios de valoraciones sobre los que trata este trabajo. La Sección 1.1 está dedicada a los espacios de Riemann-Zariski. Comenzamos con un recordatorio sobre propiedades básicas de las valoraciones. Seguidamente presentamos las principales propiedades topológicas del espacio de Riemann-Zariski $\mathrm{Z}(K|R)$ asociado a un cuerpo K y a un subanillo R del mismo. El estudio de este espacio ha recibido una atención especial en los últimos años (ver [FFL13a, FFL13b, Olb15]). En particular, explicamos que es un *espacio espectral* y consideramos $\mathrm{Z}(K|R)$ equipado con la topología constructible. A continuación nos restringimos al caso en que K es el cuerpo de funciones racionales de X y R el cuerpo base k. En este contexto geométrico, denotamos el espacio $\mathrm{Z}(K|R)$ por $\mathfrak{X}$. Tras comentar brevemente su estructura de espacio anillado, nos concentramos en el subespacio de $\mathfrak{X}$ formado por las valoraciones centradas en X, que es el límite proyectivo del sistema formado por todos los modelos propios birracionales de X, y en el espacio de Riemann-Zariski $\mathrm{RZ}(X,x)$ de X en un punto cerrado x. Exponemos algunas de las propiedades topológicas de $\mathrm{RZ}(X,x)$, como por ejemplo su conexidad, y calculamos su dimensión de Krull.

En la Sección 1.2 pasamos a los espacios de semivaloraciones. Consideramos el espacio $\mathcal{V}(A,\mathfrak{p})$ de *valoraciones normalizadas* asociado a un dominio de integridad A y a un ideal $\mathfrak{p}$ de A y ponemos de manifiesto algunas de sus propiedades topológicas. El ejemplo más significativo es el árbol valorativo. A continuación nos concentramos en el link no arquimediano normalizado $\mathrm{NL}(X,x)$ de x en X. La sección termina con el estudio de la relación entre este espacio y $\mathrm{RZ}(X,x)$. Dicha relación nos lleva a estudiar el mayor cociente Hausdorff de $\mathrm{RZ}(X,x)$.

El objetivo del Capítulo 2 es presentar las demostraciones de los Teoremas A y A'. En las Secciones 2.1 y 2.3 presentamos las pruebas de los resultados necesarios para completar dichas demostraciones y que hemos mencionado anteriormente. Puesto que la henselización de un anillo local es una herramienta fundamental, hemos incluido en la Sección 2.2 definiciones básicas y algunos resultados que utilizamos con posterioridad. El Capítulo 3 contiene una breve exposición sobre árboles y grafos. Más precisamente, en la Sección 3.2 definimos la relación de equivalencia en el conjunto de grafos de dos maneras diferentes y mostramos que coinciden. Por último, el Capítulo 4 contiene la demostración del Teorema B, que hemos resumido anteriormente.

Résumé

Les valuations sont un outil essentiel en géométrie algébrique. Historiquement elles ont joué un rôle important dans [DW82] où les valuations ont été introduites pour construire la surface de Riemann associée à un corps de fonctions algébriques d'une variable. Plus tard Zariski généralise cette idée et les valuations deviennent un ingrédient essentiel dans son approche du problème de la résolution des singularités d'une variété algébrique X. Dans [Zar44], il muni l'ensemble $\mathfrak{X}$ de tous les anneaux de valuation du corps de fonctions rationnelles K de X contenant le corps base d'une topologie et établit sa quasi-compacité. Ce fut un point clé de son programme pour la résolution. C'est également un résultat clé dans certaines tentatives récentes visant à résoudre ce problème en caractéristique positive, en suivant des nouvelles stratégies utilisant l'uniformisation locale (voir [CP08, Tei14]).

Dans ce travail nous supposons que X est une variété algébrique définie sur un corps algébriquement clos k (c'est-à-dire, un schéma séparé intègre de type fini sur k) et nous fixons un point fermé x de X. Nous initions l'étude du type d'homéomorphisme de l'espace $\mathrm{RZ}(X,x)$ formé par tous les anneaux de valuation de $\mathfrak{X}$ dominant l'anneau $\mathcal{O}_{X,x}$, muni de la topologie induite par la topologie de Zariski. Nous appelons $\mathrm{RZ}(X,x)$ *l'espace de Riemann-Zariski* de X dans x. Notre objectif est de clarifier la relation entre les propriétés topologiques de cet espace et la géométrie locale de X dans x. Notez que le cas unidimensionnel est bien compris. Si X est une courbe algébrique alors $\mathrm{RZ}(X,x)$ est en bijection avec les branches analytiques de X dans x. Cependant, la situation est plus riche en dimension supérieure.

Des préoccupations similaires sont apparues dans le cadre de la théorie des espaces analytiques tel qu'elle a été développé par Berkovich et autres après [Ber90]. En adoptant ce point de vue, on associe à X son analytifié X^{an}. Un point de X^{an} est une valeur absolue (qui donne lieu à une valuation en prenant moins le logarithme) sur le corps résiduel d'un point de X étendant la valeur absolue triviale de k. Nous pouvons considérer le sous-espace $\mathrm{L}(X,x)$ formé par tous les points de X^{an} qui se spécialisent en x à l'exception du trivial. Une propriété intéressante de cet espace, établi par Thuillier dans [Thu07], est qu'il a le type d'homotopie du complexe dual associé au diviseur exceptionnel d'une résolution des singularités de (X,x) dont le diviseur exceptionnel a des croisements normaux simples.

En fait, l'espace $\mathrm{RZ}(X,x)$ est étroitement liée à *l'entrelac non-Archimédien normalisé* $\mathrm{NL}(X,x)$ de x dans X qui est obtenu à partir de $\mathrm{L}(X,x)$ en identifiant des points définissant des valuations equivalentes (voir [Fan14a]). Si X est réduite à x, alors $\mathrm{RZ}(X,x)$ est un singleton et $\mathrm{NL}(X,x)$ est vide. Lorsque X est une courbe alors $\mathrm{RZ}(X,x)$ et $\mathrm{NL}(X,x)$ sont des espaces finis de la même cardinalité. Nous pouvons donc supposer que les variétés algébriques que nous considérons sont de dimension au moins deux. Notez que $\mathrm{NL}(X,x)$ est un espace compact tandis que $\mathrm{RZ}(X,x)$ est jamais séparé. En général, il y a une

surjection continue canonique de $\mathrm{RZ}(X, x)$ dans $\mathrm{NL}(X, x)$, et celui-ci est le plus grand quotient séparé du premier dans le cas des surfaces normales (ce n'est plus vrai en dimension supérieure). Une preuve détaillée de ces faits est donnée aux paragraphes 1.2.3 et 1.2.4 respectivement.

Nous abordons d'abord le cas régulier. Nous appelons d la dimension de X. Rappelons que la dimension de Krull d'un espace topologique est le supremum des longueurs de toutes les chaînes de sous-espaces fermés irréductibles de l'espace. Observez que $\mathrm{NL}(X, x)$ a dimension de Krull zéro car il est Hausdorff. Nous regardons à la place sa dimension de recouvrement tel que définie dans [Pea75], et nous montrons que $\mathrm{NL}(X, x)$ a pour dimension de recouvrement $d-1$. Si un autre point fermé $y \in Y$ d'une variété algébrique définie sur le même corps k est donné, et $\mathrm{NL}(X, x)$ est homéomorphe à $\mathrm{NL}(Y, y)$, alors ce résultat sur la dimension implique que X et Y ont la même dimension. La réciproque est également vraie. Supposons que X et Y ont tous les deux dimension d. D'après le Théorème de Cohen, il existe un isomorphisme de k-algèbres σ du completé $\widehat{\mathcal{O}}_{X,x}$ de l'anneau local $\mathcal{O}_{X,x}$ dans celui de $\mathcal{O}_{Y,y}$ (ils sont isomorphes à l'anneau des séries formelles en d variables avec des coefficients dans k). Cela nous permet de définir une bijection naturelle entre les semivaluations de $\widehat{\mathcal{O}}_{Y,y}$ et celles de $\widehat{\mathcal{O}}_{X,x}$ par composition avec σ. Par une semivaluation nous entendons ici une application de l'anneau dans $[0, +\infty]$ vérifient les axiomes des valuations mais qui peut envoyer à l'infini des éléments non nuls de l'anneau.

Observez que $\mathcal{O}_{X,x}$ et $\mathcal{O}_{Y,y}$ ne sont pas forcément isomorphes comme k-algèbres, même quand ils ont le même corps de fractions. Supposons que X est non singulière et prenons deux points fermés différents $x, x' \in X$. S'il existe un tel isomorphisme entre les anneaux locaux de X en ces points, nous pouvons trouver un isomorphisme $U \to U'$ qui envoie x dans x', où $U, U' \subset X$ sont des voisinages ouverts de x et x' respectivement. Ainsi, nous aurions une application birationnelle de X dans elle même en envoyant x dans x'. Si X est de type général et si le corps de base est algébriquement clos de caractéristique nulle, le groupe d'automorphismes birationnels de X est fini (voir [HMX13] pour une borne). Dans ce cas, il existe seulement un nombre fini de points x' pour lesquels $\mathcal{O}_{X,x}$ pourrait être isomorphe comme k-algèbre à $\mathcal{O}_{X,x'}$.

Revenons maintenant à notre problème. Observons qu'il a été réduit à l'étude des extensions d'une semivaluation de $\mathcal{O}_{X,x}$ à son complété. En fait, on peut montrer qu'un point ν de $\mathrm{NL}(X, x)$ définit d'une manière canonique une semivaluation convenablement normalisée $\widehat{\nu}$ de $\widehat{\mathcal{O}}_{X,x}$ dont la restriction à k est triviale. Il suffit de définir $\widehat{\nu}(f) = \lim_{n\to+\infty} \nu(f_n)$, où $(f_n)_{n=1}^{\infty}$ est une suite de Cauchy dans $\mathcal{O}_{X,x}$ qui converge vers f. Ce résultat est vrai lorsque le point x est analytiquement irréductible, c'est-à-dire, si $\widehat{\mathcal{O}}_{X,x}$ est intégre.

Notre premier résultat principal est le suivant:

Théorème A. *Soient X, Y deux variétés algébriques définies sur le même corps algébriquement clos k. Pour tous points fermés réguliers $x \in X$, $y \in Y$, les espaces* $\mathrm{NL}(X, x)$ *et* $\mathrm{NL}(Y, y)$ *sont homéomorphes si et seulement si X et Y ont la même dimension.*

En dimension deux on peut être plus précis. Un modèle topologique pour $\mathrm{NL}(\mathbf{A}^2_{\mathbf{C}}, 0)$ a déjà été proposé dans [FJ04, Section 3.2.3]. Dans cette monographie l'entrelac non-Archimédien normalisé de l'origine dans le plan affine complexe est dénommé *l'arbre valuatif.* Cet espace porte une structure affine canonique qui permet de faire de l'analysis convexe dessus et qui trouve des applications intéressantes en dynamique et en analyse complexe dans [FJ07, FJ05]. Plus précisément, il a une structure *d'arbre réel enraciné non métrique* (voir [FJ04, Definition 3.1 et Theorem 3.14] et [Nov14]). Grosso modo, ceci signifie que c'est un espace topologique où deux points différents sont reliés par un unique intervalle réel. Cette structure a été étendu en [Gra07] au cas d'un point fermé régulier d'une surface. Le type d' homéomorphisme d'une courbe arbitraire de Berkovich est également traité dans [HLP14] sous une hypothèse de dénombrabilité du corps de base. Étant donné que $\mathrm{NL}(\mathbf{A}^2_k, 0)$ est homéomorphe à la boule unité fermée sur le corps $k(\!(t)\!)$ (muni de sa valuation discrete), leur résultat montre que $\mathrm{NL}(\mathbf{A}^2_k, 0)$ est une dendrite universelle de Ważewski quand k est dénombrable.

Passons maintenant au cadre de Riemann-Zariski. Il est connu que $\mathfrak{X}$ a la même dimension de Krull que X. Nous montrons que $\mathrm{RZ}(X, x)$ a pour dimension de Krull $d-1$. Encore une fois ce fait implique que deux variétés X et Y définies sur k ont la même dimension lorsque $\mathrm{RZ}(X, x)$ et $\mathrm{RZ}(Y, y)$ sont homéomorphes pour certains points fermés réguliers $x \in X$, $y \in Y$. La réciproque est vraie aussi, cependant dans cette situation la démonstration est plus compliquée. Si l'on essaie de reproduire la preuve donnée dans le cas des entrelacs non-Archimédiens normalisés, nous sommes amenés à étudier les extensions d'une valuation de K dont l'anneau de valuation domine $\mathcal{O}_{X,x}$ au complété formel de l'anneau.

Lorsque x est un point analytiquement normal d'une surface (c'est-à-dire, le complété $\widehat{\mathcal{O}}_{X,x}$ est normal), il est montré dans [Spi90b] qu'une valuation de $\mathrm{RZ}(X, x)$ s'étend de manière unique à une valuation du corps des fractions de $\widehat{\mathcal{O}}_{X,x}$ dont l'anneau de valuation domine cet anneau. Ce résultat s'applique à notre situation car tout point régulier est analytiquement normal. Cependant cette affirmation à propos des extensions des valuations n'est plus vraie en dimension supérieure à deux. Une telle extension existe toujours, mais il n'est pas difficile de construire des exemples explicites où elle n'est pas unique. En fait, soit elle s'étend de façon unique soit il y a une infinité de telles extensions (voir [HS91]).

Par conséquent, on doit suivre une stratégie différente. Rappelons que l'hensélisé $\widetilde{\mathcal{O}}_{X,x}$ de $\mathcal{O}_{X,x}$ est la limite inductive du système des $\mathcal{O}_{X,x}$-algèbres locales étales equirésiduelles. Prenons un voisinage ouvert affine $U \subseteq X$ de x donné par une k-algèbre de type fini $A = k[T_1, \ldots, T_n]/I$ de dimension d. Le lemme de normalisation de Noether affirme qu'il existe des éléments $x_1, \ldots, x_d$ de A, qui sont algébriquement indépendants sur k, tels que A est un $k[x_1, \ldots, x_d]$-module de type fini et $\mathfrak{m} \cap k[x_1, \ldots, x_d] = (x_1, \ldots, x_d)$, où $\mathfrak{m}$ est l'idéal maximal de A correspondant au point x. En outre, chaque x_i peut être choisi pour être la classe résiduelle d'une combinaison k-linéaire générique $\sum_{1 \le i \le n} \alpha_{ij} T_j$ modulo I (voir [Eis95, Theorem 13.3]). On peut montrer qu'il est également possible de les choisir de manière à ce que l'application tangente $\mathrm{T}_{U,x} \to \mathrm{T}_{\mathbf{A}^d_k, 0}$ soit injective. Alors le morphisme fini de U vers $\mathbf{A}^d_k$ est étale dans le point x. Ensuite $\mathcal{O}_{U,x}$ est une $\mathcal{O}_{\mathbf{A}^d_k, 0}$-algèbre locale étale equirésiduelle et donc $\widetilde{\mathcal{O}}_{X,x}$ et l'hensélisé de $k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$ coincident.

En conséquence, pour tout point fermé régulier $y \in Y$ d'une variété algébrique de dimension d définie sur k, nous pouvons trouver un isomorphisme de k-algèbres σ de $\widetilde{\mathcal{O}}_{X,x}$ vers l'hensélisé $\widetilde{\mathcal{O}}_{Y,y}$ de $\mathcal{O}_{Y,y}$. Notons $\widetilde{K}$ le corps de fractions de $\widetilde{\mathcal{O}}_{X,x}$ (notez que $\widetilde{\mathcal{O}}_{X,x}$ est un sous-anneau de $\widehat{\mathcal{O}}_{X,x}$, qui est intègre par hypothèse). La remarque précédente nous permet de définir une bijection naturelle entre les valuations de $\widetilde{K}$ dont l'anneau de valuation domine $\widetilde{\mathcal{O}}_{Y,y}$ et celles dont l'anneau de valuation domine $\widetilde{\mathcal{O}}_{X,x}$, par composition avec σ. La question qui se pose naturellement maintenant est de savoir si une valuation de $\mathrm{RZ}(X,x)$ définit d'une manière canonique une valuation de $\widetilde{K}$ dont l'anneau de valuation domine $\widetilde{\mathcal{O}}_{X,x}$. Il se trouve que la réponse est affirmative. Afin de montrer ce fait nous nous appuyons sur un résultat profond de [HOST12] sur l'extension des valuations à une extension locale étale d'anneaux, sous l'hypothèse de l'excellence de l'anneau local.

Notre deuxième résultat principal est le suivant:

Théorème A'. *Soient X, Y deux variétés algébriques définies sur le même corps algébriquement clos k. Pour tous points fermés réguliers $x \in X$, $y \in Y$, les espaces* $\mathrm{RZ}(X,x)$ *and* $\mathrm{RZ}(Y,y)$ *sont homéomorphes si et seulement si X et Y ont la même dimension.*

Deux conséquences de ce théorème sont particulièrement remarquables. D'un côté, ce résultat indique que le type d'homéomorphisme de $\mathrm{RZ}(X,x)$ et $\mathrm{NL}(X,x)$ ne dépend que de la dimension de la variété X et du corps de base k. D'autre part, en supposant l'existence de résolution de singularités, il révèle la structure *auto-homéomorphe* de $\mathrm{RZ}(X,x)$ en considérant un système projectif de variétés non singulières. Cette propriété est également observée dans l'arbre valuatif et dans l'espace des places réels de $L(y)$ où $L = \mathbf{R}(\!(t^{\mathbf{Q}})\!)$ (voir [FJ04, Theorem 6.51] et [Kuh13, Corollary 21] respectivement). Nous précisons cette notion dans la Section 2.3.

Ensuite nous considérons un point singulier x d'une surface algébrique normale X. Tout d'abord notons que les espaces $\mathrm{RZ}(X,x)$ et $\mathrm{NL}(X,x)$ ont plus de structure qu'une topologie. Ce sont des espaces annelés en anneaux locaux. Le second porte une structure analytique naturelle localement modelée sur des espaces affinoïdes sur $k(\!(t)\!)$. Ces structures $k(\!(t)\!)$-analytiques locales ne sont pas canoniques et ne peuvent pas en général être recollées pour en obtenir une globale. Cette structure a été étudiée dans [Fan14a] et il a montré qu'elle détermine le complété de l'anneau local $\mathcal{O}_{X,x}$ (voir [Fan14a, Corollary 4.14]). Cependant, ce sont seulement les espaces topologiques sous-jacents ce qui nous intéresse ici. Nous introduisons maintenant les notions nécessaires pour énoncer notre résultat principal concernant le type d'homéomorphisme de $\mathrm{RZ}(X,x)$ et $\mathrm{NL}(X,x)$ dans ce cas.

Par un *graphe* nous entendons un graphe connexe fini avec au moins un sommet, sans boucles et sans arêtes multiples. Rappelons qu'un graphe Γ est un objet purement combinatoire qui peut être vu comme un CW-complexe unidimensionnel fini. Pour être précis, nous munissons l'ensemble des sommets V de Γ et l'ensemble de ses arêtes E de la topologie discrète, et l'intervalle unité $[0,1]$ de la topologie induite par la topologie standard de la droite réelle. L'espace topologique $|\Gamma|$, que nous appelons la *réalisation topologique* de Γ, est l'espace quotient obtenu de la réunion disjointe $\mathrm{V} \sqcup (\mathrm{E} \times [0,1])$ par les identifications $v \sim (e,0)$ et $v' \sim (e,1)$ données par l'incidence des sommets et arêtes.

Nous disons qu'un graphe est un *arbre* si sa réalisation topologique est simplement connexe. D'après [Sta83] nous associons à tout graphe son noyau (voir aussi la définition du squelette d'un quasipolyhedron donnée dans [Ber90]). Le *noyau d'un graphe* Γ qui n'est pas un arbre, est le sous-graphe de Γ obtenu en supprimant de façon récursive un sommet de degré un et l'arête qui lui est incidente, jusqu'à ce qu'il ne reste pas de sommets de degré un. Rappelons que le degré d'un sommet est le nombre d'arêtes qui lui sont incidentes.

On note le noyau de Γ par $\mathrm{Core}(\Gamma)$.

Soit Γ un graphe qui n'est pas un arbre. Remarquez que si Γ n'a pas de sommet de degré un, alors Γ est son propre noyau. Notez également que $|\Gamma|$ admet une rétraction par déformation sur $|\mathrm{Core}(\Gamma)|$. Le complémentaire de $|\mathrm{Core}(\Gamma)|$ dans $|\Gamma|$ est l'ensemble des points de $|\Gamma|$ qui admettent un voisinage ouvert dont la fermeture est un arbre et dont la frontière est réduite à un sommet de Γ. Nous pouvons donc imaginer Γ comme son noyau avec un certain nombre d'arbres disjoints attachés à lui.

Nous introduisons maintenant la relation d'équivalence dans l'ensemble des graphes sur lasquelle repose la caractérisation de notre résultat. Deux graphes Γ et Γ' sont *équivalents* si ils sont tous les deux des arbres, ou bien aucun ne l'est et $|\mathrm{Core}(\Gamma)|$ est homéomorphe à $|\mathrm{Core}(\Gamma')|$. Observez que cette relation d'équivalence est plus stricte que l'équivalence d'homotopie. Les trois graphes, comprenant deux triangles partageant un sommet, deux triangles partageant un côté, et un segment avec un triangle attaché à chaque extremité, ont tous des réalisations topologiques homotopiquement équivalentes, mais ils ne sont pas deux à deux équivalents.

Rappelons qu'un morphisme birationnel $\pi_{X'} : X' \to X$ est une *bonne résolution* si X' est lisse et le lieu exceptionnel $E_{X'} = \pi_{X'}^{-1}(x)_{\mathrm{red}}$ est un diviseur à croissements normaux, tel que ses composantes irréductibles sont lisses et l'intersection de deux d'entre elles est au plus un point. À toute bonne résolution est associée son *graphe dual* $\Gamma_{X'}$ dont les sommets sont en bijection avec les composantes irréductibles de $E_{X'}$ et où deux sommets sont adjacents si et seulement si les composantes irréductibles correspondantes de $E_{X'}$ s'intersectent.

Notre résultat principal dans ce cadre est le suivant:

Théorème B. *Soient $x \in X$ et $y \in Y$ des points singuliers de deux surfaces normales définies sur un corps algébriquement clos k, et $\Gamma_{X'}$, $\Gamma_{Y'}$ les graphes duaux associés à deux bonnes résolutions de (X, x) et (Y, y) respectivement. Les affirmations suivantes sont équivalentes:*

1. *Les espaces* $\mathrm{RZ}(X, x)$ *et* $\mathrm{RZ}(Y, y)$ *sont homéomorphes.*

2. *Les espaces* $\mathrm{NL}(X, x)$ *et* $\mathrm{NL}(Y, y)$ *sont homéomorphes.*

3. *Les graphes* $\Gamma_{X'}$ *et* $\Gamma_{Y'}$ *sont équivalents.*

Ce théorème implique que les espaces de valuations $\mathrm{RZ}(X, x)$ et $\mathrm{NL}(X, x)$ associés à *toute* singularité rationnelle de surface (X, x) sont homéomorphes à $\mathrm{RZ}(\mathbf{A}_k^2, 0)$ et $\mathrm{NL}(\mathbf{A}_k^2, 0)$ respectivement. Afin d'obtenir des informations plus précises sur la singularité (X, x), il sera nécessaire d'explorer des structures plus fines de $\mathrm{RZ}(X, x)$.

Comme nous l'avons dit plus haut, $\mathrm{NL}(X, x)$ est le plus grand quotient séparé de $\mathrm{RZ}(X, x)$. Par conséquent, si *1* est vérifié alors $\mathrm{NL}(X, x)$ est homéomorphe à $\mathrm{NL}(Y, y)$. Nous allons maintenant expliquer brièvement notre stratégie pour prouver que *2* implique *3* et que *3* implique *1*.

La réalisation topologique de toute graphe dual $\Gamma_{X'}$ peut être vue comme un sous-espace fermé de $\mathrm{NL}(X, x)$ et il existe une *rétraction* continue $\mathrm{r}_{X'} : \mathrm{NL}(X, x) \to |\Gamma_{X'}|$ (voir

[Fav10]). L'observation clé est la suivante: toute fibre $\mathrm{r}_{X'}^{-1}(\nu)$ de l'application $\mathrm{r}_{X'}$ est un arbre dont la frontière est réduite à ν. Une preuve de cette affirmation découle de [FJ04, Theorem 6.51]. En fait, il est demontré dans [Fan14a, Proposition 9.5 (i)] que la fibre $\mathrm{r}_{X'}^{-1}(\nu)$ est un disque analytique lorsque elle est munie de sa structure analytique canonique. Nous entendons ici par *arbre* un espace topologique qui est homéomorphe à un arbre réel enraciné non métrique dans le sens de [FJ04]. Si un graphe est un arbre, alors il l'est aussi dans ce sens.

Nous définissons le *noyau* de $\mathrm{NL}(X,x)$ comme l'ensemble de tous les points de $\mathrm{NL}(X,x)$ qui n'admettent pas un voisinage ouvert dont la fermeture est un arbre et dont la frontière est réduite à un unique point de $\mathrm{NL}(X,x)$, et on le note par $\mathrm{Core}\,(\mathrm{NL}(X,x))$. Dans [Ber90] le noyau est appellé squelette. Remarquez que, par définition, $\mathrm{Core}\,(\mathrm{NL}(X,x))$ est vide si et seulement si $\mathrm{NL}(X,x)$ est un arbre. Nous montrons que, étant donnée une bonne résolution $\pi_{X'} : X' \to X$, l'espace $\mathrm{NL}(X,x)$ est un arbre si et seulement si $\Gamma_{X'}$ est un arbre. En outre, si aucun n'est un arbre alors nous avons $\mathrm{Core}\,(\mathrm{NL}(X,x)) = |\mathrm{Core}\,(\Gamma_{X'})|$ en tant que sous-espaces de $\mathrm{NL}(X,x)$. On en déduit que *2* implique *3*.

Enfin, voici une esquisse de la preuve du fait que *3* implique *1*, qui est la partie la plus délicate de la démonstration du Théorème B. Nous commençons avec deux bonnes résolutions $\pi_{X'} : X' \to X$ et $\pi_{Y'} : Y' \to Y$, et supposons que leurs graphes duaux sont équivalents. Notre objectif est de construire un homéomorphisme de $\mathrm{RZ}(X,x)$ sur $\mathrm{RZ}(Y,y)$. Nous construisons d'abord deux bonnes résolutions $\pi_{X''} : X'' \to X$ et $\pi_{Y''} : Y'' \to Y$ qui se factorisent par $\pi_{X'}$ et $\pi_{Y'}$ respectivement, et de telle sorte que $\Gamma_{X''}$ et $\Gamma_{Y''}$ sont des graphes isomorphes. Cet isomorphisme détermine une bijection naturelle entre les composantes irréductibles $\{E_i\}_{i=1}^m$ de $E_{X''}$ et celles, disons $\{D_i\}_{i=1}^m$, de $E_{Y''}$. Nous envoyons la valuations divisorielle de $\mathrm{RZ}(X,x)$ définie par E_i à la valuation divisorielle de $\mathrm{RZ}(Y,y)$ définie par D_i. Ainsi, afin de définir une bijection de $\mathrm{RZ}(X,x)$ en $\mathrm{RZ}(Y,y)$, il suffit de se concentrer sur les valuations ayant comme centre dans X'' un point fermé. Pour ce faire, nous choisissons une bijection σ de l'ensemble des points fermés de $E_{X''}$ sur celui de $E_{Y''}$ tel que $\sigma(E_i \cap E_j) = D_i \cap D_j$ et $\sigma(E_i) \subseteq D_i$. L'idée est d'appliquer le Théorème A' pour obtenir un homéomorphisme de $\mathrm{RZ}(X'',x'')$ en $\mathrm{RZ}(Y'',\sigma(x''))$. La construction de la bijection de $\mathrm{RZ}(X,x)$ en $\mathrm{RZ}(Y,y)$ en utilisant cette idée nécessite d'une étude locale attentive aux points de $E_{X''}$. Le fait qu'il soit un homéomorphisme résulte de l'examen des comportements des suites de centres et de leurs images par σ.

Nous décrivons maintenant le contenu des chapitres:

Dans le Chapitre 1, nous introduisons les espaces de valuations que nous traitons dans ce travail. La Section 1.1 est consacrée aux espaces de Riemann-Zariski. Nous commençons par rappeler quelques faits basiques sur les valuations. Ensuite, nous présentons les principales propriétés topologiques de l'espace de Riemann-Zariski $\mathrm{Z}(K|R)$ associé à un corps K et un sous-anneau R de K. L'étude de cet espace a reçu dans ces dernières années une attention spéciale (voir [FFL13a, FFL13b, Olb15]). Nous expliquons en particulier qu'il est un *espace spectral* et nous considérons aussi $\mathrm{Z}(K|R)$ muni de sa *topologie constructible*. Dans la suite nous nous limitons au cas où K est le corps des fonctions rationnelles de X et R le corps de base k. Dans ce contexte géométrique l'espace $\mathrm{Z}(K|R)$ est noté par $\mathfrak{X}$. Nous discutons brièvement sa structure d'espace annelé. Ensuite, nous nous concentrons sur le sous-espace de $\mathfrak{X}$ formé de toutes les valuations ayant un centre dans X (qui est la limite projective du système projectif de tous les modèles propres birationnels de X), et

l'espace de Riemann-Zariski $\mathrm{RZ}(X, x)$ de X en un point fermé x. Nous exposons certaines propriétés topologiques de $\mathrm{RZ}(X, x)$, par exemple sa connexité, et nous calculons sa dimension de Krull.

Dans la Section 1.2 nous nous tournons vers les espaces de semivaluations. Nous considérons l'espace $\mathcal{V}(A, \mathfrak{p})$ des *semivaluations normalisées* associé à un anneau intègre A et un idéal $\mathfrak{p}$ de A, et developpons certaines de ses caractéristiques topologiques. L'exemple le plus significatif est l'arbre valuatif. Ensuite, nous nous concentrons sur l'entrelac non-Archimédien normalisé $\mathrm{NL}(X, x)$ de x dans X. La section se termine avec l'étude du lien entre cet espace et $\mathrm{RZ}(X, x)$. Cette relation nous amène à étudier le plus grand quotient séparé de $\mathrm{RZ}(X, x)$.

Le but du Chapitre 2 est de donner les démonstrations des Théorèmes A et A'. Dans les Sections 2.1 et 2.3 nous présentons des preuves détaillées des résultats qui sont nécessaires pour compléter ces démonstrations et que nous avons déjà mentionnés. Étant donné que l'hensélisé d'un anneau local apparaît comme un outil essentiel, dans la Section 2.2 nous incluons les définitions de base et certains résultats utiles dans la suite. Le Chapitre 3 présente une brève discussion sur les arbres et les graphes. Plus précisément, dans la Section 3.2 on définit la relation d'équivalence sur l'ensemble des graphes de deux façons différentes et nous montrons qu'elles reviennent au même. Enfin, le Chapitre 4 contient la preuve du Théorème B.

Chapter 1

Spaces of valuations

The purpose of this chapter is to introduce the valuations spaces considered in this work. In Section 1.1 we deal with valuations and Riemann-Zariski spaces. Then we turn into semivaluations and analytification of algebraic varieties in Section 1.2. The main differences between these two points of view are the presence in the first of valuations of rank larger than one and the fact that a semivaluation may take the value infinity at some non zero elements.

1.1 Riemann-Zariski spaces

In this section we follow the tradition initiated by Zariski. We present the most significant topological properties of Riemann-Zariski spaces. Next we restrict ourselves to the geometrical context and look at the Riemann-Zariski space associated to an algebraic variety X. We concentrate on the study of the fiber $\mathrm{RZ}(X,x)$ of this space above a closed point $x \in X$.

1.1.1 Basics on valuations

An integral domain R is a *valuation ring* if for all nonzero elements $x, y \in R$, either x divides y or y divides x in R. This implies that R is a local ring. If R has K as fraction field, then it is said to be a valuation ring of K. The abelian group $\Phi_R := K^*/R^*$, where $*$ stands for the group of units, is totally ordered by the order induced by the division relation on R and the canonical group homomorphism $\nu_R : K^* \to \Phi_R$, where the operation of Φ_R is now noted additively, verifies that $\nu_R(x+y) \geq \min\{\nu_R(x), \nu_R(y)\}$ for all $x, y \in K^*$. Every group homomorphism ν from K^* to a totally ordered abelian group which satisfies this last condition is called a *valuation* of K. If such a valuation ν is given, the set $R_\nu := \{x \in K^* \;/\; \nu(x) \geq 0\} \cup \{0\}$ is a valuation ring of K whose maximal ideal is $m_\nu := \{x \in K^* \;/\; \nu(x) > 0\} \cup \{0\}$. Classically, two valuations of K which differ by an order-preserving group isomorphism are called *equivalent* and they are identified. The valuations ν and ν_{R_ν} are equivalent, so valuations of K (up to equivalence) and valuations rings of K are related by a natural bijection and in the sequel we make no difference between them. A valuation ν of K is non-trivial if $\nu(f) \neq 0$ for some $f \in K^*$, otherwise it is called *trivial*.

Associated to a valuation ν of K are its *value group* $\Phi_\nu := \Phi_{R_\nu}$ and its *residue field* $k_\nu := R_\nu/m_\nu$. We may also consider an element $+\infty$ greater than any element of Φ_ν, extend in the natural way the group law on Φ_ν to $\Phi_\nu \cup \{+\infty\}$ and set $\nu(0) = +\infty$. We define the *rank* of ν, $\operatorname{rk}\nu$, as the rank of its value group, i.e. the ordinal type of the chain of proper isolated subgroups of Φ_ν; and the *rational rank* of ν, $\operatorname{rrk}\nu$, as the dimension of the $\mathbf{Q}$-vector space $\Phi_\nu \otimes_{\mathbf{Z}} \mathbf{Q}$. If we consider a subfield k of K contained in R_ν, then we have a natural embedding $k \hookrightarrow k_\nu$. We denote by $\dim\nu$ the transcendence degree of this field extension and call it the *dimension* of ν. Abhyankar's inequality (see [Abh56, Lemma 1]) states that $\operatorname{rrk}\nu + \dim\nu \leq \text{tr.deg}_k K$ whenever $\text{tr.deg}_k K$ is finite.

Let ν be a valuation of K of finite rank $\operatorname{rk}\nu > 1$ and Ψ a proper isolated subgroup of $\Phi = \Phi_\nu$, $\Psi \neq (0)$. Then the quotient group Φ/Ψ is naturally totally ordered (an element of the quotient is non negative if it corresponds to a non negative element of Φ) and the canonical epimorphism $\Phi \to \Phi/\Psi$ is a monotone non decreasing map of totally ordered groups. The composed morphism $\nu' : K^* \to \Phi \to \Phi/\Psi$ is a valuation of K such that $m_{\nu'} \subsetneq m_\nu \subsetneq R_\nu \subsetneq R_{\nu'}$. Moreover, every valuation ring of K containing R_ν can be obtained in this way. The integral domain $R_{\overline{\nu}} = R_\nu/m_{\nu'}$ is a valuation ring of the residue field $k_{\nu'}$ and its associated value group $\Phi_{\overline{\nu}}$ is isomorphic to Ψ. Note that its maximal ideal is $m_{\overline{\nu}} = m_\nu/m_{\nu'}$ and its residue field $k_{\overline{\nu}}$ equals k_ν. We say that ν is *composite* with ν' and $\overline{\nu}$ and we write $\nu = \nu' \circ \overline{\nu}$. We refer to [ZS60] for details and more basic facts concerning isolated subgroups and composite valuations.

1.1.2 The Riemann-Zariski space $\mathrm{Z}(K|R)$

Given a field K and a subring R of K, the *Riemann-Zariski space* $\mathrm{Z}(K|R)$ is the set of valuations rings of K that contain R, equipped with the *Zariski topology.* This topology is obtained by taking the subsets

$$E(A) := \{\nu \in \mathrm{Z}(K|R) \;/\; A \subset R_\nu\}\,,$$

when A ranges over the family of all finite subsets of K, as a basis of open sets. If $A = \{f_1, \ldots, f_m\}$ then we write $E(A) = E(f_1, \ldots, f_m)$. The Riemann-Zariski space is always a quasi-compact space, but it is in general very far from being Hausdorff: given $\nu \in \mathrm{Z}(K|R)$, its closure in $\mathrm{Z}(K|R)$ is the set of all valuations ν' in $\mathrm{Z}(K|R)$ such that $R_{\nu'} \subseteq R_\nu$ ([ZS60] Ch. VI §17, Theorem 40 and Theorem 38, respectively).

More generally, $\mathrm{Z}(K|R)$ is a *spectral space.* That is, apart from being a quasi-compact topological space, it also satisfies the three following properties:

1. It is a Kolmogorov space (i.e. for any two different points in the space, there is an open set which contains one of these points but not the other):

   If ν and ν' are two distinct valuations of K then either $R_\nu \not\subset R_{\nu'}$ or $R_{\nu'} \not\subset R_\nu$. Without loss of generality we may suppose that there exists $f \in K^*$ such that $\nu(f) \geq 0$ and $\nu'(f) < 0$. Hence we see that there exists an open set of $\mathrm{Z}(K|R)$ containing one of the valuations but not containing the other.

2. The family of quasi-compact open subsets is closed under finite intersection and forms a basis for the open sets:

   Given $f_1, \ldots, f_m \in K$, $E(f_1, \ldots, f_m)$ is homeomorphic to the Riemann-Zariski space $\mathrm{Z}(K|R[f_1, \ldots, f_m])$, so that any basic open subset of $\mathrm{Z}(K|R)$ is in fact quasi-compact. It suffices now to observe that any quasi-compact open subset of $\mathrm{Z}(K|R)$ is a finite union of basic open subsets and that $E(A) \cap E(A') = E(A \cup A')$ for any pair A, A' of finite subsets of K.

3. Every non empty irreducible closed subset F contains a point whose closure is F (such a point is called a generic point and it is unique if the property 1 holds):

   Recall that a non empty subset of a topological space is irreducible if it is not the union of two non empty closed subsets of the space. The proof given in [DFF87] in the special case when K is the fraction field of R can be adapted in a straightforward way to the general case.

Lemma 1.1 ([DFF87], Lemma 4.2 (a)). *Let F be a non empty irreducible closed subset of $Z(K|R)$. Any valuation ν of* $\mathrm{Z}(K|R)$ *such that R_ν is contained in $\bigcup_{\mu\in F} R_\mu$ belongs to F.*

Proof. We proceed by contradiction. Suppose that there exists $\nu \in \mathrm{Z}(K|R)$ such that $R_\nu \subseteq \bigcup_{\mu\in F} R_\mu$ and $\nu \notin F$. Let us first point out that if $\nu \in E(f)$ for some $f \in K$ then $f \in R_\nu$ and, by hypothesis, there is a valuation ring of F which also contains f, so the open subset $E(f)$ is not contained in $\mathrm{Z}(K|R) \setminus F$.

Since F is closed, we can find a finite subset A of K such that $\nu \in E(A) \subseteq \mathrm{Z}(K|R) \setminus F$. By the previous remark we may assume that $A = \{f_1, \ldots, f_m\}$ with $m \geq 2$. Then the subset F is contained in the finite union of at least two closed subsets of $\mathrm{Z}(K|R)$, $F \subseteq \mathrm{Z}(K|R) \setminus E(A) = \bigcup_{1\leq i\leq m}(\mathrm{Z}(K|R) \setminus E(f_i))$. Neither of these subsets contains F since $\nu \in E(f_i)$ for every $i \in \{1, \ldots, m\}$. This means that F is not an irreducible subset of $\mathrm{Z}(K|R)$ and ends the proof. □

Proposition 1.2 ([DFF87], Lemma 4.2 (b) and Proposition 4.3). *Let F be a non empty irreducible closed subset of* $\mathrm{Z}(K|R)$. *For any valuation $\nu \in F$, the subset $\mathcal{I} = \bigcap_{\mu\in F} m_\mu$ of R_ν is a prime ideal of R_ν and the localization $(R_\nu)_{\mathcal{I}}$ is a valuation ring of K which belongs to F and whose closure in* $\mathrm{Z}(K|R)$ *is F.*

Proof. Let us fix $\nu \in F$ and show that $\mathcal{I}$ is an ideal of R_ν. It is clear that $\mathcal{I}$ is closed under addition.

Note that for any $f, g \in K$ such that $fg \in \mathcal{I}$ we have $\mu(f^{-1}g^{-1}) = -\mu(fg) < 0$ for every $\mu \in F$ and thus $F \cap E(f^{-1}, g^{-1}) = \emptyset$. This means that F is contained in the union of the closed subsets $B(f) = \mathrm{Z}(K|R) \setminus E(f^{-1})$ and $B(g) = \mathrm{Z}(K|R) \setminus E(g^{-1})$. If neither f nor g is in $\mathcal{I}$, then F is contained neither in $B(f)$ nor in $B(g)$. This gives us a contradiction with the irreducibility of F, hence either f or g must belong to $\mathcal{I}$.

Take $x \in R_\nu$ and $y \in \mathcal{I}$. Since $x^{-1}(xy) \in \mathcal{I}$ and $x^{-1} \notin \mathcal{I}$, the previous remark applied to $f = x^{-1}$ and $g = xy$ implies that $xy \in \mathcal{I}$. We conclude that $\mathcal{I}$ is a prime ideal of R_ν. Let us denote V the localization of R_ν with respect to $\mathcal{I}$. Recall that V belongs to $\mathrm{Z}(K|R)$ and has $\mathcal{I}$ as maximal ideal ([Vaq00], Proposition 3.3). Given $\mu \in F$, by definition $\mathcal{I} \subseteq m_\mu$ and we easily deduce that $K \setminus V \subseteq K \setminus R_\mu$. Hence F is contained in the closure of V in $\mathrm{Z}(K|R)$. Finally, let us verify that $V \in F$.

By Lemma 1.1 it suffices to prove that every element in V belongs to some valuation ring of F. Given a nonzero element $f \in V$, if $f \in K \setminus R_\mu$ for every $\mu \in F$, then $f^{-1} \in \bigcap_{\mu\in F} m_\mu$. This contradicts the maximality of $\mathcal{I}$ in V and ends the proof. □

Spectral spaces have been characterized in [Hoc69] as the topological spaces that are homeomorphic to the spectrum of some commutative ring equipped with the Zariski topology. In our situation we have the following explicit construction of such a commutative ring ([HK10, Corollary 2.9] and [FFL13a, Corollary 3.6]):

If t is an indeterminate, given $\nu \in Z(K|R)$ we may denote ν' the *Gauss valuation* extending ν, that is, the valuation ν' of $\mathrm{Z}(K(t)|R)$ defined by $\nu'(f) = \min_{1\le i\le s}\{\nu(a_i)\}$ for each nonzero polynomial $f = a_0 + a_1 t + \ldots + a_s t^s \in K[t]$. The Riemann-Zariski space $\mathrm{Z}(K|R)$ is then homeomorphic to the spectrum of the subring $\mathrm{Kr}(K|R) = \bigcap_{\nu\in \mathrm{Z}(K|R)} R_{\nu'}$ of $K(t)$ (if R is an integrally closed domain and K is its fraction field, this result was already proven in [DF86, Theorem 2]).

Let X be an arbitrary topological space. A *net* in X is a family $(x_i)_{i\in I}$ of elements of X indexed by a directed set I. Given $x \in X$ and a net $(x_i)_{i\in I}$ in X, we say that $(x_i)_{i\in I}$ converges to x, and we write $x_i \to x$, if for any open neighborhood U of x there exists $i_0 \in I$ such that $x_i \in U$ for any $i \ge i_0$. Recall that a map $f : X \to Z$ between two topological spaces is continuous at $x \in X$ if, and only if, for any net $(x_i)_{i\in I}$ in X such that $x_i \to x$ in X we have $f(x_i) \to f(x)$ in Z.

The following lemma is a direct consequence of the definition of the topology of the Riemann-Zariski space.

Lemma 1.3. *Let $\nu \in Z(K|R)$ and $(\nu_i)_{i\in I}$ be a net in $\mathrm{Z}(K|R)$, $\nu_i \to \nu$ if and only if for any $f \in R_\nu$ there exists $i_0 \in I$ such that $f \in \bigcap\limits_{i\ge i_0} R_{\nu_i}$.*

Proof. Take $f \in K$ such that $f \in R_\nu$. Since $\nu \in E(f)$, if $\nu_i \to \nu$ then we can find $i_0 \in I$ such that $\nu_i \in E(f)$ for any $i \ge i_0$, i.e. $f \in R_{\nu_i}$ for any $i \ge i_0$. Let us now show the converse.

Given an open neighborhood U of ν there exist $f_1, \ldots, f_m \in K$ such that $\nu \in E(f_1, \ldots, f_m)$ and $E(f_1, \ldots, f_m) \subseteq U$. Since $E(f_1, \ldots, f_m) = \bigcap_{j=1}^m E(f_j)$, by hypothesis for every j in $\{1, \ldots, m\}$ we can find $i_{0,j} \in I$ such that $f_j \in \bigcap_{i\ge i_{0,j}} R_{\nu_i}$. We define $i_0 \in I$ to be an upper bound of $\{i_{0,j}\}_{j=1}^m \subseteq I$. Then $f_j \in \bigcap_{i\ge i_0} R_{\nu_i}$ for every $j \in \{1, \ldots, m\}$. It follows that $\nu_i \in \bigcap_{j=1}^m E(f_j)$ for any $i \ge i_0$ and therefore $\nu_i \to \nu$. □

1.1.3 Constructible topology and center map

Let X be an arbitrary topological space and let $\mathcal{K}(X)$ be the subalgebra of the Boolean algebra of subsets of X generated by the collection of all open quasi-compact subsets of X (i.e. $\mathcal{K}(X)$ is the smallest set of subsets of X that contains all open quasi-compact subsets and is stable under finite intersections and under taking complements). We call *constructible sets* the elements of $\mathcal{K}(X)$ and X^{cons} the topological space obtained by considering the *constructible topology* on X, that is the topology on X having the constructible sets as a basis of open sets.

Lemma 1.4. *Let X be a spectral space. Then X^{cons} is a Hausdorff space.*

Proof. Given two different points $x, y \in X$, since X is a Kolmogorov space, we can assume that there is an open subset of X which contains x but not y. Every open subset of X can be written as a union of constructible sets of X, so we can find $U \in \mathcal{K}(X)$ such that $x \in U$ and $y \in X \setminus U$. Both U and $X \setminus U$ are open subsets of X^{cons}, thus x and y have disjoint neighborhoods. □

If X is a spectral space, then its constructible sets are precisely the finite unions of subsets $U \cap (X \setminus V)$ with U, V open quasi-compact subsets of X. Moreover, X^{cons} is a *Stone space*

([Hoc69], Proposition 4), i.e. a quasi-compact Hausdorff totally disconnected topological space. Hence $\mathrm{Z}(K|R)^{\mathrm{cons}}$ has all these properties and a basis for its open sets is given by

$$\{E(A)\ /\ A \subset K \text{ finite}\} \cup \left\{ E(A_0) \bigcap \left(\bigcap_{i=1}^{m} (\mathrm{Z}(K|R) \setminus E(A_i)) \right) \Big/ A_0, \ldots, A_m \subset K, \text{ finite} \right\} \tag{1.1}$$

Recall that the order of a family of subsets, not all empty, of a topological space X is the largest n (if it exists) such that the family contains $n+1$ elements with non empty intersection. If such an integer n does not exist, then the order is said to be $+\infty$. The *covering dimension* of X is the least n such that any finite open cover of X has a refinement of order not exceeding n, or $+\infty$ if there is no such an integer.

The covering dimension of any non empty quasi-compact Hausdorff totally disconnected space is zero (see [Pea75, Ch. 3, Proposition 1.3]). As a consequence of this result, the space $\mathrm{Z}(K|R)^{\mathrm{cons}}$ has covering dimension zero.

Given two spectral spaces X and Y, a continuous map $X \to Y$ is a *spectral map* if the inverse images of quasi-compact open subsets are quasi-compact. Assuming that this is condition satisfied, the inverse image of any constructible set of Y is a constructible set of X and thus the map is also continuous if we endow X and Y with the constructible topologies. In our situation, given a subring R of a field K, we get a map $\mathrm{c}_R : \mathrm{Z}(K|R) \to \operatorname{Spec} R$ sending any valuation ν to its *center* in R, which is by definition the prime ideal $m_\nu \cap R$ of R. We call this map the *center map*.

Proposition 1.5. *Endowing* $\operatorname{Spec} R$ *with the Zariski topology, the center map* c_R *is a spectral map.*

Proof. A basic open subset of $\operatorname{Spec} R$ is of the form $D_f = \{\mathfrak{p} \in \operatorname{Spec} R\ /\ f \notin \mathfrak{p}\}$ where f is a nonzero element of R. Note that $\mathrm{c}_R^{-1}(D_f) = E(f^{-1})$. Therefore c_R is continuous. Moreover, since any quasi-compact open subset U of $\operatorname{Spec} R$ is a finite union of basic open subsets D_f, we can write its inverse image under c_R as a finite union of quasi-compact subsets of $\mathrm{Z}(K|R)$ and thus $\mathrm{c}_R^{-1}(U)$ is itself quasi-compact. □

Lemma 1.6. *Let K be a field, R a subring of K and $\nu, \nu_1 \in \mathrm{Z}(K|R)$ with* $\operatorname{rk} \nu_1 = 1$ *and $R_\nu \subsetneq R_{\nu_1}$. The center of ν_1 in R is $\{f \in R\ /\ \forall \alpha \in \Phi\ \exists\, m \in \mathbf{Z}_{>0}$ such that $m\,\nu(f) > \alpha\}$, where Φ denotes the value group of ν.*

Proof. Given $f \in R$, $\nu_1(f) \geq 0$. We need to prove that $\nu_1(f) > 0$ if, and only if, for any $\alpha \in \Phi$ there exists a positive integer m such that $m\,\nu(f) > \alpha$.

Let us suppose first that $\nu_1(f) > 0$. Take $\alpha = \nu(g)$ for some $g \in K^*$. Since $\nu_1(f) > 0$ and Φ_{ν_1} is an Archimedean group, we can find $m \in \mathbf{Z}_{>0}$ such that $m\,\nu_1(f) > \nu_1(g)$. We get $\nu_1(f^m g^{-1}) > 0$ and thus $\nu(f^m g^{-1}) > 0$. This implies that $m\,\nu(f) > \alpha$.

We proceed now by contraposition. Let us suppose that $\nu_1(f) = 0$ and show that there exists $\alpha \in \Phi$ such that $m\,\nu(f) \leq \alpha$ for any $m \in \mathbf{Z}_{>0}$. Denote Ψ the isolated subgroup of Φ associated to ν_1 and consider $\alpha \in \Phi \setminus \Psi$, $\alpha > 0$. Since $\nu_1(f) = 0$, we have that $\nu(f) \in \Psi$. If $m\,\nu(f) > \alpha$ for some $m \in \mathbf{Z}_{>0}$, the fact that Ψ is convex implies that α must also belong to Ψ. This contradicts the choice of α and finishes the proof. □

1.1.4 The Riemann-Zariski space associated to an algebraic variety

We restrict ourselves to the case where K is the function field of an algebraic variety X over a field k, i.e. X is an integral separated scheme of finite type over k, and we denote by $\mathfrak{X}$ the Riemann-Zariski space $\mathrm{Z}(K|k)$. If X is a point then $\mathfrak{X}$ is reduced to the trivial valuation ([ZS60] Ch. VI §4, Theorem 5, Corollary 1), so we always assume that the transcendence degree of K over k is positive. For the moment we do not make any other assumption about the field k.

Given a valuation $\nu \in \mathfrak{X}$, the valuative criterion of separatedness (see [Har77, Theorem 4.3]) states that there exists at most one scheme-theoretic point $x \in X$ such that R_ν *dominates* the local ring $\mathcal{O}_{X,x}$, that is $\mathcal{O}_{X,x} \subseteq R_\nu$ and the maximal ideal m_ν of R_ν intersects $\mathcal{O}_{X,x}$ in its maximal ideal $\mathfrak{m}_{X,x}$. If such a point exists, it is called the *center* of ν in X (we may also consider the irreducible closed set $\overline{\{x\}}$ as the center) and we say that ν is *centered* in X. When $X = \operatorname{Spec} R$ is an affine variety, the center of a valuation in X is nothing but its center in R as it was defined in Subsection 1.1.3.

The set $\mathrm{RZ}(X)$ of all valuations $\nu \in \mathfrak{X}$ such that ν is centered in X is the (finite) union of the open sets $E(x_{i,1}, \ldots, x_{i,n(i)})$ where $\left\{\operatorname{Spec} k[x_{i,1}, \ldots, x_{i,n(i)}]\right\}_{i=1}^{s}$ is an open affine covering of X, therefore it is a quasi-compact open subset of $\mathfrak{X}$ and, as a consequence of the valuative criterion of properness ([Har77], Theorem 4.7), is equal to the whole Riemann-Zariski space if and only if X is a complete variety. We endow $\mathrm{RZ}(X)$ with the topology induced by the Zariski topology of $\mathfrak{X}$.

Example 1.7. Let k be a field and $X = \operatorname{Spec} k[t]$ the affine line. The valuation rings of $k(t)$ which contain the base field k are of two kinds. One the one hand we have the local rings $R_{\nu_p} = k[t]_{(p(t))}$ where $p(t) \in k[t]$ is an irreducible polynomial. On the other hand we have the valuation ring $R_{\nu_\infty} = \{p(t)/q(t) \;/\; \deg p(t) \leq \deg q(t)\}$. Note that $k[t]$ is contained in R_{ν_p} for any $p(t)$, but not in R_{ν_∞}. We have $\mathrm{RZ}(X) \subsetneq \mathfrak{X}$.

A birational morphism $\pi : Y \to X$ induces an isomorphism between the function fields of X and Y, so we can identify their Riemann-Zariski spaces. Moreover, if ν is an element of $\mathrm{RZ}(Y)$ then it also belongs to $\mathrm{RZ}(X)$ and the image by π of the center of ν in Y is precisely the center of ν in X. In the case that π is also proper then it follows from the valuative criterion of properness that $\mathrm{RZ}(Y)$ and $\mathrm{RZ}(X)$ coincide.

If $\pi_Y : Y \to X$ and $\pi_{Y'} : Y' \to X$ are proper birational morphisms, we say that Y' *dominates* Y if the birational map $\pi_Y^{-1} \circ \pi_{Y'} : Y' \to Y$ is a morphism. Given two such pairs (Y, π_Y) and $(Y', \pi_{Y'})$ we can always find a third one dominating both (we may consider their birational join, see [Spi90a, Definition 1.4.1] and [ZS60, Ch. VI §17 Lemma 6]), so the set $\mathcal{M}$ of all the pairs (Y, π_Y) is a projective system indexed by the domination relation. We have the following fundamental result:

Theorem 1.8 (Zariski). *The natural map from* $\mathrm{RZ}(X)$ *to the projective limit* $\varprojlim_{Y \in \mathcal{M}} Y$ *which corresponds to sending a valuation* ν *to its center on each algebraic variety in* $\mathcal{M}$*, is a homeomorphism when we endow* $\varprojlim_{Y \in \mathcal{M}} Y$ *with the projective limit topology (i.e. the coarsest topology for which all the projection maps are continuous when we equip each variety in* $\mathcal{M}$ *with the Zariski topology).*

Even though we focus on the topology, the Riemann-Zariski space $\mathfrak{X}$ has a richer structure than just topology. In particular it admits a *locally ringed space structure*, introduced in [Hir64]. The structure sheaf on $\mathfrak{X}$ is the sheaf of rings whose sections on any open subset $U \subseteq \mathfrak{X}$ are given by $\mathcal{O}_{\mathfrak{X}}(U) = \bigcap_{\nu \in U} R_\nu$ and where the restriction maps are the inclusion maps. The local ring $\mathcal{O}_{\mathfrak{X},\nu}$ at a valuation ν is the corresponding valuation ring R_ν. As locally ringed spaces, the open $\mathrm{RZ}(X)$ endowed with the induced ringed structure is isomorphic to the projective limit $\varprojlim_{Y \in \mathcal{M}} Y$ (see [KS08, Theorem 3.2.5]).

Remark 1.9. The locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is not in general locally affine. Note that the ring of regular functions on a basic open subset $U = E(f_1, \ldots, f_m) \subsetneq \mathfrak{X}$ is the intersection of all valuation rings of K containing the k-algebra $A = k[f_1, \ldots, f_m]$. By [ZS60, Ch. VI §5 Theorem 6], that is the integral closure $\overline{A}$ of A in K. The ring $\overline{A}$ is also a finitely generated k-algebra ([Eis95], Corollary 13.13) and in particular, it is noetherian. If $(U, \mathcal{O}_{\mathfrak{X}}|_U)$ is isomorphic to an affine scheme, then the localization of $\overline{A}$ with respect to any prime ideal must be a valuation ring of K. Since $\overline{A}$ is noetherian, $\overline{A}$ must be a Dedekind domain with fraction field K. Therefore we deduce that the Riemann-Zariski space is not generally a scheme (it is always the case when $d = 1$ since in this case the Riemann-Zariski space is a complete non-singular model of the curve). We refer to [Olb15, Theorem 6.1] for a characterization of the subspaces of the Riemann-Zariski space $\mathrm{Z}(K|R)$ that are affine schemes.

1.1.5 The Riemann-Zariski space $\mathrm{RZ}(X, x)$ of X at x

We keep the notations of Subsection 1.1.4.

Definition 1.10 (Riemann-Zariski space of X at x)**.** *Given a closed point $x \in X$, we denote by* $\mathrm{RZ}(X, x)$ *the set of all valuations of RZ(X) whose center in X is x, equipped with the induced topology. We call this space the Riemann-Zariski space of X at x.*

Note that $\mathrm{RZ}(X, x)$ is a closed subspace of $\mathrm{RZ}(X)$ and it is therefore itself quasi-compact.

Let us first explore the case dim $X = 1$ and fix a closed point $x \in X$. If x is a regular point then $\mathcal{O}_{X,x}$ is a regular local ring of dimension one with fraction field K and therefore a valuation ring of K. The space $\mathrm{RZ}(X, x)$ is thus reduced to a unique point. If x is singular, then $\mathrm{RZ}(X, x)$ is in one-to-one correspondence with the set of maximal ideals of the integral closure of $\mathcal{O}_{X,x}$ in K.

Recall that the *Krull dimension* of a topological space Z is the supremum in the extended real line of the lengths of all chains of irreducible closed subspaces of Z. A chain

$$\emptyset \subsetneq Z_0 \subsetneq \ldots \subsetneq Z_l \subseteq Z$$

is of length l. The only irreducible subspaces of a non-empty Hausdorff space are the singletons, so the dimension of any Hausdorff topological space is zero.

Proposition 1.11. *For any closed point $x \in X$,* $\dim \mathrm{RZ}(X, x) = \dim X - 1$.

Proof. We have already pointed out that dim $\mathrm{RZ}(X, x) = 0$ whenever X is one-dimensional. Suppose now that the dimension of X is $d > 1$.

Let $\pi : \widetilde{X} \to X$ be the normalization of the blowing-up of X at the closed point $x \in X$. For any reduced irreducible component E of $\pi^{-1}(x)$, $\mathcal{O}_{\widetilde{X},E}$ is a one-dimensional discrete valuation ring of K that dominates $\mathcal{O}_{X,x}$ and whose residue field is isomorphic to the function field of E. Let us fix such a component E_1 and denote by ν_1 the corresponding rank one valuation and choose a valuation $\overline{\nu}_1$ of the residue field k_{ν_1} such that $k \subseteq R_{\overline{\nu}_1}$ and $\mathrm{tr.deg}_k k_{\overline{\nu}_1} = d-2$. If we denote by $p : R_{\nu_1} \to k_{\nu_1}$ the canonical projection, then $R_{\nu_2} := p^{-1}(R_{\overline{\nu}_1})$ is a valuation ring of K dominating $\mathcal{O}_{X,x}$ and contained in R_{ν_1}. We may iterate this construction until we have built a sequence of composite valuations $\nu_{i+1} = \nu_i \circ \overline{\nu}_i$ for $1 \leq i \leq d-1$ in $\mathrm{RZ}(X,x)$ such that

$$\emptyset \subsetneq \overline{\{\nu_d\}} \subseteq \overline{\{\nu_{d-1}\}} \subseteq \ldots \subseteq \overline{\{\nu_1\}} \subseteq \mathrm{RZ}(X,x),$$

where the bar means closure in $\mathrm{RZ}(X,x)$. Since $\{\nu_i\}$ is irreducible in $\mathrm{RZ}(X,x)$ so is its closure. It follows from the fact that $\mathrm{RZ}(X,x)$ is a Kolmogorov space that two different valuations must have different closures and thus the dimension of $\mathrm{RZ}(X,x)$ is at least $d-1$.

The Riemann-Zariski space $\mathfrak{X}$ has dimension d ([Vaq00], Proposition 7.8), so that we have the inequalities $\dim \mathrm{RZ}(X,x) \leq \dim \mathrm{RZ}(X) \leq d$. Since $\mathfrak{X}$ is the closure of the trivial valuation, it is an irreducible space and as a consequence so is $\mathrm{RZ}(X)$. By our assumption on the point x, $\mathrm{RZ}(X,x)$ is a proper closed subset of $\mathrm{RZ}(X)$ and therefore the dimension of $\mathrm{RZ}(X,x)$ must be strictly less than the dimension of $\mathrm{RZ}(X)$. The only possibility is then $\dim \mathrm{RZ}(X,x) = d-1$ and $\dim \mathrm{RZ}(X) = d$. □

The closure of a rank one valuation is a maximal irreducible subspace of $\mathrm{RZ}(X,x)$. Hence $\mathrm{RZ}(X,x)$ is not irreducible unless $d=1$ and $\mathrm{RZ}(X,x)$ is a singleton. However:

Proposition 1.12 ([Tem13], Theorem 2.4.2)**.** *The space $\mathrm{RZ}(X,x)$ is connected provided that $\mathcal{O}_{X,x}$ is analytically irreducible, that is, its completion $\widehat{\mathcal{O}}_{X,x}$ is an integral domain.*

Proof. When $\mathcal{O}_{X,x}$ is analytically irreducible, the fiber over x of the normalization morphism $\widetilde{X} \to X$ is reduced to a closed point $\tilde{x}$ ([Gro65], 7.8.3-(vii)) and $\mathrm{RZ}(X,x)$ is precisely $\mathrm{RZ}(\widetilde{X},\tilde{x})$. We may thus concentrate on the normal case.

Let us assume that X is normal and proceed by contradiction. Suppose that $\mathrm{RZ}(X,x)$ can be presented as the union of two disjoint nonempty open subsets $U, U' \subsetneq \mathrm{RZ}(X,x)$. Since $\mathrm{RZ}(X,x)$ is quasi-compact and any closed subspace of a quasi-compact space is also quasi-compact, each of these open subsets must be a finite union of basic open sets. Let us consider an affine open subset V of X. We write U as a finite union of basic open subsets of $\mathrm{RZ}(X,x)$ and we define $\{f_1/g_1, \ldots, f_n/g_n\}$ to be the (finite) set of all rational functions of X appearing that expression, with $f_i, g_i \in \mathcal{O}_X(V)$. In an analogous way, we take rational functions on X defining U', represented by $h_1/l_1, \ldots, h_m/l_m$ in V. Consider the blowing-up $\pi : Y \to X$ of X with respect to the coherent sheaf of ideals in $\mathcal{O}_X$ extending the ideal $\prod_{1\leq i\leq n}(f_i,g_i) \cdot \prod_{1\leq j\leq m}(h_j,l_j)$ of $\mathcal{O}_X(V)$. The fiber $\pi^{-1}(x)$ is the union of $\mathrm{c}_Y(U)$ and $\mathrm{c}_Y(U')$, where c_Y denotes the map which sends a valuation of $\mathrm{RZ}(Y)$ to its center in Y. This union is, by construction, disjoint. The fact that c_Y is a closed map (this can be deduced from [ZS60] Ch. VI §17, Lemma 1 and Lemma 4) implies that $\pi^{-1}(x)$ is not connected and give us a contradiction with Zariski connectedness Theorem (see [Gro61, Proposition 4.3.5]). □

A topological space Z is called a *Fréchet-Urysohn space* if whenever $z \in Z$ is in the closure of a subset A of Z, there is a sequence of points in A which converge to z.

Lemma 1.13. *Let $f : Z \to Y$ a map between two topological spaces. If Z is a Fréchet-Urysohn space then f is continuous if, and only if, given any point z in Z and any sequence $(z_n)_{n=1}^{\infty}$ in Z converging to z there exists a subsequence $(z_{\gamma(n)})_{n=1}^{\infty}$ such that $f(z_{\gamma(n)}) \to f(z)$ in Y.*

Proof. If f is continuous at a point $z \in Z$ and $(z_n)_{n=1}^{\infty}$ is a sequence in Z such that $z_n \to z$, then $f(z_n) \to f(z)$ in Y and thus we can take $\gamma(n) = n$ for any $n \geq 1$. We prove now the converse. Let us take $F \subseteq Y$ a closed subset and show that its inverse image under f is closed in Z. Given a point z in the closure of $f^{-1}(F)$ in Z, since Z is a Fréchet-Urysohn space, we can find a sequence $(z_n)_{n=1}^{\infty}$ in $f^{-1}(F)$ converging to z. By hypothesis there exists a subsequence $(z_{\gamma(n)})_{n=1}^{\infty}$ such that $f(z_{\gamma(n)}) \to f(z)$ in Y. This implies that $f(z)$ belongs to the closure of F in Y, and therefore to F. We conclude that $z \in f^{-1}(F)$ and this ends the proof. $\square$

According to [Fav15, Theorem 3.1], if X is a normal projective (this hypothesis is in fact superfluous) algebraic variety defined over a field k and x is a closed point of X, then $\mathrm{RZ}(X, x)$ is a Fréchet-Urysohn space.

We shall state the following theorem to sum up.

Theorem 1.14. *Let x be a closed point of an algebraic variety X of dimension $d \geq 1$ defined over a field k. The space $\mathrm{RZ}(X, x)$ consisting of the set of valuations of the function field of X whose center in X is x, endowed with the Zariski topology, is a quasi-compact space of dimension $d-1$. In addition, $\mathrm{RZ}(X, x)$ is also a connected Fréchet-Urysohn space provided that X is normal.*

1.2 Spaces of normalized semivaluations

In this section we adopt the point of view of the theory of analytic spaces as developed by Berkovich and others after [Ber90] and we associate to any algebraic variety X its analytification. As we did in Section 1.1, we focus on the fiber of this space above a closed point $x \in X$ and consider the space $\mathrm{NL}(X, x)$. We end this section by establishing the relation between $\mathrm{RZ}(X, x)$ and $\mathrm{NL}(X, x)$.

1.2.1 Basics on semivaluations

Let A be an integral domain. A *semivaluation* of A is a map $\nu : A \to \mathbf{R} \cup \{+\infty\}$ which satisfies $\nu(0) = +\infty$, $\nu(1) = 0$ and $\nu(fg) = \nu(f) + \nu(g)$, $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ for any $f, g \in A$. Note that a semivaluation of A extends to a valuation of the fraction field of A if and only if it takes the value $+\infty$ only at zero. In general, the set $\mathfrak{s}_\nu$ of all elements sent to $+\infty$ by ν is a prime ideal of A and ν defines a valuation of the fraction field of the integral domain $A/\mathfrak{s}_\nu$. If a field k contained in A is given, then all the semivaluations of A are assumed to extend the trivial valuation of k.

We choose here to adopt an additive point of view. Observe that we may also speak in terms of multiplicative (non-Archimedean) seminorms of A. A *multiplicative seminorm* of A is a map $|\cdot| : A \to \mathbf{R}_{\geq 0}$ such that $|0| = 0$, $|1| = 1$ and $|fg| = |f||g|$, $|f+g| \leq \max\{|f|, |g|\}$ for any $f, g \in A$. A semivaluation ν of A corresponds to the multiplicative seminorm of A defined by $e^{-\nu}$. Conversely, if $|\cdot|$ is a multiplicative seminorm of A then $-\ln(|\cdot|)$ defines a semivaluation of A. A multiplicative seminorm is called an *absolute value* if $|f| = 0$ implies $f = 0$ (it gives rise to a valuation of the fraction field of A, hence it has a valuation ring naturally associated). The *trivial* absolute value of A sends any non-zero element of A to 1.

Example 1.15. Let k be a field. Consider the polynomial ring in one variable $A = k[t]$. For any real number ε such that $0 < \varepsilon < 1$, the map $\eta_\varepsilon : A \to \mathbf{R}_{\geq 0}$ defined by $\eta_\varepsilon(f) = \varepsilon^{\mathrm{ord}_t f}$ for all non zero $f \in A$ and $\eta_\varepsilon(0) = 0$, is an absolute value of A. The semivaluation associated to η_ε is $\nu_\varepsilon = -\ln \varepsilon \, \mathrm{ord}_t$. Observe that they are all equivalent as valuations of $k(t)$ and thus they are identified with the order of vanishing at the origin. However when considered as semivaluations they are pairwise different.

We say that a semivaluation of A is *centered* if it takes values in $[0, +\infty]$. Given such a semivaluation ν, we denote by Γ_ν the semigroup $\nu(A) \setminus \{+\infty\}$.

Lemma 1.16. *Let A be a noetherian integral domain. For any centered semivaluation ν of A, Γ_ν is well-ordered.*

Proof. Since Γ_ν is a subsemigroup of $\mathbf{R}_{\geq 0}$, it is totally ordered. Let S be a nonempty subset of Γ_ν and let us show that S has a least element. Consider the set of elements $f \in A$ such that $\nu(f) \in S$ and denote by I the ideal of A which it generates. Since A is noetherian, I is finitely generated. We may choose $f_1, \ldots, f_s$ verifying $\nu(f_i) \in S$ such that $I = (f_1, \ldots, f_s)A$. Then $\alpha = \min \{\nu(f_i)\}_{i=1}^s \in S$ and $\alpha \leq \beta$ for any $\beta \in S$. □

We endow the set of all centered semivaluations of A with the *topology of pointwise convergence* and we denote the space obtained by $\mathcal{V}(A)$. This topology has for a basis of open sets finite intersections of subsets of the form $\{\nu \in \mathcal{V}(A) \,/\, a < \nu(f) < b\}$ where a and b are nonnegative real numbers and f belongs to A. In other words, it is the topology induced by the product topology in $[0, +\infty]^A = \prod_j Y_j$, where each Y_j is a copy of $[0, +\infty]$ and the product is indexed by A. Observe that a net $(\nu_i)_{i \in I}$ of semivaluations in $\mathcal{V}(A)$ converges to $\nu \in \mathcal{V}(A)$ in the topology of pointwise convergence if and only if for each f in $[0, +\infty]$, $(\nu_i(f))_{i \in I}$ converges to $\nu(f)$ in $[0, +\infty]$.

Lemma 1.17. *The space $\mathcal{V}(A)$ is compact.*

Proof. The space $[0, +\infty]^A$ is Hausdorff and in addition it is quasi-compact by Tychonoff's Theorem (see [Mun00, Theorem 37.3]). Hence $\mathcal{V}(A)$ is Hausdorff, and in order to prove its quasi-compactness it suffices to show that it is closed in $[0, +\infty]^A$.

Let $(\nu_i)_{i \in I}$ be a net in $[0, +\infty]^A$ and $\nu \in [0, +\infty]^A$ such that $\nu_i \in \mathcal{V}(A)$ for all $i \in I$ and $\nu_i \to \nu$. Then we have that $\nu_i(f) \to \nu(f)$ in $[0, +\infty]$ for any $f \in A$. Since $\nu_i(f) \geq 0$ for all i, we get $\nu(f) \geq 0$. It is straightforward to verify that $\nu(0) = +\infty$, $\nu(1) = 0$ and $\nu(fg) = \nu(f) + \nu(g)$ for any $f, g \in A$. Moreover, if k is a field contained in A and $\nu_i(f) = 0$ for all $f \in k^*$, then $\nu(f) = 0$. To conclude that $\mathcal{V}(A)$ is closed and end the proof, it remains to show that $\nu(f + g) \geq \min \{\nu(f), \nu(g)\}$ for any $f, g \in A$.

Take $f, g \in A$ and denote $M_i = \min \{\nu_i(f), \nu_i(g)\}$ for any $i \in I$. The map from the product $[0, +\infty] \times [0, +\infty]$ to $[0, +\infty]$ defined by $(a, b) \mapsto \min \{a, b\}$, is continuous. The net $((\nu_i(f), \nu_i(g)))_{i \in I}$ converges to $(\nu(f), \nu(g))$, thus $M_i \to \min \{\nu(f), \nu(g)\}$ in $[0, +\infty]$. Since $\nu_i(f + g) \geq M_i$ for all $i \in I$, this yields $\nu(f + g) \geq \min \{\nu(f), \nu(g)\}$. □

From now on we suppose that A is a noetherian integral domain and $\mathfrak{p}$ is an ideal of A. For each $\nu \in \mathcal{V}(A)$ we set $\nu(\mathfrak{p}) = \min\{\nu(f) \,/\, f \in \mathfrak{p}\}$. If $\{f_1, \ldots, f_m\}$ is a system of generators of $\mathfrak{p}$, then $\nu(\mathfrak{p}) = \min\{\nu(f_i)\}_{i=1}^m$. For every $i \in \{1, \ldots, m\}$ the map from $\mathcal{V}(A)$ to $[0, +\infty]$ defined by $\nu \mapsto \nu(f_i)$ is continuous. Hence we also get a continuous map when sending $\nu \in \mathcal{V}(A)$ to $\nu(\mathfrak{p})$.

The *center* of $\nu \in \mathcal{V}(A)$ is the prime ideal $\{f \in A \,/\, \nu(f) > 0\}$ of A. Note that the center of ν contains the ideal $\mathfrak{s}_\nu$. By the previous remark, the subset $\mathcal{V}(A, \mathfrak{p})$ of $\mathcal{V}(A)$ consisting of all semivaluations ν normalized by the condition $\nu(\mathfrak{p}) = 1$, is closed. We endow this subset with the induced topology and call it the *space of normalized semivaluations (with respect to* $\mathfrak{p}$*)*. Since $\mathcal{V}(A)$ is compact, the space $\mathcal{V}(A, \mathfrak{p})$ is also compact. In addition, every semivaluation ν such that $0 < \nu(\mathfrak{p}) < +\infty$ (i.e. such that $\mathfrak{p}$ is contained in the center of ν but not in $\mathfrak{s}_\nu$) is proportional to a unique normalized semivaluation in $\mathcal{V}(A, \mathfrak{p})$.

Lemma 1.18. *Let A be an integral domain containing a field k and $\mathfrak{m}$ a maximal ideal of A. Then $\mathcal{V}(A, \mathfrak{m})$ is homeomorphic to $\mathcal{V}(A_\mathfrak{m}, \mathfrak{m}A_\mathfrak{m})$.*

Proof. Let us denote by $\jmath : A \to A_\mathfrak{m}$ the canonical ring homomorphism. For any ν in $\mathcal{V}(A_\mathfrak{m}, \mathfrak{m}A_\mathfrak{m})$ we define $\varphi(\nu) = \nu \circ \jmath$, which is a well-defined element of $\mathcal{V}(A, \mathfrak{m})$. Observe that given a semivaluation ν of $\mathcal{V}(A, \mathfrak{m})$, the condition $\nu(\mathfrak{m}) = 1$ and the maximality of $\mathfrak{m}$ imply that $\mathfrak{s}_\nu \subsetneq \mathfrak{m} = \{f \in A \,/\, \nu(f) > 0\}$. In particular, $\nu(g) = 0$ for any $g \in A$ which is not in $\mathfrak{m}$. The map $\varphi : \mathcal{V}(A_\mathfrak{m}, \mathfrak{m}A_\mathfrak{m}) \to \mathcal{V}(A, \mathfrak{m})$ is surjective. Indeed given $\nu \in \mathcal{V}(A, \mathfrak{m})$, the map $\nu' : A_\mathfrak{m} \to [0, +\infty]$ defined by $\nu'(f/g) = \nu(f)$ is a suitably normalized semivaluation of $A_\mathfrak{m}$ such that $\varphi(\nu') = \nu$. It is just as straightforward to verify that φ is also injective and continuous. Since the source and the target of φ are compact spaces, φ is an homeomorphism. $\square$

Finally we restrict our attention to centered semivaluations of the formal power series ring $\mathbf{C}[[x, y]]$. We denote the maximal ideal of $\mathbf{C}[[x, y]]$ by $\mathfrak{m}$.

Based on the fundamental work of Zariski, Spivakovsky gives in [Spi90b] the first complete classification of the valuations of the fraction field of a regular 2-dimensional local domain A whose valuation rings dominate A. This classification is taken up again in [FJ04] when $A = \mathbf{C}[[x, y]]$. This monograph gives different interpretations of valuations in the analytic, geometric and algebraic contexts and offers a new point of view describing the structure of the space $\mathcal{V} = \mathcal{V}(A, \mathfrak{m})$. Of particular importance to our work is the fact that $\mathcal{V}$ is naturally a rooted nonmetric $\mathbf{R}$-tree (see Example 3.3). The space $\mathcal{V}$ is thus called *the valuative tree.*

Let us point out here that we may define a different normalization on the subset of $\mathcal{V}$ consisting of the semivaluations ν such that $0 < \nu(\mathfrak{m}) < +\infty$. More precisely, we may consider those semivaluations taking the value 1 on z where $z \in \mathfrak{m}$ and $\operatorname{ord} z = 1$. This subset together with the valuation ord_z and equipped with the induced topology is denoted by $\mathcal{V}_z$. Recall that given a non zero $f \in A$, $\operatorname{ord}_z(f)$ is the largest power of z which divides f. The space $\mathcal{V}_z$ is called *the relative valuative tree.*

Lemma 1.19 ([FJ04], Lemma 3.59)**.** *The spaces $\mathcal{V}$ and $\mathcal{V}_z$ are homeomorphic.*

Proof. Take $\nu \in \mathcal{V}$. If $\nu(z) < +\infty$ then we set $\varphi(\nu) = \nu/\nu(z) \in \mathcal{V}_z$. Otherwise $\mathfrak{s}_\nu$ is the ideal generated by z and ν is the semivaluation of A defined by intersection multiplicity with $\{z = 0\}$. In this case we set $\varphi(\nu) = \mathrm{ord}_z$. The map φ is continuous and bijective, hence a homeomorphism. □

1.2.2 The normalized non-Archimedean link $\mathrm{NL}(X, x)$ of x in X

From the rest of this section we denote by X an algebraic variety defined over an algebraically closed field k, i.e. X is an integral separated scheme of finite type over k. We denote by $|\cdot|_0$ the trivial absolute value of k. We follow the notations introduced in [Fan14b].

We associate to X its *analytification* X^{an} in the sense of [Ber90] which is defined as follows. Consider the set of all pairs $(\xi, |\cdot|)$ where ξ is a point of X (not necessarily closed) and $|\cdot|$ is an absolute value of the residue field $\kappa(\xi)$ of X at ξ extending $|\cdot|_0$. The space X^{an} consist of this set equipped with the weakest topology such that:

1. The natural projection $\imath : X^{\mathrm{an}} \to X$ is continuous, and
2. for any open subset $U \subseteq X$ and any $f \in \mathcal{O}_X(U)$, the map $\imath^{-1}(U) \to \mathbf{R}_{\geq 0}$ which sends $(\xi, |\cdot|)$ to $|\bar{f}|$ is continuous, where $\bar{f}$ denotes the residue class of f in $\kappa(\xi)$.

The topological space X^{an} is always Hausdorff, and it is compact if and only if X is a complete variety (see [Ber90, Theorem 3.5.3]). If X is a point then X^{an} is reduced to the trivial absolute value, so we always assume that the transcendence degree of K over k is positive.

The *residue field* of X^{an} at a point $\mathrm{x} = (\xi, |\cdot|)$ is the completion of $\kappa(\xi)$ with respect to the absolute value $|\cdot|$. We denote it by $\mathscr{H}(\mathrm{x})$ and its valuation ring by $\mathscr{H}(\mathrm{x})^o$. The extension $\kappa(\xi) \hookrightarrow \mathscr{H}(\mathrm{x})$ induces a morphism $\mathrm{Spec}\,\mathscr{H}(\mathrm{x}) \to X$. If it extends to a morphism $\mathrm{Spec}\,\mathscr{H}(\mathrm{x})^o \to X$ then we say that x has a center in X (it follows from the valuative criterion of separateness that this morphism is unique whenever it exists). We denote the image of the closed point of $\mathrm{Spec}\,\mathscr{H}(\mathrm{x})^o$ under this morphism by $\mathrm{sp}_X(\mathrm{x})$ and we call it the *center* of x in X. We denote by $X^{\beth}$ the set of points in X^{an} which have a center in X and we endow it with the induced topology. By the valuative criterion of properness, $X^{\beth}$ and X^{an} coincide if and only if X is a complete variety.

The *specialization map* $\mathrm{sp}_X : X^{\beth} \to X$ which sends any point of $X^{\beth}$ to its center in X is an *anticontinuous* map (i.e. the inverse image of any open subset of X is closed in $X^{\beth}$). Recall that its analogous in the Riemann-Zariski setting, the center map $\mathrm{RZ}(X) \to X$, is continuous. Furthermore, for all $\mathrm{x} \in X^{\beth}$ the point $\mathrm{sp}_X(\mathrm{x})$ belongs to the closure of $\imath(\mathrm{x})$ in X.

Remark 1.20. Suppose that $X = \mathrm{Spec}\,A$, where A is a finitely generated k-algebra. A point $\mathrm{x} = (\xi, |\cdot|)$ of X^{an} belongs to $X^{\beth}$ if $|\bar{f}| \leq 1$ for any $f \in A$, where $\bar{f}$ denotes the residue class of f in A/ξ. If this is satisfied, then $\mathrm{sp}_X(\mathrm{x}) = \left\{ f \in A \,/\, |\bar{f}| < 1 \right\}$. Observe that ξ is contained in $\mathrm{sp}_X(\mathrm{x})$, which means exactly that $\mathrm{sp}_X(\mathrm{x})$ is in the closure of ξ.

The space $X^{\beth}$ is in fact homeomorphic to $\mathcal{V}(A)$. Indeed, a point $(\xi, |\cdot|) \in X^{\beth}$ gives rise to the centered semivaluation

$$A \to A/\xi \xrightarrow{|\cdot|} [0, 1] \xrightarrow{-\ln} [0, +\infty]$$

of A. Conversely, with the notations of Subsection 1.2.1, a semivaluation ν of $\mathcal{V}(A)$ defines a prime ideal $\mathfrak{s}_\nu$ of A and an absolute value $|\cdot|$ of $\kappa(\mathfrak{s}_\nu)$ by setting $e^{-\nu(\bar{f})}$ for any $\bar{f} \in A/\mathfrak{s}_\nu$. This gives a one-to-one correspondence. Continuity follows directly from the definition of the topologies:

The weakest topology on X^{an} which satisfies condition 2 also verifies condition 1. If we take a non zero $f \in A$ and denote by $D_f = \{\xi \in X \mathbin{/} f \notin \xi\}$ the basic open subset of X defined by f, then the inverse image of D_f under the map $\imath$ is the set of all $(\xi, |\cdot|) \in X^{\mathrm{an}}$ such that $|\bar{f}| > 0$.

Finally, take a non zero $f \in A$. The inverse image of $X \setminus D_f$ under sp_X is the set of all points $(\xi, |\cdot|)$ of $X^{\beth}$ such that $|\bar{f}| < 1$, which is an open subset of $X^{\beth}$. Hence the map sp_X is anticontinuous.

Let us fix a subvariety Z of X. For any $\mathrm{x} \in X^{\beth}$, if x belongs to $\imath^{-1}(Z)$ then it is also in $\mathrm{sp}_X^{-1}(Z)$. We define $\mathrm{L}(X,Z) = \mathrm{sp}_X^{-1}(Z) \setminus (X^{\beth} \cap \imath^{-1}(Z))$, which is an open subset of $X^{\beth}$, and we endow it with the induced topology from the topology of $X^{\beth}$. In [Fan14b] the space $\mathrm{L}(X,Z)$ is referred to as the *non-Archimedean link of Z in X*.

Let $\pi : X' \to X$ be a proper morphism which induces an isomorphism from the open subset $X' \setminus \pi^{-1}(Z)$ to the open subset $X \setminus Z$. We have a well-defined mapping from $\mathrm{L}(X', \pi^{-1}(Z))$ to $\mathrm{L}(X,Z)$ by sending $\mathrm{x}' = (\xi', |\cdot|)$ to $\mathrm{x} = (\pi(\xi'), |\cdot|)$. The center of x in X is $\pi(\mathrm{sp}_{X'}(\mathrm{x}'))$. Take now a point $\mathrm{x} = (\xi, |\cdot|)$ of $\mathrm{L}(X,Z)$. Since $X' \setminus \pi^{-1}(Z) \to X \setminus Z$ is an isomorphism, there exists a unique morphism from $\operatorname{Spec} \mathscr{H}(\mathrm{x})$ to X' making commutative the diagram

$$\begin{tikzcd} \operatorname{Spec} \mathscr{H}(\mathrm{x}) \arrow[r] \arrow[d] & X' \arrow[d, "\pi"] \\ \operatorname{Spec} \mathscr{H}(\mathrm{x})^o \arrow[r, "\varphi_{\mathrm{x}}"] & X \end{tikzcd}$$

where φ_{x} comes from the fact that $\mathrm{x} \in X^{\beth}$. By the valuative criterion of properness, there exists a unique morphism $\overline{\varphi}_{\mathrm{x}}$ from $\operatorname{Spec} \mathscr{H}(\mathrm{x})^o$ to X' such that $\varphi_{\mathrm{x}} = \pi \circ \overline{\varphi}_{\mathrm{x}}$. Therefore the point $\mathrm{x}' = (\pi^{-1}(\xi), |\cdot|)$ belongs to $\mathrm{L}(X', \pi^{-1}(Z))$. This defines a bijective map from $\mathrm{L}(X', \pi^{-1}(Z))$ to $\mathrm{L}(X,Z)$ which is in fact an homeomorphism (see [Thu07, Proposition 11.1]). For any $\mathrm{x} \in \mathrm{L}(X,Z)$, we say that $\mathrm{sp}_{X'}(\mathrm{x}')$ is the center of x in X'. Observe that the map $\mathrm{sp}_{X'} : \mathrm{L}(X,Z) \to X'$ is anticontinuous.

The space $\mathrm{L}(X,Z)$ was introduced in [Thu07], where it is called the generic fiber of the formal completion of X along Z. If Z is the singular locus of X, then [Thu07, Proposition 4.7] shows that $\mathrm{L}(X,Z)$ is homotopy equivalent to the dual complex associated to the exceptional divisor of a resolution of singularities of (X,Z) whose exceptional divisor has simple normal crossings. This results holds whenever the base field k is perfect.

From now on we concentrate in the case where Z is reduced to a closed point x of X. Then there is only one point in X^{an} of the form $(x, |\cdot|)$, namely $|\cdot|$ must be $|\cdot|_0$. Hence $\mathrm{L}(X,x) = \mathrm{sp}_X^{-1}(x) \setminus \{(x, |\cdot|_0)\}$.

Remark 1.21. Let $U = \operatorname{Spec} A$ be an open affine neighborhood of x in X and $\mathfrak{m}$ the maximal ideal of A corresponding the point x. For any $\mathrm{x} = (\xi, |\cdot|) \in \mathrm{L}(X,x)$, since $\mathrm{sp}_X(\mathrm{x}) = x$ belongs to the closure of ξ in X, the point ξ belongs to U. The homeomorphism from $U^{\beth}$ to $\mathcal{V}(A)$ given in Remark 1.20 induces an homeomorphism from $\mathrm{L}(X,x)$

to $\{\nu \in \mathcal{V}(A) \,/\, 0 < \nu(\mathfrak{m}) < +\infty\}$ equipped with the topology induced by the topology of $\mathcal{V}(A)$.

We now identify two points of the space $\mathrm{L}(X, x)$ if they define the same valuation. More precisely, we consider the following action of $\mathbf{R}_{>0}$ on $\mathrm{L}(X, x)$: given $\lambda \in \mathbf{R}_{>0}$ and a point $(\xi, |\cdot|)$ of $\mathrm{L}(X, x)$, we define $\lambda \cdot (\xi, |\cdot|) = (\xi, |\cdot|^{\lambda})$.

Definition 1.22 (Normalized non-Archimedean link of x in X)**.** *Given a closed point $x \in X$, the normalized non-Archimedean link of x in X, denoted by $\mathrm{NL}(X, x)$, is the quotient of $L(X, x)$ by the group action defined above, endowed with the quotient topology.*

In [Fan14a] the space $\mathrm{NL}(X, x)$ is introduced for an arbitrary subvariety of X and endowed in a richer structure. This space carries a natural analytic structure locally modeled on affinoid spaces over $k(\!(t)\!)$. However, these local $k(\!(t)\!)$-analytic structures are not canonical and cannot in general be glued to get a global one. Here we only concern ourselves with topology.

Lemma 1.23. *If $U = \operatorname{Spec} A$ is an open affine neighborhood of x in X and $\mathfrak{m}$ the maximal ideal of A defining the point x, then $\mathrm{NL}(X, x)$ is homeomorphic to the space of normalized semivaluations $\mathcal{V}(A, \mathfrak{m})$.*

Proof. By Remark 1.21 it suffices to show that $\mathcal{V}(A, \mathfrak{m})$ is homeomorphic to the quotient $\mathcal{L}/\mathbf{R}_{>0}$ of the space $\mathcal{L} = \{\nu \in \mathcal{V}(A) \,/\, 0 < \nu(\mathfrak{m}) < +\infty\}$ by the action of $\mathbf{R}_{>0}$ given by multiplication: $(\lambda \cdot \nu)(f) = \lambda\, \nu(f)$ for all $f \in A$.
We define a surjective map φ from $\mathcal{L}$ to $\mathcal{V}(A)$ as follows. Given $\nu \in \mathcal{L}$ we set $\varphi(\nu) = \frac{\nu}{\nu(\mathfrak{m})}$. Since the map from $\mathcal{V}(A)$ to $[0, +\infty]$ defined as $\nu \mapsto \nu(\mathfrak{m})$ is continuous, the map φ is also continuous. Furthermore, $\varphi(\nu) = \varphi(\nu')$ if and only if ν and ν' are proportional, so φ descends to the quotient. Therefore we have a map $\overline{\varphi} : \mathcal{L}/\mathbf{R}_{>0} \to \mathcal{V}(A)$ which is continuous and bijective. Its inverse map is the composition of the embedding $\mathcal{V}(A) \hookrightarrow \mathcal{L}$ and the quotient map $\mathcal{L} \to \mathcal{L}/\mathbf{R}_{>0}$. Hence φ is an homeomorphism. $\square$

In view of Lemma 1.18, $\mathrm{NL}(X, x)$ is also homeomorphic to the space of normalized semivaluations $\mathcal{V}(\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$.

Finally, we compute the covering dimension of $\mathrm{NL}(X, x)$. When X is a curve, any semivaluation of $\mathrm{NL}(X, x)$ is in fact a valuation of the function field K of X whose valuation ring dominates $\mathcal{O}_{X,x}$. Thus $\mathrm{NL}(X, x)$ is a finite space in bijection with $\mathrm{RZ}(X, x)$ (each point corresponds to a maximal ideal of the integral closure of $\mathcal{O}_{X,x}$ in K). Since $\mathrm{NL}(X, x)$ is Hausdorff, it is a discrete topological space and every open cover of the space has a refinement consisting of disjoint open sets, so that its covering dimension is zero. More generally we have:

Proposition 1.24. *The covering dimension of $\mathrm{NL}(X, x)$ is not greater than $\dim X - 1$.*

Proof. Let us denote by d the dimension of X. According to [Mun00, Theorem 32.2] every compact (quasi-compact and Hausdorff) space is normal. Hence $\mathrm{NL}(X, x)$ is a normal space. If $\mathrm{NL}(X, x) = \bigcup_{i \geq 1} F_i$ where each F_i is a closed subspace of covering dimension not exceeding $d - 1$, then by [Pea75, Ch. 3, Theorem 2.5], the covering dimension of $\mathrm{NL}(X, x)$ is not greater than $d - 1$. Let us show that such a family $\{F_i\}_{i \geq 1}$ exists.

Take an open affine neighborhood $U = \operatorname{Spec} A$ of x in X and a system of generators $\{f_1, \dots, f_s\}$ of the maximal ideal $\mathfrak{m}$ corresponding to the point x. For all $1 \le i \le s$, we define $F_i = \{\nu \in \mathrm{NL}(X, x) \,/\, \nu(f_i) = 1\}$. Since all the maps $\nu \mapsto \nu(f_i)$ are continuous, $\{F_i\}_{i=1}^{s}$ is a family of closed subset of $\mathrm{NL}(X, x)$. We have $\mathrm{NL}(X, x) = \bigcup_{i=1}^{s} F_i$, so in order to end the proof it suffices to show that F_i has covering dimension at most $d - 1$ for all $i \in \{1, \dots, s\}$.

Let us take an integer i, $1 \le i \le s$. The regular function $f_i \in A$ induces a morphism $U \to \operatorname{Spec} k[t] = \mathbf{A}^1_k$ so that A can be regarded as a $k[t]$-module. We set $B = A \otimes_{k[t]} k(\!(t)\!)$. Note that $f_i \otimes 1 = 1 \otimes t$ in B. Since A is a k-algebra of finite type, B is a $k(\!(t)\!)$-algebra of finite type. We denote $V = \operatorname{Spec} B$ and consider its analytification V^{an} in the sense of Berkovich. By [Ber90, Theorem 3.4.8(iv)], the covering dimension of V^{an} is equal to the dimension of V, which is $d - 1$. Therefore the covering dimension of any closed subspace of V^{an} is less or equal than $d - 1$ (see [Pea75, Proposition 1.5]). To complete the proof we now show that $F = F_i$ can be identified with a closed subspace of V^{an}.

By definition, V^{an} is the set of semivaluations of B which extend the valuation ord_t of $k(\!(t)\!)$, endowed with the topology of pointwise convergence. In other words, the underlying set of V^{an} consists of all semivaluations ν of B that are trivial on k^* and satisfy $\nu(1 \otimes t) = 1$. We consider the closed subspace

$$W = \left\{\nu \in V^{\beth} \,/\, \nu(f_j \otimes 1) \ge 1 \;\forall j \in \{1, \dots, s\}\,,\; j \ne i\right\}$$

of V^{an}. We claim that W is homeomorphic to F. Indeed, the map φ from W to F which sends $\nu \in W$ to the semivaluation of A defined by $g \mapsto \nu(g \otimes 1)$ is continuous. Furthermore, φ is a bijection. If ν is a semivaluation of A lying in F then it extends in a unique way to a semivaluation of B. This extension is defined by $\widetilde{\nu}(g \otimes 1) = \nu(g)$ and $\widetilde{\nu}(1 \otimes t) = \widetilde{\nu}(f_i \otimes 1) = 1$. We have $\widetilde{\nu}(f_j \otimes 1) = \nu(f_j) \ge 1$ for all $j \in \{1, \dots, s\}$, $j \ne i$, hence $\widetilde{\nu} \in W$. Since W is quasi-compact and F is Hausdorff, the map φ is a homeomorphism. $\square$

Corollary 1.25. *If the point x is either a regular point or an isolated singularity of X and we assume the existence of resolution of singularities, the covering dimension of* $\mathrm{NL}(X, x)$ *is* $\dim X - 1$.

Proof. We keep the notations of the proof of Proposition 1.24. We know that the covering dimension of $\mathrm{NL}(X, x)$ is at most $d - 1$. Since covering dimension is monotone on closed subspaces (see [Pea75, Proposition 1.5]), in order to prove the statement it suffices to find a closed subspace F of $\mathrm{NL}(X, x)$ whose covering dimension is $d - 1$.

Let $\pi : X' \to X$ be a proper birational map such that X' is non-singular, π is an isomorphism over $X \setminus \{x\}$ and the exceptional divisor $E = \operatorname{supp} \pi^{-1}(x)$ is a simple normal crossings divisor. Without loss of generality, we may assume that there exist d irreducible components $E_1, \dots, E_d$ of the exceptional divisor such that $\bigcap_{i=1}^{d} E_i$ is reduced to a point x'. We denote by $b_1, \dots, b_d$ its multiplicities. Pick local coordinates $(z_1, \dots, z_d)$ at the point x' such that $E_i = \{z_i = 0\}$. Since the point x' is regular, the completion of the local ring of X' at x' is isomorphic as k-algebra to $k[[z_1, \dots, z_d]]$. Any $\beta = (\beta_1, \dots, \beta_d) \in \mathbf{R}^d_{\ge 0}$ such that $\sum_{i=1}^{d} \beta_i b_i = 1$ gives rise to a valuation ν_β of $\mathrm{NL}(X, x)$. It suffices to set $\nu_\beta(0) = +\infty$ and

$$\nu_\beta(f) = \min\left\{\sum_{i=1}^{d} \beta_i \alpha_i \,/\, c_\alpha \ne 0\right\},$$

if $f \in A_{\mathfrak{m}}$ is written as $\sum_\alpha c_\alpha z^\alpha$ in $k[[z_1, \dots, z_d]]$.

We define a map φ from the simplex $\Delta = \left\{(y_1, \ldots, y_d) \in \mathbf{R}^d_{\geq 0} \;/\; \sum_{i=1}^d y_i = 1\right\}$ to $\mathrm{NL}(X, x)$ as follows. Given $y = (y_1, \ldots, y_d) \in \Delta$, we set $\varphi(y) = \nu_{\beta(y)}$ where $\beta(y) = (y_1/b_1, \ldots, y_d/b_d)$. The map φ is injective and continuous (see [JM12] for details). The fact that Δ is quasi-compact implies that $\varphi(\Delta)$ is quasi-compact and thus closed in $\mathrm{NL}(X, x)$. Therefore φ yields a homeomorphism between Δ and a closed subspace $F = \varphi(\Delta)$ of $\mathrm{NL}(X, x)$. Since Δ has covering dimension $d - 1$, the subspace F has also covering dimension $d - 1$. □

1.2.3 The canonical map from $\mathrm{RZ}(X, x)$ to $\mathrm{NL}(X, x)$

Let X be an algebraic variety defined over an algebraically closed field k and $x \in X$ a closed point. The aim of this subsection is to show that there exists a natural surjective continuous map from $\mathrm{RZ}(X, x)$ to $\mathrm{NL}(X, x)$.

Given a valuation $\nu \in \mathrm{RZ}(X, x)$ of rank $r \geq 1$, we consider the maximal chain of valuation rings of K containing R_ν, $R_\nu = R_{\nu_r} \subsetneq R_{\nu_{r-1}} \subsetneq \ldots \subsetneq R_{\nu_1}$. Let us choose j to be the smallest integer $i \in \{1, \ldots, r\}$ such that ν_i is centered in x. If $j = 1$ then we set

$$\pi(\nu)(f) = \frac{\nu_1(f)}{\nu_1(\mathfrak{m}_{X,x})}$$

for any non zero $f \in \mathcal{O}_{X,x}$. Let us now suppose $r > 1$ and $j > 1$. We define then $\pi(\nu)$ as follows.

The quotient ring $R_{\nu_j}/m_{\nu_{j-1}}$ is a valuation ring of the residue field of the valuation ν_{j-1}. It corresponds to the rank one valuation $\overline{\nu}_{j-1}$ such that $\nu_j = \nu_{j-1} \circ \overline{\nu}_{j-1}$. By definition of the integer j, its center in the ring $\mathcal{O}_{X,x}/(\mathcal{O}_{X,x} \cap m_{\nu_{j-1}})$ is $\mathfrak{m}_{X,x}/(\mathcal{O}_{X,x} \cap m_{\nu_{j-1}})$. Therefore, given a non zero $f \in \mathcal{O}_{X,x}$, by setting

$$\pi(\nu)(f) = \begin{cases} \overline{\nu}_{j-1}(\bar{f}) & \text{if } f \notin m_{\nu_{j-1}} \\ +\infty & \text{otherwise} \end{cases}$$

where $\bar{f}$ denotes the residue class of f in $\mathcal{O}_{X,x}/(\mathcal{O}_{X,x} \cap m_{\nu_{j-1}})$, we define a semivaluation $\pi(\nu)$ on X centered in x. After division by a suitable constant we obtain an element of $\mathrm{NL}(X, x)$ that, by abuse of notation, we will also call $\pi(\nu)$.

Let us introduce some notations. Given $\nu \in \mathrm{RZ}(X, x)$, in the sequel we will denote by ν_* the valuation ν_j and when $j > 1$, ν'_* the valuation ν_{j-1} and $\overline{\nu}_*$ the rank one valuation $\overline{\nu}_{j-1}$.

Proposition 1.26. *The map $\pi : \mathrm{RZ}(X, x) \to \mathrm{NL}(X, x)$ is surjective and continuous.*

We will make use of the following result to prove Proposition 1.26.

Lemma 1.27. *Let X be an algebraic variety and Y, Z two subvarieties of X, neither one containing the other. Let $X' \to X$ be the blowing up of X at $Y \cap Z$ (defined as the sum $\mathcal{I}_Y + \mathcal{I}_Z$ of the ideal sheaves). Then the strict transforms of Y and Z have empty intersection.*

Proof. Without loss of generality we may assume that X is affine. Suppose that $\mathcal{I}_Y$ is generated by $\{f_i\}_{i=1}^n$ and $\mathcal{I}_Z$ by $\{g_j\}_{j=1}^m$. The strict transform of Y is empty at a point x' of X' where $(\mathcal{I}_Y + \mathcal{I}_Z)\cdot\mathcal{O}_{X',x'} = (f_i)\mathcal{O}_{X',x'}$ for some i. Similarly, the strict transform of Z is empty when this ideal is generated by g_j for some j. Therefore the strict transforms of Y and Z are disjoint. □

We are now in position to prove Proposition 1.26.

Proof of Proposition 1.26. Let us denote by R the local ring of X at the point x and $\mathfrak{m}$ its maximal ideal. We call K its fraction field.

The map π is surjective. For any $v \in \mathrm{NL}(X,x)$, $\mathfrak{s}_v = v^{-1}(+\infty)$ is a prime ideal of R. If this ideal is reduced to zero, then v extends in a unique way to a rank one valuation of K and the image by π of this valuation is v. Otherwise v induces a valuation $\overline{v}$ of the fraction field of the quotient ring $R/\mathfrak{s}_v$ whose center in this ring is the maximal ideal $\mathfrak{m}/\mathfrak{s}_v$. The choice of a valuation ν' of K such that $R_{\nu'}$ dominates the localization $R_{\mathfrak{s}_v}$ gives us a composite valuation $\nu'\circ\overline{v}$ of K with center x in X and whose image by π is v. Therefore π is a surjective mapping.

The map π is continuous. To prove that π is continuous it suffices to show that the inverse image of any open set of the form $\{v \in \mathrm{NL}(X,x) \;/\; a < v(f) < b\}$, where $a, b \in \mathbf{Q}$, $1 \le a < b$ and $f \in \mathfrak{m}$, is open in $\mathrm{RZ}(X,x)$. The openness of such a subset will follow once we have proved that the subsets $U_{>\alpha} = \{\nu \in \mathrm{RZ}(X,x) \;/\; \alpha < \pi(\nu)(f) \le +\infty\}$ and $U_{<\alpha} = \{\nu \in \mathrm{RZ}(X,x) \;/\; \pi(\nu)(f) < \alpha\}$ are open subsets of $\mathrm{RZ}(X,x)$ for any rational number $\alpha \ge 1$ and any non zero $f \in \mathfrak{m}$. We shall only prove that $U_{>\alpha}$ is open. The same arguments show that $U_{<\alpha}$ is also open. In order to prove that $U_{>\alpha}$ is open, for any $\nu \in U_{>\alpha}$ we construct an open set $\mathcal{U} \subseteq \mathrm{RZ}(X,x)$ included in $U_{>\alpha}$ that contains ν.

Pick $\alpha = p/q$ with p, q coprime integers, $p \ge q > 0$ and a non zero $f \in \mathfrak{m}$. We denote by $\phi : X' \to \operatorname{Spec} R$ the blowing-up of $\operatorname{Spec} R$ at its closed point. Let ν be a valuation in $U_{>\alpha}$.

Case 1: $\pi(\nu)(f) = +\infty$. Let $\eta : Y \to X'$ be the normalized blowing-up of $(\mathfrak{m}^N + (f))\cdot\mathcal{O}_{X'}$, with $N > \alpha$. The composed morphism $\psi = \phi\circ\eta : Y \to X$ is a proper birational morphism that is an isomorphism over $X \setminus \{x\}$ and $\psi^{-1}(x)$ is purely of codimension one. Let $\mathrm{c} : \mathrm{RZ}(X,x) \to Y$ be the continuous map which associates to any valuation in $\mathrm{RZ}(X,x)$ its center in Y.

Observe that $y = \mathrm{c}(\nu)$ is contained the strict transform of $\{f = 0\}$. Indeed, the hypothesis on $\pi(\nu)(f)$ implies that the center z' of ν'_* in X' is contained in $\{f = 0\}$. Since ν'_* is not centered in the point $x \in X$, we deduce that z' is not contained in the center of the blowing-up η. Hence $\mathrm{c}(\nu'_*)$ (which is the strict transform of z') is contained in the strict transform of $\{f = 0\}$ and therefore y is also contained in that strict transform.

Pick any irreducible component E of $\psi^{-1}(x)$ that contains y. Let $g \in \mathcal{O}_{X,x}$ be such that $(g)\cdot\mathcal{O}_{Y,y} = \mathfrak{m}\cdot\mathcal{O}_{Y,y}$. Then $(g^N)\cdot\mathcal{O}_{Y,y} = (\mathfrak{m}^N + (f))\cdot\mathcal{O}_{Y,y}$, since otherwise f would generate $(\mathfrak{m}^N + (f))\cdot\mathcal{O}_{Y,y}$ and this is impossible because the strict transform of $\{f = 0\}$ contains y. We conclude that g^N divides f in $\mathcal{O}_{Y,y}$. Therefore

$$\nu_E(\mathfrak{m}) = \nu_E(g) \le \frac{\nu_E(f)}{N} < \frac{\nu_E(f)}{\alpha},$$

where ν_E denotes the divisorial valuation defined by E. This means that all the irreducible components of $\psi^{-1}(x)$ containing the point y verify $\pi(\nu_E) \in U_{>\alpha}$.

Take an open neighborhood $U \subset Y$ of y which is strictly contained in $E_{\mathcal{G}} \setminus (E_{\mathcal{G}} \cap D)$, where $E_{\mathcal{G}}$ is the union of all the irreducible components of $\psi^{-1}(x)$ containing y and D is the union of the remaining ones. Let us prove that $\mathcal{U} = \mathrm{c}^{-1}(U)$ is an open subset of $\mathrm{RZ}(X, x)$ which satisfies the desired properties.

It is an open set since U is open and c is continuous, and it contains ν by construction. Finally we show that $\mathcal{U}$ is contained in $U_{>\alpha}$. Take $\mu \in \mathcal{U}$ and set $z = \mathrm{c}(\mu_*)$. The center of μ in Y belongs to U, so that z is also in U. Since the center of μ_* in X is the point x, there exists E in $E_{\mathcal{G}}$ such that z belongs to E. Pick $g \in \mathcal{O}_{X,x}$ such that $(g) \cdot \mathcal{O}_{Y,z} = \mathfrak{m} \cdot \mathcal{O}_{Y,z}$. We have $\pi(\nu_E) \in U_{>\alpha}$, so $\nu_E(f^p/g^q) > 0$ and f^p/g^q belongs to $\mathfrak{m}_{Y,z}$ (note that f^p/g^q is without indeterminacy). In particular, $\mu_*(f^p/g^q) > 0$ and we deduce that $\pi(\mu)(f) > \alpha$ as required.

Case 2: $\pi(\nu)(f) < +\infty$. We replace the birational morphism $\eta : Y \to X'$ above by the normalized blowing-up of $(\mathfrak{m}^p + (f^q)) \cdot \mathcal{O}_{X'}$. Abusing notation, we denote by η this birational morphism. Write $\psi : Y \to X$ for the composed birational morphism as before and $c : \mathrm{RZ}(X, x) \to Y$ for the center map.

Denote $y_* = \mathrm{c}(\nu_*)$ and pick $g \in \mathcal{O}_{X,x}$ such that $(g) \cdot \mathcal{O}_{Y,y_*} = \mathfrak{m} \cdot \mathcal{O}_{Y,y_*}$. By construction either f^q/g^p or g^p/f^q are regular at the point y_*. Since $\nu \in U_{>\alpha}$ we deduce that f^q/g^p must vanish at y_*.

We say that an irreducible component E of $\psi^{-1}(x)$ is *good* when the image by the map π of the divisorial valuation ν_E defined by E belongs to $U_{>\alpha}$. It is *bad* otherwise. We may assume that y_* belongs at least to one good component. Otherwise it suffices to consider the blowing-up $Y' \to Y$ of Y with respect to the sheaf of ideals defining y_*. The center of ν_* in Y' must be in the newly created exceptional divisor, all of whose components are good because f^q/g^p has to vanish on them.

Let $E_{\mathcal{G}}$ (resp. $E_{\mathcal{B}}$) be the union of all good (resp. bad) components containing y_* and D the union of the irreducible components of $\psi^{-1}(x)$ which do not contain y_*. We denote by $\mathcal{I}_{\mathcal{G}}$ (resp. $\mathcal{I}_{\mathcal{B}}$) the sheaf of ideals defining $E_{\mathcal{G}}$ (resp. $E_{\mathcal{B}}$), and for any integer $l \geq 1$ we consider the normalized blowing-up $\phi_l : Y_l \to Y$ of the sheaf of ideals $\mathcal{I}_{\mathcal{G}}^l + \mathcal{I}_{\mathcal{B}}$.

Claim: The center of ν_* in Y_l does not belong to the strict transform of any bad component at least for any l large enough.

Proof of the claim. We fix $l \geq 1$, and suppose that the center z of ν_* in Y_l belongs to the strict transform of some bad component. This means that $(\mathcal{I}_{\mathcal{G}}^l + \mathcal{I}_{\mathcal{B}}) \cdot \mathcal{O}_{Y_l,z} = \mathcal{I}_{\mathcal{G}}^l \cdot \mathcal{O}_{Y_l,z}$. Denote $\mathcal{J}_{\mathcal{B}} = \mathcal{I}_{\mathcal{B}} \cdot \mathcal{O}_{Y_l,z}$ and $\mathcal{J}_{\mathcal{G}} = \mathcal{I}_{\mathcal{G}} \cdot \mathcal{O}_{Y_l,z}$. This gives $\nu_*(\mathcal{J}_{\mathcal{B}}) \geq l\, \nu_*(\mathcal{J}_{\mathcal{G}}) > 0$. If ν_* is a rank one valuation, then l is bounded and this ends the proof. Suppose that ν_* has rank larger than one. Since ν'_* is not centered in $x \in X$, we have $\nu'_*(\mathcal{J}_{\mathcal{B}}) = \nu'_*(\mathcal{J}_{\mathcal{G}}) = 0$. This implies that $\nu_*(\mathcal{J}_{\mathcal{B}})$ and $\nu_*(\mathcal{J}_{\mathcal{G}})$ belong to the convex subgroup of Φ_{ν_*} where $\overline{\nu}_*$ takes its values. This subgroup has rank one, so we conclude again that l is bounded. □

Let us fix l for which the claim applies. Then we consider the complement U in Y_l of the union of the strict transforms of $E_{\mathcal{B}}$ and D. Let

$$\mathcal{U} = \{\mu \in \mathrm{RZ}(X, x) \text{ such that the center of } \mu \text{ in } Y_l \text{ belongs to } U\}\,.$$

This is an open set since U is open, and it contains ν since the center of ν is included in the one of ν_* which belongs to U. In order to complete the proof, we now show that $\mathcal{U}$ is contained in $U_{>\alpha}$.

Take $\mu \in \mathcal{U}$ and denote by z the center of μ_* in Y_l. The center of μ in Y_l belongs to U, so that z is also in U. Since the center in X of μ_* is the point x, we deduce that either z belongs to the strict transform of E for some E in $E_{\mathcal{G}}$ or z belongs to the exceptional locus of ϕ_l. Suppose that the first happens. Pick $g \in \mathcal{O}_{X,x}$ such that $(g)\cdot\mathcal{O}_{Y_l,z} = \mathfrak{m}\cdot\mathcal{O}_{Y_l,z}$. Since E is a good component, $\nu_E(f^p/g^q) > 0$ and f^p/g^q belongs to $\mathfrak{m}_{Y_l,z}$ (note that f^p/g^q is without indeterminacy). We get $\pi(\mu)(f) > \alpha$ as required. If z belongs to the exceptional locus of ϕ_l, then $\mathrm{c}(\mu_*)$ is $\phi_l(z) \in E_{\mathcal{G}}$ and we also have $\pi(\mu)(f) > \alpha$. $\square$

1.2.4 Largest Hausdorff quotient of $\mathrm{RZ}(X, x)$: the normal surface case

The purpose of this subsection is to prove the following:

Proposition 1.28. *If x is a normal point of a surface X, the space $\mathrm{NL}(X,x)$ is the largest Hausdorff quotient of $\mathrm{RZ}(X,x)$.*

Given $\nu, \nu' \in \mathrm{RZ}(X,x)$, we write $\nu \sim \nu'$ if there exists $\mu \in \mathrm{RZ}(X,x)$ such that ν and ν' are both in the closure of μ. The binary relation $\sim$ on $\mathrm{RZ}(X,x)$ is obviously reflexive and symmetric. It is in fact an equivalence relation:
Suppose that $\{\nu,\nu'\} \subseteq \overline{\{\mu\}}$ and $\{\nu',\nu''\} \subseteq \overline{\{\mu'\}}$, where the bar means closure in $\mathrm{RZ}(X,x)$. Then $R_{\nu'} \subseteq R_\mu$ and $R_{\nu'} \subseteq R_{\mu'}$. Since the valuation rings of K containing any fixed valuation ring form a chain with respect to set inclusion, either $R_\mu \subseteq R_{\mu'}$ or $R_{\mu'} \subseteq R_\mu$. Without loss of generality we may assume that $R_\mu \subseteq R_{\mu'}$ and thus $\{\nu,\nu''\} \subseteq \overline{\{\mu'\}}$.

Let us denote $q : \mathrm{RZ}(X,x) \to \mathrm{RZ}(X,x)/\!\sim$ the quotient map, which sends a valuation to the equivalence class containing it, and $\mathrm{RZ}(X,x)^\sim$ the quotient space $\mathrm{RZ}(X,x)/\!\sim$ (i.e. the set of equivalence classes of elements of $\mathrm{RZ}(X,x)$ equipped with the finest topology which makes q continuous).

Consider $\nu,\nu' \in \mathrm{RZ}(X,x)$ with $q(\nu) = q(\nu')$. By definition there exists $\mu \in \mathrm{RZ}(X,x)$ such that ν and ν' belong to the closure of μ. It follows directly from the definition of the map π in Proposition 1.26 that $\pi(\nu) = \pi(\nu') = \pi(\mu)$, so π factors through q. Let us denote $\tilde{\pi}$ the unique continuous map from $\mathrm{RZ}(X,x)^\sim$ to $\mathrm{NL}(X,x)$ satisfying $\tilde{\pi} \circ q = \pi$.

Proposition 1.29. *In the hypothesis of Proposition 1.28, the spaces $\mathrm{RZ}(X,x)^\sim$ and $\mathrm{NL}(X,x)$ are homeomorphic.*

Proof. Since $\mathrm{RZ}(X,x)^\sim$ is the image of $\mathrm{RZ}(X,x)$ by q and $\mathrm{RZ}(X,x)$ is quasi-compact, $\mathrm{RZ}(X,x)^\sim$ is also quasi-compact. Thus $\tilde{\pi}$ is a continuous surjective map from a quasi-compact space to a Hausdorff space, and therefore to establish the result it is enough to show that $\tilde{\pi}$ is injective.
Suppose that $\pi(\nu) = \pi(\nu') = v$ for some $\nu,\nu' \in \mathrm{RZ}(X,x)$. Let us prove that $\nu \sim \nu'$. If v is a valuation of K then by construction ν and ν' are both in the closure of v, hence $\nu \sim \nu'$. We assume now that $\mathfrak{s}_v = v^{-1}(+\infty)$ is not reduced to zero. By the hypothesis on the dimension, the valuations ν and ν' have necessarily rank two. Writing $\nu = \nu_1 \circ \overline{\nu}_1$ give us that $\mathfrak{s}_v = \mathcal{O}_{X,x} \cap m_{\nu_1} \subsetneq \mathfrak{m}_{X,x}$. The localization $(\mathcal{O}_{X,x})_{\mathfrak{s}_v}$ is a local ring of dimension one with fraction field K. The fact that $\mathcal{O}_{X,x}$ is integrally closed implies that $(\mathcal{O}_{X,x})_{\mathfrak{s}_v}$

is also integrally closed, and therefore a valuation ring of K. Since it is dominated by R_{ν_1} we must have $(\mathcal{O}_{X,x})_{\mathfrak{s}_v} = R_{\nu_1}$. The same arguments apply to ν', so we conclude that there exists $\mu \in \mathrm{RZ}(X)$ (but which does not belong to $\mathrm{RZ}(X,x)$) such that $\nu = \mu \circ \overline{\nu}$ and $\nu' = \mu \circ \overline{\nu'}$. Now it suffices to observe that $\pi(\nu) = \pi(\nu')$ means that the valuations $\overline{\nu}$ and $\overline{\nu'}$ are the same when restricted to the subring $\mathcal{O}_{X,x}/(\mathcal{O}_{X,x} \cap m_\mu)$ of the residue field k_μ. Thus $\overline{\nu} = \overline{\nu'}$ and then $\nu = \nu'$. □

This enables us to complete the proof of Proposition 1.28.

Proof of Proposition 1.28. The space $\mathrm{NL}(X,x)$ is homeomorphic to $\mathrm{RZ}(X,x)^\sim$ by Proposition 1.29. Since it is Hausdorff, in order to prove the corollary, we need to prove that any continuous map from $\mathrm{RZ}(X,x)$ into a Hausdorff space factors uniquely through $q : \mathrm{RZ}(X,x) \to \mathrm{RZ}(X,x)^\sim$ (see [Mac71] Ch. V, 9, Proposition 2). This property follows from the fact that in dimension two the equivalence relation $\sim$ on $\mathrm{RZ}(X,x)$ translates into the following: two valuations of $\mathrm{RZ}(X,x)$ are equivalent if they belong to the closure of the same divisorial valuation of $\mathrm{RZ}(X,x)$ (hence they cannot be separated by disjoint open sets and as a consequence they have the same the image under any continuous map to Hausdorff space). □

The map $\tilde{\pi}$ is not in general injective as illustrated by the following example:

Example 1.30. Consider $R = \mathbf{C}[x_1,x_2,x_3]_{(x_1,x_2,x_3)}$ and denote by K its fraction field. Recall that by the order of a polynomial at x_i we refer to the largest power of x_i which divides the polynomial. We define a rank two valuation v_2 of K by setting

$$\nu_2(f) = (\nu_1(f), \mathrm{ord}_{x_2} f_1(0,x_2,x_3)) \in \mathbf{Z}^2_{\mathrm{lex}}$$

for any nonzero $f \in \mathbf{C}[x_1,x_2,x_3]$, where $\nu_1(f) = \mathrm{ord}_{x_1} f$ and $f_1 = x_1^{-\nu_1(f)} f$. Similarly, we define $\nu_1'(f) = \mathrm{ord}_{x_2} f$ and $\nu_2'(f) = (\nu_1'(f), \mathrm{ord}_{x_1} f_2(x_1,0,x_3)) \in \mathbf{Z}^2_{\mathrm{lex}}$. Observe that ν_2 and ν_2' have both center in R the prime ideal $(x_1,x_2)R$, and residue field isomorphic to $\mathbf{C}(x_3)$. Let us denote $\overline{\nu}$ the x_3-adic valuation of $\mathbf{C}(x_3)$. Then we get two valuations of K of rank three, say $\nu = \nu_2 \circ \overline{\nu}$ and $\nu' = \nu_2' \circ \overline{\nu}$, whose center in R is the maximal ideal of R. By construction $\nu \not\sim \nu'$ and $\pi(\nu) = \pi(\nu')$. Indeed, their image by π is the semivaluation of R which maps $f \in R$ to infinity if $f \in (x_1,x_2)R$, and otherwise to $\mathrm{ord}_{x_3} f(0,0,x_3)$.

Chapter 2

Homeomorphism type in the regular case

In this chapter we prove Theorem A and Theorem A'. The organization of the chapter is as follows.

Let x be an analytically irreducible closed point of an algebraic variety X defined over an algebraically closed field. In the first section we show that a point in $\mathrm{NL}(X,x)$ defines in a canonical way a suitably normalized semivaluation on the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring of X at x whose restriction to the base field is trivial. We then rely on this observation to prove Theorem A. In the second section we present some results concerning the henselization of a local ring which are needed in the sequel. Finally, in the third section we discuss briefly the problem of extending in a unique way a valuation of $\mathrm{RZ}(X,x)$ to a valuation of the fraction field of $\widehat{\mathcal{O}}_{X,x}$ dominating that ring and prove Theorem A', using the fact that the extension to the henselization is unique.

2.1 Proof of Theorem A

For the rest of this section, X is an algebraic variety defined over an algebraically closed field k. We consider a closed point $x \in X$ at which X is analytically irreducible and denote by R the local ring of X at x and by $\mathfrak{m}$ its maximal ideal. By assumption the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is an integral domain. We call $\widehat{\mathfrak{m}}$ the maximal ideal of $\widehat{R}$.

Proposition 2.1. *There is a one-to-one correspondence between* $\mathrm{NL}(X,x)$ *and the set of all centered semivaluations* $\hat{\nu} : \widehat{R} \to [0,+\infty]$ *extending the trivial valuation of* k *and normalized by the condition* $\hat{\nu}(\widehat{\mathfrak{m}}) = 1$.

Proof. Let ν be a semivaluation of $\mathrm{NL}(X,x)$ and consider $\Gamma_\nu = \nu(R) \setminus \{+\infty\}$. Recall that ν is a centered semivaluation of R such that $\nu_{|k^*} = 0$ and $\nu(\mathfrak{m}) = 1$ (see Lemma 1.18 and Lemma 1.23). Take a nonzero $f \in \widehat{R}$ and a Cauchy sequence $(f_n)_{n=1}^{\infty}$ in R converging to f. If the sequence $(\nu(f_n))_{n=1}^{\infty}$ is not bounded above by an element of Γ_ν, then we set $\hat{\nu}(f) = +\infty$. Now suppose that there exists an upper bound in Γ_ν for the sequence $(\nu(f_n))_{n=1}^{\infty}$. Consider the subset

$$\Lambda = \{\beta \in \Gamma_\nu \,/\, \forall\, n\, \exists\, n' > n \text{ such that } \nu(f_{n'}) \leq \beta\}\,.$$

Lemma 1.16 implies that Γ_ν is well ordered. By hypothesis Λ is not empty, so we may consider the smallest element α of Λ. If $\alpha = 0$ then we deduce that $\nu(f_n) = 0$ for all n

large enough and we set $\hat{\nu}(f) = 0$. Assume that $\alpha > 0$. Since Γ_ν does not contain any infinite bounded sequence (see [HOST12, Lemma 2.2] and [CT08, Lemma 3.1]), the set $\{\beta \in \Gamma_\nu \,/\, \beta < \alpha\}$ is finite. Let $\alpha' \in \Gamma_\nu$ be the immediate predecessor of α. By definition of α, the element α' does not belong to Λ. Hence $\nu(f_n) > \alpha'$ for all n large enough, that is, $\nu(f_n) \geq \alpha$ for $n \gg 0$. We deduce that $\nu(f_n) = \alpha$ for all n large enough and we set $\hat{\nu}(f) = \alpha$.

The definition of $\hat{\nu}$ does not depend on the choice of the Cauchy sequence. Moreover, if $(\nu(f_n))_{n=1}^{\infty}$ is not bounded then it tends to infinity. It is straightforward to verify that $\hat{\nu}(f) = \lim_{n\to+\infty} \nu(f_n)$ defines a semivaluation on $\widehat{R}$ having the desired properties.

In order to end the proof we need to show the uniqueness of $\hat{\nu}$. Let $\hat{\mu} : \widehat{R} \to [0, +\infty]$ be a semivaluation verifying the conditions of the statement. For any $f \in \widehat{R}$ we can find a Cauchy sequence $(f_n)_{n=1}^{\infty}$ in R converging to f such that $f - f_n \in \widehat{\mathfrak{m}}^n$ for any $n \geq 1$. Since $\hat{\mu}(\widehat{\mathfrak{m}}) = 1$, then for any n we have the inequalities $\hat{\mu}(f - f_n) \geq n$ and

$$\hat{\mu}(f) \geq \min\{\hat{\mu}(f - f_n), \hat{\mu}(f_n)\} \geq \min\{n, \nu(f_n)\}.$$

If $\hat{\mu}(f) = \alpha \in \mathbf{R}$ then it follows that $\nu(f_n) = \alpha$ for every $n > \alpha$. Suppose now that $\hat{\mu}(f) = +\infty$. If the sequence $(\nu(f_n))_{n=1}^{\infty}$ is bounded above then there exists β such that $\hat{\mu}(f - f_n) = \min\{\hat{\mu}(f), \nu(f_n)\} = \nu(f_n) \leq \beta$ for all $n \geq 1$, which is a contradiction. Hence $\hat{\mu}(f) = \hat{\nu}(f)$ for all $f \in \widehat{R}$. □

We denote by $\widehat{\mathrm{NL}}(X, x)$ the space of normalized semivaluations $\mathcal{V}(\widehat{R}, \widehat{\mathfrak{m}})$ defined in Subsection 1.2.1. That is, $\widehat{\mathrm{NL}}(X, x)$ is the space consisting of the set of all semivaluations $\hat{\nu} : \widehat{R} \to [0, +\infty]$ which are trivial on k and such that $\hat{\nu}(\widehat{\mathfrak{m}}) = 1$, equipped with the topology of pointwise convergence. Recall that $\widehat{\mathrm{NL}}(X, x)$ is compact. As a consequence of Proposition 2.1 we have the following:

Corollary 2.2. *The spaces* $\mathrm{NL}(X, x)$ *and* $\widehat{\mathrm{NL}}(X, x)$ *are homeomorphic.*

Proof. The map from $\widehat{\mathrm{NL}}(X, x)$ to $\mathrm{NL}(X, x)$ defined by $\hat{\nu} \mapsto \hat{\nu} \circ \imath$, where $\imath : R \hookrightarrow \widehat{R}$ is the inclusion, is a continuous bijection from a quasi-compact space into a Hausdorff space. □

The normalized non-Archimedean link $\mathrm{NL}(\mathbf{A}^2_{\mathbf{C}}, 0)$ of the origin in the affine plane over $\mathbf{C}$ is thus homeomorphic to the valuative tree $\mathcal{V}$ of [FJ04]. A topological model for this space is proposed in [FJ04, Section 3.2.3]. It turns out that the homeomorphism type of $\mathrm{NL}(\mathbf{A}^2_k, 0)$ depends only on the cardinality of the base field k, when k is an algebraically closed field of characteristic zero.

Next we address the proof of the first main result of this chapter.

Theorem A. *Let X, Y be two algebraic varieties defined over the same algebraically closed field k. For all regular closed points $x \in X$, $y \in Y$, the spaces* $\mathrm{NL}(X, x)$ *and* $\mathrm{NL}(Y, y)$ *are homeomorphic if and only if X and Y have the same dimension.*

Proof. It follows from Proposition 1.25 that X and Y have the same dimension whenever $\mathrm{NL}(X, x)$ and $\mathrm{NL}(Y, y)$ are homeomorphic. Conversely, under our assumptions on the points x and y, if X and Y have the same dimension then the formal completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as k-algebras. Hence $\widehat{\mathrm{NL}}(X, x)$ and $\widehat{\mathrm{NL}}(Y, y)$ are naturally homeomorphic. Corollary 2.2 implies that $\mathrm{NL}(X, x)$ and $\mathrm{NL}(Y, y)$ are homeomorphic. □

2.2 Henselization of a local ring

Throughout this section, A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. We will give a quick review of the definitions and results concerning the henselization that we need for the understanding of the rest of the chapter.

The ring A is *henselian* if it satisfies *Hensel's lemma*. That is, given a monic polynomial $P \in A[t]$, if there exists a factorization $\overline{P} = \overline{F}\,\overline{G}$ in $k[t]$ with $\overline{F}, \overline{G}$ relative prime monic polynomials, then there exists $F, G \in A[t]$ reducing to $\overline{F}$ and $\overline{G}$ respectively and such that $P = FG$. For some equivalent definitions of being henselian, we refer to [LM02, Nag62b, Ray70].

Hensel's Lemma does not hold for most of the rings found in algebraic geometry. Consider for instance $A = k[x]_{(x)}$ where k is a field. Then $P(t) = t^2 + t + x \in A[t]$ is irreducible but $\overline{P}(t) = t(t+1)$ in $k[t]$. However any complete local ring is henselian (one may drops the noetherian assumption, see [Nag62b, Theorem 30.3]) and thus A can be embedded in a henselian local ring.

The *henselization* of A is a henselian local ring $\widetilde{A}$ with a local homomorphism $\imath : A \to \widetilde{A}$ which has the following universal property: for any local homomorphism $\varphi : A \to B$ with B a henselian local ring, there exists a unique local homomorphism $\widetilde{\varphi} : \widetilde{A} \to B$ such that $\widetilde{\varphi} \circ \imath = \varphi$.

We give now an explicit construction of $\widetilde{A}$. Recall that an A-algebra is *finite* if it is a finitely generated module over A. We say that a local A-algebra B is *étale* if the following conditions are satisfied:

- B is the localization of a finite A-algebra at a prime ideal lying over $\mathfrak{m}$,
- B is a flat A-module,
- B is unramified (i.e. the maximal ideal of B is $\mathfrak{m}B$ and the residue field $B/\mathfrak{m}B$ is a finite separable extension of k).

If the residue field of B is k then B is called an *equiresidual local étale A-algebra*. We have the following characterization:

Proposition 2.3 ([LM02], Proposition 13.1)**.** *Let B be a local A-algebra. B is an equiresidual local étale A-algebra if and only if B is the localization of a finite A-algebra and the natural inclusion $A \hookrightarrow B$ induces an isomorphism $\widehat{A} \to \widehat{B}$ of formal completions.*

Example 2.4. Consider an A-algebra $B = (A[t]/(f(t)))_{\mathfrak{p}}$ where $f(t) = t^n + \ldots + a_1 t + a_0$ with $a_0 \in m$ and $a_1 \notin m$, and $\mathfrak{p}$ is a maximal ideal of $A[t]/(f(t))$ containing the class of t modulo $(f(t))$. Then B is an equiresidual local étale A-algebra. In fact, any equiresidual local étale A-algebra is of this form (see [LM02, Corollaire 12.29]).

Let I be the set of isomorphism classes of equiresidual local étale A-algebras. Consider for each $i \in I$ a representative B_i. Set $i \leq j$ if there exists an homomorphism of A-algebras $f_{ij} : B_i \to B_j$. Denote by $\mathfrak{n}_i$ and $\mathfrak{n}_j$ the maximal ideals of B_i and B_j respectively. Then we have $f_{ij}(\mathfrak{n}_i) = f_{ij}(\mathfrak{m}B_i) \subseteq \mathfrak{m}B_j = \mathfrak{n}_j$, thus f_{ij} is a local homomorphism. Moreover, since the formal completions of B_i and B_j are isomorphic, the map f_{ij} is injective. One can also show that such a homomorphism f_{ij} is unique.

Proposition 2.5 ([LM02], Propositions 13.4 and 13.6)**.** *The relation $\leq$ is an order relation in I and $(I, \leq)$ is a directed set. The inductive limit $\widetilde{A} = \bigcup_{i \in I} B_i$ of the direct system (B_i, f_{ij}) is a henselization of A.*

We denote by $\widetilde{\mathfrak{m}}$ the maximal ideal of the henselization $\widetilde{A}$ of A. We now list some properties of $\widetilde{A}$ (for their proof we refer once more to the literature). First of all, the local homomorphism $\imath : A \to \widetilde{A}$ is faithfully flat, $\widetilde{\mathfrak{m}} = \mathfrak{m}\widetilde{A}$ and the homomorphism $k \to \widetilde{A}/\widetilde{\mathfrak{m}}$ is an isomorphism. The ring $\widetilde{A}$ is also noetherian and has the same formal completion as A. Furthermore, A is reduced (resp. integrally closed) if and only if $\widetilde{A}$ has the same property.

Example 2.6. Let A be an excellent local ring and denote by $\widehat{A}$ its formal completion. Suppose that A is *analytically normal* (i.e. $\widehat{A}$ is a normal domain). Denote by $\widehat{A}^{\text{alg}}$ the ring of all elements in $\widehat{A}$ which are algebraic over A. Since $\widetilde{A}$ is a direct limit of algebraic extensions, $\widetilde{A}$ is contained in $\widehat{A}^{\text{alg}}$. In fact, $\widetilde{A} = \widehat{A}^{\text{alg}}$. It is proved in [Nag62b, Corollary 44.3] that $\widetilde{A}$ is algebraically closed in $\widehat{A}$. That is, any element in $\widehat{A}$ which is algebraic over $\widetilde{A}$ is already in $\widetilde{A}$. Therefore $\widehat{A}^{\text{alg}}$ is contained in $\widetilde{A}$.

In particular, given a field k, $k[[x_1, \ldots, x_d]]^{\text{alg}}$ is the henselization of the localization of the polynomial ring $k[x_1, \ldots, x_d]$ at the maximal ideal generated by $x_1, \ldots, x_d$.

The following result is well-known. The proof we present allows us to show Corollary 2.8, which will be a key point in the proof of Theorem B.

Proposition 2.7. *Let X, Y be two algebraic varieties of the same dimension defined over an algebraically closed field k. If $x \in X$ and $y \in Y$ are regular closed points, then the henselizations of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as k-algebras.*

Proof. Suppose that X has dimension d. Take an open affine neighborhood $U \subseteq X$ of x and denote by A the finitely generated k-algebra $\mathcal{O}_X(U)$. Then the point x corresponds to a maximal ideal $\mathfrak{m} = (x_1, \ldots, x_d)A$ of A and $\mathcal{F} = \{x_1, \ldots, x_d\}$ is a minimal system of generators of $A_{\mathfrak{m}}$. The family $\mathcal{F}$ is algebraically independent over k. Indeed, a non trivial polynomial relation between $x_1, \ldots, x_d$ would give a non trivial polynomial relation between their initial forms, which is not possible since $A_{\mathfrak{m}}$ is regular. Hence we have a polynomial ring $B = k[x_1, \ldots, x_d]$ contained in A. Let us denote by $\mathfrak{p}$ the maximal ideal of B generated by $\mathcal{F}$. We now show that $A_{\mathfrak{m}}$ is a equiresidual local étale $B_{\mathfrak{p}}$-algebra.

The formal completion of $A_{\mathfrak{m}}$ is by construction isomorphic to the formal power series ring $k[[x_1, \ldots, x_d]]$. It remains to show that $A_{\mathfrak{m}}$ is a localization of a finite $B_{\mathfrak{p}}$-algebra (see Proposition 2.3).

We begin by considering $A' = A \otimes_B B_{\mathfrak{p}}$, which is a finitely generated $B_{\mathfrak{p}}$-algebra. The ideal $\mathfrak{m}' = \mathfrak{m} \otimes_B B_{\mathfrak{p}} + A \otimes_B \mathfrak{p}B_{\mathfrak{p}}$ of A' is maximal since $A'/\mathfrak{m}' \cong k \otimes_B k \cong k$. In addition, $\mathfrak{m}'$ is the unique prime ideal of A' lying over $\mathfrak{p}B_{\mathfrak{p}}$. According to [Gro61, Corollaire 4.4.7], there exists a finite $B_{\mathfrak{p}}$-algebra C and a maximal ideal $\mathfrak{n}$ of C such that $\mathfrak{n} \cap B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$ and $C_{\mathfrak{n}}$ is isomorphic to $A'_{\mathfrak{m}'}$ as $B_{\mathfrak{p}}$-algebra. We have reduced the problem to show that $A_{\mathfrak{m}}$ and $A'_{\mathfrak{m}'}$ are isomorphic as $B_{\mathfrak{p}}$-algebras.

In order to see this, we consider the multiplicative closed subsets $\mathfrak{b} = B \setminus \mathfrak{p}$ and $\mathfrak{a} = A \setminus \mathfrak{m}$ of A. Note that $\mathfrak{b} \subseteq \mathfrak{a}$. As B-modules, $A' = A \otimes_B B_{\mathfrak{p}} = \mathfrak{b}^{-1}(A \otimes_B B) = \mathfrak{b}^{-1}A$. So we have the following commutative diagram

$$\begin{tikzcd} A \arrow[r, hook, "\imath_{A'}"] \arrow[d, hook] & \mathfrak{b}^{-1}A = A' \arrow[dl, hook] \arrow[d, hook] \\ (\mathfrak{a}\mathfrak{b})^{-1}A = A_{\mathfrak{m}} \arrow[r, "\psi"] & S^{-1}A' \end{tikzcd}$$

where S is the image by $\imath_{A'}$ of $\mathfrak{a}$ and ψ is an isomorphism of rings. Observe that an element of S can be written as $s/1 = s \otimes 1$ in A' with $s \in A$, $s \notin \mathfrak{m}$. Then the image of $s \otimes 1$ in $A'/\mathfrak{m}'$ is not zero. This implies that $\mathfrak{m}'$ and S are disjoint. We claim that A' is a local ring. If this holds, then $S^{-1}A' = A'_{\mathfrak{m}'}$ as A-algebras and we are done.

Finally, we verify that A' is local. Suppose that $\mathfrak{q}$ is a maximal ideal of A' and take $z \in \mathfrak{q}$. Then we can find $t \in A \setminus \mathfrak{b}$ such that $zt \in A$. If zt is a unit of A then it is also a unit of A', which contradicts the maximality of $\mathfrak{q}$. Therefore zt is not a unit of A. Since A is local, $zt \in \mathfrak{m}$ and we can write $z = x/t$ in A' with $x \in \mathfrak{m}$. Therefore $\mathfrak{q} \subseteq \mathfrak{m}A' \subseteq \mathfrak{m}'$, which means that $\mathfrak{q} = \mathfrak{m}'$.

This ends the proof of the fact that $A_{\mathfrak{m}}$ is an equiresidual local étale $B_{\mathfrak{p}}$-algebra. Then the henselization of $A_{\mathfrak{m}}$ is isomorphic to that of $B_{\mathfrak{p}}$. We deduce that the henselizations of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are both isomorphic as k-algebras to the henselization of the local ring of $\mathbf{A}_k^d$ at the origin. $\square$

Corollary 2.8. *Let X, Y be two algebraic varieties of dimension d defined over the same algebraically closed field k. Let $x \in X$, $y \in Y$ be regular closed points. For any pair of systems of local coordinates $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_d\}$ of x and y respectively, there exists an isomorphism of k-algebras $\widetilde{\sigma} : \widetilde{\mathcal{O}}_{X,x} \to \widetilde{\mathcal{O}}_{Y,y}$ such that $\sigma(x_i) = y_i$ for all i.*

Proof. In the hypothesis of the statement we have picked open affine neighborhoods $U \subseteq X$ and $V \subseteq Y$ of x and y respectively and $\{x_1, \ldots, x_d\} \subseteq \mathcal{O}_X(U)$, $\{y_1, \ldots, y_d\} \subseteq \mathcal{O}_Y(V)$ such that $\mathfrak{m}_{X,x} = (x_1, \ldots, x_d)\mathcal{O}_{X,x}$ and $\mathfrak{m}_{Y,y} = (y_1, \ldots, y_d)\mathcal{O}_{Y,y}$. Let σ be the isomorphism of k-algebras from $k[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$ to $k[y_1, \ldots, y_d]_{(y_1,\ldots,y_d)}$ defined by $\sigma(x_i) = y_i$ for all $i = 1, \ldots, d$. The universal property of the henselization allows us to extend σ to an isomorphism of k-algebras $\widetilde{\sigma}$ from the henselization of $k[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$ to that of $k[y_1, \ldots, y_d]_{(y_1,\ldots,y_d)}$. By Proposition 2.7, those henselian local rings are $\widetilde{\mathcal{O}}_{X,x}$ and $\widetilde{\mathcal{O}}_{Y,y}$ respectively. This ends the proof. $\square$

2.3 Proof of Theorem A'

We now turn to the Riemann-Zariski situation. In this section we shall work in the same setting as at the beginning of Section 2.1. We consider an analytically irreducible closed point $x \in X$ of an algebraic variety defined over an algebraically closed field k. We keep the same notations and denote by $\widehat{K}$ the fraction field of $\widehat{R}$.

We begin by studying the behavior of valuations under the passage to formal completion in a particular situation.

Theorem 2.9 ([Spi90b], Theorem 3.1)**.** *With the same notations, let us suppose that R has dimension two and is analytically normal (i.e. its formal completion $\widehat{R}$ is a normal domain). Given a valuation $\nu \in \mathrm{RZ}(X, x)$ there exists a unique valuation $\widehat{\nu}$ of $\widehat{K}$ such that $\widehat{\nu}_{|K} = \nu$ and $R_{\widehat{\nu}}$ dominates $\widehat{R}$.*

Proof. Let us denote by Φ the value group Φ_ν of ν. By Abhyankar's inequality either $\operatorname{rk}\nu = 1$ or $\operatorname{rk}\nu = 2$. If $\operatorname{rk}\nu = 1$ then Φ is isomorphic as ordered group to a subgroup of $\mathbf{R}$. Otherwise it follows from [Abh56, Corollary 1] that Φ is isomorphic as ordered group to $\mathbf{Z}^2_{\text{lex}}$.

If Φ is a subgroup of $\mathbf{R}$, Proposition 2.1 applied to ν gives then a semivaluation $\widehat{\nu}$ of $\widehat{R}$ such that $\widehat{\nu}(\widehat{\mathfrak{m}}) = \min\{\widehat{\nu}(f) \,/\, f \in \widehat{\mathfrak{m}}\} > 0$. If $\widehat{\nu}$ takes the value $+\infty$ only at zero, then it defines a valuation of $\widehat{K}$ satisfying the desired properties. Otherwise $\widehat{\mathfrak{p}} = \widehat{\nu}^{-1}(+\infty)$ is a prime ideal of $\widehat{R}$ of height one and $\widehat{\nu}$ induces a valuation $\overline{\widehat{\nu}}$ of the fraction field of $\widehat{R}/\widehat{\mathfrak{p}}$ whose valuation ring dominates that ring. Since $\widehat{R}$ is normal, $\widehat{R}_{\widehat{\mathfrak{p}}}$ is a regular local ring of dimension one. Hence $\widehat{R}_{\widehat{\mathfrak{p}}} = \widehat{R}_{\widehat{\nu}_1}$ for a discrete valuation $\widehat{\nu}_1$ of $\widehat{K}$ of rank one. The composite valuation $\widehat{\nu} = \widehat{\nu}_1 \circ \overline{\widehat{\nu}}$ satisfies the required properties.

Assume now that $\Phi = \mathbf{Z}^2_{\text{lex}}$ and write $\nu = \nu_1 \circ \overline{\nu}$, $\mathfrak{p} = m_{\nu_1} \cap R$. Let $(0) \subsetneq \Phi' \subsetneq \Phi$ be the chain of isolated subgroups of Φ. Recall that Φ_{ν_1} is isomorphic as ordered group to Φ/Φ' and $\Phi_{\overline{\nu}}$ to Φ'. The ring R is noetherian, so we may use exactly the same arguments given in the proof of Lemma 1.16 to show that the semigroup $\Gamma = \nu(R) \setminus \{+\infty\}$ is well-ordered. We set $\gamma = \min\{\nu(f) \,/\, f \in \mathfrak{m}\}$ and consider the smallest group Φ_1 in the chain such that $\gamma \in \Phi_1$. Observe that Φ_1 is either Φ' or Φ. We also set $\Gamma_1 = \Gamma \cap \Phi_1$. We have $\gamma \in \Gamma_1$.

For the rest of the proof, by $f_n \to f$ we mean that $(f_n)_{n=1}^\infty$ is a Cauchy sequence in R which converges to $f \in \widehat{R}$. We set

$$\widehat{\mathfrak{p}} = \left\{ f \in \widehat{R} \,/\, \exists\, f_n \to f \text{ such that } (\nu(f_n))_{n=1}^\infty \text{ is not bounded by an element of } \Phi_1 \right\}.$$

If $f \in \widehat{R}$ belongs to $\widehat{\mathfrak{p}}$ then $(\nu(f_n))_{n=1}^\infty$ is not bounded by an element of Φ_1 for all $f_n \to f$. The set $\widehat{\mathfrak{p}}$ is in fact a prime ideal of $\widehat{R}$.

Case 1. Let us first study the situation $\Phi_1 = \Phi'$. On the one hand, this equivalent to the existence of $f \in \mathfrak{m}$ such that $\nu_1(f) = 0$. Thus $\mathfrak{p} \subsetneq \mathfrak{m}$. On the other hand, we have $\widehat{\mathfrak{p}} \cap R = \mathfrak{p}$. It follows that $\mathfrak{p}\widehat{R} = \widehat{\mathfrak{p}}$ is a prime ideal of $\widehat{R}$ of height one. Since $\widehat{R}$ is normal, $\widehat{R}_{\widehat{\mathfrak{p}}}$ is a regular local ring of dimension one. Moreover its maximal ideal is generated by some $t \in \mathfrak{p}$. Then any $f \in \widehat{R}$ can be written uniquely as $t^s(g/h)$ with $s \in \mathbf{Z}_{\geq 0}$ and $g, h \in \widehat{R} \setminus \widehat{\mathfrak{p}}$. Next we define the restriction of $\widehat{\nu}$ to $\widehat{R} \setminus \widehat{\mathfrak{p}}$.

The semigroup Γ_1 equals $\overline{\nu}(R/\mathfrak{p}) \setminus \{+\infty\}$. Take $f \in \widehat{R} \setminus \widehat{\mathfrak{p}}$ and $f_n \to f$. Then there exists $\varphi \in \Phi_1$ such that $\nu(f_n) \leq \varphi$ for all n. Note that in particular $\nu(f_n) \in \Phi_1$ for all n (because Φ_1 is convex). The element φ belongs to the set

$$\Lambda = \{\beta \in \Gamma_1 \,/\, \forall\, n\, \exists\, n' > n \text{ such that } \nu(f_{n'}) \leq \beta\}.$$

Let α be the smallest element of Λ (Γ_1 is well-ordered). Since Φ_1 is archimedean, Γ_1 contains no infinite bounded sequences (see [CT08, Lemma 3.1]) and $\alpha, \gamma \in \Gamma_1$, we can repeat the arguments given in the proof of Proposition 2.1 to show that $\nu(f_n) = \alpha$ for all n large enough. The value α is independent of the choice of the Cauchy sequence. We set $\widehat{\nu}(f) = \alpha$.

For an arbitrary element f of $\widehat{R}$, we define $\widehat{\nu}(f) = \widehat{\nu}(t^s(g/h)) = s\,\nu(t) + \widehat{\nu}(g) - \widehat{\nu}(h)$. We obtain in this way a valuation $\widehat{\nu}$ of $\widehat{K}$ with the required properties.

Case 2. Finally we address the situation $\Phi_1 = \Phi$. This is equivalent to $\mathfrak{m} \subseteq m_{\nu_1}$, and thus to $\mathfrak{p} = \mathfrak{m}$. Therefore the semigroup $\Gamma_{\nu_1} = \nu_1(R) \setminus \{+\infty\}$ is well-ordered and contains no infinite bounded sequences. We associate to any $\varphi \in \Gamma$ the ideals of R, $\mathcal{P}_\varphi = \{f \in R \,/\, \nu(f) \geq \varphi\}$ and $\mathcal{P}^+_\varphi = \{f \in R \,/\, \nu(f) > \varphi\}$. According to [ZS60, Appendix

3, Corollary of Lemma 4], for any $\varphi_1 \in \Gamma_{\nu_1}$ the subset $\left\{\varphi \in \Gamma \,/\, \mathcal{P}^+_{\varphi_1} \subseteq \mathcal{P}_\varphi \subseteq \mathcal{P}_{\varphi_1}\right\}$ is finite. We deduce that the set $\{\varphi \in \Gamma \,/\, \varphi \leq \beta\}$ is also finite for any $\beta \in \Gamma$. Hence if $f \in \widehat{R} \setminus \widehat{\mathfrak{p}}$ and $f_n \to f$, the sequence $(\nu(f_n))_{n=1}^{\infty}$ takes a finite number of values in Φ. Let us prove that this sequence must be stationary.

We proceed by contradiction. Suppose that there exists $\beta, \beta' \in \Phi$, $\beta < \beta'$, such that β and β' are attained infinitely many times by the sequence $(\nu(f_n))_{n=1}^{\infty}$. Let n be a positive integer. Then there exists $n_0 \geq 1$ such that $f_i - f_j \in \mathfrak{m}^n$ for any $i, j \geq n_0$. Take $m, m' \geq n_0$ such that $\nu(f_m) = \beta$ and $\nu(f_{m'}) = \beta'$. We have $\beta = \min\{\beta, \beta'\} = \nu(f_i - f_j) \geq n\gamma$. Therefore $\beta \geq n\gamma$ for any $n \geq 1$. We find a contradiction since $\gamma \notin \Phi'$.

This limit value is fact independent of the choice of the Cauchy sequence. This enables us to define a valuation $\widehat{\nu}$ of the fraction field of $\widehat{R}/\widehat{\mathfrak{p}}$ with value group Φ and such that $R_{\widehat{\nu}}$ dominates $\widehat{R}/\widehat{\mathfrak{p}}$. Since $\widehat{R}$ has dimension two and it is noetherian, $\widehat{\mathfrak{p}}$ must be the zero ideal. We get a valuation $\widehat{\nu}$ of $\widehat{K}$ satisfying the desired properties. □

Remark 2.10. In the proof of Theorem 2.9 given in [Spi90b], for any $f \in \widehat{R}$ which is not in $\widehat{\mathfrak{p}}$, $\widehat{\nu}(f)$ is defined to be $\limsup_{n\to+\infty} \nu(f_n)$ for any Cauchy sequence $(f_n)_{n=1}^{\infty}$ in R which converges to f. This upper limit is a well defined element of Γ_1 since Γ_1 is well-ordered and $(\nu(f_n))_{n=1}^{\infty}$ is bounded in Φ_1. We show that the sequence $(\nu(f_n))_{n=1}^{\infty}$ is stationary.

The point of the following example is to emphasize that some invariants of the valuation, in particular the rank, are not preserved under the extension.

Example 2.11. Consider $R = \mathbf{C}[x, y]_{(x,y)}$ and denote by K its fraction field. Let $w(x) = \sum_{i\geq 1} a_i x^i$ be a series in $\mathbf{C}[[x]]$ which is transcendental over $\mathbf{C}(x)$. Consider the prime ideal $\widehat{\mathfrak{p}}$ of $\widehat{R}$ generated by $h(x, y) = y - w(x)$. Observe that by construction $\widehat{\mathfrak{p}} \cap R = (0)$. Therefore we have an injection $\phi : R \hookrightarrow \widehat{R}/\widehat{\mathfrak{p}}$. Since $\widehat{R}/\widehat{\mathfrak{p}}$ is a regular local ring of dimension one, it is a discrete valuation ring of its fraction field. We may consider ν the restriction of this valuation to K. In other words, the ring $\widehat{R}/\widehat{\mathfrak{p}}$ is isomorphic to $\mathbf{C}[[t]]$ and ϕ corresponds to sending x to t and y to $w(t)$. Hence $\nu(f) = \mathrm{ord}_t f(t, w(t))$ for any nonzero $f \in R$.

In order to extend ν to a valuation $\hat{\nu}$ as desired, one needs to choose a valuation of the fraction field of $\widehat{R}$ dominating the localization $\widehat{R}_{\widehat{\mathfrak{p}}}$. But again this last one is a regular local ring of dimension one, so there is a unique valuation ν' we can choose: the order of vanishing at the maximal ideal $\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}$. For any nonzero $f \in \widehat{R}$,

$$\hat{\nu}(f) = (\nu'(f), \mathrm{ord}_t \bar{f}(t, w(t))) \in \mathbf{Z}^2_{\mathrm{lex}},$$

where $\bar{f}(x, y) = h(x, y)^{-\nu'(f)} f(x, y)$.

The previous example suggest the following:

Example 2.12. Consider $R = \mathbf{C}[x, y, z]_{(x,y,z)}$ and $K = \mathbf{C}(x, y, z)$. Let $w(x) = \sum_{i\geq 1} a_i x^i$ and $u(x) = \sum_{i\geq 1} b_i x^i$ be two series in $\mathbf{C}[[x]]$ which are algebraically independent over $\mathbf{C}(x)$. Recall that this means that there is no non trivial polynomial $P \in \mathbf{C}(x)[t_1, t_2]$ such that $P(w(x), u(x)) = 0$. By [MS39, Lemma 1], there exists an infinite number of such formal power series. Let $\widehat{\mathfrak{p}}$ be the prime ideal of $\widehat{R}$ generated by $h_1(x, y, z) = y - w(x)$ and $h_2(x, y, z) = z - u(x)$. As in Example 2.11 we have $\widehat{\mathfrak{p}} \cap R = (0)$ and an injection $R \hookrightarrow \widehat{R}/\widehat{\mathfrak{p}} \cong \mathbf{C}[[t]]$ by sending x to t, y to $w(t)$ and z to $u(t)$. Let ν be the restriction to

K of the t-adic valuation $\bar{\nu}$ of $\mathbf{C}((t))$. To construct an extension $\hat{\nu}$ of ν one needs again to choose a valuation of the fraction field of $\widehat{R}$ dominating the localization $\widehat{R}_{\widehat{\mathfrak{p}}}$. But now $\widehat{R}_{\widehat{\mathfrak{p}}}$ is two-dimensional so we have infinitely many valuations with which to compose $\bar{\nu}$.

One should note that, unlike the case of $\mathrm{NL}(X,x)$, extending a valuation of $\mathrm{RZ}(X,x)$ to a valuation of $\widehat{K}$ whose valuation ring dominates $\widehat{R}$ can not be done in general in an unique way. The approach which led to the proof of Theorem A seems difficult to carry out successfully in the Riemann-Zariski setting. Instead, we focus on the extension of a valuation to the henselization $\widetilde{R}$ of R. We are assuming that $\widehat{R}$ is an integral domain, so $\widetilde{R}$ is also an integral domain. We call $\widetilde{K}$ the fraction field of $\widetilde{R}$.

Since R is a local noetherian domain which is excellent we are under the hypothesis of the following theorem:

Theorem 2.13 ([HOST12], Theorem 7.1)**.** *Let A be a local noetherian excellent domain with fraction field K and ν a valuation of K whose valuation ring dominates A. Let A^e be a local étale A-algebra contained in the henselization of A. There exists a unique prime ideal H of A^e such that $H \cap A = (0)$ and ν extends to a valuation ν^e of the fraction field of A^e/H whose valuation ring dominates that ring. Furthermore, the following properties are also satisfied: the extension ν^e is unique, the group of values is preserved and H is a minimal prime of A^e.*

Take a valuation $\nu \in \mathrm{RZ}(X,x)$. Let us apply the previous result to ν. Consider a local étale R-algebra R^e contained in $\widetilde{R}$. Observe that R^e is also an integral domain. We denote by K^e its fraction field. Since the zero-ideal is the unique minimal prime of R^e, the last assertion of Theorem 2.13 implies that $H = (0)$. We conclude that ν extends in a unique way to a valuation $\nu^e \in \mathrm{Z}(K^e|k)$ whose valuation ring dominates R^e and having the same value group as ν.

We denote by $\widetilde{\mathrm{RZ}}(X,x)$ the subspace of the Riemann-Zariski space $Z(\widetilde{K}|k)$ consisting of the set of all valuation rings of $\widetilde{K}$ dominating $\widetilde{R}$, endowed with the topology induced by the Zariski topology.

Remark 2.14. Given $\nu \in \mathrm{RZ}(X,x)$, there exists a unique valuation $\widetilde{\nu} \in \widetilde{\mathrm{RZ}}(X,x)$ such that $\widetilde{\nu}_{|K} = \nu$. To see this, take a nonzero $f \in \widetilde{R}$. Since $\widetilde{R}$ is the inductive limit of the system of equiresidual local étale R-algebras (see Proposition 2.5), there exists such a R-algebra, say R^e, such that $f \in R^e$. We define $\widetilde{\nu}(f) = \nu^e(f)$, where ν^e is the valuation of the fraction field of R^e whose existence guarantees Theorem 2.13. Since R^e is a localization of a finite R-algebra, R^e is excellent (excellence is preserved by localization and any finitely generated algebra over an excellent ring is excellent). If $R^e \hookrightarrow R^{e'}$ then we deduce that $\nu^{e'}(g) = \nu^e(g)$ for all $g \in R^e$. Therefore $\widetilde{\nu}$ is well defined and gives rise to a valuation of $\widetilde{\mathrm{RZ}}(X,x)$. The uniqueness of $\widetilde{\nu}$ follows directly from Theorem 2.13.
Observe that by construction ν and its extension $\widetilde{\nu}$ have the same value group.

Proposition 2.15. *The spaces $\mathrm{RZ}(X,x)$ and $\widetilde{\mathrm{RZ}}(X,x)$ are homeomorphic.*

Proof. By Remark 2.14, the map $\rho : \widetilde{\mathrm{RZ}}(X,x) \to \mathrm{RZ}(X,x)$ which sends a valuation μ to its restriction $\mu_{|K}$ to the field K is bijective. Moreover, it is clearly a continuous map because $R_{\mu_{|K}} = K \cap R_\mu$ for any such a valuation μ. In order to prove that ρ is open, one only needs to check that $\rho(E(f))$ is open in $\mathrm{RZ}(X,x)$ for every $f \in \widetilde{K}$.

Pick an element $f \in \widetilde{K}$. Since $K \hookrightarrow \widetilde{K}$ is an algebraic field extension, we can consider the minimal polynomial $p(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0 \in K[t]$ of f. The set V of all valuations $\nu \in \mathrm{RZ}(X,x)$ such that $a_i \in R_\nu$ for all $i \in \{0,\ldots,n-1\}$ is contained in $\rho(E(f))$. Indeed, given $\nu \in V$, if $\widetilde{\nu}$ is its extension to $\widetilde{\mathrm{RZ}}(X,x)$ then we have the inclusions $R[a_0,\ldots,a_{n-1}] \subseteq R_\nu \subseteq R_{\widetilde{\nu}}$. Since f is integral over $R[a_0,\ldots,a_{n-1}]$ this yields $f \in R_{\widetilde{\nu}}$, that is, $\nu \in \rho(E(f))$. Conversely, $\rho(E(f)) \subseteq V$. To see this, let us take $\mu \in \widetilde{\mathrm{RZ}}(X,x)$ such that $f \in R_\mu$ and show that $\mu_{|K} \in V$. We need to prove that $a_i \in R_\mu$ for all $i \in \{0,\ldots,n-1\}$.

Let $L = K(f,\alpha_1,\ldots,\alpha_{n-1})$ be the splitting field of $p(t)$ and $\overline{\mu}$ an extension of $\mu_{|K}$ to L. The coefficients of $p(t)$ are symmetric polynomials functions of the roots $f,\alpha_1,\ldots,\alpha_{n-1}$, therefore to conclude that $a_i \in R_\mu$ for any $i \in \{0,\ldots,n-1\}$ is sufficient to verify that $\alpha_j \in R_{\overline{\mu}}$ for all $j \in \{1,\ldots,n-1\}$.

According to [ZS60, Ch. VI §7, Corollary 3] every extension of μ to the field L can be written as $\overline{\mu}\circ\sigma$ for σ in the Galois group $\mathrm{Gal}(L|K)$. The uniqueness of the extension to $\widetilde{K}$ implies the uniqueness of the extension to the subfield $K(f)$, so $\overline{\mu}(\sigma(f)) = \mu(f)$ for every $\sigma \in \mathrm{Gal}(L|K)$ and $\overline{\mu}(\alpha_j) \geq 0$ for all $j \in \{1,\ldots,n\}$. $\square$

Remark 2.16. Let us denote $Z = \mathrm{RZ}(X,x)$ and $\widetilde{Z} = \widetilde{\mathrm{RZ}}(X,x)$ and consider both spaces as locally ringed spaces (see Subsection 1.1.4). Observe that the morphism $\rho : \widetilde{Z} \to Z$ is not an isomorphism of locally ringed spaces since $\mathcal{O}_{\widetilde{Z},\tilde{\nu}} = R_{\tilde{\nu}}$ is not necessarily isomorphic to $\mathcal{O}_{Z,\rho(\tilde{\nu})} = R_{\tilde{\nu}} \cap K$.

The same result holds for the constructible topology.

Proposition 2.17. *Viewing* $\mathrm{RZ}(X,x)$ *as subspace of* $Z(K|k)^{cons}$ *and* $\widetilde{\mathrm{RZ}}(X,x)$ *as subspace of* $Z(\widetilde{K}|k)^{cons}$, *the spaces* $\mathrm{RZ}(X,x)$ *and* $\widetilde{\mathrm{RZ}}(X,x)$ *are homeomorphic.*

Proof. Let us go back to the bijective map $\rho : \widetilde{\mathrm{RZ}}(X,x) \to \mathrm{RZ}(X,x)$ defined in the proof of Proposition 2.15. It follows directly from the description (1.1) of the basic open sets for the constructible topology that ρ is continuous for these topologies. Since $\mathrm{Z}(K|k)^{\mathrm{cons}}$ is Hausdorff, to see that ρ is a homeomorphism it is enough to prove that $\widetilde{\mathrm{RZ}}(X,x)$ is quasi-compact.

The constructible topology of $\mathrm{Z}(\widetilde{K}|\widetilde{R})$ is exactly the one induced by the constructible topology of $\mathrm{Z}(\widetilde{K}|k)$. Therefore the quasi-compactness of $\widetilde{\mathrm{RZ}}(X,x)$ as subspace of $\mathrm{Z}(\widetilde{K}|k)^{\mathrm{cons}}$ is equivalent to the property of being closed in $\mathrm{Z}(\widetilde{K}|\widetilde{R})^{\mathrm{cons}}$ (because $\mathrm{Z}(\widetilde{K}|\widetilde{R})^{\mathrm{cons}}$ is compact). In order to end the proof we show now that $\widetilde{\mathrm{RZ}}(X,x)$ has this property.

On the one hand, observe that the maximal ideal $\widetilde{\mathfrak{m}}$ of $\widetilde{R}$ is closed in $\mathrm{Spec}\,\widetilde{R}$. As a consequence it is also closed in $(\mathrm{Spec}\,\widetilde{R})^{\mathrm{cons}}$. On the other hand, $\widetilde{\mathrm{RZ}}(X,x)$ is the inverse image under the center map $c_{\widetilde{R}} : \mathrm{Z}(\widetilde{K}|\widetilde{R})^{\mathrm{cons}} \to (\mathrm{Spec}\,\widetilde{R})^{\mathrm{cons}}$ of $\widetilde{\mathfrak{m}}$. The fact that $\widetilde{\mathrm{RZ}}(X,x)$ is closed in $\mathrm{Z}(\widetilde{K}|\widetilde{R})^{\mathrm{cons}}$ follows from Proposition 1.5. $\square$

We now prove the second main result of this chapter.

Theorem A'. *Let* X,Y *be two algebraic varieties defined over the same algebraically closed field* k. *For all* $x \in X$, $y \in Y$ *regular closed points, the spaces* $\mathrm{RZ}(X,x)$ *and* $\mathrm{RZ}(Y,y)$ *are homeomorphic if and only if* X *and* Y *have the same dimension.*

Proof. If $\mathrm{RZ}(X,x)$ and $\mathrm{RZ}(Y,y)$ are homeomorphic then it follows by Proposition 1.11 that X and Y have the same dimension. Let us now prove the converse.

As the assumptions of Proposition 2.7 are satisfied, the henselizations of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as k-algebras. We have then a natural homeomorphism between $\widetilde{\mathrm{RZ}}(X,x)$ and $\widetilde{\mathrm{RZ}}(Y,y)$. To end the proof it suffices to apply Proposition 2.15. □

Remark 2.18. Take $\nu \in \mathrm{RZ}(X,x)$ and consider its extension $\widetilde{\nu} \in \widetilde{\mathrm{RZ}}(X,x)$. In view of [ZS60, Ch. VI §6, Corollary 1], the transcendence degree of the field extension $k_\nu \hookrightarrow k_{\widetilde{\nu}}$ is not greater than $\mathrm{tr.deg}_K \widetilde{K}$. Since the field extension $K \hookrightarrow \widetilde{K}$ is algebraic, we get $\mathrm{tr.deg}_k k_\nu = \mathrm{tr.deg}_k k_{\widetilde{\nu}}$. Therefore the homeomorphism given in Theorem A' preserves not just the rank and the rational rank (recall that $\Phi_\nu = \Phi_{\widetilde{\nu}}$) but also the dimension of the valuation.

When we can freely make use of the existence of resolutions of singularities, Theorem A' allows us to show that, in the regular case, the Riemann-Zariski space of X at x has the following property: we can observe anywhere in $\mathrm{RZ}(X,x)$ small patterns of the space that look like the whole $\mathrm{RZ}(X,x)$. More precisely, following [CD94] we say that a topological space Z is a *self-homeomorphic space* if for any open subset $U \subseteq Z$ there is a subset $V \subseteq U$ such that V is homeomorphic to Z.

Corollary 2.19. *Let X be an algebraic variety defined over an algebraically closed field k of characteristic zero. If $x \in X$ is a regular closed point, then* $\mathrm{RZ}(X,x)$ *is self-homeomorphic.*

Proof. Suppose that X has dimension $d > 1$ (otherwise the result is clear). Theorem A' implies that $\mathrm{RZ}(X,x)$ is homeomorphic to the Riemann-Zariski space of the d-dimensional affine space over k at the origin. Therefore it suffices to show that $Z = \mathrm{RZ}(\mathbf{A}^d_k, 0)$ is self-homeomorphic.

To see this, take U an open subset of Z. Without loss of generality we may assume that U is a basic open subset, that is, $U = \{\nu \in Z \ / \ f_1/g_1, \dots, f_m/g_m \in R_\nu\}$ where f_i and g_i are polynomials in $k[x_1, \dots, x_d]$ and $g_i \neq 0$ for all $i = 1, \dots, m$. We need to show that there exists $V \subseteq U$ homeomorphic to Z. Let $\psi : Y \to X$ be the blowing-up of $\mathbf{A}^d_k$ with respect to the ideal $(x_1, \dots, x_d) \cdot \prod_{1 \le i \le m} (f_i, g_i)$ of $k[x_1, \dots, x_d]$. Pick a resolution of singularities $\pi' : X' \to Y$ and denote $\pi = \psi \circ \pi'$. We choose a closed point $x' \in \pi^{-1}(0)$ in an affine chart $W \subseteq X'$ such that $f_i/g_i \in \mathcal{O}_{X'}(W)$ for all $i = 1, \dots, m$. By construction $\mathrm{RZ}(X', x') \subseteq U$. Since x' is regular, using again Theorem A' we see that $\mathrm{RZ}(X', x')$ is homeomorphic to Z. Hence it suffices to take $V = \mathrm{RZ}(X', x')$ to complete the proof. □

Chapter 3

Graphic tools

This chapter provides a short introduction to trees and graphs. They are both important tools in the treatment of the two-dimensional case (Theorem B). Firstly, the normalized non-Archimedean link $\mathrm{NL}(X,x)$ associated to a normal surface singularity contains a family of subsets, each of which carries a tree structure (even the whole space may carry such a structure). We specify in the first section what we mean by a tree and we show some properties useful for the sequel. Secondly, the complement of the union of the trees of the family evoked above deprived of their roots turns out to be a finite graph. Its equivalence class, under a certain equivalence relation, determines the homeomorphism type of both the Riemann-Zariski space $\mathrm{RZ}(X,x)$ and $\mathrm{NL}(X,x)$. In the second section we concentrate on graphs, giving in particular the definition of this equivalence relation.

3.1 Trees

For us a tree is a rooted non-metric $\mathbf{R}$-tree in the sense of [FJ04] (we refer for details to Sections 3.1 and 7.2).

A *tree* is a topological space consisting of a partially ordered set $(\mathcal{T},\leq)$ such that:

- There exists a unique smallest element τ_0 in $\mathcal{T}$ (called the *root* of $\mathcal{T}$),
- if $\tau\in\mathcal{T}$, then $\{\sigma\in\mathcal{T}\ /\ \sigma\leq\tau\}$ is isomorphic (as ordered set) to a real interval,
- every totally ordered convex subset of $\mathcal{T}$ is isomorphic (as ordered set) to a real interval,
- every non-empty subset of $\mathcal{T}$ admits an infimum in $\mathcal{T}$,

which is equipped with the *weak tree topology*, described as follows. Given two elements τ,τ' in $\mathcal{T}$, we denote by $\tau\wedge\tau'$ the infimum of $\{\tau,\tau'\}$ and we call the subset

$$[\tau,\tau']=\{\sigma\in\mathcal{T}\ /\ \tau\wedge\tau'\leq\sigma\leq\tau\}\cup\{\sigma\in\mathcal{T}\ /\ \tau\wedge\tau'\leq\sigma\leq\tau'\}$$

a *segment.* If $\tau\in\mathcal{T}$, we define an equivalence relation on the set $\mathcal{T}\setminus\{\tau\}$ by setting $\tau'\equiv_\tau\tau''$ if and only if $[\tau,\tau']\cap[\tau,\tau'']\neq\{\tau\}$. The equivalence classes are called the *tangent vectors* at τ and each of them determines an open subset of $\mathcal{T}$, $U_\tau(\tau')=\{\sigma\in\mathcal{T}\setminus\{\tau\}\ :\ \sigma\equiv_\tau\tau'\}$. The weak tree topology is the topology generated by all these subsets when τ ranges over $\mathcal{T}$. Thus an open subset of $\mathcal{T}$ is a union of finite intersections of subsets of the form $U_\tau(\tau')$.

Given two different points τ, τ' of a tree $(\mathcal{T}, \leq)$, for any $\sigma \in [\tau, \tau'] \setminus \{\tau, \tau'\}$, $U_\sigma(\tau)$ and $U_\sigma(\tau')$ are disjoint open neighborhoods of τ and τ', thus $\mathcal{T}$ is Hausdorff. Furthermore, the segment $[\tau, \tau']$ endowed with the induced topology from that of $\mathcal{T}$ is homeomorphic to $[0, 1]$ endowed with the induced topology from that of $\mathbf{R}$:

Indeed, the second axiom allows us to define a bijective mapping $\phi : [0, 1] \to [\tau, \tau']$ such that their restrictions to $[0, \phi^{-1}(\tau \wedge \tau')]$ and $[\phi^{-1}(\tau \wedge \tau'), 1]$ are isomorphism of ordered sets. It is clear that the inverse image of a proper basic open subset of $[\tau, \tau']$ under ϕ is a real interval of the form $[0, a)$ or $(a, 1]$ with $0 \leq a \leq 1$, thus ϕ is continuous. Since $[0, 1]$ is quasi-compact and $[\tau, \tau']$ is Hausdorff, we conclude that ϕ is an homeomorphism.

Therefore any tree is arcwise connected. Moreover it is *uniquely arcwise connected.* That is, for every two of its points there is exactly one arc in the space joining these points.

Lemma 3.1. *Let $(\mathcal{T}, \leq)$ be a tree and τ, τ' two different points of $\mathcal{T}$. The image of any injective continuous mapping $\gamma : [0, 1] \to \mathcal{T}$ with $\gamma(0) = \tau$ and $\gamma(1) = \tau'$ is the segment $[\tau, \tau']$.*

Proof. Consider $\gamma([0, 1])$ equipped with the induced topology from that of $\mathcal{T}$. Since $\gamma : [0, 1] \to \gamma([0, 1])$ is an homeomorphism, if we have $[\tau, \tau'] \subseteq \gamma([0, 1])$ then the inverse image of $[\tau, \tau']$ under γ is a connected subset of $[0, 1]$ containing 0 and 1. Hence we have $[\tau, \tau'] = \gamma([0, 1])$ as desired. Let us therefore show that $[\tau, \tau'] \subseteq \gamma([0, 1])$.

We can assume that τ is the root of $\mathcal{T}$ (we may need to redefine the partial ordering on $\mathcal{T}$, see 3.1.2 of [FJ04]). In view of [FJ04, Corollary 7.9], the mapping f which sends $t \in [0, 1]$ to $\gamma(t) \wedge \tau' \in [\tau, \tau']$ is continuous. Take $\sigma \in [\tau, \tau']$ different from τ and τ'. Define Σ to be the set of all $t \in [0, 1]$ such that $f(t) = \sigma$. On the one hand, Σ is nonempty thanks to the Intermediate Value Theorem. On the other hand, Σ is closed, so $s = \inf \Sigma$ belongs to Σ. In order to complete the proof it suffices to show that $\gamma(s) = \sigma$.

We proceed now by contradiction to prove that $\gamma(s) = \sigma$. Suppose that $\gamma(s) \neq \sigma$. The basic open subset $U_\sigma(\gamma(s))$ of $\mathcal{T}$ is mapped to σ by f, hence $U = U_\sigma(\gamma(s)) \cap \gamma([0, 1])$ is an open subset of $\gamma([0, 1])$ containing $\gamma(s)$ such that $f(U) = \{\sigma\}$. It follows that there exists an open neighborhood of s in $[0, 1]$ whose image by f is reduced to σ, contradicting the minimality of s. □

Corollary 3.2. *Any arcwise connected subspace of a tree is a tree.*

Proof. Let $(\mathcal{T}, \leq)$ be a tree and $\mathcal{S} \subseteq \mathcal{T}$ an arcwise connected subspace of $\mathcal{T}$. As previously noted, we can assume that the root τ_0 of $\mathcal{T}$ belongs to $\mathcal{S}$. Since $\mathcal{S}$ is arcwise connected, by Lemma 3.1, the segment $[\tau_0, \sigma]$ of $\mathcal{T}$ is contained in $\mathcal{S}$ for any $\sigma \in \mathcal{S}$. It is straightforward to deduce from this that $\mathcal{S}$ together with the restriction of the partial ordering $\leq$ satisfies the axioms of a tree and that the weak tree topology it carries coincide with the topology induced from that of $(\mathcal{T}, \leq)$. □

Example 3.3. Let us go back to the valuative tree $\mathcal{V}$. We use the same notations as in Section 1.2 and call $A = \mathbf{C}[[x, y]]$ and $\mathfrak{m}$ the maximal ideal of A. Recall that $\mathcal{V}$ is the set of semivaluations $\nu : A \to [0, +\infty]$ such that $\nu_{|\mathbf{C}^*} = 0$ and normalized by the condition $\nu(\mathfrak{m}) = 1$, equipped with the topology of pointwise convergence.

It is shown in [FJ04, Section 3.2] (see also [Nov14]) that $\mathcal{V}$ is a tree with the partial ordering $\leq$ defined by

$$\nu \leq \nu' \text{ if and only if } \nu(f) \leq \nu'(f) \text{ for all } f \in \mathbf{C}[[x,y]].$$

The proof relies on the encoding of valuations by key polynomials (see [FJ04, Ch. 2]) and it is quite technical in nature. A second approximation to the tree structure of $\mathcal{V}$ is possible taking into account that a valuation of the fraction field of A whose valuation ring dominates A corresponds univocally to a sequence of *infinitely near points over the origin* (see [FJ04, Chapter 6]).

Two remarks follow easily from the definition of the partial order. First observe that the normalization we have chosen makes $\nu_{\mathfrak{m}}$ the smallest element of $\mathcal{V}$, where $\nu_{\mathfrak{m}}(f) = \operatorname{ord} f = \max\{n \,/\, f \in \mathfrak{m}^n\}$, all non zero $f \in A$. Hence $\nu_{\mathfrak{m}}$ is the root of $(\mathcal{V}, \leq)$. Furthermore, any irreducible $\phi \in \mathfrak{m}$ gives rise to a semivaluation ν_ϕ of $\mathcal{V}$ by considering the intersection multiplicity divided by $\operatorname{ord} \phi$. If we suppose that $\nu \geq \nu_\phi$ then $\nu(\phi) \geq \nu_\phi(\phi) = +\infty$, so that $\mathfrak{s}_\nu = (\phi)A$. Since the quotient domain $A/(\phi)A$ is one dimensional, we conclude that $\nu = \nu_\phi$. Therefore the *curve valuations* ν_ϕ are maximal elements of $\mathcal{V}$. We refer to [FJ04, Proposition 3.20] for a complete picture of $\mathcal{V}$.

In turn, the relative valuative tree $\mathcal{V}_z$ is also a tree. Given $\nu, \nu' \in \mathcal{V}_z$, the partial order relation is defined by setting $\nu \leq_z \nu'$ if and only if $\nu(f) \leq \nu'(f)$ for all $f \in \mathbf{C}[[x,y]]$. Note that the valuation ord_z is the root of $(\mathcal{V}_z, \leq_z)$. We refer to [FJ04, Section 3.9] for details.

3.2 Graphs

There are several definitions of a graph. Here we have adopted the view point of [Ser80].

A *graph* Γ consists of two sets $V(\Gamma)$ and $E(\Gamma)$, whose elements are respectively called the *vertices* and the *edges* of Γ, and two maps:

- $E(\Gamma) \to E(\Gamma)$, $e \mapsto \bar{e}$, such that $e \neq \bar{e}$ and $\bar{\bar{e}} = e$
- $E(\Gamma) \to V(\Gamma)$, $e \mapsto \iota(e)$

Therefore any edge $e \in E(\Gamma)$ comes with a *reverse edge* $\bar{e}$. For any edge e we call $\iota(e)$ and $\iota(\bar{e})$ the *endpoints* of e. We also say that e is *incident* to $\iota(e)$ and $\iota(\bar{e})$, or e joins $\iota(e)$ to $\iota(\bar{e})$. Two vertices are *adjacent* if there exists an edge incident to both (a vertex may be adjacent to itself).

Given $u, v \in V(\Gamma)$, a *path* of length $n \geq 1$ joining u to v is a sequence of vertices and edges of Γ of the form $u = v_0, e_1, v_1, e_2, \ldots, e_n, v_n = v$ where $v_{i-1} = \iota(e_i)$ and $v_i = \iota(\bar{e}_i)$ for $i = 1, \ldots, n$. If $e_i \neq \bar{e}_{i+1}$ for $i = 1, \ldots, n-1$, then the path is *reduced*. By convention we shall call a path of length zero any sequence of the form u, u where u is a vertex of Γ. A path of length zero is always reduced. A graph is *connected* if any two vertices can be joined by a path. Throughout this section by graph we mean a connected graph which is in addition *finite*, which means that its sets of vertices and edges are both finite.

A graph Γ is a purely combinatorial object, however it can also be regarded as a finite one-dimensional CW-complex (see [Chi01]). In order to do this, we endow $V(\Gamma)$ and $E(\Gamma)$

with the discrete topology and the unit interval $[0,1]$ with the induced topology from that of $\mathbf{R}$. The topological space $|\Gamma|$, which we call the *topological realization* of Γ, is the quotient space of the disjoint union $V(\Gamma) \sqcup (E(\Gamma) \times [0,1])$ under the identifications $(e,0) \sim \iota(e)$ and $(e,t) \sim (\bar{e}, 1-t)$ for any $e \in E(\Gamma)$ and $t \in [0,1]$. Let us denote by q the quotient map and, for any $e \in E(\Gamma)$, call $|e| = q(\{e\} \times [0,1])$ an *edge* of $|\Gamma|$. Then $|e| = |\bar{e}|$ and any edge of $|\Gamma|$ is homeomorphic either to $[0,1]$ or the unit circle $\mathbf{S}^1$.

The *degree* of a vertex v of a graph Γ is the number of edges e of Γ such that $\iota(e) = v$. It corresponds to the number of connected components of a small punctured neighborhood of v in $|\Gamma|$, $\left(\bigcup_{\iota(e)=v} q\left(\{e\} \times [0, t_e)\right)\right) \setminus \{v\}$ where $0 < t_e < 1$.

A *morphism* from a graph Γ to a graph Γ' is a mapping γ from $V(\Gamma) \cup E(\Gamma)$ to $V(\Gamma') \cup E(\Gamma')$ which sends vertices to vertices and edges to edges in such a way that $\gamma(\bar{e}) = \overline{\gamma(e)}$ and $\gamma(\iota(e)) = \iota(\gamma(e))$ for any $e \in E(\Gamma)$. Note that this implies that $\gamma(\iota(\bar{e})) = \iota(\overline{\gamma(e)})$ for any $e \in E(\Gamma)$. An *isomorphism* of graphs is a bijective morphism of graphs. We say that a graph Γ is a *subgraph* of Γ' if $V(\Gamma) \subseteq V(\Gamma')$, $E(\Gamma) \subseteq E(\Gamma')$ and the inclusion $V(\Gamma) \cup E(\Gamma) \hookrightarrow V(\Gamma') \cup E(\Gamma')$ is a morphism. If this is verified, then we have a natural closed embedding $|\Gamma| \hookrightarrow |\Gamma'|$.

We define now an operation on a graph whose result is still a graph but which does not induce a morphism. Given a graph Γ, a *subdivision of an edge* e of Γ consists of the addition of a new vertex v to $V(\Gamma)$, the addition of new edges e', e'' (and their reverses) to $E(\Gamma)$, joining $\iota(e)$ to v and v to $\iota(\bar{e})$ respectively, and the deletion of e and $\bar{e}$. We say that a graph Γ' is a *subdivision* of Γ if there exists a finite sequence $\Gamma = \Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_n = \Gamma'$ where each arrow represents either an isomorphism of graphs or an edge subdivision.

Lemma 3.4. *Let Γ, Γ' be two graphs. The topological realizations of Γ and Γ' are homeomorphic if and only if there exist a subdivision Σ of Γ and a subdivision Σ' of Γ' such that Σ and Σ' are isomorphic.*

Proof. The topological realization of a given graph is homeomorphic to the topological realization of any graph obtained by subdivision of one of its edges, therefore it is clear that two graphs $|\Gamma|$ and $|\Gamma'|$ are homeomorphic if they have isomorphic subdivisions.

Let us now show the converse. If $|\Gamma|$ and $|\Gamma'|$ are homeomorphic to the unit circle $\mathbf{S}^1$ then the result is clear, so let us suppose that this is not the case. We define Δ (resp. Δ') to be a graph without vertices of degree two and whose topological realization is homeomorphic to $|\Gamma|$ (resp. $|\Gamma'|$). Observe that such a Δ (resp. Δ') exists since Γ (resp. Γ') is not homeomorphic to $\mathbf{S}^1$: we just need to perform a finite number of operations, any of them being the inverse of an edge subdivision, starting from Γ (resp. Γ'). Pick a homeomorphism $f : |\Delta| \to |\Delta'|$. Since all the vertices of $|\Delta|$ and $|\Delta'|$ are of degree different from two, f restricts to a bijection between $V(\Delta)$ and $V(\Delta')$ which respects the degree. Take for instance an edge e of Δ with endpoints u, v both of degree greater than two. Then $f(|e|)$ is a closed connected subset of $|\Delta'|$ with $f(|e|) \cap V(\Delta') = \{f(u), f(v)\} = f(\partial|e|) = \partial f(|e|)$, so there must be an edge of Δ' with endpoints $f(u)$ and $f(v)$. The same is true on the other cases. In fact f induces a bijection $\eta : E(\Delta) \to E(\Delta')$ such that f maps the endpoints of any edge e of Δ to the endpoints of $\eta(e)$. It suffices now to subdivide m times every edge of Δ and every edge of Δ' where $m = \max\{\mathrm{card}(V(\Gamma)) - \mathrm{card}(V(\Delta)), \mathrm{card}(V(\Gamma')) - \mathrm{card}(V(\Delta'))\}$ to find isomorphic subdivisions of Γ and Γ'. $\square$

A graph Γ is a *tree* if given any two vertices u, v of Γ there exists a unique reduced path joining u to v. The topological realization of Γ is then a tree in the sense introduced before. More precisely, the choice of a vertex of Γ determines a unique tree structure on $|\Gamma|$:

Let us choose a vertex of Γ and call it v_0. Take a point $p = q((e,t)) = q((\bar{e}, 1-t))$ of $|\Gamma|$, $p \neq v_0$. There exists a unique edge $b \in \{e, \bar{e}\}$ for which we can find a reduced path $v_0, e_1, v_1, \ldots, v_{n-1} = \iota(b), e_n = b, v_n = \iota(\bar{b})$. We may assume that $b = e$. Since Γ is a tree, we have in addition that any two vertices in the path are different, so it induces an injective continuous mapping $\alpha_p : [0,1] \to |\Gamma|$ such that $\alpha_p(0) = v_0$ and $\alpha_p(1) = p$. We set $\alpha_{v_0}(u) = v_0$ for all $u \in [0,1]$. It suffices to declare $p \leq p'$ if and only if $\alpha_p([0,1]) \subseteq \alpha_{p'}([0,1])$.

Given a graph Γ which is not a tree, following the notations of [Sta83] we associate to Γ its *core*.

Definition 3.5 (The core of a graph)**.** *Let Γ be a graph which is not a tree. If Γ has no vertex of degree one, then Γ is its own core; otherwise its core is the subgraph of Γ obtained by repeatedly deleting a vertex of degree one and the edges incident to it (which are exactly two, one being the reverse of the other) until no more vertices of degree one remain.*

We may thus think of Γ as its core, denoted $\mathrm{Core}\,(\Gamma)$, with some disjoint trees attached to it (see Figure 3.1).

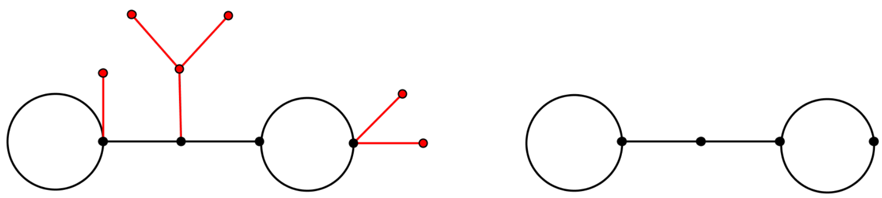

Figure 3.1: A graph and its core on the right.

Lemma 3.6. *Let Γ be a graph which is not a tree. The complement of $|\mathrm{Core}\,(\Gamma)\,|$ in $|\Gamma|$ is the set of points $p \in |\Gamma|$ which admit an open neighborhood $U \subsetneq |\Gamma|$ whose closure $\overline{U}$ in $|\Gamma|$ is a tree and whose boundary ∂U is reduced to a vertex of Γ.*

Proof. Take $p \in |\Gamma|$. Let us suppose first that $U \subsetneq |\Gamma|$ is an open neighborhood of p verifying the hypothesis of the Lemma. Note that $\overline{U} = U \sqcup \{v\}$ for some $v \in V(\Gamma)$. Since the boundary of U is reduced to v, if $q(\{e\} \times (a,b))$ is contained in U for some $0 \leq a < b \leq 1$ and $e \in E(\Gamma)$, then $|e|$ is entirely contained in $\overline{U}$. From this and the fact that a tree is connected it follows that there exists a subgraph Γ' of Γ such that $\overline{U} = |\Gamma'|$. The graph Γ' is clearly a tree. Moreover, recall that if $u \in V(\Gamma)$ belongs to U then $|e| \subseteq U$ for any edge e of Γ such that $\iota(e) = u$ and $\iota(\bar{e}) \neq v$. Therefore we have that $E(\Gamma') \cap E(\mathrm{Core}\,(\Gamma)) = \emptyset$ and $V(\Gamma') \cap V(\mathrm{Core}\,(\Gamma))$ is either empty or equal to $\{v\}$, which implies that $p \notin |\mathrm{Core}\,(\Gamma)\,|$.

Assume now that p does not belong to the topological realization of $\mathrm{Core}\,(\Gamma)$. Consider $e \in E(\Gamma) \setminus E(\mathrm{Core}\,(\Gamma))$ such that $p \in |e|$. There exists a unique subgraph Γ' of Γ such that $e \in E(\Gamma')$, the graph Γ' has a unique vertex v' of degree one which is in addition an endpoint of e, and $\mathrm{Core}\,(\Gamma) = \mathrm{Core}\,(\Gamma')$. We may suppose that $v' = \iota(\bar{e})$. We can find a unique path $v_0, e_1, v_1, \ldots, v_n, e_n = e, v_n = v'$ in Γ' of length $n \geq 1$ where $v_0 \in V(\mathrm{Core}\,(\Gamma))$ and $v_i, e_i \notin E(\mathrm{Core}\,(\Gamma))$ for $1 \leq i \leq n$. The connected component of $|\Gamma| \setminus \{v_0\}$ which contains $|e|$ is an open neighborhood of p, its closure is a tree and its boundary is the point v_0. $\square$

Remark 3.7. Note also that Γ admits a deformation retraction to $\mathrm{Core}\,(\Gamma)$.

We define two elementary modifications, which we call *collapses*, in a fixed graph Γ:

- Removing a vertex of Γ of degree one and the edges incident to it.
- Provided that $V(\Gamma)$ is not reduced to a single point, removing a vertex v of Γ of degree two and every edge incident to it and adding a new pair of edges $e, \bar{e}$ with endpoints the vertices that were adjacent to v.

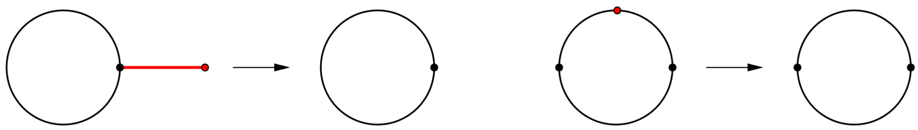

Figure 3.2: Examples of collapses.

The inverse operations of both types of collapse are called *expansions* and we also consider them, and any isomorphism of graphs, as elementary modifications. Remark that an expansion which is the inverse of a collapse of the second kind described above is an edge subdivision. The other kind of expansion consists on choosing a vertex u of Γ and adding a new vertex v to $V(\Gamma)$ and two edges $e, \bar{e}$ to $E(\Gamma)$ such that $\iota(e) = u$ and $\iota(\bar{e}) = v$.

A *modification* is a finite sequence $\Gamma = \Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_n = \Gamma'$ where each arrow represents an elementary modification.

Definition 3.8 (Equivalent graphs). *We say that two graphs Γ and Γ' are equivalent if there exists a modification which transforms Γ into Γ'.*

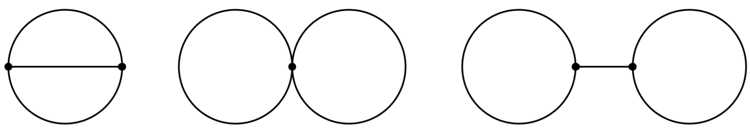

Figure 3.3: Equivalence is finer than homotopy.

An elementary modification produces a graph whose topological realization is homotopy equivalent to that of the graph, but equivalence and homotopy are two different notions: the three graphs in Figure 3.3 have homotopy equivalent topological realizations but they are not pairwise equivalent (the latter statement will follow from Proposition 3.11).

Remark 3.9. Let Γ be a graph. Γ is a tree if, and only if, it is equivalent to a graph which is reduced to a unique vertex (without any edges).

Lemma 3.10. *Let Γ, Γ' be two graphs. If Γ and Γ' are equivalent, then there exist two finite sequences of expansions $\Gamma = \Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_n$ and $\Gamma' = \Gamma'_0 \to \Gamma'_1 \to \ldots \to \Gamma'_m$ such that Γ_n and Γ'_m are isomorphic.*

Proof. Assume that Γ and Γ' are not isomorphic graphs. Let $f : \Gamma = \Gamma_0 \to \ldots \to \Gamma_n = \Gamma'$ be a modification. Without loss of generality, we can assume that any two graphs in the sequence are not isomorphic. Suppose that there is at least a collapse and an expansion in the sequence (otherwise the result is clear). The basic idea is that all the expansions can be done first.

Let us concentrate on the simplest interesting modification: the composition of a collapse c and an expansion ϵ, $\Gamma \xrightarrow{c} \Gamma_1 \xrightarrow{\epsilon} \Gamma'$ (the result is also clear in the symmetric situation). Denote by p the vertex of Γ removed by c and by p' the vertex of Γ' added to Γ_1 by ϵ.

Suppose that p' is of degree one. Since c is a collapse, there exists a unique vertex $v \in V(\Gamma)$, $v \neq p$ such that $\epsilon(c(v))$ is adjacent to p'. If v is not adjacent to p then it is clear that we get Γ' from Γ adding first p' and then an edge that joins it to v (and its reverse edge) and finally collapsing p in the resulting graph. If v is adjacent to p then p has degree two, otherwise we would have $\epsilon = c^{-1}$. Thus there exists $u \in V(\Gamma)$ different from p (but maybe equal to v) and a reduced path u, e, p, e', v in Γ. Again Γ' can be obtained from Γ by performing first an expansion and then a collapse.

If the degree of p' is two, then ϵ is the subdivision of an edge e_1 of Γ_1. Since c is a collapse, there exists a unique pair of vertices $u, v \in V(\Gamma)$, both different from p, such that e_1 joins $c(u)$ to $c(v)$. Note that the case $u = v$ is not excluded. The vertex u must be adjacent to v, otherwise ϵ would be c^{-1}. Moreover,

$$\operatorname{card}\left(\{e \in E(\Gamma) \ / \ \iota(e) = u, \iota(\bar{e}) = v\}\right) \leq \operatorname{card}\left(\{e \in E(\Gamma_1) \ / \ \iota(e) = c(u), \iota(\bar{e}) = c(v)\}\right)$$

because c is a collapse. The fact that $\epsilon \circ c$ is not an isomorphism implies that the previous inequality is an equality, that is, there does not exist a reduced path in Γ of the form u, e, p, e', v. Removing an edge which joins u to v and its reverse edge, adding p' and two edges joining p' to u and p' to v respectively and their reverse edges, and finally collapsing p in the graph that we obtain, transforms Γ into Γ'.

Therefore, there exists an expansion $\Gamma \xrightarrow{\epsilon'} \Gamma'_1$ and a collapse $\Gamma'_1 \xrightarrow{c'} \Gamma'$ such that $\epsilon \circ c = c' \circ \epsilon'$ and it suffices to take $\Delta = \Gamma'_1$.

Suppose now that $n \geq 3$. If the set

$$I = \{i \in \{1, \ldots, n-1\} \ / \ \Gamma_{i-1} \to \Gamma_i \text{ is a collapse and } \Gamma_i \to \Gamma_{i+1} \text{ is an expansion}\}$$

is empty, then the assertion follows immediately. Otherwise we define $r = \min I$. As we remarked before, we can exchange the subsequence $\Gamma_{r-1} \to \Gamma_r \to \Gamma_{r+1}$ by the composition of an expansion $\Gamma_{r-1} \to \Gamma'_r$ and a collapse $\Gamma'_r \to \Gamma_{r+1}$. After that one may need to clean up the sequence $f' : \Gamma \to \ldots \to \Gamma_{r-1} \to \Gamma'_r \to \Gamma_{r+1} \to \Gamma'$ of isomorphisms. We now repeat the whole process, replacing the modification f by f'. After a finite number of steps we end up with I empty. ☐

Proposition 3.11. *Two graphs Γ and Γ' are equivalent if, and only if, either they are both trees or they are not and $|\mathrm{Core}\,(\Gamma)\,|$ is homeomorphic to $|\mathrm{Core}\,(\Gamma')\,|$.*

Proof. By Remark 3.9, any two trees are equivalent and, given any two equivalent graphs Γ and Γ', Γ is a tree if and only if Γ' is a tree. So let us assume that Γ and Γ' are not trees. Since any graph (which is not a tree) is equivalent to its core, Γ and Γ' are equivalent if, and only if, so are their cores. Therefore it suffices to show that this last statement is equivalent to $|\mathrm{Core}\,(\Gamma)\,|$ and $|\mathrm{Core}\,(\Gamma')\,|$ being homeomorphic.

In view of Lemma 3.4, if the topological realizations of $\mathrm{Core}\,(\Gamma)$ and $\mathrm{Core}\,(\Gamma')$ are homeomorphic then there exists a graph Δ such that $\mathrm{Core}\,(\Gamma) \to \Delta$ and $\mathrm{Core}\,(\Gamma') \to \Delta$, where each arrow denotes a finite sequence of edge subdivisions and graph isomorphisms. Thus $\mathrm{Core}\,(\Gamma)$ is equivalent to $\mathrm{Core}\,(\Gamma')$ by definition.

Suppose now that $\mathrm{Core}\,(\Gamma)$ and $\mathrm{Core}\,(\Gamma')$ are equivalent. According to Lemma 3.10, there exists a graph Δ which is obtained from $\mathrm{Core}\,(\Gamma)$ and $\mathrm{Core}\,(\Gamma')$ by finite sequences of expansions ϵ and ϵ' respectively. The modifications $\mathrm{Core}\,(\Gamma) \xrightarrow{\epsilon} \Delta \to \mathrm{Core}\,(\Delta)$ and $\mathrm{Core}\,(\Gamma') \xrightarrow{\epsilon'} \Delta \to \mathrm{Core}\,(\Delta)$ are necessarily compositions of edge subdivisions and graph isomorphisms and the graph $\mathrm{Core}\,(\Delta)$ is a subdivision of $\mathrm{Core}\,(\Gamma)$ and $\mathrm{Core}\,(\Gamma')$. Hence the topological realizations of $\mathrm{Core}\,(\Gamma)$ and $\mathrm{Core}\,(\Gamma')$ are homeomorphic. $\square$

Remark 3.12. Two finite simplicial complexes are *simple-homotopy equivalent* if there exists a finite sequence of elementary simplicial collapses and expansions from one to the other. Such a sequence is called a simple homotopy equivalence. A modification of graphs in the sense we introduced is an example of simple-homotopy equivalence (for the general definition we refer to [Coh73, Chapter I, §2]). Any two simple-homotopy equivalent simplicial complexes are homotopy equivalent. The converse is not true in general. In [Whi50] Whitehead associated to any finite simplicial complex Y a group $\mathrm{Wh}(Y)$, defined in terms of the fundamental group $\pi_1(Y)$, and assigned to each homotopy equivalence $f : X \to Y$ a unique element $\tau(f) \in \mathrm{Wh}(Y)$ which is 0 if and only if f is a simple-homotopy equivalence. If Y is a graph, then $\pi_1(|Y|)$ is a free group. It is known that in this case $\mathrm{Wh}(Y)$ is trivial (see [BHS64] and [Sta65]). Therefore the notions of homotopy and simple-homotopy equivalence coincide in our situation and hence an equivalence of graphs is stronger than a simple-homotopy equivalence.

Chapter 4

Homeomorphism type in the normal surface singularity case

This chapter is divided into two sections. The first section is devoted to the normalized non-Archimedean link $\mathrm{NL}(X,x)$ for a normal surface singularity $x \in X$. We explain how the topological realization of certain graphs associated to resolutions of singularities of X embed in the space $\mathrm{NL}(X,x)$ and then how it can be described as the limit of those simplicial complexes. Next we detect a small piece of $\mathrm{NL}(X,x)$, its core, of crucial importance for our purposes. In the second section we prove Theorem B.

4.1 The core of $\mathrm{NL}(X,x)$

Throughout this section x is a singular point of a normal algebraic surface X defined over an algebraically closed field k. We denote by K its function field. Our presentation of the normalized non-Archimedean link $\mathrm{NL}(X,x)$ of x in X as a limit of simplicial complexes follows [Fav10].

We say that a proper birational map $\pi_{X'} : X' \to X$ is a *good resolution* if X' is smooth and the exceptional locus $E_{X'} = \pi_{X'}^{-1}(x)_{\mathrm{red}}$ is a divisor with normal crossing singularities such that its irreducible components are smooth and the intersection of any two of them is at most a point. A good resolution always exists.

Recall that since a good resolution $\pi_{X'} : X' \to X$ is proper and induces an isomorphism from the open subset $X' \setminus E_{X'}$ to the open subset $X \setminus \{x\}$, any semivaluation of $\mathrm{NL}(X,x)$ admits a center in X'. Moreover, the map $\mathrm{sp}_{X'} : \mathrm{NL}(X,x) \to X'$ which sends any semivaluation to its center in X' is anticontinuous. We refer to Subsection 1.2.2 for details. In particular, the subset $U(p) = \{\nu \in \mathrm{NL}(X,x) \;/\; \mathrm{sp}_{X'}(\nu) = p\}$ is open in $\mathrm{NL}(X,x)$ for any closed point $p \in E_{X'}$. We keep this notation for the rest of this section.

Remark 4.1. Let $\pi_{X'} : X' \to X$ be a good resolution and E an irreducible component of the exceptional locus $E_{X'}$. Given a closed point $p \in E$ which is a regular point of $E_{X'}$, the closure of $U(p)$ in $\mathrm{NL}(X,x)$ is $U(p) \bigsqcup \left\{ b_E^{-1}\nu_E \right\}$, where ν_E is the divisorial valuation defined by E and $b_E = \nu_E(\mathfrak{m}_{X,x})$. To see this, let us take $\nu \in \mathrm{NL}(X,x) \setminus U(p)$, $\nu \neq b_E^{-1}\nu_E$, and show that there exists an open neighborhood $V \subseteq \mathrm{NL}(X,x)$ of ν which does not intersect $U(p)$. If the center of ν in X' is a closed point $p' \in E_{X'}$, then $p' \neq p$ and it suffices to consider $V = U(p')$. Otherwise, assuming that $E_{X'}$ is not irreducible, the center of ν in X' is an irreducible component of $E_{X'}$ different from E. Then it suffices to define V to be the inverse image of $E_{X'} \setminus W$ under the specialization map $\mathrm{sp}_{X'} : \mathrm{NL}(X,x) \to X'$, where W is an open neighborhood of p in E not containing the singular points of $E_{X'}$.

To any good resolution $\pi_{X'} : X' \to X$ we attach a graph, its *dual graph* $\Gamma_{X'}$, whose vertices are in bijection with the irreducible components of $E_{X'}$ and where two vertices are adjacent if and only if the corresponding irreducible components of $E_{X'}$ intersect. A dual graph has no loops ($\iota(e) \neq \iota(\bar{e})$ for any edge e) and no multiple edges (if two different edges e, e' have the same endpoints then $e' = \bar{e}$).

The topological realization of any dual graph $\Gamma_{X'}$ can be embedded into $\mathrm{NL}(X,x)$ as a closed set in the following way:

Any vertex of $\Gamma_{X'}$ corresponds to a unique irreducible component E of the exceptional locus $E_{X'}$, so we can identify every vertex with a normalized divisorial valuation in $\mathrm{NL}(X,x)$, $b_E^{-1}\nu_E$ where $b_E = \nu_E(m_{X,x})$. By abuse of notation, throughout this subsection we continue to write ν_E for the normalized valuation $b_E^{-1}\nu_E$. We consider now an edge $|e|$ of $|\Gamma_{X'}|$ whose endpoints are identified with the irreducible components E and E' of $E_{X'}$. Pick local coordinates (z, z') at the point p of X' where E and E' intersect such that $E = \{z = 0\}$ and $E' = \{z' = 0\}$. Since p is a regular point of X' the completion $\widehat{\mathcal{O}}_{X',p}$ is isomorphic as k-algebra to $k[[z, z']]$. Thus given $f \in \mathcal{O}_{X',p}$, $f \neq 0$, we can write $f = \sum c_{i,j} z^i (z')^j$ with $c_{i,j} \in k$. For any $t \in [0,1]$ we define a monomial valuation of K by setting

$$\nu_t(f) = \min \left\{ \frac{t}{b_E} i + \frac{1-t}{b_{E'}} j \Big/ c_{i,j} \in k^* \right\}$$

and $\nu_t(0) = +\infty$. We identify the point $q((e,t)) \in |e|$ with the quasi-monomial valuation in $\mathrm{NL}(X,x)$ defined by $\nu_t(f)$ for all $f \in \mathcal{O}_{X,x}$. This construction, independent of the choice of local coordinates, gives an injective continuous mapping $|\Gamma_{X'}| \hookrightarrow \mathrm{NL}(X,x)$. In the sequel we may not distinguish between $|\Gamma_{X'}|$ and its image in $\mathrm{NL}(X,x)$.

For any good resolution $\pi_{X'} : X' \to X$ there exists a naturally defined continuous retraction map $\mathrm{r}_{X'} : \mathrm{NL}(X,x) \to |\Gamma_{X'}|$:

Given a semivaluation $\nu \in \mathrm{NL}(X,x)$, if $\nu \in |\Gamma_{X'}|$ then we define $\mathrm{r}_{X'}(\nu) = \nu$. Suppose that ν does not belong to $|\Gamma_{X'}|$ and denote p the center of ν in X' (which is a closed point of X'). If p is the intersection of two irreducible components E and E' of $E_{X'}$, we take local coordinates (z, z') at p such that $E = \{z = 0\}$ and $E' = \{z' = 0\}$ and we map ν to the quasi-monomial $\mathrm{r}_{X'}(\nu) \in |\Gamma_{X'}|$ corresponding to the unique monomial valuation $\nu_t \in U(p)$ such that $\nu_t(z) = \nu(z)$ and $\nu_t(z') = \nu(z')$. Otherwise the point p belongs to a single irreducible component E of $E_{X'}$ and we set $\mathrm{r}_{X'}(\nu) = \nu_E$.

If $\pi_{X''} : X'' \to X$ is a good resolution which dominates $\pi_{X'} : X' \to X$, then the restriction $\mathrm{r}_{X',X'}$ of $\mathrm{r}_{X''}$ to $|\Gamma_{X''}|$ is the natural continuous retraction we may consider from $|\Gamma_{X''}|$ to $|\Gamma_{X'}|$. Let us make this more precise:

Given a good resolution $\pi : Z \to X$, by [Nag62a, Theorem 4.3] Z can be embedded as an open dense subset of a complete surface $\overline{Z}$. Let $\widetilde{Z}$ be a resolution of singularities of $\overline{Z}$. Since $\widetilde{Z}$ is a non-singular complete surface, $\widetilde{Z}$ is in fact projective (see [Zar58]). Therefore Z is quasi-projective. Observe now that $\pi_{X'} \circ \pi_{X''}^{-1}$ is a birational morphism between two non singular quasi-projective surfaces and therefore it is the composition of a sequence of blowing-ups of points (this fact follows from the factorization theorem of birational morphisms between non-singular projective surfaces [Sha77, Theorem 5] and

Nagata's compactification Theorem). Let us assume that $\pi_{X'} \circ \pi_{X''}^{-1}$ is the blowing up of a point $p \in E_{X'}$ (in the general case the retraction is defined recursively). The graph $\Gamma_{X''}$ is obtained from $\Gamma_{X'}$ by the elementary subdivision of an edge of $\Gamma_{X'}$ whenever p is a singular point of $E_{X'}$. Thus in this case the topological realizations of $\Gamma_{X'}$ and $\Gamma_{X''}$ are homeomorphic and the retraction $|\Gamma_{X''}| \to |\Gamma_{X'}|$ is nothing but the identity map. If p is a non singular point of $E_{X'}$, then $\Gamma_{X''}$ is obtained from $\Gamma_{X'}$ by an expansion which is the inverse of a collapse of the first kind. The topological realization of $\Gamma_{X''}$ is that of $\Gamma_{X'}$ with a new segment $|e|$ attached to a vertex $v \in V(\Gamma_{X'})$. In this case the map $|\Gamma_{X''}| \to |\Gamma_{X'}|$ sends every point of $|e|$ to v and restricts to the identity on $|\Gamma_{X'}|$.

Furthermore, these maps are compatible (i.e. $\mathrm{r}_{X'} = \mathrm{r}_{X',X'} \circ \mathrm{r}_{X''}$) and the induced continuous mapping $\mathrm{NL}(X,x) \to \varprojlim |\Gamma_{X'}|$ is an homeomorphism.

The following proposition is a consequence of results of [FJ04]:

Proposition 4.2. *Let* $\pi_{X'} : X' \to X$ *be a good resolution and* E *an irreducible component of the exceptional locus* $E_{X'}$*. For any closed point* $p \in E$ *which is a regular point of* $E_{X'}$*, the closure of* $U(p)$ *in* $\mathrm{NL}(X,x)$ *is a tree whose boundary is reduced to the normalized divisorial valuation* ν_E *associated to* E*.*

Proof. We denote by $\overline{U}(p)$ the closure of $U(p)$ in $\mathrm{NL}(X,x)$. By Remark 4.1, $\overline{U}(p)$ equals $U(p) \bigsqcup \{\nu_E\}$. Since $U(p)$ is an open subset of $\mathrm{NL}(X,x)$, the boundary of $\overline{U}(p)$ is reduced to the semivaluation ν_E. We now prove that $\overline{U}(p)$ is a tree.

First of all observe that we are not assuming that $\pi_{X'}$ factors through the normalized blowing up of $x \in X$, so we could have some embedded components. Therefore we write the pull-back of the coherent sheaf of ideals $\mathfrak{m}$ of $\mathcal{O}_X$ defining the point x as $\mathcal{O}_{X'}(-C) \otimes_{\mathcal{O}_{X'}} \mathcal{I}$, where C is a divisor on X' with supp $C = E_{X'}$ and $\mathcal{I}$ is a coherent sheaf of ideals in $\mathcal{O}_{X'}$ with finite co-support (that is, for any affine chart U of X', $\mathrm{Spec}(\mathcal{O}_{X'}(U)/\mathcal{I}(U))$ is a finite set of closed points). Choose local coordinates (z,z') at p such that $E = \{z = 0\}$. The ideal $\mathcal{I}_p$ of $\mathcal{O}_{X',p}$ is either a primary ideal or the ring $\mathcal{O}_{X',p}$.

Suppose that $\mathcal{I}_p$ is a primary ideal of $\mathcal{O}_{X',p}$. Then $\mathfrak{m}_{X,x}\mathcal{O}_{X',p} = (z)^{b_E} \cdot \mathcal{I}_p$. Hence in the open subset $U(p)$ the normalization $\nu(\mathfrak{m}_{X,x}) = 1$ translates into $\nu(z)b_E + \nu(\mathcal{I}_p) = 1$. We denote by $\widehat{\mathcal{I}}_p$ the extension of $\mathcal{I}_p$ in $k[[z,z']]$. Passing to the completion, we can identify $U(p)$ with the subspace of $\mathcal{V}(k[[z,z']])$ consisting of all semivaluations $\nu : k[[z,z']] \to [0,+\infty]$ whose restriction to k^* is trivial, which are centered in the maximal ideal (z,z') and such that $\nu(z)b_E + \nu(\widehat{\mathcal{I}}_p) = 1$. If $\nu(z) = 0$ for some $\nu \in U(p)$, since there exists $n \geq 1$ such that $z^n \in \mathcal{I}_p$ we would get $0 = \nu(z^n) \geq \nu(\mathcal{I}_p) = 1$, which is a contradiction. Therefore $\nu(z) > 0$ for all $\nu \in U(p)$ and we have a well defined map φ from $\overline{U}(p)$ to the relative valuative tree $\mathcal{V}_z$. It suffices to define $\varphi(\nu_E) = \mathrm{ord}_z$ and

$$\varphi(\nu) = \frac{b_E\,\nu}{1 - \nu(\widehat{\mathcal{I}}_p)} = \frac{\nu}{\nu(z)}$$

for any $\nu \in U(p)$. In the case where $\mathcal{I}_p$ is the ring $\mathcal{O}_{X',p}$, we have $\nu(\mathcal{I}_p) = 0$ and we may consider $\varphi : \overline{U}(p) \to \mathcal{V}_z$ defined exactly as before. We claim that φ is an homeomorphism. Indeed, the map from $\mathcal{V}_z$ to $\overline{U}(p)$ which sends ord_z to ν_E and $\nu \in \mathcal{V}_z \setminus \{\mathrm{ord}_z\}$ to $\frac{\nu}{\nu(\mathfrak{m}_{X,x})}$ is the inverse map of φ. Since $\mathrm{NL}(X,x)$ and $\mathcal{V}_z$ are both endowed with the topology of pointwise convergence, it is straightforward to verify that they are both continuous maps.

According to [FJ04, Proposition 3.6.1], $\mathcal{V}_z$ is a tree rooted at ord_z. From this fact and the existence of φ we deduce that $\overline{U}(p)$ is a tree (rooted at ν_E) and this finishes the proof. □

The key observation is the following:

Proposition 4.3. *Let $\Gamma_{X'}$ be the dual graph associated to a good resolution $\pi_{X'} : X' \to X$. Any fiber $\mathrm{r}_{X'}^{-1}(\nu)$ of the natural retraction $\mathrm{r}_{X'} : \mathrm{NL}(X,x) \to |\Gamma_{X'}|$ is a tree whose boundary is reduced to the semivaluation ν.*

Proof. Let ν be a semivaluation in $|\Gamma_{X'}|$. Assume first that ν is a vertex of $\Gamma_{X'}$ and denote by E the irreducible component of the exceptional locus $E_{X'}$ which determines ν. Consider the set F of all closed points $p \in E$ which are not singular points of $E_{X'}$. The fiber $\mathrm{r}_{X'}^{-1}(\nu)$ can be written as $\{\nu\} \bigsqcup_{p\in F} U(p)$, where $U(p)$ is the open subset of $\mathrm{NL}(X,x)$ of semivaluations whose center in X' is p. By Proposition 4.2, the closure $\overline{U}(p)$ of $U(p)$ in $\mathrm{NL}(X,x)$ is a tree whose boundary is reduced to the normalized divisorial valuation $\nu = \nu_E$. Therefore $\mathrm{r}_{X'}^{-1}(\nu)$ is a bunch of trees sharing a unique point ν which is the boundary of every tree in the family. In fact, the fiber of $\mathrm{r}_{X'}$ above ν is itself a tree, as we explain next.

Take $\mu, \mu' \in \mathrm{r}_{X'}^{-1}(\nu)$. Abusing notation, we declare $\mu \le \mu'$ if there exists $p \in F$ such that $\mu, \mu' \in \overline{U}(p)$ and $\mu \le \mu'$ in $\overline{U}(p)$. This defines a tree structure on $\mathrm{r}_{X'}^{-1}(\nu)$. Moreover, any basic open subset for the weak tree topology is also open for the induced topology from the topology of $\mathrm{NL}(X,x)$. To see this, consider μ and μ' two different semivaluations in $\mathrm{r}_{X'}^{-1}(\nu)$ and denote by W the basic open subset in $(\mathrm{r}_{X'}^{-1}(\nu), \le)$ determined by the tangent vector at μ corresponding to μ'. If $\mu \in U(p)$ and $\mu' \in U(p')$ with $p \neq p'$, then

$$W = U_\mu(\nu) \bigcup_{q\in F,\, q\neq p} U(q), \tag{4.1}$$

where $U_\mu(\nu)$ is open in $U(p)$. If $\mu, \mu' \in U(p)$ then $W = U_\mu(\mu') \subseteq U(p)$ if $\mu \le \mu' < \nu$ in $U(p)$; otherwise W can be written as in (4.1). In any case W is open in $\mathrm{NL}(X,x)$. Now observe that the fiber $\mathrm{r}_{X'}^{-1}(\nu)$ is closed, so it is quasi-compact as subspace of $\mathrm{NL}(X,x)$. Since any tree is Hausdorff, we can conclude that $\mathrm{r}_{X'}^{-1}(\nu)$ and $(\mathrm{r}_{X'}^{-1}(\nu), \le)$ are homeomorphic, which implies that the fiber of $\mathrm{r}_{X'}$ above ν is a tree.

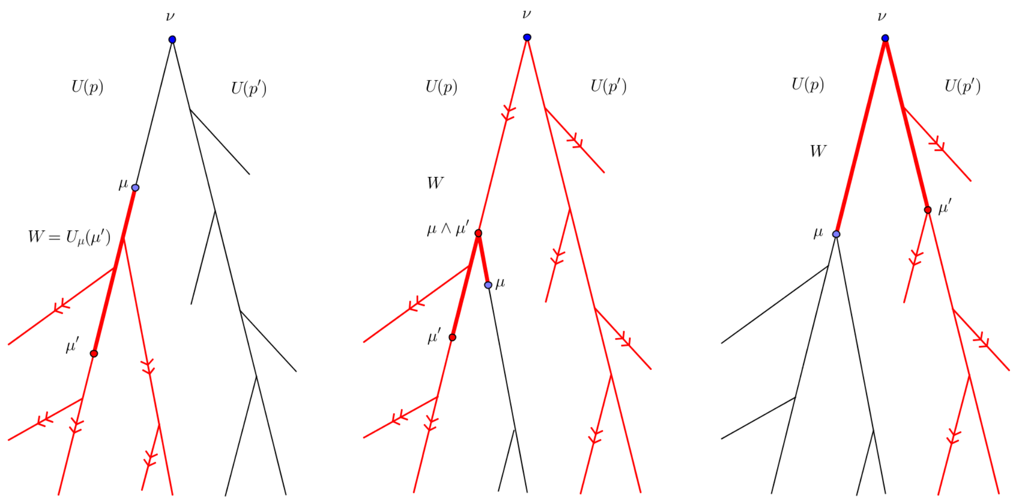

Figure 4.1: The basic open subset W is drawn in red

Suppose now that ν belongs to the interior of an edge of $|\Gamma_{X'}|$. If ν corresponds to an irrational in the interval $(0,1)$ then $\mathrm{r}_{X'}^{-1}(\nu) = \{\nu\}$ and the statement is true in this case. Otherwise, ν is a quasi-monomial valuation of rational rank one. This means that ν is a divisorial valuation. Hence there exists a finite sequence of blowing up of points $\pi : X'' \to X'$ such that $\pi_{X''} = \pi_{X'} \circ \pi$ is a good resolution and the center of ν in X'' is a prime divisor $E \subseteq E_{X''}$. Since $\mathrm{r}_{X''}^{-1}(\nu) = \mathrm{r}_{X'}^{-1}(\nu)$, we have reduced the problem to the first case we treated above. □

Corollary 4.4. *The normalized non-Archimedean link* $\mathrm{NL}(X, x)$ *is a tree if, and only if, the dual graph associated to any good resolution is a tree.*

Proof. Let $\pi_{X'} : X' \to X$ be a good resolution. Since $|\Gamma_{X'}| \subseteq \mathrm{NL}(X, x)$ is arcwise connected, by Corollary 3.2, if $\mathrm{NL}(X, x)$ is a tree then $|\Gamma_{X'}|$ is also a tree. Let us now show the converse.

Assume that the dual graph associated to any good resolution is a tree. Take a dual graph $\Gamma_{X'}$ and choose a tree structure $(|\Gamma_{X'}|, \leq)$ of $|\Gamma_{X'}|$. Proposition 4.3 allow us to equip $\mathrm{NL}(X, x)$ with a tree structure. Abusing notation, given two semivaluations ν, ν' in $\mathrm{NL}(X, x)$ we declare $\nu \leq \nu'$ if and only if one of the following conditions is satisfied:

- $\nu, \nu' \in |\Gamma_{X'}|$ and $\nu \leq \nu'$ in $|\Gamma_{X'}|$
- $\nu \in |\Gamma_{X'}|$, $\nu' \notin |\Gamma_{X'}|$ and $\nu \leq \mathrm{r}_{X'}(\nu')$ in $|\Gamma_{X'}|$
- $\nu, \nu' \notin |\Gamma_{X'}|$, $\mathrm{r}_{X'}(\nu) = \mathrm{r}_{X'}(\nu')$ and $\nu \leq \nu'$ in $\mathrm{r}_{X'}^{-1}(\mu)$ where $\mu = \mathrm{r}_{X'}(\nu)$.

It is straightforward to verify that $(\mathrm{NL}(X, x), \leq)$ satisfies the four axioms of a tree, so we concentrate on the topology. Since $\mathrm{NL}(X, x)$ is quasi-compact and $(\mathrm{NL}(X, x), \leq)$ is Hausdorff, to show that these two spaces are homeomorphic and end the proof it suffices to prove that any basic open set of $(\mathrm{NL}(X, x), \leq)$ is open in $\mathrm{NL}(X, x)$. Let us take $\nu, \nu' \in \mathrm{NL}(X, x)$, $\nu \neq \nu'$, and show that $U_\nu(\nu')$ is open in $\mathrm{NL}(X, x)$.

We suppose first that $\nu \in |\Gamma_{X'}|$. If $\nu = \mathrm{r}_{X'}(\nu')$ then $U_\nu(\nu')$ is nothing else than the basic open set in $\mathrm{r}_{X'}^{-1}(\nu)$ determined by the tangent vector at ν corresponding to ν'. Otherwise $U_\nu(\nu')$ is the inverse image under $\mathrm{r}_{X'}$ of the basic open set in $|\Gamma_{X'}|$ determined by the tangent vector at ν which corresponds to $\mathrm{r}_{X'}(\nu')$. Therefore, in both cases, $U_\nu(\nu')$ is open in $\mathrm{NL}(X, x)$.

Suppose now that $\nu \notin |\Gamma_{X'}|$ and set $\mu = \mathrm{r}_{X'}(\nu)$. If $\mu = \mathrm{r}_{X'}(\nu')$ and $\nu < \nu'$ then $U_\nu(\nu')$ is just the basic open set in $\mathrm{r}_{X'}^{-1}(\mu)$ determined by the tangent vector at ν corresponding to ν'. Otherwise $U_\nu(\nu')$ can be written as the union of $\mathrm{r}_{X'}^{-1}(|\Gamma_{X'}| \setminus \{\mu\})$ and the basic open set in $\mathrm{r}_{X'}^{-1}(\mu)$ determined by the tangent vector at ν defined by μ. Again in both cases we have that $U_\nu(\nu')$ is open in $\mathrm{NL}(X, x)$. □

We define the *core* of $\mathrm{NL}(X, x)$ in a way analogous what to we did for graphs (see the topological characterization given in Lemma 3.6). In [Ber90, p. 76] the core is referred to as the skeleton.

Definition 4.5 (The core of $\mathrm{NL}(X, x)$)**.** *The core of the normalized non-Archimedean link* $\mathrm{NL}(X, x)$ *of x in X is the set of all semivaluations in* $\mathrm{NL}(X, x)$ *which do not admit a proper open neighborhood whose closure is a tree and whose boundary is reduced to a single semivaluation of* $\mathrm{NL}(X, x)$*. We denote it* $\mathrm{Core}\,(\mathrm{NL}(X, x))$.

Observe that by definition $\mathrm{Core}\,(\mathrm{NL}(X, x))$ is empty if and only if $\mathrm{NL}(X, x)$ is a tree.

Lemma 4.6. *If $\pi_{X'} : X' \to X$ is a good resolution, then* $\mathrm{Core}\,(\mathrm{NL}(X,x)) \subseteq |\Gamma_{X'}|$.

Proof. Take $\nu \in \mathrm{NL}(X,x)$ and suppose that $\nu \notin |\Gamma_{X'}|$. Then $\mathrm{r}_{X'}(\nu)$ is different from ν. Set $\mu = \mathrm{r}_{X'}(\nu)$. In view of Proposition 4.3, $\mathrm{r}_{X'}^{-1}(\mu) \setminus \{\mu\}$ is an open neighborhood of ν such that its closure $\mathrm{r}_{X'}^{-1}(\mu)$ is a tree and its boundary is reduced to μ. This means that $\nu \notin \mathrm{Core}\,(\mathrm{NL}(X,x))$. □

However one can be more specific:

Proposition 4.7. *Let $\pi_{X'} : X' \to X$ be a good resolution. If* $\mathrm{NL}(X,x)$ *is not a tree, then* $\mathrm{Core}\,(\mathrm{NL}(X,x)) = |\mathrm{Core}\,(\Gamma_{X'})\,|$ *as subspaces of* $\mathrm{NL}(X,x)$.

Proof. Observe that under the assumptions of the proposition the topological realization of $\Gamma_{X'}$ is not a tree (see Corollary 4.4).

Take $\nu \in \mathrm{NL}(X,x)$ and suppose first that $\nu \in \mathrm{Core}\,(\mathrm{NL}(X,x))$. We know by Lemma 4.6 that $\nu \in |\Gamma_{X'}|$. We proceed by contradiction. Suppose $\nu \notin |\mathrm{Core}\,(\Gamma_{X'})\,|$. According to Lemma 3.6 there exists an open neighborhood $W \subsetneq |\Gamma_{X'}|$ of ν such that its closure $\mathrm{Cl}(W)$ in $|\Gamma_{X'}|$ is a tree and its boundary is reduced to a vertex ν' of $\Gamma_{X'}$. Since the retraction $\mathrm{r}_{X'}$ is continuous, $U = \mathrm{r}_{X'}^{-1}(W) \subsetneq \mathrm{NL}(X,x)$ is an open neighborhood of ν. The closure $\overline{U}$ of U in $\mathrm{NL}(X,x)$ is contained in $\mathrm{r}_{X'}^{-1}(\mathrm{Cl}(W)) = \mathrm{r}_{X'}^{-1}(\nu') \bigsqcup U$. In fact, $\overline{U} = \{\nu'\} \bigsqcup U$ because $\mathrm{r}_{X'}^{-1}(\nu') \setminus \{\nu'\}$ is open in $\mathrm{NL}(X,x)$. Imitating the proof of Corollary 4.4, we see that $\overline{U}$ in $\mathrm{NL}(X,x)$ inherits a natural tree structure from that of $\mathrm{Cl}(W)$ and $\mathrm{r}_{X'}^{-1}(\mu)$ for any $\mu \in W$. Hence ν does not belong to the core of $\mathrm{NL}(X,x)$ and we get a contradiction. This proves that $\mathrm{Core}\,(\mathrm{NL}(X,x))$ is contained in $|\mathrm{Core}\,(\Gamma_{X'})\,|$.

In order to finish the proof it suffices to check that $\nu \notin |\mathrm{Core}\,(\Gamma_{X'})\,|$ whenever $\nu \in |\Gamma_{X'}|$ and $\nu \notin \mathrm{Core}\,(\mathrm{NL}(X,x))$. Suppose that ν satisfies these two conditions. Take a proper open subset U of $\mathrm{NL}(X,x)$ which contains ν and such that its closure $\overline{U}$ in $\mathrm{NL}(X,x)$ is a tree and its boundary is reduced to a semivaluation ν'. Since $\nu \in |\Gamma_{X'}|$, $W = U \cap |\Gamma_{X'}|$ is a non-empty open subset of $|\Gamma_{X'}|$. If $W = |\Gamma_{X'}|$ then $|\Gamma_{X'}| \subseteq \overline{U}$ and by Corollary 3.2, $\Gamma_{X'}$ would be a tree, contradicting the hypothesis of the lemma. Thus $W \subsetneq |\Gamma_{X'}|$. Let us denote Z the closure of W in $|\Gamma_{X'}|$. The connectedness of $|\Gamma_{X'}|$ implies that W is not closed. We have $W \subsetneq Z \subseteq \overline{U} \cap |\Gamma_{X'}| = (U \cap |\Gamma_{X'}|) \bigsqcup (\{\nu'\} \cap |\Gamma_{X'}|)$. We deduce from this that ν' must belong to $|\Gamma_{X'}|$ and $Z = \overline{U} \cap |\Gamma_{X'}| = W \bigsqcup \{\nu'\}$. Indeed, if $\nu' \notin |\Gamma_{X'}|$ then we would get $Z \subseteq U \cap |\Gamma_{X'}| = W$, which is a contradiction.

By enlarging U slightly if necessary, we may choose ν' to be a vertex of $\Gamma_{X'}$. If Z is a tree (as subspace of $|\Gamma_{X'}|$) then from Lemma 3.6 it would follow that $\nu \notin |\mathrm{Core}\,(\Gamma_{X'})\,|$ and this would end the proof. Let us prove that Z is a tree.

Recall that we see $|\Gamma_{X'}|$ as a subspace of $\mathrm{NL}(X,x)$, so the subspace topology that Z inherits from $|\Gamma_{X'}|$ is the same as the one it inherits from $\overline{U}$. Since $\overline{U}$ is a tree, if Z is arcwise connected then Corollary 3.2 holds and Z is also a tree. Therefore it suffices to take $p \in W$ arbitrary and show that there exists a path γ in Z from p to ν'. Suppose first that p belongs to the interior of an edge $|e|$ of $|\Gamma_{X'}|$. The only boundary point of Z is ν' thus $|e|$ is entirely contained in Z. If the edge e is incident to ν' then it is easy to define such a path γ. Otherwise it suffices to join either $\iota(e)$ or $\iota(\bar{e})$ to ν' by a path in Z. Hence we can concentrate on the case when p is a vertex of $\Gamma_{X'}$. Suppose that $p \in V(\Gamma_{X'})$. Since the boundary of Z is reduced to ν', $\bigcup_{\iota(e)=p} |e|$ must be contained in Z. Note that in particular any vertex of $\Gamma_{X'}$ adjacent to p is also in Z. If p is adjacent to ν' then the edge of $\Gamma_{X'}$ joining p to ν' induces the desired path. Otherwise the problem is reduced to finding a path in Z from a vertex adjacent to p to ν'. The graph $\Gamma_{X'}$ is finite and connected, so the existence of such a path γ is guaranteed. This shows that Z is tree and enables us to complete the proof. □

One can also show that the fiber $\mathrm{r}_{X'}^{-1}(\nu)$ is in fact an analytic disk when endowed with its canonical analytic structure (see [Fan14a, Proposition 9.5 (i)]).

Finally, let us now summarize what we have seen in Corollary 4.4 and Proposition 4.7:

Proposition 4.8. *Let $\pi_{X'} : X' \to X$ be a good resolution. The space $NL(X,x)$ is a tree if and only if $\Gamma_{X'}$ is a tree. If neither is a tree, we have* $\mathrm{Core}\,(NL(X,x)) = |\mathrm{Core}\,(\Gamma_{X'})\,|$ *as subspaces of $NL(X,x)$.*

4.2 Proof of Theorem B

The purpose of this section is to give the proof of the main result of this chapter:

Theorem B. *Let $x \in X$ and $y \in Y$ be singular points of normal algebraic surfaces defined over an algebraically closed field k and $\Gamma_{X'}$, $\Gamma_{Y'}$ the dual graphs associated to two good resolutions of (X,x) and (Y,y) respectively. The following statements are equivalent:*

1. *The spaces* $\mathrm{RZ}(X,x)$ *and* $\mathrm{RZ}(Y,y)$ *are homeomorphic.*

2. *The spaces $NL(X,x)$ and $NL(Y,y)$ are homeomorphic.*

3. *The graphs $\Gamma_{X'}$ and $\Gamma_{Y'}$ are equivalent.*

We start by presenting some lemmas needed for the proof. For the rest of this section we shall assume that X and Y are algebraic surfaces defined over the same algebraically closed field k. Recall that, given two regular closed points $x \in X$ and $y \in Y$, the choice of an isomorphism between the henselizations of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ gives us an homeomorphism between $\mathrm{RZ}(X,x)$ and $\mathrm{RZ}(Y,y)$ (see Theorem A').

Lemma 4.9. *Suppose that X and Y are non singular. Let E, D be prime divisors in X and Y respectively and $x \in E$, $y \in D$ two closed points. Let $\sigma : \widetilde{\mathcal{O}}_{Y,y} \to \widetilde{\mathcal{O}}_{X,x}$ be an isomorphism between the henselizations of the local rings which sends the equation of D to the equation of E. For any valuation $\nu \in \mathrm{RZ}(X,x)$, $R_\nu \subseteq R_{\nu_E}$ if and only if $R_{\varphi(\nu)} \subseteq R_{\nu_D}$, where $\varphi : \mathrm{RZ}(X,x) \to \mathrm{RZ}(Y,y)$ denotes the homeomorphism induced by σ.*

Proof. Let us first consider an arbitrary algebraic variety X defined over k and $x \in X$ a regular closed point. Keeping the notations of Section 2.3, given $\nu \in \mathrm{RZ}(X,x)$, ν and its extension $\widetilde{\nu} \in \widetilde{\mathrm{RZ}}(X,x)$ have the same value group Φ. Assume that Φ has rank larger than one. By the description given in Lemma 1.6, the center $\mathfrak{q}$ in $\mathcal{O}_{X,x}$ of the rank one valuation with which ν is composite coincides with $\widetilde{\mathfrak{q}} \cap \mathcal{O}_{X,x}$, where $\widetilde{\mathfrak{q}}$ is the center in $\widetilde{\mathcal{O}}_{X,x}$ of the rank one valuation with which $\widetilde{\nu}$ is composite. Moreover, $\mathfrak{q} = m_{X,x}$ if and only if $\widetilde{\mathfrak{q}}$ is the maximal ideal of $\widetilde{\mathcal{O}}_{X,x}$. Recall that any prime ideal of height one of a UFD is principal. In particular when $\dim \mathcal{O}_{X,x} = 2$, we deduce that if $\mathfrak{q}$ is generated by an element $f \in \mathcal{O}_{X,x}$ then $\widetilde{\mathfrak{q}}$ is generated by an element $\tilde{f} \in \widetilde{\mathcal{O}}_{X,x}$ dividing f in $\widetilde{\mathcal{O}}_{X,x}$.

In the hypothesis of the lemma, we have picked local coordinates (u, v) at x and (u', v') at y such that $E = \{u = 0\}$, $D = \{u' = 0\}$ and $\sigma(u') = u$. Let us take a valuation $\nu \in \mathrm{RZ}(X, x)$ and suppose that $R_\nu \subseteq R_{\nu_E}$ holds. Then $\mathrm{rk}\,\nu = 2$ (note that $\nu \neq \nu_E$) and hence $\mathrm{rk}\,\varphi(\nu) = 2$. We consider $\mu \in \mathrm{RZ}(Y)$ such that $R_{\varphi(\nu)} \subsetneq R_\mu$. Let us show that $\mu = \nu_D$. The converse is proved in an analogous way.

Denote by $\widetilde{\varphi(\nu)}$ the extension of $\varphi(\nu)$ to $\widetilde{\mathrm{RZ}}(Y, y)$. Applying the remarks made at the beginning of the proof to ν, we can write the center in $\widetilde{\mathcal{O}}_{Y,y}$ of the rank one valuation with which $\widetilde{\varphi(\nu)}$ is composite as $\tilde{\mathfrak{a}} = (\sigma^{-1}(\tilde{u}))\widetilde{\mathcal{O}}_{Y,y}$, for some $\tilde{u} \in \widetilde{\mathcal{O}}_{X,x}$ dividing u in $\widetilde{\mathcal{O}}_{X,x}$. In addition, $\widetilde{\varphi(\nu)}$ is not centered in the maximal ideal $\tilde{\mathfrak{m}}_{Y,y}$. Taking again into account the remarks made at the beginning we see that the center of μ in $\mathcal{O}_{Y,y}$ is $\mathfrak{a} = \tilde{\mathfrak{a}} \cap \mathcal{O}_{Y,y} \subsetneq \mathfrak{m}_{Y,y}$. Since $\sigma^{-1}(\tilde{u})$ divides $\sigma^{-1}(u)$ in $\widetilde{\mathcal{O}}_{Y,y}$ and $u' = \sigma^{-1}(u)$, we deduce that u' belongs to $\mathfrak{a}$. It suffices now to observe that $\mathfrak{a}$ is a principal ideal and u' is irreducible in order to conclude that $\mu = \nu_D$. □

We might state the following lemma in terms of nets in $\mathrm{RZ}(X)$, but for our purposes it suffices to deal with sequences.

Lemma 4.10. *Suppose that the surface X is non singular. Let E be a prime divisor in X and $(\nu_n)_{n=1}^\infty$ a sequence of valuations in $\mathrm{RZ}(X)$. If the center x_n of ν_n in X belongs to E for any n and $x_i \neq x_j$ for $i \neq j$, then $(\nu_n)_{n=1}^\infty$ is convergent. In addition, $\nu_n \to \nu$ in $\mathrm{RZ}(X)$ if and only if ν is in the closure of the divisorial valuation associated to E.*

Proof. A sequence $(\nu_n)_{n=1}^\infty$ satisfying the hypothesis of the lemma converges to the divisorial valuation ν_E. Indeed, given any f in the function field of X with $\nu_E(f) \geq 0$, for n large enough, x_n is not a pole of f and thus $\nu_n(f) \geq 0$. Recall that by Lemma 1.3 this means that $\nu_n \to \nu_E$. If ν is a valuation of $\mathrm{RZ}(X)$ in the closure of ν_E, $\nu \neq \nu_E$, and U is an open neighborhood of ν, then $\nu_E \in U$. Since $(\nu_n)_{n=1}^\infty$ converges to ν_E, $\nu_n \in U$ for n large enough. Therefore, $\nu_n \to \nu$. We will proceed by contradiction to finish the proof of the second assertion.

Take $(\nu_n)_{n=1}^\infty$ satisfying the assumptions about the sequence of centers and suppose that $\nu_n \to \nu$ where $\nu \in \mathrm{RZ}(X)$ is not in the closure of ν_E. We denote by x the center of ν in X. Note that the continuity of the map which sends a valuation of $\mathrm{RZ}(X)$ to its center in X implies that $x_n \to x$ in X. Moreover, x must be a closed point of E. To see this, observe that either x is a closed point of X or the generic point ξ of a prime divisor D, $D \neq E$, of X. Consider the open subset $U = X \setminus E$. By hypothesis $x_n \notin U$ for all $n \geq 1$. If x is a closed point of X and $x \notin E$ then U is an open neighborhood of x in X and this contradicts $x_n \to x$. If $x = \xi$ then $x_n \to y$ for all $y \in \overline{\{\xi\}} = D$. Take a closed point $y \in D \setminus E$, then U is an open neighborhood of y in X and this contradicts $x_n \to y$.

Since ν does not belong to the closure of ν_E in $\mathrm{RZ}(X)$, it satisfies either $\mathrm{rk}\,\nu = 1$ or $R_\nu \subsetneq R_{\nu_1}$ for some rank one valuation $\nu_1 \in \mathrm{RZ}(X)$, $\nu_1 \neq \nu_E$. Let us now study both possibilities.

Suppose that ν is a rank one valuation. Pick local coordinates (u, v) at x such that $E = \{u = 0\}$ and a rational function f on X regular at x. Since the value group of ν is archimedean, we can find a positive integer m such that $\nu(f^m/u) \geq 0$. On the other hand, the hypothesis made on the sequence of centers implies that, for n large enough, f is a unit of $\mathcal{O}_{X,x_n}$ and therefore $\nu_n(f^m/u) = -\nu_n(u) < 0$. We see that $(\nu_n)_{n=1}^\infty$ does not converge to ν.

Now suppose that ν is a rank two valuation composite with a rank one valuation ν_1 different from ν_E. Consider a finite composition $\pi : X' \to X$ of point blowups above x

such that the center C of ν_1 in X' and the strict transform of E are disjoint. The sequence $(\pi^{-1}(x_n))_{n=1}^{\infty}$ of centers in X' does not converge to the center of ν in X', because this center is a closed point of C. Hence we conclude that $(\nu_n)_{n=1}^{\infty}$ does not converge to ν and this ends the proof. $\square$

We are now in position to prove Theorem B.

Proof of 1 ⇒ 2. Assume that $\mathrm{RZ}(X,x)$ and $\mathrm{RZ}(Y,y)$ are homeomorphic. By Proposition 1.28, $\mathrm{NL}(X,x)$ and $\mathrm{NL}(Y,y)$ are also homeomorphic. $\square$

Proof of 2 ⇒ 3. Suppose that $\mathrm{NL}(X,x)$ and $\mathrm{NL}(Y,y)$ are homeomorphic. If $\mathrm{NL}(X,x)$ is a tree then $\mathrm{NL}(Y,y)$ must also be a tree. According to Corollary 4.4, $\Gamma_{X'}$ and $\Gamma_{Y'}$ are both trees and thus they are equivalent graphs. Suppose that both normalized non-Archimedean links are not trees. The definition of the core is purely topological, hence we have a natural homeomorphism between the cores of $\mathrm{NL}(X,x)$ and $\mathrm{NL}(Y,y)$ when equipped with their respective induced topologies. Since neither $\Gamma_{X'}$ nor $\Gamma_{Y'}$ are trees (again by Corollary 4.4) we are led to consider their cores. By Proposition 4.7, we conclude that $|\mathrm{Core}\,(\Gamma_{X'})\,|$ and $|\mathrm{Core}\,(\Gamma_{Y'})\,|$ are homeomorphic. Therefore $\Gamma_{X'}$ and $\Gamma_{Y'}$ are equivalent graphs. $\square$

Proof of 3 ⇒ 1. Suppose that $\Gamma_{X'}$ and $\Gamma_{Y'}$ are equivalent graphs. Our goal is to construct an homeomorphism φ from $\mathrm{RZ}(X,x)$ to $\mathrm{RZ}(Y,y)$. We begin by the case where there exists an isomorphism of graphs $\tau : \Gamma_{X'} \to \Gamma_{Y'}$. In Step 1 we assume that both exceptional locus are irreducible, while in Step 2 we treat the case of any two isomorphic graphs. Next we address the general case.

Step 1. Let us assume first that the exceptional loci of $\pi_{X'}$ and $\pi_{Y'}$, denoted E and D respectively, are both irreducible. Note that E and D have both the same cardinality as the field k. Indeed, since E is a proper normal curve over k, we have a finite flat surjective morphism from E to $\mathbf{P}^1_k$ of degree $n = [L : k(t)]$ where L denotes the function field of E, and thus an injection $E \hookrightarrow \mathbf{P}^1_k \times \{1,\ldots,n\}$. The cardinality of E is bounded by the cardinalities of $\mathbf{P}^1_k$ and $\mathbf{P}^1_k \times \{1,\ldots,n\}$, which both equals the cardinality of the field k.

We define a bijective map $\varphi : \mathrm{RZ}(X,x) \to \mathrm{RZ}(Y,y)$ as follows. The divisorial valuation associated to E is sent to the divisorial valuation associated to D, that is, $\varphi(\nu_E) = \nu_D$. We choose a bijection σ between the closed points of E and those of D and, for every closed point $z \in E$ an isomorphism $\sigma_z : \widetilde{\mathcal{O}}_{Y',\sigma(z)} \to \widetilde{\mathcal{O}}_{X',z}$ between the henselizations of the local rings which maps the local equation of D to that of E (see Corollary 2.8). A valuation $\nu \in \mathrm{RZ}(X',z)$ is sent by φ to its image in $\mathrm{RZ}(Y',\sigma(z))$ by the homeomorphism induced by σ_z (see Theorem A'). Let us prove that φ is continuous. Observe that by construction, φ^{-1} will be also continuous.

According to [Fav15, Theorem 3.1], $\mathrm{RZ}(X,x)$ is a Fréchet-Urysohn space. Thus the continuity of φ will follow if, given a sequence of valuations $(\nu_n)_{n=1}^{\infty}$ in $\mathrm{RZ}(X,x)$ converging to a valuation $\nu \in \mathrm{RZ}(X,x)$, we can extract a subsequence such that $(\varphi(\nu_{\gamma(n)}))_{n=1}^{\infty}$ converges to $\varphi(\nu)$ (see Lemma 1.13). For any positive integer n, we denote by x_n the center of ν_n in X'. Note that the sequence $(x_n)_{n=1}^{\infty}$ converges to the center x' of ν in X'.

First suppose that there exists $z \in E$ and $n_0 \geq 1$ such that $x_n = z$ for $n \geq n_0$. If z is a closed point of E, then the sequence $(x_n)_{n=1}^{\infty}$ has z as unique limit and therefore $x' = z$. We have that $(\nu_n)_{n=n_0}^{\infty} \subseteq \mathrm{RZ}(X',z)$ and $\nu \in \mathrm{RZ}(X',z)$. This yields $\varphi(\nu_n) \to \varphi(\nu)$ because

φ restricted to $\mathrm{RZ}(X', z)$ is continuous. Suppose now that z is the generic point of E, that is, $\nu_n = \nu_E$ for all $n \geq n_0$. If x' is also the generic point of E then $\nu = \nu_E$ and is clear that $\varphi(\nu_n) \to \varphi(\nu)$. Otherwise x' is a closed point of E and, by Lemma 1.3, we are then in the situation $R_\nu \subsetneq R_{\nu_E}$. If this is the case, then Lemma 4.9 implies $R_{\varphi(\nu)} \subsetneq R_{\nu_D}$. Since $(\varphi(\nu_n))_{n=1}^\infty$ converges to ν_D (apply again Lemma 1.3), it also converges to any valuation in the closure of ν_D, so $\varphi(\nu_n) \to \varphi(\nu)$. This ends the proof in the case where the sequence of centers $(x_n)_{n=1}^\infty$ is stationary.

Suppose now that sequence of centers does not stabilize and we can extract a subsequence $(\nu_{\gamma(n)})_{n=1}^\infty$ of valuations where all the centers are different. Then $(\nu_{\gamma(n)})_{n=1}^\infty$ satisfies the assumptions of Lemma 4.10. Since this sequence also converges to ν, the valuation ν is in the closure of ν_E, and by Lemma 4.9, $\varphi(\nu)$ is in the closure of ν_D. Observe that by construction the centers of $(\varphi(\nu)_{\gamma(n)})_{n=1}^\infty$ are also pairwise distinct. Applying again Lemma 4.10 to the sequence $(\varphi(\nu_{\gamma(n)}))_{n=1}^\infty$ we conclude that $\varphi(\nu_{\gamma(n)}) \to \varphi(\nu)$.

If the sequence of centers does not stabilize but we are not in the previous situation, then there exists a finite number of different points $z_1, \ldots, z_l$ of E ($l \geq 2$) such that $x_n \in \{z_1, \ldots, z_l\}$ for all n large enough and any point is visited by the sequence infinitely many times. Since $x_n \to x'$, we deduce that $l = 2$, one point is the generic point of E and the other one is x' (which must be a closed point of E). Hence we can extract a subsequence $(\nu_{\gamma(n)})_{n=1}^\infty$ of valuations in $\mathrm{RZ}(X', x')$ which converges to $\nu \in \mathrm{RZ}(X', x')$. The continuity of φ restricted to $\mathrm{RZ}(X', x')$ implies that $\varphi(\nu_{\gamma(n)}) \to \varphi(\nu)$. This ends the proof of the continuity of φ and the proof of Step 1.

Step 2. Suppose now that the exceptional loci of $\pi_{X'}$ and $\pi_{Y'}$ both have $m \geq 2$ irreducible components and that there exists a graph isomorphism $\tau : \Gamma_{X'} \to \Gamma_{Y'}$. The isomorphism τ determines a natural enumeration of those irreducible components, say $E_1, \ldots, E_m$ and $D_1, \ldots, D_m$ respectively. We fix a bijection $\sigma : \bigcup_{i=1}^m E_i \to \bigcup_{i=1}^m D_j$ between the closed points such that $\sigma(E_i \cap E_j) = D_i \cap D_j$ for any $i, j \in \{1, \ldots, m\}$, $i \neq j$, such that E_i intersects E_j and $\sigma(E_i) \subseteq D_i$ for $1 \leq i \leq m$. For any closed point z of the exceptional locus $E_{X'} = \bigcup_{i=1}^m E_i$, we choose an isomorphism between the henselizations of the local rings $\sigma_z : \widetilde{\mathcal{O}}_{Y',\sigma(z)} \to \widetilde{\mathcal{O}}_{X',z}$ that sends the local equation of every D_i passing through $\sigma(z)$ to the local equation of the corresponding component E_i in $E_{X'}$ (see Corollary 2.8). We define a bijection φ from $\mathrm{RZ}(X, x)$ to $\mathrm{RZ}(Y, y)$ exactly as we did before. That is, by means of the homeomorphism at the level of valuation spaces determined by each σ_z and setting $\varphi(\nu_{E_i}) = \nu_{D_i}$ for $1 \leq i \leq m$.

In order to check the continuity of φ we follow the same idea as in Step 1. Let us take a sequence of valuations $(\nu_n)_{n=1}^\infty$ in $\mathrm{RZ}(X, x)$ converging to a valuation $\nu \in \mathrm{RZ}(X, x)$. We denote by x' the center of ν in X' and by x_n the center of ν_n in X' for any $n \geq 1$. We differentiate again three possibilities for the sequence $(x_n)_{n=1}^\infty$ of centers. In fact, we can find $i \in \{1, \ldots, m\}$ such that one of the following situations holds:

- There exists $z \in E_i$ and $n_0 \geq 1$ such that $x_n = z$ for $n \geq n_0$.
- We can extract a subsequence of valuations where all the centers in X' are different and belong to E_i.
- We can extract a subsequence of valuations where all the centers in X' are equal to x' which is in addition a closed point of E_i.

It suffices now to repeat the same arguments used in the proof of the case of one prime divisor in each good resolution to show that there exists a subsequence $(\nu_{\gamma(n)})_{n=1}^\infty$ such that $(\varphi(\nu_{\gamma(n)}))_{n=1}^\infty$ converges to $\varphi(\nu)$.

General case. If $\Gamma_{X'}$ and $\Gamma_{Y'}$ are not isomorphic graphs then by Lemma 3.10 there exist a graph Δ and two finite sequences of expansions which transform $\Gamma_{X'}$ and $\Gamma_{Y'}$ into Δ. Let us suppose that one of the dual graphs is isomorphic to Δ, for instance $\Gamma_{X'}$. Then by hypothesis we can get from Y' and after a finite number of blowing up of points above the point y, a good resolution $\pi_{\widetilde{Y}} : \widetilde{Y} \to Y$ which factors through $\pi_{Y'}$ and such that $\Gamma_{\widetilde{Y}}$ is isomorphic to $\Gamma_{X'}$. We are now in the case treated above. If neither $\Gamma_{X'}$ nor $\Gamma_{Y'}$ are isomorphic to Δ, we just need to do the previous construction starting from both good resolutions. $\square$

It is proven in [Kol14, Theorem 2] that any finite simplicial complex of dimension one can be obtained as the dual graph associated to a good resolution of an isolated surface singularity. In particular, for any connected finite graph Γ there exists a normal surface singularity (X, x) and a good resolution $X' \to X$ such that $\Gamma_{X'}$ is isomorphic to Γ. Taking into account this result, the proof of Theorem B may be simplified arguing by induction on the length of the modification which transforms $\Gamma_{X'}$ into $\Gamma_{Y'}$.

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