Departamento de Análisis Matemático

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TEOREMAS DE LIOUVILLE PARA ECUACIONES Y SISTEMAS ELÍPTICOS NO LINEALES

Liouville theorems for nonlinear elliptic equations and systems

TESIS DOCTORAL

Autor

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Dirigida por

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Don Jorge José García Melián, profesor titular de Matemática Aplicada del Departamento de Análisis Matemático, **INFORMA**

- Que es tutor del alumno Miguel Ángel Burgos Pérez, quien ha desarrollado bajo su dirección la tesis doctoral titulada *Teoremas de Liouville para ecuaciones y sistemas elípticos no lineales.*
- Que mediante el presente documento da su visto bueno a la defensa de dicha tesis.

La Laguna, 2 de diciembre de 2016

Fdo: Jorge J. García Melián

A mi madre

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Introduction

This memoir is concerned with Liouville theorems for nonlinear elliptic equations and systems. Roughly speaking, a Liouville theorem is a result about nonexistence of certain kind of solutions (positive, bounded, radial...) for those equations, usually when posed in unbounded domains.

Perhaps the first and most celebrated nonlinear Liouville theorem was the one obtained in [40], where the model equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N \tag{I.1}$$

with $p > 1$ was considered. It was shown there that equation (I.1) does not admit any positive, classical solution provided that ($N \geq 3$) and

$$1 < p < \frac{N+2}{N-2}.$$

The so-called "critical" exponent $p_S = \frac{N+2}{N-2}$ turns out to be optimal for nonexistence, since it can be proved that radially symmetric, positive classical solutions of (I.1) exist when $p \geq \frac{N+2}{N-2}$ (cf. for instance Section I.9 in [67]). As a matter of fact, solutions can be classified in the critical case $p = p_S$: it was shown in [22] that any positive solution of (I.1) is given by

$$u(x) = \left(\frac{\mu}{1 + \mu^2 |x - x_0|^2} \right)^{\frac{N-2}{2}}, \tag{I.2}$$

for some $\mu > 0$ and $x_0 \in \mathbb{R}^N$ and is therefore radially symmetric with respect to some point x_0.

It should be remarked also that when $N = 2$ there is no restriction on the exponent p, since it is well-known that positive, superharmonic functions in $\mathbb{R}^2$ are necessarily constant.

Problem (I.1) was subsequently investigated in [24], where a simpler proof of the previous nonexistence result was given. The proof in [24] has a different flavor, and it is based on the well-known method of moving planes, requiring the use of the Kelvin transform.

An important step to fully understand nonlinear Liouville theorems for elliptic equations is to analyze problem (I.1) when the power function is replaced by a more general nonlinearity, that is,

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N \tag{I.3}$$

where $f : (0, +\infty) \to \mathbb{R}$ is positive and, say, locally Lipschitz continuous. In this regard, one of the main results in [11] stated that, if $N \geq 3$ and f verifies the "subcriticality" condition

$$\frac{f(s)}{s^{\frac{N+2}{N-2}}} \quad \text{is nonincreasing in } (0, +\infty),$$

then either u is a constant or – up to a constant factor – $f(t)$ is the critical power $t^{\frac{N+2}{N-2}}$ and u is of the form (I.2) for some $\mu > 0$, $x_0 \in \mathbb{R}^N$. The restrictions that f is nonnegative and locally Lipschitz continuous in $(0, +\infty)$ were later replaced in [52] by the only requirement that f is locally bounded in $(0, +\infty)$. While the main tool in [11] was the moving planes method, in [52] the method of moving spheres was used. Both of them rely heavily on the Kelvin transform.

Observe that in the previous results the operator involved in the respective equations is always the Laplacian. This fact is related with the use of the Kelvin transform in the proofs, and it makes them hard to generalize when other operators are taken into account. However, some nonexistence results have been obtained for other second order operators.

Indeed, a natural extension of equation (I.3) is obtained when the Laplacian is replaced by the p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, that is,

$$-\Delta_p u = f(u) \text{ in } \mathbb{R}^N, \tag{I.4}$$

where $p > 1$ and $f \in C^1$ is positive in $(0, +\infty)$. Some Liouville theorems for positive, weak solutions of this problem were proved in [71]. Among other things, it is shown there that problem (I.4) does not admit any positive solution if $f(t) = t^q$, where $p < N$ and $1 < q < \frac{N(p-1)+p}{N-p}$. Some more general nonlinearities termed there as "subcritical" were also considered.

All the nonexistence results alluded to above are concerned with positive solutions of nonlinear equations (let us also mention [34], where stable solutions of (I.3) are considered irrespective of their sign). However, there are also nonexistence results which hold even for supersolutions, which will be of particular importance to us in this memoir.

As a first outstanding achievement, it was shown in [39] that problem (I.1) does not admit any positive supersolution provided that $N \geq 3$ and

$$1 < p \leq \frac{N}{N-2}. \tag{I.5}$$

This range of exponents is again sharp for nonexistence. In this case, positive supersolutions can be explicitly constructed when $p > \frac{N}{N-2}$.

An important feature with respect to this result is that it remains valid when the underlying domain $\mathbb{R}^N$ is replaced by an exterior domain. An exterior domain is of the form $\mathbb{R}^N \setminus K$, where K is a compact, simply connected set in $\mathbb{R}^N$. It clearly follows that solutions (resp. supersolutions) of any of the above equations in an exterior domain are also solutions (resp. supersolutions) in $\mathbb{R}^N \setminus B_{R_0}$ for some $R_0 > 0$, where B_{R_0} stands for the ball of radius R_0 centered at the origin. Therefore the obtention of nonexistence theorems in exterior domains can always be restricted to the latter ones.

As far as we know, the first nonexistence result in exterior domains was obtained in [12], where the $p-$Laplacian version of problem (I.1) in $\mathbb{R}^N \setminus B_{R_0}$ was analyzed. When $p = 2$, Theorem 1.3 there applies and shows that there are no positive solutions in the regime given by (I.5). It should be stressed that Liouville theorems for supersolutions are somehow more robust with regard to perturbations of the operators. Let us quote for instance [46], where the problem

$$-\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = u^p$$

for a general second order uniformly elliptic operator was considered, or [13] and [55], which dealt in particular with positive supersolutions of

$$-\operatorname{div} A\left(x, u, \nabla u\right) = u^p \tag{I.6}$$

in exterior domains with $N \geq 3$ and $p > 1$, where A is a function verifying certain positivity and growth restrictions.

A great step forward in the understanding of Liouville theorems for supersolutions was given in [5]. Problem (I.3), when posed in an exterior domain, i.e.

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{I.7}$$

was considered for a general continuous positive nonlinearity f (as a matter of fact, the problem was analyzed when the Laplacian is substituted by a quite general fully nonlinear elliptic operator). The main novelty is to show, with the use of methods which essentially involve only the comparison principle, that Liouville theorems for supersolutions in exterior domains depend only on the behavior of the nonlinearity f at zero. Specifically, it was proved there that if $N \geq 3$ and f verifies

$$\liminf_{t\to 0^+} \frac{f(t)}{t^{\frac{N}{N-2}}} > 0, \tag{I.8}$$

then equation (I.7) does not admit any positive supersolution. Some results were also obtained when $N = 2$.

The main disadvantage of condition (I.8) is that it is not expected to be optimal. However, the optimal condition for nonexistence of positive supersolutions in exterior domains was later obtained in [3]. When $N \geq 3$, it was shown there that a necessary and sufficient condition for the existence of such supersolutions is

$$\int_0^\delta \frac{f(t)}{t^{\frac{2(N-1)}{N-2}}} dt < +\infty \tag{I.9}$$

for some $\delta > 0$. A condition of a similar type was also obtained when $N = 2$. An advantage of the methods used in [3] is that they allow to study much more general problems than (I.7). These generalizations include the Laplacian with weights and gradient terms, the p-Laplacian operator and uniformly elliptic fully nonlinear operators with radial symmetry.

The problem of obtaining Liouville theorems for positive supersolutions has been extensively studied in more general settings, which include the p-Laplacian or fully nonlinear operators (cf. for instance [6], [14], [23], [27], [46] and [48]). Let us mention in passing that, aside equations posed in $\mathbb{R}^N$ or in exterior domains, other unbounded domains have also been frequently considered in the literature of Liouville theorems, like cone-like domains and in particular the half-space. However, we will not be concerned with this type of domains in the present work (see [5], [8], [9], [10], [28], [41], [44] or [47] for a survey).

Most if not all of the results in the above discussion deal with operators which have some kind of homogeneity, which some times is essential in the proofs. Thus a natural question related to (I.7) is to know how the nonexistence results are modified when the equation is perturbed in order that the homogeneity is destroyed. This was the motivation in [2], where the problem

$$-\Delta u + |\nabla u|^q = \lambda f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{I.10}$$

was studied in the regime $N \geq 3$, $q > 1$, and the nonlinearity f is as above continuous and positive in $(0, +\infty)$. The number λ is a positive parameter, which turns out to be important in the discussion of nonexistence of supersolutions in some regimes.

Let us mention that this problem had been considered in the case $f(t) = t^p$, $p > 0$ in [25], [59], [69] and [73] (the problem for the p-Laplacian was studied in [37] and [38]). See also [68] for a survey on the problem. However, the authors of these papers were mainly interested in the existence and nonexistence of *radially symmetric solutions* in $\mathbb{R}^N$.

One of the important observations in [2] is that positive supersolutions of (I.10) can be classified into two types: those which blow up at infinity, that is, verify

$$\lim_{|x|\to+\infty} u(x) = +\infty$$

uniformly and those which do not. This phenomenon is caused by the presence of the gradient term in the equation. Under the assumption that f can be compared to a power near zero

$$\liminf_{s\to 0} \frac{f(s)}{s^p} > 0, \tag{I.11}$$

or near infinity,

$$\liminf_{s\to+\infty} \frac{f(s)}{s^p} > 0, \tag{I.12}$$

where $p > 0$, several nonexistence results were obtained, which turn out to be optimal precisely when f is given by a power. We will not give precise results here, but only mention that, when considering supersolutions which do not blow up at infinity, the relation between q and $\frac{N}{N-1}$ is very important. In particular, when $q > \frac{N}{N-1}$, the nonexistence region coincides with the one in (I.5). As for the supersolutions blowing up at infinity, nonexistence does not even depend on N.

The general aim of the first part of the present memoir (Chapters 1 and 2) is to further explore the influence of a gradient term in Liouville theorems for nonlinear equations like the ones considered above. We will be concerned with several equations which can be seen as generalizations of (I.7).

The first generalization we will analyze is obtained when a gradient term appears multiplying the nonlinearity, that is,

$$-\Delta u = f(u)|\nabla u|^q \text{ in } \mathbb{R}^N \setminus B_{R_0}, \tag{I.13}$$

where $N \geq 2$ and $q > 0$. The nonlinearify f is a continuous function defined in $[0, +\infty)$ and positive in $(0, +\infty)$, in the same spirit as before. By refining the arguments in [2], we perform a classifications of all positive supersolutions into four types, which depend on the monotonicity of the function $m(R) = \inf_{|x|=R} u(x)$ for large R.

Then we obtain necessary and sufficient conditions in order to have supersolutions of each of these types. As a consequence, we also obtain Liouville theorems for supersolutions depending on the values of N, q and on some integrability properties on f at zero or infinity. We also adapt these results to the case where the equation is posed in the whole space $\mathbb{R}^N$.

Let us mention that this precise problem has been studied in [14] for some more general operators when $0 < q < 1$, but only the case $f(t) = t^p$ is dealt with there. Also, the more general problem

$$-\mathrm{div}(h(x)g(u)A(|\nabla u|)\nabla u) \geq f(x, u, \nabla u) \quad \text{in } \mathbb{R}^N$$

was considered in [36] under several assumptions on the involved functions, but the non-linearity $f(x, u, \nabla u)$ is essentially like $u^p|\nabla u|^q$ for some $p, q > 0$, so that again only the power case was previously known.

Our method of proof is inspired in [3]. Assuming that a positive supersolution of (I.13) exists, we show that there exists a positive, *radially symmetric solution* of (I.13) which is of the same type. In what follows, we will refer to this procedure as a "radial reduction". By means of a change of variables which involves the fundamental solution of the Laplacian, the radial problem is then transformed into a one-dimensional one which is precisely analyzed.

In Chapter 2 we consider problem (I.10). Our aim is to "fine tune" the nonexistence results obtained in [2] to include some more general nonlinearities f. Recall that the essential assumption in [2] is that f can be compared to a power function, either near zero or near infinity (cf. hypotheses (I.11) and (I.12)).

Thus our more relevant objective is to study nonlinearities which are compared to a logarithmic perturbation of the critical power, and to obtain nonexistence regions in terms of the parameters involved. It is interesting to mention that in the regime $q > \frac{N}{N-1}$, the gradient term is again not relevant for the question of existence of supersolutions not blowing up at infinity.

In the second part of this work, we deal with Liouville theorems for positive supersolutions of some elliptic systems. Perhaps the most important of such systems is a natural generalization of problem (I.1), given by

$$\begin{cases} -\Delta u = v^p \\ -\Delta v = u^s \end{cases} \quad \text{in } \mathbb{R}^N, \tag{I.14}$$

where $p, s > 0$ and $N \geq 3$. This system has been widely studied, and some nonexistence theorems for positive solutions have been obtained. However, the full expected Liouville theorem is nowadays still not proved. Indeed, the so-called *Lane-Emden conjecture* states that there do not exist positive solutions of (I.14) when

$$\frac{1}{p+1} + \frac{1}{s+1} > \frac{N-2}{N}.$$

At the best of our knowledge, aside the radially symmetric case, which was proved in [56], only the cases $N = 3$ and $N = 4$ are completely proved (see [63], [70] and [72]). Nevertheless, some more restrictive conditions have been obtained which imply nonexistence of positive solutions (we refer to [21] or [30]; see also references in [72]).

As for supersolutions of (I.14), it was shown in [56] (see also [70]) that there do not exist positive supersolutions when $0 < ps \leq 1$ or $ps > 1$ and

$$N - 2 \leq \max\left\{\frac{2(p+1)}{ps-1}, \frac{2(q+1)}{ps-1}\right\}.$$

Without being exhaustive with the references, let us finally mention that some generalizations of (I.14) have been considered in [7], [13], [15], [26], [29], [32], [50], [57], [62], [65], [66] and [75]. The purpose of all these works was to deal with systems where the Laplacian was replaced by a more general operator or the power nonlinearities by some more general functions, which can be compared to a power.

Our first incursion into the systems realm will be made in Chapter 3. The objective there is to analyze the effect of introducing a gradient term into (I.14), in the spirit of (I.10). More precisely, we will consider the problem

$$\begin{cases} -\Delta u + |\nabla u|^q = \lambda f(v) \\ -\Delta v + |\nabla v|^q = \mu g(u) \end{cases} \tag{I.15}$$

in exterior domains of $\mathbb{R}^N$, where $q > 1$ and the functions f and g behave like a power near zero or infinity. We show that positive supersolutions do not exist in some ranges of the parameters involved, which are optimal in the special case where $f(v) = v^p$ and $g(u) = u^s$, with $p, s > 0$. The case $q = 1$ in (I.15) will be subsequently analyzed in Chapter 4.

Our final contribution deals with another generalization of system (I.14). In our opinion, it would be desirable to obtain necessary and sufficient conditions for the existence of positive supersolutions in exterior domains when the power functions there are replaced by general nondecreasing functions $f(v)$ and $g(u)$. However, we content ourselves for the moment with the analysis of the special case

$$\begin{cases} -\Delta u = v \\ -\Delta v = g(u) \end{cases} \quad \text{in } \mathbb{R}^N \setminus B_{R_0},$$

which is equivalent to the fourth order problem

$$(-\Delta)^2 u = g(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}. \tag{I.16}$$

This will be the subject of Chapter 5.

As we have already discussed, nonlinear Liouville theorems have been frequently analyzed for equations of second order. However, when it comes to a fourth order problem as (I.16), the situation is fairly different and the results already available are far from being complete. Let us mention in this regard the work [74], where the problem with a power function:

$$(-\Delta)^2 u = u^p \quad \text{in } \mathbb{R}^N \tag{I.17}$$

with $p > 1$ and $N \geq 5$ has been studied. A Liouville theorem for positive solutions was obtained there, revealing the importance of the critical exponent $\frac{N+4}{N-4}$. As for positive supersolutions, the only results known to us are those in [56] and [58], which state that the range of nonexistence is $1 < p \leq \frac{N}{N-4}$.

But when the underlying domain $\mathbb{R}^N$ is replaced by an exterior domain and the power nonlinearity by a more general one, no results seem to be available in the literature. Assuming the function g is nondecreasing and continuous in $[0, +\infty)$ and positive in $(0, +\infty)$, we give a necessary and sufficient condition for the existence of positive classical supersolutions u of (I.17) which additionally verify $-\Delta u > 0$, which resembles condition (I.9) obtained in [3] for the second order case. It is to be noted that positive supersolutions not verifying the condition $-\Delta u > 0$ seem to always exist, irrespective of the behavior of f at zero.

Taking advantage of these results, problem (I.16) is also analyzed when it is posed in $\mathbb{R}^N$, and a Liouville theorem is obtained.

As a final remark, let us comment that all our proofs rely on the radial reduction introduced above, followed by a careful examination of the radially symmetric version of each problem.

To conclude the Introduction, let us also mention that we will be dealing in the rest of this work with several notions of supersolutions, which suit well to the problem at hand. However, with some more effort, other perhaps more general notions could be considered as well.

Chapter 1

Scalar equation with gradient terms: Multiplicative term

The purpose of this chapter is to analyze the existence and nonexistence of supersolutions of the elliptic problem

$$-\Delta u = f(u)|\nabla u|^q \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{1.1.1}$$

where $N \geq 2$, and $q > 0$. The nonlinearify f is a continuous function defined in $[0, +\infty)$ and positive in $(0, +\infty)$. In particular, we are interested in obtaining Liouville type theorems for (1.1.1).

In this chapter we will always be dealing with continuous weak supersolutions, that is, functions $u \in H^1_{\text{loc}}(\mathbb{R}^N \setminus B_{R_0}) \cap C(\mathbb{R}^N \setminus B_{R_0})$ verifying

$$\int_{\mathbb{R}^N \setminus B_{R_0}} \nabla u \nabla \phi \geq \int_{\mathbb{R}^N \setminus B_{R_0}} f(u)|\nabla u|^q \phi$$

for every nonnegative $\phi \in C_0^\infty(\mathbb{R}^N \setminus B_{R_0})$.[1]

1.1 Classification of supersolutions

As we said in the Introduction, our analysis of positive supersolutions u of problem (1.1.1) is based upon the fact that the function

$$m(R) = \inf_{|x|=R} u(x),$$

which is positive for $R > R_0$, is monotone for large R (see Lemma 1.3.3 in Section 1.3). With no loss of generality we can always assume $m(R)$ is monotone if $R > R_0$. This allows to classify all possible supersolutions into four types:

[1] The results in this chapter are contained in [19].

Type 1: $m(R)$ is decreasing and $\lim_{R\to+\infty} m(R) = 0$;

Type 2: $m(R)$ is increasing and $\lim_{R\to+\infty} m(R) = \ell$ for some $\ell > 0$;

Type 3: $m(R)$ is decreasing and $\lim_{R\to+\infty} m(R) = \ell$ for some $\ell > 0$;

Type 4: $m(R)$ is increasing and $\lim_{R\to+\infty} m(R) = +\infty$.

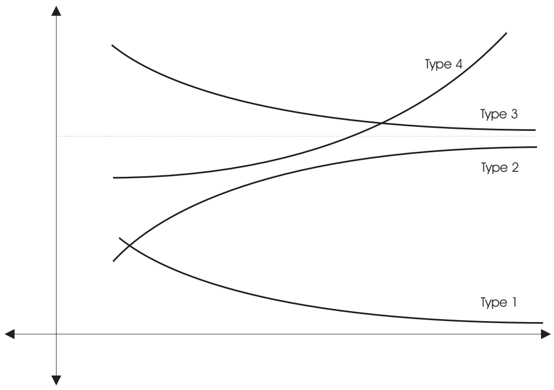

Figure 1.1. The four types of supersolutions for problem (1.1.1).

We are intentionally excluding from our classification all constant solutions of (1.1.1). Also, it is important to observe that the singular nature of the problem allows the existence of supersolutions which are not constant, but are *eventually constant* in the sense that they are constant for $|x| > R_1$ for some $R_1 > R_0$, at least when $0 < q < 1$ (see Remark 1.3.4 in Section 1.3). Thus these supersolutions are also excluded from our classification. Let us mention in passing that this phenomenon does not seem to be possible for $q \geq 1$.

1.2 Main results

The existence of each type of supersolution depends first of all on the dimension N. The cases $N \geq 3$ and $N = 2$ –as in multiple well-known situations– are genuinely different, as can be seen for instance from the fact that supersolutions of type 4 are never possible when $N \geq 3$, while for $N = 2$, the only possible types are 2 and 4 (which are the nondecreasing ones). See Lemma 1.3.5 below.

Let us begin with the case of higher dimensions $N \geq 3$. Since f is assumed to be positive in $(0, +\infty)$, the existence of supersolutions of types 2 and 3 does not really depend on f, and it is only related to the relative values of q and N. However, when considering supersolutions of type 1, the relevant condition is

$$\int_0^\delta \frac{f(t)}{t^\theta}\,dt < +\infty, \tag{1.2.1}$$

for some $\delta > 0$, where

$$\theta = \frac{(2-q)(N-1)}{N-2}. \tag{1.2.2}$$

This condition resembles the one found in [3] for the case $q = 0$ (and actually reduces to that one in this particular case). See equation (I.9) in the Introduction.

Our results for problem (1.1.1) in the case $N \geq 3$ can be summarized as follows.

Theorem 1.1. *Assume $N \geq 3$ and $f \in C([0,+\infty))$ is positive in $(0,+\infty)$. Then:*

a) *If $q > \frac{N}{N-1}$, there exist positive supersolutions of (1.1.1) of types 1, 2 and 3.*

b) *When $1 \leq q \leq \frac{N}{N-1}$, no supersolutions of (1.1.1) of type 3 exist, while there always exist supersolutions of type 2. Supersolutions of type 1 exist if and only if (1.2.1) holds.*

c) *For $0 < q < 1$, there never exist supersolutions of (1.1.1) of types 2 and 3, while supersolutions of type 1 exist if and only if (1.2.1) holds.*

Moreover, positive supersolutions of type 4 never exist in this case.

As a consequence of the statements above, we have the following Liouville theorem for (1.1.1):

Corollary 1.2.1 (Liouville theorem)**.** *Assume $N \geq 3$ and $f \in C([0,+\infty))$ is positive in $(0,+\infty)$. If $q < 1$ and (1.2.1) does not hold, then every positive supersolution of problem (1.1.1) is eventually constant.*

As for the case $N = 2$, our results are:

Theorem 1.2. *Assume $N = 2$ and $f \in C([0,+\infty))$ is positive in $(0,+\infty)$. Then:*

a) *If $q \geq 2$, there exist positive supersolutions of (1.1.1) of types 2 and 4.*

b) *When $1 \leq q < 2$, there always exist supersolutions of type 2. Supersolutions of type 4 exist if and only if there exist $a, M > 0$ such that*

$$\int_M^\infty e^{at} f(t)\,dt < +\infty. \tag{1.2.3}$$

c) *For* $0 < q < 1$, *there never exist supersolutions of* (1.1.1) *of type 2, while supersolutions of type 4 exist if and only if* (1.2.3) *holds.*

Moreover, positive supersolutions of types 1 and 3 never exist in this case.

Corollary 1.2.2 (Another Liouville theorem)**.** *Assume* $N = 2$ *and let* $f \in C([0, +\infty))$ *be positive in* $(0, +\infty)$. *If* $q < 1$ *and* (1.2.3) *does not hold, then every positive supersolution of problem* (1.1.1) *is eventually constant.*

Of particular interest in (1.1.1) is the special case where f is a power, $f(t) = t^p$, $p > 0$, that is,

$$-\Delta u = u^p |\nabla u|^q \quad \text{in } \mathbb{R}^N \setminus B_{R_0}. \tag{1.2.4}$$

The above theorems directly apply to this case to obtain for instance: when $N \geq 3$, if $0 < q < 1$ and $0 < p \leq \frac{N-q(N-1)}{N-2}$, then every positive supersolution of (1.2.4) is eventually constant; for $N = 2$, the nonexistence of not eventually constant positive supersolutions holds if $0 < q < 2$. Both results are sharp, and coincide with those in [36] when the equation is considered in $\mathbb{R}^N$ (for $N \geq 3$). See Figure 1.2.

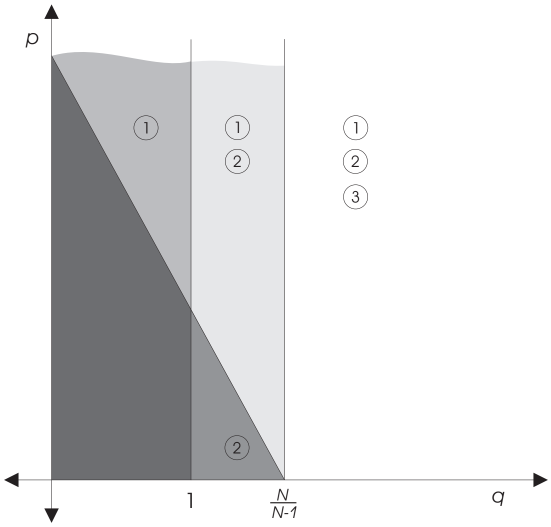

Figure 1.2. Regions of existence of supersolutions when $f(t) = t^p$ and $N \geq 3$. The region of nonexistence is shown in dark grey.

Observe that all nonexistence results stated above apply equally to the equation in (1.1.1) when it is posed in the whole $\mathbb{R}^N$, namely

$$-\Delta u = f(u)|\nabla u|^q \quad \text{in } \mathbb{R}^N, \tag{1.2.5}$$

where $N \geq 3$ (it is well known that the only positive, superharmonic functions in $\mathbb{R}^2$ are constants, so that the case $N = 2$ is uninteresting). However, it is to be stressed that the maximum principle implies $m(R) = \inf_{|x|\leq R} u(x)$ for all positive supersolutions, so that the function $m(R)$ is always strictly decreasing, unless u is constant. Therefore, only supersolutions of types 1 and 3 are possible, and it also follows that eventually constant supersolutions which are not constant do not exist.

Regarding the existence, if u is a positive supersolution of (1.1.1) of one of these types, then $m(R)$ is decreasing in $(R_0, +\infty)$. Hence, it is easily checked that the function $\tilde{u} = \min\{u, m(R_0)\}$ is a weak $H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ supersolution of (1.2.5). Therefore,

Theorem 1.3 (Classification in $\mathbb{R}^N$)**.** *Assume* $N \geq 3$ *and* $f \in C([0, +\infty))$ *is positive in* $(0, +\infty)$*. Then:*

a) *If* $q > \frac{N}{N-1}$*, there exist positive supersolutions of* (1.2.5) *of types 1 and 3.*

b) *When* $0 < q \leq \frac{N}{N-1}$*, no supersolutions of* (1.2.5) *of type 3 exist, while supersolutions of type 1 exist if and only if* (1.2.1) *holds.*

The corresponding Liouville theorem is:

Corollary 1.2.3 (Liouville Theorem in $\mathbb{R}^N$)**.** *Assume that* $N \geq 3$ *and* $f \in C([0, +\infty))$ *is positive in* $(0, +\infty)$*. If* $0 < q \leq \frac{N}{N-1}$ *and* (1.2.1) *does not hold, then the only positive supersolutions of problem* (1.2.5) *are constants.*

To conclude the discussion of our results, let us mention that the very interesting question of existence and nonexistence of positive *solutions* of (1.2.5) seems to be very delicate for general functions f. However, as a consequence of the uniqueness theorems for ode's, the only radially symmetric, positive solutions of (1.2.5) are constants if f is locally Lipschitz in $(0, +\infty)$ and $q \geq 1$. The case $0 < q < 1$ seems more difficult to deal with, even with radial symmetry. We refer to [43] for an Emden-Fowler analysis in the particular case $f(t) = t^p$, $p > 0$.

1.3 Preliminary properties

In this section we consider several questions related to the classification of positive supersolutions of (1.1.1) and the reduction to the radial setting. We begin with an extension of a result in [3] (see also [2]), which gives sense to the classification introduced in the previous section. Remember that, for a positive supersolution u of (1.1.1) we always denote for $R > R_0$

$$m(R) = \inf_{|x|=R} u(x).$$

When u is nonzero, it follows from the strong maximum principle that m is strictly positive. The main property we need with regard to this function concerns its monotonicity.

Lemma 1.3.1. *Let $u \in H^1_{\rm loc}(\mathbb{R}^N \setminus B_{R_0}) \cap C(\mathbb{R}^N \setminus B_{R_0})$ be a nonnegative function verifying $-\Delta u \geq 0$ in $\mathbb{R}^N \setminus B_{R_0}$ in the weak sense. Then, there exists $R_1 > R_0$ such that the function $m(R)$ is monotone for $R > R_1$.*

Proof. For $R_2 > R_1 > R_0$, consider the annulus $A(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$. Thanks to the maximum principle applied to u in this annulus we obtain

$$\min_{A(R_1,R_2)} u = \min\{m(R_1), m(R_2)\}.$$

As an immediate consequence, we obtain that the function $\min_{A(R_1,R_2)} u$ is increasing in R_1 and decreasing in R_2 whenever $R_2 > R_1 > R_0$.

We claim that for arbitrary values $S_3 > S_2 > S_1 > R_0$ it is not possible to have $m(S_2) < \min\{m(S_1), m(S_3)\}$, that is, the function m does not achieve local strict minima. Suppose that the claim is not true, so we have that $m(S_2) < \min\{m(S_1), m(S_3)\}$ for some $S_3 > S_2 > S_1 > R_0$. As a consequence of the above discussion, the function

$$h(\delta) = \min\{m(S_2 - \delta(S_2 - S_1)), m(S_2 + \delta(S_3 - S_2))\}$$

is decreasing in $0 \leq \delta \leq 1$, so $m(S_2) = h(0) \geq h(1) = \min\{m(S_1), m(S_3)\}$ and we arrive at a contradiction.

There are three possibilities: either $m(R)$ is nonincreasing, or it is nondecreasing, or it is nondecreasing in (R_0, R_1) for some $R_1 > R_0$ and then nonincreasing in $(R_1, +\infty)$. Whatever the case, $m(R)$ is monotone for $R > R_1$. $\square$

Remark 1.3.2. The previous lemma still holds for weak supersolutions u of general operators L whenever there exists a maximum principle for L. See, for instance, [64] for equations with gradient terms.

As a consequence of the strong maximum principle it is possible to show that the function $m(R)$ is indeed strictly monotone.

Lemma 1.3.3. *Let $u \in H^1_{\rm loc}(\mathbb{R}^N \setminus B_{R_0}) \cap C(\mathbb{R}^N \setminus B_{R_0})$ be a nonnegative function verifying $-\Delta u \geq 0$ in $\mathbb{R}^N \setminus B_{R_0}$ in the weak sense. Then, there exists $R_1 > R_0$ such that the function $m(R)$ is either strictly increasing, or strictly decreasing or constant in $(R_1, +\infty)$.*

Proof. According to Lemma 1.3.1, there exists $R_0' \geq R_0$ such that $m(R)$ is monotone for $R > R_0'$. As we have already observed, we can always assume $R_0' = R_0$.

Suppose first that m is nondecreasing. Let us show that if there exists an interval $[R_1, R_2]$ where m is constant, then m is constant for $R > R_1$. Indeed, assume on the contrary the existence of $R_3 > R_2$ such that $m(R_3) > m(R_1)$. For any $R \in [R_1, R_2]$ the function u attains an interior minimum in the annulus $A(R, R_3)$. Thus by the strong maximum principle u has to be constant in $A(R, R_3)$, contradicting that $m(R_3) > m(R_1)$. We deduce that either m is constant for $R > R_0$ or m is strictly increasing in $(R_0, +\infty)$.

A similar argument shows that, if m is nonincreasing and it is constant in an interval $[R_1, R_2]$, then $m = m(R_1)$ if $R \in (R_0, R_1)$. Assume such an interval exists (if not m is strictly decreasing and we are done) and m is not constant for $R > R_2$. Choose R_2 with the property that the interval $[R_1, R_2]$ is the maximal interval where m is constant. Then m is decreasing for $R > R_2$. If not, there would exist an interval $[R_3, R_4]$ with $R_3 > R_2$ and $m(R) = m(R_3) < m(R_2)$ if $R \in [R_3, R_4]$. But this would imply m is constant for $R < R_4$, a contradiction. Therefore either m is strictly decreasing for $R > R_2$ or m is constant for $R > R_2$. This concludes the proof. $\square$

Remark 1.3.4. As mentioned in Section 1.1 of this chapter, if $q < 1$ there always exist positive nonconstant supersolutions of (1.1.1) which are constant for large $|x|$. Indeed, fix $\lambda > 0$ and denote $M = \sup_{[0,\lambda]} f(t)$. Consider the function $v = \lambda - A(R_0 + \delta - |x|)^\alpha$ in $R_0 < |x| < R_0 + \delta$, where $\alpha = (2-q)/(1-q) > 1$ and $A, \delta > 0$. It will be a supersolution in the interval $(R_0, R_0 + \delta)$ provided that

$$\alpha - 1 - (N-1)\frac{R_0 + \delta - r}{r} \geq M\alpha^{q-1}A^{q-1}, \quad R_0 < r < R_0 + \delta \tag{1.3.1}$$

where $r = |x|$, as usual. Since $(R_0 + \delta - r)/r \leq \delta/R_0$, because the function $(R_0 + \delta - r)/r$ is decreasing in r, we have that

$$\alpha - 1 - (N-1)\frac{R_0 + \delta - r}{r} \geq \alpha - 1 - (N-1)\frac{\delta}{R_0}.$$

Then the inequality (1.3.1) can always be achieved if $\delta < R_0(\alpha - 1)/(N-1)$, and then A is chosen large enough (remember that $q < 1$). Finally, the function

$$u(x) = \begin{cases} v(x) & R_0 < |x| < R_0 + \delta \\ \lambda & |x| \geq R_0 + \delta \end{cases}$$

will be a positive C^1 supersolution of (1.1.1).

Our next step is to shed some light into the question of existence of supersolutions. Depending on the dimension N, only some types of supersolutions are possible, as the following lemma shows.

Lemma 1.3.5. *Assume $q > 0$ and $f \in C([0, +\infty))$ is nonnegative. When $N = 2$, there do not exist positive supersolutions of* (1.1.1) *of types 1 and 3, while for $N \geq 3$, no supersolutions of type 4 exist.*

Proof. Assume u is a positive, not eventually constant supersolution of (1.1.1) and consider initially $N \geq 3$. Choose $R_2 > R_1 > R_0$ and define the function

$$\Phi(x) = \frac{m(R_1) - m(R_2)}{R_1^{2-N} - R_2^{2-N}}(|x|^{2-N} - R_2^{2-N}) + m(R_2) \tag{1.3.2}$$

for $x \in A(R_1, R_2)$. Observe that Φ is harmonic in $A(R_1, R_2)$, with $u \geq \Phi$ on $\partial A(R_1, R_2)$. Therefore, by the maximum principle it follows that $u \geq \Phi$ in $A(R_1, R_2)$. In particular, for every $R \in (R_1, R_2)$,

$$\begin{aligned} m(R) &\geq \frac{m(R_1) - m(R_2)}{R_1^{2-N} - R_2^{2-N}}(R^{2-N} - R_2^{2-N}) + m(R_2) \\ &= m(R_2)\left(\frac{R_1^{2-N} - R^{2-N}}{R_1^{2-N} - R_2^{2-N}}\right) + m(R_1)\left(\frac{R^{2-N} - R_2^{2-N}}{R_1^{2-N} - R_2^{2-N}}\right). \end{aligned} \tag{1.3.3}$$

Now assume that u is of type 4, that is, $\lim_{R\to+\infty} m(R) = +\infty$. Letting $R_2 \to +\infty$ in (1.3.3) with fixed R and R_1 we arrive at a contradiction.

Next we analyze the case $N = 2$. We consider the function which is obtained by replacing in the definition of Φ the power $2 - N$ by a logarithm, that is

$$\Psi(x) = \frac{m(R_1) - m(R_2)}{\log R_1 - \log R_2}(\log|x| - \log R_2) + m(R_2) \tag{1.3.4}$$

for $x \in A(R_1, R_2)$. As before, $u \geq \Psi$ in $A(R_1, R_2)$, so that

$$\begin{aligned} m(R) &\geq \frac{m(R_1) - m(R_2)}{\log R_1 - \log R_2}(\log R - \log R_2) + m(R_2) \\ &= m(R_2)\left(\frac{\log R_1 - \log R}{\log R_1 - \log R_2}\right) + m(R_1)\left(\frac{\log R - \log R_2}{\log R_1 - \log R_2}\right) \end{aligned} \tag{1.3.5}$$

for $R \in (R_1, R_2)$. Assume u is not of type 4, so that $\lim_{R\to+\infty} m(R) = \ell \in [0, +\infty)$ since $m(R)$ is monotone and bounded. We can let $R_2 \to +\infty$ in (1.3.5) to obtain $m(R) \geq m(R_1)$. Since R and R_1 are arbitrary, it follows that m is nondecreasing. Therefore u has to be of type 2. The proof is concluded. □

Finally, we show that when $N \geq 3$, and for a suitable range of q, it is always possible to reduce problem (1.1.1) to a radial setting. The case $N = 2$ is slightly different (see Remark 1.3.7).

Lemma 1.3.6. *Assume $N \geq 3$, $0 < q \leq 2$ and $f \in C([0, +\infty))$ is positive in $(0, +\infty)$. If there exists a positive supersolution u of (1.1.1), then there exists $R_1 > R_0$ and a positive, radially symmetric solution $z \in C^1(\mathbb{R}^N \setminus B_{R_1})$ of (1.1.1) in $\mathbb{R}^N \setminus B_{R_1}$* which is of the same type as u.

Proof. For $R_2 > R_1 > R_0$, consider the harmonic function Φ given by (1.3.2) in $A(R_1, R_2)$. To stress the dependence of Φ on R_2 we will temporarily denote it by Φ_{R_2}. As before, $u \geq \Phi_{R_2}$ in $A(R_1, R_2)$. Now we analyze the problem

$$\begin{cases} -\Delta z = f(z)|\nabla z|^q & \text{in } A(R_1, R_2) \\ \quad z = \Phi_{R_2} & \text{on } \partial A(R_1, R_2). \end{cases} \tag{1.3.6}$$

It is clear that u is a supersolution of (1.3.6) while Φ_{R_2} is a subsolution. Therefore, by the method of sub and supersolutions (see Theorem A.1 in the Appendix), there exists a minimal solution z_{R_2} of (1.3.6), which is radially symmetric, hence a classical solution.

Now we would like to pass to the limit as $R_2 \to +\infty$. Observe that the inequalities $0 < z_{R_2} \leq u$, give local bounds for the set $\{z_{R_2}\}_{R_2 > R_1}$. Since $q \leq 2$, we can also obtain local bounds for the gradient of the solutions using Theorem 3.1 in Section 4.3, Chapter IV of [49]. Then it is standart to get local $C^{1,\alpha}$ bounds (cf. [42]), so that by means of a diagonal procedure we obtain a sequence $R_{2,n} \to +\infty$ such that $z_{R_{2,n}} \to z$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \overline{B_{R_1}})$. In particular, we obtain that z is a radially symmetric weak solution of the equation $-\Delta z = f(z)|\nabla z|^q$ in $\mathbb{R}^N \setminus \overline{B_{R_1}}$. Since $\Phi_{R_2} \leq z_{R_2} \leq u$, letting $R_2 \to +\infty$ and employing the radial symmetry of z we have

$$0 < \frac{m(R_1) - \ell}{R_1^{2-N}}\, r^{2-N} + \ell \leq z(r) \leq m(r), \qquad r > R_1, \tag{1.3.7}$$

where $\ell = \lim_{R \to +\infty} m(R)$, which is finite by Lemma 1.3.5. Now reaching the desired conclusion is easy: if u is of type 1, then $\ell = 0$, so that $z(r) \to 0$ as well, therefore z is decreasing for large r and it is of type 1. If u is of type 2 then $\ell > 0$ and $z(r) \leq m(r) < \ell$ for $r > R_1$, and we obtain $\lim_{r \to +\infty} z(r) = \ell$ by (1.3.7). Hence z is of type 2. Finally, when u is of type 3 we have $m(R_1) > \ell$, so that again by (1.3.7) $z > \ell$ and $\lim_{r \to +\infty} z(r) = \ell$; thus z is of type 3. □

Remark 1.3.7. When $N = 2$ and there exists a positive supersolution of (1.1.1), it is equally possible to obtain a positive, radially symmetric solution z as in Lemma 1.3.6. Unfortunately, it does not seem possible to show with the same methods that z is of the same type as u.

1.4 Proofs in the case $N \geq 3$

The purpose of the present section is to prove Theorem 1.1. For the sake of clarity, we split the proof into a series of lemmas, dealing in turn with each type of supersolutions. Of course we will assume throughout the section that $N \geq 3$ and $f \in C([0,+\infty))$ is positive in $(0,+\infty)$.

We begin by considering the case $q \geq 2$.

Lemma 1.4.1. *Assume $q \geq 2$. Then there exist positive, radially symmetric solutions of* (1.1.1) *of types 1, 2 and 3.*

Proof. Since we will be looking for radially symmetric solutions, we assume $u(x) = z(r)$, $r = |x|$, so that we need to solve the equation

$$-z'' - \frac{N-1}{r} z' = f(z)|z'|^q \quad \text{for large } r. \tag{1.4.1}$$

With the change of variables $s = r^{2-N}/(N-2)$, $z(r) = w(s)$ (which involves the fundamental solution of the Laplacian), the equation (1.4.1) gets transformed into

$$-w'' = cs^\nu f(w)|w'|^q \quad \text{for small } s,$$

where $c = (N-2)^\nu$ and $\nu = -\theta = \frac{(q-2)(N-1)}{N-2} \geq 0$. Note that the previous change of variable transform the inverval $(R_0, +\infty)$ in $(0, s_0)$. In addition, the monotonicity of the functions z and w is reversed. For $\lambda > 0, \mu \in \mathbb{R}$ or $\lambda = 0$, $\mu > 0$, we consider the Cauchy problem:

$$\begin{cases} -w'' = cs^\nu f(w)|w'|^q & s > 0, \\ w(0) = \lambda \\ w'(0) = \mu. \end{cases}$$

Since f is continuous and $\nu \geq 0$, it follows by Cauchy-Peano's theorem that there exists at least a local solution w of this problem, defined in an interval $[0, s_0]$ for some small positive s_0. This solution is in addition positive if s_0 is small enough. Thus there exists a positive, radially symmetric solution u of (1.1.1) in the complement of a ball. Finally, observe that u is of type 1 when $\lambda = 0$, $\mu > 0$, of type 2 when $\lambda > 0$, $\mu < 0$ and of type 3 if $\lambda > 0$, $\mu > 0$. The proof is concluded. □

We will introduce next a technical lemma. It will be handy in order to prove the nonexistence of solutions for certain one-dimensional problems which will arise along this memoir, showing that integral conditions like (1.2.1) are necessary for the existence of solutions.

Lemma 1.4.2. *Assume $\eta > 1$, and let G be a positive, continuous function defined in $(0, \delta]$ and verifying*

$$\int_0^\delta G(t)dt = +\infty. \tag{1.4.2}$$

If there exist a positive, nondecreasing function $\beta \in C^1(0, s_0]$ with $\lim_{s\to 0^+} \beta(s) = 0$, $\beta(s_0) \leq \delta$ and a measurable function α with $\alpha(s) \geq \alpha_0 > 0$ almost everywhere in $[0, s_0]$, which verify

$$\alpha(s) \geq \int_s^{s_0} G(\beta(t))\alpha^\eta(t)\beta'(t)dt, \quad \textit{for almost all} \;\; s \in (0, s_0], \tag{1.4.3}$$

then there exists $s_1 \in (0, s_0)$ such that $\alpha \equiv +\infty$ almost everywhere in $[0, s_1]$.

Proof. Using (1.4.3) and the fact that $\alpha \geq \alpha_0$ in $(0, s_0]$, we obtain

$$\alpha(s) \geq \alpha_0^\eta \int_s^{s_0} G(\beta(t))\beta'(t)dt = \alpha_0^\eta \int_{\beta(s)}^{\beta(s_0)} G(\tau)d\tau. \tag{1.4.4}$$

Since $\lim_{s\to 0+} \beta(s) = 0$, we deduce from (1.4.2) that $\lim_{s\to 0+} \alpha(s) = +\infty$. Thus $\alpha(s) \geq 1$ for small s and this means that, diminishing s_0 if necessary we may always assume $\alpha_0 = 1$.

Define

$$H(z) = \int_z^{\beta(s_0)} G(\tau)d\tau, \quad z \in (0, \beta(s_0)].$$

Then, according to (1.4.4), we see that $\alpha(s) \geq H(\beta(s))$ in $(0, s_0]$. Using this in (1.4.3) we get

$$\begin{aligned}\alpha(s) &\geq \int_s^{s_0} G(\beta(t))H(\beta(t))^\eta \beta'(t)dt \\ &= \int_{\beta(s)}^{\beta(s_0)} G(\tau)H(\tau)^\eta d\tau \\ &= -\int_{\beta(s)}^{\beta(s_0)} H(\tau)^\eta H'(\tau)d\tau = \frac{H(\beta(s))^{\eta+1}}{\eta+1}\end{aligned}$$

in $(0, s_0]$. Observe that $H(\beta(s_0)) = 0$ has been used. We can iterate this procedure to obtain two sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ given by $a_1 = 1$, $a_k = \eta a_{k-1} + 1$, $b_1 = 1$, $b_k = b_{k-1}^\eta a_k$, such that

$$\alpha(s) \geq \frac{H(\beta(s))^{a_k}}{b_k} \quad \text{in } (0, s_0) \text{ for every } k. \tag{1.4.5}$$

It is not difficult to see that

$$a_k = \sum_{i=0}^{k-1} \eta^i = \frac{\eta^k - 1}{\eta - 1},$$

and we obtain

$$\eta^{k-1} \le a_k \le C_1 \eta^{k-1}, \tag{1.4.6}$$

for some positive constant $C_1 > 0$ which does not depend on k. It then follows that

$$b_k \le C_1 b_{k-1}^{\eta} \eta^{k-1},$$

and iterating this inequality we obtain

$$b_k \le C_1^{\sum_{i=0}^{k-2} \eta^i} \eta^{\sum_{i=0}^{k-2} \eta^i (k-(i+1))}. \tag{1.4.7}$$

We notice that the sum in the last exponent is an arithmetic-geometric sum, so that

$$\sum_{i=0}^{k-2} \eta^i (k-(i+1)) = -\frac{k-1}{\eta-1} + \frac{\eta^k - \eta}{(\eta-1)^2} = \frac{\eta^k - k\eta + k - 1}{(\eta-1)^2}. \tag{1.4.8}$$

It follows from (1.4.7) and (1.4.8) that

$$b_k \le C_2^{\eta^{k-2}} \tag{1.4.9}$$

for every k, for some constant $C_2 > 1$, independent of k.

Finally, we notice that, from (1.4.2) and the definition of H, it is possible to choose $s_1 \in (0, s_0)$ such that $H(\beta(s)) \ge 2C_2$ in $(0, s_1)$. Therefore, from (1.4.5), (1.4.6) and (1.4.9) we deduce

$$\alpha(s) \ge 2^{\eta^{k-1}} \quad \text{in } (0, s_1) \text{ for every } k.$$

Letting $k \to +\infty$, we see that $\alpha \equiv +\infty$ in $[0, s_1]$, as was to be shown. This concludes the proof of the lemma. □

We next turn to the subquadratic case $0 < q < 2$, and consider supersolutions of type 1. The proof of our next lemma is an adaptation of that of Theorem 6 in [3].

Lemma 1.4.3. *Assume $0 < q < 2$. Then there exist positive supersolutions of* (1.1.1) *of type 1 if and only if*

$$\int_0^{\delta} \frac{f(t)}{t^{\theta}}\, dt < +\infty \tag{1.4.10}$$

for some $\delta > 0$, where

$$\theta = \frac{(2-q)(N-1)}{N-2}.$$

Proof. Assume first that (1.4.10) does not hold, and there exists a positive supersolution of type 1 of (1.1.1). By Lemma 1.3.6 there exists $R_1 > R_0$ and a positive, radially symmetric solution z of (1.1.1) in $\mathbb{R}^N \setminus B_{R_1}$ of type 1. We make the change of variables

$s = r^{2-N}/(N-2)$, $w(s) = z(r)$ in the ordinary differential equation (1.4.1) satisfied by z. Then the function w is nondecreasing and verifies, for some $s_0 > 0$,

$$\begin{cases} -w'' = cs^{-\theta} f(w)(w')^q & s \in (0, s_0) \\ w(0) = 0, \end{cases} \tag{1.4.11}$$

where $c = (N-2)^{-\theta}$. We claim that $w' > 0$ in $(0, s_0)$. Indeed, if $w'(s_1) = 0$ for some $s_1 \in (0, s_0)$, then since $w'' \leq 0$, we would obtain $w'(s) = 0$ for $s \in (s_1, s_0)$, so that w would be constant in (s_1, s_0), against our assumptions. Thus $w' > 0$.

Also, the mean value theorem gives $w(s) = w'(\xi)s \geq w'(s)s$, where ξ is some point in the interval $(0, s)$. Hence

$$0 < w'(s) \leq \frac{w(s)}{s} \quad \text{in } (0, s_0). \tag{1.4.12}$$

The monotonicity of w' implies that $w(s) \geq C_0 s$ for some $C_0 > 0$ and every $s \in (0, s_0)$. We divide the equation in (1.4.11) by $(w')^{q-1}$ and integrate in (s, s_0) to arrive at

$$(w'(s))^{2-q} \geq (2-q)c \int_s^{s_0} \frac{f(w(t))}{t^\theta} w'(t) dt$$

for every $s \in (0, s_0)$. Hence, using (1.4.12):

$$(w'(s))^{2-q} \geq (2-q)c \int_s^{s_0} \frac{f(w(t))}{w(t)^\theta} \left(\frac{w(t)}{t}\right)^\theta w'(t) dt \geq (2-q)c \int_s^{s_0} \frac{f(w(t))}{w(t)^\theta} \left(w'(t)\right)^\theta w'(t) dt$$

for $s \in (0, s_0)$. So we can apply Lemma 1.4.2 with $\alpha(t) = (w'(t))^{2-q}$, $G(t) = (2-q)cf(t)/t^\theta$, $\eta = \frac{\theta}{2-q} = \frac{N-1}{N-2} > 1$ and $\beta(t) = w(t)$ and we get a contradiction. This shows that condition (1.4.10) is necessary for existence.

Next, let us prove the converse implication and assume that (1.4.10) holds. For $\lambda > 0$, consider the Cauchy problem

$$\begin{cases} -w'' = cs^{-\theta} f(w)(w')^q & s \in (0, s_0) \\ w(0) = 0 \\ w'(0) = \lambda. \end{cases} \tag{1.4.13}$$

Observe that any positive solution of (1.4.13) gives rise, with the change of variables $s = r^{2-N}/(N-2)$, $w(s) = v(r)$, to a positive, radially symmetric solution of (1.1.1) of type 1. Therefore our proof is reduced to show that there actually exists a positive solution of (1.4.13) when λ is small enough.

Denote $z_\lambda(s) = \lambda s$. In the Banach space $X = \{z \in C^1[0, s_0] : \ z(0) = 0\}$ endowed with the standard C^1 norm $|z|_{C^1} = \max\{|z|_\infty, |z'|_\infty\}$, consider the set $B = \{z \in X :$

$|z - z_\lambda|_{C^1} \le \frac{\lambda}{2}\}$, which is closed and convex, and for $z \in B$ define the integral operator

$$Tz(s) = \lambda s - c\int_0^s \int_0^t \tau^{-\theta} f(z(\tau))|z'(\tau)|^q \, d\tau \, dt, \quad s \in [0, s_0].$$

We claim that T is well-defined, maps B into B and is compact. To show the first two assertions, notice that for every $z \in B$, we have the inequalities $\frac{\lambda}{2}s \le z \le \frac{3\lambda}{2}s$, $\frac{\lambda}{2} \le z' \le \frac{3\lambda}{2}$ in $[0, s_0]$, so that

$$\begin{aligned} |(Tz)'(s) - \lambda| &= c\int_0^s \tau^{-\theta} f(z(\tau))|z'(\tau)|^q d\tau \\ &\le c\left(\frac{3\lambda}{2}\right)^{q+\theta-1} \int_0^s \frac{f(z(\tau))}{z(\tau)^\theta} z'(\tau) d\tau \\ &= c\left(\frac{3\lambda}{2}\right)^{q+\theta-1} \int_0^{z(s)} \frac{f(t)}{t^\theta} dt \\ &\le c\left(\frac{3\lambda}{2}\right)^{q+\theta-1} \int_0^{\frac{3\lambda}{2}s} \frac{f(t)}{t^\theta} dt \le \frac{\lambda}{2} \end{aligned} \tag{1.4.14}$$

in $[0, s_0]$, taking λ small enough (observe that $\theta + q - 1 > 1$ since $N \ge 3$ and $0 < q < 2$). Moreover, integrating in (1.4.14) we also see that

$$\begin{aligned} |Tz(s) - \lambda s| &= c\int_0^s \int_0^t \tau^{-\theta} f(z(\tau))|z'(\tau)|^q \, d\tau dt \le c\int_0^{s_0} \int_0^t \tau^{-\theta} f(z(\tau))|z'(\tau)|^q \, d\tau dt \\ &\le cs_0\left(\frac{3\lambda}{2}\right)^{q+\theta-1} \int_0^{\frac{3\lambda}{2}s_0} \frac{f(t)}{t^\theta} dt \le \frac{\lambda}{2}, \end{aligned}$$

provided λ is further diminished if necessary. Thus T is well defined and maps B into B.

To show that T is compact, let $\{z_n\}_{n=1}^\infty$ be an arbitrary sequence and denote $w_n = Tz_n$. It follows by (1.4.14) that $\{w_n'\}_{n=1}^\infty$ is uniformly bounded in $[0, s_0]$, so that $\{w_n\}$ is equicontinuous and uniformly bounded, and we may assume that $w_n \to w$ uniformly in $[0, s_0]$, for some $w \in C[0, s_0]$. We claim that $w \in C^1[0, s_0]$ and $w_n' \to w'$ uniformly in $[0, s_0]$.

Observe that $|w_n''(s)| = s^{-\theta} f(z_n(s))$ is uniformly bounded in compacts of $(0, s_0]$. Hence by means of Arzelá-Ascoli's theorem and a diagonal argument we may assume that also $w_n' \to \bar{w}$ uniformly in compacts of $(0, s_0]$ for some $\bar{w} \in C(0, s_0]$. We readily get that $w \in C^1(0, s_0]$ and $\bar{w} = w'$. But the convergence is indeed uniform in $[0, s_0]$ (defining $w'(0) = \lambda$). To prove it, take $\varepsilon > 0$. By (1.4.14) with z substituted by z_n:

$$|w_n'(s) - \lambda| \le c\left(\frac{3\lambda}{2}\right)^{q+\theta-1} \int_0^{\frac{3\lambda}{2}s} \frac{f(t)}{t^\theta} dt,$$

and the same is true for $w'(s)$ by passing to the limit. Hence

$$|w_n'(s) - w'(s)| \leq 2c \left(\frac{3\lambda}{2}\right)^{q+\theta-1} \int_0^{\frac{3\lambda}{2}\delta} \frac{f(t)}{t^\theta} dt \leq \varepsilon$$

provided $s \in [0, \delta]$ for $\delta < s_0$ small enough. Since $w_n' \to w'$ uniformly in $[\delta, s_0]$, we also have $|w_n'(s) - w'(s)| \leq \varepsilon$ for $s \in [\delta, s_0]$ if n is large enough. Thus $w_n' \to w'$ uniformly in $[0, s_0]$, that is, $w_n \to w$ in X and T is compact.

The continuity of T is shown by using a similar argument. Hence we can apply Schauder's fixed point theorem to obtain a fixed point $w \in B$ of T, which is a solution of (1.4.13) with $w > 0$ in $(0, s_0]$. As already observed, this concludes the proof of the lemma. □

Remark 1.4.4. It can be easily deduced from the above proof that condition (1.4.10) is necessary and sufficient for existence of solutions even if f is not defined at zero.

We conclude our sequence of lemmas dealing with supersolutions of type 2.

Lemma 1.4.5. *There exist positive supersolutions of* (1.1.1) *of type 2 if and only if* $q \geq 1$.

Proof. Assume first $q < 1$. Let u be a positive supersolution of (1.1.1) of type 2. By Lemma 1.3.6, there exists a positive, radial solution z of (1.1.1) of the same type defined in $\mathbb{R}^N \setminus B_{R_1}$ for some $R_1 > R_0$. Thus z verifies (1.4.1) in Lemma 1.4.1, while $z' \geq 0$ for large r and $\lim_{r\to+\infty} z(r) = \ell > 0$. We perform the same change of variables as before $s = r^{2-N}/(N-2)$ and $w(s) = z(r)$. Then, for some small positive s_0 and some $c > 0$:

$$\begin{cases} -w'' = cs^{-\theta} f(w)|w'|^q & s \in (0, s_0), \\ w(0) = \ell, \end{cases}$$

where now $w' \leq 0$. Arguing as in Lemma 1.4.3 we can show that $w' < 0$ in $(0, s_0)$. Letting $v = \ell - w$, we have that

$$\begin{cases} v'' = cs^{-\theta} f(\ell - v)(v')^q & s \in (0, s_0), \\ v(0) = 0. \end{cases}$$

Observe also that $v' > 0$. Then, since f is strictly positive in a neighborhood of ℓ, we obtain that $cf(\ell - v) \geq C > 0$ for small s and for some constant D. After dividing the previous equation by $(v')^q$ and integrating between s and s_0 for $s \in (0, s_0)$:

$$0 \leq \frac{(v')^{1-q}}{1-q} \leq -C\frac{s^{1-\theta}}{\theta - 1} + D.$$

We arrive at a contradiction by letting s go to zero, since $\theta > 1$ in this case.

When $q \geq 1$, it is easy to construct a positive supersolution of (1.1.1) with radial symmetry. Indeed, we can look for u in the form $u(x) = \ell - e^{-\alpha|x|}$, where $\alpha > 0$ is chosen suitably and $\ell > 0$ is arbitrary. It is not hard to check that u will be a supersolution of (1.1.1) provided that

$$\alpha - \frac{N-1}{|x|} \geq f(\ell - e^{-\alpha|x|})\alpha^{q-1}e^{(1-q)\alpha|x|}$$

for large $|x|$. Denoting $M = \sup_{t\in[0,\ell]} f(t)$, the previous inequality is a consequence of

$$\alpha - \frac{N-1}{R_0} \geq M\alpha^{q-1}e^{(1-q)\alpha R_0}.$$

This inequality can be easily achieved for fixed R_0 by choosing α large enough. The proof is concluded. □

We finally turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. First of all observe that, by Lemma 1.3.5, there do not exist positive supersolutions of type 4 when $N \geq 3$. Also, part a) is a direct consequence of Lemma 1.4.1 when $q \geq 2$.

On the other hand, all the assertions dealing with supersolutions of types 1 and 2 are already proved in Lemmas 1.4.3 and 1.4.5, respectively. Observe that when $\frac{N}{N-1} < q < 2$ we have $0 < \theta < 1$, in addition f is defined at 0 by our hypotheses, hence (1.2.1) always holds. Thus, only the statements regarding supersolutions of type 3 need to be shown. But observe that u is a supersolution of type 3 and $\ell = \lim_{R\to+\infty} m(R) \in (0, +\infty)$, if and only if $v = u - \ell$ is a positive supersolution of type 1 for the problem

$$-\Delta v = g(v)|\nabla v|^q \quad \text{in } \mathbb{R}^N \setminus B_{R_1}$$

for some $R_1 > R_0$, where $g(t) = f(t - \ell)$. By Lemma 1.4.3, such supersolutions exist if and only if

$$\int_0^\delta \frac{g(t)}{t^\theta}\, dt < +\infty$$

for some small positive δ. Since $g(0) = f(\ell) > 0$, the integral is convergent when $q > \frac{N}{N-1}$, independent of f. This shows the assertions on positive supersolutions of type 3 in parts a), b) and c). The proof of Theorem 1.1 is concluded. □

1.5 Positive supersolutions in two dimensions

In the last section of this chapter we deal with positive supersolutions of (1.1.1) in the planar case $N = 2$. It is worthy of mention that the reduction to a radial setting as in Lemma 1.3.6 is not possible, so a slightly different approach is needed. In most proofs we will actually reduce the problem to a radial one, but in a finite annulus $A(R_1, R_2)$. This implies that some further estimates are needed before we can let $R_2 \to +\infty$.

By Lemma 1.3.5 we know that only supersolutions of types 2 and 4 are possible. Let us begin with the former ones.

Lemma 1.5.1. *Problem* (1.1.1) *admits a positive supersolution of type 2 if and only if* $q \geq 1$.

Proof. The proof is similar to that of Lemma 1.4.5 in Section 1.4. Indeed, the construction of positive supersolutions of (1.1.1) of type 2 when $q \geq 1$ is exactly the same as in that lemma, just setting $N = 2$.

Thus we assume in what follows that $q < 1$ and show that no positive supersolutions of (1.1.1) of type 2 exist. Assume for a contradiction that there is one such u. Choose $R_1 > R_0$ and for $R_2 > R_1$, consider the problem

$$\begin{cases} -\Delta v = f(v)|\nabla v|^q & \text{in } A(R_1, R_2) \\ \quad v = \Psi & \text{on } \partial A(R_1, R_2), \end{cases} \tag{1.5.1}$$

where Ψ is given by (1.3.4) in Lemma 1.3.5 (recall that Ψ is harmonic in $A(R_1, R_2)$ and $\Psi(R_i) = m(R_i)$ for $i = 1, 2$). By the same arguments as in the proof of Lemma 1.3.6, there exists a radially symmetric solution $z \in C^1[R_1, R_2]$ of this problem verifying $\Psi \leq z \leq u$ in $A(R_1, R_2)$ (the solution z depends of course on R_2, but we are not making this dependence explicit for brevity).

By the maximum principle, and since m is an increasing function, we have $z \geq m(R_1)$. Moreover, since z is radially symmetric,

$$z(r) \leq m(r) \leq m(R_2) \leq \lim_{R\to+\infty} m(R) =: \ell$$

if $R_1 \leq r \leq R_2$, where $\ell \in (0, +\infty)$. It also follows that z attains its maximum at R_2.

Let $\gamma = \min_{t\in[m(R_1),\ell]} f(t) > 0$. We deduce that z verifies

$$-z'' - \frac{1}{r}z' \geq \gamma|z'|^q \quad \text{if } R_1 < r < R_2. \tag{1.5.2}$$

With the change of variables $s = \log r$, $w(s) = z(r)$, we reduce (1.5.2) to the problem

$$\begin{cases} -w'' \geq \gamma e^{(2-q)s}|w'|^q & s \in (s_1, s_2) \\ \; w(s_1) = m(R_1) \\ \; w(s_2) = m(R_2), \end{cases}$$

where $s_i = \log R_i$, $i = 1, 2$.

Let us see that $w' > 0$ in (s_1, s_2). Indeed, if we had $w'(\tilde{s}) = 0$ for some $\tilde{s} \in (s_1, s_2)$, using the monotonicity of w' we would arrive at $w'(s) \leq 0$ for every $s \in (\tilde{s}, s_2)$. Since w attains its maximum at s_2, this would imply that $w(s) = m(R_2)$ if $s \in (\tilde{s}, s_2)$. This in turn yields $v(r) = m(R_2)$ for $r \in (e^{\tilde{s}}, R_2)$, which leads to $m(r) = m(R_2)$ if $r \in (e^{\tilde{s}}, R_2)$, which is not possible because $m(r)$ is strictly increasing according to Lemma 1.3.3.

Thus $w' > 0$. Dividing by $(w')^q$ and integrating between s_1 and s_2, we have

$$\begin{aligned} w'(s_1)^{1-q} &\geq w'(s_1)^{1-q} - w'(s_2)^{1-q} \\ &\geq \gamma \frac{1-q}{2-q} (e^{(2-q)s_2} - e^{(2-q)s_1}) \\ &= \gamma \frac{1-q}{2-q} (R_2^{2-q} - R_1^{2-q}). \end{aligned} \tag{1.5.3}$$

Next, we claim that

$$w'(s_1) \leq C \tag{1.5.4}$$

for a positive constant C independent of R_2. Indeed, let $M = \sup_{t \in [m(R_1), m(R_1+1)]} f(t)$. Then

$$-z'' - \frac{1}{r} z' \leq M|z'|^q \text{ if } R_1 < r < R_1 + 1.$$

With the change of variables $s = \log r$, $w(s) = z(r)$ we have, if $\delta > 0$ is small:

$$-w'' \leq M e^{(2-q)(s_1+\delta)} |w'|^q = K|w'|^q, \qquad s \in (s_1, s_1 + \delta).$$

Dividing by $(w')^{q-1}$ and integrating between s_1 and s we obtain:

$$\begin{aligned} w'(s)^{2-q} &\geq w'(s_1)^{2-q} - (2-q)K(w(s) - w(s_1)) \\ &\geq w'(s_1)^{2-q} - (2-q)Km(R_1 + 1). \end{aligned}$$

If we assume the right-hand side is positive (otherwise there is nothing to prove), we can raise to the power $\frac{1}{2-q}$ and integrate in $(s_1, s_1 + \delta)$ to have

$$m(R_1 + 1) \geq (w'(s_1)^{2-q} - (2-q)Km(R_1 + \delta))^{\frac{1}{2-q}} \delta.$$

This shows (1.5.4).

Coming back to (1.5.3), we have:

$$\gamma \frac{1-q}{2-q} (R_2^{2-q} - R_1^{2-q}) \leq C^{1-q}.$$

Letting $R_2 \to +\infty$, we obtain a contradiction, which shows that no positive supersolutions of (1.1.1) of type 2 exist. The proof is concluded. □

The analysis of supersolutions of type 4 is slightly different since necessary and sufficient conditions for their existence are only needed when $q \in (0, 2)$ (in light of Lemma 1.4.1). Let us deal with this case next.

Lemma 1.5.2. *Assume* $0 < q < 2$. *Then there exist positive supersolutions of* (1.1.1) *of type 4 if and only if*

$$\int_M^\infty e^{at} f(t)\, dt < +\infty \tag{1.5.5}$$

for some $M, a > 0$.

Proof. Assume (1.5.5) does not hold and there exists a positive supersolution u of (1.1.1) of type 4. We argue similarly as in the proof of Lemma 1.5.1. Let v be the solution of problem (1.5.1) obtained there. Performing in the radial version of (1.5.1) the same change of variables $s = \log r$, we arrive at

$$\begin{cases} -w'' = e^{(2-q)s} f(w)|w'|^q & s \in (s_1, s_2) \\ w(s_1) = m(R_1) \\ w(s_2) = m(R_2), \end{cases}$$

where $s_i = \log R_i$, $i = 1, 2$. It is equally proved that $w' > 0$ in (s_1, s_2) and that (1.5.4) holds. Hence we may divide the equation by $(w')^{q-1}$ and integrate between s_1 and s_2 to get

$$\frac{w'(s_1)^{2-q}}{2-q} \geq \int_{s_1}^{s_2} e^{(2-q)t} f(w(t))w'(t)\, dt.$$

With the aid of (1.5.4), this inequality gives

$$\int_{s_1}^{s_2} e^{(2-q)t} f(w(t))w'(t)\, dt \leq \frac{C^{2-q}}{2-q}. \tag{1.5.6}$$

Next, we claim that $w(s) \leq Ks$ for $s \in (s_1, s_2)$, where K does not depend on R_2. Indeed, since $w'' \leq 0$, we deduce using (1.5.4):

$$\begin{aligned} w(s) &\leq m(R_1) + w'(s_1)(s - s_1) \leq m(R_1) + Cs \\ &\leq \left(\frac{m(R_1)}{\log R_1} + C \right) s = Ks, \end{aligned}$$

where K does not depend on R_2. This shows the claim.

Coming back to (1.5.6), since $s \geq \frac{1}{K} w(s)$, we see that

$$\int_{m(R_1)}^{m(R_2)} e^{\frac{2-q}{K}\tau} f(\tau)\, d\tau = \int_{s_1}^{s_2} e^{\frac{2-q}{K} w(t)} f(w(t))w'(t)\, dt \leq \frac{C^{2-q}}{2-q}.$$

Letting $R_2 \to +\infty$ we obtain that (1.5.5) holds with $M = m(R_1)$ and $a = (2-q)/(1-q)$, against the assumption. Thus no positive supersolutions of (1.1.1) of type 4 can exist.

To conclude the proof, let us assume that (1.5.5) holds. Our intention is to show that the problem

$$\begin{cases} -w'' = e^{(2-q)s} f(w)|w'|^q & \text{in } (s_0, +\infty) \\ \lim\limits_{s\to+\infty} w(s) = +\infty \end{cases} \tag{1.5.7}$$

admits a positive solution. The change of variables $s = \log r$, $w(s) = v(r)$ will then provide with a positive, radially symmetric solution of (1.1.1) which is of type 4.

For this sake, we use again Schauder's fixed point theorem, but with some differences with respect to the proof of Lemma 1.4.3. Consider the Banach space $\widetilde{X} = \{z \in C^1[s_0, +\infty) : \ \|z\| < +\infty\}$, where

$$\|z\| = \max\left\{ \sup_{[s_0,+\infty)} \frac{|z(s)|}{s}, \sup_{[s_0,+\infty)} |z'(s)| \right\},$$

and the set $\widetilde{B} = \{z \in \widetilde{X} : \ \|z - z_\lambda\| \le \frac{\lambda}{2}\}$, where $z_\lambda(s) = \lambda s$ and $\lambda > 0$ is fixed. Define the operator

$$Tz(s) = \lambda s - \int_{s_0}^{s} \int_{s_0}^{t} e^{(2-q)\tau} f(z(\tau))|z'(\tau)|^q d\tau dt, \quad s \in [s_0, +\infty).$$

Let us prove that T maps $\widetilde{B}$ into $\widetilde{B}$ if λ is chosen large enough. To begin with, assume $\lambda \ge 2(2-q)/a$ where a is as in (1.5.5). Taking into account that $z(s) \ge \frac{\lambda}{2} s$ and $\frac{\lambda}{2} \le z'(s) \le \frac{3\lambda}{2}$ in $[s_0, +\infty)$ for every $z \in \widetilde{B}$:

$$\begin{aligned} |(Tz)'(s) - \lambda| &= \int_{s_0}^{s} e^{(2-q)\tau} f(z(\tau)) z'(\tau)^q d\tau \\ &\le \left(\frac{3\lambda}{2}\right)^{q-1} \int_{s_0}^{s} e^{az(\tau)} f(z(\tau)) z'(\tau) d\tau \\ &= \left(\frac{3\lambda}{2}\right)^{q-1} \int_{z(s_0)}^{z(s)} e^{at} f(t) dt \\ &\le \left(\frac{3\lambda}{2}\right)^{q-1} \int_{\frac{\lambda}{2}s_0}^{\infty} e^{at} f(t) dt \le \frac{\lambda}{2}, \end{aligned} \tag{1.5.8}$$

if λ is chosen large enough, since $0 < q < 2$ and $z(s) \ge \lambda s/2 \ge (2-q)s/a$. On the other hand,

$$\begin{aligned} \left|\frac{Tz(s) - \lambda s}{s}\right| &= \left|\frac{1}{s} \int_{s_0}^{s} \int_{s_0}^{t} e^{(2-q)\tau} f(z(\tau))|z'(\tau)|^q d\tau dt\right| \\ &\le \frac{1}{s} \int_{s_0}^{s} \frac{\lambda}{2} dt \le \frac{\lambda}{2} \frac{s - s_0}{s} \le \frac{\lambda}{2}, \end{aligned}$$

for every $s > s_0$, hence T is well-defined and $T(\widetilde{B}) \subset \widetilde{B}$.

Let us finally show that T is compact (as before, the continuity of T is shown by arguing in a similar way). Take an arbitrary sequence $\{z_n\}_{n=1}^{\infty} \subset \widetilde{B}$ and let $w_n = Tz_n$. By Arzelá-Ascoli's theorem and a diagonal argument, since the sequences $\{Tz_n\}$, $\{(Tz_n)'\}$ and $\{(Tz_n)''\}$ are locally uniformly bounded, we may assume $w_n \to w$, $w_n' \to w'$ uniformly on compact sets for some function $w \in C^1[s_0, +\infty)$.

Let us show that the convergence $w_n' \to w'$ is actually uniform in $[s_0, +\infty)$. Indeed, observe that, if we fix $s_1 > s_0$, take $s > s_1$, and argue as in (1.5.8) we arrive at

$$|w_n'(s) - w_n'(s_1)| = \int_{s_1}^{s} e^{(2-q)\tau} f(z_n(\tau))|z_n'(\tau)|^q d\tau \leq \left(\frac{3\lambda}{2}\right)^{q-1} \int_{\frac{\lambda}{2}s_1}^{\infty} e^{at} f(t)dt.$$

A similar equality holds for w', by passing to the limit. Hence

$$|w_n'(s) - w'(s)| \leq |w_n'(s_1) - w'(s_1)| + 2\left(\frac{3\lambda}{2}\right)^{q-1} \int_{\frac{\lambda}{2}s_1}^{\infty} e^{at} f(t)dt$$

for every $s > s_1$. Next take $\varepsilon > 0$. Choosing s_1 large enough we have the last term less than $\frac{\varepsilon}{2}$. Taking n large enough we also have $|w_n'(s_1) - w'(s_1)| \leq \frac{\varepsilon}{2}$, hence $|w_n'(s) - w'(s)| \leq \varepsilon$ if $s > s_1$. Since this inequality also holds in $[s_0, s_1]$ for large enough n, we obtain that $w_n' \to w'$ uniformly in $[s_0, +\infty)$. Hence

$$\frac{|w_n(s) - w(s)|}{s} \leq \frac{1}{s} \int_{s_0}^{s} |w_n'(t) - w'(t)|dt \to 0$$

uniformly in $[s_0, +\infty)$, as $n \to +\infty$. This shows that T is compact in $\widetilde{B}$.

Therefore we can apply Schauder's fixed point theorem to obtain that T has a fixed point w in $\widetilde{B}$, which is a solution of (1.5.7) verifying $\frac{\lambda}{2}s \leq w(s) \leq \frac{3\lambda}{2}s$. Observe that this implies that w is positive and $\lim_{s\to+\infty} w(s) = +\infty$. As remarked above, this function provides with a positive, radially symmetric solution of (1.1.1) of type 4. The proof is concluded. □

We finally consider the only left case where $q \geq 2$. It is worth mentioning that, although positive supersolutions can always be constructed, they can be of different types, depending on the behavior at infinity of their derivatives. However, we are not exploring this distinction further.

Lemma 1.5.3. *Assume $q \geq 2$. Then there always exist positive supersolutions of* (1.1.1) *of type 4.*

Proof. As in the proof of Lemma 1.5.2, it suffices to obtain a positive supersolution of the problem

$$\begin{cases} -w'' = e^{-(q-2)s} f(w)|w'|^q & \text{in } (s_0, +\infty) \\ \lim\limits_{s\to+\infty} w(s) = +\infty. \end{cases} \tag{1.5.9}$$

for some $s_0 > 0$. We consider first the case $q = 2$. Observe that since $f(t) > 0$, we always have

$$\int_\lambda^\infty e^{F(t)}\, dt = +\infty \tag{1.5.10}$$

for every $\lambda > 0$, where $F(t) = \int_0^t f(\tau)d\tau$. In this case, equation (1.5.9) can be easily integrated. Indeed, a positive, increasing solution of problem (1.5.9) is given implicitly by

$$\int_\lambda^{w(s)} e^{F(t)} dt = s - s_0,$$

for $s > s_0$, where λ is positive and arbitrary. This equation can be solved because of condition (1.5.10), which also gives $\lim_{s\to+\infty} w(s) = +\infty$. Observe in passing that $w'(s) = e^{-F(w(s))}$, so that when $\lim_{t\to+\infty} F(t) = +\infty$ we also obtain $\lim_{s\to+\infty} w'(s) = 0$.

Now we turn to the case $q > 2$. Assume first that f verifies a condition similar to (1.5.5): there exist $a, M > 0$ such that

$$\int_M^\infty e^{-at} f(t)\, dt < +\infty. \tag{1.5.11}$$

In the space $\widetilde{X}$ introduced in the proof of Lemma 1.5.2, consider again the set $\widetilde{B} = \{z \in \widetilde{X} : \ \|z - z_\lambda\| \le \frac{\lambda}{2}\}$, where $z_\lambda(s) = \lambda s$ and $\lambda > 0$ is fixed. On $\widetilde{B}$ define the operator

$$Tz(s) = \lambda s - \int_{s_0}^s \int_{s_0}^t e^{-(q-2)\tau} f(z(\tau))|z'(\tau)|^q d\tau dt, \quad s \in [s_0, +\infty).$$

We can argue in a completely similar way as in (1.5.8), except that now the exponent in the exponential is negative, to obtain that, for $\lambda \le 2/(3a)$:

$$\begin{aligned} |(Tz)'(s) - \lambda| &= \int_{s_0}^s e^{-(q-2)\tau} f(z(\tau)) z'(\tau)^q d\tau \\ &\le \left(\frac{3\lambda}{2}\right)^{q-1} \int_{s_0}^s e^{-az(\tau)} f(z(\tau)) z'(\tau) d\tau \\ &= \left(\frac{3\lambda}{2}\right)^{q-1} \int_{z(s_0)}^{z(s)} e^{-at} f(t) dt \\ &\le \left(\frac{3\lambda}{2}\right)^{q-1} \int_{\frac{\lambda}{2}s_0}^{\infty} e^{-at} f(t) dt. \end{aligned}$$

This difference can be made less than or equal to $\frac{\lambda}{2}$ provided λ is chosen small enough, since $q > 2$. The rest of the proof in this case is essentially the same as that of Lemma 1.5.2 and therefore will be omitted.

To conclude the proof, consider the case where f does not satisfy (1.5.11). In particular, f verifies $\lim_{t\to+\infty} F(t) = +\infty$ where F is again the primitive of f. Now let w be the solution of (1.5.9) with $q = 2$, obtained at the beginning of the proof. Since $\lim_{s\to+\infty} w'(s) = 0$, it follows that

$$-w'' = f(w)(w')^2 \geq e^{-(q-2)s} f(w)(w')^q, \quad s \geq s_0,$$

if s_0 is large enough, so that w is a positive supersolution of (1.5.9). The proof of Lemma 1.5.3 is concluded. $\square$

Proof of Theorem 1.2. It is immediate with the use of Lemmas 1.5.1, 1.5.2 and 1.5.3. $\square$

Chapter 2

Scalar equation with gradient terms: Additive term

In this chapter we are dealing with the nonlinear elliptic problem

$$-\Delta u + |\nabla u|^q = \lambda f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{2.1.1}$$

where $N \geq 2$, $q > 1$, f is continuous, nondecreasing function and $\lambda > 0$ is a parameter.

Positive supersolutions of (2.1.1) can be classified much as in Chapter 1, with the only difference that only types 1 and 4 are now possible.

It can be shown that the function $m(R)$ is ultimately monotone, therefore the limit $l = \lim_{R \to +\infty} m(R)$ always exists. However, only $l = 0$ and $l = +\infty$ are admissible in the present situation (see Lemma 2 in [2]). That is, positive supersolutions of types 2 and 3 are not possible as in problem (1.1.1) in Chapter 1. This is related to the fact that all constants are solutions to the latter problem, while this is not true for (2.1.1)

Due to our approach in this chapter, though, we will not need such a fine classification of supersolutions. It will be enough for our purposes to distinguish between supersolutions which blow up at infinity, that is, verifying

$$\lim_{|x| \to \infty} u(x) = +\infty \tag{2.1.2}$$

uniformly (type 4), and those which do not.

In the present chapter, we will be dealing with weak supersolutions $u \in C^1(\mathbb{R}^N \setminus B_{R_0})$, that is, functions verifying

$$\int_{\mathbb{R}^N \setminus B_{R_0}} \nabla u \nabla \phi + \int_{\mathbb{R}^N \setminus B_{R_0}} |\nabla u|^q \phi \geq \lambda \int_{\mathbb{R}^N \setminus B_{R_0}} f(u)\phi$$

for every nonnegative $\phi \in C_0^\infty(\mathbb{R}^N \setminus B_{R_0})$.[1]

[1]The results in this chapter are contained in [1].

For the sake of clarity we focus during most part of the chapter on supersolutions which do not blow up, and consider supersolutions verifying (2.1.2) in Section 2.6.

2.1 Main results

Let us now turn to the explicit statements of our results. For the first of them, that in which $1 < q < \frac{N}{N-1}$, the following condition on f will be relevant:

There exists $\beta \in \mathbb{R}$ such that the following limit exists (possibly being infinity):

$$l = \lim_{t \to 0+} \frac{f(t)}{t^{\frac{q}{2-q}}(-\log t)^{\beta}}. \tag{2.1.3}$$

In spite of requiring this condition, it is interesting to remark that with the same methods nonexistence results can be obtained for slightly more general functions f (see the precise statement in Theorem 2.4 in Section 2.4).

Theorem 2.1. *Assume $N \geq 2$ and $1 < q < \frac{N}{N-1}$. Let $f : (0, +\infty) \to \mathbb{R}$ be nondecreasing, continuous and positive, and suppose that condition* (2.1.3) *holds. Then:*

a) *If $0 < l < +\infty$, then*

   i) *If $\beta > 0$, then there are no positive supersolutions of* (2.1.1) *for any $\lambda > 0$.*

   ii) *For $\beta = 0$, there exists $\lambda^* > 0$ such that, for $\lambda < \lambda^*$ problem* (2.1.1) *admits positive supersolutions, while for $\lambda > \lambda^*$ no supersolutions exist.*

   iii) *When $\beta < 0$, there exists a positive supersolution of* (2.1.1) *for every $\lambda > 0$.*

b) *If $l = +\infty$ and $\beta \geq 0$, there are no positive supersolutions of* (2.1.1) *for any $\lambda > 0$.*

c) *If $l = 0$ and $\beta \leq 0$, there exists a positive supersolution of* (2.1.1) *for every $\lambda > 0$.*

Remark 2.1.1. Observe that the value of β in (2.1.3) (see also (2.1.4) below) is not unique, since we allow the limit to be zero or infinity. In particular, for a given nonlinearity we may apply Theorem 2.1 a) i) or b) (resp. a) iii) and c)) above using different choices for β.

When $q \geq \frac{N}{N-1}$, the exponent $\frac{q}{2-q}$ does not play a role any more, and it has to be replaced by the "natural" exponent for the Laplacian, namely $\frac{N}{N-2}$. Thus instead of requiring condition (2.1.3) we will impose:

There exists $\beta \in \mathbb{R}$ such that the following limit exists (possibly being infinity):

$$l = \lim_{t\to 0+} \frac{f(t)}{t^{\frac{N}{N-2}}(-\log t)^{\beta}}. \tag{2.1.4}$$

Let us state first what could be termed as "critical" case, $q = \frac{N}{N-1}$. It is to be remarked that the threshold value of β for nonexistence in Theorem 2.1 shifts from $\beta = 0$ to the unexpected $\beta = \frac{N}{N-2}$ in the power of the logarithm. Let us also mention that the case $N = 2$ and $q \geq 2$ is slightly different, since all solutions turn out to verify (2.1.2) (see the proof of Theorem 2.5 in Section 2.6). Therefore, the contents of Theorems 2.2 and 2.3 below are only meaningful when $N \geq 3$, since they only deal with supersolutions that do not blow up at infinity.

Theorem 2.2. *Assume $N \geq 3$ and $q = \frac{N}{N-1}$. Let $f : (0, +\infty) \to \mathbb{R}$ be nondecreasing, continuous and positive, and suppose that condition* (2.1.4) *holds. Then:*

a) *If $0 < l < +\infty$, then*

   i) *If $\beta > \frac{N}{N-2}$, then there are no positive supersolutions of* (2.1.1) *for any $\lambda > 0$.*

   ii) *For $\beta = \frac{N}{N-2}$, there exists $\lambda^* > 0$ such that, for $\lambda < \lambda^*$ problem* (2.1.1) *admits positive supersolutions, while for $\lambda > \lambda^*$ no supersolutions exist.*

   iii) *When $\beta < \frac{N}{N-2}$, there exists a positive supersolution of* (2.1.1) *for every $\lambda > 0$.*

b) *If $l = +\infty$ and $\beta \geq \frac{N}{N-2}$, there are no positive supersolutions of* (2.1.1) *for any $\lambda > 0$.*

c) *If $l = 0$ and $\beta \leq \frac{N}{N-2}$, there exists a positive supersolution of* (2.1.1) *for every $\lambda > 0$.*

When q is larger than $\frac{N}{N-1}$ the relevant hypothesis on f is again (2.1.4). In this case, the threshold value of β for nonexistence lowers down dramatically to -1. It is to be noted that the parameter λ in (2.1.1) becomes of no importance in this regime.

Theorem 2.3. *Assume $N \geq 3$ and $q > \frac{N}{N-1}$. Let $f : (0, +\infty) \to \mathbb{R}$ be nondecreasing, continuous and positive, and suppose that condition* (2.1.4) *holds. Then:*

a) *If $0 < l < +\infty$, then*

   i) *If $\beta \geq -1$, there are no positive supersolutions of* (2.1.1) *for any $\lambda > 0$.*

ii) *When* $\beta < -1$*, there exists a positive supersolution of* (2.1.1) *for every* $\lambda > 0$.

b) *If* $l = +\infty$ *and* $\beta \geq -1$*, there are no positive supersolutions of* (2.1.1) *for any* $\lambda > 0$.

c) *If* $l = 0$ *and* $\beta < -1$*, there exists a positive supersolution of* (2.1.1) *for every* $\lambda > 0$.

Let us mention explicitly that the condition of the existence of the limit can be slightly relaxed for the nonexistence results above: the limit can be replaced by the inferior limit.

In this regard, observe that the condition $l > 0$ with $\beta \geq -1$ is a generalization of (I.8) in the Introduction (cf. [2]).

After presenting the results about the nonexistence of positive supersolutions of (2.1.1), let us compare the regions of nonexistence obtained for the equation (2.1.1) with the region of nonexistence for the equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{2.1.5}$$

when, in both cases, $f(t) = t^p$. Obviously if u is a positive supersolution of (2.1.5), then it is also a supersolution of (2.1.1), so the expected critical exponent for problem (2.1.1) must be less than the critical exponent for problem (2.1.5). Recall that the optimal condition for nonexistence for (2.1.5) is $p \leq \frac{N}{N-2}$ (see for instance [3]).

First, when $1 < q \leq \frac{N}{N-1}$, Theorem 2.1 implies nonexistence for $p \leq \frac{q}{2-q}$ (depending on λ when $p = \frac{q}{2-q}$). So, as we said before, the presence of the gradient term naturally causes a decrease in the nonexistence exponent. However, the situation is different when $q > \frac{N}{N-1}$. According to Theorem 2.3, if we consider $p \leq \frac{N}{N-2}$, then existence of positive supersolutions is not possible, which means that the result of nonexistence for equation (2.1.1) is exactly the same as the result of nonexistence for (2.1.5) (remember we are dealing with supersolutions which do not blow up at infinity). It follows that the presence of the gradient is irrelevant when $q > \frac{N}{N-1}$. Actually, observe that the nonexistence result in Theorem 2.3 does not depend on q.

It is also worthy of mention that whenever a positive supersolution of problem (2.1.1) can be constructed in an exterior domain in all cases above, this supersolution can be extended to be defined in the whole $\mathbb{R}^N$ (the key is that all the constructed supersolutions go to zero at infinity). This is easily done as follows: assume u is a supersolution in $\mathbb{R}^N \setminus B_{R_0}$ going to zero at infinity. Choose any unit vector e and $R > 2R_0$. Set $\delta = \min_{B_{2R_0} \setminus B_{R_0}} u > 0$. If R is large enough, then $u(x \pm 2Re) \leq \delta$ in B_{2R_0}, this implying that $u(x) \leq u(x - 2Re)$ in $B_{2R_0}(2Re)$. Thus $v(x) = \min\{u(x), u(x - 2Re)\}$ is a positive (weak) supersolution defined in the whole $\mathbb{R}^N$. The only disadvantage is that the supersolution so constructed does not belong to $C^1(\mathbb{R}^N)$, only to $H^1_{\text{loc}}(\mathbb{R}^N)$.

2.2 Reduction to the radial setting

In this section we perform the reduction of equation (2.1.1) to its radially symmetric version

$$-z'' - \frac{N-1}{r} z' + |z'|^q = \lambda f(z) \qquad \text{in } (R_0, +\infty).$$

This reduction is carried out in two stages. First, we show that the existence of a positive supersolution of (2.1.1) always implies the existence of a radially symmetric positive supersolution. Second, if u does not blow up, then it is possible to show the existence of a radially symmetric positive *solution* which does not blow up either. Since our main tool is the method of sub and supersolutions, which is not completely straightforward for problem (2.1.1) when $q > 2$, some extra care is needed in this case.

Lemma 2.2.1. *Assume $N \geq 2$, $q > 1$ and let $f : (0, +\infty) \to \mathbb{R}$ be positive, nondecreasing and continuous. Suppose there exists a positive supersolution u of problem* (2.1.1)*. Then there exists a positive radial supersolution $z \in C^2(\mathbb{R}^N \setminus B_{R_0})$ of* (2.1.1)*. In addition, if u does not blow up at infinity, then z is bounded, while if u blows up at infinity, z is bounded from below.*

Proof. For $R > R_0$, let $m(R) = \inf_{|x|=R} u(x)$. Choose $R_1 > R_0$ and consider the problem

$$\begin{cases} -\Delta v + |\nabla v|^q = \lambda f(m(|x|)) & \text{in } A(R_0, R_1) \\ \quad v = 0 & \text{on } \partial A(R_0, R_1), \end{cases}$$

where $A(R_0, R_1) := \{x \in \mathbb{R}^N : \ R_0 < |x| < R_1\}$. According to Theorem III.1 in [54], there exists a unique (positive) solution $v_{R_1} \in W^{2,\theta}(A(R_0, R_1))$ of this problem with $\theta > N$, which as a consequence is radially symmetric. Setting $z_{R_1} = \min\{m(R_0), m(R_1)\} + v_{R_1}$, we obtain a radially symmetric solution of

$$\begin{cases} -\Delta z + |\nabla z|^q = \lambda f(m(|x|)) & \text{in } A(R_0, R_1) \\ \quad z = \min\{m(R_0), m(R_1)\} & \text{on } \partial A(R_0, R_1). \end{cases}$$

Now the monotonicity of f implies that $f(u) \geq f(m(|x|))$, so that u is a supersolution of this problem and by comparison $u \geq z_{R_1}$ in $A(R_0, R_1)$, since $u \geq z_{R_1}$ on $\partial A(R_0, R_1)$. In particular, the set $\{z_{R_1}\}_{R_1 > R_0}$ is bounded. Using part 2 in Theorem A.1 in [51], we obtain local bounds for $\{|\nabla z_{R_1}|\}_{R_1 > R_0}$ in L^θ for every $\theta > 1$. This yields that $\{\Delta z_{R_1}\}_{R_1 > R_0}$ is locally bounded in L^θ for every $\theta > 1$, and hence by classical regularity $\{z_{R_1}\}_{R_1 > R_0}$ is locally bounded in $C^{1,\alpha}$ for every $\alpha \in (0, 1)$ (see [42]). Thus we may select a sequence $R_{1,n} \to +\infty$ such that $z_{R_{1,n}} \to z$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \overline{B_{R_0}})$, and then z is a nonnegative radially symmetric solution of $-\Delta z + |\nabla z|^q = f(m(|x|))$ in $|x| > R_0$, with

$$\min\{m(R_0), \lim_{R_1 \to +\infty} m(R_1)\} \le z(x) \le u(x). \tag{2.2.1}$$

By bootstrapping and since z is radially symmetric, we deduce that z is actually a classical solution. On the other hand, $m(|x|) \ge z(x)$, and then $f(m(|x|)) \ge f(z(x))$, and z is a positive, radially symmetric supersolution of (2.1.1).

Finally, when u does not blow up at infinity we have the function m bounded, hence z is bounded, while if u blows up at infinity then from (2.2.1) we see that $z \ge m(R_0)$ in $|x| > R_0$. This concludes the proof. □

When the supersolution u does not blow up at infinity it is indeed possible to obtain a radially symmetric, positive *solution* in an exterior domain. This is very convenient since working with solutions is somewhat easier than with supersolutions. As we have already remarked, the main difficulty lies in the case $q > 2$. The next lemma is mainly aimed at such situation.

Lemma 2.2.2. *Assume $N \ge 2$, $q > 1$ and if $N = 2$ additionally assume $q \le 2$. For every $M > 0$ there exists $\widetilde{R} = \widetilde{R}(M) > 0$ such that for every $R_1, R_2 > \widetilde{R}$ with $R_2 \ge 2R_1$ and every $a, b > 0$ with $a, b \le M$ there exists a radially symmetric, positive function $\phi \in C^2(\overline{A(R_1, R_2)})$ verifying*

$$-\Delta\phi + |\nabla\phi|^q \le 0 \qquad \text{in } R_1 < |x| < R_2$$

and $\phi(R_1) = a$, $\phi(R_2) = b$.

Proof. When $1 < q \le 2$, the problem

$$\begin{cases} -\Delta\phi + |\nabla\phi|^q = 0 & \text{in } R_1 < |x| < R_2 \\ \phi(R_1) = a \\ \phi(R_2) = b. \end{cases}$$

can easily be solved with no restrictions on a nor b with the aid of the classical method of sub and supersolutions (cf. [4]), since $\underline{\phi} = \min\{a, b\}$ is a subsolution while $\overline{\phi} = \max\{a, b\}$ is a supersolution.

Thus only the case $q > 2$ and $N \ge 3$ deserves special attention. Consider the problem

$$\begin{cases} -\Delta\phi + |\nabla\phi|^2 = 0 & \text{in } R_1 < |x| < R_2 \\ \phi(R_1) = a \\ \phi(R_2) = b. \end{cases} \tag{2.2.2}$$

With the standard change of variables $\phi = -\log w$, problem (2.2.2) is equivalent to

$$\begin{cases} \Delta w = 0 & \text{in } R_1 < |x| < R_2 \\ w(R_1) = e^{-a} \\ w(R_2) = e^{-b}, \end{cases} \tag{2.2.3}$$

whose explicit solution is given by

$$w(x) = \frac{e^{-a} - e^{-b}}{R_1^{2-N} - R_2^{2-N}}(|x|^{2-N} - R_2^{2-N}) + e^{-b}.$$

Assume $a > b$, the other case being treated in a completely similar way. By the maximum principle (or direct computation), $w \geq e^{-a}$ in $R_1 < |x| < R_2$, so that

$$\begin{aligned} |\nabla \phi| = \frac{|\nabla w|}{w} &\leq (N-2)\,\frac{e^{a-b} - 1}{R_1^{2-N} - R_2^{2-N}}\,|x|^{1-N} \\ &\leq (N-2)\,\frac{e^{a-b} - 1}{(1 - 2^{2-N})R_1} \leq (N-2)\,\frac{e^{M}}{(1 - 2^{2-N})\widetilde{R}} \leq 1 \end{aligned}$$

in $R_1 < |x| < R_2$, provided $\widetilde{R}$ is taken large enough (depending on M). Thus since $q \geq 2$, $-\Delta \phi + |\nabla \phi|^q \leq -\Delta \phi + |\nabla \phi|^2 = 0$ in $R_1 < |x| < R_2$, as was to be shown. $\square$

Lemma 2.2.3. *Assume $N \geq 3$, $q > 1$ and if $N = 2$ additionally assume $q \leq 2$. Let $f : (0, +\infty) \to \mathbb{R}$ be positive, nondecreasing and continuous and suppose there exists a positive supersolution u of problem* (2.1.1) *which does not blow up at infinity. Then there exists a positive, bounded, radially symmetric solution $z \in C^2(\mathbb{R}^N \setminus B_{R_1})$ of* (2.1.1) *in $\mathbb{R}^N \setminus B_{R_1}$ for some $R_1 > R_0$.*

Proof. By Lemma 2.2.1, there exists a positive, bounded, radially symmetric supersolution v of (2.1.1). Let $M = \sup v$ and choose $\widetilde{R} = \widetilde{R}(M)$ as given by Lemma 2.2.2. For $R_1 > \widetilde{R}$ and $R_2 > 2R_1$, consider the problem

$$\begin{cases} -\Delta z + |\nabla z|^q = \lambda f(z) & \text{in } A(R_1, R_2) \\ z = v(R_1) & \text{on } |x| = R_1 \\ z = v(R_2) & \text{on } |x| = R_2. \end{cases} \tag{2.2.4}$$

Let ϕ be the function given by Lemma 2.2.2 with $a = v(R_1)$, $b = v(R_2)$. Then clearly ϕ is a subsolution of (2.2.4) while v is a supersolution and by comparison $\phi \leq v$ in $A(R_1, R_2)$. Since $\phi = v$ on $\partial A(R_1, R_2)$, we can use the method of sub and supersolutions (see Theorem A.2 in the Appendix) to obtain a positive solution z_{R_2} of (2.2.4) verifying $\phi \leq z_{R_2} \leq v$.

In the present situation we could pass to the limit as in Lemma 2.2.1. However it could very well happen that the limit function is identically zero. To overcome this problem we obtain appropriate bounds for the solutions up to the boundary ∂B_{R_1}. To obtain local bounds for the gradient, we use Lemma A.1 in the Appendix, together with Remark A.2: for any $R > 2R_1$, we have that $|z'_{R_2}(r)| \le C$ for $r \in [2R_1, R]$ and $R_2 > R$. In order to get bounds up to R_1, we use the second part of Lemma A.1, so that we need to control the derivative $z'_{R_2}(R_1)$. Notice that $\phi(R_1) = z_{R_2}(R_1) = v(R_1)$, hence

$$0 \ge \phi'(R_1) \ge z'_{R_2}(R_1) \ge v'(R_1).$$

Lemma A.1 and Remark A.2 again will give that $|z'_{R_2}(r)| \le C$ for $r \in [R_1, 2R_1]$, and therefore in $[R_1, R]$.

Hence, we may pass to the limit in a standard way to obtain that, for a subsequence $R_2 \to +\infty$, $\lim_{R_2 \to +\infty} z_{R_2} = z$ in $C^1_{\rm loc}[R_1, +\infty)$, where z is a nonnegative, radial solution of (2.1.1) in $\mathbb{R}^N \setminus B_{R_1}$. Since the convergence is up to ∂B_{R_1}, we see that $z(R_1) = v(R_1)$, which implies that z is nontrivial, hence strictly positive by the strong maximum principle. This concludes the proof. □

We finally include a further result on the existence of radially symmetric solutions for problem (2.1.1) when g is replaced by a nonlinearity which is comparable to f near zero.

Lemma 2.2.4. *Assume $N \ge 3$, $q > 1$ and if $N = 2$ additionally assume $q \le 2$. Let $f, g : (0, +\infty) \to \mathbb{R}$ be positive, nondecreasing and continuous and suppose that*

$$l = \liminf_{t \to 0+} \frac{f(t)}{g(t)} > 0 \tag{2.2.5}$$

(the case $l = +\infty$ not being excluded). If there exists a positive supersolution u of problem (2.1.1) *which does not blow up at infinity, then for every $\mu \in (0, \lambda l)$ there exists $R_2 > R_0$ and a positive, radially symmetric solution h of*

$$-\Delta h + |\nabla h|^q = \mu g(h) \quad \text{in } \mathbb{R}^N \setminus B_{R_2}. \tag{2.2.6}$$

which verifies $\lim_{r \to +\infty} h(r) = 0$.

Proof. By Lemma 2.2.3, there exists a positive, radially symmetric solution z of (2.1.1) in $\mathbb{R}^N \setminus B_{R_1}$ for some $R_1 > R_0$. By Lemma 2.3.1 below (whose proof is independent of this one) $\lim_{r \to +\infty} z(r) = 0$. Using hypothesis (2.2.5), for every $\mu \in (0, \lambda l)$, there exists $R_2 > R_1$ such that $\lambda f(z(r)) \ge \mu g(z(r))$ in $r \ge R_2$.

Thus z is a supersolution of problem (2.2.6) and we may apply Lemma 2.2.3 again to obtain a positive, radially symmetric solution h of (2.2.6) verifying $h \le z$. It follows in particular that $\lim_{r \to +\infty} h(r) = 0$. □

2.3 Some properties of radial solutions

Since in the previous section we have reduced our problem to a radially symmetric one, it is now the turn to discuss a couple of properties of positive, radially symmetric, bounded solutions of (2.1.1). We begin by analyzing the behaviour at infinity of such solutions.

Lemma 2.3.1. *Assume $N \geq 2$, $q > 1$ and let $f : (0, +\infty) \to \mathbb{R}$ be continuous and positive. Suppose that $z \in C^2(R_1, +\infty)$ is a positive, bounded solution of*

$$-z'' - \frac{N-1}{r} z' + |z'|^q = \lambda f(z) \qquad \text{in } (R_1, +\infty). \tag{2.3.1}$$

Then there exists $R_2 > R_1$ such that $z' < 0$ for $r > R_2$ and $\lim_{r\to+\infty} z(r) = \lim_{r\to+\infty} z'(r) = 0$.

Proof. As we already pointed out, it follows as a consequence of Lemma 1 in [2] that there exists $R_2 > R_1$ such that z is monotone for $r > R_1$, hence $z'(r)$ is either nonnegative or nonpositive for $r > R_2$. Notice that, if z' vanishes at some point $r_0 > R_1$, then it has a local minimum or maximum at r_0, so that $z''(r_0) = 0$ and we obtain $f(z(r_0)) = 0$, a contradiction. Hence, either $z' < 0$ or $z' > 0$.

Let us rule out the second possibility. Assume then that z is increasing. Since z is bounded, we can apply Lemma S in the Appendix to this chapter (Section 2.7) to obtain a sequence $r_n \to +\infty$ and a number $\theta > 0$ such that $z(r_n) \to \theta$, $r_n z'(r_n) \to 0$ and $\lim_{n\to\infty} r_n^2 z''(r_n) \geq 0$. In particular, $z'(r_n) \to 0$, and from (2.3.1):

$$z''(r_n) \to -\lambda f(\theta) < 0,$$

a contradiction. Thus z cannot be increasing.

It follows that $z' < 0$ for large r. Let us finally see that $\lim_{r\to+\infty} z(r) = 0$. Assume for a contradiction that there exists $\delta > 0$ with $z > \delta$. Define the nonnegative function

$$E(r) = \frac{z'(r)^2}{2} + \lambda F(z(r)) + \int_{R_2}^{r} |z'(s)|^{q+1} ds,$$

where $F(t) = \int_\delta^t f(s)ds$. The function E is decreasing for $r > R_2$, since

$$\begin{aligned} E'(r) &= z'(r)z''(r) + \lambda f(z(r))z'(r) + |z'(r)|^{q+1} \\ &= z'(r)(z''(r) + \lambda f(z(r)) - |z'(r)|^q) = -\frac{N-1}{r} z'(r)^2 < 0. \end{aligned}$$

Thus E has a nonnegative limit. We deduce that

$$\frac{z'(r)^2}{2} + \int_{R_2}^{r} |z'(s)|^{q+1} ds$$

has a limit, so that the integral $\int_{R_2}^{r} |z'(s)|^{q+1} ds$ is convergent. Thus $z'(r)$ has a limit, which has to be zero. From (2.3.1) the function $z''(r)$ also has a limit, which is also zero, since $z(r)$ is monotone and bounded. Hence, $z(r) \to 0$, a contradiction. This concludes the proof. $\square$

Remark 2.3.2. It becomes clear from the above proof that if $f(0) > 0$ then no positive solutions of (2.1.1) not blowing up at infinity may exist. Therefore we will implicitly assume almost everywhere in what follows that $f(0) = 0$.

Next we will obtain lower bounds for radially symmetric, positive supersolutions of the equation $-\Delta u + |\nabla u|^q = 0$. It turns out that the lower bounds depend on the relation between q and $\frac{N}{N-1}$. The following lemma will be also useful in the next chapter.

Lemma 2.3.3. *Assume $q > 1$, and let $z \in C^2(R_1, +\infty)$ be positive and verify*

$$-z'' - \frac{N-1}{r} z' + |z'|^q \geq 0 \qquad \text{in } (R_1, +\infty) \tag{2.3.2}$$

together with $z'(r) < 0$, $\lim_{r\to+\infty} z(r) = 0$. Then there exists positive constants, all of them denoted by C, such that for $r \geq R_1$:

a) *If $N \geq 2$ and $1 < q < \frac{N}{N-1}$ then:*

$$|z'| \geq C r^{-\frac{1}{q-1}}, \qquad z \geq C r^{-\frac{2-q}{q-1}}.$$

b) *If $N \geq 3$ and $q = \frac{N}{N-1}$ then:*

$$|z'| \geq C r^{1-N} (\log r)^{1-N}, \qquad z \geq C r^{2-N} (\log r)^{1-N}.$$

c) *If $N \geq 3$ and $q > \frac{N}{N-1}$ then:*

$$|z'| \geq C r^{1-N}, \qquad z \geq C r^{2-N}.$$

Proof. Setting $Z = -r^{N-1} z'$, we have

$$Z' + r^{-(q-1)(N-1)} Z^q \geq 0.$$

In case a) it follows that $(q-1)(N-1) < 1$, so that, dividing by Z^q and integrating in (r_0, r) for some $r > r_0 > R_1$:

$$-\frac{Z^{1-q}}{q-1} + \frac{r^{1-(q-1)(N-1)}}{1-(q-1)(N-1)} \geq -C. \tag{2.3.3}$$

It follows that $Z \geq Cr^{-\frac{1}{q-1}+N-1}$, i.e. $-z' \geq Cr^{-\frac{1}{q-1}}$. Integrating in $(r, +\infty)$ and using $z(+\infty) = 0$ we obtain the desired inequality.

In case b), after integration we obtain

$$-\frac{Z^{1-q}}{q-1} + \log r \geq -C,$$

so that $Z \geq C(\log r)^{-\frac{1}{q-1}}$, that is $-z' \geq Cr^{1-N}(\log r)^{1-N}$. Integrating in $(r, +\infty)$ and using l'Hôpital rule we arrive at

$$z(r) \geq C \int_r^\infty s^{1-N}(\log s)^{1-N} ds \geq Cr^{2-N}(\log r)^{1-N}.$$

In case c), when $q > \frac{N}{N-1}$ equation (2.3.3) only gives $Z^{1-q} \leq C$, so that $-z' \geq Cr^{1-N}$ and then $z \geq Cr^{2-N}$ after integration. □

Remark 2.3.4. A careful examination of the proof above shows that a finer information is actually available: the constants can be estimated from below. As an illustration, if $q < \frac{N}{N-1}$, then any positive solution w of the inequality (2.3.2) is such that

$$\liminf_{r \to +\infty} r^{\frac{1}{q-1}} |z'(r)| \geq \left(\frac{q-1}{N - q(N-1)} \right)^{-\frac{1}{q-1}}$$

and

$$\liminf_{r \to +\infty} r^{\frac{2-q}{q-1}} z(r) \geq \frac{q-1}{2-q} \left(\frac{q-1}{N - q(N-1)} \right)^{-\frac{1}{q-1}}.$$

When $q = \frac{N}{N-1}$, we have

$$\liminf_{r \to +\infty} \frac{z(r)}{r^{2-N}(\log r)^{1-N}} \geq \frac{(N-1)^{N-1}}{N-2}.$$

These inequalities will be used in some proofs later on.

2.4 Proofs in the case $1 < q < \frac{N}{N-1}$

This section is devoted to analyze problem (2.1.1) in the case where $1 < q < \frac{N}{N-1}$. Throughout the section we will *only* deal with supersolutions which do not blow up at infinity.

We begin by stating and proving a slightly modified version of the nonexistence part in Theorem 2.1. Although we are requiring more regularity on f, hypothesis (2.1.3) can be replaced by a more general one.

Theorem 2.4. *Assume $N \geq 2$, $1 < q < \frac{N}{N-1}$ and let $f : (0, +\infty) \to \mathbb{R}$ be nondecreasing, C^1 and positive. Suppose in addition that*

$$l_1 := \liminf_{s \to 0+} s^{\frac{q-1}{2-q}} \int_s^\delta \frac{d\tau}{f(\tau)^{\frac{1}{q}}} < +\infty \tag{2.4.1}$$

for some small $\delta > 0$. Then:

a) If $l_1 = 0$, problem (2.1.1) *does not admit positive supersolutions for any $\lambda > 0$.*

b) If $l_1 > 0$, there exists $\lambda^* > 0$ such that problem (2.1.1) *does not admit positive supersolutions for $\lambda > \lambda^*$.*

Proof. Assume there exists a positive supersolution u of (2.1.1). By Lemma 2.2.3, there exists a positive, bounded, radially symmetric solution z of $-\Delta z + |\nabla z|^q = \lambda f(z)$ in $\mathbb{R}^N \setminus B_{R_1}$ for some $R_1 > R_0$. Thus z verifies

$$-z'' - \frac{N-1}{r} z' + |z'|^q = \lambda f(z) \qquad \text{for } r > R_1. \tag{2.4.2}$$

By Lemma 2.3.1, we obtain that $z' < 0$ for large r and $\lim_{r \to +\infty} z(r) = 0$.

Consider the function

$$H(r) = \lambda f(z(r)) - a|z'(r)|^q \quad \text{for } r > R_1,$$

where $a > 1$ is to be chosen (cf. [69]). Since f is nondecreasing,

$$\begin{aligned} H'(r) &= -\lambda f'(z)|z'| + aq|z'|^{q-1} \left(\frac{N-1}{r}|z'| + |z'|^q - \lambda f(z) \right) \\ &\leq aq|z'|^{q-1} \left(\frac{N-1}{r}|z'| + |z'|^q - \lambda f(z) \right). \end{aligned}$$

Taking into account the lower estimates for $|z'|$ given by part a) of Lemma 2.3.3, we deduce that, at any point r_0 where H vanishes ($\lambda f(z(r_0)) = a|z'(r_0)|^q$), we have

$$H'(r_0) \leq aq|z'|^q \left(\frac{N-1}{r_0} - (a-1)|z'|^{q-1} \right) \leq \frac{aq|z'|^q}{r_0} (N - 1 - (a-1)C) < 0,$$

provided a is taken large enough. Observe that C can be taken independently of z (cf. Remark 2.3.4), hence a is also independent of z. Thus H may vanish at most once, and in particular it has constant sign for large r.

If $H(r) > 0$ for large r, then $\lambda f(z(r)) > a|z'|^q$. Thus, by part a) of Lemma 2.3.3

$$\begin{aligned} -z'' &> a|z'|^q - \frac{N-1}{r}|z'| - |z'|^q = |z'|^q \left(a - 1 - \frac{N-1}{r}|z'|^{1-q} \right) \\ &\geq |z'|^q \left(a - 1 - \frac{N-1}{C} \right) = |z'|^q. \end{aligned}$$

Since $-z' > 0$, it is well-known that $-z'$ blows up in finite time because $q > 1$ and we get a contradiction.

Then we necessarily have $H(r) < 0$ for large r. This translates into

$$-\frac{z'(r)}{f(z(r))^{\frac{1}{q}}} \geq \lambda^{\frac{1}{q}} a^{-\frac{1}{q}}.$$

Integrating between r_0 and r where $r > r_0$ are large enough:

$$\int_{z(r)}^{z(r_0)} \frac{d\tau}{f(\tau)^{\frac{1}{q}}} \geq \lambda^{\frac{1}{q}} a^{-\frac{1}{q}} (r - r_0) \geq C\lambda^{\frac{1}{q}} r,$$

where C is independent of z. Now we use the lower estimate for z given by part a) of Lemma 2.3.3 to obtain:

$$\frac{1}{r} \int_{Cr^{-\alpha}}^{z(r_0)} \frac{d\tau}{f(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}}$$

where $\alpha = \frac{2-q}{q-1}$. Setting $s = Cr^{-\alpha}$, and letting $s \to 0+$:

$$l_1 = \liminf_{s \to 0+} s^{\frac{q-1}{2-q}} \int_{s}^{z(r_0)} \frac{d\tau}{f(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}},$$

where C is independent of z (Remark 2.3.4). Thus when $l_1 = 0$, there are no positive supersolutions for any value of λ, while if $0 < l_1 < +\infty$, no positive supersolutions exist provided λ is larger than $l_1^q C^{-q}$. This concludes the proof. □

Finally, we proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. a) In this case $l \in (0, +\infty)$. By Lemma 2.2.4, there exists $R_1 > R_0$ and a positive, radially symmetric, bounded solution of

$$-\Delta z + |\nabla z|^q = \mu g(z) \text{ in } |x| > R_1 \tag{2.4.3}$$

for every $\mu \in (0, \lambda l)$, where $g(t) := t^{\frac{q}{2-q}}(-\log t)^{\beta}$. It is not hard to check that condition (2.4.1) on g holds if and only if $\beta \geq 0$. Moreover, when $\beta > 0$ we obtain $l_1 = 0$. Thus, according to Theorem 2.4, there are no positive supersolutions of (2.4.3) for any $\mu > 0$ if $\beta > 0$, and there are no positive supersolutions for large μ if $\beta = 0$. Thus the same is true for problem (2.1.1). This shows i) and the second part of ii).

To show the rest of part a), we will construct explicit radial supersolutions. Consider first (2.1.1) with $f(t)$ replaced by $t^{\frac{q}{2-q}}(-\log t)^{\beta}$ with $\beta < 0$. Let $u = r^{-\alpha}(\log r)^{\gamma}$, where

$\alpha = \frac{2-q}{q-1}$ and $\gamma > 0$ is to be chosen. The function u will be a supersolution provided that

$$\begin{aligned} &-\alpha(\alpha+2-N)(\log r)^{\gamma} + \gamma(2\alpha+2-N)(\log r)^{\gamma-1} \\ &-\gamma(\gamma-1)(\log r)^{\gamma-2} + (\log r)^{\gamma q}|\alpha - \gamma(\log r)^{-1}|^{q} \\ &\quad \geq \lambda(\log r)^{\frac{\gamma q}{2-q}+\beta}\left(\alpha - \gamma\frac{\log(\log r)}{\log r}\right)^{\beta}. \end{aligned}$$

For sufficiently large r, only the leading terms are important, hence it suffices to have $\gamma q > \frac{\gamma q}{2-q} + \beta$, which is equivalent to

$$\gamma < -\frac{\beta}{q}\,\frac{2-q}{q-1}.$$

This election is possible since $\beta < 0$. Thus u is a supersolution in $|x| > R$ for sufficiently large R. Using Lemma 2.2.4, we see that (2.1.1) admits a supersolution for every $\lambda > 0$.

When $\beta = 0$, it is already known that (2.1.1) has positive supersolutions for small λ (see [2]). Defining

$$\lambda^* = \sup\{\lambda > 0 : \text{ (2.1.1) has a positive supersolution}\},$$

which is finite by the above proof, and taking into account that a supersolution of (2.1.1) for some λ is also a supersolution for smaller values of λ, we deduce that for $\lambda < \lambda^*$ there exist positive supersolutions while for $\lambda > \lambda^*$ no positive supersolutions exist. This concludes the proof of part a).

The proof of parts b) and c) is a direct consequence of the previous discussion and Lemma 2.2.4. □

2.5 The case $q \geq \frac{N}{N-1}$

In this section we will consider the proof of Theorems 2.2 and 2.3 (thus we are only dealing with supersolutions not verifying (2.1.2)). Before proceeding to the actual proofs, we will make a preliminary discussion which is somewhat similar to the previous case. We begin by considering a positive, bounded, radially symmetric solution z of

$$-z'' - \frac{N-1}{r}z' + |z'|^q = \lambda g(z) \qquad \text{for } r > R_1, \tag{2.5.1}$$

where $g(t) := t^{\frac{N}{N-2}}(-\log t)^{\beta}$, which in addition verifies $z' < 0$ for large r and $\lim_{r\to+\infty} z(r) = 0$. Consider the function

$$H(r) = \lambda g(z(r)) - 2|z'(r)|^q \quad \text{for } r > R_1. \tag{2.5.2}$$

At any point $r_0 > R_1$ where H vanishes, we have

$$\begin{aligned} H'(r_0) &= -\lambda g'(z)|z'| + 2q|z'|^{q-1}\left(\frac{N-1}{r_0}|z'| - |z'|^q\right) \\ &\leq |z'|\left(-\lambda g'(z) + \frac{2q(N-1)}{r_0}|z'|^{q-1}\right) \\ &\leq |z'|\left(-\lambda g'(z) + \frac{2^{\frac{1}{q}}q(N-1)}{r_0}\lambda^{\frac{q-1}{q}}g(z)^{\frac{q-1}{q}}\right). \end{aligned} \tag{2.5.3}$$

Now observe that $g'(t) \geq Ct^{\frac{2}{N-2}}(-\log t)^\beta$ for some positive C and $t \in (0, \delta]$. Thus

$$\begin{aligned} \frac{H'(r_0)}{|z'|} &\leq \left(-C\lambda z^{\frac{2}{N-2}}(-\log z)^\beta + \frac{C}{r_0}\lambda^{\frac{q-1}{q}} z^{\frac{q-1}{q}\frac{N}{N-2}}(-\log z)^{\beta\frac{q-1}{q}}\right) \\ &\leq z^{\frac{(q-1)N}{q(N-2)}}(-\log z)^{\beta\frac{q-1}{q}}\left(-C\lambda z^{\frac{1}{N-2}\left(2-\frac{(q-1)N}{q}\right)}(-\log z)^{\frac{\beta}{q}} + \frac{C}{r_0}\lambda^{\frac{q-1}{q}}\right). \end{aligned}$$

From now on, the proof varies depending on whether $q = \frac{N}{N-1}$ or $q > \frac{N}{N-1}$. Thus, let us consider both cases separately. Since the case $q > \frac{N}{N-1}$ is simpler, let us consider it first.

Proof of Theorem 2.3. The proofs of b) and c) follow from that of a) as in Theorem 2.1, hence we only show a). Let us begin with the proof of nonexistence when $\beta \geq -1$. By employing Lemma 2.2.4, we may restrict to $g(t) = t^{\frac{N}{N-2}}(-\log t)^\beta$. Further, let us observe that if z is a solution of (2.5.1) then it is also a supersolution for every value of $q' < q$, since $\lim_{r\to+\infty}|z'(r)| = 0$. Thus we may always assume $q < \frac{N}{N-2}$.

Consider the function H given by (2.5.2). According to (2.5.3), at any point where H vanishes we need to consider the sign of

$$G(r) := -C\lambda z^{\frac{1}{N-2}\left(2-\frac{(q-1)N}{q}\right)}(-\log z)^{\frac{\beta}{q}} + \frac{C}{r}\lambda^{\frac{q-1}{q}}.$$

Taking into account that the exponent of z in the above expression is positive and using Lemma 2.3.3 c) we obtain

$$\begin{aligned} G(r) &\leq -C\lambda r^{-2+\frac{(q-1)N}{q}}(\log r)^{\frac{\beta}{q}} + \frac{C}{r}\lambda^{\frac{q-1}{q}} \\ &= \frac{1}{r}\left(-C\lambda r^{-1+\frac{(q-1)N}{q}}(\log r)^{\frac{\beta}{q}} + C\lambda^{\frac{q-1}{q}}\right) \\ &< 0 \end{aligned}$$

for large r, since $q > \frac{N}{N-1}$. Thus $H'(r) < 0$ at points where H vanishes, and so it has at most one zero. Hence H has constant sign for large r. We need to discuss both possibilities

$H(r) > 0$ for large r or $H(r) < 0$ for large r in turn. When $H(r) > 0$ for large r, we have $|z'|^q \leq \frac{\lambda g(z)}{2}$, so that

$$-z'' - \frac{N-1}{r} z' \geq \frac{1}{2}\lambda g(z)$$

for large r. This contradicts Theorem 1 in [3], since

$$\int_0^\delta \frac{\tau^{\frac{N}{N-2}}(-\log\tau)^\beta}{\tau^{2\frac{N-1}{N-2}}} d\tau = +\infty \tag{2.5.4}$$

for $\beta \geq -1$.

When $H(r) < 0$, we just argue as in the proof of Theorem 2.1. We arrive at

$$\int_{w(r)}^{w(r_0)} \frac{d\tau}{g(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}} r,$$

for $r > r_0$ large enough. Choosing r_0 such that $z(r_0) \leq 1/2$ and using the lower bound provided by Lemma 2.3.3, we have

$$\int_{Cr^{2-N}}^{\frac{1}{2}} \frac{d\tau}{g(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}} r,$$

which is equivalent, letting $s = Cr^{2-N}$, to

$$s^{\frac{1}{N-2}} \int_s^{\frac{1}{2}} \frac{d\tau}{g(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}}.$$

Now using that $g(t) = t^{\frac{N}{N-2}}(-\log t)^\beta$, the integral can be estimated to obtain

$$s^{\frac{q(N-1)-N}{q(N-2)}}(-\log s)^{-\frac{\beta}{q}} \geq C\lambda^{\frac{1}{q}}.$$

Letting $s \to 0^+$ we obtain a contradiction since $q > \frac{N}{N-1}$. Thus no positive supersolutions may exist when $\beta \geq -1$.

On the other hand, when $\beta < -1$, the integral in (2.5.4) is convergent. Thus by Theorem 1 in [3] there exists a solution w of

$$-\Delta w = \lambda f(w) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

for some $R_0 > 0$. Obviously this function w is also a supersolution of (2.1.1). This concludes the proof. $\square$

Proof of Theorem 2.2. As before, we only prove part a) and consider $g(t) = t^{\frac{N}{N-2}}(-\log t)^\beta$. Let us begin with the proofs of nonexistence in i) and ii). Recall that in this case $q = \frac{N}{N-1}$.

Assuming a positive supersolution exists and letting z be the radial solution obtained with the use of Lemma 2.2.4, if the function H given by (2.5.2) vanishes at r_0 then by (2.5.3)

$$H'(r_0) \leq |z'| z^{\frac{1}{N-2}} (-\log z)^{\frac{\beta}{N}} \left(-C\lambda z^{\frac{1}{N-2}} (-\log z)^{\frac{\beta(N-1)}{N}} + \frac{C}{r_0} \lambda^{\frac{1}{N}} \right).$$

Next, observe that, using Lemma 2.3.3 b) and Remark 2.3.4, there exists a constant C, which can be chosen independently of λ, such that $z \geq C r^{2-N} (\log r)^{1-N}$ for $r \geq r_1$ (of course r_1 will depend upon λ, but this will cause no inconveniences in what follows). We deduce

$$H'(r_0) \leq \frac{|z'| z^{\frac{1}{N-2}} (-\log z)^{\frac{\beta}{N}}}{r_0} \left(-C\lambda (\log r_0)^{(N-1)\left(\frac{\beta}{N} - \frac{1}{N-2}\right)} + C \right).$$

If $\beta > \frac{N}{N-2}$, it is clear that $H'(r_0) < 0$, provided r_0 is large enough. When $\beta = \frac{N}{N-2}$, it suffices to take λ large enough to achieve the same conclusion. Thus we obtain that the function H has a constant sign for large r, both when $\beta > \frac{N}{N-2}$ or when $\beta = \frac{N}{N-2}$ and λ is large.

The case where $H(r) > 0$ is ruled out as in the proof of Theorem 2.3. Thus we assume in what follows that $H(r) < 0$ for large r. We proceed as in the proofs of Theorems 2.1 and 2.3 to obtain

$$\int_{z(r)}^{z(r_0)} \frac{d\tau}{g(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}} r,$$

for $r > r_0$ large enough. Choosing r_0 such that $z(r_0) \leq 1/2$ and using the lower bound provided by Lemma 2.3.3 b, we have

$$\int_{Cr^{2-N}(\log r)^{1-N}}^{\frac{1}{2}} \frac{d\tau}{g(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}} r.$$

Now set $s = Cr^{2-N}(\log r)^{1-N}$. It follows that $-\log s \geq C \log r$ for some positive constant C, hence $s \geq Cr^{2-N}(-\log s)^{1-N}$. This readily gives $r \geq C s^{-\frac{1}{N-2}} (-\log s)^{-\frac{N-1}{N-2}}$. Hence for small positive s:

$$s^{\frac{1}{N-2}} (-\log s)^{\frac{N-1}{N-2}} \int_s^{\frac{1}{2}} \frac{d\tau}{g(\tau)^{\frac{1}{q}}} \geq C\lambda^{\frac{1}{q}}. \tag{2.5.5}$$

The integral can be estimated and (2.5.5) gives

$$(-\log s)^{(N-1)\left(\frac{1}{N-2} - \frac{\beta}{N}\right)} \geq C\lambda^{\frac{1}{q}}$$

for small s. When $\beta > \frac{N}{N-2}$ we immediately obtain a contradiction letting $s \to 0+$. If $\beta = \frac{N}{N-2}$ we deduce that λ cannot be too large, hence no positive solutions exist when λ is large. This concludes the nonexistence part in the proof of a).

As for the existence parts in ii) and iii), it suffices to show that a positive, radial supersolution can be constructed for large r if $\beta < \frac{N}{N-2}$ and $\lambda > 0$ or $\beta = \frac{N}{N-2}$ and λ is small. We look for a supersolution in the form $u = Ar^{2-N}(\log r)^\gamma$ for some $A > 0$, $\gamma \in \mathbb{R}$. It suffices to have

$$\gamma(N-2) - \gamma(\gamma-1)(\log r)^{-1} + A^{q-1}(\log r)^{\frac{\gamma N}{N-1}+1-\gamma} \left|N-2-\gamma(\log r)^{-1}\right|^q \geq \lambda A^{\frac{2}{N-2}}(\log r)^{\frac{\gamma N}{N-2}+\beta+1-\gamma}\left(N-2-\frac{\log A}{\log r}-\gamma\frac{\log(\log r)}{\log r}\right)^\beta \tag{2.5.6}$$

for large r. If $\beta < \frac{N}{N-2}$, we can choose

$$-(N-1) < \gamma < -\frac{\beta(N-1)(N-2)}{N},$$

and (2.5.6) will hold for large r irrespective of the value of $A > 0$. When $\beta = \frac{N}{N-2}$, we need to choose $\gamma = 1 - N$ and (2.5.6) reduces to

$$\gamma(N-2) - \gamma(\gamma-1)(\log r)^{-1} + A^{q-1}\left|N-2-\gamma(\log r)^{-1}\right|^q \geq \lambda A^{\frac{2}{N-2}}\left(N-2-\frac{\log A}{\log r}-\gamma\frac{\log(\log r)}{\log r}\right)^\beta,$$

which will hold for large r if $\gamma(N-2) + A^{q-1}(N-2)^q > \lambda(N-2)^\beta A^{\frac{2}{N-2}}$. For a given $\lambda > 0$, such a value of A can be found provided that

$$0 < \lambda < \sup_{x>0} \frac{\gamma(N-2) + x^{q-1}(N-2)^q}{(N-2)^\beta x^{\frac{2}{N-2}}}.$$

Observe that this supremum is finite since $\gamma < 0$ and $q = \frac{N}{N-1} < \frac{N}{N-2}$. Hence the function u will be a supersolution with this choice of A. This concludes the proof. □

2.6 Supersolutions which blow up at infinity

To conclude with the analysis of (2.1.1), we are dealing with positive supersolutions which blow up at infinity, that is, which verify

$$\lim_{|x|\to+\infty} u(x) = +\infty.$$

Since, from now on, the parameter λ will not play any role, we are setting $\lambda = 1$, and will consider the problem

$$-\Delta u + |\nabla u|^q = f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}. \tag{2.6.1}$$

Our main result of the section is the following:

Theorem 2.5. *Assume* $f : (0, +\infty) \to \mathbb{R}$ *is positive,* C^1 *and nondecreasing. If the condition*

$$\int_M^\infty \frac{d\tau}{f(\tau)^{\frac{1}{q}}} < +\infty \tag{2.6.2}$$

holds for some $M > 0$*, then problem* (2.6.1) *does not admit positive supersolutions blowing up at infinity. Moreover, if* $N = 2$ *and* $q \geq 2$ *problem* (2.6.1) *does not admit any positive supersolutions.*

Remark 2.6.1. As in the previous sections, we can test condition (2.6.2) with the class of functions $f(t) = t^p (\log t)^\beta$, for $p > 0$ and $\beta \in \mathbb{R}$ (and functions which behave like these at infinity). It is not difficult to check that (2.6.2) holds when $p > q$ or $p = q$ and $\beta > q$. Thus no supersolutions of (2.6.1) which blow up at infinity can exist in these cases.

However, when $p < q$ or $p = q$ and $\beta \leq q$, it is not hard to construct positive supersolutions blowing up at infinity. Setting $u = e^v$ and looking for radially symmetric solutions v, it suffices to have

$$-v'' - (v')^2 - \frac{N-1}{r} v' + e^{(q-1)v} (v')^q \geq e^{(p-1)v} v^\beta$$

for large r. If $p < q$, we may simply take $v = r$, while for $p = q$ and $\beta \leq q$, we set $v = e^{\alpha r}$ with an arbitrary $\alpha > 1$.

Before coming to the proof of Theorem 2.5, we need an additional result which will be handy in the case $N = 2$ and $q \geq 2$.

Lemma 2.6.2. *If* $N = 2$ *and* $q \geq 2$*, there exists a solution* ϕ *of*

$$-\phi'' - \frac{\phi'}{r} + |\phi'|^q = 0, \quad in \quad (1, \infty)$$

such that $\lim_{r\to\infty} \phi(r) = -\infty$ *and* $\phi'(r) < 0$ *for large* r.

Proof. Defining $W = -r\phi'$ we see that W has to satisfy

$$W' + r^{1-q} |W|^q = 0.$$

If $q = 2$ we obtain the particular solution $W(r) = 1/\log r$, thus we can take $\phi(r) = -\log(\log r)$. When $q > 2$, an integration gives $W(r) = (\frac{q-1}{q-2}(1 - r^{2-q}))^{-\frac{1}{q-1}}$, thus

$$\phi(r) = -\int_1^r \frac{W(s)}{s} ds$$

is the required function. The lemma follows. □

Proof Theorem 2.5. Assume first that (2.6.1) has a positive supersolution which blows up at infinity. By Lemma 2.2.1, there exists a positive, radially symmetric supersolution $z = z(r)$ of

$$-z'' - \frac{N-1}{r}z' + |z'|^q = f(z) \qquad \text{in } r > R_0$$

which is bounded from below. Thanks to Lemma 1 in [2], there exists $R_1 > R_0$ such that z is monotone for $r > R_1$.

If z were bounded, we may apply Lemma 2.3.1 to deduce that $\lim_{r\to+\infty} z(r) = 0$, contradicting that z is bounded from below. Hence z is not bounded, and it has to be increasing for $r > R_1$ with $\lim_{r\to+\infty} z(r) = +\infty$. As in the proof of Lemma 2.3.1, we also deduce that z' does not vanish for $r > R_1$, hence $z' > 0$ in $r > R_1$.

Consider the function

$$H(r) = f(z(r)) - 2z'(r)^q, \quad r > R_1.$$

If $H(r_0) = 0$ for some $r_0 > R_1$, we have

$$H'(r_0) \geq f'(z)z' + \frac{2(N-1)q}{r_0}(z')^q + 2q(z')^{2q-1} > 0,$$

thus H can vanish at most once for $r > R_1$, and in particular it has constant sign for sufficiently large r. Let us first rule out the case $H(r) > 0$ for large r. Observe that this implies

$$-z'' \geq \frac{1}{2}f(z)$$

for large r. We may integrate between r_0 and r for $r > r_0$, both large enough, and obtain

$$-z'(r) \geq -z'(r_0) + \frac{1}{2}\int_{r_0}^{r} f(z(t))dt \geq -z'(r_0) + \frac{1}{2}f(z(r_0))(r - r_0).$$

Integrating once again between r_0 and r:

$$-z(r) \geq -z(r_0) - z'(r_0)(r - r_0) + \frac{1}{4}f(z(r_0))(r - r_0)^2,$$

which implies that $z(r) \to -\infty$ as $r \to +\infty$, a contradiction.

So we may assume in what follows that $H(r) < 0$ for large r, that is $(z')^q \geq \frac{1}{2}f(z)$. This implies, after integrating:

$$\int_{z(r_0)}^{z(r)} \frac{d\tau}{f(\tau)^{\frac{1}{q}}} \geq 2^{-\frac{1}{q}}(r - r_0)$$

for $r > r_0$ large enough. Letting $r \to +\infty$ we arrive at

$$\int_{z(r_0)}^{\infty} \frac{d\tau}{f(\tau)^{\frac{1}{q}}} = +\infty$$

which contradicts condition (2.6.2).

Finally, consider the case $N = 2$ and $q \geq 2$ and let us show that all radially symmetric solutions blow up at infinity. Suppose by contradiction that z is bounded so that Lemma 2.3.1 implies $\lim_{r\to+\infty} z(r) = 0$ and $z' < 0$ for large r.

Take the function ϕ given by Lemma 2.6.2 and fix a large r_0. Choose a value $0 < a < 1$ such that $z'(r_0)/\phi'(r_0) > a$ (which is possible since $\phi' < 0$) and let b be such that $z(r_0) = a\phi(r_0) + b$. Observe that $\underline{\phi} := a\phi + b$ is a subsolution of

$$-z'' - \frac{1}{r}z' + |z'|^q = f(z),$$

and we also have $\lim_{r\to\infty} \underline{\phi}(r) = -\infty$. Thus using comparison we have $\underline{\phi} \leq z$ in $(r_0, +\infty)$. Since $\underline{\phi}(r_0) = z(r_0)$, then $\underline{\phi}'(r_0) \leq z'(r_0)$ which contradicts the choice of a. This contradiction shows that z blows up at infinity, as was to be shown. This concludes the proof. □

2.7 Appendix to Chapter 2

We include in this Appendix a slightly modified version of Lemma S in [69]:

Lemma 2.7.1. *Suppose that $z \in C^2(R_0, +\infty)$ where $R_0 \geq 0$ and $w(r) = r^\alpha z(r)$ is bounded for large enough r and some $\alpha \in \mathbb{R}$. Then there exists a sequence $\{r_n\}_{n\geq 1}$ with $r_n \to +\infty$ such that*

i) $\displaystyle\lim_{n\to+\infty} w(r_n) = \theta := \liminf_{r\to+\infty} w(r)$,

ii) $\displaystyle\lim_{n\to+\infty} r_n^{\alpha+1} z'(r_n) = -\alpha\theta$,

iii) $\displaystyle\lim_{n\to+\infty} r_n^{\alpha+2} z''(r_n) \geq \alpha(\alpha+1)\theta$.

The last limit may be $+\infty$.

Proof. We distinguish three cases.
Case 1). The function w is increasing for large r and $w(r) \to \theta$ when r goes to $+\infty$. Then $w' \geq 0$ for large r and

$$\liminf_{r\to+\infty} rw' = 0, \tag{2.7.1}$$

otherwise w would grow at least logarithmically against the assumption that it is bounded. Now, two situations are possible:

1.1) The function rw' decreases monotonically to 0. Then, $(rw')' \leq 0$ and

$$\limsup_{r \to +\infty} r(rw')' = 0,$$

for the same reasons as (2.7.1), that is, there exists a sequence $\{r_n\}_{n \geq 1}$ with $r_n \to +\infty$ such that $r_n^2 w''(r_n) + r_n w'(r_n) \to 0$ which implies $r_n^2 w''(r_n) \to 0$.

1.2) The function rw' decreases to 0 along a sequence of local minima $\{r_n\}$ of rw'. That is, there exists a sequence $\{r_n\}$ with $r_n \to +\infty$ such that $r_n w'(r_n) \to 0$ and $r_n w''(r_n) + w'(r_n) = 0$, so again $r_n^2 w''(r_n) \to 0$.

In both situations 1.1) and 1.2) we have $r_n w'(r_n), r_n^2 w''(r_n) \to 0$. Now by simple computation,

$$\begin{aligned} r^{\alpha+1} z' &= -\alpha w + rw', \\ r^{\alpha+2} z'' &= \alpha(\alpha+1)w - 2\alpha r w' + r^2 w''. \end{aligned} \tag{2.7.2}$$

Letting $n \to +\infty$ through the sequence $\{r_n\}$ in (2.7.2) we arrive at the desired conclusion. Observe that i) is always true because w is monotone.

Case 2). The function w is decreasing for large r and $w(r) \to \theta$ when r goes to $+\infty$. The proof is similar to case 1), replacing w by $-w$.

Case 3). The function w oscillates for large r, and along a sequence of local minima $\{r_n\}$, we have $w(r_n) \to \theta$. That is, there exists a sequence $\{r_n\}_{n \geq 1}$ with $r_n \to +\infty$ such that

$$\lim_{n \to +\infty} w(r_n) = \theta, \ w'(r_n) = 0 \ \text{ and } \ w''(r_n) \geq 0.$$

Hence by (2.7.2),

$$\begin{aligned} r_n^{\alpha+1} z'(r_n) &= -\alpha w(r_n), \\ r_n^{\alpha+2} z''(r_n) &\geq \alpha(\alpha+1) w(r_n). \end{aligned}$$

The proof of the lemma is concluded after taking a subsequence of $\{r_n\}$ which ensures that $\lim_{n \to +\infty} r_n^{\alpha+2} z''(r_n)$ exists (possibly being infinite). $\square$

Chapter 3

Systems with superlinear growth in the gradient

We continue our analysis of Liouville theorems in the presence of gradient terms by considering next the system:

$$\begin{cases} -\Delta u + |\nabla u|^q &= \lambda f(v) \\ -\Delta v + |\nabla v|^q &= \mu g(u), \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{3.1.1}$$

where $N \geq 2$, $R_0 > 0$, $q > 1$, f, g are nondecreasing positive functions defined in $(0, +\infty)$ and $\lambda, \mu > 0$ are parameters. In the next chapter we will study the same system but with $q = 1$.

We will be dealing throughout with weak positive supersolutions $(u, v) \in C^1(\mathbb{R}^N \setminus B_{R_0})^2$ of (3.1.1), that is, functions verifying

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N \setminus B_{R_0}} \nabla u \nabla \phi + \int_{\mathbb{R}^N \setminus B_{R_0}} |\nabla u|^q \phi &\geq \lambda \displaystyle\int_{\mathbb{R}^N \setminus B_{R_0}} f(v)\phi \\ \displaystyle\int_{\mathbb{R}^N \setminus B_{R_0}} \nabla v \nabla \psi + \int_{\mathbb{R}^N \setminus B_{R_0}} |\nabla v|^q \psi &\geq \mu \displaystyle\int_{\mathbb{R}^N \setminus B_{R_0}} g(u)\psi \end{cases}$$

for arbitrary nonnegative $\phi, \psi \in C_0^\infty(\mathbb{R}^N \setminus B_{R_0})$.[1]

3.1 Nonexistence results

As was shown in Chapter 2 for the scalar equation

$$-\Delta u + |\nabla u|^q = \lambda f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0},$$

[1]The results in this chapter are contained in [18].

the presence of the gradient term allows the existence of different types of positive supersolutions. The same is true when considering positive supersolutions of (3.1.1). Thus it makes sense to distinguish between those supersolutions which blow up at infinity, that is,

$$\lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = +\infty, \tag{3.1.2}$$

uniformly and those which do not. It is worth remarking that if (u, v) is a positive supersolution of (3.1.1), then u blows up at infinity if and only if v does (see Lemma 3.2.1 below).

We begin by considering supersolutions which do not verify (3.1.2). For this kind of supersolutions, the behavior of f and g at zero is important; hence we assume that

$$\theta_1 := \liminf_{t\to 0} \frac{f(t)}{t^p} > 0, \quad \theta_2 := \liminf_{t\to 0} \frac{g(t)}{t^s} > 0, \tag{3.1.3}$$

for some $p, s > 0$.

It turns out that the existence of positive supersolutions of (3.1.1) depends on the relative values of q and the exponent $\frac{N}{N-1}$, in the same spirit as in the scalar equation. Thus we split our nonexistence results in three different regimes depending on whether $1 < q < \frac{N}{N-1}$, $q = \frac{N}{N-1}$ or $q > \frac{N}{N-1}$.

Before stating our main achievements when $1 < q \le \frac{N}{N-1}$ let us introduce, for $ps > q^2$, the positive exponents

$$\alpha = \frac{q(p+q)}{ps-q^2}, \qquad \beta = \frac{q(s+q)}{ps-q^2}, \tag{3.1.4}$$

which along with their relative position with respect to $\gamma = \frac{2-q}{q-1}$ give the regions of nonexistence of positive supersolutions. Without loss of generality, and given the symmetry of the problem, we can always assume that $p \ge s$.

Theorem 3.1. *Assume $N \ge 2$ and $1 < q < \frac{N}{N-1}$. Let $f, g : (0, +\infty) \to \mathbb{R}$ be nondecreasing, continuous and positive functions, and suppose that condition* (3.1.3) *holds with $p \ge s > 0$. Then:*

a) *If $ps \le q^2$, then there are no positive supersolutions of* (3.1.1) *which do not blow up at infinity.*

b) *If $ps > q^2$ and $\alpha > \gamma$, then there are no positive supersolutions of* (3.1.1) *which do not blow up at infinity.*

c) *If* $ps > q^2$, $\alpha = \beta = \gamma$, *and any pair of positive numbers* $(t_1, t_2) > 0$ *verifies at least one of the following inequalities*

$$\begin{aligned} \alpha(N-2-\alpha)t_1 + \alpha^q t_1^q &< \lambda\theta_1 t_2^p, \\ \beta(N-2-\beta)t_2 + \beta^q t_2^q &< \mu\theta_2 t_1^s, \end{aligned} \tag{3.1.5}$$

then there are no positive supersolutions of (3.1.1) *which do not blow up at infinity.*

d) *If* $ps > q^2$, $\beta < \alpha = \gamma$, *and any pair of positive numbers* $(t_1, t_2) > 0$ *verifies at least one of the following inequalities*

$$\left\{ \begin{aligned} \alpha(N-2-\alpha)t_1 + \alpha^q t_1^q &< \lambda\theta_1 t_2^p, \\ \beta^q t_2^q &< \mu\theta_2 t_1^s, \end{aligned} \right. \tag{3.1.6}$$

then there are no positive supersolutions of (3.1.1) *which do not blow up at infinity.*

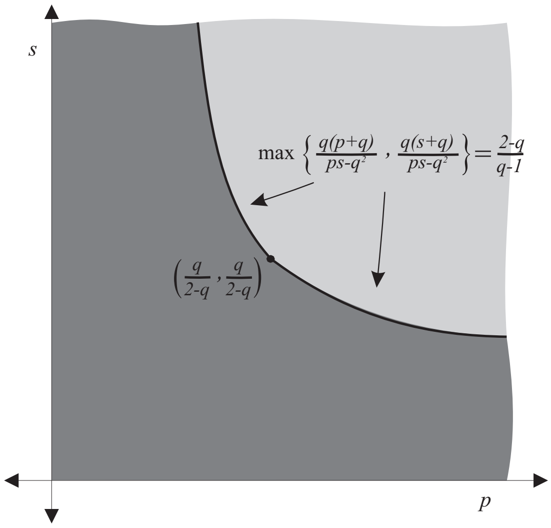

Figure 3.1. Regions when $1 < q < \frac{N}{N-1}$. Nonexistence in dark grey, existence in light grey and nonexistence depending on λ and μ on the black curve.

Theorem 3.2. *Assume* $N \geq 3$ *and* $q = \frac{N}{N-1}$. *Let* $f, g : (0, +\infty) \to \mathbb{R}$ *be nondecreasing, continuous and positive functions, and suppose that condition* (3.1.3) *holds with* $p \geq s$. *Then:*

a) *If* $ps \leq \left(\frac{N}{N-1}\right)^2$, *then there are no positive supersolutions of* (3.1.1) *which do not blow up at infinity.*

b) *If* $ps > \left(\frac{N}{N-1}\right)^2$ *and* $\alpha > N-2$ *then there are no positive supersolutions of* (3.1.1) *which do not blow up at infinity.*

We consider next the remaining case $q > \frac{N}{N-1}$. The situation is slightly more complicated here, and nonexistence results depend on the relation between two pairs of exponents, which are

$$\overline{\alpha} = \frac{2(p+1)}{ps-1}, \qquad \overline{\beta} = \frac{2(s+1)}{ps-1}, \tag{3.1.7}$$

and

$$\widehat{\alpha} = \frac{q(p+2)}{ps-q}, \qquad \widehat{\beta} = \frac{q+2s}{ps-q}, \tag{3.1.8}$$

respectively. It is worth remarking that the latter exponents play a role only in the regime $ps > q$.

Remark 3.1.1. Observe that the exponents $\widehat{\alpha}$, $\widehat{\beta}$ are not symmetric if we interchange p and s. This is due to the fact that we are assuming $p \geq s$. When $p < s$ then the exponents $\widehat{\alpha} = \frac{q(p+2)}{ps-q}$, $\widehat{\beta} = \frac{q+2s}{ps-q}$ have to be replaced by $\widehat{\alpha}' = \frac{q+2p}{ps-q}$, $\widehat{\beta}' = \frac{q(s+2)}{ps-q}$.

Theorem 3.3. *Assume* $N \geq 3$ *and* $q > \frac{N}{N-1}$. *Let* $f, g : (0, +\infty) \to \mathbb{R}$ *be nondecreasing, continuous and positive functions, and suppose that condition* (3.1.3) *holds with* $p \geq s$. *Then:*

a) *If* $ps \leq 1$, *then there are no positive supersolutions of* (3.1.1) *which do not blow up at infinity.*

b) *If* $ps > 1$, $\overline{\alpha} \geq N-2$ *and* $\overline{\beta} \geq \gamma$, *then there are no positive supersolutions of* (3.1.1) *which do not blow up at infinity.*

c) *If* $ps > 1$, $\overline{\alpha} \geq N-2$, $\overline{\beta} < \gamma$ *and* $\widehat{\alpha} \geq N-2$, *then there are no positive supersolutions of* (3.1.1) *which do not blow up at infinity.*

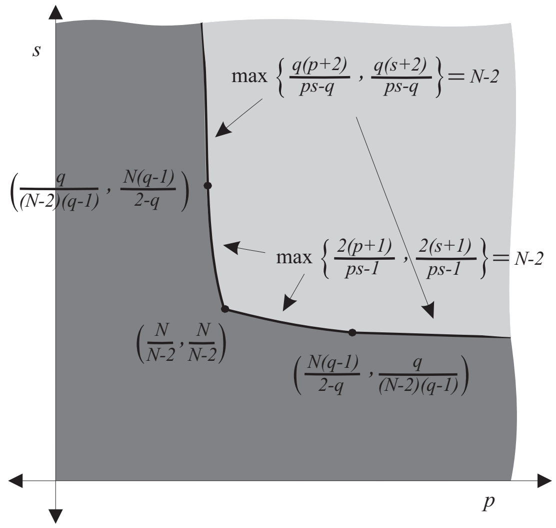

Figure 3.2. Regions when $q > \frac{N}{N-1}$. Nonexistence in dark grey and existence in light grey.

Remark 3.1.2.
a) It will be seen that $ps > 1$ and $\overline{\beta} \leq \gamma$ imply $ps > q$, so the exponents in (3.1.8) are well-defined and positive (see the beginning of the proof of Lemma 3.6.2 below).
b) The case $N = 2$ is excluded from Theorems 3.2 and 3.3 since all positive supersolutions of the system (3.1.1) when $N = 2$ and $q \geq 2$ verify (3.1.2), see the proof of Theorem 2.5 in Chapter 2. Thus it will be included in our next theorem.

Finally, we consider positive supersolutions which verify (3.1.2). In this case, only the behavior of f and g at infinity is important. We assume that

$$\liminf_{t\to+\infty} \frac{f(t)}{t^p} > 0, \quad \liminf_{t\to+\infty} \frac{g(t)}{t^s} > 0, \tag{3.1.9}$$

for some $p, s > 0$.

Theorem 3.4. *Assume $N \geq 2$. Let $f, g : (0, +\infty) \to \mathbb{R}$ be nondecreasing, continuous and positive functions, and suppose that condition* (3.1.9) *holds for some $p, s > 0$. If $ps > q^2$, then there are no positive supersolutions of* (3.1.1) *which blow up at infinity.*

It is important to remark that our results are essentially optimal, in the sense that if $f(t) = t^p$ and $g(t) = t^s$, $p, s > 0$ then supersolutions can be explicitly constructed in all cases not covered by Theorems 3.1, 3.2, 3.3 and 3.4; see Section 3.9. The same is true with the assumptions

$$\limsup_{t\to 0} \frac{f(t)}{t^p} < +\infty, \ \limsup_{t\to 0} \frac{g(t)}{t^s} < +\infty$$

for supersolutions not blowing up at infinity and

$$\limsup_{t\to +\infty} \frac{f(t)}{t^p} < +\infty, \ \limsup_{t\to +\infty} \frac{g(t)}{t^s} < +\infty$$

for those which do.

It is also worthy of mention that the main differences between system (3.1.1) and equation (2.1.1) appear in the range $q > \frac{N}{N-1}$. With regard to the equation, the gradient term does not play any role when $q > \frac{N}{N-1}$. That is, the region of nonexistence depends on the relative position of the critical exponent $\frac{2}{p-1}$ with respect to $N-2$, and it does not depend on q (see [2]). But when dealing with system (3.1.1) there are two exponents involved, given in (3.1.7). If one of them is greater than $N-2$, then it is still relevant if the other one is greater than or less than γ. The appearance of the exponents $\widehat{\alpha}$, $\widehat{\beta}$ in (3.1.8) is a consequence of this phenomenon. According to Theorem 3.3, the gradient is not relevant when $q \geq 2$ for the system.

3.2 Classification of supersolutions

We introduce a similar notation as the one in Chapter 1. For a function $z \in C^1(\mathbb{R}^N \setminus B_{R_0})$, we denote

$$m_z(R) = \min_{|x|=R} z(x), \qquad R > R_0, \tag{3.2.1}$$

since it is convenient to explicitly express the dependence of $m_z(R)$ on the function z. According to Lemma 1 in [2] we obtain that there exists $R_1 > R_0$ such that the function $m_z(R)$ is monotone for $R > R_1$. We will assume troughout that if a positive supersolution (u, v) of (3.1.1) exist, then R_0 has been chosen such that $m_u(R)$ and $m_v(R)$ are monotone for $R > R_0$.

The following lemma allows us to distinguish between supersolutions which blow up at infinity, that is, which verify (3.1.2) and supersolutions which do not blow up. The proof is based on a device introduced in [27] and later refined in [35].

Lemma 3.2.1. *Assume $q > 1$, and let f, g be continuous and nondecreasing positive functions defined in $(0, +\infty)$. If $(u, v) \in C^1(\mathbb{R}^N \setminus B_{R_0})^2$ is a positive supersolution of (3.1.1) and $m_u(R)$, $m_v(R)$ are given by (3.2.1), then, either*

a) *The functions* m_u *and* m_v *are bounded, nonincreasing and converge to zero as* $R \to \infty$. *In addition, if* f *and* g *verify* (3.1.3) *then there exists* $C > 0$ *such that*

$$\begin{aligned} m_v^p(2R) &\leq C\left(\frac{m_u(R)}{R^2} + \frac{m_u^q(R)}{R^q}\right), \\ m_u^s(2R) &\leq C\left(\frac{m_v(R)}{R^2} + \frac{m_v^q(R)}{R^q}\right), \end{aligned}$$

for $R > R_0$.

Or,

b) *The functions* $m_u(R)$ *and* $m_v(R)$ *are nondecreasing and diverge to infinity when* $R \to \infty$.

Proof. Choose a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi = 0$ in $(-\infty, 1) \cup (4, +\infty)$ and $\phi = 1$ in $[2, 3]$. For $R > R_0$ consider the function

$$w(x) = u(x) - m_u(2R)\phi\left(\frac{|x|}{R}\right) \quad |x| > R_0.$$

Observe that there exists a point $x_R \in \partial B_{2R}$ such that $w(x_R) = 0$. Since $w > 0$ in $\mathbb{R}^N \setminus B_{4R}$ and in $B_R \setminus B_{R_0}$, then w achieves a nonpositive minimum at some point $y_R \in \overline{B_{4R}} \setminus B_R$. This implies $\Delta w(y_R) \geq 0$ and $\nabla w(y_R) = 0$, so that

$$\begin{aligned} -\Delta u(y_R) + |\nabla u(y_R)|^q \leq & -\frac{m_u(2R)}{R^2}\Delta\phi\left(\frac{|x|}{R}\right) \\ & + \frac{m_u^q(2R)}{R^q}\left|\nabla\phi\left(\frac{|x|}{R}\right)\right|^q \leq C\left(\frac{m_u(2R)}{R^2} + \frac{m_u^q(2R)}{R^q}\right), \end{aligned}$$

where C is a positive constant which depends on ϕ (we still use the convention that the letter C denotes positive constants, not necessarily the same everywhere). Since (u, v) is a supersolution of (3.1.1) it follows that

$$f(v(y_R)) \leq C\left(\frac{m_u(2R)}{R^2} + \frac{m_u^q(2R)}{R^q}\right). \tag{3.2.2}$$

Assume first that $m_u(R)$ is bounded. Then (3.2.2) gives $\lim_{R\to+\infty} f(v(y_R)) = 0$. Since f is positive and nondecreasing in $(0, +\infty)$, we have

$$\lim_{R\to+\infty} v(y_R) = 0.$$

By using the monotonicity of m_v, we deduce that m_v is nonincreasing for large R and $\lim_{R\to+\infty} m_v(R) = 0$. Arguing in a similar way for v it follows that also $\lim_{R\to+\infty} m_u(R) =$

0 and m_u is nondecreasing for large R. This shows the first part of a). Moreover, if f and g verify (3.1.3) we see that $f(v(y_R)) \geq Cv^p(y_R) \geq Cm_v^p(4R)$. Hence

$$m_v^p(4R) \leq C\left(\frac{m_u(2R)}{R^2} + \frac{m_u^q(2R)}{R^q}\right)$$

for large R. In the same way

$$m_u^s(4R) \leq C\left(\frac{m_v(2R)}{R^2} + \frac{m_v^q(2R)}{R^q}\right)$$

for large R, and the proof of a) follows by replacing $2R$ by R and enlarging C if necessary.

The remaining case, contained in part b), arises when both m_u and m_v are unbounded. But then by monotonicity we immediately have that m_u and m_v are nondecreasing for large R, so that $\lim_{R\to+\infty} m_u(R) = \lim_{R\to+\infty} m_v(R) = +\infty$. This concludes the proof. $\square$

Remark 3.2.2.
a) The inequalites given by Lemma 3.2.1 are a byproduct of our proof, but they will not be used in obtaining lower or upper estimates for positive supersolutions. However, a version of these inequalites is useful in the proof of one of our auxiliary results, Theorem 3.5 in Section 3.6.
b) Observe that the proof of Lemma 3.2.1 remains valid when $q = 1$.

3.3 Reduction to the radial setting

In this section we show that the existence of a positive supersolution of (3.1.1) implies the existence of a positive radially symmetric supersolution, and in most cases even the existence of a positive radially symmetric *solution*.

Our first result in this direction is the analogue of Lemma 2.2.1 in Chapter 2.

Lemma 3.3.1. *Assume $N \geq 2$, $q > 1$ and let $f, g : (0, +\infty) \to \mathbb{R}$ be positive, nondecreasing and continuous functions. Suppose there exists a positive supersolution (u, v) of* (3.1.1)*. Then there exists a positive radial supersolution $(z, w) \in C^2(\mathbb{R}^N \setminus B_{R_0})^2$ of* (3.1.1)*. In addition, if (u, v) does not blow up at infinity, then z, w, z' and w' go to 0 as $R \to +\infty$, and if (u, v) blows up at infinity, then (z, w) also blows up at infinity.*

Proof. Choose $R_1 > R_0$ and introduce the annulus $A(R_0, R_1) := \{x \in \mathbb{R}^N : \ R_0 < |x| < R_1\}$. For $R > R_0$, we take the functions m_u, m_v given by (3.2.1) and we consider the uncoupled problems

$$\begin{cases} -\Delta z + |\nabla z|^q = \lambda f(m_v(|x|)) & \text{in } A(R_0, R_1), \\ z = 0 & \text{on } \partial A(R_0, R_1), \end{cases} \tag{3.3.1}$$

and
$$\begin{cases} -\Delta w + |\nabla w|^q = \mu g(m_u(|x|)) & \text{in } A(R_0,R_1), \\ w = 0 & \text{on } \partial A(R_0,R_1). \end{cases} \tag{3.3.2}$$
The function 0 is a subsolution for both problems. Then by Theorem III.1 in [54], there exist unique positive solutions $z_{R_1}, w_{R_1} \in W^{2,\theta}(A(R_0,R_1))$ of (3.3.1) and (3.3.2), respectively, where $\theta > N$ is arbitrary. Note that the functions z_{R_1}, w_{R_1} are radially symmetric because of uniqueness.

Setting $\widehat{z}_{R_1} = \min\{m_u(R_0), m_u(R_1)\} + z_{R_1}$, $\widehat{w}_{R_1} = \min\{m_v(R_0), m_v(R_1)\} + w_{R_1}$, we obtain that $\widehat{z}_{R_1}$ and $\widehat{w}_{R_1}$ are radially symmetric solutions of
$$\begin{cases} -\Delta z + |\nabla z|^q = \lambda f(m_v(|x|)) & \text{in } A(R_0,R_1), \\ z = \min\{m_u(R_0), m_u(R_1)\} & \text{on } \partial A(R_0,R_1), \end{cases} \tag{3.3.3}$$
and
$$\begin{cases} -\Delta w + |\nabla w|^q = \mu g(m_u(|x|)) & \text{in } A(R_0,R_1), \\ w = \min\{m_v(R_0), m_v(R_1)\} & \text{on } \partial A(R_0,R_1), \end{cases} \tag{3.3.4}$$
respectively. Next, observe that the monotonicity of f and g implies that $f(u) \geq f(m_u(|x|))$ and $g(v) \geq g(m_v(|x|))$, so that u is a supersolution of (3.3.3) and v is a supersolution of (3.3.4). Then by comparison $u \geq \widehat{z}_{R_1}$ and $v \geq \widehat{w}_{R_1}$ in $A(R_0,R_1)$, so that the sets $\{\widehat{z}_{R_1}\}_{R_1>R_0}$ and $\{\widehat{w}_{R_1}\}_{R_1>R_0}$ are bounded.

By part 2 of Theorem A.1 in [51] we obtain local bounds for $\{\nabla \widehat{z}_{R_1}\}_{R_1>R_0}$ and $\{\nabla \widehat{w}_{R_1}\}_{R_1>R_0}$ in L^θ for every $\theta > 1$. This yields that $\{\Delta \widehat{z}_{R_1}\}_{R_1>R_0}$ and $\{\Delta \widehat{w}_{R_1}\}_{R_1>R_0}$ are locally bounded in L^θ for every $\theta > 1$. Hence by classical regularity and Sobolev embeddings it turns out that $\{\widehat{z}_{R_1}\}_{R_1>R_0}$ and $\{\widehat{w}_{R_1}\}_{R_1>R_0}$ are locally bounded in $C^{1,\alpha}$ for every $\alpha \in (0,1)$. By means of a diagonal argument we thus obtain a sequence $\{R_{1,n}\}_{n\geq 1}$ such that $\lim_{n\to+\infty} R_{1,n} = +\infty$, $\lim_{n\to+\infty} \widehat{z}_{R_{1,n}} = z$ and $\lim_{n\to+\infty} \widehat{w}_{R_{1,n}} = w$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \overline{B_{R_0}})$.

If we denote $l_u = \lim_{R\to+\infty} m_u(R)$ and $l_v = \lim_{R\to+\infty} m_v(R)$, then z is a radially symmetric solution of
$$-\Delta z + |\nabla z|^q = \lambda f(m_v(|x|)) \quad \text{in } \mathbb{R}^N \setminus B_{R_0},$$
with $\min\{m_u(R_0), l_u\} \leq z \leq m_u$, and w is a radially symmetric solution of
$$-\Delta w + |\nabla w|^q = \mu g(m_u(|x|)) \quad \text{in } \mathbb{R}^N \setminus B_{R_0},$$
with $\min\{m_v(R_0), l_v\} \leq w \leq m_v$. By bootstrapping and using that z, w are radially symmetric, we deduce that z, w are indeed classical solutions. Since f and g are nondecreasing we obtain that $f(m_v(|x|)) \geq f(w)$ and $g(m_u(|x|)) \geq g(z)$, so that (z, w) is a

radially symmetric nonnegative supersolution of the system

$$\begin{cases} -\Delta z + |\nabla z|^q &= \lambda f(w), \\ -\Delta w + |\nabla w|^q &= \mu g(z), \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}. \tag{3.3.5}$$

The strong maximum principle implies $z, w > 0$ (observe that z, w are nontrivial). This shows the first part of the lemma.

Now assume (u, v) does not blow up at infinity. By Lemma 3.2.1.a), $l_u = l_v = 0$. Since $z \le m_u$, $w \le m_v$ then $\lim_{r\to+\infty} z(r) = \lim_{r\to+\infty} w(r) = 0$. Let us show that $z'(r) \to 0$ as $r \to +\infty$.

Take an arbitrary sequence $\{x_n\} \subset \mathbb{R}^N \setminus B_{R_0}$ with $|x_n| \to +\infty$ and consider the function $\overline{z}_n = z(x_n + y)$ for $|y| < |x_n| - R_0$. The function $\overline{z}_n$ verifies the equation $-\Delta \overline{z}_n + |\nabla \overline{z}_n|^q = \lambda f(m_v(|x_n + y|))$. Arguing as above we see that, passing to a subsequence, $\overline{z}_n \to \overline{z}$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ with $-\Delta \overline{z} + |\nabla \overline{z}|^q = 0$. Using that $\overline{z}_n(0) = z(x_n) \to 0 = \overline{z}(0)$, the strong maximum principle yields $\overline{z} = 0$ in $\mathbb{R}^N$, thus $\nabla z(x_n) = \nabla z_n(0) \to 0$. Since the sequence $\{x_n\}$ was arbitrary we have shown that $z(r)$ and $z'(r)$ go to 0 when $r \to +\infty$, and the same is true for w and w' with a similar proof.

Finally, suppose (u, v) blows up at infinity. By Lemma 3.2.1.a), $l_u = l_v = +\infty$. Since $m_u(R_0) = \min\{m_u(R_0), l_u\} \le z$, $m_v(R_0) = \min\{m_v(R_0), l_v\} \le w$ then z, w are bounded from below. Hence by Lemma 3.2.1, we get that (z, w) must blow up at infinity. $\square$

Remark 3.3.2. The fact that z, z', w, w' go to 0 as $r \to +\infty$ for positive radial supersolutions which do not blow up at infinity allows us to assume that $q < 2$ for the proofs of nonexistence, since if (z, w) is a supersolution of (3.1.1) then it is a supersolution of

$$\begin{cases} -\Delta z + |\nabla z|^{q'} &= \lambda f(w) \\ -\Delta w + |\nabla w|^{q'} &= \mu g(z) \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_1}$$

for every $q' < q$, if R_1 is large enough.

Lemma 3.3.3. *Assume $N \ge 2$, $q > 1$ and if $N = 2$ additionally assume $q \le 2$. Let $f, g : (0, +\infty) \to \mathbb{R}$ be positive, nondecreasing and continuous functions verifying* (3.1.3) *and suppose there exists a positive supersolution (u, v) of* (3.1.1) *which does not blow up at infinity. Then for every $\nu_1 \in (0, \theta_1)$, $\nu_2 \in (0, \theta_2)$, there exist $R_1 > R_0$ and a positive, bounded, radially symmetric solution (z, w) of*

$$\begin{cases} -\Delta z + |\nabla z|^q &= \lambda \nu_1 w^p \\ -\Delta w + |\nabla w|^q &= \mu \nu_2 z^s, \end{cases} \qquad \textit{in } \mathbb{R}^N \setminus B_{R_1}. \tag{3.3.6}$$

Proof. By Lemma 3.3.1 there exists a positive, bounded, radially symmetric supersolution $(\overline{u}, \overline{v})$ of (3.1.1) verifying $\lim_{r\to+\infty} \overline{u} = \lim_{r\to+\infty} \overline{v} = 0$. Taking $\nu_1 \in (0,\theta_1)$, $\nu_2 \in (0,\theta_2)$, we may use condition (3.1.3) to deduce that there exists $R_0' > R_0$ such that $f(v) \geq \nu_1 v^p$, $g(u) \geq \nu_2 u^s$ if $r > R_0'$. This shows that $(\overline{u}, \overline{v})$ is actually a supersolution of (3.3.6), for every $R_1 > R_0'$.

We now use Lemma 2.2.2: let $M = \sup_{r\geq 2R_0} \overline{u}(r) + \overline{v}(r)$. Then there exists $\widetilde{R} > R_0'$ (depending on M) such that for every $R_1, R_2 > \widetilde{R}$ with $R_2 > 2R_1$, we can find radially symmetric, positive functions $\underline{u}_{R_2}, \underline{v}_{R_2}$ verifying

$$\begin{cases} -\Delta z + |\nabla z|^q & \leq & 0 & \text{in } A(R_1,R_2), \\ z & = & \overline{u} & \text{on } \partial A(R_1,R_2), \end{cases}$$

and

$$\begin{cases} -\Delta z + |\nabla z|^q & \leq & 0 & \text{in } A(R_1,R_2), \\ z & = & \overline{v} & \text{on } \partial A(R_1,R_2), \end{cases}$$

respectively. Observe that, since $\overline{u}(R_1) > \overline{u}(R_2)$, we deduce from the maximum principle that $\underline{u}_{R_2} \leq \overline{u}(R_1)$ in $A(R_1,R_2)$, hence $\underline{u}'_{R_2}(R_1) \leq 0$, and in a similar fashion $\underline{v}'_{R_2}(R_1) \leq 0$. From now on, we fix $R_1 > \widetilde{R}$ and take R_2 such that $R_2 > 2R_1$.

The comparison principle yields $\underline{u}_{R_2} \leq \overline{u}$, $\underline{v}_{R_2} \leq \overline{v}$ in $A(R_1,R_2)$, so we are in a position to use the method of sub and supersolutions, Theorem A.4 in the Appendix, to obtain a radial solution (z_{R_2}, w_{R_2}) of the problem

$$\begin{cases} -\Delta z + |\nabla z|^q & = & \lambda \nu_1 w^p, & \\ -\Delta w + |\nabla w|^q & = & \mu \nu_2 z^s, & \text{in } A(R_1,R_2), \\ z & = & \overline{u}, & \\ w & = & \overline{v}, & \text{on } \partial A(R_1,R_2). \end{cases} \tag{3.3.7}$$

Since $0 \leq z_{R_2} \leq \overline{u}$ and $0 \leq w_{R_2} \leq \overline{v}$ we get global bounds for $\{z_{R_2}\}$ and $\{w_{R_2}\}$.

Arguing in a similar way as in Lemma 2.2.3, we obtain that (for some sequence $R_2 \to +\infty$) $z_{R_2} \to z$, $w_{R_2} \to w$, where (z,w) is a positive solution of (3.1.1). This concludes the proof. □

3.4 Lower bounds

In this section we provide some new lower estimates for radially symmetric, bounded positive supersolutions of some equations involving the operator $-\Delta u + |\nabla u|^q$. In Section 2.3 some lower estimates were obtained for positive radial supersolutions of the equation

$$-\Delta u + |\nabla u|^q = 0 \text{ in } \mathbb{R}^N \setminus B_{R_0}. \tag{3.4.1}$$

However, the estimates given in Lemma 2.3.3 can be improved when the right-hand side in (3.4.1) is a power function, rather than zero. Since estimates for the gradient will not be used in the arguments to follow, we give a nonradial proof, which is based on a Hadamard three circles type argument, in the spirit of [27] (see also [2]). We recall that $\gamma = \frac{2-q}{q-1}$.

Lemma 3.4.1. *Assume $q > 1$, $C > 0$ and let u be a positive function verifying*

$$-\Delta u + |\nabla u|^q \geq C|x|^{-\theta-2} \ \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{3.4.2}$$

where $\gamma \leq \theta < N-2$. Then there exists a positive constant $C' > 0$ such that

$$u(x) \geq C'|x|^{-\theta} \qquad \text{for } |x| > R_0.$$

Proof. We first remark that since $\gamma \leq \theta < N-2$, then the function $\psi(x) = A|x|^{-\theta} + B$ is a subsolution of (3.4.2) if $A > 0$ is small enough and $B \in \mathbb{R}$. Actually it suffices that $0 < A \leq A_0$, where A_0 is the unique positive solution of the equation $\theta A_0(N-2-\theta) + \theta^q A_0^q R_0^{-q(\theta+1)+\theta+2} = C$.

The proof of the lemma reduces to show that $\liminf_{|x|\to+\infty} m_u(|x|)|x|^\theta > 0$. If this were not true, there would exist a sequence $R_n \to +\infty$ such that $m_u(R_n)R_n^\theta \to 0$. Choosing n_0 such that $m_u(R_{n_0})R_{n_0}^\theta < A_0$ then the function

$$\phi(x) = \frac{m_u(R_{n_0}) - m_u(R_n)}{R_{n_0}^{-\theta} - R_n^{-\theta}}(|x|^{-\theta} - R_n^{-\theta}) + m_u(R_n)$$

is a subsolution of (3.4.2) in the annulus $A(R_{n_0}, R_n)$ for large enough n, with $\phi = m_u$ on $\partial A(R_{n_0}, R_n)$. By comparison we get $\phi \leq u$ in $A(R_{n_0}, R_n)$. Letting $n \to +\infty$, we obtain

$$m_u(|x|)|x|^\theta \geq m_u(R_{n_0})R_{n_0}^\theta \ \text{ in } \ |x| > R_{n_0}.$$

Taking x_j such that $|x_j| = R_j$ and letting $j \to +\infty$ we arrive at a contradiction. Hence $\liminf_{R\to+\infty} m_u(R)R^\theta > 0$, which immediately gives $u(x) \geq C'|x|^{-\theta}$. $\square$

Lemma 3.4.2. *Assume $1 < q < 2$, $C > 0$ and let u be a positive function verifying*

$$-\Delta u + |\nabla u|^q \geq C|x|^{-(\theta+1)q} \ \ \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{3.4.3}$$

where $\theta \leq \gamma$. Then there exists a positive constant $C' > 0$ such that

$$u(x) \geq C'|x|^{-\theta}, \qquad \text{for } |x| > R_0.$$

Proof. The proof is essentially the same as that of Lemma 3.4.1. The only significative difference is that, since $\theta \leq \gamma$, then the function $\psi(x) = A|x|^{-\theta} + B$ is a subsolution of (3.4.3) if $0 < A \leq A_0$ where A_0 verifies $\theta A_0(N-2-\theta)R_0^{-\theta-2+(\theta+1)q} + \theta^q A_0^q = C$, and $B \in \mathbb{R}$. $\square$

Remark 3.4.3. When
$$-\Delta u \geq C|x|^{-N} \quad \text{in } \mathbb{R}^N \setminus B_{R_0},$$
it can be proved in a similar way as in Lemma 3.4.1 that $u(x) \geq C|x|^{2-N} \log|x|$ for some $C > 0$ and large $|x|$. We only have to take into account that $\phi(x) = A|x|^{2-N}\log|x|$ is a subsolution of the equation for suitable values of A (see [27]). This lower estimate will be used later on, during the proof of Theorem 3.5.

3.5 Upper bounds when $1 < q < \frac{N}{N-1}$

In this section we obtain upper estimates for bounded radial solutions (z, w) of the system (3.1.1) when $1 < q < \frac{N}{N-1}$. The proof is inspired on a device introduced in [69], which has already been used in some proofs in Chapter 2.

Lemma 3.5.1. *Assume $N \geq 2$ and $1 < q < \frac{N}{N-1}$. Suppose that (z, w) is a positive radial solution of the system*
$$\left\{ \begin{array}{rcl} -z'' - \dfrac{N-1}{r} z' + |z'|^q & = & cw^p, \\ -w' - \dfrac{N-1}{r} w' + |w'|^q & = & dz^s, \end{array} \right. \qquad \text{in } (R_0, +\infty), \tag{3.5.1}$$
where $p, s > 0$ and $c, d > 0$. Then:

a) The case $ps < q^2$ is not possible.

b) If $ps = q^2$, then there exist positive constants $C_1, C_2 > 0$ such that
$$z \leq C_1 r^{-\frac{p}{q}} e^{-C_2 r}, \quad w \leq C_1 r^{-\frac{s}{q}} e^{-C_2 r}, \qquad r \geq R_0.$$

c) If $ps > q^2$, then there exists a positive constant $C > 0$ such that
$$z \leq C r^{-\alpha}, \quad w \leq C r^{-\beta}, \qquad r \geq R_0,$$
where $\alpha = \frac{q(p+q)}{ps-q^2}$ and $\beta = \frac{q(s+q)}{ps-q^2}$.

Proof. We introduce the functions

$$H_1(r) = cw^p(r) - a|z'(r)|^q,$$
$$H_2(r) = dz^s(r) - a|w'(r)|^q,$$

for $r > R_0$ where $a > 1$ will be chosen later. Let $r_0 > R_0$ be any point where $H_1(r_0) = 0$, i.e. $cw^p(r_0) = a|z'(r_0)|^q$. Using that w is nonincreasing, we obtain that

$$\begin{aligned} H_1'(r_0) &= pcw^{p-1}(r_0)w'(r_0) + aq|z'(r_0)|^{q-1}\left(\frac{N-1}{r_0}|z'(r_0)| + |z'(r_0)|^q - cw(r_0)^p\right) \\ &\le aq|z'(r_0)|^q\left(\frac{N-1}{r_0} - (a-1)|z'(r_0)|^{q-1}\right). \end{aligned} \tag{3.5.2}$$

As a consequence of the lower estimate for $|z'|$ given by part a) of Lemma 2.3.3 we have $|z'(r_0)|^{q-1} \ge Cr_0^{-1}$, so that

$$H_1'(r_0) \le \frac{aq|z'(r_0)|^q}{r_0}(N-1-(a-1)C) < 0,$$

if we choose, say $a = (N-1)/C + 2$. It follows that H_1 may vanish at most once, so it has constant sign for large r. It can then be shown as in the proof of Theorem 2.4 that $H_1 < 0$ for large r and similarly $H_2 < 0$.

Therefore we have

$$\begin{cases} -z'(r) > \nu w^{\frac{p}{q}}(r), \\ -w'(r) > \nu z^{\frac{s}{q}}(r), \end{cases}$$

for large enough r and some $\nu > 0$. We introduce the change of variable $Z(t) = z(r)$ and $W(t) = w(r)$, where

$$t = H(r) := \int_r^{+\infty} \nu w^{\frac{p}{q}}(\tau)d\tau.$$

The function H is well defined since

$$\int_r^{+\infty} \nu w^{\frac{p}{q}}(\tau)d\tau < \int_r^{+\infty} -z'(\tau)d\tau = z(r) < +\infty. \tag{3.5.3}$$

Observe that Z and W are defined in $(0,\delta)$ for some $\delta > 0$ and verify $Z(0) = W(0) = 0$. By (3.5.3), we see that $Z(t) > t$. In addition,

$$W'(t) = -\frac{w'(r)}{\nu w^{\frac{p}{q}}(r)} > \frac{z^{\frac{s}{q}}(r)}{w^{\frac{p}{q}}(r)} = \frac{Z^{\frac{s}{q}}(t)}{W^{\frac{p}{q}}(t)} > \frac{t^{\frac{s}{q}}}{W^{\frac{p}{q}}(t)}.$$

Hence, $W^{\frac{p}{q}}(t)W'(t) > t^{\frac{s}{q}}$ and integrating between 0 and t we obtain

$$W(t) \ge Ct^{\frac{s+q}{p+q}}, \ t \in (0,\delta).$$

Then coming back to the original variables this means that $w(r) \geq CH^{\frac{s+q}{p+q}}(r)$ for r larger than some r_0. Since $H'(r) = -w^{\frac{p}{q}}(r)$, this yields

$$-H'(r) = w^{\frac{p}{q}}(r) \geq CH^{\frac{p(s+q)}{q(p+q)}}(r),\ r \geq r_0$$

Integrating between r_0 and r we get

$$-\int_{r_0}^{r} H^{-\frac{p(s+q)}{q(p+q)}}(\tau)H'(\tau)d\tau \geq C(r-r_0). \tag{3.5.4}$$

We have to distinguish three cases depending on the relative position between the exponent $\frac{p(s+q)}{q(p+q)}$ and 1.
First case: $\frac{p(s+q)}{q(p+q)} < 1$. This condition is equivalent to $ps < q^2$. By (3.5.4),

$$-\frac{H^{1-\frac{p(s+q)}{q(p+q)}}(r) - H^{1-\frac{p(s+q)}{q(p+q)}}(r_0)}{1-\frac{p(s+q)}{q(p+q)}} \geq C(r-r_0). \tag{3.5.5}$$

Letting $r \to +\infty$ and taking into account that $H(r) \to 0$ we arrive at a contradiction. This shows a).
Second case: $\frac{p(s+q)}{q(p+q)} = 1$. This condition is equivalent to $ps = q^2$. By (3.5.4),

$$-\log H(r) + \log H(r_0) \geq C(r-r_0).$$

Thus, $H(r) \leq e^{-Cr}$. In addition,

$$H(r) = \int_r^{+\infty} \nu w^{\frac{p}{q}}(\tau)d\tau \geq \nu \int_r^{2r} w^{\frac{p}{q}}(\tau)d\tau \geq \nu r w^{\frac{p}{q}}(2r), \tag{3.5.6}$$

so that $w^{\frac{p}{q}}(2r) \leq C_1 r^{-1} e^{-C_2 r}$. A similar reasoning gives the upper bound for z. This shows b).
Third case: $\frac{p(s+q)}{q(p+q)} > 1$. This condition is equivalent to $ps > q^2$. By (3.5.4),

$$H^{1-\frac{p(s+q)}{q(p+q)}}(r) \geq Cr,$$

hence $H(r) \leq Cr^{-\frac{q(p+q)}{ps-q^2}}$. Since $H(r) \geq \nu r w^{\frac{p}{q}}(2r)$, by (3.5.6) we have

$$w^{\frac{p}{q}}(2r) \leq Cr^{-\frac{q(p+q)}{ps-q^2}-1} = Cr^{-\frac{p(s+q)}{ps-q^2}},$$

thus $w(r) \leq Cr^{-\frac{q(s+q)}{ps-q^2}} = Cr^{-\beta}$. A similar reasoning gives the upper bound for z. This concludes the proof. $\square$

Remark 3.5.2. An examination of the proof above shows that the upper estimates just obtained are still valid for positive radially symmetric, bounded *supersolutions* of (3.5.1). Specifically, the only significant change would appear in (3.5.2), which would read as

$$H_1'(r_0) \quad \leq \quad pcw^{p-1}(r_0)w'(r_0) + aq|z'(r_0)|^{q-1}\left(\frac{N-1}{r_0}|z'(r_0)| + |z'(r_0)|^q - cw(r_0)^p\right),$$

and the rest goes without any change.

3.6 Upper bounds when $q > \frac{N}{N-1}$

This section is devoted to obtain upper estimates for positive solutions of (3.1.1) which do not blow up at infinity when $q > \frac{N}{N-1}$. In contrast with the previous section, the fundamental tool here will be the doubling lemma in [63]. This will allow in turn to deal with general solutions in some cases and not merely with radially symmetric ones.

We assume throughout the section without loss of generality that

$$p \geq s.$$

We also assume $ps > 1$. The following exponents will be relevant in our discussion:

$$\overline{\alpha} = \frac{2(p+1)}{ps-1},\ \overline{\beta} = \frac{2(s+1)}{ps-1},\ \widehat{\alpha} = \frac{q(p+2)}{ps-q} \text{ and } \widehat{\beta} = \frac{q+2s}{ps-q}.$$

Recall that $\gamma = \frac{2-q}{q-1}$.

Lemma 3.6.1. *Assume $N \geq 3$, $q > \frac{N}{N-1}$, and let $c, d > 0$. Suppose also that $\overline{\alpha} \geq N-2$ and $\overline{\beta} \geq \gamma$. If (u, v) is a positive solution of the problem*

$$\begin{cases} -\Delta u + |\nabla u|^q &= cv^p, \\ -\Delta v + |\nabla v|^q &= du^s, \end{cases} \quad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{3.6.1}$$

such that

$$\lim_{|x| \to +\infty} u(x) = \lim_{|x| \to +\infty} v(x) = 0, \tag{3.6.2}$$

then there exists a positive constant $C > 0$ such that

$$u \leq C|x|^{-\overline{\alpha}}, \quad v \leq C|x|^{-\overline{\beta}}, \quad \text{for large } |x|.$$

Our next task is to obtain a similar result as that in Lemma 3.6.1 for a different range of exponents. It is worthy of mention that the passage to the limit required in the proof is slightly more delicate, since the second order term vanishes, therefore making it difficult to deal with the limit equation in its full generality. This is the reason why we restrict ourselves again to the context of radially symmetric solutions.

Lemma 3.6.2. *Assume $N \geq 3$, $q > \frac{N}{N-1}$, and let $c, d > 0$. Suppose in addition that $\overline{\alpha} \geq N-2$, $\overline{\beta} < \gamma$ and $\widehat{\alpha} \geq N-2$. If (u, v) is a positive, bounded, radially symmetric solution of the problem*

$$\begin{cases} -\Delta u + |\nabla u|^q &= cv^p, \\ -\Delta v + |\nabla v|^q &= du^s, \end{cases} \quad \text{in } \mathbb{R}^N \setminus B_{R_0},$$

such that $\lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0$*, then there exists a positive constant* $C > 0$ *such that*

$$u \le C|x|^{-\widehat{\alpha}}, \quad v \le C|x|^{-\widehat{\beta}}, \qquad \textit{for large } |x|.$$

We have already remarked that the proof of the upper estimates just stated relies in the doubling lemma in [63]. When applying this lemma and introducing a suitable scaling we end up with positive solutions of several systems. When $\overline{\beta} > \gamma$ the system so obtained is

$$\begin{cases} -\Delta u &= cv^p, \\ -\Delta v &= du^s, \end{cases} \qquad \text{in } \mathbb{R}^N.$$

As commented in the Introduction, Liouville theorems for solutions of this system are not fully understood. In our present context, however, it suffices to have the corresponding nonexistence result for supersolutions, which has been obtained in [56] (see also [65] and [72] for references). It was shown there that if $\overline{\alpha} \ge N-2$ then no positive supersolutions exist (remember our assumption $p \ge s$).

When $\overline{\beta} = \gamma$, the system which arises is

$$\begin{cases} -\Delta u &= cv^p, \\ -\Delta v + |\nabla v|^q &= du^s, \end{cases} \qquad \text{in } \mathbb{R}^N. \tag{3.6.3}$$

We are not aware of any Liouville theorem for this system, so our next intention before coming to the proof of Lemmas 3.6.1 and 3.6.2 is to obtain one.

Theorem 3.5. *Assume* $N \ge 3$, $q > \frac{N}{N-1}$, $ps > 1$, $\overline{\alpha} \ge N-2$, $\overline{\beta} = \gamma$ *and* $c, d > 0$*. Then system* (3.6.3) *does not admit positive supersolutions.*

Proof. Let us begin with a remark about the exponents. We claim that under our assumptions we have $ps > q$. To show this, assume on the contrary that $ps \le q$. Then

$$(ps-1)(2-q) \le (q-1)(2-q) < q-1 < 2(q-1)(s+1),$$

that is, $\overline{\beta} = 2(s+1)/(ps-1) > (2-q)/(q-1) = \gamma$, which is not possible by hypothesis.

Suppose that (u, v) is a positive supersolution of (3.6.3). Next we consider the functions m_u, m_v given by (3.2.1). As a consequence of the maximum principle,

$$\inf_{|x|=R} u(x) = \inf_{|x|\le R} u(x) \quad \text{and} \quad \inf_{|x|=R} v(x) = \inf_{|x|\le R} v(x),$$

so that these functions are nonincreasing. Arguing as in the first part of the proof of Lemma 3.2.1 we obtain for positive R the following inequalities

$$\begin{aligned} m_v^p(2R) &\le C\frac{m_u(R)}{R^2}, \\ m_u^s(2R) &\le C\left(\frac{m_v(R)}{R^2} + \frac{m_v^q(R)}{R^q}\right). \end{aligned} \tag{3.6.4}$$

On the other hand, since u is superharmonic in $\mathbb{R}^N$, the function $R^{N-2}m_u(R)$ is increasing (see Corollary 3.1 in [27]; the proof of this fact is actually a modification of that in Lemma 3.4.1).

Therefore, there exists a constant $C > 0$ such that $m_u(2R) \geq Cm_u(R)$, for $R > 0$. Hence, after replacing $2R$ by R in the first inequality of (3.6.4) we obtain:

$$\begin{aligned} m_v^p(R) &\leq C\frac{m_u(R)}{R^2}, \\ m_u^s(R) &\leq C\left(\frac{m_v(R)}{R^2} + \frac{m_v^q(R)}{R^q}\right), \end{aligned} \tag{3.6.5}$$

for $R > 0$. We define

$$M_u(R) = R^{\overline{\alpha}}m_u(R) \text{ and } M_v(R) = R^{\overline{\beta}}m_v(R).$$

Then, by (3.6.5), since $\overline{\beta} + 2 = (\overline{\beta} + 1)q = \overline{\alpha}s$ and $p\overline{\beta} = \overline{\alpha} + 2$, we have

$$\begin{aligned} M_v^p(R) &\leq CM_u(R), \\ M_u^s(R) &\leq C(M_v(R) + M_v^q(R)). \end{aligned}$$

We deduce $M_v^{ps}(R) \leq C(M_v(R) + M_v^q(R))$. Since $ps > q > 1$, we see that $M_v(R)$ is bounded and therefore, $M_u(R)$ is also bounded. This means that there exists $C > 0$ such that

$$m_u(R) \leq CR^{-\overline{\alpha}},\ m_v(R) \leq CR^{-\overline{\beta}} \text{ for large } R.$$

In addition, $m_u(R) \geq CR^{2-N}$ because $R^{N-2}m_u(R)$ is increasing, so that $u(x) \geq C|x|^{2-N}$ for some $C > 0$.

It follows that $-\Delta v + |\nabla v|^q \geq C|x|^{-s(N-2)} \geq C|x|^{-\overline{\beta}-2}$, since $(N-2)s \leq \overline{\alpha}s = \overline{\beta} + 2$. By Lemma 3.4.1, we obtain that $m_v(R) \geq CR^{-\overline{\beta}}$. Thus,

$$\begin{aligned} CR^{2-N} &\leq m_u(R) \leq CR^{-\overline{\alpha}}, \\ CR^{-\overline{\beta}} &\leq m_v(R) \leq CR^{-\overline{\beta}}, \end{aligned} \tag{3.6.6}$$

for large R. If $\overline{\alpha} > N-2$ we get a contradiction, so assume that $\overline{\alpha} = N-2$. This would imply

$$-\Delta u \geq Cv^p \geq C|x|^{-p\overline{\beta}} = C|x|^{-N} \text{ in } \mathbb{R}^N \setminus B_{R_1}, \tag{3.6.7}$$

for large enough R_1. Hence, Remark 3.4.3 would give, for large R, that $m_u(R) \geq CR^{2-N}\log R$, which contradicts the upper estimate in the first line of (3.6.6) since $\overline{\alpha} = N-2$ now. The theorem is proved. □

Proof of Lemma 3.6.1. First of all, the assumption that u, v decay to zero at infinity implies that they are bounded. Then arguing as in the proof of Lemma 3.3.1 we see that both $|\nabla u|$ and $|\nabla v|$ are also bounded in, say $\mathbb{R}^N \setminus B_{2R_0}$. It also follows by that proof that $|\nabla u|$ and $|\nabla v|$ go to zero at infinity.

We claim that there exists $C > 0$ such that for any z with $|z| > 2R_0$

$$\begin{aligned} &u(x) \le C d_z^{-\overline{\alpha}}(x), && v(x) \le C d_z^{-\overline{\beta}}(x), \\ &|\nabla u(x)| \le C d_z^{-(\overline{\alpha}+1)}(x), && |\nabla v(x)| \le C d_z^{-(\overline{\beta}+1)}(x), \end{aligned} \tag{3.6.8}$$

for any $x \in B\left(z, \dfrac{|z|}{2}\right)$ where $d_z(x) = \operatorname{dist}\left(x, \partial B\left(z, \dfrac{|z|}{2}\right)\right)$.

We introduce the function

$$M(x) = u^{\frac{1}{\overline{\alpha}}}(x) + v^{\frac{1}{\overline{\beta}}}(x) + |\nabla u(x)|^{\frac{1}{\overline{\alpha}+1}} + |\nabla v(x)|^{\frac{1}{\overline{\beta}+1}}$$

and suppose that the claim is not true. Then there exist two sequences $\{z_n\}_{n\ge1}$ and $\{y_n\}_{n\ge1}$ such that $|z_n| > 2R_0$, $y_n \in B(z_n, |z_n|/2)$ and

$$M(y_n) > 2n d_{z_n}^{-1}(y_n).$$

By Lemma 5.1 in [63] there exists a sequence $\{x_n\}_{n\ge1}$ such that $x_n \in B(z_n, |z_n|/2)$, $M(x_n) > 2n d_{z_n}^{-1}(x_n)$ and

$$M(y) \le 2M(x_n) \ \text{ for } |y - x_n| \le n M^{-1}(x_n).$$

Since $u, v, |\nabla u|, |\nabla v|$ are bounded, we have $2n d_{z_n}^{-1}(x_n) \le C$, hence $d_{z_n}(x_n) \to +\infty$ as $n \to +\infty$. Therefore $|x_n| \to +\infty$, which implies that $M(x_n) \to 0$, since all the functions involved in the definition of M go to zero at infinity.

We take $\theta_n = M^{-1}(x_n) \to +\infty$ and define for $y \in B_n$, the ball with radius n centered at the origin,

$$\begin{aligned} \overline{u}_n(y) &= \theta_n^{\overline{\alpha}} u(x_n + \theta_n y), \\ \overline{v}_n(y) &= \theta_n^{\overline{\beta}} v(x_n + \theta_n y). \end{aligned}$$

The functions $\overline{u}_n, \overline{v}_n$ are well defined. In addition,

$$\overline{u}_n^{\frac{1}{\overline{\alpha}}} + \overline{v}_n^{\frac{1}{\overline{\beta}}} + |\nabla \overline{u}_n|^{\frac{1}{\overline{\alpha}+1}} + |\nabla \overline{v}_n|^{\frac{1}{\overline{\beta}+1}} \le 2 \ \text{ in } B_n, \tag{3.6.9}$$

and

$$\overline{u}_n^{\frac{1}{\overline{\alpha}}}(0) + \overline{v}_n^{\frac{1}{\overline{\beta}}}(0) + |\nabla \overline{u}_n(0)|^{\frac{1}{\overline{\alpha}+1}} + |\nabla \overline{v}_n(0)|^{\frac{1}{\overline{\beta}+1}} = 1.$$

They also verify the system

$$\left\{\begin{array}{rcl} -\Delta\overline{u}_n+\theta_n^{\overline{\alpha}+2-(\overline{\alpha}+1)q}|\nabla\overline{u}_n|^q & = & c\overline{v}_n^p, \\ -\Delta\overline{v}_n+\theta_n^{\overline{\beta}+2-(\overline{\beta}+1)q}|\nabla\overline{v}_n|^q & = & d\overline{u}_n^s, \end{array}\right. \quad \text{in } B_n. \tag{3.6.10}$$

Now we have to distinguish two cases:
i) $\overline{\beta}>\frac{2-q}{q-1}$. We have that $\overline{\alpha}+2-(\overline{\alpha}+1)q<0$ and $\overline{\beta}+2-(\overline{\beta}+1)q<0$. Since $\overline{u}_n$, $\overline{v}_n$, $|\nabla\overline{u}_n|$ and $|\nabla\overline{v}_n|$ are locally bounded by (3.6.9), standard elliptic estimates imply that $\overline{u}_n$ and $\overline{v}_n$ are locally bounded in $C^{1,\alpha}$. Choosing a subsequence in the standard way we may assume that $(\overline{u}_n,\overline{v}_n)\to(\overline{u},\overline{v})$ in $C^1_{\text{loc}}(\mathbb{R}^n)$ with

$$\left\{\begin{array}{rcl} -\Delta\overline{u} & = & c\overline{v}^p, \\ -\Delta\overline{v} & = & d\overline{u}^s, \end{array}\right. \quad \text{in } \mathbb{R}^N, \tag{3.6.11}$$

and

$$\overline{u}^{\frac{1}{\overline{\alpha}}}(0)+\overline{v}^{\frac{1}{\overline{\beta}}}(0)+|\nabla\overline{u}(0)|^{\frac{1}{\overline{\alpha}+1}}+|\nabla\overline{v}(0)|^{\frac{1}{\overline{\beta}+1}}=1,$$

which means that the pair $(\overline{u},\overline{v})$ is nontrivial, therefore positive by the strong maximum principle. By bootstrapping it actually follows that $\overline{u},\overline{v}\in C^\infty(\mathbb{R}^N)$. Since $\overline{\alpha}\geq N-2$, the system (3.6.11) does not admit positive solutions, and we get a contradiction. This shows the claim when $\overline{\beta}>\frac{2-q}{q-1}$.
ii) $\overline{\beta}=\frac{2-q}{q-1}$. Now, system (3.6.10) becomes

$$\left\{\begin{array}{rcl} -\Delta\overline{u}_n+\theta_n^{\overline{\alpha}+2-(\overline{\alpha}+1)q}|\nabla\overline{u}_n|^q & = & c\overline{v}_n^p, \\ -\Delta\overline{v}_n+|\nabla\overline{v}_n|^q & = & d\overline{u}_n^s, \end{array}\right. \quad \text{in } B_n. \tag{3.6.12}$$

Letting again $n\to+\infty$ in (3.6.12) as before we obtain that $(\overline{u}_n,\overline{v}_n)\to(\overline{u},\overline{v})$ in $C^1_{\text{loc}}(\mathbb{R}^n)$ with

$$\left\{\begin{array}{rcl} -\Delta\overline{u} & = & c\overline{v}^p, \\ -\Delta\overline{v}+|\nabla\overline{v}|^q & = & d\overline{u}^s, \end{array}\right. \quad \text{in } \mathbb{R}^N,$$

and

$$\overline{u}^{\frac{1}{\overline{\alpha}}}(0)+\overline{v}^{\frac{1}{\overline{\beta}}}(0)+|\nabla\overline{u}(0)|^{\frac{1}{\overline{\alpha}+1}}+|\nabla\overline{v}(0)|^{\frac{1}{\overline{\beta}+1}}=1.$$

By Theorem 3.5 this is not possible. This shows the claim when $\overline{\beta}=\frac{2-q}{q-1}$.

To conclude the proof of the lemma, it suffices to take $x=z$ in (3.6.8) to arrive at $u\leq C|x|^{-\overline{\alpha}}$ and $v\leq C|x|^{-\overline{\beta}}$, for a positive constant C that does not depend on x. □

Proof of Lemma 3.6.2. Let us start with a remark about the exponents involved. If we assume that $\widehat{\alpha}\geq N-2$, $p\geq s$, $ps>1$ and $\overline{\beta}<\gamma<\overline{\alpha}$ then we claim that $ps>q$ and

$\widehat{\beta} < \gamma < \widehat{\alpha}$. The first part of this claim is proved in the same way as at the beginning of Theorem 3.5. It is then easy to check that $\overline{\beta} < \gamma$ if and only if $\widehat{\beta} < \gamma$. Indeed, a calculation shows that

$$\frac{\widehat{\beta} - \gamma}{\overline{\beta} - \gamma} = \frac{ps - 1}{ps - q}.$$

Note that $\gamma < N - 2 \le \widehat{\alpha}$ because $q > \frac{N}{N-1}$.

The proof of the upper estimates is similar to the proof of Lemma 3.6.1 but of course with the important restriction that we now work with the radial version of system (3.6.1), namely

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + |u'|^q &= cv^p, \\ -v'' - \dfrac{N-1}{r}v' + |v'|^q &= du^s, \end{cases} \tag{3.6.13}$$

for $r \ge R_0$. Let us remark that, by the same reason as in the proof of Lemma 3.6.1, we have that u, v, u', v' go to zero at infinity.

We are going to prove that if (u, v) is a positive bounded radial solution of (3.6.13), then there exists $C > 0$ such that for any $\rho > 2R_0$

$$\begin{aligned} &u(r) \le C d_\rho^{-\widehat{\alpha}}(r), \\ &|u'(r)| \le C d_\rho^{-(\widehat{\alpha}+1)}(r), \\ &v(r) \le C d_\rho^{-\widehat{\beta}}(r) \end{aligned} \tag{3.6.14}$$

for $r \in I_\rho = (\rho/2, 3\rho/2)$ and $d_\rho(r) = \mathrm{dist}(r, \partial I_\rho) = \min\{r - \rho/2, 3\rho/2 - r\}$.

Suppose that the claim is not true. Then there exist sequences $\{\rho_n\}_{n\ge1}$, $\{\tau_n\}_{n\ge1}$ with $\rho_n > 2R_0$, $\tau_n \in (\rho_n/2, 3\rho_n/2)$ and

$$M(\tau_n) := u^{\frac{1}{\widehat{\alpha}}}(\tau_n) + |u'(\tau_n)|^{\frac{1}{\widehat{\alpha}+1}} + v^{\frac{1}{\widehat{\beta}}}(\tau_n) > 2n d_{\rho_n}^{-1}(\tau_n).$$

By Lemma 5.1 in [63], there exists a sequence $\{r_n\}_{n\ge1}$ such that $r_n \in (\rho_n/2, 3\rho_n/2)$, $M(r_n) > 2n d_{\rho_n}^{-1}(r_n)$ and

$$M(r) \le 2M(r_n) \quad \text{if } |r - r_n| \le n M^{-1}(r_n).$$

We take $\theta_n = M^{-1}(r_n)$ and introduce the scaled functions

$$\begin{aligned} \overline{u}_n(\tau) &= \theta_n^{\widehat{\alpha}} u(r_n + \theta_n \tau), \\ \overline{v}_n(\tau) &= \theta_n^{\widehat{\beta}} v(r_n + \theta_n \tau), \end{aligned}$$

for $|\tau| \le n$. Iit follows similarly as before that $d_{\rho_n}(r_n) \to +\infty$. Then $r_n \to +\infty$ and $M(r_n) \to 0$, so that $\theta_n \to +\infty$.

It is not hard to check that $(\overline{u}_n, \overline{v}_n)$ is a solution of the system

$$\begin{cases} -\overline{u}_n'' - \dfrac{N-1}{\frac{r_n}{\theta_n}+\tau}\overline{u}_n' + \theta_n^{\widehat{\alpha}+2-(\widehat{\alpha}+1)q}|\overline{u}_n'|^q &= c\overline{v}_n^p, \\ -\theta_n^{(\widehat{\beta}+1)q-\widehat{\beta}-2}\left(\overline{v}_n'' + \dfrac{N-1}{\frac{r_n}{\theta_n}+\tau}\overline{v}_n'\right) + |\overline{v}_n'|^q &= d\overline{u}_n^s, \end{cases} \tag{3.6.15}$$

for $|\tau| \le n$.

Before taking limits in (3.6.15) let us prove that $r_n/\theta_n \to +\infty$. For this sake, observe that $M(r_n)d_{\rho_n}(r_n) > 2n \to +\infty$, and

$$d_{\rho_n}(r_n) = \min\left\{r_n - \frac{\rho_n}{2}, \frac{3\rho_n}{2} - r_n\right\} \le r_n.$$

Then, $r_n/\theta_n = M(r_n)r_n \ge M(r_n)d_{\rho_n}(r_n) \to +\infty$, as claimed.

Next, notice that

$$\overline{u}_n^{\frac{1}{\widehat{\alpha}}} + \overline{v}_n^{\frac{1}{\widehat{\beta}}} + |\overline{u}_n'|^{\frac{1}{\widehat{\alpha}+1}} \le 2 \quad \text{for } |\tau| \le n,$$

and

$$\overline{u}_n^{\frac{1}{\widehat{\alpha}}}(0) + \overline{v}_n^{\frac{1}{\widehat{\beta}}}(0) + |\overline{u}_n'(0)|^{\frac{1}{\widehat{\alpha}+1}} = 1. \tag{3.6.16}$$

Using the first equation of (3.6.15), we deduce that $\overline{u}_n''$ is locally bounded, hence, there exists $\overline{u} \in C^1(\mathbb{R})$ such that

$$\lim_{n\to+\infty} \overline{u}_n = \overline{u} \quad \text{in } C^1_{\text{loc}}(\mathbb{R}).$$

By Lemma A.1 in the Appendix (cf. also Remark A.2), $|\overline{v}_n'|$ is also locally bounded, so that there exists $\overline{v} \in C(\mathbb{R})$ such that

$$\lim_{n\to+\infty} \overline{v}_n = \overline{v} \quad \text{in } C_{\text{loc}}(\mathbb{R}).$$

Thus we may pass to the limit in (3.6.16) to obtain

$$\overline{u}^{\frac{1}{\widehat{\alpha}}}(0) + \overline{v}^{\frac{1}{\widehat{\beta}}}(0) + |\overline{u}'(0)|^{\frac{1}{\widehat{\alpha}+1}} = 1, \tag{3.6.17}$$

which shows that $(\overline{u}, \overline{v})$ is nontrivial. Passing to the limit as $n \to +\infty$ in the first equation of (3.6.15) and taking into account that, by our hypotheses $\widehat{\alpha} + 2 - (\widehat{\alpha}+1)q < 0$, we obtain

$$-\overline{u}'' = c\overline{v}^p \quad \text{in } \mathbb{R} \tag{3.6.18}$$

in the weak sense, and standard regularity implies that the equation is verified in the classical sense. On the other hand, taking limits in the second equation of (3.6.15) and using that $(\widehat{\beta}+1)q - \widehat{\beta} - 2 < 0$ we have

$$|\overline{v}'|^q = d\overline{u}^s \quad \text{in } \mathbb{R} \tag{3.6.19}$$

in the viscosity sense (cf. [45]). We claim that this is impossible. By (3.6.18), we see that $\overline{u}$ is concave, but since $\overline{u}$ is nonnegative and bounded, it has to be constant. Hence $\overline{v} = 0$ by (3.6.18), so that, by (3.6.19), we get $\overline{u} = 0$ which contradicts (3.6.17). This shows (3.6.14).

Taking $r = \rho$ in (3.6.14) we obtain that $u \leq Cr^{-\widehat{\alpha}}$ and $v \leq Cr^{-\widehat{\beta}}$ for large r and a positive constant C which does not depend on r. $\square$

3.7 Proofs of Theorems 3.1, 3.2 and 3.3

In this section we collect the proofs of our main theorems of this chapter. After the preliminary work done the proofs are reduced to compare the lower and upper estimates for a positive radial solution of (3.1.1) to arrive at a contradiction. In some cases, we do not have an immediate contradiction between these estimates, so an extra analysis has to be carried out, which involves Lemma S in Section 2.7 of Chapter 2.

Before proceeding to the proof of Theorem 1, we recall that the relevant exponents in the case $1 < q < \frac{N}{N-1}$ are $\gamma = \frac{2-q}{q-1}$, $\alpha = \frac{q(p+q)}{ps-q^2}$ and $\beta = \frac{q(s+q)}{ps-q^2}$. The exponents α and β verify $q(\alpha+1) = p\beta$ and $q(\beta+1) = s\alpha$.

Proof of Theorem 3.1. Assume there exists a positive supersolution (u, v) of (3.1.1) and choose $\varepsilon > 0$. By Lemma 3.3.3, there exists a positive, bounded, radially symmetric solution (z, w) of

$$\left\{ \begin{array}{rcl} -\Delta z + |\nabla z|^q & = & \lambda(\theta_1 - \varepsilon) w^p \\ -\Delta w + |\nabla w|^q & = & \mu(\theta_2 - \varepsilon) z^s, \end{array} \right. \quad \text{in } \mathbb{R}^N \setminus B_{R_1}, \tag{3.7.1}$$

for some $R_1 > R_0$.

By Lemma 3.5.1.a) we arrive at an immediate contradiction if $ps < q^2$. If $ps = q^2$, by Lemmas 2.3.3.a) and 3.5.1.b), we deduce

$$Cr^{-\gamma} \leq z \leq C_1 r^{-\frac{p}{q}} e^{-C_2 r},$$

which is a contradiction for large r. This shows part a).

If $ps > q^2$ and $\alpha > \gamma$ then by Lemmas 2.3.3.a) and 3.5.1.c) we have that

$$Cr^{-\gamma} \leq z \leq Cr^{-\alpha},$$

which is again a contradiction for large r. This shows b).

Now assume that $\alpha = \beta = \gamma$. By Lemmas 2.3.3.a) and 3.5.1.c), we have for large r,

$$C_1 r^{-\alpha} \leq z(r) \leq C_2 r^{-\alpha} \quad \text{and} \quad C_1 r^{-\beta} \leq w(r) \leq C_2 r^{-\beta}.$$

We take $t_1 = \liminf_{r\to+\infty} z(r) r^\alpha$ and $t_2 = \liminf_{r\to+\infty} w(r) r^\beta$, both limits being finite and positive. By Lemma S, there exist sequences $\{r_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ with $r_n, s_n \to +\infty$, and

$$\begin{array}{rclrcl}
\lim_{n\to+\infty} r_n^\alpha z(r_n) & = & t_1, & \lim_{n\to+\infty} s_n^\beta w(s_n) & = & t_2, \\
\lim_{n\to+\infty} r_n^{\alpha+1} z'(r_n) & = & -\alpha t_1, & \lim_{n\to+\infty} s_n^{\beta+1} w'(s_n) & = & -\beta t_2, \\
\lim_{n\to+\infty} r_n^{\alpha+2} z''(r_n) & \geq & \alpha(\alpha+1) t_1, & \lim_{n\to+\infty} s_n^{\beta+2} w''(s_n) & \geq & \beta(\beta+1) t_2.
\end{array}$$

By multiplying the first equation in the radial version of (3.7.1) by $r_n^{\alpha+2}$ and passing to the limit (taking into account that $\alpha+2 = q(\alpha+1) = p\beta$) we obtain

$$\alpha(N-2-\alpha)t_1 + \alpha^q t_1^q \geq \lambda(\theta_1 - \varepsilon) t_2^p.$$

Arguing similarly with the second equation in (3.7.1) evaluated at s_n we find

$$\beta(N-2-\beta)t_2 + \beta^q t_2^q \geq \mu(\theta_2 - \varepsilon) t_1^s.$$

We can now let $\varepsilon \to 0$ to obtain a contradiction with (3.1.5). This shows c).

The proof of d) is similar. Assume that $\beta < \alpha = \gamma$ and (3.1.6). By Lemmas 2.3.3.a) and 3.5.1.c)

$$C_1 r^{-\alpha} \leq z(r) \leq C_2 r^{-\alpha} \quad \text{and} \quad C_1 r^{-\gamma} \leq w(r) \leq C_2 r^{-\beta}.$$

Our immediate aim is to improve the lower inequality for w. Notice that

$$-\Delta w + |\nabla w|^q \geq C z^s \geq C r^{-s\alpha} = C r^{-q(\beta+1)},$$

so by Lemma 3.4.2, we obtain $w \geq C r^{-\beta}$ for large r. Hence it makes sense to define $t_1 = \liminf_{r\to+\infty} z(r) r^\alpha$ and $t_2 = \liminf_{r\to+\infty} w(r) r^\beta$, both limits being finite and positive.

By Lemma S, there exist sequences $\{r_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ with $r_n, s_n \to +\infty$, and

$$\begin{array}{rclrcl}
\lim_{n\to+\infty} r_n^\alpha z(r_n) & = & t_1, & \lim_{n\to+\infty} s_n^\beta w(s_n) & = & t_2, \\
\lim_{n\to+\infty} r_n^{\alpha+1} z'(r_n) & = & -\alpha t_1, & \lim_{n\to+\infty} s_n^{\beta+1} w'(s_n) & = & -\beta t_2, \\
\lim_{n\to+\infty} r_n^{\alpha+2} z''(r_n) & \geq & \alpha(\alpha+1) t_1, & \lim_{n\to+\infty} s_n^{\beta+2} w''(s_n) & \geq & \beta(\beta+1) t_2.
\end{array}$$

Arguing as before we have

$$\alpha(N-2-\alpha)t_1 + \alpha^q t_1^q \geq \lambda(\theta_1 - \varepsilon) t_2^p. \tag{3.7.2}$$

On the other hand,

$$-w'' - \frac{N-1}{r} w' + |w'|^q \geq \mu(\theta_2 - \varepsilon) z^s,$$

and multiplying by $s_n^{q(\beta+1)}$ we obtain that

$$s_n^{q(\beta+1)-\beta-2}\left(-s_n^{\beta+2}w''(s_n)-(N-1)s_n^{\beta+1}w'(s_n)\right)+|s_n^{\beta+1}w'(s_n)|^q \geq \\ \mu(\theta_2-\varepsilon)(s_n^\alpha z(s_n))^s. \tag{3.7.3}$$

Observe that $q(\beta+1)-\beta-2<0$. Then passing to the limit as $n\to+\infty$ in (3.7.3) we arrive at

$$\beta^q t_2^q \geq \mu(\theta_2-\varepsilon)t_1^s. \tag{3.7.4}$$

Finally, letting $\varepsilon\to 0$ in (3.7.2) and (3.7.4) we contradict (3.1.6). This concludes the proof. $\square$

We consider next the case $q>\frac{N}{N-1}$. Recalling that $p\geq s$, the relevant exponents are $\gamma=\frac{2-q}{q-1}$, $\overline{\alpha}=\frac{2(p+1)}{ps-1}$, $\overline{\beta}=\frac{2(s+1)}{ps-1}$, $\widehat{\alpha}=\frac{q(p+2)}{ps-q}$ and $\widehat{\beta}=\frac{q+2s}{ps-q}$. These exponents are defined by means of the following pair of systems

$$\left\{\begin{array}{rcl}\overline{\alpha}+2&=&p\overline{\beta}\\ \overline{\beta}+2&=&s\overline{\alpha}\end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{rcl}\widehat{\alpha}+2&=&p\widehat{\beta}\\ q(\widehat{\beta}+1)&=&s\widehat{\alpha}.\end{array}\right.$$

Proof of Theorem 3.3. Assume (u,v) is a positive supersolution of (3.1.1) which does not blow up at infinity. Then, by Lemma 3.3.3 there exists a positive, radial supersolution (z,w) of the system

$$\left\{\begin{array}{rcl}-\Delta z+|\nabla z|^q&=&cw^p,\\ -\Delta w+|\nabla w|^q&=&dz^s,\end{array}\right. \tag{3.7.5}$$

in $\mathbb{R}^N\setminus B_{R_1}$ for suitable $R_1>R_0$ and positive values of c and d. Observe that we can always assume that $q<2$ (cf. Remark 3.3.2). By Lemma 2.3.3.c), we have that

$$z,w\geq Cr^{2-N}. \tag{3.7.6}$$

We first show b). Suppose that $\overline{\alpha}\geq N-2$ and $\overline{\beta}\geq\gamma$. By Lemma 3.6.1 we obtain

$$z\leq Cr^{-\overline{\alpha}},\quad w\leq Cr^{-\overline{\beta}}.$$

If $\overline{\alpha}>N-2$ we get a contradiction with the lower estimate in (3.7.6), so we may suppose that $\overline{\alpha}=N-2$. We have:

$$-\Delta w+|\nabla w|^q\geq Cr^{-s(N-2)}=Cr^{-s\overline{\alpha}}=Cr^{-(\overline{\beta}+2)},$$

and by Lemma 3.4.1, $w \geq Cr^{-\overline{\beta}}$ for large r follows. Thus we have obtained so far

$$\begin{array}{rcl} C_1 r^{-\overline{\alpha}} & \leq z \leq & C_2 r^{-\overline{\alpha}}, \\ C_1 r^{-\overline{\beta}} & \leq w \leq & C_2 r^{-\overline{\beta}}, \end{array}$$

where all involved constants are positive. We can apply Lemma S, in order to get a number θ and a sequence $\{r_n\}_{n\geq 1}$ such that $r_n \to +\infty$,

$$\begin{array}{rcl} \lim\limits_{n\to+\infty} r_n^{\overline{\alpha}} z(r_n) & = & \theta, \\ \lim\limits_{n\to+\infty} r_n^{\overline{\alpha}+1} z'(r_n) & = & -\overline{\alpha}\theta, \\ \lim\limits_{n\to+\infty} r_n^{\overline{\alpha}+2} z''(r_n) & \geq & \overline{\alpha}(\overline{\alpha}+1)\theta. \end{array}$$

Multiplying the first equation in the radial version of (3.7.5) by $r^{\overline{\alpha}+2}$ and evaluating at r_n we obtain that

$$-r_n^{\overline{\alpha}+2} z''(r_n) - (N-1) r_n^{\overline{\alpha}+1} z'(r_n) + r_n^{\overline{\alpha}+2} |z'(r_n)|^q = c w^p(r_n) r_n^{\overline{\alpha}+2} = c(w(r_n) r_n^{\overline{\beta}})^p \geq C. \tag{3.7.7}$$

Note that

$$r_n^{\overline{\alpha}+2} |z'(r_n)|^q = r_n^{\overline{\alpha}+2-q(\overline{\alpha}+1)} |r_n^{\overline{\alpha}+1} z'(r_n)|^q \to 0,$$

so that taking limits in (3.7.7) we get

$$0 = \overline{\alpha}\theta(N-2-\overline{\alpha}) = -\overline{\alpha}(\overline{\alpha}+1)\theta + (N-1)\overline{\alpha}\theta \geq C > 0,$$

and we arrive at a contradiction. This shows b).

Next we show c). Suppose that $\overline{\alpha} \geq N-2$, $\overline{\beta} < \gamma$ and $\widehat{\alpha} \geq N-2$. Then by Lemma 3.6.2 we obtain

$$z \leq Cr^{-\widehat{\alpha}}, \ w \leq Cr^{-\widehat{\beta}}.$$

If $\widehat{\alpha} > N-2$ we get a contradiction with the lower estimate (3.7.6), so we can suppose $\widehat{\alpha} = N-2$. We have:

$$-\Delta w + |\nabla w|^q \geq Cr^{-s(N-2)} = Cr^{-s\widehat{\alpha}} = Cr^{-q(\widehat{\beta}+1)}$$

and by Lemma 3.4.2, $w \geq Cr^{-\widehat{\beta}}$ for large r. Thus

$$\begin{array}{rcl} C_1 r^{-\widehat{\alpha}} & \leq z \leq & C_2 r^{-\widehat{\alpha}}, \\ C_1 r^{-\widehat{\beta}} & \leq w \leq & C_2 r^{-\widehat{\beta}}, \end{array}$$

where all involved constants are positive and we finish as before with the aid of Lemma S: there exist a number θ and a sequence $\{r_n\}_{n\geq 1}$ such that $r_n \to +\infty$,

$$\begin{array}{rcl}\lim\limits_{n\to+\infty} r_n^{\widehat{\alpha}} z(r_n) & = & \theta, \\ \lim\limits_{n\to+\infty} r_n^{\widehat{\alpha}+1} z'(r_n) & = & -\widehat{\alpha}\theta \\ \lim\limits_{n\to+\infty} r_n^{\widehat{\alpha}+2} z''(r_n) & \geq & \widehat{\alpha}(\widehat{\alpha}+1)\theta.\end{array}$$

Multiplying the first equation of the radial version if (3.7.5) by $r^{\widehat{\alpha}+2}$ and evaluating at r_n we obtain that

$$\begin{aligned}-r_n^{\widehat{\alpha}+2} z''(r_n) - (N-1) r_n^{\widehat{\alpha}+1} z'(r_n) + r_n^{\widehat{\alpha}+2} |z'(r_n)|^q \\ = c w^p(r_n) r_n^{\widehat{\alpha}+2} = c(w(r_n) r_n^{\widehat{\beta}})^p \geq C.\end{aligned} \tag{3.7.8}$$

Then taking limits in (3.7.8) we get:

$$0 = \widehat{\alpha}\theta(N-2-\widehat{\alpha}) = -\widehat{\alpha}(\widehat{\alpha}+1)\theta + (N-1)\widehat{\alpha}\theta \geq C > 0,$$

and we arrive at a contradiction. This contradiction shows c).

To conclude the proof only a) remains to be shown, so suppose $ps \leq 1$. Since $w \to 0$ we deduce

$$\left\{\begin{array}{rcl} -\Delta z + |\nabla z|^q & \geq & c w^{p+t}, \\ -\Delta w + |\nabla w|^q & \geq & d z^s, \end{array}\right. \quad \text{in } \mathbb{R}^N \setminus B_{R_1}, \tag{3.7.9}$$

for any $t > 0$, for a conveniently large value of R_1. Fix $\varepsilon > 0$ and choose t such that $(p+t)s = 1+\varepsilon$. The exponents involved in the nonexistence of supersolutions for the system (3.7.9) are:

$$\overline{\alpha} = \frac{2(p'+1)}{p's-1} = \frac{2(p+t+1)}{\varepsilon},\ \overline{\beta} = \frac{2(s+1)}{p's-1} = \frac{2(s+1)}{\varepsilon},$$

where $p' = p+t$. Choosing ε small enough we have that $\overline{\alpha}, \overline{\beta} > N-2$, so we may apply b) to get that system (3.7.5) does not admit positive supersolutions which do not blow up at infinity. $\square$

We complete the proofs of our Liouville theorems for system (3.1.1) by considering the threshold value $q = \frac{N}{N-1}$. In this case, we take advantage of Theorem 3.1, and perturb $q = \frac{N}{N-1}$ to obtain the desired nonexistence theorem. In this regard, it is important to recall that if there exists a positive supersolution which does not blow up at infinity of (3.1.1) and $q' < q$, then there also exists a supersolution of (3.1.1) with q replaced by q' (see Remark 3.3.2).

Proof of Theorem 3.2. Assume that (u, v) is a positive supersolution of (3.1.1) which does not blow up at infinity for $q = \frac{N}{N-1}$ and $p \geq s$.

Suppose that $ps > q^2$ and $\alpha > \gamma = N - 2$, then by continuity there exists $q' < \frac{N}{N-1}$ such that $ps > (q')^2$ and $\alpha' = q'(p + q')/(ps - (q')^2) > (2 - q')/(q' - 1) = \gamma'$, so applying Theorem 3.1.b) we get a contradiction. This shows b).

Suppose that $ps < q^2$. Again there exists $q' < \frac{N}{N-1}$ such that $ps < (q')^2$ and by Theorem 3.1.a) we get a contradiction.

If $ps = q^2$, we fix $\varepsilon > 0$ and take $q' = q - \varepsilon$. Then $ps > (q')^2$ and $q' < \frac{N}{N-1}$. In order to use Theorem 3.1.b) we have to check that $\alpha = q'(p + q')/(ps - (q')^2) > (2 - q')/(q' - 1)$, but

$$\alpha = \frac{(q - \varepsilon)(p + q - \varepsilon)}{ps - q^2 - \varepsilon^2 + 2\varepsilon q} = \frac{(q - \varepsilon)(p + q - \varepsilon)}{\varepsilon(2q - \varepsilon)}.$$

Taking $\varepsilon > 0$ small enough we obtain that $\alpha > (2 - q')/(q' - 1)$ and we can use Theorem 3.1.b) to get a contradiction. This concludes the proof. $\square$

3.8 Supersolutions blowing up at infinity

In this section we deal with positive supersolutions of (3.1.1) which blow up at infinity. For this type of supersolutions the reduction to radial solutions is not needed. Actually, the reduction performed in Lemma 3.3.1 is sufficient. The proof of Theorem 3.4 is similar to the proof of Lemma 3.5.1.

Proof of Theorem 3.4. Assume that (u, v) is a positive supersolution of (3.1.1) which blows up at infinity. Then by Lemma 3.3.1 there exist $R_1 > R_0$ and a positive radial supersolution (z, w) of (3.1.1) with $z', w' > 0$ and $z, w \to +\infty$ as $r \to +\infty$. By (3.1.9), we also have

$$\begin{aligned} -z'' + (z')^q &\geq Cw^p, \\ -w'' + (w')^q &\geq Cz^s \end{aligned}$$

for $r > R_1$ and some $C > 0$.

We consider the functions

$$\begin{aligned} H_1(r) &= Cw^p - 2(z')^q, \\ H_2(r) &= Cz^s - 2(w')^q, \end{aligned}$$

for $r > R_1$. Assume $r_0 > R_1$ is such that $H_1(r_0) = 0$. It is easily checked that

$$\begin{aligned} H_1'(r_0) &= Cpw^{p-1}w' - 2q(z')^{q-1}z'' \\ &\geq Cpw^{p-1}w' + 2q(z')^{q-1}\left(Cw^p - (z')^q\right) \\ &= Cpw^{p-1}w' + 2q(z')^{2q-1} > 0. \end{aligned}$$

Hence H_1 may vanish at most once, so it has constant sign for large r. We claim that $H_1 < 0$ for large r. Suppose that $H_1 > 0$ for large r; then there exists $r_0 \geq R_1$ such that

$$-z'' \geq \frac{C}{2} w^p \geq \delta \text{ for } r \geq r_0.$$

By integrating twice between r_0 and r it follows that

$$z(r) \leq z(r_0) + z'(r_0)(r - r_0) - \frac{\delta}{2}(r - r_0)^2 \to -\infty,$$

which is a contradiction. This shows the claim. It is shown similarly that $H_2 < 0$ for large r.

Thus we have

$$\begin{array}{rcl} z' & > & C w^{\frac{p}{q}}, \\ w' & > & C z^{\frac{s}{q}} \end{array} \tag{3.8.1}$$

for a different constant C. By means of a scaling we can suppose in the rest of the proof that $C = 1$.

We introduce the change of variable $t = H(r) := \int_{r_1}^{r} w^{p/q}(\tau) d\tau$, where r_1 is large and fixed, and we denote $Z(t) = z(r)$ and $W(t) = w(r)$. Observe that Z, W are defined in $(t_0, +\infty)$ for some $t_0 > 0$ with $\lim_{t\to+\infty} Z(t) = \lim_{t\to+\infty} W(t) = +\infty$. By (3.8.1), we have $Z(t) \geq t$. In addition,

$$\frac{dW}{dt} = \frac{w'}{w^{p/q}} > \frac{z^{s/q}}{w^{p/q}} \geq \frac{t^{s/q}}{W^{p/q}}. \tag{3.8.2}$$

Then, integrating (3.8.2) we obtain that

$$W(t) \geq C t^{\frac{s+q}{p+q}}, \text{ for } t \geq t_0.$$

That is,

$$H'(r) = w^{p/q}(r) = W^{p/q}(t) \geq C t^{\frac{p(s+q)}{q(p+q)}} = C H^{\frac{p(s+q)}{q(p+q)}}(r).$$

But $\frac{p(s+q)}{q(p+q)} > 1$ because $ps > q^2$. This implies that H blows up in finite time and we get a contradiction. The proof is concluded. □

3.9 Optimality of the nonexistence results

In this section we will show that all nonexistence theorems of this chapter are essentially sharp.

We begin considering supersolutions not blowing up at infinity. Thus we assume

$$\nu_1 = \limsup_{t\to 0} \frac{f(t)}{t^p} < +\infty, \quad \nu_2 = \limsup_{t\to 0} \frac{g(t)}{t^s} < +\infty \tag{3.9.1}$$

and (p, q, s) does not verify any of the hypotheses in Theorems 3.1, 3.2 and 3.3 (observe that θ_1 and θ_2 in (3.1.5) and (3.1.6) have to be replaced by ν_1 and ν_2). Then positive supersolutions can be explicitly constructed. They are all of the form

$$u = A|x|^{-a}, \quad v = B|x|^{-b}, \tag{3.9.2}$$

where A, B, a, b are positive numbers. By condition (3.9.1), it is sufficient to have, for large $|x|$, the inequalities

$$\begin{aligned} a(N-2-a)A|x|^{-(a+2)} + a^q A^q |x|^{-q(a+1)} &\geq CB^p |x|^{-pb}, \\ b(N-2-b)B|x|^{-(b+2)} + b^q B^q |x|^{-q(b+1)} &\geq DA^s |x|^{-sa}, \end{aligned} \tag{3.9.3}$$

where we have set $C = \lambda(\nu_1 + \eta)$, $D = \mu(\nu_2 + \eta)$ and $\eta > 0$ is chosen as small as desired. Thus a distinction into cases is mandatory.

First, suppose $1 < q < \frac{N}{N-1}$, $p \geq s$ and $ps > q^2$. We set $a = \alpha$, $b = \beta$ (given by (3.1.4)). If $\beta \leq \alpha < \gamma$, it is sufficient to choose A, B verifying

$$\begin{aligned} \alpha^q A^q &> CB^p, \\ \beta^q B^q &> DA^s. \end{aligned} \tag{3.9.4}$$

It can be checked that

$$A = \left(\frac{\alpha^{q^2} \beta^{pq}}{(2C)^q (2D)^p} \right)^{\frac{1}{ps-q^2}}, \quad B = \left(\frac{\alpha^{sq} \beta^{q^2}}{(2C)^s (2D)^q} \right)^{\frac{1}{ps-q^2}}, \tag{3.9.5}$$

is a possible election for A and B to verify (3.9.4).

When $\beta = \alpha = \gamma$ then (3.9.3) is implied by

$$\begin{aligned} \alpha(N-2-\alpha)A + \alpha^q A^q &\geq \lambda(\nu_1 + \eta)B^p, \\ \beta(N-2-\beta)B + \beta^q B^q &\geq \mu(\nu_2 + \eta)A^s, \end{aligned}$$

while for $\beta < \alpha = \gamma$, it is sufficient to solve the system of inequalities

$$\begin{aligned} \alpha(N-2-\alpha)A + \alpha^q A^q &\geq \lambda(\nu_1 + \eta)B^p, \\ \beta^q B^q &> \mu(\nu_2 + \eta)A^s. \end{aligned}$$

This shows that Theorem 3.1 is sharp.

With regard to Theorem 3.2, suppose next that $q = \frac{N}{N-1}$, $p \geq s$, $ps > q^2$ and $\alpha \leq N - 2 = \gamma$. In the same way as before it can be proved that positive supersolutions of (3.1.1) of the form (3.9.2) can be found, by simply choosing $a = \alpha$, $b = \beta$ and A and B as in (3.9.5). Thus Theorem 3.2 is also sharp.

Finally, suppose that $q > \frac{N}{N-1}$, $p \geq s$ and $ps > 1$. Two further cases have to be distinguished: the first one arises when $\overline{\alpha} < N-2$. In that case, it is enough to take $a = \overline{\alpha}$, $b = \overline{\beta}$ given by (3.1.7) and A, B verifying

$$\begin{array}{rcl} \overline{\alpha}(N-2-\overline{\alpha})A & \geq & CB^p, \\ \overline{\beta}(N-2-\overline{\beta})B & \geq & DA^s. \end{array} \tag{3.9.6}$$

It suffices to choose, for instance,

$$A = \left(\frac{\overline{\alpha}(N-2-\overline{\alpha})\overline{\beta}^p(N-2-\overline{\beta})^p}{CD^p} \right)^{\frac{1}{ps-1}},$$
$$B = \left(\frac{\overline{\alpha}^s(N-2-\overline{\alpha})^s\overline{\beta}(N-2-\overline{\beta})}{C^sD} \right)^{\frac{1}{ps-1}}.$$

We remark that the pair (u, v) is a positive supersolution for (3.1.1) also when $q \geq 2$ (they are indeed a supersolution of the system (3.6.11), where the gradient term does not appear).

The second case is obtained when $\overline{\alpha} \geq N-2$, $\overline{\beta} < \gamma$ and $\widehat{\alpha} < N-2$. It is easy to check that in this case we necessarily have $\widehat{\beta} < \gamma < N-2$. So, it is now enough to take $a = \widehat{\alpha}$, $b = \widehat{\beta}$ given by (3.1.8) and solve the system of inequalities

$$\begin{array}{rcl} \widehat{\alpha}(N-2-\widehat{\alpha})A & \geq & CB^p, \\ \widehat{\beta}^q B^q & \geq & DA^s, \end{array}$$

one of whose solutions is

$$A = \left(\frac{\widehat{\alpha}^q(N-2-\widehat{\alpha})^q\widehat{\beta}^{pq}}{C^qD^p} \right)^{\frac{1}{ps-q}}, \quad B = \left(\frac{\widehat{\alpha}^s(N-2-\widehat{\alpha})^s\widehat{\beta}^q}{C^sD} \right)^{\frac{1}{ps-q}}.$$

Notice that $ps > q$ by our hypotheses (see the beginning of the proof of Lemma 3.6.2). This shows that Theorem 3.3 is optimal.

With regard to supersolutions of (3.1.1) which blow up at infinity, we next assume:

$$\limsup_{t\to+\infty} \frac{f(t)}{t^p} < +\infty, \ \limsup_{t\to+\infty} \frac{g(t)}{t^s} < +\infty$$

with positive p, s verifying $ps \leq q^2$. In this case, it is easily checked that $(e^{\alpha r}, e^{\beta r})$ is a positive supersolution of (3.1.1) for large R_0 whenever $\alpha = \frac{\beta p}{q}$ and β is large enough.

Chapter 4

Systems with linear growth in the gradient

We will analyze next the elliptic system considered in the previous chapter, but now we are setting $q = 1$, and we will restrict attention only to power functions, that is

$$\begin{cases} -\Delta u + |\nabla u| &= \lambda v^p, \\ -\Delta v + |\nabla v| &= \mu u^s, \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}. \tag{4.1.1}$$

where $p, s > 0$ and $\lambda, \mu > 0$.[1]

We have chosen to restrict to power nonlinearities because in this case we will be able to give existence and nonexistence results all in one. Of course, almost everything remains unchanged if we consider the more general problem

$$\begin{cases} -\Delta u + |\nabla u| &= \lambda f(v) \\ -\Delta v + |\nabla v| &= \mu g(u), \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0},$$

where f and g are positive, nondecreasing, continuous functions behaving like a power near zero or infinity.

4.1 Existence and nonexistence results

It goes without saying that in the present situation, supersolutions can also be divided into two types: those which verify

$$\lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = +\infty \tag{4.1.2}$$

and those which do not. We begin by considering the latter ones.

[1] The results in this chapter are contained in [17].

Theorem 4.1. *Assume $N \geq 2$. Then:*

a) *If $ps < 1$, then there are no positive supersolutions of* (4.1.1) *which do not blow up at infinity for any $\lambda, \mu > 0$.*

b) *If $ps > 1$, then there exists a positive supersolution of* (4.1.1) *which does not blow up at infinity for every $\lambda, \mu > 0$.*

c) *If $ps = 1$, then there exists a value $\Sigma(p) > 0$ such that, if $\lambda\mu^p < \Sigma$ problem* (4.1.1) *admits positive supersolutions which do not blow up at infinity, while for $\lambda\mu^p > \Sigma$ no such supersolutions exist.*

A similar statement holds for positive supersolutions which verify (4.1.2).

Theorem 4.2. *Assume $N \geq 2$. Then:*

a) *If $ps < 1$, then there exists a positive supersolution of* (4.1.1) *which blows up at infinity for every $\lambda, \mu > 0$.*

b) *If $ps > 1$, then there are no positive supersolutions of* (4.1.1) *which do blow up at infinity for any $\lambda, \mu > 0$.*

c) *If $ps = 1$, then there exists a value $\Sigma(p) > 0$ such that, if $\lambda\mu^p < \Sigma$ problem* (4.1.1) *admits positive supersolutions which blow up at infinity, while for $\lambda\mu^p > \Sigma$ no such supersolutions exist.*

It is to be noted that the proofs follow essentially the same lines as in the previous chapter, with some obvious modifications.

4.2 Lower and upper estimates

We obtain in what follows some estimates for radially symmetric positive solutions of (4.1.1). They are essentially different from those obtained in Lemma 2.3.3.

Lemma 4.2.1. *Let $z \in C^2(R_0, +\infty)$ be positive and verify*

$$-z'' - \frac{N-1}{r} z' + |z'| \geq 0 \text{ in } (R_0, +\infty). \tag{4.2.1}$$

Then:

a) *If $z'(r) < 0$ and $\lim_{r\to+\infty} z(r) = 0$, there exists a positive constant C such that*

$$z \geq C r^{1-N} e^{-r}$$

for large r.

b) *If $z'(r) > 0$, there exists a positive constant C such that*

$$z \leq Ce^r$$

for large r.

Proof. We consider first case a). Multiply the equation (4.2.1) by r^{N-1} and define $w(r) = -r^{N-1}z'(r)$. Then, we obtain

$$\frac{w'}{w} + 1 \geq 0 \quad \text{ in } (R_0, +\infty).$$

By integrating between R_0 and r, we have that

$$w \geq Ce^{-r} \quad \text{ in } (R_0, +\infty),$$

where $C > 0$. Hence,

$$-z' \geq Cr^{1-N}e^{-r}$$

for large r. We now integrate between r and $+\infty$ and use the fact that $\lim_{r\to+\infty} z(r) = 0$, to get

$$z(r) = -\int_r^{+\infty} z'(s)ds \geq C\int_r^{+\infty} s^{1-N}e^{-s}ds,$$

for large enough r. Then a) follows because, using l'Hôpital rule

$$\int_r^{+\infty} s^{1-N}e^{-s}ds \sim r^{1-N}e^{-r}$$

as $r \to +\infty$.

As for b), assume that $z' > 0$. Then (4.2.1) immediately implies

$$-\frac{z''}{z'} + 1 \geq 0.$$

By integrating twice between R_0 and r, we obtain

$$z \leq Ce^r.$$

This concludes the proof. □

4.3 Proof of Theorems 4.1 and 4.2

The proof of Theorems 4.1 and 4.2 is also based on obtaining appropiate bounds which will be in contradiction with the ones given in Lemma 4.2.1. We start our way to the proofs with two lemmas where only nonexistence results for system (4.1.1) are obtained.

As in the previous chapter, it is possible to show that the existence of a positive supersolution of (4.1.1) implies the existence of a radially symmetric supersolution of the same type. That is, Lemma 3.3.1 is still valid for problem (4.1.1), with the only difference that the gradient bounds of [51] have to be replaced by those in Chapter 4 of [49].

Lemma 4.3.1. *Assume $N \geq 2$. Then:*

a) *If $ps < 1$, then there are no positive supersolutions of* (4.1.1) *which do not blow up at infinity.*

b) *If $ps = 1$ and*

$$\lambda\mu^p > \frac{1}{p^p}, \tag{4.3.1}$$

then there are no positive supersolutions of (4.1.1) *which do not blow up at infinity.*

Proof. Assume that (u, v) is a positive supersolution of (4.1.1) which does not blow up at infinity. Then, as remarked above, there exists a radial positive supersolution (z, w) of (4.1.1) which does not blow up at infinity. In addition, the functions z and w verify $z', w' < 0$ and $\lim_{r\to+\infty} z(r) = \lim_{r\to+\infty} w(r) = 0$. Hence, the pair (z, w) is a supersolution of the system

$$\left\{ \begin{array}{rcl} -z'' - \dfrac{N-1}{r} z' - z' & = & \lambda w^p, \\ -w' - \dfrac{N-1}{r} w' - w' & = & \mu z^s, \end{array} \right. \qquad \text{in } (R_0, +\infty).$$

We introduce the functions

$$H_1(r) = \lambda w^p(r) + a z'(r),$$
$$H_2(r) = \mu z^s(r) + a w'(r),$$

for $r > R_0$ where $a > 1$ will be chosen later. Let $r_0 > R_0$ be any point where $H_1(r_0) = 0$, i.e. $\lambda w^p(r_0) = -a z'(r_0)$. Using that w is nonincreasing, we obtain that

$$\begin{array}{rcl} H_1'(r_0) & \leq & \lambda p w^{p-1}(r_0) w'(r_0) - a \left(\dfrac{N-1}{r_0} z'(r_0) + z'(r_0) + \lambda w^p(r_0) \right) \\ & \leq & -a z'(r_0) \left(\dfrac{N-1}{R_0} - a + 1 \right). \end{array}$$

If we choose for instance $a = N/R_0 + 1$, we see that $H_1'(r_0) < 0$. This implies that the function H_1 may vanish at most once, and in particular it has a constant sign for large r. Observe that for large enough R_0 the value a can be chosen as close to 1 as we want.

We claim that $H_1 < 0$ for large r. Suppose not. Then $\lambda w^p > -az'$ for large r, so that

$$-z'' > \left(-a + \frac{N-1}{r} + 1\right) z' \geq -Cz',$$

where $C > 0$. Integrating twice between r_0 and r yields

$$-z(r) > Ce^{Cr} + D,$$

for large r. We arrive at a contradiction when $r \to +\infty$. Thus we conclude that $H_1 < 0$, and in the same way it is shown that $H_2 < 0$.

Therefore we have proved that, for large enough r,

$$\begin{cases} -z'(r) > c_1 w^p(r), \\ -w'(r) > c_2 z^s(r), \end{cases}$$

where $c_1 = \lambda/a$ and $c_2 = \mu/a$. Next, we introduce the change of variable

$$t = H(r) := \int_r^{+\infty} c_2 z^s(\tau) d\tau,$$

and we denote $Z(t) = z(r)$ and $W(t) = w(r)$. The function H is well defined since

$$\int_r^{+\infty} c_2 z^s(\tau) d\tau < \int_r^{+\infty} -w'(\tau) d\tau = w(r) < +\infty. \tag{4.3.2}$$

Observe that Z, W are defined in an interval of the form $(0, \delta)$ for some $\delta > 0$ and verify $Z(0) = W(0) = 0$. By (4.3.2), we see that $W(t) > t$. In addition,

$$Z'(t) = -\frac{z'(r)}{c_2 z^s(r)} > \frac{c_1 w^p(r)}{c_2 z^s(r)} = \frac{c_1 W^p(t)}{c_2 Z^s(t)} > \frac{c_1 t^p}{c_2 Z^s(t)}.$$

Hence, $Z^s(t) Z'(t) > c_1/c_2 t^p$ and integrating between 0 and t we obtain

$$Z(t) \geq \left(\frac{c_1(s+1)}{c_2(p+1)} t^{p+1}\right)^{\frac{1}{s+1}}.$$

Rewriting this inequality in the original variables we have

$$z(r) \geq \left(\frac{c_1(s+1)}{c_2(p+1)} H^{p+1}(r)\right)^{\frac{1}{s+1}}.$$

Since $H'(r) = -c_2 z^s(r)$, we see that

$$-H'(r) \geq c_2 \left(\frac{c_1(s+1)}{c_2(p+1)} \right)^{\frac{s}{s+1}} H^{\frac{s(p+1)}{s+1}}(r).$$

Integrating between r_0 and r we get

$$-\int_{r_0}^{r} H^{-\frac{s(p+1)}{s+1}} H' d\tau \geq D(r - r_0), \tag{4.3.3}$$

where

$$D = c_2 \left(\frac{c_1(s+1)}{c_2(p+1)} \right)^{\frac{s}{s+1}}.$$

Next, we have to distinguish two cases depending on the relative position of the exponents $\frac{s(p+1)}{s+1}$ and 1.

First case: $\frac{s(p+1)}{s+1} < 1$. This condition is equivalent to $ps < 1$. By (4.3.3),

$$-\frac{H^{\frac{1-ps}{s+1}}(r) - H^{\frac{1-ps}{s+1}}(r_0)}{\frac{1-ps}{s+1}} \geq D(r - r_0).$$

Letting $r \to +\infty$ and taking into account that $H(r) \to 0$ we arrive at a contradiction. This shows a).

Second case: $\frac{s(p+1)}{s+1} = 1$. This condition is equivalent to $ps = 1$. By (4.3.3),

$$-\log H(r) + \log H(r_0) \geq D(r - r_0).$$

Hence, $H(r) \leq Ce^{-Dr}$, where $C > 0$. Then,

$$\int_{r}^{+\infty} c_2 z^s(\tau) d\tau \leq Ce^{-Dr}. \tag{4.3.4}$$

On the other hand, Lemma 4.2.1 implies

$$\int_{r}^{+\infty} z^s(\tau) d\tau \geq \int_{r}^{+\infty} C\tau^{s(1-N)} e^{-s\tau} d\tau \geq Cr^{s(1-N)} e^{-sr}, \tag{4.3.5}$$

for large r, where we have used again l'Hôpital rule for the last inequality. Combining (4.3.4) and (4.3.5), we see that necessarily

$$D = c_2 \left(\frac{c_1(s+1)}{c_2(p+1)} \right)^{\frac{s}{s+1}} \leq s. \tag{4.3.6}$$

Using that $ps = 1$ and the definition of c_1 and c_2 it is not hard to see that (4.3.6) is equivalent to

$$\lambda \mu^p \leq a^{p+1} \frac{1}{p^p}.$$

Taking into account that a can be chosen as close to 1 as we want, we arrive at a contradiction with (4.3.1). This concludes the proof. $\square$

Our next result is the analogue of Lemma 4.3.1 for supersolutions which blow up at infinity. The proof is entirely similar to that of Lemma 4.3.1, with the only difference that Lemma 4.2.1 gives upper bounds for the solutions, while lower bounds are obtained with the introduction of the functions H_1 and H_2. We will not give the details.

Lemma 4.3.2. *Assume $N \geq 2$. Then:*

a) *If $ps > 1$, then there are no positive supersolutions of* (4.1.1) *which blow up at infinity.*

b) *If $ps = 1$ and*

$$\lambda\mu^p > \frac{1}{p^p}, \tag{4.3.7}$$

then there are no positive supersolutions of (4.1.1) *which blow up at infinity.*

Our next step is intended to show the existence of the value Σ in the statement of Theorems 4.1 and 4.2 when $ps = 1$. It states that the existence of a positive supersolution of

$$\begin{cases} -\Delta u + |\nabla u| &= \lambda_0 v^p, \\ -\Delta v + |\nabla v| &= \mu_0 u^s, \end{cases} \tag{4.3.8}$$

for some pair (λ_0, μ_0) implies that the system admits a positive supersolution for any pair (λ, μ) such that $\lambda\mu^p \leq \lambda_0\mu_0^p$. Moreover, the supersolution so obtained blows up at infinity if and only if (u, v) does.

Lemma 4.3.3. *Assume $N \geq 2$ and $ps = 1$. Let (u, v) be a positive supersolution of system* (4.3.8) *for some $\lambda_0, \mu_0 > 0$. Then, for every pair (λ, μ) such that*

$$\lambda_0\mu_0^p \geq \lambda\mu^p > 0 \tag{4.3.9}$$

there exists a pair $(\widetilde{u}, \widetilde{v})$ such that

$$\begin{cases} -\Delta\widetilde{u} + |\nabla\widetilde{u}| &\geq \lambda\widetilde{v}^p, \\ -\Delta\widetilde{v} + |\nabla\widetilde{v}| &\geq \mu\widetilde{u}^s, \end{cases} \quad \text{in } \mathbb{R}^N \setminus B_{R_0}.$$

Moreover, $(\widetilde{u}, \widetilde{v})$ blows up at infinity if and only if (u, v) does.

Proof. Let $\alpha = \lambda_0/\lambda$, so that $\mu_0\alpha^s \geq \mu$. Consider the functions $\widetilde{u}$, $\widetilde{v}$ defined by

$$\widetilde{u} = \frac{1}{\alpha}u, \ \widetilde{v} = v.$$

Then it is easily seen that $(\widetilde{u}, \widetilde{v})$ is a supersolution of system (4.3.8) with λ_0, μ_0 replaced by λ, μ. Indeed,

$$\begin{cases} -\Delta\widetilde{u} + |\nabla\widetilde{u}| &\geq \dfrac{\lambda_0}{\alpha}\widetilde{v}^p &= \lambda\widetilde{v}^p, \\ -\Delta\widetilde{v} + |\nabla\widetilde{v}| &\geq \mu_0\alpha^s\widetilde{u}^s &\geq \mu\widetilde{u}^s, \end{cases}$$

in $\mathbb{R}^N \setminus B_{R_0}$, and the lemma is proved. □

We finally come to the proof of Theorem 4.1. We will not give that of Theorem 4.2, which is entirely similar (modeled on Lemma 4.3.2).

Proof of Theorem 4.1. Case a) is an immediate consequence of Lemma 4.3.1. As for b), it is easy to check that $(e^{-\alpha|x|}, e^{-\beta|x|})$ is a positive supersolution of (4.1.1) for large R_0 whenever $\alpha = \frac{\varepsilon(p+1)}{ps-1}$, $\beta = \frac{\varepsilon(s+1)}{ps-1}$ and $\varepsilon > 0$ is small enough.

So only part c) remains to be proved. Thus assume $ps = 1$. It is not hard to check that, when λ, μ are small enough, the pair $(e^{-\alpha|x|}, e^{-\beta|x|})$ is a supersolution of (4.1.1) when $\alpha = p\beta$ and $\beta > 0$ is small. On the other hand, by Lemma 4.3.3, if problem (4.1.1) admits a positive supersolution which does not blow up at infinity for some $\lambda_0, \mu_0 > 0$, then the same is true for every values $\lambda, \mu > 0$ such that $\lambda\mu^p \leq \lambda_0\mu_0^p$. Therefore, it makes sense to define

$$\Sigma = \sup \left\{ \lambda\mu^p : \begin{array}{c} \text{problem (4.1.1) admits a positive supersolution} \\ \text{which does not blow up at infinity} \end{array} \right\}.$$

By the above discussion and Lemma 4.3.1, it follows that Σ is finite and positive, while it is clear by its very definition that there exists a positive supersolution of (4.1.1) which does not blow up at infinity whenever $\lambda\mu^p < \Sigma$, while no such supersolutions exist when $\lambda\mu^p > \Sigma$. The proof is concluded. $\square$

Chapter 5

Fourth order equations

Our purpose in the present chapter is to obtain Liouville theorems for positive supersolutions of the problem

$$(-\Delta)^2 u = g(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{5.1.1}$$

where g is a continuous function in $[0,+\infty)$ which is positive in $(0,+\infty)$, as in previous situations.

In order to reduce technicalities to a minimum, we will be dealing now with classical supersolutions $u \in C^4(\mathbb{R}^N \setminus B_{R_0})$, verifying the equation in (5.1.1) pointwise.[1]

5.1 Nonexistence results

As we have already pointed out, equation (5.1.1) is equivalent to the system

$$\begin{cases} -\Delta u = v \\ -\Delta v = g(u), \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{5.1.2}$$

We follow the same approach as in the rest of our Liouville theorems: reduction to a radial setting and obtention of nonexistence results for radially symmetric solutions. However, this procedure cannot be followed for arbitrary functions g, since it is well-known that the method of sub and supersolutions for systems strongly depends on the monotonicity of the involved functions (cf. for instance Chapter 8 in [61]).

Therefore, we will assume in our main results that g is nondecreasing, which will make system (5.1.2) of cooperative type and allow us to perform the desired radial reduction.

[1]The results in this chapter are contained in [20].

Theorem 5.1. *Assume g is continuous and nondecreasing in $[0,+\infty)$ and positive in $(0,+\infty)$. When $1 \le N \le 4$, problem (5.1.1) does not admit any positive, classical supersolution u verifying $-\Delta u > 0$ in $\mathbb{R}^N \setminus B_R$. If $N \ge 5$, then such supersolutions exist if and only if*

$$\int_0^\delta \frac{g(s)}{s^{\frac{2(N-2)}{N-4}}} ds < +\infty \tag{5.1.3}$$

for some $\delta > 0$.

Remark 5.1.1.
a) The sufficiency of condition (5.1.3) for the existence of positive supersolutions of (5.1.1) does not require g to be nondecreasing. Actually, it will be shown in Section 5.3 that condition (5.1.3) is necessary and sufficient for the existence of positive, *radially symmetric solutions* of (5.1.1), without the monotonicity assumption.
b) The requirement that the supersolutions verify $-\Delta u > 0$ is by no means superfluous. When the nonlinearity g verifies

$$\limsup_{t\to 0+} \frac{g(t)}{t} < +\infty \qquad \text{or} \qquad \limsup_{t\to +\infty} \frac{g(t)}{t} < +\infty, \tag{5.1.4}$$

then positive supersolutions u of (5.1.1) can be constructed, which do not verify $-\Delta u > 0$ (cf. Remark 5.4.1 in Section 5.4). This applies in particular to the nonlinearity $g(t) = t^p$ where $p > 0$ is arbitrary.

As already observed, the condition $-\Delta u > 0$ for positive supersolutions is important in order to obtain a Liouville theorem, and examples can be constructed where the supersolutions do not enjoy this property. However, when problem (5.1.1) is posed in $\mathbb{R}^N$ instead of an exterior domain, it can be sometimes concluded that all positive supersolutions verify $-\Delta u > 0$. Thus we restrict our attention in what follows to the problem

$$(-\Delta)^2 u = g(u) \qquad \text{in } \mathbb{R}^N, \tag{5.1.5}$$

where g is as above.

The following result is fairly general.

Theorem 5.2 (Liouville theorem in $\mathbb{R}^N$)**.** *Assume $N \ge 5$. In addition to the hypotheses verified by g in Theorem 5.1, suppose that g is convex and*

$$\liminf_{t\to +\infty} \frac{g(t)}{t^p} > 0 \tag{5.1.6}$$

for some $p > 1$. If

$$\int_0^\delta \frac{g(s)}{s^{\frac{2(N-2)}{N-4}}} ds = +\infty,$$

then problem (5.1.5) does not admit positive classical supersolutions.

As customary, an important particular case of Theorem 5.2 is obtained when we set $g(t) = t^p$, $p > 1$. In this way, we obtain a different proof of one of the results in [58] (see Example 5.2 there).

Corollary 5.1.1. *Assume $N \geq 5$ and $1 < p \leq \frac{N}{N-4}$. Then the problem*

$$(-\Delta)^2 u = u^p \qquad \text{in } \mathbb{R}^N$$

does not admit positive classical supersolutions.

5.2 Reduction to the radial setting

The purpose of this section is to show that, when g is nondecreasing, the existence of a positive supersolution u of (5.1.1) with the additional property $-\Delta u > 0$ implies the existence of a radially symmetric positive solution of the same problem with the same property. We know that, by enlarging R_0 in (5.1.1) we can always assume that u is smooth up to ∂B_{R_0}, and the inequality in (5.1.1) is verified on ∂B_{R_0}. We will tacitly assume that this has been done everywhere.

Lemma 5.2.1. *Assume $N \geq 2$ and g is nondecreasing and positive in $(0, +\infty)$. If there exists a positive supersolution u of (5.1.1) with $-\Delta u > 0$ in $\mathbb{R}^N \setminus B_{R_0}$, then there exists a radially symmetric positive supersolution z of (5.1.1) which also verifies $-\Delta z > 0$ in $\mathbb{R}^N \setminus B_{R_0}$.*

Proof. Denote $v = -\Delta u$. Then it is clear that the pair (u, v) is a positive supersolution of the system

$$\begin{cases} -\Delta u = v \\ -\Delta v = g(u), \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{5.2.1}$$

which is of cooperative type, since g is nondecreasing. Our intention is to show that (5.2.1) admits a positive, radially symmetric solution with the use of the method of sub and supersolutions.

For $R \geq R_0$, we consider the functions:

$$m_u(R) = \min_{|x|=R} u(x), \qquad m_v(R) = \min_{|x|=R} v(x) \tag{5.2.2}$$

(cf. Chapter 3). Now, for $R_1 > R_0$ consider once again the annulus $A(R_0, R_1) = \{x \in$

$\mathbb{R}^N : \ R_0 < |x| < R_1\}$ and the boundary value problem:

$$\begin{cases} \begin{rcases} -\Delta u = v \\ -\Delta v = g(u) \end{rcases} \text{in } A(R_0, R_1) \\ u_{|\partial B_{R_0}} = m_u(R_0) \\ v_{|\partial B_{R_0}} = m_v(R_0) \\ u_{|\partial B_{R_1}} = m_u(R_1) \\ v_{|\partial B_{R_1}} = m_v(R_1). \end{cases} \tag{5.2.3}$$

By definition (5.2.2) we see that (u, v) is a positive supersolution of (5.2.3). However, since g is not smooth enough to directly use the method of sub and supersolutions, we approximate it as follows.

Let $a = \inf_{A(R_0,R_1)} u > 0$, $b = \sup_{A(R_0,R_1)} u < +\infty$ and choose a sequence of nondecreasing smooth functions g_n such that

$$\sup_{[a,b]} |g_n - g| < \frac{1}{n}. \tag{5.2.4}$$

Observe that, if n is large enough, it follows by the positivity of g in $[a, b]$ that $g_n > \frac{1}{n}$ in $[a, b]$. Now consider the problem

$$\begin{cases} \begin{rcases} -\Delta u = v \\ -\Delta v = g_n(u) - \frac{1}{n} \end{rcases} \text{in } A(R_0, R_1) \\ u_{|\partial B_{R_0}} = m_u(R_0) \\ v_{|\partial B_{R_0}} = m_v(R_0) \\ u_{|\partial B_{R_1}} = m_u(R_1) \\ v_{|\partial B_{R_1}} = m_v(R_1). \end{cases} \tag{5.2.5}$$

As a consequence of (5.2.4) we have that (u, v) is a positive supersolution of (5.2.5). Let us construct a comparable subsolution $(\underline{u}, \underline{v})$. For this aim, denote by $\Phi(r)$ the fundamental solution of the Laplacian in $\mathbb{R}^N$, namely:

$$\Phi(r) = \begin{cases} r^{2-N} & N \neq 2 \\ \log r & N = 2. \end{cases}$$

We define:

$$\underline{u}(x) = \frac{m_u(R_0) - m_u(R_1)}{\Phi(R_0) - \Phi(R_1)} \left(\Phi(|x|) - \Phi(R_1)\right) + m_u(R_1)$$
$$\underline{v}(x) = \frac{m_v(R_0) - m_v(R_1)}{\Phi(R_0) - \Phi(R_1)} \left(\Phi(|x|) - \Phi(R_1)\right) + m_v(R_1).$$

It easily follows that $(\underline{u}, \underline{v})$ verifies the boundary conditions in (5.2.5). Moreover, since both $\underline{u}$ and $\underline{v}$ are harmonic functions and $a \leq \underline{u} \leq b$, we see that $(\underline{u}, \underline{v})$ is a subsolution of (5.2.5).

In addition, $-\Delta u \geq 0 = -\Delta \underline{u}$ in $A(R_0, R_1)$ with $u \geq m_u(R_0) = \underline{u}$ on $|x| = R_0$ and $u \geq m_u(R_1) = \underline{u}$ on $|x| = R_1$, so that the maximum principle gives $u \geq \underline{u}$ in $A(R_0, R_1)$ and similarly $v \geq \underline{v}$ in $A(R_0, R_1)$. Hence $(u, v) \geq (\underline{u}, \underline{v})$ in $A(R_0, R_1)$ with the usual ordering.

Thus we may use the method of sub and supersolutions (cf. Theorem A.3 in Appendix) to obtain the existence of a minimal radially symmetric positive solution (z_n, w_n) of (5.2.5) in the interval between the sub and the supersolution.

The next step will be to pass to the limit as $n \to +\infty$ in (5.2.5). First observe that since both $\{z_n\}$ and $\{w_n\}$ are bounded, it is possible to obtain $C^{1,\alpha}$ bounds in $\overline{A(R_0, R_1)}$ by means of standard regularity. Thus by selecting a suitable sequence we see that $(z_n, w_n) \to (z, w)$ in $C^1(\overline{A(R_0, R_1)})^2$, and we may pass to the limit in (5.2.5) to obtain that (z, w) is a positive, weak solution of (5.2.3). To stress the dependence of this solution with respect to R_1, we will denote it by (z_{R_1}, w_{R_1}).

Now a similar argument allows us to pass to the limit as $R_1 \to +\infty$ in (5.2.3), to obtain a nonnegative, radially symmetric weak solution (z, w) of (5.2.1). Since the convergence can be ensured up to ∂B_{R_0}, we have in particular $z = m_u(R_0)$, $w = m_v(R_0)$ there, hence (z, w) is nontrivial. By the strong maximum principle (z, w) is positive.

Observe finally that since (z, w) is radially symmetric, it follows that indeed $(z, w) \in C^2(\mathbb{R}^N \setminus B_{R_0})$, therefore $z \in C^4(\mathbb{R}^N \setminus B_{R_0})$. The proof is concluded. $\square$

Remark 5.2.2. When the function g is not nondecreasing, the method of sub and supersolutions can only be used with several restrictions on g. Indeed, it can be shown that, for a given bounded domain Ω, there exists $L_0 > 0$ such that if g is Lipschitz with Lipschitz constant less than L_0 then there exists a solution between $\underline{u}$ and u. However, it is to be expected that in the case $\Omega = A(R_0, R_1)$, we have $L_0 \to 0$ as $R_1 \to +\infty$.

This problem arises because of the failure of the maximum principle for systems which are not of cooperative type. See for instance [31] for some insight into this problem.

5.3 Proofs in the radially symmetric case

In this section, we will restrict to positive solutions of the radially symmetric version of problem (5.1.1). Since we are only taking into account solutions u verifying in addition

$-\Delta u > 0$, we may consider instead positive solutions of the system

$$\begin{cases} -\Delta u = v \\ -\Delta v = g(u) \end{cases} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{5.3.1}$$

and its radially symmetric version:

$$\begin{cases} -(r^{N-1}z')' = r^{N-1}w \\ -(r^{N-1}w')' = r^{N-1}g(z) \end{cases} \qquad r \geq R_0. \tag{5.3.2}$$

Therefore we will only consider throughout the section positive, classical solutions of system (5.3.2).

We begin by analyzing system (5.3.2) in the case of lower dimensions, that is $1 \leq N \leq 4$. The existence of positive solutions can be easily ruled out by means of more or less standard arguments.

Lemma 5.3.1. *Assume* $1 \leq N \leq 4$ *and* $g \geq 0$. *Then problem* (5.3.2) *does not admit any positive classical solution.*

Proof. We consider first the case $N = 2$ ($N = 1$ is a trivial variation of this and the proof will not be given). We claim that w is nondecreasing. To show it, assume on the contrary that $w'(r_0) < 0$ for some $r_0 > R_0$. Since g is positive, we have $(rw')' \leq 0$ for $r > r_0$. Integrating twice between r_0 and an arbitrary $r > r_0$ we see that

$$w(r) \leq r_0 w'(r_0)(\log r - \log r_0) + w(r_0),$$

which implies $w(r) \to -\infty$ as $r \to +\infty$, against the assumption. Therefore, $w'(r) \geq 0$ for every $r > R$. Since w is positive, we deduce that $w(r) \geq c > 0$ for large r. Then $(rz')' \leq -cr$, and an integration gives

$$z(r) \leq -\frac{c}{4}r^2 + A\log r + B,$$

for some constants A, B, which is a contradiction with the positivity of z. Therefore no positive supersolutions exist in this case.

Next, assume $N \geq 3$. Since w is superharmonic, it is well-known (cf. proof of Theorem 3.5 in Chapter 3, and also Lemma 3.1 in [56] for a direct proof) that the function $r^{N-2}w$ is nondecreasing. Thus there exists $c > 0$ such that $w(r) \geq cr^{2-N}$ when r is large enough. This in turn gives $-(r^{N-1}z')' \geq cr$. Upon integration we obtain, when $N = 3$:

$$z(r) \leq -\frac{c}{2}r + Ar^{-1} + B,$$

and for $N = 4$:

$$z(r) \leq -\frac{c}{2}\log r + Ar^{-2} + B,$$

where A, B are constants. In both cases we arrive at a contradiction when r goes to infinity. This concludes the proof. $\square$

We now turn to the case $N \geq 5$. Theorem 5.1 is a consequence of the following result, which completely solves the question of existence of positive solutions of (5.3.2). Observe that the monotonicity of g is not needed here.

Theorem 5.3. *Assume $N \geq 5$ and g is continuous in $[0, +\infty)$ and positive in $(0, +\infty)$. Then problem (5.3.2) admits a positive classical solution (z, w) if and only if (5.1.3) holds.*

The first step to prove Theorem 5.3 is to obtain some preliminary properties of positive solutions of (5.3.2). This is the content of our next lemma.

Lemma 5.3.2. *Assume $N \geq 5$. Then if (z, w) is a positive solution of (5.3.2), there exists $R_1 > R_0$ such that z and w are decreasing for $r > R_1$ and*

$$\lim_{r\to+\infty} z(r) = \lim_{r\to+\infty} w(r) = 0.$$

Proof. Since z and w are superharmonic, it follows from Lemma 1.3.1 that z and w are monotone for large r. Moreover, $r^{N-1}z'(r)$ is a nonincreasing function so that, after integration:

$$z(r) \leq Ar^{2-N} + B$$

for some constants A, B. Thus z is bounded and the same is true for w. It makes then sense to denote

$$\ell_1 = \lim_{r\to+\infty} z(r) \quad \text{and} \quad \ell_2 = \lim_{r\to+\infty} w(r),$$

which are nonnegative real numbers. We next claim that $\ell_2 = 0$. If not, we would have $(r^{N-1}z'(r))' \sim -\ell_2 r^{N-1}$ as $r \to +\infty$, and an integration would give

$$z(r) \sim -\frac{\ell_2}{2N}r^2 + Cr^{2-N} + D \quad \text{as } r \to +\infty,$$

for some constants C, D, which is a contradiction. Then we must have $\ell_2 = 0$.

Similarly, if $\ell_1 > 0$, then $(r^{N-1}w'(r))' \sim -g(\ell_1)r^{N-1}$ and since $g(\ell_1) > 0$ we would obtain as above that v is not positive for large r. This contradiction shows that $\ell_1 = 0$, concluding the proof of the lemma. $\square$

In order to prove Theorem 5.3, it is more convenient to introduce the change of variables $s = r^{2-N}$, $Z(s) = z(r)$, $W(s) = w(r)$. Then, using Lemma 5.3.2 we see that both Z and W are increasing functions in an interval $(0, s_0)$, for some suitably small s_0, and they vanish at zero. In addition, it is not hard to see that:

$$\begin{cases} -Z'' = cs^{-\gamma}W \\ -W'' = cs^{-\gamma}g(Z) \qquad 0 < s < s_0, \\ Z(0) = W(0) = 0 \end{cases} \tag{5.3.3}$$

where $\gamma = \frac{2(N-1)}{N-2}$ and $c > 0$. Since we are assuming henceforth that $N \geq 5$, it is an easy matter to check that

$$2 < \gamma < 3. \tag{5.3.4}$$

We now prove the main result of this section, Theorem 5.3. The sufficiency of condition (5.1.3) for existence of positive solutions is shown by means of an application of Schauder's fixed point theorem to a suitable operator in a suitable Banach space, but this procedure is considerably more involved than in Chapter 1. The necessity is achieved with an iteration argument which resembles the one used in Chapter 1, and uses as a main ingredient Lemma 1.4.2 there.

Proof of Theorem 5.3. We begin by showing the necessity of condition (5.1.3). Thus assume there exists a positive classical solution (z, w) of (5.3.2). By the previous discussion, there exists a positive solution (Z, W) of (5.3.3). Moreover, both Z and W are increasing and concave by Lemma 5.3.2. Therefore

$$\frac{Z(s)}{s} \geq Z'(s) \quad \text{and} \quad \frac{W(s)}{s} \geq W'(s) \quad \text{for } s \in (0, s_0]. \tag{5.3.5}$$

We claim that there exists a positive constant C such that

$$W'(s)^2 \geq C \int_s^{s_0} \frac{g(Z(\tau))}{Z(\tau)^\nu} W'(\tau)^\nu Z'(\tau) d\tau \tag{5.3.6}$$

for every $s \in (0, s_0]$, where $\nu = 2(N-2)/(N-4)$. Once this is proved, we can apply Lemma 1.4.2 to obtain a contradiction, which shows that (5.1.3) is necessary for existence.

To prove the claim, we integrate the first equation in (5.3.3) and use (5.3.5) to obtain

$$\begin{aligned} \frac{Z(s)}{s} &\geq Z'(s) = Z'(s_0) + c\int_s^{s_0} t^{-\gamma} W(t) dt \geq cW(s) \int_s^{s_0} t^{-\gamma} dt \\ &\geq Cs^{1-\gamma} W(s) \geq Cs^{2-\gamma} W'(s) \end{aligned}$$

for $s \in (0, s_0]$. Therefore

$$Z(s) \geq C s^{3-\gamma} W'(s) \qquad \text{in } (0, s_0]. \tag{5.3.7}$$

We now multiply the second equation in (5.3.3) by W' and integrate in $(0, s_0]$. This yields

$$W'(s)^2 \geq 2c \int_s^{s_0} \frac{g(Z(\tau))}{\tau^\gamma} W'(\tau) d\tau. \tag{5.3.8}$$

Using (5.3.7) and (5.3.5) we have:

$$\begin{aligned} \frac{g(Z(\tau))}{\tau^\gamma} W'(\tau) &= \frac{g(Z(\tau))}{Z(\tau)^\nu} \frac{Z(\tau)^{\nu-1}}{\tau^\gamma} Z(\tau) W'(\tau) \\ &\geq C \frac{g(Z(\tau))}{Z(\tau)^\nu} \tau^{(3-\gamma)(\nu-1)-\gamma} Z(\tau) W'(\tau)^\nu \\ &\geq C \frac{g(Z(\tau))}{Z(\tau)^\nu} Z'(\tau) W'(\tau)^\nu, \end{aligned} \tag{5.3.9}$$

where we have used that $\nu = \frac{2}{3-\gamma}$. Plugging inequality (5.3.9) into (5.3.8) we obtain (5.3.6), concluding the proof of the claim.

To show that condition (5.1.3) is also sufficient for the existence of solutions of (5.3.3), we use Schauder's fixed point theorem. Introduce the space

$$X = \left\{ (Z, W) : \; Z \in C[0, s_0] \cap C^1(0, s_0], \; W \in C^1[0, s_0] \text{ and } \|Z\| < \infty \right\},$$

where

$$\|Z\| = \sup_{(0,s_0]} (s^{\gamma-2} |Z'(s)| + s^{\gamma-3} |Z(s)|).$$

It is not hard to show that X is a Banach space when provided with the natural product norm

$$\|(Z, W)\|_X = \|Z\| + \|W\|_{C^1}.$$

Observe also that $Z(0) = 0$ whenever $(Z, W) \in X$. Our intention in what follows is to show that, under condition (5.1.3), the operator

$$T(Z, W)(s) = \left(\mu s + \int_0^s \int_t^{s_0} \tau^{-\gamma} W(\tau) d\tau dt, \; \mu s - \int_0^s \int_0^t \tau^{-\gamma} g(Z(\tau)) d\tau dt \right),$$

is well defined and has a fixed point in a suitable subset of X, where $\mu > 0$ is small enough. Notice that fixed points of T provide with solutions of (5.3.3).

Let $W_\mu(s) = \mu s$ and

$$Z_\mu(s) = \mu s + \int_0^s \int_t^{s_0} \tau^{-\gamma} W_\mu(\tau) d\tau dt = \mu s + \frac{\mu}{\gamma - 2} \left(\frac{1}{(3-\gamma)} s^{3-\gamma} - s_0^{2-\gamma} s \right).$$

For positive constants C_1 and C_2 to be chosen below, define

$$h(s) = C_1 \int_0^{C_2 s^{3-\gamma}} \frac{g(\tau)}{\tau^\nu} d\tau. \tag{5.3.10}$$

The function h is well-defined by (5.1.3). Moreover, it is continuous and verifies $h(0) = 0$ (recall that $\gamma < 3$ by (5.3.4)). Now introduce the subset of X given by

$$B = \left\{ (Z, W) \in X : \begin{array}{c} \|Z - Z_\mu\| \le a\mu, \ \|W - W_\mu\|_{C^1} \le b\mu, \\ |W'(s) - \mu| \le h(s), \ s \in (0, s_0], \ W(0) = 0 \end{array} \right\},$$

where a, b are positive constants less than one which will be selected later on. It is clear that B is a closed, convex subset of X, and it is also worthy of mention that for every Z such that $(Z, W) \in B$, we have

$$Z \le \mu \left(\frac{1}{(\gamma - 2)(3 - \gamma)} + a \right) s^{3-\gamma} + \mu s \le K_1 \mu s^{3-\gamma} \tag{5.3.11}$$

for every $s \in (0, s_0]$, where $K_1 > 0$ does not depend on μ. Moreover, it also follows that for such Z:

$$Z' \ge \mu + \frac{\mu}{\gamma - 2} \left(s^{2-\gamma} - s_0^{2-\gamma} \right) - a\mu s^{2-\gamma}. \tag{5.3.12}$$

Let us refine this inequality. If we take $0 < a < \min \left\{ \frac{1}{\gamma-2}, s_0^{\gamma-2} \right\}$, it is easy to check that

$$\inf_{s \in (0, s_0]} \left\{ \left(1 - \frac{s_0^{\gamma-2}}{\gamma - 2} \right) s^{\gamma-2} + \frac{1}{\gamma - 2} - a \right\} > 0.$$

Hence, according to (5.3.12) there exists $C > 0$ such that

$$Z' \ge \mu C s^{2-\gamma}, \quad s \in (0, s_0]. \tag{5.3.13}$$

Observe that inequality (5.3.13) also implies that $Z > 0$ in $(0, s_0]$ for every such Z, and it similarly follows that $W(s) \ge \mu(1 - b)s > 0$ in $(0, s_0]$.

We claim that T is a well-defined, compact operator on B which verifies $T(B) \subset B$. The existence of a fixed point (z, w) of T in B is then a consequence of Schauder's fixed point theorem, and the proof would be concluded by noticing that with our choice of a and b both Z and W are positive and (Z, W) is a solution of (5.3.3).

For simplicity, denote $T(Z, W) = (T_1(Z, W), T_2(Z, W))$. Observe first that

$$\begin{aligned} |T_1'(Z, W)(s) - Z_\mu'(s)| &\le \int_s^{s_0} \tau^{-\gamma} |W(\tau) - W_\mu(\tau)| d\tau \\ &\le b\mu \int_s^{s_0} t^{1-\gamma} d\tau \le \frac{b\mu}{\gamma - 2} s^{2-\gamma} \end{aligned}$$

for $s \in (0, s_0]$. Hence, upon integration

$$|T_1(Z,W)(s) - Z_\mu(s)| \leq \frac{b\mu}{(\gamma-2)(3-\gamma)} s^{3-\gamma}$$

for $s \in (0, s_0]$, and this leads to

$$\|T_1(Z,W) - Z_\mu\| \leq \frac{b\mu(4-\gamma)}{(\gamma-2)(3-\gamma)} \leq a\mu,$$

provided that we choose b small enough. Fix such a b. On the other hand:

$$\begin{aligned} |T_2'(Z,W)(s) - W_\mu'(s)| &= \int_0^s \tau^{-\gamma} g(Z(\tau)) d\tau \\ &= \int_0^s \frac{g(Z(\tau))}{Z(\tau)^\nu} \left(\frac{Z(\tau)}{\tau^{3-\gamma}}\right)^\nu \frac{\tau^{2-\gamma}}{Z'(\tau)} Z'(\tau) d\tau, \end{aligned}$$

where we have used that $\nu = \frac{2}{3-\gamma}$. Now we take into account the inequalities (5.3.11) and (5.3.13) to arrive at

$$\left(\frac{Z(\tau)}{\tau^{3-\gamma}}\right)^\nu \frac{\tau^{2-\gamma}}{Z'(\tau)} \leq K_2 \mu^{\nu-1},$$

for every $\tau \in (0, s_0]$, where $K_2 > 0$ is a constant that does not depend on μ. Thus

$$|T_2'(Z,W)(s) - W_\mu'(s)| \leq K_2 \mu^{\nu-1} \int_0^{Z(s)} \frac{g(t)}{t^\nu} dt \leq K_2 \mu^{\nu-1} \int_0^{K_1 \mu s^{3-\gamma}} \frac{g(t)}{t^\nu} dt.$$

At this stage, we choose the constants C_1 and C_2 in the definition (5.3.10) of h. Setting $C_1 = K_2 \mu^{\nu-1}$, $C_2 = K_1 \mu$, we see that:

$$|T_2'(Z,W)(s) - W_\mu'(s)| \leq h(s), \tag{5.3.14}$$

and integrating this inequality

$$|T_2(Z,W)(s) - W_\mu(s)| \leq \int_0^s h(t) dt \leq h(s)s, \tag{5.3.15}$$

since h is increasing. Notice that also $h(s) \leq C\mu^{\nu-1}$ in $(0, s_0]$, for a positive constant C which does not depend on μ provided that, say, $\mu < 1$. Therefore, since $\nu > 2$, it is possible to have $\|T_2(Z,W) - W_\mu\|_{C^1} \leq b\mu$ if μ is small enough. To summarize, we have shown that $T(B) \subset B$ when μ is chosen small and in particular T is well defined.

Now we will show that the operator T is compact on B. Let $\{(Z_n, W_n)\} \subset B$. From the invariance of B under T we see that both $\{\|T_1(Z_n, W_n)\|\}$ and $\{\|T_2(Z_n, W_n)\|_{C^1}\}$ remain bounded. This also implies that $\{T_1(Z_n,W_n)\}$ is bounded in $[0,s_0]$, while $\{T_1'(Z_n,W_n)\}$ will only be bounded on compact sets of $(0,s_0]$. Moreover,

$$\begin{aligned} T_1''(Z_n,W_n)(s) &= -s^{-\gamma}W_n(s)\\ T_2''(Z_n,W_n)(s) &= -s^{-\gamma}g(Z_n(s)), \end{aligned}$$

so that $\{T_1''(Z_n,W_n)(s)\}$ and $\{T_2''(Z_n,W_n)(s)\}$ are bounded in compact sets of $(0,s_0]$. Therefore, up to a subsequence and by means of a diagonal argument we may assume that

$$\begin{aligned} T_1(Z_n,W_n) &\to x \quad \text{in } C^1_{\text{loc}}(0,s_0]\\ T_2(Z_n,W_n) &\to y \quad \text{in } C^1_{\text{loc}}(0,s_0]. \end{aligned} \tag{5.3.16}$$

To conclude the proof, we only need to show that we indeed have the convergence $T(Z_n,W_n)\to(x,y)$ in X.

First of all, by (5.3.14) and (5.3.15) we see that $|T_2'(Z_n,W_n)(s)-W_\mu'(s)|\le h(s)$ and $|T_2(Z_n,W_n)(s)-W_\mu(s)|\le h(s)s$ for every $s\in(0,s_0]$. Passing to the limit we have $|y'(s)-W_\mu'(s)|\le h(s)$ and $|y(s)-W_\mu(s)|\le h(s)s$ in $(0,s_0]$. Hence

$$\left.\begin{aligned} |T_2'(Z_n,W_n)(s)-y'(s)| &\le 2h(s)\\ |T_2(Z_n,W_n)(s)-y(s)| &\le 2h(s)s \end{aligned}\right. \quad \text{in } (0,s_0].$$

Choose $\varepsilon>0$. Since $h(0)=0$, there exists $\delta>0$ such that

$$\sup_{s\in(0,\delta]}\left(|T_2(Z_n,W_n)(s)-y(s)|+|T_2'(Z_n,W_n)(s)-y'(s)|\right)\le\varepsilon.$$

On the other hand, by (5.3.16) we have that $T_2'(Z_n,W_n)\to y'$ and $T_2(Z_n,W_n)\to y$ uniformly in $[\delta,s_0]$. Therefore, if n is large enough:

$$\|T_2(Z_n,W_n)-y\|_{C^1}\le\varepsilon.$$

To obtain a similar inequality for the first components, observe that $|W_n'(s)-W_\mu'(s)|\le h(s)$, whence for $m>n$ we see that

$$\begin{aligned} |W_n'(s)-W_m'(s)| &\le 2h(s),\\ |W_n(s)-W_m(s)| &\le 2h(s)s \end{aligned}$$

for every $s\in(0,s_0]$. This yields:

$$\begin{aligned} |T_1'(Z_n,W_n)(s)-T_1'(Z_m,W_m)(s)| &\le \int_s^{s_0}\tau^{-\gamma}|W_n(\tau)-W_m(\tau)|d\tau\\ &\le 2\int_s^{s_0}\tau^{1-\gamma}h(\tau)d\tau. \end{aligned}$$

Now let $\varepsilon > 0$ be as above and choose $\eta \in (0, s_0]$. Letting $m \to \infty$ in the last expression allows us to write:

$$\begin{aligned}|T_1'(Z_n, W_n)(s) - x'(s)| &\le 2\int_s^\eta \tau^{1-\gamma} h(\tau)d\tau + 2\int_\eta^{s_0} \tau^{1-\gamma} h(\tau)d\tau \\ &\le 2h(\eta)\frac{s^{2-\gamma}}{\gamma-2} + 2\int_\eta^{s_0} \tau^{1-\gamma} h(\tau)d\tau\end{aligned}$$

whenever $s \in (0, \eta]$. Hence

$$s^{\gamma-2}|T_1'(z_n, w_n)(s) - x'(s)| \le \frac{2}{\gamma-2} h(\eta) + C(\eta)s^{\gamma-2},$$

for every $s \in (0, \eta]$, where $C(\eta)$ depends only on η. Choosing η small enough, we can have the first term in the above sum less than $\varepsilon/2$, say. If we now restrict further s to the interval $(0, \delta]$, where $\delta < \eta$ is small, we get

$$\sup_{s\in(0,\delta]} s^{\gamma-2}|T_1'(Z_n, W_n)(s) - x'(s)| \le \varepsilon.$$

The supremum in the interval $[\delta, s_0]$ can be controlled as above, with the use of (5.3.16). After an integration, we see that $\|T_1(Z_n, W_n) - x\| \le \varepsilon$ if n is large enough. Therefore

$$\|T(Z_n, W_n) - (x, y)\|_X \le \varepsilon$$

if n is large enough. Thus the operator T is compact on B, as was to be shown. The proof is concluded. □

5.4 Completion of proofs

In this section we collect the proofs of our main results in this chapter, Theorems 5.1 and 5.2. Corollary 5.1.1 follows at once from Theorem 5.2, so its proof will not be given.

Proof of Theorem 5.1. After the results of the previous sections, the proof is almost immediate. Since g is nondecreasing, we see thanks to Lemma 5.2.1 that the existence of a positive supersolution u of (5.1.1) with $-\Delta u > 0$ is equivalent to the existence of a positive, radially symmetric solution of the system (5.3.2). Then Theorem 5.1 is a direct consequence of Theorem 5.3. □

Proof of Theorem 5.2. We claim that every positive classical supersolution of (5.1.5) verifies $-\Delta u > 0$ in $\mathbb{R}^N$, so that Theorem 5.1 can be applied. The proof of this fact is an adaptation of the proof of Theorem 3.1 in [74], and proceeds by contradiction with an iteration argument which is somehow similar to that in the proof of Lemma 1.4.2.

Let $v = -\Delta u$ and assume that $v(x_0) < 0$ for some $x_0 \in \mathbb{R}^N$. With no loss of generality, we may assume that $x_0 = 0$. For $r > 0$ consider the averages

$$\overline{u}(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u(y) d\sigma(y),$$

and $\overline{v}(r)$ with the same meaning. Since $u \in C^4(\mathbb{R}^N)$, we have $\overline{u}, \overline{v} \in C^2[0, +\infty)$. Moreover, by the convexity of g and Jensen's inequality it is not hard to see that

$$\begin{aligned} -(r^{N-1}\overline{u}')' &= r^{N-1}\overline{v} \\ -(r^{N-1}\overline{v}')' &\geq r^{N-1}g(\overline{u}) \end{aligned} \tag{5.4.1}$$

for $r > 0$.

It is well-known that

$$\overline{v}'(r) = \frac{1}{|\partial B_r|} \int_{B_r} \Delta v(y) dy \leq -\frac{1}{|\partial B_r|} \int_{B_r} g(u(y)) dy < 0.$$

Hence $\overline{v}(r) \leq \overline{v}(0) = v(0) < 0$. Whence, integrating the first equation in (5.4.1) twice between 0 and r we arrive at

$$\overline{u}(r) \geq u(0) - \frac{\overline{v}(0)}{2N} r^2 \geq c_0 r^2 \quad \text{for } r > 0, \tag{5.4.2}$$

where $c_0 > 0$. It is worthy of mention that also

$$\overline{u}'(r) = -\frac{1}{r^{N-1}} \int_0^r t^{N-1} \overline{v}(t) dt > 0.$$

We will fix a value $r_0 > 0$ such that $\overline{u}(r) \geq 1$, say, when $r \geq r_0$. On the other hand, by our assumption (5.1.6), we see that there exists $a > 0$ such that $g(t) \geq at^p$ when $t \geq 1$. Hence

$$g(\overline{u}(r)) \geq a\overline{u}(r)^p \qquad \text{for } r \geq r_0. \tag{5.4.3}$$

Our next aim is to show that there exist sequences of positive numbers $\{c_k\}$, $\{\alpha_k\}$ and $\{r_k\}$ with the property that

$$\overline{u}(r) \geq c_k r^{\alpha_k} \qquad \text{when } r \geq r_k, \tag{5.4.4}$$

where r_0 is as fixed above, the value of c_0 coincides with that given in (5.4.2) and $\alpha_0 = 2$.

Inequality (5.4.4) is shown by induction on k. Thus assume that $\overline{u}(r) \geq c_k r^{\alpha_k}$ if $r \geq r_k$. Using (5.4.3) and the second equation in (5.4.1), we see after integrating between r_k and r that

$$-r^{N-1}\overline{v}'(r) \geq \frac{ac_k^p}{\alpha_k p + N} \left(r^{\alpha_k p + N} - r_k^{\alpha_k p + N} \right) \geq \frac{ac_k^p}{2(\alpha_k p + N)} r^{\alpha_k p + N},$$

provided that $r \geq 2^{\frac{1}{\alpha_k p+N}} r_k$. Integrating once again, it follows similarly that

$$-\overline{v}(r) \geq \frac{a c_k^p}{4(\alpha_k p+N)(\alpha_k p+2)} r^{\alpha_k p+2}$$

if $r \geq 2^{\frac{1}{\alpha_k p+2}} 2^{\frac{1}{\alpha_k p+N}} r_k$. We deduce in particular that

$$-\overline{v}(r) \geq \frac{a c_k^p}{4(\alpha_k p+N)^2} r^{\alpha_k p+2} =: d_k r^{\alpha_k p+2}$$

if $r \geq 2^{\frac{2}{\alpha_k p+2}} r_k$. We now introduce this inequality in the first equation of (5.4.1) and integrate to obtain

$$\overline{u}'(r) \geq \frac{d_k}{2(\alpha_k p+N+2)} r^{\alpha_k p+3}$$

if $r \geq 2^{\frac{1}{\alpha_k p+N+2}} 2^{\frac{2}{\alpha_k p+2}} r_k$, and similarly

$$\overline{u}(r) \geq \frac{d_k}{4(\alpha_k p+N+2)^2} r^{\alpha_k p+4}$$

if $r \geq 2^{\frac{4}{\alpha_k p+2}} r_k$. Thus, we see that (5.4.4) holds true with

$$\alpha_{k+1} = \alpha_k p + 4 \tag{5.4.5}$$

$$c_{k+1} = \frac{a c_k^p}{16(\alpha_k p+N+2)^4} \tag{5.4.6}$$

$$r_{k+1} = 2^{\frac{4}{\alpha_k p+2}} r_k. \tag{5.4.7}$$

Our next purpose is to analyze the previous relations to obtain properties of the sequences $\{\alpha_k\}$, $\{c_k\}$ and $\{r_k\}$. The first remark in this regard is the following: by (5.4.4) with $k=1$ we see that $\overline{u}(r) \geq c_1 r^{2p+4}$ if $r \geq r_1$, so that $\liminf_{r\to+\infty} \overline{u}(r)/r^2 = +\infty$. Thus, by taking r_0 larger if necessary, we can always have c_0 as large as desired.

As for (5.4.5), it is a linear difference equation with constant coefficients, whose solution can be easily found to be

$$\alpha_k = \frac{2(p+1)}{p-1} p^k - \frac{4}{p-1}.$$

Thus there exist positive constants μ_1 and μ_2 not depending on k such that

$$\mu_1 p^k \leq \alpha_k \leq \mu_2 p^k$$

for every $k \in \mathbb{N}$. This inequality readily implies

$$r_k \leq r_\infty := 2^{\sum_{k=1}^{\infty} \frac{4}{\mu_1 p^{k+1}}} r_0 < +\infty,$$

and also

$$c_{k+1} \geq \lambda \frac{c_k^p}{p^{4k}},$$

for every $k \in \mathbb{N}$ and some $\lambda > 0$ independent of k (observe also that λ is independent of the choice of c_0 above). Setting $d_k = \log c_k$, we see that d_k verifies the linear difference inequality

$$d_{k+1} \geq \log \lambda + p d_k - 4k \log p,$$

which can be used to prove by induction that

$$d_k \geq \left(d_0 - \frac{B - \log \lambda}{p-1} \right) p^k + Bk + \frac{B - \log \lambda}{p-1} \tag{5.4.8}$$

for every k, where $B = 4 \log p/(p-1)$. Recalling that c_0 (hence d_0) can be taken as large as desired, (5.4.8) implies the existence of positive constants A and θ such that

$$c_k \geq A p^{\frac{4k}{p-1}} e^{\theta p^k}$$

for every k. Summing up, we have obtained that

$$\overline{u}(r) \geq A p^{\frac{4k}{p-1}} e^{\theta p^k} r^{\alpha_k} \qquad \text{whenever } r > r_\infty,$$

and we arrive at a contradiction by fixing such an r and letting $k \to +\infty$.

This contradiction shows that $v \geq 0$ in $\mathbb{R}^N$, and the strong maximum principle implies $v > 0$ in $\mathbb{R}^N$, that is $-\Delta u > 0$ in $\mathbb{R}^N$. The proof is concluded with a simple application of Theorem 5.1. □

Remark 5.4.1. As observed in Remarks 5.1.1 b), it is possible to construct positive supersolutions of problem (5.1.1) not verifying the condition $-\Delta u > 0$, at least when g verifies (5.1.4):

$$\limsup_{t \to 0+} \frac{g(t)}{t} < +\infty \qquad \text{or} \qquad \limsup_{t \to +\infty} \frac{g(t)}{t} < +\infty.$$

To see this, we will look for a supersolution of the form

$$u(r) = e^{\alpha r}, \qquad r > 0,$$

where $\alpha \in \mathbb{R}$ and $r = |x|$. It is not hard to check that

$$(-\Delta)^2 u = \left(\alpha^4 + \frac{2(N-1)}{r} \alpha^3 + \frac{(N-1)(N-3)}{r^2} \alpha^2 - \frac{(N-1)(N-3)}{r^3} \alpha \right) e^{\alpha r}.$$

Thus, if $r \geq R_0$ and $|\alpha|$ is large enough, we see that

$$(-\Delta)^2 u \geq \frac{1}{2} \alpha^4 e^{\alpha r}.$$

Hence to have a positive supersolution of (5.1.1) it suffices to have

$$e^{-\alpha r} g(e^{\alpha r}) \leq \frac{1}{2}\alpha^4.$$

This is certainly possible by choosing $|\alpha|$ large enough, with the only prevention that, if the first condition in (5.1.4) holds then we take $\alpha < 0$, while if the second holds we have to choose $\alpha > 0$. Observe finally that the supersolution so constructed verifies $-\Delta u = -(\alpha^2 + \frac{N-1}{r}\alpha)e^{\alpha r} < 0$ if r is large enough.

Appendix

In this Appendix we collect, for the reader's convenience, some results which have been used in the memoir. They are concerned with the obtention of bounds for the gradient of solutions and the method of sub and supersolutions.

A.1 Bounds for the gradient

The next lemma is a simplified version of the proof of Theorem IV.1 in [54], adapted to deal with one-dimensional situations.

Lemma A.1. *Assume $q > 1$, $c < d$ and let $u \in C^2(c,d)$ be a function such that*

$$-\varepsilon u'' + \nu |u'|^q \le k \quad \text{in } (c,d) \tag{A.1}$$

with $\nu, k > 0$ and $\varepsilon \in (0,1]$. Then for every $[c', d'] \subset (c,d)$ there exists $C = C(q, c', d', \nu) > 0$ such that

$$|u'| \le C(k+1)^{\frac{1}{q}} \quad \text{in } [c', d'].$$

In addition, under the same hypotheses, if $u \in C^1[c,d]$ and $|u'(c)|$, $|u'(d)| \le M$ then

$$|u'| \le \max \left\{ M, \left(\frac{k}{\nu} \right)^{\frac{1}{q}} \right\} \quad \text{in } [c,d].$$

Proof. We will assume with no loss of generality that $\nu = 1$. Choose $\xi \in C_0^\infty(c,d)$ such that $0 \le \xi \le 1$ and $\xi \equiv 1$ in $[c', d']$. We take $\phi(r) = \xi^\theta(r)$, with $\theta > 0$, and

$$z(r) = \phi(r)(u'(r))^2.$$

Since z is nonnegative and has compact support in (c,d), it reaches a maximum at a point $x_0 \in (c,d)$. Hence $z'(x_0) = 0$, which yields

$$u''(x_0) = -\frac{\phi'(x_0) u'(x_0)}{2\phi(x_0)}$$

(observe that necessarily $\phi(x_0) > 0$). By (A.1),

$$\left|\frac{z(x_0)}{\phi(x_0)}\right|^{\frac{q}{2}} = |u'(x_0)|^q \le k + \varepsilon u''(x_0) = k - \frac{\varepsilon\phi'(x_0)u'(x_0)}{2\phi(x_0)} \le k + \frac{\varepsilon|\phi'(x_0)|z^{\frac{1}{2}}(x_0)}{2\phi^{\frac{3}{2}}(x_0)}.$$

Thus,

$$\left(\frac{z(x_0)}{\phi(x_0)}\right)^q \le \left(k + \frac{\varepsilon|\phi'(x_0)|z^{\frac{1}{2}}(x_0)}{2\phi^{\frac{3}{2}}(x_0)}\right)^2 \le 2k^2 + \frac{\varepsilon^2|\phi'(x_0)|^2 z(x_0)}{2\phi^3(x_0)},$$

so that,

$$z^q(x_0) \le 2k^2\phi^q(x_0) + \frac{1}{2}|\phi'(x_0)|^2\phi^{q-3}(x_0)z(x_0).$$

Next, observe that

$$(\phi')^2\phi^{q-3} = \theta^2\xi^{2\theta-2}(\xi')^2\xi^{\theta(q-3)} \le C\xi^{\theta(q-1)-2}.$$

Hence, if we choose, say, $\theta = 2/(q-1)$ we obtain

$$z^q(x_0) \le 2k^2 + Cz(x_0) \le 2k^2 + \frac{1}{2}z^q(x_0) + C.$$

Then, $z^q(x_0) \le 4k^2 + C$, therefore $(\phi(u')^2)^q \le 4k^2 + C$ in (c,d) and $(u')^{2q} \le 4k^2 + C$ in (c',d'), so that

$$|u'| \le (4k^2 + C)^{\frac{1}{2q}} \le C(k+1)^{\frac{1}{q}} \quad \text{in } [c',d'].$$

This shows the first part of the lemma. To prove the second, assume that $u \in C^1[c,d]$ and there exists $M > 0$ such that $|u'(c)|, |u'(d)| \le M$. Then, if u' reaches a minimum or maximum at $x_0 \in (c,d)$ we have that $u''(x_0) = 0$, and it immediately follows that $|u'|^q \le k$. Hence the conclusion. □

Remark A.2. The previous lemma is not only applicable to the one-dimensional problem, but also to its radially symmetric $N-$dimensional version, namely:

$$-\varepsilon\left(u'' + \frac{N-1}{r}u'\right) + \nu|u'|^q \le k \qquad \text{in } (c,d),$$

at least in the case where $d > c > 0$. Indeed, using that

$$\frac{N-1}{r}|u'| \le \frac{\nu}{2}|u'|^q + C,$$

we obtain $-\varepsilon u'' + \frac{\nu}{2}|u'|^q \le k + C$ in (c,d), and Lemma A.1 can be applied.

A.2 The method of sub and supersolutions

We introduce next some instances of this well-known method which are suitable for our purposes. In all cases, we will consider only radially symmetric solutions in the annulus $A(R_0, R_1) = \left\{x \in \mathbb{R}^N : R_0 < |x| < R_1\right\}$, where $R_1 > R_0 > 0$. Proofs will only be given in those cases where we could not find a pertinent reference.

We begin by considering the problem:

$$\begin{cases} -\Delta u &= f(u)|\nabla u|^q & \text{in } A(R_0, R_1), \\ u &= c_1 & \text{on } \partial B_{R_0}, \\ u &= c_2 & \text{on } \partial B_{R_1}, \end{cases} \tag{A.2}$$

where $0 < q \leq 2$, $f \in C(\mathbb{R})$, and $c_1, c_2 \in \mathbb{R}$.

In this context, we require that both the subsolution and the supersolution belong to $H^1(A(R_0, R_1)) \cap L^\infty(A(R_0, R_1))$.

Theorem A.1. *Assume $f \in C(\mathbb{R})$, $0 < q \leq 2$, $c_1, c_2 \in \mathbb{R}$ and that there exist a weak subsolution $\underline{u}$ and a weak supersolution $\overline{u}$ of* (A.2) *with $\underline{u} \leq \overline{u}$ in $A(R_0, R_1)$. Then there exist a minimal and a maximal weak solution of* (A.2) *in the order interval $[\underline{u}, \overline{u}]$. Moreover, if $\underline{u}$ (resp. $\overline{u}$) is radially symmetric then so is the minimal (resp. maximal) weak solution.*

Proof. The existence of a weak solution of (A.2) in the interval $[\underline{u}, \overline{u}]$ is given by Theorem 2.1 in [16]. It is worth remarking that the proof can easily be reduced to the case $c_1 = c_2 = 0$ (see also [33]). Thus we are only proving that there exists a minimal weak solution (the existence of a maximal weak solution is similarly proved). This proof is taken from [60]. We define

$$B = \inf\left\{\int_{A(R_0,R_1)} u : u \in [\underline{u}, \overline{u}] \text{ is a solution of (A.2)}\right\}.$$

Of course, B is well defined and by definition, there exists a sequence of weak solutions $\{u_n\}_{n=1}^\infty \subset [\underline{u}, \overline{u}]$ such that $\int_{A(R_0,R_1)} u_n \to B$. Since $\{u_n\}$ is a uniformly bounded sequence we can use the results in Section 4.4, Chapter IV of [49] to get that the sequence $\{\nabla u_n\}$ is also uniformly bounded. Thus, by passing to a subsequence, we may assume that $u_n \to u$ weakly in $H^1(A(R_0, R_1))$ and uniformly in $\overline{A(R_0, R_1)}$), where $u \in [\underline{u}, \overline{u}]$ is a weak solution of (A.2) with $\int_{A(R_0,R_1)} u = B$. We claim that u is the minimal weak solution.

Let $v \in [\underline{u}, \overline{u}]$ be an arbitrary weak solution of (A.2). Since $w = \min\{u, v\}$ is a supersolution of (A.2) with $w \geq \underline{u}$, there exists a weak solution $z \in [\underline{u}, w]$ of (A.2). But

then $z \leq w \leq u$ and
$$B \leq \int_{A(R_0,R_1)} z \leq \int_{A(R_0,R_1)} u = B.$$
If follows that $u = z \leq w \leq v$. This concludes the first part of the proof.

To show the last assertion in the statement, assume $\underline{u}$ is radially symmetric, and let $u_{\min}$ denote the minimal weak solution given by the first part of the lemma. Take an arbitrary rotation Q and let

$$\overline{\overline{u}}(x) = \min\{u_{\min}(Qx), \overline{u}(x)\}.$$

It is easily seen that $\overline{\overline{u}}$ is a supersolution, which verifies, by the radial symmetry of $\underline{u}$ that $\overline{\overline{u}} \geq \underline{u}$. Then, there exists a weak solution w in the order interval $[\underline{u}, \overline{\overline{u}}] \subset [\underline{u}, \overline{u}]$, hence $w \geq u_{\min}$. On the other hand $w(x) \leq u_{\min}(Qx)$, and it follows that

$$u_{\min}(x) \leq u_{\min}(Qx).$$

Since Q is an arbitrary rotation, this shows that $u_{\min}$ is radially symmetric. □

Next, we provide a similar statement as Theorem A.2 for the related problem

$$\begin{cases} -\Delta u + |\nabla u|^q &= f(u) \quad \text{in } A(R_0, R_1), \\ u &= c_1 \quad \text{on } \partial B_{R_0}, \\ u &= c_2 \quad \text{on } \partial B_{R_1}, \end{cases} \tag{A.3}$$

where as above, $q > 1$, $c_1, c_2 \in \mathbb{R}$ and $f \in C(\mathbb{R})$.

It is important to stress that the classical results in the method of sub and supersolutions (cf. for instance [4], [16] and [33]) require $q \leq 2$. To deal with arbitrary values of $q > 1$ we make use of the results in [53] and [54].

Theorem A.2. *Assume $q > 1$, $f \in C(\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. If there exists a subsolution $\underline{u} \in C^2(\overline{A(R_0, R_1)})$ and a supersolution $\overline{u} \in C^2(\overline{A(R_0, R_1)})$ of (A.3) such that $\underline{u} = \overline{u}$ on $\partial A(R_0, R_1)$, then there exists a solution $u \in [\underline{u}, \overline{u}]$ of (A.3). Moreover, if $\underline{u}, \overline{u}$ are radially symmetric then so is u.*

Proof. It is a direct consequence of Theorem III.1 in [53]. The proofs there only deals with sub and supersolutions vanishing on $\partial A(R_0, R_1)$, but it is easily seen that it can be adapted to the present situation since $\underline{u} = \overline{u}$ on the boundary. In addition, let us remark that $f \in C^1(\mathbb{R})$ is assumed there. However, this restriction can be relaxed to $f \in C(\mathbb{R})$ in view of the arguments leading to Theorem III.1 in [54]. □

A.3 Sub and supersolutions for systems

It is now the turn to consider a version of the previous theorems for the elliptic systems that have been studied in this work.

We first analyze the system

$$\begin{cases} -\Delta u = f(v) & \\ -\Delta v = g(u) & \text{in } A(R_0, R_1) \\ u_{|\partial B_{R_0}} = c_1 & \\ v_{|\partial B_{R_0}} = d_1 & \\ u_{|\partial B_{R_1}} = c_2 & \\ v_{|\partial B_{R_1}} = d_2, & \end{cases} \tag{A.4}$$

where f, g are nondecreasing functions and $c_1, c_2, d_1, d_2 \in R$. The following result is a simplified version of Theorem 4.1 in Chapter 8 in [61].

Theorem A.3. *Assume f, g are nondecreasing and locally Lipschitz functions and let $c_1, c_2, d_1, d_2 \in \mathbb{R}$. If there exists a subsolution $(\underline{u}, \underline{v}) \in (C^2(A(R_0, R_1)) \cap C(\overline{A(R_0, R_1)}))^2$ and a supersolution $(\overline{u}, \overline{v}) \in (C^2(A(R_0, R_1)) \cap C(\overline{A(R_0, R_1)}))^2$ of* (A.4) *such that $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$ in $A(R_0, R_1)$ and $\underline{u} = \underline{v}$ and $\overline{u} = \overline{v}$ on $\partial A(R_0, R_1)$, then there exist a minimal and a maximal solution of* (A.4) *between the sub and supersolution.*

Finally, we deal with a system which involves gradient terms as in (A.3):

$$\begin{cases} -\Delta u + |\nabla u|^q = f(v) & \\ -\Delta v + |\nabla v|^q = g(u) & \text{in } A(R_0, R_1) \\ u_{|\partial B_{R_0}} = c_1 & \\ v_{|\partial B_{R_0}} = d_1 & \\ u_{|\partial B_{R_1}} = c_2 & \\ v_{|\partial B_{R_1}} = d_2, & \end{cases} \tag{A.5}$$

where $q > 1$ and the functions $f, g \in C(\mathbb{R})$ are nondecreasing.

Theorem A.4. *Assume $q > 1$, $f, g \in C^1(\mathbb{R})$ are nondecreasing functions and there exist a subsolution $(\underline{u}, \underline{v}) \in C^1(A(R_0, R_1))^2$ and a supersolution $(\overline{u}, \overline{v}) \in C^1(A(R_0, R_1))^2$ of* (A.5). *If $\overline{u}, \overline{v}, \underline{u}, \underline{v}$ are radially symmetric functions and verify $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ in $A(R_0, R_1)$ together with $\underline{u} = \overline{u}$ and $\underline{v} = \overline{v}$ on $\partial A(R_0, R_1)$, then there exists a classical radially symmetric solution (u, v) of* (A.5) *with $\underline{u} \le u \le \overline{u}$ and $\underline{v} \le v \le \overline{v}$ in $A(R_0, R_1)$.*

Proof. We begin by solving the system

$$\begin{cases} -\Delta u_1 + |\nabla u_1|^q &= f(\underline{v}) \\ -\Delta v_1 + |\nabla v_1|^q &= g(\underline{u}) \qquad \text{in } A(R_0,R_1) \\ u_{|\partial B_{R_0}} = c_1 \\ v_{|\partial B_{R_0}} = d_1 \\ u_{|\partial B_{R_1}} = c_2 \\ v_{|\partial B_{R_1}} = d_2, \end{cases} \tag{A.6}$$

Note that system A.6 is indeed uncoupled. For instance, regarding the equation involving u_1 we need to solve

$$\begin{cases} -\Delta w + |\nabla w|^q &= f(\underline{v}), \quad \text{in } A(R_0,R_1), \\ w &= h(x), \quad \text{on } \partial A(R_0,R_1), \end{cases} \tag{A.7}$$

where

$$h(x) = \begin{cases} c_1 & |x| = R_0 \\ c_2 & |x| = R_1. \end{cases}$$

The function f is nondecreasing, so it is clear that $\overline{u}$ is a supersolution of this problem, while $\underline{u}$ is a subsolution. By Theorem III.1 in [54], we can ensure the existence of a solution u_1 verifying $\underline{u} \le u_1 \le \overline{u}$. In addition, the solution u_1 is radially symmetric, so we deduce that it is a classical solution. It follows in a similar way that $\underline{v} \le v_1 \le \overline{v}$. Now we can define (u_2, v_2) by replacing $\underline{u}$ and $\underline{v}$ in (A.6) by u_1 and v_1, respectively. By the monotocity of f we have

$$-\Delta u_2 + |\nabla u_2|^q = f(v_1) \ge f(\underline{v}) = -\Delta u_1 + |\nabla u_1|^q,$$

so that by comparison $u_2 \ge u_1$ since $u_1 = u_2 = \underline{u}$ on $\partial A(R_0,R_1)$.

By induction we obtain two sequences of positive radially symmetric functions $\{u_n\}_{n\ge1}$, $\{v_n\}_{n\ge1}$ such that

$$\begin{aligned} \underline{u} \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le \overline{u}, \\ \underline{v} \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le \overline{v}, \end{aligned} \tag{A.8}$$

in $A(R_0,R_1)$. Let us obtain appropriate bounds for the solutions in order to pass to the limit. By (A.8), we have that $|u_n|, |v_n| \le C$, and

$$-u_n'' - \frac{N-1}{r} u_n' + |u_n'|^q \le C.$$

In addition, $u_n(R_0) = \underline{u}(R_0) = \overline{u}(R_0)$ and $\underline{u} \le u_n \le \overline{u}$, thus $\underline{u}'(R_0) \le u_n'(R_0) \le \overline{u}'(R_0)$. Similarly, $\overline{u}'(R_1) \le u_n'(R_1) \le \underline{u}'(R_1)$. We may then apply Lemma A.1 (see also Remark A.2) to obtain that $|u_n'| \le C$ in $\overline{A(R_0,R_1)}$, and in the same way $|v_n'| \le C$ in $\overline{A(R_0,R_1)}$.

As a consequence, $|u_n''|, |v_n''| \le C$. Hence we deduce that $u_n \to u$ and $v_n \to v$ in $C^1(\overline{A(R_0, R_1)})$ where (u, v) is a solution of (A.5) in the weak sense. But u and v are radially symmetric, so that (u, v) is a classical solution and $\underline{u} \le u \le \overline{u}$, $\underline{v} \le v \le \overline{v}$. $\square$

Remark A.3. Both results above can be adapted to deal with nonlinearities f, g depending on both variables provided that f is nondecreasing in v and locally Lipschitz in u and g is nondecreasing in u and locally Lipschitz in v.

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