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Analytical tools to study the criticality at the outer boundary of potential centers

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Introduction

The main interest of this memoir is contained in the framework of the qualitative theory of differential equations. We can consider the origin of the theoretical study of differential equations at the end of the 17th century with Isaac Newton, when he tried to understand and model the observations given by Kepler's laws. Motivated by the simplicity of such models, the first interest for the mathematician community in this area was to integrate differential equations, dealing with a relatively complete theory for linear ordinary differential equations but finding difficulties in the study of nonlinear systems. As a first attempt nonlinear systems where studied by using perturbation methods. It was at the end of the 19th century when Poincaré showed that perturbation methods might not yield correct results in all cases due to the non-convergence of the formal series used. Also Liouville proved the impossibility to solve generically differential equations by means of combination of elementary functions. Motivated by those difficulties was Poincaré who started a new approach in the study of differential equations through the mixing of analysis and geometry giving rise to the born of the qualitative theory of differential equations that we know today.

This qualitative analysis of differential equations started by Poincaré at the end of the 19th century was followed at the beginning of the 20th century by several works of Lyapunov, Birkhoff, Andronov and Arnold among others, that are considered the origin of this field. This qualitative approach is nowadays the main field in the study of differential equations and from the last 30 years up to now an incredible amount of research has been developed in this direction.

Motivated by the variation of physical parameters on the models given by differential equations it is important the study of the behaviour of the system and its solutions when the parameters vary. A change of the parameters could imply a change in the qualitative properties of the system. Dynamic bifurcation theory studies when this changes occur and it is one of the most important topics in the qualitative theory of differential equations. One of the most important problems in this field is known as the *Hilbert's sixteenth problem*, proposed by Hilbert in his inaugural talk at the *International Congress of Mathematics* in Paris (1900) as a part of a list of 23 problems of several areas in mathematics.

The problem had two parts: the first one about the classification of ovals defined by a polynomial equation, the second one about the limit cycles of a polynomial vector field. This second part is concerned with the upper bound of the number of limit cycles in polynomial vector fields of degree less or equal than n. This problem attracted the major part of attention of the community and it remains unsolved. Related with the Hilbert's sixteenth problem there exist several weaker versions. For instance, Roussarie [53] gives a formulation of the *existential Hilbert's sixteenth problem*:

> *For any $n \geqslant 2$, there exists a number $H(n) < \infty$, such that any vector field of degree $\leqslant n$ has less than $H(n)$ limit cycles.*

In order to tackle this problem, Roussarie introduced the notion of limit periodic set and cyclicity, which is the bound of the number of limit cycles that bifurcates from it. He conjectured that the cyclicity is finite for any analytic perturbation inside the family of polynomial vector fields of degree n.

Related with this important problem we find the main topic of this memoir: the *period function*, which gives the period of each periodic orbit surrounding a center. However, the beginnings of the history of the work on period functions starts even before the modern analysis. It was in 1632 when Galileo discovered isochronicity for small oscillations of a pendulum, that is all the oscillations have the same period. Later Huygens in 1673 observed that the pendulum clock has a monotone period function and also described the first nonlinear isochronous pendulum using the motion of a particle on a cycloid under the gravity action: the cycloidal pendulum.

As all the fields in qualitative theory of differential equations, since the last 30 years it has been an explosion of advanced new research involving the period function and questions about its behaviour have been extensively studied by a large number of authors. Some of those works where firstly focused on giving sufficient conditions for the period function to be monotone. This was motivated by the fact that monotonicity is a non-degeneracy condition for the bifurcation of subharmonic solutions of periodically forced Hamiltonian systems. We reefer to the excellent book of Chow and Hale [12] in this issue. Also monotonicity of the period function is a sufficient condition for the existence and uniqueness of solution for some boundary value problems (see, for instance, [8, 58]). As we advanced before, the questions surrounding the period function are also related with the Hilbert's sixteenth problem (see, [4, 17, 53, 66]) and the weakened version proposed by Arnold [2].

Similarly as in the case of limit cycles we find the study of bifurcation of critical periodic orbits. There is a huge amount of literature concerning the bifurcation of critical periodic orbits and, more generically, the behaviour of the period function as parameters vary. It deserves to be mentioned in the first place the work of Chicone and Jacobs [11] which gives general results concerning the bifurcation of critical periodic orbits for families of planar analytic vector fields with a nondegenerated center at the origin. In this work bifurcations from the center of the period annulus are studied. Related with the motivation of finding sufficient conditions of monotonicity we find the work of Schaaf [57], which deals with criteria of monotonicity for the period function of planar Hamiltonian vector fields; and the works of Chicone [7] and Chow and Wang [13], which study the monotonicity for potential planar systems. Also the finiteness of the number of critical periodic orbits for planar analytic vector fields has been studied, for instance, by Chicone and Dumortier [10].

Recent works on period function are concerned with isochronicity (for instance, [15,33, 42–44]), bounding the number of critical periods (see, [18,35]) and bifurcations of critical periodic orbits (see, [11,19,20,27,54,55,59]) among others. An especial mention deserves the study of the period function of quadratic centers. The literature classify these centers in four different families: Hamiltonian, reversible Q_3^R, codimension four Q_4 and generalized Lotka-Volterra systems Q_3^{LV}. Chicone conjectured in [9] that reversible centers have at most two critical periods, and the centers of the three other families have monotonic period function. In this direction, Coppel and Gavrilov [16] proved that Hamiltonian quadratic centers are monotonous and Zhao [67] proved the same property for Q_4 centers. Concerning the Q_3^{LV} family, only partial results have been proved in the recent years. It has to be quoted, for instance, works [52,56,64] where the authors proved monotonicity of the hypersurface inside the family Q_3^{LV} corresponding to the classical Lotka-Volterra centers, and [62] where the same property was proved for other hypersurfaces in the family Q_3^{LV}. Finally, the reversible quadratic centers can be brought, by an affine transformation and a constant rescaling of time, to Loud's centers [34] which has attracted a lot of attention in the recent years (see, for instance, [41,46,61,63]).

Strongly related with the main topic of this work, the bifurcation of critical periodic orbits from the outer boundary of the period annulus has also been studied. We refer the reader to the series of papers [14,38–41,45,46] in this direction. However the techniques used in these works go generally through the extension of the vector field at infinity by means of the *Poincaré compactification*. The polycycle in this situation consists of regular trajectories and singular points with a hyperbolic sector, which after a desingularization process give rise to saddles and saddle-nodes. It is here where the use of normal forms of such singular points allows to obtain an asymptotic development of the period function near the outer boundary.

The ultimate goal in the study of the period function in a family of centers is to give a global bifurcation diagram. That is, to give a partition of the parameter space in such a way the corresponding period functions of two parameters belonging to the same connected component of the partition are qualitatively the same. Mardešić, Marín and Villadelprat in [40] prove that study the global bifurcation diagram of the period function is equivalent to study the local bifurcation diagram at the inner boundary of the period annulus, at the interior of the period annulus, and at the outer boundary of the period annulus. This result is a consequence of the definition given in [40] for a parameter to be a bifurcation value. In the present work we do not use that definition but another one which allows us to quantify the maximal number of critical periodic orbits that may bifurcate from it. However, our definition will not adapt to the case of the interior of the period annulus in general.

In this memoir we use the notion of *critical periodic orbit* instead of critical period. The reason for this is twofold. Firstly because critical period is semantically ambiguous. Secondly because critical periodic orbit refers to a geometric object in the period annulus and this makes more evident the underlying parallelisms with the study of limit cycles. This enables us to introduce the notion of *criticality*, which is an adaptation of the cyclicity, its counterpart in the study of limit cycles. For a family $\{X_\mu\}_{\mu\in\Lambda}$ of analytic planar differential systems with a center at p_μ and a fixed parameter $\hat{\mu} \in \Lambda$, we define the criticality of the pair $(p_{\hat{\mu}}, X_{\hat{\mu}})$ as the maximal number of critical periodic orbits of X_μ that tend to $p_{\hat{\mu}}$ as $\mu \to \hat{\mu}$. We shall define it by means of the Hausdorff distance between non-empty compact subsets of $\mathbb{R}^2$. We also give the corresponding definition of criticality at the outer boundary of the period annulus. In this case, since the period annulus may be unbounded, we consider the compactification of $\mathbb{R}^2$ into $\mathbb{RP}^2$ in order that the outer boundary is well defined. Consequently, the criticality in this situation is given by means of the Hausdorff distance between non-empty compact subsets of $\mathbb{RP}^2$. As we advanced previously, we introduce a different notion of local regular and local bifurcation value of the period function with respect to the one given in [40]. This definition will be given in terms of the criticality and it will be only addressed to the previous cases. Therefore we shall say that a fixed parameter is a local regular value of the period function if the corresponding criticality is zero. Otherwise it will be a local bifurcation value of the period function. Particularly this notion shows the parallelism between the study of limit cycles and the study of the period function. At this point we remark that our approach does not allow us to define the notion of regular value at the interior of the period annulus in terms of the criticality. For instance, it may happen that an isochronous center has criticality zero, when it should be a local bifurcation value of the period function for any reasonable definition. For this reason here we will give a definition of criticality at the interior of the period annulus which is only addressed to centers that are isochronous. This definition was already introduced by Garijo and Villadelprat [18]. The techniques, and even the definition, for centers not being isochronous is beyond the scope of this memoir. This kind of phenomenon is analogous to the so-called *blue-sky* bifurcation of semi-stable limit cycle, which is its counterpart in the context of Hilbert's sixteenth problem (see for instance [50]).

Our main interest in this memoir is to study the bifurcation of critical periodic orbits from the outer boundary of the period annulus. According with the notion of criticality, we shall study the number of critical periodic orbits of a continuous center X_μ that can emerge of disappear from the outer boundary of the period annulus as we move slightly the parameter. More concretely, we are concerned with continuous families of planar analytic potential systems $X_\mu = -y\partial_x + V_\mu'(x)\partial_y$ that have a non-degenerated center at the origin with the periodic orbits inside the energy levels of the Hamiltonian function $H_\mu(x,y) = \frac{1}{2}y^2 + V_\mu(x)$. If we denote by $\mathscr{P}_\mu$ the period annulus of X_μ, we have that $H_\mu(\mathscr{P}_\mu) = (0, h_0(\mu))$, where h_0 may be either finite or infinite. The period function $T(h)$ of the periodic orbit γ_h inside the energy level $\{H(x,y) = h\}$ is given by the Abelian integral

$$T(h) = \int_{\gamma_h} \frac{dx}{y}.$$

The derivative of the period function $T'(h)$ is also given by an Abelian integral and we are interested in its zeros near $h = h_0(\mu)$, which correspond to critical periodic orbits near the outer boundary of the period annulus.

The tools we develop in this memoir allow to tackle the problem in the following two situations: either $h_0(\mu) = +\infty$ for all $\mu \approx \hat{\mu}$, or $h_0(\mu) < +\infty$ for all $\mu \approx \hat{\mu}$. We do not treat the case in which in any neighbourhood of $\hat{\mu}$ there are μ_1 and μ_2 with $h(\mu_1) = +\infty$ and $h(\mu_2) < +\infty$. Theorems C and E are addressed to cases $h_0 = +\infty$ and $h_0 < +\infty$, respectively. In this two main results, for a fixed $\hat{\mu} \in \Lambda$, we give sufficient conditions in order that the criticality at the outer boundary of the period annulus is less or equal than n for $n \in \mathbb{N} \cup \{0\}$. The main idea in both cases is to find some analytic functions $\phi_\mu^i(h)$, $i = 1, 2, \ldots, n$, verifying that there exist $\delta, \varepsilon > 0$ such that if $\|\mu - \hat{\mu}\| < \delta$ then $(\phi_\mu^1, \phi_\mu^2, \ldots, \phi_\mu^n, T_\mu')$ is an extended complete Chebyshev system (ECT-system for short, see Definition 1.4.1) on the interval $(h_0(\mu) - \varepsilon, h_0(\mu))$. This implies in particular that $T_\mu'(h)$ has at most n zeroes for $h \in (h_0(\mu) - \varepsilon, h_0(\mu))$, counted with multiplicities, for all $\mu \approx \hat{\mu}$ and, accordingly, the criticality is bounded by n. We choose different kind of functions ϕ_μ^i depending if the energy at the outer boundary considered is either finite or infinite, but in both situations we take them simple enough in order that $(\phi_\mu^1, \phi_\mu^2, \ldots, \phi_\mu^n)$ is an ECT-system on $(0, h_0(\mu))$. To this end, the main idea will be to guarantee that the Wronskian of $(\phi_\mu^1, \phi_\mu^2, \ldots, \phi_\mu^n, T_\mu')$ is non-vanishing for $h \approx h_0(\mu)$ and for $\mu \approx \hat{\mu}$. On the other hand, Theorem D, which is addressed to the finite energy case, give sufficient conditions for a parameter to be a regular value of the period function at the outer boundary and also gives a criteria to show that at most one critical periodic orbit bifurcates from the outer boundary. Although Theorem E is more general in terms of the upper-bound of the criticality, Theorem D is interesting because simpler hypothesis are needed. Chapter 2 is devoted to the proof of these three results.

As we already discussed, the main interest is then to guarantee that the Wronskian of $(\phi_\mu^1, \phi_\mu^2, \dots, \phi_\mu^n, T_\mu')$ is non-vanishing for $h \approx h_0(\mu)$ and for $\mu \approx \hat{\mu}$. In this regard we dedicate Chapter 1 to the development of some analytic tools with this aim in view. Given an analytic function $f_\mu : [0, \infty) \to \mathbb{R}$ depending continuously on a d-dimensional parameter μ, we will be concerned with the uniform upper-bound of the number of zeroes near infinity of the function $\mathscr{F}[f_\mu]$, where

$$\mathscr{F}[f](x) := \int_0^{\frac{\pi}{2}} f(x \sin\theta) d\theta.$$

We point out that the derivative of the period function of a potential analytic vector field can be written in terms of the operator under consideration for the particular continuous family of analytic functions $f_\mu(z) = z(g_\mu^{-1})''(z) - z(g_\mu^{-1})''(-z)$, $z \in [0, h_0(\mu))$, where $g_\mu(x) := \operatorname{sgn}(x)\sqrt{V_\mu(x)}$. For this reason, in this chapter our aim is to complete the function $\mathscr{F}[f_\mu]$ with some analytic functions $\{x^{\nu_i(\mu)}\}$, $i = 1, \dots, n$, where $\nu_1, \dots, \nu_2$ are real continuous function in the parameters, in such a way $(x^{\nu_1(\mu)}, \dots, x^{\nu_n(\mu)}, \mathscr{F}[f_\mu](x))$ form an ECT-system in an interval of the form $(M, +\infty)$, where $M > 0$ is independent on the parameters. As we already commented, this is equivalent to show that the Wronskian $W[x^{\nu_1(\mu)}, \dots, x^{\nu_n(\mu)}, \mathscr{F}[f_\mu](x)]$ has no zeros in $(M, +\infty)$. In order to obtain the desired upper bounds, the delicate point will be as usual to guarantee the uniformity with respect to the parameters of the function. In this direction, the techniques developed in this chapter are concerned with the asymptotic behaviour at infinity of this Wronskian. Theorem A shows that if $f_\mu : [0, +\infty) \to \mathbb{R}$ is an analytic function such that $f_\mu(x) = x^{\alpha(\mu)}(\Delta(\mu) + r_\mu(x))$ with $\Delta(\mu) \neq 0$ and $r_\mu(x)$ tending to zero uniformly as x tends to infinity then, under some hypothesis on f_μ and α, the function $\mathscr{F}[f_\mu]$ has a similar behaviour. This result will be generalized in Theorem B by considering the differential operator

$$\mathscr{L}_{\boldsymbol{\nu}_n}[f](x) := \frac{W[x^{\nu_1}, \dots, x^{\nu_n}, f](x)}{x^{\sum_{i=1}^n (\nu_i - 1)}}.$$

We show in Theorem B that under some assumptions in the asymptotic behaviour of the function $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]$ near infinity, the function $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[\mathscr{F}[f_\mu]]$ can be written locally for $\mu \approx \hat{\mu}$ as $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[\mathscr{F}[f_\mu]](x) = x^{\alpha(\mu)}(\Delta(\mu) + r_\mu(x))$ with $\Delta(\mu) \neq 0$ and $r_\mu(x)$ tending to zero uniformly as x tends to infinity. Particularly, the function $\mathscr{F}[f_\mu]$ has no more than n isolated zeroes, counted with multiplicities, in an interval $(M, +\infty)$ with $M > 0$.

Our testing ground in this memoir is the two-parametric family of potential differential systems given by

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = (1+x)^p - (1+x)^q, \end{cases}$$

which has a non-degenerated center at the origin for all parameters $\mu := (q, p)$ varying inside $\Lambda := \{(q, p) \in \mathbb{R}^2 : p > q\}$. We became interested in this problem because of the previous results by Miyamoto and Yagasaki on the issue. Both authors proved, see [47], that the period function is monotonous when $q = 1$ and $p \in \mathbb{N}$. As it often occurs, they came across the period function when studying the solutions of an elliptic Neumann problem and needed this monotonicity property to prove a bifurcation result. Later Yagasaki improved the result showing in [65] the monotonicity of the period function for $q = 1$ and any $p \in \mathbb{R}$ with $p > 1$. Concerning this family, in Theorem F we shall prove some regions in the space of parameters where the period function is monotonous. This result extends the previous ones by Miyamoto and Yagasaki. Moreover, in Theorem G we deal with the study of the bifurcation of critical periodic orbits from the inner boundary of the period annulus. Particularly we show that the first period constant of the center at the origin of the system under consideration is given by

$$\Delta_1(q, p) := 2p^2 + 2q^2 + 7pq - p - q - 1.$$

The parameters outside the hyperbola $\{\Delta_1 = 0\}$ are local regular values of the period function at the center. The hyperbola consists of local bifurcation values and we prove that its criticality is exactly one. On the other hand, in Theorem H we prove that the criticality at the interior of the period annulus is one for the isochronous centers of the family under consideration when we perturb the parameter by a germ of analytic curve.

Finally, we use the criteria developed in Chapter 2 to study the criticality at the outer boundary of the period annulus. We prove in Theorem I that the parameters

$$\Gamma_B := \{\mu \in \Lambda : q = 0\} \cup \{\mu \in \Lambda : p = 1, q \leqslant -1\} \cup \{\mu \in \Lambda : p + 2q + 1 = 0, q \geqslant -1\}$$

correspond to local bifurcation values at the outer boundary. We also show that almost all point out of Γ_B correspond to local bifurcation value and we prove that the criticality is one for some parameters in Γ_B. The combination of all these results will lead us to propose a conjectural bifurcation diagram for the global behaviour of the period function of the system under consideration. All these results are collected in Chapter 3.

The results of this memoir have led to three works. The first one have appeared in *Journal of Differential Equations* with the title "The criticality of centers of potential systems at the outer boundary". The second one, entitled "Study of the period function of a two-parameter family of centers" collects almost all the results in Chapter 3 and it has been recently submitted. Finally the third one, which is devoted to the study of the asymptotic behaviour of the Wronskian function described before, is entitled "Analytic tools to bound the criticality at the outer boundary of the period annulus" and it is also under submission process.

CHAPTER 1

Relating the zeroes of f and $\mathscr{F}[f]$

In this chapter we consider the integral operator $\mathscr{F} : \mathscr{C}^\omega\big([0,a)\big) \longrightarrow \mathscr{C}^\omega\big([0,a)\big)$ defined by $\mathscr{F}[f](x) := \int_0^{\frac{\pi}{2}} f\big(x\sin\theta\big)d\theta$. Our main interest is to give a uniform upper-bound of the number of zeroes near $a = +\infty$ of the function $\mathscr{F}[f_\mu]$ in terms of the behaviour at infinity of a continuous family of analytic function $\{f_\mu\}$. This will be done by embedding the function $\mathscr{F}[f_\mu]$ into an ECT-system on an interval of the form $(M, +\infty)$ with $M > 0$.

1.1 Introduction

Our initial goal when we study families of planar potential systems is to investigate the maximum number of critical periodic orbits that bifurcate from the outer boundary of the period annulus when we move sightly the parameter.

We consider an analytic potential system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = V'(x), \end{cases}$$

with a non-degenerated center at the origin. The period function associated to the center can be written using the Hamiltonian function $H(x,y) = \frac{1}{2}y^2 + V(x)$ as

$$T(h) = \int_{\gamma_h} \frac{dx}{y} = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h}\sin\theta)d\theta$$

where $g(x) := \operatorname{sgn}(x)\sqrt{V(x)}$ and the definite integral follows by using the polar coordinates that brings the oval $\gamma_h \subset \{\frac{1}{2}y^2 + V(x) = h\}$ to the circle of radius $\sqrt{h}$.

Suppose now that the function V depends continuously on a parameter $\mu \in \Lambda \subset \mathbb{R}^d$, so we deal with a family of differential systems $X_\mu = -y\partial_x + V_\mu'(x)\partial_y$. Following the obvious notation, we compute the derivative with respect to the energy of the above definite integral

$$T_\mu'(h) = \frac{1}{\sqrt{2h}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h}\sin\theta)\sin\theta d\theta = \frac{1}{\sqrt{2h}}\int_0^{\frac{\pi}{2}} \mathcal{P}[z(g_\mu^{-1})''(z)]\big|_{z=\sqrt{h}\sin\theta}\, d\theta$$

where $\mathcal{P}[f](x) := f(x) + f(-x)$ is twice the even part of the function f. Then the bifurcation problem that we are interested in turns into give a uniform upper-bound of the number of zeroes of the function $T_\mu'(h)$ near the energy corresponding to the outer boundary of the period annulus. As usual, the delicate point will be to guarantee the uniformity with respect to the parameters of the system.

This chapter is devoted to provide analytic tools in order to tackle with the above bifurcation problem from a general point of view. We consider the operator

$$\mathscr{F} : \mathscr{C}^\omega\big([0,a)\big) \longrightarrow \mathscr{C}^\omega\big([0,a)\big)$$

with $a \in \mathbb{R}^+ \cup \{+\infty\}$ defined by

$$\mathscr{F}[f](x) := \int_0^{\frac{\pi}{2}} f\big(x\sin\theta\big)d\theta.$$

Here $\mathscr{C}^\omega([0,a))$ stands for the set of analytic functions on $(0,a)$ that can be analytically extended to $x = 0$. We point out that the derivative of the period function can be written in terms of the operator under consideration. That is, $\sqrt{2}h^2T'_\mu(h^2) = \mathscr{F}[f_\mu](h)$ with $f_\mu(x) = \mathcal{P}[x(g_\mu)''(x)]$. This motivates us to study the number of zeroes of $\mathscr{F}[f_\mu]$. Particularly, we are concerned with the uniform upper-bound of the number of isolated zeroes, counted with multiplicities, of the function $\mathscr{F}[f_\mu](x)$ for $x \approx a$ in terms of the family $\{f_\mu\}$.

We focus our study in the case $a = +\infty$. This study is motivated dynamically by the expression of the derivative of the period function when we consider analytic planar potential systems for which the energy level at the outer boundary is infinite. The case when the energy level at the outer boundary is finite will be reduced to the previous one, so in all this chapter we shall assume $f \in \mathscr{C}^\omega([0,+\infty))$.

As a first attempt, in Section 1.2 and Section 1.3 we investigate the first term of the asymptotic expansion of the function $\mathscr{F}[f_\mu](x)$ at $x = +\infty$ uniformly on the parameter in terms of the asymptotic behaviour of $f_\mu(x)$ at $x = +\infty$. One of the main result of this section, which is presented in Theorem A, particularly shows that under some hypothesis on the family $\{f_\mu\}$ we can ensure the existence of a continuous function $\alpha(\mu)$ in $\hat{\mu}$ such that the limit of the function $x^{\alpha(\mu)}\mathscr{F}[f_\mu](x)$ as (x,μ) tends to $(+\infty,\hat{\mu})$ exists and it is different from zero. That is, the function under consideration can be written as $\mathscr{F}[f_\mu](x) = x^{-\alpha(\mu)}(\Delta(\mu) + r_\mu(x))$ with a remainder $r_\mu(x)$ tending to zero uniformly as x tends to infinity. This result particularly shows that $\mathscr{F}[f_\mu](x)$ has no zeros for $x \approx +\infty$ and $\mu \approx \hat{\mu}$ provided that $\Delta(\hat{\mu}) \neq 0$. More concretely Section 1.2 is devoted to this study when f is not parameter dependent. In Section 1.3 we use the results obtained Section 1.2 in order to deal with the parameter dependent case.

Directly related with the bounding of the number of zeroes of functions are the so-called *extended complete Chebyshev systems* (ECT-systems for short, see Definition 1.4.1). The main idea in order to give a uniform upper-bound of the number of zeroes of $\mathscr{F}[f_\mu]$ near infinity is to find some analytic real functions $\phi_\mu^1,\ldots,\phi_\mu^n$ satisfying that $(\phi_\mu^1,\ldots,\phi_\mu^n)$ is an ECT-system and that there exist $\varepsilon, M > 0$ such that if $\|\mu-\hat{\mu}\| < \varepsilon$ then $(\phi_\mu^1,\ldots,\phi_\mu^n,\mathscr{F}[f_\mu])$ form an ECT-system on the interval $(M,+\infty)$. This particularly shows that the function $\mathscr{F}[f_\mu](x)$ has at most n isolated zeroes in $(M,+\infty)$, counted with multiplicities, for all $\mu \approx \hat{\mu}$. As we shall see, Lemma 1.4.3 shows that, if $(\phi_\mu^1,\ldots,\phi_\mu^n)$ is an ECT-system, $(\phi_\mu^1,\ldots,\phi_\mu^n,\mathscr{F}[f_\mu])$ form an ECT-system on the interval $(M,+\infty)$ if and only if the function $W[\phi_\mu^1,\ldots,\phi_\mu^n,\mathscr{F}[f_\mu]](x)$ has no zeroes in $(M,+\infty)$ for all $\mu \approx \hat{\mu}$. In this chapter we consider the functions $x^{\nu_1(\mu)},\ldots,x^{\nu_n(\mu)}$ where $\nu_1,\ldots,\nu_n$ are real continuous functions in a neighbourhood of $\hat{\mu}$. The ECT-system $(x^{\nu_1(\mu)},\ldots,x^{\nu_n(\mu)})$ is the simplest one we can consider in order to embed $\mathscr{F}[f_\mu]$ into an ECT-system. Theorem B is addressed in this direction and shows that under some hypothesis on the behaviour of the function $W[x^{\nu_1(\mu)},\ldots,x^{\nu_n(\mu)},f_\mu(x)]$ at infinity, we can ensure that $W[x^{\nu_1(\mu)},\ldots,x^{\nu_n(\mu)},\mathscr{F}[f_\mu](x)]$ has no zeros near infinity. The main goal in Section 1.4 is to prove Theorem B using

Theorem A. In Chapter 2 more details about the role of this Wronskian in the study of the criticality at the outer boundary of the period annulus are given.

1.2 First term of the asymptotic expansion of $\mathscr{F}[f]$

For $a \in \mathbb{R}^+ \cup \{+\infty\}$, from now on we shall denote by $\mathscr{C}^\omega([0,a))$ the set of analytic functions on $(0,a)$ that can be analytically extended to $x=0$ (analytic in $[0,a)$ for short). That is, $f \in \mathscr{C}^\omega([0,a))$ if there exist $\varepsilon > 0$ and an analytic function $\hat{f}$ in $(-\varepsilon, a)$ such that $\hat{f} \equiv f$ in $[0,a)$, i.e. $\hat{f}$ is a real-analytic continuation of f in $(-\varepsilon, a)$.

Definition 1.2.1. We define the integral operator $\mathscr{F} : \mathscr{C}^\omega\big([0,a)\big) \longrightarrow \mathscr{C}^\omega\big([0,a)\big)$ by

$$\mathscr{F}[f](x) := \int_0^{\frac{\pi}{2}} f\big(x \sin\theta\big) d\theta,$$

where $a \in \mathbb{R}^+ \cup \{+\infty\}$. □

In this chapter we are concerned with the asymptotic behaviour at $a = +\infty$ of the function $\mathscr{F}[f](x)$ in terms of the asymptotic behaviour of $f(x)$. We begin by introducing precisely the notion of "asymptotic behaviour".

Definition 1.2.2. Let f be an analytic function on (a,b). We say that f is *quantifiable at b by α with limit ℓ* in case that:

(i) If $b \in \mathbb{R}$, then $\lim_{x\to b^-} f(x)(b-x)^\alpha = \ell$ and $\ell \neq 0$.

(ii) If $b = +\infty$, then $\lim_{x\to+\infty} \frac{f(x)}{x^\alpha} = \ell$ and $\ell \neq 0$.

We call α the *quantifier* of f at b. We shall use the analogous definition at a. □

The integral $\int_0^{\frac{\pi}{2}} \sin^\alpha \theta d\theta$ is convergent for all $\alpha > -1$. In what follows we shall denote its value by $\mathscr{G}(\alpha)$. It is well known, see for instance [1], that

$$\mathscr{G}(\alpha) := \int_0^{\frac{\pi}{2}} \sin^\alpha \theta d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)} = \frac{1}{2} B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right), \tag{1.1}$$

where Γ and B are respectively the Gamma and Beta functions.

Proposition 1.2.3. *Let f be an analytic function on $[0,+\infty)$. Assume that f is quantifiable at $+\infty$ by α with limit a. If $\alpha > -1$, then the function $\mathscr{F}[f]$ is quantifiable at $+\infty$ by α with limit $a\mathscr{G}(\alpha)$.*

Proof. Consider a given $\varepsilon > 0$. Since f is quantifiable at $+\infty$ by α with limit a, there exists $M > 0$ such that

$$\left|f(z)z^{-\alpha} - a\right| < \frac{\varepsilon}{2\mathscr{G}(\alpha)} \text{ for all } z > M. \tag{1.2}$$

Moreover, due to the continuity of f, there exists $K > 0$ such that $|f(z)| < K$ for all $z \in [0, M]$. Then, for any $x > 0$,

$$\frac{1}{x^\alpha}\int_0^{\arcsin(\frac{M}{x})} |f(x\sin\theta)|\, d\theta \leqslant \frac{K}{x^\alpha}\arcsin\left(\frac{M}{x}\right).$$

On account of $\alpha > -1$, $\lim_{x\to+\infty}\frac{K}{x^\alpha}\arcsin(M/x) = 0$. Hence we can take $x_1 > 0$ such that

$$\left|\frac{1}{x^\alpha}\int_0^{\arcsin(\frac{M}{x})} f(x\sin\theta)d\theta\right| < \frac{\varepsilon}{4} \text{ for all } x > x_1.$$

Similarly $\lim_{x\to+\infty}\int_0^{\arcsin(M/x)} a\,|\sin\theta|^\alpha\, d\theta = 0$, so there exists $x_2 > 0$ such that

$$\left|\int_0^{\arcsin(\frac{M}{x})} a\sin^\alpha\theta d\theta\right| < \frac{\varepsilon}{4} \text{ for all } x > x_2.$$

Taking $x_3 = \max\{x_1, x_2\}$, from the two previous inequalities we obtain that

$$\left|\int_0^{\arcsin(\frac{M}{x})}\left(\frac{f(x\sin\theta)}{x^\alpha} - a\sin^\alpha\theta\right)d\theta\right| < \frac{\varepsilon}{2}$$

for all $x > x_3$. In addition, due to $x\sin\theta \in (M, x)$ for $\theta \in \big(\arcsin(M/x), \pi/2\big)$, from the inequality in (1.2) we get

$$\left|\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\left(\frac{f(x\sin\theta)}{(x\sin\theta)^\alpha} - a\right)\sin^\alpha\theta\right| < \frac{\varepsilon}{2\mathscr{G}(\alpha)}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\sin^\alpha\theta d\theta < \frac{\varepsilon}{2\mathscr{G}(\alpha)}\int_0^{\frac{\pi}{2}}\sin^\alpha\theta d\theta = \frac{\varepsilon}{2}$$

for all $x > M$. Finally, taking $x_4 = \max\{x_3, M\}$, the combination of the two previous inequalities gives

$$\begin{aligned}\left|\frac{\mathscr{F}[f](x)}{x^\alpha} - a\mathscr{G}(\alpha)\right| &= \left|x^{-\alpha}\int_0^{\frac{\pi}{2}} f(x\sin\theta)d\theta - a\mathscr{G}(\alpha)\right| = \left|\int_0^{\frac{\pi}{2}}\left(\frac{f(x\sin\theta)}{x^\alpha} - a\sin^\alpha\theta\right)d\theta\right| \\ &\leqslant \left|\int_0^{\arcsin(\frac{M}{x})}\left(\frac{f(x\sin\theta)}{x^\alpha} - a\sin^\alpha\theta\right)d\theta\right| + \left|\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\left(\frac{f(x\sin\theta)}{(x\sin\theta)^\alpha} - a\right)\sin^\alpha\theta d\theta\right| < \varepsilon\end{aligned}$$

for all $x > x_4$. This proves the result on account of Definition 1.2.2. ∎

The previous result shows that if f is quantifiable at infinity by $\alpha > -1$ then $\mathscr{F}[f]$ inherits this behaviour at infinity. Particularly, when $\alpha > 0$, both functions tend to infinity with the same order. We shall consider next the case $\alpha \leqslant -1$, so in particular when f tends to zero at infinity. To this end the following definitions are needed:

Definition 1.2.4. Given an analytic function f on $[0,+\infty)$ we define, for all $n \geqslant 1$,

$$f_n(z) := f_{n-1}(z)z^2 + z\int_0^z f_{n-1}(t)dt,$$

where we set $f_0 := f$. □

The following result provides a formula that relates $\mathscr{F}[f]$ and $\mathscr{F}[f_n]$.

Lemma 1.2.5. *Let f be an analytic function on $[0,+\infty)$. Then for any $n \in \mathbb{N}$ we have that $\mathscr{F}[f_n](x) = x^{2n}\mathscr{F}[f](x)$ for all $x > 0$.*

Proof. Let us fix $x > 0$ and note that if h is any analytic function on $[0,x]$, then the change of variable $u = x\sin\theta$ gives

$$\mathscr{F}[h](x) = \int_0^{\pi/2} h(x\sin\theta)d\theta = \int_0^x \frac{h(u)}{\sqrt{x^2-u^2}}du. \tag{1.3}$$

Let $n \in \mathbb{N}$ and set $g(z) := \frac{1}{z}\int_0^z f_{n-1}(t)dt$. Then, integrating by parts,

$$\int_0^x g(u)u^2\frac{du}{\sqrt{x^2-u^2}} = \int_0^x \left(g'(u)u + g(u)\right)\sqrt{x^2-u^2}du = \int_0^x f_{n-1}(u)\sqrt{x^2-u^2}du. \tag{1.4}$$

Some easy manipulations show that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \big(f_{n-1}+g\big)(x\sin\theta)\sin^2\theta d\theta &= \frac{1}{x^2}\int_0^x \big(f_{n-1}+g\big)(u)\frac{u^2du}{\sqrt{x^2-u^2}}\\ &= \frac{1}{x^2}\int_0^x f_{n-1}(u)\left(\frac{u^2}{\sqrt{x^2-u^2}} + \sqrt{x^2-u^2}\right)du\\ &= \int_0^x \frac{f_{n-1}(u)du}{\sqrt{x^2-u^2}} = \int_0^{\frac{\pi}{2}} f_{n-1}(x\sin\theta)d\theta,\end{aligned}$$

where in the first and fourth equalities we use equality (1.3) with $h(z) = z^2\big(f+g\big)(z)$ and $h(z) = f(z)$, respectively, while in the second one we use (1.4). Hence, on account of Definition 1.2.4 we have $\mathscr{F}[f_n](x) = x^2\mathscr{F}[f_{n-1}](x)$ for all $x > 0$. Then, recursively, $\mathscr{F}[f_n](x) = x^{2n}\mathscr{F}[f_0](x)$ for all $x > 0$ as we desired. ■

Next result shows that if f is quantifiable at $+\infty$ by $\alpha = -1$, then $\mathscr{F}[f]$ is not quantifiable in the sense of Definition 1.2.2.

Proposition 1.2.6. *Let f be an analytic function on $[0,+\infty)$ quantifiable at $+\infty$ by $\alpha = -1$ with limit a. Then the function $\mathscr{F}[f]$ satisfies $\lim_{x\to+\infty}\frac{x\mathscr{F}[f](x)}{\ln x} = a$.*

Proof. Consider a given $\varepsilon > 0$. Since f is quantifiable at infinity by $\alpha = -1$ with limit a, there exists $M > 0$ such that $|zf(z) - a| < \varepsilon/6$ for all $z \geqslant M$. Moreover, due to f is continuous, there exists $K > 0$ such that $|f(z)| \leqslant K$ for all $z \in [0, M]$. Therefore

$$\frac{x}{\ln x}\int_0^{\arcsin(\frac{M}{x})} |f(x\sin\theta)|d\theta < K\frac{x}{\ln x}\arcsin\left(\frac{M}{x}\right)$$

for all $x > M$. This shows that $\lim_{x\to+\infty}\frac{x}{\ln x}\int_0^{\arcsin(\frac{M}{x})}|f(x\sin\theta)|d\theta = 0$ and so there exists $x_0 > M$ satisfying that

$$\frac{x}{\ln x}\int_0^{\arcsin(\frac{M}{x})} |f(x\sin\theta)|d\theta < \frac{\varepsilon}{3}$$

for all $x > x_0$. On the other hand, since one can verify that

$$\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}} \frac{1}{\sin\theta}d\theta = \ln\left(\frac{x+\sqrt{x^2-M^2}}{M}\right)$$

for all $x \geqslant M$, we have that $\lim_{x\to+\infty}\frac{1}{\ln x}\int_{\arcsin(M/x)}^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta} = 1$. Accordingly there exists $x_1 > x_0$ such that

$$\left|\frac{1}{\ln x}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta}\right| < 2$$

and

$$\left|\frac{1}{\ln x}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta} - 1\right| < \frac{\varepsilon}{3|a|}$$

for all $x > x_1$. Taking these inequalities into account we get that if $x > x_1$ then

$$\begin{aligned}
\left|\frac{x}{\ln x}\mathscr{F}[f](x) - a\right| &< \frac{x}{\ln x}\int_0^{\arcsin(\frac{M}{x})}|f(x\sin\theta)|d\theta + \left|\frac{x}{\ln x}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}} f(x\sin\theta)d\theta - a\right| \\
&< \frac{\varepsilon}{3} + \left|\frac{1}{\ln x}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\frac{f(x\sin\theta)x\sin\theta - a + a}{\sin\theta}d\theta - a\right| \\
&< \frac{\varepsilon}{3} + \frac{1}{\ln x}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\frac{|f(x\sin\theta)x\sin\theta - a|}{\sin\theta}d\theta \\
&\quad + |a|\left|\frac{1}{\ln x}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\frac{1}{\sin\theta}d\theta - 1\right| \\
&< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This completes the proof of the result. ∎

According to the previous results the cases $\alpha = -1$ and $\alpha > -1$ are completely different with regard to the asymptotic behaviour of $\mathscr{F}[f]$ at infinity. The following results clarify that $\alpha = -1$ is a threshold in this sense. As we shall see, to study the case $\alpha < -1$ it is required to take the momenta of f into account. Before state the next results we introduce some new notation.

Definition 1.2.7. Given an analytic function f on $[0, +\infty)$ we call

$$M_n[f] := \int_0^{+\infty} x^{2n-2} f(x) dx$$

the n-th *momentum* of f, whenever it is well defined. □

Although the integral $M_n[f]$ is not the classical n-momentum of f, we call them momentum because the obvious similitude. In fact, $M_n[f]$ is strongly related with the classical $(2n-2)$-momentum. A sufficient condition in order that the n-th momentum of f is well-defined is that f is quantifiable at infinity by $\alpha < 1 - 2n$.

For $\alpha < -1$ let us take $n \in \mathbb{N}$ such that $\alpha + 2n \in [-1, 1)$. Then for $j = 1, 2, \ldots, n$, we define $\alpha_j := \prod_{i=1}^{j} \frac{\alpha+2i}{\alpha+2i-1}$.

Lemma 1.2.8. *Let f be an analytic function on $[0, +\infty)$ quantifiable at $+\infty$ by α with limit a and let f_j be defined as in Definition 1.2.4. Assume that $\alpha < -1$ and let $n \in \mathbb{N}$ be such that $\alpha + 2n \in [-1, 1)$. Then the following hold:*

(*a*) *If $M_1[f_0] = M_1[f_1] = \ldots = M_1[f_{k-1}] = 0$ for some $1 \leqslant k < n$, then f_j is quantifiable at $+\infty$ by $\alpha + 2j$ with limit $a\alpha_j$ for all $j = 1, 2, \ldots, k$.*

(*b*) *If $M_1[f_0] = M_1[f_1] = \ldots = M_1[f_{n-1}] = 0$ and $\alpha + 2n \neq 0$, then f_n is quantifiable at $+\infty$ by $\alpha + 2n$ with limit $a\alpha_n$.*

Proof. To show (*a*) assume that $M_1[f_0] = M_1[f_1] = \ldots = M_1[f_{k-1}] = 0$ for some $k < n$. We will prove inductively that

$$\lim_{z \to +\infty} \frac{f_j(z)}{z^{\alpha+2j}} = a\alpha_j \text{ for all } j = 1, 2, \ldots, k.$$

We begin with the base case $j = 1$. From Definition 1.2.4 we get, using $f_0 = f$, that

$$\frac{f_1(z)}{z^{\alpha+2}} = \frac{f(z)}{z^{\alpha}} + \frac{1}{z^{\alpha+1}} \int_0^z f(t) dt.$$

The assumption on f implies that $\lim_{z \to +\infty} \frac{f(z)}{z^{\alpha}} = a$. Moreover, the hypothesis $M_1[f_0] = 0$ and $\alpha < -1$ imply that $\lim_{z \to +\infty} \frac{1}{z^{\alpha+1}} \int_0^z f(t) dt$ is a $0/0$ indeterminate limit. Thus, by applying Hôpital's Rule, this limit is equal to $\frac{a}{\alpha+1}$. Consequently

$$\lim_{z \to +\infty} \frac{f_1(z)}{z^{\alpha+2}} = a\frac{\alpha+2}{\alpha+1} = a\alpha_1,$$

which is a real number different from zero because $\alpha+2 \neq 0$ (Here we use that $\alpha+2n<1$ and $k<n$). So the case $j=1$ follows. Suppose now that the result holds for $j<k$ and let us show its validity for $j+1$. We have

$$\frac{f_{j+1}(z)}{z^{\alpha+2(j+1)}} = \frac{f_j(z)z^2}{z^{\alpha+2(j+1)}} + \frac{z\int_0^z f_j(t)dt}{z^{\alpha+2(j+1)}}.$$

By induction hypothesis, $\lim_{z\to+\infty} \dfrac{f_j(z)}{z^{\alpha+2j}} = a\alpha_j$. On the other hand, by assumption, $M_1[f_j] = \int_0^{+\infty} f_j(t)dt = 0$ and $\alpha+2j+1<0$, so the second function above is again a $0/0$ indeterminate form as z tends to $+\infty$. Then by applying Hôpital's Rule we get

$$\lim_{z\to+\infty} \frac{\int_0^z f_j(t)dt}{z^{\alpha+2j+1}} = \lim_{z\to+\infty} \frac{f_j(z)}{(\alpha+2j+1)z^{\alpha+2j}} = \frac{a\alpha_j}{\alpha+2j+1}.$$

Hence $\lim_{z\to+\infty} \frac{f_{j+1}(z)}{z^{\alpha+2(j+1)}} = a\alpha_j \frac{\alpha+2(j+1)}{\alpha+2j+1} = a\alpha_{j+1}$, as desired, and this proves (a). To show (b), by using the same arguments we obtain that $\lim_{z\to+\infty} \frac{f_n(z)}{z^{\alpha+2n}} = a\alpha_{n-1}\frac{\alpha+2n}{\alpha+2n-1} = a\alpha_n$, which is a number different from zero due to $\alpha+2n \neq 0$. This completes the proof of the result. ■

Next result provides a useful tool for the computation of the momentum $M_1[f_n]$.

Lemma 1.2.9. *Let f be an analytic function on $[0,+\infty)$ quantifiable at $+\infty$ by $\alpha<-1$ with limit a and let f_j be defined in Definition 1.2.4. Let us take any $n \geqslant 2$ satisfying $\alpha+2n<1$ and assume that $M_1[f_0] = M_1[f_1] = \cdots = M_1[f_{n-2}] = 0$. Then*

$$M_1[f_{n-1}] = c_n M_n[f],$$

where $c_1 := 1$ and $c_n := \prod_{k=1}^{n-1}\left(1-\frac{1}{2k}\right)$ for $n \geqslant 2$.

Proof. On account of Lemma 1.2.8, since $M_1[f_0] = M_1[f_1] = \cdots = M_1[f_{n-2}] = 0$ and $f_0 = f$ is quantifiable at $+\infty$ by $\alpha < -2n+1$, we have that the functions $f_{n-(k+1)}$ are quantifiable at $+\infty$ by $\alpha+2(n-k-1)$. Then, for any $k \in \{1,2,\dots,n-1\}$, integrating by parts we get

$$\begin{aligned}\int_0^{+\infty} x^{2(k-1)} f_{n-k}(x)dx &= \int_0^{+\infty}\left(x^{2k} f_{n-k-1}(x) + x^{2k-1}\int_0^x f_{n-k-1}(u)du\right)dx \\ &= \left(1-\frac{1}{2k}\right)\int_0^{+\infty} x^{2k} f_{n-(k+1)}(x)dx + \lim_{x\to+\infty}\frac{x^{2k}}{2k}\int_0^x f_{n-(k+1)}(u)du,\end{aligned}$$

where we used on the second equality that $f_{n-(k+1)}$ is continuous at the origin. Since $f_{n-(k+1)}$ is quantifiable at $+\infty$ by $\alpha+2(n-k-1)$ and $M_1[f_{n-k-1}] = 0$, by the Hôpital's Rule we obtain

$$\lim_{x\to+\infty}\frac{x^{2k}}{2k}\int_0^x f_{n-(k+1)}(u)du = \lim_{x\to+\infty} -\frac{f_{n-(k+1)}(x)}{4k^2x^{-2k-1}} = 0.$$

Therefore
$$\int_0^{+\infty} x^{2(k-1)} f_{n-k}(x)dx = \left(1-\frac{1}{2k}\right)\int_0^{+\infty} x^{2k} f_{n-(k+1)}(x)dx$$
and, using this equality iteratively,
$$M_1[f_{n-1}] = \int_0^{+\infty} f_{n-1}(x)dx = \frac{1}{2}\int_0^{+\infty} x^2 f_{n-2}(x)dx = \ldots = c_n\int_0^{+\infty} x^{2(n-1)} f_0(x)dx.$$
This proves the result by Definition 1.2.7. ■

The previous result gives an equivalent relation between the first momentum of f_{n-1} and the n-th momentum of f when all the previous momentum vanish. In this situation, both momentum are proportional by a positive factor. As we shall see, it will be useful to consider $M_1[f_{n-1}]$ in some of the proofs whereas we shall use $M_n[f]$ on the statements for the sake of simplicity in the computations when we apply the results.

Proposition 1.2.10. *Let f be an analytic function in $[0,+\infty)$ quantifiable at $+\infty$ by $\alpha < -1$ with limit a. Let us take $n \in \mathbb{N}$ such that $\alpha + 2n \in [-1, 1)$. Then the following hold:*

(a) *If $M_1[f] = M_2[f] = \cdots = M_{j-1}[f] = 0$ and $M_j[f] \neq 0$ for some $1 \leqslant j \leqslant n$, then $\mathscr{F}[f]$ is quantifiable at $+\infty$ by $1-2j$ with limit $\prod_{i=1}^{j-1}\left(1-\frac{1}{2i}\right)M_j[f]$.*

(b) *If $M_1[f] = M_2[f] = \cdots = M_n[f] = 0$ and $\alpha + 2n \notin \{0, -1\}$, then $\mathscr{F}[f]$ is quantifiable at $+\infty$ by α with limit $a\mathscr{G}(\alpha+2n)\prod_{i=1}^{n}\frac{\alpha+2i}{\alpha+2i-1}$.*

(c) *If $M_1[f] = M_2[f] = \cdots = M_n[f] = 0$ and $\alpha + 2n = -1$, then $\mathscr{F}[f]$ is not quantifiable at $+\infty$ and $\lim_{x\to+\infty}\frac{x^{2n+1}}{\ln x}\mathscr{F}[f](x) = a\prod_{i=1}^{n}\frac{\alpha+2i}{\alpha+2i-1}$.*

Proof. Let us prove (a). Since $M_1[f] = M_2[f] = \cdots = M_{j-1}[f] = 0$ and $M_j[f] \neq 0$, by Lemma 1.2.9 we have that $M_1[f_0] = M_1[f_1] = \cdots = M_1[f_{j-2}] = 0$ and $M_1[f_{j-1}] = c_j M_j[f]$ where $c_j := \prod_{i=1}^{j-1}\left(1-\frac{1}{2i}\right)$. Then, by Lemma 1.2.8, f_{j-1} is quantifiable at infinity by $\alpha + 2(j-1)$ which is strictly smaller than -1. Consequently,
$$\lim_{z\to+\infty}\frac{f_j(z)}{z} = \lim_{z\to+\infty}\frac{f_j(z)}{z} z f_{j-1}(z) + \int_0^z f_{j-1}(t)dt = M_1[f_{j-1}] = c_j M_j[f] \neq 0.$$
Then
$$\lim_{x\to+\infty} x^{2j-1}\mathscr{F}[f](x) = \lim_{x\to+\infty}\frac{1}{x}\mathscr{F}[f_j](x) = c_j M_j[f] \neq 0,$$
where the first equality follows by Lemma 1.2.5 and the second one by applying Proposition 1.2.3 to f_j on account of the previous limit. This proves (a).

To show (b) we note that, again by Lemma 1.2.5, $x^{2n}\mathscr{F}[f](x) = \mathscr{F}[f_n](x)$. Due to $M_1[f] = \cdots = M_n[f] = 0$ by Lemma 1.2.9 we have $M_1[f_0] = \cdots = M_1[f_{n-1}] = 0$. Let us denote $\alpha_n := \prod_{i=1}^{n} \frac{\alpha+2i}{\alpha+2i-1}$ for the sake of shortness. Then, since $\alpha \notin \{-2n, -2n-1\}$, f_n is quantifiable at $+\infty$ by $\alpha+2n > -1$ with limit $a\alpha_n$ thanks to Lemma 1.2.8. Thus, by Proposition 1.2.3, $\mathscr{F}[f_n]$ is also quantifiable at $+\infty$ by $\alpha+2n$ with limit $a\alpha_n\mathscr{G}(\alpha+2n)$. Accordingly, from the above equality we get that $\mathscr{F}[f]$ is quantifiable at $+\infty$ by α with limit $a\alpha_n\mathscr{G}(\alpha+2n)$ and so (b) follows.

Finally let us show (c). By the previous argumentation, f_n is quantifiable at $+\infty$ by $\alpha+2n=-1$ with limit $a\alpha_n$ thanks to Lemma 1.2.8. Thus, by applying Proposition 1.2.6,

$$\lim_{x\to+\infty} \frac{x}{\ln x}\mathscr{F}[f_n](x) = a\alpha_n \neq 0,$$

and hence, using Lemma 1.2.5 once again $\lim_{x\to+\infty} \frac{x^{2n+1}}{\ln x}\mathscr{F}[f](x) = a\alpha_n$. This shows (c) and completes the proof of the result. ■

We point out that the hypothesis $M_1[f] = M_2[f] = \ldots = M_{j-1}[f] = 0$ in the previous statement is void if $j=1$.

Remark 1.2.11. Notice that the previous result deals with all the possible values of the quantifier α (even when $\mathscr{F}[f]$ turns to be not quantifiable) except by the case when $M_1[f] = M_2[f] = \cdots = M_n[f] = 0$ and $\alpha = -2n$. We want to remark that the hypothesis of f to be quantifiable by $\alpha = -2n$ in this case is not enough to stablish the quantifier of $\mathscr{F}[f]$ at infinity. In fact, it is not even possible to say if it is quantifiable or not. For instance, let us consider the following three examples:

$$f(z) = \begin{cases} \frac{1}{z^2} & z \geqslant 1 \\ 4z-3 & z \in [0,1) \end{cases},$$

$$g(z) = \begin{cases} \frac{1}{z^2} + \frac{9}{10z^4} & z \geqslant 1 \\ \frac{32}{5}z - \frac{9}{2} & z \in [0,1) \end{cases},$$

$$h(z) = \begin{cases} \frac{1}{z^2} + \frac{1}{z^3} & z \geqslant 1 \\ 7z-5 & z \in [0,1) \end{cases}.$$

All these functions are quantifiable at infinity by $\alpha = -2$ and it is a computation to show that the first momentum of the three functions vanishes. One can verify that $\mathscr{F}[f]$ and $\mathscr{F}[g]$ are quantifiable at infinity by -3 and by -5 respectively, and that $\mathscr{F}[h]$ is not quantifiable since

$$\lim_{x\to+\infty} \frac{x^3}{\ln x}\mathscr{F}[h](x) = \frac{1}{2}.$$

With these examples we pretend to emphasize that the previous result is sharp in terms of quantifying $\mathscr{F}[f]$ at infinity with the only information of the quantifier of f at infinity. □

1.3 First term of the asymptotic expansion of the family $\mathscr{F}[f_\mu]$

In this section we generalize the previous results to a family of functions depending continuously on parameters. First of all we extend the previous notion of quantifiable behaviour to this situation and we define the notion of continuous family of analytic functions.

Definition 1.3.1. Let Λ be an open subset of $\mathbb{R}^d$ and suppose that, for each $\mu \in \Lambda$, f_μ is an analytic function on some real interval I_μ. Suppose furthermore that the map $(x,\mu) \longmapsto f_\mu(x)$ is continuous on $\{(x,\mu) \in \mathbb{R} \times \Lambda : x \in I_\mu\}$. Then we say that $\{f_\mu\}_{\mu\in\Lambda}$ is a *continuous family of analytic functions* on I_μ. □

Definition 1.3.2. Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions defined in $I_\mu = \big(a(\mu), b(\mu)\big)$. Assume that b is either a continuous function from Λ to $\mathbb{R}$ or $b(\mu) = +\infty$ for all $\mu \in \Lambda$. Given $\hat{\mu} \in \Lambda$ we shall say that $\{f_\mu\}_{\mu\in\Lambda}$ is *continuously quantifiable* in $\hat{\mu}$ at $b(\mu)$ by $\alpha(\mu)$ with limit ℓ if there exists an open neighbourhood U of $\hat{\mu}$ such that f_μ is quantifiable at $b(\mu)$ by $\alpha(\mu)$ for all $\mu \in U$ and, moreover,

(*i*) In case that $b(\hat{\mu}) < +\infty$, then $\lim_{(x,\mu)\to(b(\hat{\mu}),\hat{\mu})} f_\mu(x)\big(b(\mu) - x\big)^{\alpha(\mu)} = \ell$ and $\ell \neq 0$.

(*ii*) In case that $b(\hat{\mu}) = +\infty$, then $\lim_{(x,\mu)\to(+\infty,\hat{\mu})} \frac{f_\mu(x)}{x^{\alpha(\mu)}} = \ell$ and $\ell \neq 0$.

We shall use the analogous definition for the left endpoint of I_μ. □

Remark 1.3.3. Notice that the map $\alpha : U \to \mathbb{R}$ that appears in the above definition is continuous at $\hat{\mu}$. Indeed, if α is not continuous then there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \alpha(\mu_n) = \alpha(\hat{\mu}) + \kappa$ with $\kappa \neq 0$. Then, for instance in case that $b(\hat{\mu}) = +\infty$, we will have

$$\ell = \lim_{(x,n)\to(+\infty,+\infty)} \frac{f_{\mu_n}(x)}{x^{\alpha(\mu_n)}} = \lim_{x\to+\infty} \left(\lim_{n\to+\infty} \frac{f_{\mu_n}(x)}{x^{\alpha(\mu_n)}} \right) = \lim_{x\to+\infty} \frac{f_{\hat{\mu}}(x)}{x^{\alpha(\hat{\mu})+\kappa}},$$

which, on account of $\ell \neq 0$, contradicts the fact that, by definition, $\lim_{x\to+\infty} \frac{f_{\hat{\mu}}(x)}{x^{\alpha(\hat{\mu})}}$ is finite and different from zero. □

From now on we shall assume that $\{f_\mu\}_{\mu\in\Lambda}$ is a continuous family of analytic functions in $[0,+\infty)$ continuously quantifiable in $\hat{\mu} \in \Lambda$ at $+\infty$ by $\alpha : \Lambda \to \mathbb{R}$ with limit $a(\hat{\mu})$. That is, for all μ in a neighbourhood of $\hat{\mu}$, f_μ is quantifiable by $\alpha(\mu)$ with limit $a(\mu)$ and

$$\lim_{(z,\mu)\to(+\infty,\hat{\mu})} \frac{f_\mu(z)}{z^{\alpha(\mu)}} = a(\hat{\mu}) \neq 0.$$

Following the same strategy as in the previous section, our aim is to investigate if the family $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable at infinity assuming that the family $\{f_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable by $\alpha(\mu)$. The purpose of this study is essentially the uniformity of the limit with respect to the parameter.

Lemma 1.3.4. *Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions on $[0,+\infty)$. Assume that $\{f_\mu\}_{\mu\in\Lambda}$ and $\{f'_\mu\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ and $\beta(\mu)$ with limit a and b, respectively, and that $\alpha(\hat{\mu})\neq 0$. Then $\beta=\alpha-1$ and $b=\alpha(\hat{\mu})a$.*

Proof. The result follows by using Hôpital's Rule and the uniqueness of the quantifier (see Remark 1.3.3). ∎

The next result is the analogous to Proposition 1.2.3 for the parametric case and in its statement $\mathscr{G}$ is the function defined in (1.1).

Theorem 1.3.5. *Consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of analytic functions on $[0,+\infty)$ continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit a. If $\alpha(\hat{\mu})>-1$, then the family $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit $a\mathscr{G}(\alpha(\hat{\mu}))$.*

Proof. On account of Remark 1.3.3 and the fact that $\alpha(\hat{\mu})>-1$, there exists a compact neighbourhood K_1 of $\hat{\mu}$ such that $\alpha(\mu)>-1$ for all $\mu\in K_1$. Consequently $\int_0^{\frac{\pi}{2}}(\sin\theta)^{\alpha(\mu)}d\theta=\mathscr{G}\big(\alpha(\mu)\big)$ for all $\mu\in K_1$. Let us take $N:=\max\{\mathscr{G}\big(\alpha(\mu)\big);\mu\in K_1\}$, which is well defined since $\mu\longmapsto\mathscr{G}\big(\alpha(\mu)\big)$ is continuous. Consider a given $\varepsilon>0$. Since $\{f_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit a, there exist $M>0$ and a compact neighbourhood $K_2\subset K_1$ of $\hat{\mu}$ such that

$$\left|f_\mu(z)z^{-\alpha(\mu)}-a\right|<\frac{\varepsilon}{4N} \tag{1.5}$$

for all $z>M$ and $\mu\in K_2$. We have on the other hand, for any $x>0$,

$$\begin{aligned}\left|\int_0^{\frac{\pi}{2}}\left(\frac{f_\mu(x\sin\theta)}{x^{\alpha(\mu)}}-a(\sin\theta)^{\alpha(\hat{\mu})}\right)d\theta\right| &\leqslant \left|\int_0^{\frac{\pi}{2}}\left(\frac{f_\mu(x\sin\theta)}{x^{\alpha(\mu)}}-a(\sin\theta)^{\alpha(\mu)}\right)d\theta\right| \\ &\quad+\left|\int_0^{\frac{\pi}{2}}a\left((\sin\theta)^{\alpha(\mu)}-(\sin\theta)^{\alpha(\hat{\mu})}\right)d\theta\right|.\end{aligned} \tag{1.6}$$

Since $\mu\longmapsto\mathscr{G}\big(\alpha(\mu)\big)$ is continuous, there exists a compact neighbourhood $K_3\subset K_2$ of $\hat{\mu}$ such that

$$\left|\int_0^{\frac{\pi}{2}}a\left((\sin\theta)^{\alpha(\mu)}-(\sin\theta)^{\alpha(\hat{\mu})}\right)d\theta\right|=|a|\left|\mathscr{G}\big(\alpha(\mu)\big)-\mathscr{G}\big(\alpha(\hat{\mu})\big)\right|<\frac{\varepsilon}{2} \tag{1.7}$$

for all $\mu \in K_3$. Let us take $x_1 > 1$ and let us denote $R := \max\{|f_\mu(z)|\,;(z,\mu) \in [0,M]\times K_3\}$ and $\hat{\alpha} := \min\{\alpha(\mu) : \mu \in K_3\}$. Then

$$\frac{1}{x^{\alpha(\mu)}}\int_0^{\arcsin(\frac{M}{x})} |f_\mu(x\sin\theta)|\,d\theta \leqslant \frac{R}{x^{\alpha(\mu)}}\arcsin\left(\frac{M}{x}\right) \leqslant \frac{R}{x^{\hat{\alpha}}}\arcsin\left(\frac{M}{x}\right)$$

for all $x > x_1$ and $\mu \in K_3$. Due to $\hat{\alpha} > -1$, $\lim_{x\to+\infty}\frac{K}{x^{\hat{\alpha}}}\arcsin(M/x) = 0$, so there exists $x_2 > \max\{x_1, M\}$ satisfying

$$\frac{1}{x^{\alpha(\mu)}}\int_0^{\arcsin(\frac{M}{x})} |f_\mu(x\sin\theta)|\,d\theta < \frac{\varepsilon}{8} \tag{1.8}$$

for all $x > x_2$ and $\mu \in K_3$. There exists in addition $x_3 > x_2$ such that

$$\left|a\int_0^{\arcsin(\frac{M}{x})}(\sin\theta)^{\alpha(\mu)}d\theta\right| \leqslant \left|a\int_0^{\arcsin(\frac{M}{x})}(\sin\theta)^{\hat{\alpha}}d\theta\right| < \frac{\varepsilon}{8} \tag{1.9}$$

for all $x > x_3$ and $\mu \in K_3$, where in the first inequality we use that $0 < \sin\theta < 1$, while in the second one we take $\hat{\alpha} > -1$ and $\lim_{x\to+\infty}\arcsin(M/x) = 0$ into account. The triangular inequality combined with (1.8) and (1.9) yields to

$$\left|\int_0^{\arcsin(\frac{M}{x})}\left(\frac{f_\mu(x\sin\theta)}{x^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\mu)}\right)d\theta\right| < \frac{\varepsilon}{4} \tag{1.10}$$

for all $\mu \in K_3$ and $x > x_3$. On the other hand notice that $M < x\sin\theta < x$ for all $\theta \in \big(\arcsin(M/x), \pi/2\big)$. Thus, from (1.5), we get

$$\left|\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\left(\frac{f_\mu(x\sin\theta)}{(x\sin\theta)^{\alpha(\mu)}} - a\right)(\sin\theta)^{\alpha(\mu)}\right| \leqslant \frac{\varepsilon}{4N}\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}(\sin\theta)^{\alpha(\mu)}d\theta \leqslant \frac{\varepsilon}{4} \tag{1.11}$$

for all $x > x_3$ and $\mu \in K_3$. The combination of (1.10) and (1.11) show that

$$\begin{aligned}\left|\int_0^{\frac{\pi}{2}}\left(\frac{f_\mu(x\sin\theta)}{x^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\mu)}\right)d\theta\right| &\leqslant \left|\int_0^{\arcsin(\frac{M}{x})}\left(\frac{f_\mu(x\sin\theta)}{x^{\alpha(\mu)}} - a(\sin\theta)^{\alpha(\mu)}\right)d\theta\right| \\ &\quad + \left|\int_{\arcsin(\frac{M}{x})}^{\frac{\pi}{2}}\left(\frac{f_\mu(x\sin\theta)}{(x\sin\theta)^{\alpha(\mu)}} - a\right)(\sin\theta)^{\alpha(\mu)}d\theta\right| < \frac{\varepsilon}{2}\end{aligned}$$

for all $x > x_3$ and $\mu \in K_3$. By using the above inequality together with (1.7), from (1.6) we get

$$\left|x^{-\alpha(\mu)}\mathscr{F}[f_\mu](x) - a\mathscr{G}(\alpha(\hat{\mu}))\right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

for all $x > x_3$ and $\mu \in K_3$. This completes the proof of the result. ■

It is clear by Proposition 1.2.6 that we can not expect $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ to be continuously quantifiable when $\alpha(\hat{\mu}) = -1$ since $\mathscr{F}[f_{\hat{\mu}}]$ is not even quantifiable. The rest of this chapter is dedicated to the case $\alpha(\hat{\mu}) < -1$. With this aim in view we shall first prove some previous results. The first one illustrates a relation between the limit of a function uniformly on the parameters and the limit of several variables.

Lemma 1.3.6. *Let $a \in \mathbb{R}^+ \cup \{+\infty\}$, Λ be an open subset of $\mathbb{R}^d$ and $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions on $[0,a)$. The following statements hold:*

(*a*) *If $\lim_{x\to a} f_\mu(x) =: f_\mu(a)$ uniformly in μ, then for all $\hat\mu \in \Lambda$, $\lim_{(x,\mu)\to(a,\hat\mu)} f_\mu(x) = f_{\hat\mu}(a)$.*

(*b*) *If $\lim_{(x,\mu)\to(a,\hat\mu)} f_\mu(x) =: f_{\hat\mu}(a)$ exists for all $\hat\mu \in \Lambda$, then $\lim_{x\to a} f_\mu(x) = f_\mu(a)$ uniformly on compact subsets of Λ.*

Proof. We prove the result in the case a is finite. (The case $a = +\infty$ follows with the obvious adaptations.) In order to prove (a) let us show first the continuity of the function $\mu \longmapsto f_\mu(a)$ at some fixed $\hat\mu$. Consider a given $\varepsilon > 0$. The uniformity of the limit $\lim_{x\to a} f_\mu(x) = f_\mu(a)$ implies that there exists $\delta > 0$ such that

$$|f_\mu(x) - f_\mu(a)| < \frac{\varepsilon}{3}$$

for all $x \in (a-\delta, a)$ and $\mu \in \Lambda$. On the other hand, since $\mu \longmapsto f_\mu(x)$ is continuous, there exists a neighbourhood U of $\hat\mu$ such that

$$|f_\mu(x) - f_{\hat\mu}(x)| < \frac{\varepsilon}{3}$$

for all $\mu \in U$. Therefore, on account of the two previous inequalities and taking an auxiliary $x \in (a, a-\delta)$,

$$|f_\mu(a) - f_{\hat\mu}(a)| \leqslant |f_\mu(a) - f_\mu(x)| + |f_\mu(x) - f_{\hat\mu}(x)| + |f_{\hat\mu}(x) - f_{\hat\mu}(a)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $\mu \in U$, which proves the continuity of $\mu \longmapsto f_\mu(a)$ at $\hat\mu$. Let us show now that, under the uniformity assumption, $f_\mu(x)$ tends to $f_{\hat\mu}(a)$ as $(x,\mu) \longrightarrow (a,\hat\mu)$. Consider a given $\varepsilon > 0$. Then, since $\mu \longmapsto f_\mu(a)$ is continuous, there exists a neighbourhood U of $\hat\mu$ such that $|f_\mu(a) - f_{\hat\mu}(a)| < \frac{\varepsilon}{2}$ for all $\mu \in U$. Furthermore, thanks to the uniformity assumption, there exists $\delta > 0$ such that $|f_\mu(x) - f_\mu(a)| < \frac{\varepsilon}{2}$ for all $x \in (a-\delta, a)$ and $\mu \in U$. Consequently,

$$|f_\mu(x) - f_{\hat\mu}(a)| \leqslant |f_\mu(x) - f_\mu(a)| + |f_\mu(a) - f_{\hat\mu}(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $x \in (a-\delta, a)$ and $\mu \in U$, and this proves (a). To show (b) let us consider a compact subset K of Λ. By hypothesis $(x,\mu) \longmapsto f_\mu(x)$ extends continuously to $[0,a]\times K$, which is also compact. So the map is uniformly continuous, which clearly implies that $\lim_{x\to a} f_\mu(x) = f_\mu(a)$ is uniform on K. This proves (b) and completes the proof of the result. ∎

Following Definition 1.2.7, for each $\mu \in \Lambda$, we define $f_n(\,\cdot\,;\mu)$ setting $f_0(\,\cdot\,;\mu) := f_\mu$.

Lemma 1.3.7. *Let Λ be an open subset of $\mathbb{R}^d$ and consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of analytic functions on $[0,+\infty)$. Suppose that $\{f_{n-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\beta(\mu)$ with $\beta(\hat{\mu})<-1$. Then $M_1[f_{n-1}(\,\cdot\,;\mu)]$ is well defined and continuous on some neighbourhood of $\hat{\mu}$ and, moreover,*

$$\lim_{(z,\mu)\to(+\infty,\hat{\mu})}\int_0^z f_{n-1}(t;\mu)dt = M_1[f_{n-1}(\,\cdot\,;\mu)].$$

Proof. We claim that $\lim_{z\to+\infty}\int_0^z f_{n-1}(t;\mu)dt$ converges uniformly to $M_1[f_{n-1}(\cdot;\mu)]$ in a neighbourhood of $\hat{\mu}$. Once we prove the claim then the result will follow by (a) in Lemma 1.3.6. Consider a given $\varepsilon>0$. On account of Remark 1.3.3 we can take a compact neighbourhood K_1 of $\hat{\mu}$ such that $\beta(\mu)<-1$ for all $\mu\in K_1$. Let us denote $\hat{\beta}:=\max\{\beta(\mu);\mu\in K_1\}$, which is strictly smaller than -1. Since $\{f_{n-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\beta(\mu)$ with, let us say, limit a, there exist $\hat{z}>0$ and a compact neighbourhood $K_2\subset K_1$ of $\hat{\mu}$ such that $\left|z^{-\beta(\mu)}f_{n-1}(z;\mu)-a\right|<1$ for all $z>\hat{z}$ and $\mu\in K_2$. On the other hand, since the integral $\int_0^{+\infty}t^{\hat{\beta}}dt$ converges due to $\hat{\beta}<-1$, there exists $b>\hat{z}$ such that $\int_b^{+\infty}t^{\hat{\beta}}dt<\frac{\varepsilon}{1+|a|}$. Therefore,

$$\begin{aligned}\left|\int_c^{+\infty}f_{n-1}(t;\mu)dt\right| &\leqslant \int_c^{+\infty}\left|\frac{f_{n-1}(t;\mu)}{t^{\beta(\mu)}}-a\right|t^{\beta(\mu)}dt+|a|\int_c^{+\infty}t^{\beta(\mu)}dt\\ &<(1+|a|)\int_c^{+\infty}t^{\beta(\mu)}dt<(1+|a|)\int_c^{+\infty}t^{\hat{\beta}}dt<\varepsilon\end{aligned}$$

for all $c\in(b,+\infty)$ and $\mu\in K_2$. This proves the claim and so the result follows. ■

Proposition 1.3.8. *Let Λ be an open subset of $\mathbb{R}^d$ and consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of analytic functions on $[0,+\infty)$ continuously quantifiable in Λ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)$. Assume that for some $\hat{\mu}\in\Lambda$, $\alpha(\hat{\mu})<-1$ and let us take $n\in\mathbb{N}$ such that $\alpha(\hat{\mu})+2n\in[-1,1)$. Then, setting $\alpha_j(\mu):=\prod_{i=1}^j\frac{\alpha(\mu)+2i}{\alpha(\mu)+2i-1}$ for $j=1,2,\ldots,n$, the following assertions hold:*

(a) If $M_1[f_0(\,\cdot\,;\mu)]\equiv M_1[f_1(\,\cdot\,;\mu)]\equiv\cdots\equiv M_1[f_{k-1}(\,\cdot\,;\mu)]\equiv 0$ for some $1\leqslant k<n$, then for each $j=1,2,\ldots,k$, $\{f_j(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)+2j$ with limit $a(\mu)\alpha_j(\mu)$.

(b) If $M_1[f_0(\,\cdot\,;\mu)]\equiv M_1[f_1(\,\cdot\,;\mu)]\equiv\cdots\equiv M_1[f_{n-1}(\,\cdot\,;\mu)]\equiv 0$ and $\alpha(\hat{\mu})+2n\notin\{0,-1\}$, then $\{f_n(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)+2n$ with limit $a(\mu)\alpha_n(\mu)$.

Proof. To show (a) assume that $M_1[f_0(\,\cdot\,;\mu)]\equiv M_1[f_1(\,\cdot\,;\mu)]\equiv\cdots\equiv M_1[f_{k-1}(\,\cdot\,;\mu)]\equiv 0$ for some $1\leqslant k<n$. We will prove inductively that there exists a neighbourhood U_j of $\hat{\mu}$ such that

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})}\frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}}=a(\bar{\mu})\alpha_j(\bar{\mu})$$

for all $\bar{\mu} \in U_j$. For $j = 0$ this follows by assumption taking $U_0 = \Lambda$. For the inductive step suppose that it is true for $j-1$. By applying Lemma 1.2.8 for each fixed $\mu \in U_{j-1}$ we have

$$\lim_{z \to +\infty} \frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}} = a(\mu)\alpha_j(\mu).$$

Thus, for each fixed $\mu \in U_{j-1}$, the function $f_j(z;\mu)$ is quantifiable at $+\infty$. Let us show that is, indeed, continuously quantifiable. With this aim in view we note that

$$\frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}} = \frac{f_{j-1}(z;\mu)}{z^{\alpha(\mu)+2(j-1)}} + \frac{\int_0^z f_{j-1}(t;\mu)dt}{z^{\alpha(\mu)+2j-1}}. \tag{1.12}$$

By the induction hypothesis, $\{f_{j-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in U_{j-1} at $+\infty$ by $\alpha(\mu) + 2(j-1)$ with limit $a(\mu)\alpha_{j-1}(\mu)$. Therefore

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{f_{j-1}(z;\mu)}{z^{\alpha(\mu)+2(j-1)}} = a(\bar{\mu})\alpha_{j-1}(\bar{\mu}) \tag{1.13}$$

for all $\bar{\mu} \in U_{j-1}$. To obtain the limit of the second summand in (1.12) we use the uniform Hôpital's Rule in Proposition 4.0.2. With this aim in view note that the functions $\int_0^z f_{j-1}(t;\mu)dt$ and $z^{\alpha(\mu)+2j-1}$ are differentiable on $(0,+\infty)$ for each $\mu \in U_{j-1}$. Moreover, from (1.13), the limit of the quotient of derivatives is

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{f_{j-1}(z;\mu)}{(\alpha(\mu)+2j-1)z^{\alpha(\mu)+2j-2}} = \frac{a(\bar{\mu})\alpha_{j-1}(\bar{\mu})}{\alpha(\bar{\mu})+2j-1}$$

for all $\bar{\mu} \in U_{j-1}$ and so, by applying Lemma 1.3.6, there exists a compact neighbourhood K of $\hat{\mu}$ such that

$$\lim_{z\to+\infty} \frac{f_{j-1}(z;\mu)}{(\alpha(\mu)+2j-1)z^{\alpha(\mu)+2j-2}} = \frac{a(\mu)\alpha_{j-1}(\mu)}{\alpha(\mu)+2j-1}$$

uniformly on K. Therefore it only remains to check condition (e) in Proposition 4.0.2, i.e., that there exists $c \in (0,+\infty)$ such that, for each $x \in (c,+\infty)$,

$$\lim_{z\to+\infty} \frac{z^{\alpha(\mu)+2j-1}}{x^{\alpha(\mu)+2j-1}} = 0 \text{ and } \lim_{z\to+\infty} \frac{\int_0^z f_{j-1}(t;\mu)dt}{x^{\alpha(\mu)+2j-1}} = 0$$

uniformly on μ. In order to verify this let us take a neighbourhood U_j of $\hat{\mu}$ such that $\hat{\alpha} := \max\{\alpha(\mu)+2j-1 : \mu \in U_j\}$ is strictly smaller than -1. Then, taking $x > 1$,

$$\frac{z^{\alpha(\mu)+2j-1}}{x^{\alpha(\mu)+2j-1}} = \left(\frac{z}{x}\right)^{\alpha(\mu)+2j-1} < z^{\alpha(\mu)+2j-1} < z^{\hat{\alpha}} \longrightarrow 0 \text{ as } z \text{ tends to } +\infty,$$

and so the first limit tends to zero uniformly on U_j. We claim that the second limit is also uniform in a neighbourhood of $\hat{\mu}$. To show this we note that, by Lemma 1.3.7,

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{\int_0^z f_{j-1}(t;\mu)dt}{x^{\alpha(\mu)+2j-1}} = \frac{M_1[f_{j-1}(\,\cdot\,;\bar{\mu})]}{x^{\alpha(\bar{\mu})+2j-1}} = 0$$

for all $\bar{\mu} \in U_j$ and then the claim follows by Lemma 1.3.6. Taking U_j to be the intersection of the previous neighbourhoods we can thus apply Proposition 4.0.2 and assert that

$$\lim_{z\to+\infty} \frac{\int_0^z f_{j-1}(t;\mu)dt}{z^{\alpha(\mu)+2j-1}} = \frac{a(\mu)\alpha_{j-1}(\mu)}{\alpha(\mu)+2j-1}$$

uniformly on U_j. Consequently, by applying Lemma 1.3.6 once again,

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{\int_0^z f_{j-1}(t;\mu)dt}{z^{\alpha(\mu)+2j-1}} = \frac{a(\bar{\mu})\alpha_{j-1}(\bar{\mu})}{\alpha(\bar{\mu})+2j-1}$$

for all $\bar{\mu} \in U_j$. Then, from (1.12), the above limit together with (1.13) show that

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{f_j(z;\mu)}{z^{\alpha(\mu)+2j}} = a(\bar{\mu})\alpha_{j-1}(\bar{\mu}) + \frac{a(\bar{\mu})\alpha_{j-1}(\bar{\mu})}{\alpha(\bar{\mu})+2j-1} = a(\bar{\mu})\alpha_j(\bar{\mu}) \neq 0.$$

Therefore $f_j(z;\mu)$ is continuously quantifiable in U_j at $+\infty$ by $\alpha(\mu)+2j$ with limit $a(\mu)\alpha_j(\mu)$. This shows the inductive step and so (a) follows. The proof of (b) follows exactly the same way taking into account that $\alpha_n(\mu)$ is well defined and non-vanishing due to $\alpha(\mu)+2n \notin \{0,-1\}$ in a neighbourhood of $\hat{\mu}$. ∎

Now we are in position to prove the second main result of this section. In its statement recall that $\mathscr{G}$ is the function defined in (1.1).

Theorem 1.3.9. *Let Λ be an open subset of $\mathbb{R}^d$ and consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of analytic function on $[0,+\infty)$. Suppose that $\{f_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)$. Assume that for some $\hat{\mu} \in \Lambda$, $\alpha(\hat{\mu}) < -1$ and let us take $n \in \mathbb{N}$ such that $\alpha(\hat{\mu})+2n \in [-1,1)$. The following assertions hold:*

(a) If $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \ldots \equiv M_{j-1}[f_\mu] \equiv 0$ and $M_j[f_{\hat{\mu}}] \neq 0$ for some $1 \leqslant j \leqslant n$, then $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat{\mu}$ at $+\infty$ by $1-2j$ with limit $\prod_{i=1}^{j-1}(1-\frac{1}{2i})M_j[f_\mu]$.

(b) If $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \cdots \equiv M_n[f_\mu] \equiv 0$ and $\alpha(\hat{\mu})+2n \notin \{-1,0\}$, then $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)\mathscr{G}(\alpha(\mu)+2n)\prod_{i=1}^{n}\frac{\alpha(\mu)+2i}{\alpha(\mu)+2i-1}$.

Proof. Let us show (a). Due to $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \ldots \equiv M_{j-1}[f_\mu] \equiv 0$ and $M_j[f_{\hat{\mu}}] \neq 0$, by Lemma 1.2.9 we have $M_1[f_0(\,\cdot\,;\mu)] = M_1[f_1(\,\cdot\,;\mu)] = \cdots = M_1[f_{j-2}(\,\cdot\,;\mu)] = 0$ for all $\mu \in \Lambda$ and $M_1[f_{j-1}(\,\cdot\,;\hat{\mu})] \neq 0$. By applying Proposition 1.3.8 there exists a neighbourhood $\hat{U}$ of $\hat{\mu}$ such that $\{f_{j-1}(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{U}$ at $+\infty$ by $\alpha(\mu)+2(j-1)$ with limit $a(\mu)\alpha_{j-1}(\mu)$. Then

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} f_{j-1}(z;\mu)z = \lim_{(z,\mu)\to(+\infty,\bar{\mu})} a(\mu)\alpha_{j-1}(\mu)z^{\alpha(\mu)+2j-1} = 0$$

for any $\bar{\mu} \in \hat{U}$, due to $j \leqslant n$ and $\alpha(\mu) + 2n < 1$. Consequently, on account of Definition 1.2.4 and using Lemma 1.3.7 we get

$$\lim_{(z,\mu)\to(+\infty,\bar{\mu})} \frac{f_j(z;\mu)}{z} = \lim_{(z,\mu)\to(+\infty,\bar{\mu})} \int_0^z f_{j-1}(t;\mu)dt = M_1[f_{j-1}(\,\cdot\,;\bar{\mu})].$$

Accordingly, the family $\{f_j(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{U}$ at $+\infty$ by 1 with limit $M_1[f_{j-1}(\,\cdot\,;\mu)] = c_j M_j[f_\mu]$, where $c_j := \prod_{i=1}^{j-1}\big(1 - \frac{1}{2i}\big)$. Hence, by Lemma 1.2.5 and Theorem 1.3.5, $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{U}$ at $+\infty$ by $1-2j$ with limit $c_j M_j[f_\mu]$. This proves the validity of (a).

Let us turn now to the proof of (b). In this case, by Proposition 1.3.8, $\{f_n(\,\cdot\,;\mu)\}_{\mu\in\Lambda}$ is continuously quantifiable in a neighbourhood of $\hat{\mu}$ at $+\infty$ by $\alpha(\mu) + 2n$ with limit $a(\mu)\prod_{i=1}^n \frac{\alpha(\mu)+2i}{\alpha(\mu)+2i-1}$. Since $\alpha(\mu) + 2n > -1$, by Lemma 1.2.5 and Theorem 1.3.5 it follows that $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)\mathscr{G}(\alpha(\mu)+2n)\prod_{i=1}^n \frac{\alpha(\mu)+2i}{\alpha(\mu)+2i-1}$. So the result is proved. ■

For convenience we gather Theorems 1.3.5 and 1.3.9 in a single result.

Theorem A. *Let Λ be an open subset of $\mathbb{R}^d$ and consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of analytic functions on $[0,+\infty)$. Suppose that $\{f_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)$. The following assertions hold:*

(a) *If $\alpha(\hat{\mu}) > -1$, then $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in a neighbourhood of $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)\mathscr{G}(\alpha(\mu))$.*

(b) *If $\alpha(\hat{\mu}) < -1$, let us take $n \in \mathbb{N}$ such that $\alpha(\hat{\mu}) + 2n \in [-1,1)$. In this case:*

- $(b1)$ *If $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \ldots \equiv M_{j-1}[f_\mu] \equiv 0$ and $M_j[f_{\hat{\mu}}] \neq 0$ for some $1 \leqslant j \leqslant n$, then $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat{\mu}$ at $+\infty$ by $1-2j$ with limit $\prod_{i=1}^{j-1}\big(1-\frac{1}{2i}\big)M_j[f_{\hat{\mu}}]$.*
- $(b2)$ *If $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \cdots \equiv M_n[f_\mu] \equiv 0$ and $\alpha(\hat{\mu}) + 2n \notin \{-1,0\}$, then $\{\mathscr{F}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in some neighbourhood of $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit $a(\mu)\mathscr{G}(\alpha(\mu)+2n)\prod_{i=1}^n \frac{\alpha(\mu)+2i}{\alpha(\mu)+2i-1}$.*

We point out that the hypothesis $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \ldots \equiv M_{j-1}[f_\mu] \equiv 0$ in $(b1)$ of the previous statement is void for $j = 1$.

Theorem A particularly shows that, under assumptions on the family $\{f_\mu\}_{\mu\in\Lambda}$, the function $\mathscr{F}[f_\mu]$ is written as $\mathscr{F}[f_\mu](x) = x^{-\nu(\mu)}(\Delta(\mu)+r_\mu(x))$ where $r_\mu(x)$ tends uniformly to zero as x tends to infinity and the functions ν and Δ are continuous. This implies that there exist $\varepsilon, M > 0$ such that if $\|\mu - \hat{\mu}\| < \varepsilon$ then $\mathscr{F}[f_\mu]$ has no zeroes in $(M,+\infty)$. Next section is devoted to generalize this in terms of bounding the number of zeroes of $\mathscr{F}[f_\mu]$.

1.4 Uniform upper-bound of the zeros of $\mathscr{F}[f_\mu]$

This section is devoted to generalize the previous result Theorem A. Later we use it to prove the result concerning the criticality. Next we recall the notions of Chebyshev system and Wronskian, that will be very useful for our purposes.

Definition 1.4.1. Let $f_0, f_1, \dots f_{n-1}$ be analytic functions on an open interval $I \subset \mathbb{R}$. The ordered set $(f_0, f_1, \dots f_{n-1})$ is an *extended complete Chebyshev system* (for short, an ECT-system) on I if, for all $k = 1, 2, \dots n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on I counted with multiplicities. (Let us mention that, in these abbreviations, "T" stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev). □

Definition 1.4.2. Let $f_0, f_1, \dots, f_{k-1}$ be analytic functions on an open interval I of $\mathbb{R}$. The Wronskian of $(f_0, f_1, \dots, f_{k-1})$ at $x \in I$ is

$$W[f_0, f_1, \dots, f_{k-1}](x) = \det\left(f_j^{(i)}(x)\right)_{0 \leqslant i,j \leqslant k-1} = \begin{vmatrix} f_0(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & \cdots & f_{k-1}'(x) \\ & \vdots & \\ f_0^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

□

Next lemma is well-known (see for instance [29]). We also refer the reader to [37] for further details on ECT-systems.

Lemma 1.4.3. *Let $f_0, f_1, \dots, f_{k-1}$ be analytic functions on I. $(f_0, f_1, \dots, f_{n-1})$ is an ECT-system on I if and only if for each $k \in \{1, \dots, n\}$, $W[f_0, f_1, \dots, f_{k-1}](x) \neq 0$ for all $x \in I$.*

Definition 1.4.4. Given $\nu_1, \nu_2, \dots, \nu_n \in \mathbb{R}$ we define the differential operator

$$\mathscr{L}_{\boldsymbol{\nu}_n} : \mathscr{C}^\omega\big((0, +\infty)\big) \longrightarrow \mathscr{C}^\omega\big((0, +\infty)\big)$$

given by

$$\mathscr{L}_{\boldsymbol{\nu}_n}[f](x) := \frac{W\left[x^{\nu_1}, x^{\nu_2}, \dots, x^{\nu_n}, f(x)\right]}{x^{\sum_{i=1}^n (\nu_i - i)}}.$$

Here, and in what follows, for the sake of shortness we use the notation $\boldsymbol{\nu}_n = (\nu_1, \dots, \nu_n)$. In addition, we define $\mathscr{L}_{\boldsymbol{\nu}_0} = id$. □

The main idea will be to find continuous function $\nu_1, \ldots, \nu_n$ pairwise distinct at $\mu = \hat{\mu}$ satisfying that there exist $\varepsilon, M > 0$ such that for all $\|\mu - \hat{\mu}\| < \varepsilon$, the ordered set $(x^{\nu_1(\mu)}, \ldots, x^{\nu_n(\mu)}, \mathscr{F}[f_\mu](x))$ form an ECT-system on $(M, +\infty)$. In particular, this will imply that $\mathscr{F}[f_\mu]$ has no more than n isolated zeroes counted with multiplicities for $x \approx +\infty$ and $\mu \approx \hat{\mu}$. Since $\nu_1, \ldots, \nu_n$ are pairwise distinct at $\mu = \hat{\mu}$, $(x^{\nu_1(\mu)}, \ldots, x^{\nu_n(\mu)})$ is an ECT-system for $\mu \approx \hat{\mu}$ and all $x > 0$. Therefore, on account of Lemma 1.4.3, the property above is equivalent to show that there exist $\varepsilon, M > 0$ such that if $\|\mu - \hat{\mu}\| < \varepsilon$ then the function $x \mapsto W[x^{\nu_1(\mu)}, \ldots, x^{\nu_n(\mu)}, \mathscr{F}[f_\mu](x)]$ has no zeros in $(M, +\infty)$. The main idea of this section is then to apply Theorem A with the Wronskian function above to study its behaviour at infinity. More precisely, in this section we study under which conditions the quantifier of $\mathscr{L}_{\boldsymbol{\nu}_n}[f_\mu]$ at $x = +\infty$ enables to quantify $(\mathscr{L}_{\boldsymbol{\nu}_n} \circ \mathscr{F})[f_\mu]$ at $x = +\infty$. To this end some previous technical results about Wronskians are presented in the following lines. The first two lemmas are already known (see, respectively, [36] and [28, 51]).

Lemma 1.4.5. *Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on I. Then the following statements hold:*

(*a*) $W[f_0 \circ \varphi, \ldots, f_{n-1} \circ \varphi](x) = (\varphi'(x))^{\frac{(n-1)n}{2}} W[f_0, \ldots, f_{n-1}](\varphi(x))$ *for any analytic diffeomorphism* φ.

(*b*) $W[gf_0, \ldots, gf_{n-1}](x) = g(x)^n W[f_0, \ldots, f_{n-1}](x)$ *for any analytic function* g.

Lemma 1.4.6. *Let $f_0, f_1, \ldots, f_n$ be analytic functions on an open interval I such that $W[f_0, \ldots, f_{n-2}, f_{n-1}]$ does not vanish on I. Then*

$$\left(\frac{W[f_0, \ldots, f_{n-2}, f_n]}{W[f_0, \ldots, f_{n-2}, f_{n-1}]}\right)' = \frac{W[f_0, \ldots, f_n]\, W[f_0, \ldots, f_{n-2}]}{(W[f_0, \ldots, f_{n-2}, f_{n-1}])^2}.$$

Lemma 1.4.7. *Given $\nu_1, \nu_2, \ldots, \nu_n \in \mathbb{R}$, the following identity holds:*

$$\frac{W\left[x^{\nu_1}, x^{\nu_2}, \ldots, x^{\nu_n}\right]}{x^{\sum_{i=1}^{n}(\nu_i - i)}} = x^n \prod_{\substack{i,j=1 \\ i>j}}^{n} (\nu_i - \nu_j).$$

Proof. We prove the result by induction on n. Since the base case $n = 1$ is obvious, let us show the induction step. By applying (*b*) in Lemma 1.4.5 we get

$$W[x^{\nu_1}, \ldots, x^{\nu_{n-1}}, x^{\nu_n}] = x^{n\nu_n} W[x^{\nu_1 - \nu_n}, \ldots, x^{\nu_{n-1} - \nu_n}, 1]. \tag{1.14}$$

Let us denote $\beta_i := \nu_i - \nu_n$ for shortness. Then, using well-known properties of the determinant and the induction hypothesis,

$$\begin{aligned} W[x^{\beta_1},\ldots,x^{\beta_{n-1}},1] &= (-1)^{n-1}W[\beta_1 x^{\beta_1-1},\ldots,\beta_{n-1}x^{\beta_{n-1}-1}] \\ &= (-1)^{n-1}W[x^{\beta_1-1},\ldots,x^{\beta_{n-1}-1}]\prod_{k=1}^{n-1}\beta_k \\ &= (-1)^{n-1}x^{\sum_{i=1}^{n-1}(\beta_i-1-i)}x^{n-1}\prod_{k=1}^{n-1}\beta_k\prod_{\substack{i,j=1\\ i>j}}^{n-1}(\beta_i-\beta_j). \end{aligned}$$

Consequently, substituting the previous equality in (1.14), we have

$$\begin{aligned} W[x^{\nu_1},\ldots,x^{\nu_{n-1}},x^{\nu_n}] &= x^{n\nu_n}(-1)^{n-1}x^{\sum_{i=1}^{n-1}(\nu_i-\nu_n-1-i)}x^{n-1}\prod_{i=k}^{n-1}(\nu_k-\nu_n)\prod_{\substack{i,j=1\\ i>j}}^{n-1}(\nu_i-\nu_j) \\ &= x^n x^{\sum_{i=1}^{n}(\nu_i-i)}\prod_{\substack{i,j=1\\ i>j}}^{n}(\nu_i-\nu_j), \end{aligned}$$

where we used $\beta_i=\nu_i-\nu_n$ in the first equality and the second one follows by means of some easy manipulations. This shows the induction step and so the result is proved. ■

The previous lemma enables to write the differential operator under consideration as a quotient of Wronskians. Indeed, if ν_i for $i=1,2,\ldots,n$ are pairwise distinct, we have that

$$\mathscr{L}_{\boldsymbol{\nu}_n}[f](x) = x^n\prod_{\substack{i,j=1\\ i>j}}^{n}(\nu_i-\nu_j)\frac{W[x^{\nu_1},\ldots,x^{\nu_n},f(x)]}{W[x^{\nu_1},\ldots,x^{\nu_n}]}.$$

At this point it is worth noting that the linear ordinary differential operator

$$f\longmapsto\frac{W[\phi_1,\phi_2,\ldots,\phi_n,f]}{W[\phi_1,\phi_2,\ldots,\phi_n]}$$

has already appeared in the literature in relation with the so called “Chebyshev asymptotic scales” (see [24, 25] and references therein). Of course, it is also related with the division-derivation algorithm (see [53] for instance) due to the fact that its kernel is spanned by $\{\phi_1,\phi_2,\ldots,\phi_n\}$.

Our next result shows that the integral operator $\mathscr{F}$ and the differential operator $\mathscr{L}_{\boldsymbol{\nu}_n}$ commute. This fact is the key point in order to prove our main results.

Proposition 1.4.8. *For any given $f\in\mathscr{C}^\omega((0,+\infty))$ and $\nu_1,\ldots,\nu_n\in\mathbb{R}$, the following recurrence holds:*

$$\mathscr{L}_{\boldsymbol{\nu}_n}[f](x) = c_n\big(x\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f]'(x)-\nu_n\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f](x)\big),$$

where $c_1 := 1$ and $c_n := \prod_{i=1}^{n-1}(\nu_n - \nu_i)$ for $n \geqslant 2$. In particular, if f can be extended analytically to $x = 0$, then $\mathscr{L}_{\boldsymbol{\nu}_n}[f]$ can be extended analytically to $x = 0$. Moreover, $\mathscr{F} \circ \mathscr{L}_{\boldsymbol{\nu}_n} = \mathscr{L}_{\boldsymbol{\nu}_n} \circ \mathscr{F}$.

Proof. We can suppose that $\nu_1, \nu_2, \ldots, \nu_n$ are pairwise distinct, otherwise there is nothing to be proved. The case $n = 1$ of the recurrence is straightforward because, by definition, $\mathscr{L}_{\boldsymbol{\nu}_0} = id$ and

$$\mathscr{L}_{\boldsymbol{\nu}_1}[f](x) = \frac{W[x^{\nu_1}, f(x)]}{x^{\nu_1 - 1}} = xf'(x) - \nu_1 f(x).$$

Let us show now the case $n \geqslant 2$. To this end take any $k \in \{1, 2, \ldots, n-1\}$ and note that, by Lemma 1.4.6,

$$\left(\frac{W[x^{\nu_1}, \ldots, x^{\nu_k}, g(x)]}{W[x^{\nu_1}, \ldots, x^{\nu_{k+1}}]}\right)' = \frac{W[x^{\nu_1}, \ldots, x^{\nu_{k+1}}, g(x)]W[x^{\nu_1}, \ldots, x^{\nu_k}]}{(W[x^{\nu_1}, \ldots, x^{\nu_{k+1}}])^2}$$

for any analytic function g. Hence, some easy computations taking Lemma 1.4.7 into account show that

$$\begin{aligned} W[x^{\nu_1}, \ldots, x^{\nu_k}, g(x)]' &= \frac{W[x^{\nu_1}, \ldots, x^{\nu_{k+1}}, g(x)]}{x^{\nu_{k+1}-k}\prod_{i=1}^{k}(\nu_{k+1} - \nu_i)} \\ &\quad + \frac{1}{x}\big(\nu_{k+1} + \textstyle\sum_{i=1}^{k}(\nu_i - i)\big)W[x^{\nu_1}, \ldots, x^{\nu_k}, g(x)]. \end{aligned} \tag{1.15}$$

By Definition 1.4.4 we have on the other hand that

$$W[x^{\nu_1}, x^{\nu_2}, \ldots, x^{\nu_k}, f(x)]' = x^{\sum_{i=1}^{k}(\nu_i - i)}\mathscr{L}_{\boldsymbol{\nu}_k}[f]'(x) + \frac{1}{x}\sum_{i=1}^{k}(\nu_i - i)W[x^{\nu_1}, x^{\nu_2}, \ldots, x^{\nu_k}, f(x)].$$

Then, using (1.15) with $g = f$ together with the above equality and the definition of $\mathscr{L}_{\boldsymbol{\nu}_k}[f]$ once again, after some computations we get

$$\mathscr{L}_{\boldsymbol{\nu}_{k+1}}[f](x) = \prod_{i=1}^{k}(\nu_{k+1} - \nu_i)\big(x\mathscr{L}_{\boldsymbol{\nu}_k}[f]'(x) - \nu_{k+1}\mathscr{L}_{\boldsymbol{\nu}_k}[f](x)\big).$$

Thus, taking $k = n-1$ we obtain the recurrence in the statement for $n \geqslant 2$.

Let us turn to the proof of $\mathscr{F} \circ \mathscr{L}_{\boldsymbol{\nu}_n} = \mathscr{L}_{\boldsymbol{\nu}_n} \circ \mathscr{F}$. We show it by induction on $n \geqslant 0$ taking advantage of the recurrence we have just proved. The base case $n = 0$ is clear because $\mathscr{L}_{\boldsymbol{\nu}_0} = id$. To show the induction step we note that $\mathscr{F}[g]'(x) = \frac{1}{x}\mathscr{F}[xg'(x)]$ for any $g \in \mathscr{C}^\omega((0, +\infty))$. Thus, deriving the induction hypothesis, we get

$$0 = \mathscr{F}\big[\mathscr{L}_{\boldsymbol{\nu}_n}[g]\big]'(x) - \mathscr{L}_{\boldsymbol{\nu}_n}\big[\mathscr{F}[g]\big]'(x) = \frac{1}{x}\mathscr{F}\big[x\mathscr{L}_{\boldsymbol{\nu}_n}[g]'(x)\big] - \mathscr{L}_{\boldsymbol{\nu}_n}\big[\mathscr{F}[g]\big]'(x).$$

Therefore,

$$
\begin{aligned}
0 =& \mathscr{F}\left[x\mathscr{L}_{\boldsymbol{\nu}_n}[g]'(x)\right] - x\mathscr{L}_{\boldsymbol{\nu}_n}\left[\mathscr{F}[g]\right]'(x)\\
=& \mathscr{F}\left[\frac{1}{c_{n+1}}\mathscr{L}_{\boldsymbol{\nu}_{n+1}}[g] + \nu_{n+1}\mathscr{L}_{\boldsymbol{\nu}_n}[g]\right](x) - \left(\frac{1}{c_{n+1}}\mathscr{L}_{\boldsymbol{\nu}_{n+1}}\left[\mathscr{F}[g]\right] + \nu_{n+1}\mathscr{L}_{\boldsymbol{\nu}_n}\left[\mathscr{F}[g]\right]\right)(x)\\
=& \left(\frac{1}{c_{n+1}}\mathscr{F}\left[\mathscr{L}_{\boldsymbol{\nu}_{n+1}}[g]\right] + \nu_{n+1}\mathscr{F}\left[\mathscr{L}_{\boldsymbol{\nu}_n}[g]\right] - \frac{1}{c_{n+1}}\mathscr{L}_{\boldsymbol{\nu}_{n+1}}\left[\mathscr{F}[g]\right] - \nu_{n+1}\mathscr{L}_{\boldsymbol{\nu}_n}\left[\mathscr{F}[g]\right]\right)(x)\\
=& \frac{1}{c_{n+1}}\left(\mathscr{F}\left[\mathscr{L}_{\boldsymbol{\nu}_{n+1}}[g]\right] - \mathscr{L}_{\boldsymbol{\nu}_{n+1}}\left[\mathscr{F}[g]\right]\right)(x),
\end{aligned}
$$

where in the second equality we use twice the recurrence, taking $f = g$ and $f = \mathscr{F}[g]$, in the third one the linearity of $\mathscr{F}$, and in the fourth one the induction hypothesis. Hence $\mathscr{F}\left[\mathscr{L}_{\boldsymbol{\nu}_{n+1}}[g]\right] - \mathscr{L}_{\boldsymbol{\nu}_{n+1}}\left[\mathscr{F}[g]\right] = 0$ and so the induction step follows. This concludes the proof of the result. ■

Lemma 1.4.9. *Let f be an analytic function on $[0,+\infty)$, $\nu_1,\nu_2,\ldots,\nu_n \in \mathbb{R}$ and $\ell \in \mathbb{N}$. Let us assume that $\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f]$ is quantifiable at $+\infty$ by ξ. If $\xi < 1-2\ell$, then*

$$
M_\ell\left[\mathscr{L}_{\boldsymbol{\nu}_n}[f]\right] = c_n(1-2\ell-\nu_n)M_\ell\left[\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f]\right],
$$

where $c_1 := 1$ and $c_n := \prod_{i=1}^{n-1}(\nu_n - \nu_i)$ for $n \geqslant 2$.

Proof. By using the recurrence in Proposition 1.4.8 and the definition of the momentum,

$$
\begin{aligned}
M_\ell\left[\mathscr{L}_{\boldsymbol{\nu}_n}[f]\right] &= \int_0^{+\infty} x^{2\ell-2}\mathscr{L}_{\boldsymbol{\nu}_n}[f](x)dx\\
&= c_n\int_0^{+\infty} x^{2\ell-2}\left(x\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f]'(x) - \nu_n\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f](x)\right)dx.
\end{aligned}
$$

Since $\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f]$ is quantifiable at infinity by $\xi < 1-2\ell$, we can assert that

$$
\lim_{x\to+\infty} x^{2\ell-1}\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f](x) = 0.
$$

Moreover, by Proposition 1.4.8, $\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f]$ extends analytically to $x=0$. So, integrating by parts the previous equality,

$$
M_\ell\left[\mathscr{L}_{\boldsymbol{\nu}_n}[f]\right] = c_n(1-2\ell-\nu_n)\int_0^{+\infty} x^{2\ell-2}\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f](x)dx = c_n(1-2\ell-\nu_n)M_\ell\left[\mathscr{L}_{\boldsymbol{\nu}_{n-1}}[f]\right]
$$

and this proves the result. ■

In the following statement $\nu_1,\nu_2,\ldots,\nu_n$ are not real numbers any more but continuous functions on Λ. For shortness, we keep using the notation $\boldsymbol{\nu}_n(\mu) = \big(\nu_1(\mu),\ldots,\nu_n(\mu)\big)$.

Theorem B. *Let Λ be an open subset of $\mathbb{R}^d$ and $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions on $[0,+\infty)$. Assume that, in a neighbourhood of some fixed $\hat{\mu}\in\Lambda$, there exist $n\geqslant 0$ continuous functions $\nu_1,\nu_2,\ldots,\nu_n$, with $\nu_1(\hat{\mu}),\nu_2(\hat{\mu}),\ldots,\nu_n(\hat{\mu})$ pairwise distinct, and such that the family $\big\{\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $+\infty$ by $\xi(\mu)$ with limit $\ell(\mu)$. The following assertions hold:*

(*a*) *If $\xi(\hat{\mu})>-1$, then $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F})[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\xi(\mu)$ with limit $\ell(\mu)\mathscr{G}(\xi(\mu))$.*

(*b*) *If $\xi(\hat{\mu})<-1$, let us take $m\in\mathbb{N}$ such that $\xi(\hat{\mu})+2m\in[-1,1)$. In this case:*

- (*b*1) *If $M_1\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv M_2\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv\ldots\equiv M_{j-1}\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv 0$ and $M_j\big[\mathscr{L}_{\boldsymbol{\nu}_n(\hat{\mu})}[f_{\hat{\mu}}]\big]\neq 0$ for some $1\leqslant j\leqslant m$, then $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F})[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $1-2j$ with limit $\prod_{i=1}^{j-1}\big(1-\frac{1}{2i}\big)M_j\big[\mathscr{L}_{\boldsymbol{\nu}_n(\hat{\mu})}[f_{\hat{\mu}}]\big]$.*
- (*b*2) *If $M_1\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv M_2\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv\ldots\equiv M_m\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv 0$ and $\xi(\hat{\mu})+2m\notin\{-1,0\}$, then $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F})[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $+\infty$ by $\xi(\mu)$ with limit $\ell(\mu)\prod_{i=1}^{m}\frac{\xi(\mu)+2i}{\xi(\mu)+2i-1}\mathscr{G}(\xi(\mu)+2m)$.*

Proof. We first apply Proposition 1.4.8, which shows that $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]$ is an analytic function on $[0,+\infty)$ for each $\mu\in\Lambda$, and that

$$(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F})[f_\mu](x)=(\mathscr{F}\circ\mathscr{L}_{\boldsymbol{\nu}_n(\mu)})[f_\mu](x)=\int_0^{\frac{\pi}{2}}\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu](x\sin\theta)d\theta.$$

Then the result follows by applying Theorem A to the family $\big\{\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big\}_{\mu\in\Lambda}$. ■

We point out that the assumption $M_1\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv\ldots\equiv M_{j-1}\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]\equiv 0$ in $(b1)$ is void for $j=1$. Recall in addition that, by definition, $\mathscr{L}_{\boldsymbol{\nu}_0(\mu)}=id$. In particular, the statement of Theorem B with $n=0$ gives Theorem A.

CHAPTER 2

Criticality of potential centers at the outer boundary

We introduce the notion of critical periodic orbit of an analytic planar center. The number of critical periodic orbits that bifurcate from the outer boundary of the period annulus of a potential center is studied. We call this number the criticality at the outer boundary. Our main results provide sufficient conditions in order to bound this number.

2.1 Introduction and main definitions

In this chapter we study planar differential systems

$$\begin{cases} \dot{x} = f(x, y), \\ \dot{y} = g(x, y), \end{cases}$$

where f and g are analytic functions on some open subset U of $\mathbb{R}^2$. A singular point $p \in U$ of the vector field $X = f(x, y)\partial_x + g(x, y)\partial_y$ is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the *period annulus* of the center and it will be denoted by $\mathscr{P}$. In order to define properly the boundary of $\mathscr{P}$ we need to compactify $\mathbb{R}^2$. One of the possible compactifications is the *stereographic projection* (a.k.a. *Alexandroff compactification*) of the real plane into the sphere. In this case a single "point at infinity" is adjoined to the plane. In this work we shall embed $\mathbb{R}^2$ into $\mathbb{RP}^2$.

We recall that the *real projective plane* $\mathbb{RP}^2$ is the set of lines through the origin in $\mathbb{R}^3$. Every such line is called a *projective point*. The real plane $\mathbb{R}^2$ can be embedded into $\mathbb{RP}^2$ by considering the identification of a point p of the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ with the line L_p that joins p and the origin of $\mathbb{R}^3$ (see [60] for instance). Every line in $\mathbb{R}^3$ that passes through the origin is identified with a point in the plane with the exception of the lines in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. This identification therefore embed the affine plane $\mathbb{R}^2$ in the projective plane $\mathbb{RP}^2$. Moreover, $\mathbb{RP}^2$ contains extra projective points that do not correspond to points in $\mathbb{R}^2$. These extra projective points are called "points at infinity" and correspond to lines through the origin in $\mathbb{R}^3$ that lie in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Each point at infinity corresponds to a unique direction in the plane $\mathbb{R}^2$ (that is, where parallel lines "coincides" at infinity). It is common to represent the projective plane $\mathbb{RP}^2$ in the unit disk. We consider, for instance, the following procedure: for a given p in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$, the line L_p intersects the unit sphere $\mathbb{S}^2$ in two points, namely $\pi_+(p)$ and $\pi_-(p)$, which are the intersection with the north and the south hemisphere of $\mathbb{S}^2$, respectively. Then each projective point of $\mathbb{RP}^2$ can be represented by a point on the north hemisphere of the unit sphere with the opposite points of the equator identified. A vertical projection of the north hemisphere into the unit disk give us the desired interpretation of $\mathbb{RP}^2$ into the disk (see Figure 2.1).

Considering the previous procedure, we embed the period annulus $\mathscr{P}$ into $\mathbb{RP}^2$ and we denote by $\partial\mathscr{P}$ its boundary, which is a compact subset of $\mathbb{RP}^2$. Clearly the center p belongs to $\partial\mathscr{P}$, and in what follows we will call it the *inner boundary* of the period annulus. We also define the *outer boundary* of the period annulus to be $\Pi := \partial\mathscr{P} \setminus \{p\}$. Note that Π is a non-empty compact subset of $\mathbb{RP}^2$. We point out that the compactification of $\mathbb{R}^2$

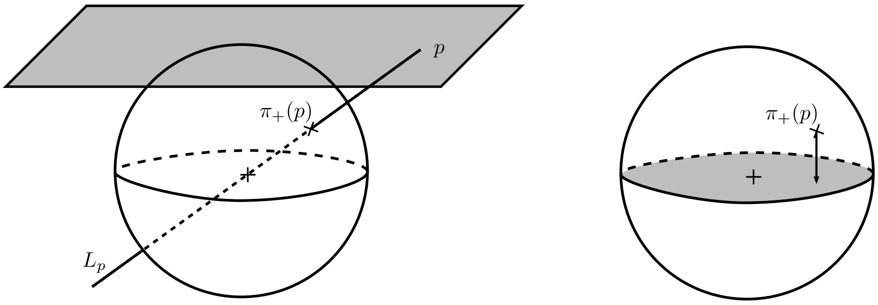

Figure 2.1: Embedding of $\mathbb{R}^2$ into $\mathbb{RP}^2$.

is a topological construction and we do not compactify the vector field itself since the functions f and g are supposed to be analytic but not polynomial, so the vector field does not extend to infinity. The *period function* of the center assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant, then the center is said to be *isochronous*. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parametrize this set. This can be done by taking an analytic transverse section to X on $\mathscr{P}$, for instance an orbit of the orthogonal vector field $X^{\perp}$. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization, then $s \longmapsto T(s) := \{\text{period of } \gamma_s\}$ is an analytic map that provides the qualitative properties of the period function that we are concerned with. In particular the existence of *critical periodic orbits*, which are main objects under study in this work.

Definition 2.1.1. Let X be an analytic planar differential system with a center p. Let $\{\gamma_s\}_{s\in(0,1)}$ be a parametrization of the periodic orbits in $\mathscr{P}$ and consider $s \longmapsto T(s)$. For a given $\hat{s} \in (0,1)$ we say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center X if $\hat{s}$ is an isolated zero of $T'(s)$ of multiplicity k. That is,

$$T'(s) = \alpha(s-\hat{s})^k + o\big((s-\hat{s})^k\big)$$

with $\alpha \neq 0$ and $k \geqslant 1$. Moreover, we say that $\hat{s}$ is a *critical period* of T. □

One can readily see that the definition of critical periodic orbit does not depend on the particular parametrization of the set of periodic orbits used. Critical periodic orbits play in the study of the period function an equivalent role to limit cycles in the framework of the Hilbert's sixteenth problem, which is a fundamental notion in qualitative theory of differential systems in the plane.

Suppose now that the vector field X depends on a parameter $\mu \in \Lambda$, where Λ is an open set of $\mathbb{R}^d$. Thus, for each $\mu \in \Lambda$, we have an analytic vector field X_μ, defined on some open subset U_μ of $\mathbb{R}^2$, with a center at p_μ. Concerning the regularity with respect to the parameter, we shall assume that $\{X_\mu\}_{\mu\in\Lambda}$ is a continuous family of planar differential systems, meaning that the map $(x, y, \mu) \to X_\mu(x, y)$ is continuous on the subset $\{(x, y, \mu); \mu \in \Lambda \text{ and } (x, y) \in U_\mu\}$ of $\mathbb{R}^{d+2}$. Fix $\hat{\mu} \in \Lambda$ and, following the notation introduced previously, let $\Pi_{\hat{\mu}}$ be the outer boundary of the period annulus $\mathscr{P}_{\hat{\mu}}$ of the center at $p_{\hat{\mu}}$ of $X_{\hat{\mu}}$.

The ultimate aim in the study of the global behaviour of the period function of a given family of centers $\{X_\mu\}_{\mu\in\Lambda}$ is to decompose the parameter space $\Lambda = \cup\Lambda_i$ in such a way that if two parameters belong to the same set Λ_i, then the corresponding period functions are qualitatively the same. The set $\cup\partial\Lambda_i$ consists of those parameters $\hat{\mu} \in \Lambda$ for which some critical periodic orbit emerges or disappears as μ tends to $\hat{\mu}$. The authors in [40] proved that there are three different places where a critical periodic orbit may bifurcate from, namely: the inner boundary of the period annulus (i.e., the center itself), the "interior" of the period annulus, or the outer boundary of the period annulus. The union of these three local bifurcation "curves" form the global bifurcation diagram of the period function. This results follows from the definition of bifurcation value of the period function that the authors give in [40]. With this definition the authors are able to define properly the concept of bifurcation in each one of the three previous situations in terms of the zeroes of the derivative of a parametrization of the period function. However, their definition does not allow them to quantify the number of critical periodic orbits that may bifurcate. In this work we introduce the notion of criticality, which is the counterpart of the notion of cyclicity in the study of limit cycles. In the present chapter, for a fixed parameter $\hat{\mu} \in \Lambda$, we define the criticality at the outer boundary of the period annulus as the maximal number of critical periodic orbits that tend to $\Pi_{\hat{\mu}}$ as $\mu \to \hat{\mu}$. This definition is given by means of the Hausdorff distance between non-empty compact subsets of $\mathbb{RP}^2$. Therefore, on account of this notion, we define a local regular value of the period function at the outer boundary as a parameter $\hat{\mu}$ such that its criticality at the outer boundary is zero. Otherwise we shall say that $\hat{\mu}$ is a local bifurcation value of the period function at the outer boundary of the period annulus. This definition, on the contrary as the one of the authors in [40], enables us to quantify the maximal number of critical periodic orbits bifurcating. As we shall comment later, we also introduce the criticality at the inner boundary and similarly we can define local regular and bifurcation value of the period function at the center. However, this definition does not allow us to define local bifurcation value at the interior of the period annulus in terms of the criticality.

The aim of the present chapter is to provide sufficient conditions in order to answer the following bifurcation problem: which is the maximum number of critical periodic orbits that can emerge or disappear from $\Pi_{\hat{\mu}}$ as we move slightly the parameter $\mu \approx \hat{\mu}$? At this point it is to be quoted some previous results on the period function closely related to the ones we are concerned with. The goal of the series of papers [38–41,45,46] is also to study the bifurcation of critical periodic orbits from the outer boundary in a family of centers. However there are some striking differences with our approach due to the fact that we deal with non-polynomial vector fields. Recall that a polynomial vector field X on $\mathbb{R}^2$ can be extended to a vector field $\hat{X}$ on the two-dimensional sphere $\mathbb{S}^2$ by means of the *Poincaré compactification*. The compactified vector field $\hat{X}$ is meromorphic on the equator of $\mathbb{S}^2$, which corresponds to the line at infinity in the original coordinates. Thus, even in case that the center has an unbounded period annulus, one can use this meromorphic extension $\hat{X}$ to study the bifurcation of critical periodic orbits from its outer boundary Π, which becomes a polycycle in $\mathbb{S}^2$. The polycycle consists of regular trajectories and singular points with a hyperbolic sector, which after the desingularization process give rise to saddles and saddle-nodes. It is here where the use of normal forms of such singular points permit to obtain an asymptotic development of the period function near Π. Computing the first non-vanishing coefficient in this development is the key tool in the mentioned series of papers in order to determine which parameters are local regular values of the period function at Π. On the contrary, the vector fields that we deal with in the present paper are not polynomial, but only analytic on some open subset U of $\mathbb{R}^2$. We compactify the set $\mathscr{P}$ in order to define its outer boundary Π in case that $\mathscr{P}$ is unbounded, but we can not compactify the vector field X itself. Furthermore, even in the case of a bounded period annulus, it may happen that the vector field X is not defined at all the points in Π. For this reason the approach that we follow must be completely different. It is also to be noted that once we have determined the local bifurcation values of the period function at the outer boundary, we aim to bound its criticality. This is also a novelty with respect to the quoted papers previously.

Our study restricts to potential systems and we shall give sufficient conditions in order to bound the criticality at the outer boundary of the period annulus. If we denote by $h_0(\mu)$ the energy level at the outer boundary of a potential system X_μ, the tools that we develop in this chapter allow to tackle the problem in the following two situations:

- either $h_0(\mu) = +\infty$ for all $\mu \approx \hat{\mu}$,
- or $h_0(\mu) < +\infty$ for all $\mu \approx \hat{\mu}$.

We do not treat the case in which in any neighbourhood of $\hat{\mu}$ there are μ_1 and μ_2 with $h(\mu_1) = +\infty$ and $h(\mu_2) < +\infty$. Due to the previous situations and the difference between the techniques used in the approach, we divide potential systems in two classes, namely, potential systems with infinite energy and potential systems with finite energy.

This section is organized in the following way. In Section 2.2 we introduce some notation and general results for both kind of potential systems. Then Section 2.3 is devoted to deal with potential systems with infinite energy level whereas Section 2.3 deals with potential systems with finite energy. In Section 2.3 we prove Theorem C, which gives sufficient conditions to bound the criticality for potential systems with infinite energy. On the other hand, Section 2.4 is divided in two independent parts: Section 2.4.1 and Section 2.4.2. The first one is dedicated to give sufficient conditions for a parameter to be a regular value of the period function at the outer boundary (see Theorem D). We also give a criterion to show that at most one critical periodic orbit bifurcates from the outer boundary. The second one is devoted to prove Theorem E, which deals with sufficient conditions to bound the criticality at the outer boundary for potential systems with finite energy as general as the results in Theorem C for the infinite energy case.

2.2 Criticality and potential systems

As we mentioned before, the present chapter is addressed to the study of the number of critical periodic orbits of X_μ that can emerge or disappear from $\Pi_{\hat{\mu}}$ as we move slightly the parameter $\mu \approx \hat{\mu}$. We call this number the criticality of the outer boundary. In order to define the criticality precisely we adapt the notion of cyclicity (cf. [3, 53]), which is its counterpart in the study of limit cycles. Before that, we introduce the notion of Hausdorff distance of metric spaces.

Definition 2.2.1. Let X and Y be two non-empty subsets of a metric space (M, d). We define the Hausdorff distance $d_H(X, Y)$ by

$$d_H(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\}.$$

□

In the following definition, d_H stands for the Hausdorff distance between sets of $\mathbb{RP}^2$.

Definition 2.2.2. Consider a continuous family $\{X_\mu\}_{\mu\in\Lambda}$ of planar analytic vector fields with a center and fix some $\hat{\mu} \in \Lambda$. Suppose that the outer boundary of the period annulus varies continuously at $\hat{\mu} \in \Lambda$, meaning that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_H(\Pi_\mu, \Pi_{\hat{\mu}}) \leqslant \varepsilon$ for all $\mu \in \Lambda$ with $\|\mu - \hat{\mu}\| \leqslant \delta$. Then, setting

$$N(\delta,\varepsilon) = \sup\{\#\text{ critical periodic orbits } \gamma \text{ of } X_\mu \text{ with } d_H(\gamma, \Pi_{\hat{\mu}}) \leqslant \varepsilon \text{ and } \|\mu - \hat{\mu}\| \leqslant \delta\},$$

we define $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) := \inf_{\delta,\varepsilon} N(\delta,\varepsilon)$ to be the *criticality* of $(\Pi_{\hat{\mu}}, X_{\hat{\mu}})$ with respect to the deformation X_μ. □

Notice that $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big)$ may be infinite but if it is finite, then it gives the maximal number of critical periodic orbits γ of X_μ that tend to $\Pi_{\hat{\mu}}$ in the Hausdorff sense as $\mu \to \hat{\mu}$. Similarly as occur in the case of limit periodic sets (see [53]) the convergence of a sequence of critical periodic orbits $\{\gamma_{\mu_n}\}_n$ to $\Pi_{\hat{\mu}}$ is equivalent to the following: for any $\varepsilon > 0$ there exists n_0 such that if $n \geqslant n_0$ then γ_{μ_n} enters the ε-neighbourhood in $\mathbb{RP}^2$ of $\Pi_{\hat{\mu}}$ and inversely, $\Pi_{\hat{\mu}}$ enters the ε-neighbourhood of γ_{μ_n}.

The assumption that the period annulus varies continuously ensures that these changes do not occur abruptly. In this regard note that $X_\mu = -y\partial_x + (x + \mu x^3 + x^5)\partial_y$, with $\mu \in \mathbb{R}$, form a continuous family of planar analytic vector fields with a center at the origin for which the outer boundary does not vary continuously at $\mu = 2$. Indeed, the period annulus $\mathscr{P}_\mu$ is the whole plane for $\mu < 2$, whereas is bounded for $\mu = 2$ (see [40] for details). The notion of criticality as defined in Definition 2.2.2 is meaningless in this situation.

Definition 2.2.3. We say that $\hat{\mu} \in \Lambda$ is a *local regular value of the period function at the outer boundary* of the period annulus if $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$. Otherwise we say that it is a *local bifurcation value of the period function at the outer boundary.* □

The notions that we have introduced so far are general. In the present chapter we consider analytic potential differential systems

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = V_\mu'(x), \end{cases}$$

depending on a parameter $\mu \in \Lambda$, where Λ is an open subset of $\mathbb{R}^d$. Here V_μ is an analytic function on a certain real interval I_μ that contains $x = 0$. In what follows sometimes we shall use the vector field notation $X_\mu := -y\partial_x + V_\mu'(x)\partial_y$ to refer to the above differential system. We suppose $V_\mu'(0) = 0$ and $V_\mu''(0) > 0$, so that the origin is a non-degenerated center and we shall denote the projection of its period annulus $\mathscr{P}_\mu$ on the x-axis by $\mathcal{I}_\mu = (x_\ell(\mu), x_r(\mu))$. Thus $x_\ell(\mu) < 0 < x_r(\mu)$. The corresponding Hamiltonian function is given by $H_\mu(x,y) = \frac{1}{2}y^2 + V_\mu(x)$, where we fix that $V_\mu(0) = 0$, and we set the energy level of the outer boundary of $\mathscr{P}_\mu$ to be $h_0(\mu)$, so that $V_\mu(\mathcal{I}_\mu) = [0, h_0(\mu))$. Note that $h_0(\mu)$ is a positive number or $+\infty$. In addition we define

$$g_\mu(x) := x\sqrt{\frac{V_\mu(x)}{x^2}},$$

which is clearly a diffeomorphism between $\mathcal{I}_\mu$ and $\big(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)}\big)$ since the potential function satisfies $V_\mu(0) = V_\mu'(0) = 0$ and $V_\mu''(0) > 0$. For each $h \in (0, h_0(\mu))$, let γ_h be the periodic orbit inside the energy level $\{\frac{1}{2}y^2 + V_\mu(x) = h\}$ and let us denote by (x_h^-, x_h^+) the projection of γ_h on the x-axis (see Figure 2.2). (Here the dependence of γ_h and $x_h^\pm$ on μ is omitted for shortness.) Taking $\gamma_h \subset \{\frac{1}{2}y^2 + V_\mu(x) = h\}$ into account, we get

$$T_\mu(h) = \int_{\gamma_h} \frac{dx}{y} = \sqrt{2}\int_{x_h^-}^{x_h^+} \frac{dx}{\sqrt{h - V_\mu(x)}}.$$

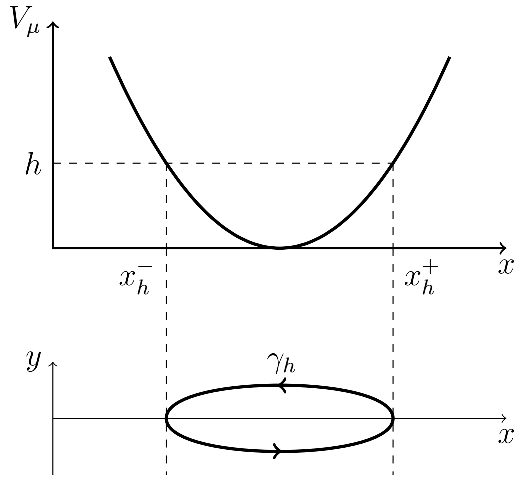

Figure 2.2: Interpretation of the periodic orbit γ_h.

We perform the change of variables given by $x = (g_\mu^{-1})(\sqrt{h}\sin\theta)$, which brings the oval $\gamma_h \subset \{\frac{1}{2}y^2 + V_\mu(x) = h\}$ to the circle of radius $\sqrt{h}$ and yields to

$$T_\mu(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})'(\sqrt{h}\sin\theta)d\theta. \tag{2.1}$$

It is well known that, for each $\mu \in \Lambda$, the function T_μ is an analytic function on $(0, h_0(\mu))$ and can be extended analytically at $h = 0$ due to the non-degeneracy of the center.

Definition 2.2.4. Following the notation introduced just before, we say that the family of potential analytic differential systems $\{X_\mu\}_{\mu\in\Lambda}$ verifies the hypothesis **(H)** in case that:

(a) For all $k \geqslant 0$, the map $(x,\mu) \longmapsto V_\mu^{(k)}(x)$ is continuous on $\{(x,\mu) \in \mathbb{R}\times\Lambda : x \in I_\mu\}$,

(b) $\mu \longmapsto x_r(\mu)$ is continuous on Λ or $x_r(\mu) = +\infty$ for all $\mu \in \Lambda$,

(c) $\mu \longmapsto x_\ell(\mu)$ is continuous on Λ or $x_\ell(\mu) = -\infty$ for all $\mu \in \Lambda$,

(d) $\mu \longmapsto h_0(\mu)$ is continuous on Λ or $h_0(\mu) = +\infty$ for all $\mu \in \Lambda$.

□

Remark 2.2.5. Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic differential system verifying **(H)**. Then the outer boundary of its period annulus varies continuously in the sense of Definition 2.2.2. Indeed, to show this let $\gamma_{h,\mu}$ be the periodic orbits of X_μ inside the energy level $\{\frac{1}{2}y^2 + V_\mu(x) = h\}$. Then

$$d_H(\Pi_\mu, \Pi_{\hat\mu}) \leqslant d_H(\gamma_{h,\hat\mu}, \Pi_{\hat\mu}) + 2d_H(\gamma_{h,\hat\mu}, \gamma_{h,\mu}) + d_H(\gamma_{h,\mu}, \Pi_\mu),$$

which tends to zero as $h \to h_0(\hat\mu)$ and $\mu \to \hat\mu$ thanks to the hypothesis (a) and (d) in **(H)** and the continuity of solutions of ordinary differential equations with respect to parameters. □

In this chapter we give sufficient conditions for families of potential systems satisfying hypothesis **(H)** in order to bound the criticality at the outer boundary of the period annulus. As we previously commented, we shall dedicate each of the following sections to the two classes of potential systems according to property (d) in Definition 2.2.4. We finish this section by proving two results that do not depend on whether the energy level at the outer boundary is finite or infinite. Before that, we introduce the Invariance Domain Theorem, which was proved by L.E.J. Brouwer in 1912 as a corollary of the Brouwer Fixed Point Theorem (see for instance [5]).

Theorem 2.2.6 (Invariance Domain Theorem)**.** *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a continuous and injective function. Then $f(U)$ is open in $\mathbb{R}^n$ and $f^{-1} : f(U) \to U$ is continuous.*

Lemma 2.2.7. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic differential systems verifying hypothesis* **(H)**. *Then the map $(z, \mu) \longmapsto g_\mu^{-1}(z)$ is continuous on the open set $\Big\{(z, \mu) \in \mathbb{R} \times \Lambda : z \in \big(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)}\big)\Big\}$.*

Proof. By the hypothesis in **(H)**, $\Omega := \{(x, \mu) \in \mathbb{R} \times \Lambda : x \in \mathcal{I}_\mu\}$ is an open subset of $\mathbb{R}^{d+1}$ and the map $G : \Omega \longrightarrow \mathbb{R}^{d+1}$ given by $G(x, \mu) = (g_\mu(x), \mu)$ is continuous. It is also injective because, for each fixed $\mu \in \Lambda$, g_μ is a diffeomorphism from $\big(x_\ell(\mu), x_r(\mu)\big)$ to $\big(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)}\big)$. Then the result follows by the Invariance Domain Theorem. ∎

Lemma 2.2.8. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic differential systems verifying hypothesis* **(H)**. *Then*

$$\lim_{z\to-\sqrt{h_0(\mu)}} g_\mu^{-1}(z) = x_\ell(\mu) \text{ and } \lim_{z\to\sqrt{h_0(\mu)}} g_\mu^{-1}(z) = x_r(\mu)$$

uniformly in compacts of Λ. Moreover, if h_0, x_ℓ and x_r are finite at $\mu = \hat{\mu}$, then the map $(z, \mu) \longmapsto g_\mu^{-1}(z)$ extends continuously to $(-\sqrt{h_0(\hat{\mu})}, \hat{\mu})$ and $(\sqrt{h_0(\hat{\mu})}, \hat{\mu})$ for all $\mu \in \Lambda$.

Proof. Let us prove the first assertion of the lemma. Consider a given compact subset K of Λ. Let us prove for instance that $\lim_{z\to\sqrt{h_0(\mu)}} g_\mu^{-1}(z) = x_r(\mu)$ uniformly on K. We consider the case when $h_0(\mu) = \infty$ and $x_r(\mu) < \infty$. Set $\delta := \min\{x_r(\mu) : \mu \in K\}$. Then for any $0 < \varepsilon < \delta$ define

$$A_\varepsilon := \max\big\{g_\mu\big(x_r(\mu) - \varepsilon\big) : \mu \in K\big\},$$

which is well defined because K is compact and $\mu \longmapsto g_\mu\big(x_r(\mu) - \varepsilon\big)$ is continuous. Thus $g_\mu(x_r(\mu) - \varepsilon) < z$ for all $z > A_\varepsilon$ and $\mu \in K$, which implies $0 < x_r(\mu) - g_\mu^{-1}(z) < \varepsilon$. This ends the proof in this case. The other cases follows in a similar way with the obvious modifications. Finally the continuity of $(z, \mu) \mapsto g_\mu^{-1}(z)$ follows from the first assertion of the lemma together with Lemma 1.3.6 and the continuity of h_0. ∎

Next two sections are concerned with the criticality at the outer boundary of potential systems verifying the hypothesis **(H)**. Section 2.3 is devoted to prove Theorem C, that deals with the case $h_0 \equiv +\infty$, whereas in Section 2.4 we prove Theorems D and E that tackle the case in which h_0 is finite.

2.3 Bounding the criticality of potential centers with infinite energy level

In this section we shall study the criticality at the outer boundary of the period annulus for families of potential systems such that $h_0(\mu) = +\infty$ for all $\mu \in \Lambda$. The main result of this section is Theorem C, which gives, for a given $n \in \mathbb{N}$, sufficient conditions in order to ensure that $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant n$. That is, the derivative of the period function has no more than n isolated zeroes near $h = \infty$. With this aim in view, the idea will be to take a non-vanishing function f and find sufficient conditions in order that fT'_μ can be embedded into the simples ECT-system we can consider, namely $(h^{\nu_1(\mu)}, h^{\nu_2(\mu)}, \dots, h^{\nu_n(\mu)})$. We precise this in the following result.

Lemma 2.3.1. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic differential systems such that $h_0 \equiv +\infty$. Assume that there exist $n \geqslant 1$ continuous functions $\nu_1, \nu_2 \dots, \nu_n$ in a neighbourhood of some fixed $\hat{\mu} \in \Lambda$ and an analytic non-vanishing function f on $(0, +\infty)$ such that*

$$\lim_{h\to+\infty} h^{\nu_n(\mu)} W[h^{\nu_1(\mu)}, \dots, h^{\nu_{n-1}(\mu)}, f(h)T'_\mu(h)] = \Delta(\mu),$$

uniformly in $\mu \approx \hat{\mu}$, and $\Delta(\hat{\mu}) \neq 0$. Then $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant n - 1$.

Proof. Note that $\nu_1, \nu_2 \dots, \nu_{n-1}$ must be pairwise distinct at $\mu = \hat{\mu}$ because $\Delta(\hat{\mu}) \neq 0$. Thus, by continuity, for each $k = 1, 2, \dots, n-1$ we have that $W[h^{\nu_1(\mu)}, \dots, h^{\nu_k(\mu)}] \neq 0$ for all $h > 0$ and $\mu \approx \hat{\mu}$. On the other hand, by the uniformity of the limit as h tends to $+\infty$ and the assumption $\Delta(\hat{\mu}) \neq 0$, there exist $M > 0$ and a neighbourhood U of $\hat{\mu}$ such that

$$W[h^{\nu_1(\mu)}, \dots, h^{\nu_{n-1}(\mu)}, f(h)T'_\mu(h)] \neq 0 \text{ for } h \in (M, +\infty) \text{ and } \mu \in U.$$

Accordingly, by applying Lemma 1.4.3 we can assert that $(h^{\nu_1(\mu)}, \dots, h^{\nu_{n-1}(\mu)}, f(h)T'_\mu(h))$ is an ECT-system on $(M, +\infty)$ for all $\mu \in U$. In particular, since f is a unity, T'_μ has no more than $n-1$ isolated zeros on $(M, +\infty)$ for $\mu \approx \hat{\mu}$, counted with multiplicities. We claim that this implies $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant n-1$, see Definition 2.2.2. To show this notice first that, by Remark 2.2.5, the outer boundary of the period annulus varies continuously. Suppose, by contradiction, that there exist n sequences $\{\gamma^k_{\mu_i}\}_{i\in\mathbb{N}}$, $k = 1, 2, \dots, n$, where each $\gamma_{\mu_i}^k$ is a critical periodic orbit of X_{μ_i}, such that $\mu_i \to \hat{\mu}$ and $d_H(\gamma_{\mu_i}^k, \Pi_{\hat{\mu}}) \to 0$ as $i \to +\infty$. Then, due to

$$d_H(\gamma_{\mu_i}^k, \Pi_{\mu_i}) \leqslant d_H(\gamma_{\mu_i}^k, \Pi_{\hat{\mu}}) + d_H(\Pi_\mu, \Pi_{\hat{\mu}}),$$

we have that $d_H(\gamma_{\mu_i}^k, \Pi_{\mu_i})$ tends to zero as $i \to +\infty$. This contradicts that, for all $\mu \in U$, T'_μ has no more than $n-1$ isolated zeroes on $(M, +\infty)$. So the claim is true and the results follows. ■

Next theorem is a non-parametric result and so the dependence on μ is omitted for the sake of shortness. It is concerned with the limit of the period function and its derivative as h tends to infinity.

Theorem 2.3.2. *Let X be an analytic potential differential system with $h_0 = +\infty$ and such that $(g^{-1})''$ is monotonous near the endpoints of the interval $(-\infty, +\infty)$. Then the following statements hold:*

(*i*) *The limits $\lim_{x\to-\infty}(g^{-1})'(x) =: a_\ell$ and $\lim_{x\to+\infty}(g^{-1})'(x) =: a_r$ exist and both $a_\ell, a_r \in [0, +\infty]$. Moreover $T(h)$ tends to $(a_\ell + a_r)\frac{\pi}{\sqrt{2}}$ as $h \to +\infty$.*

(*ii*) *The limits $\lim_{x\to-\infty}(g^{-1})''(x) =: b_\ell$ and $\lim_{x\to+\infty}(g^{-1})''(x) =: b_r$ exist. Moreover $\sqrt{h}T'(h)$ tends to $(b_\ell + b_r)\frac{\sqrt{2}}{2}$ as $h \to +\infty$ except for the cases $\{b_\ell = +\infty, b_r = -\infty\}$ and $\{b_\ell = -\infty, b_r = +\infty\}$.*

Proof. For the sake of brevity we only prove (*i*) since (*ii*) follows similarly. From the expression for the period function in (2.1) we get $T(s^2) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta$. The monotonicity of $(g^{-1})''$ near the endpoints of $(-\infty, +\infty)$ implies the same property for $(g^{-1})'$. Therefore a_ℓ (respectively, a_r) either exists or it is infinity. In addition, due to $g' > 0$, we have $a_\ell, a_r \in [0, +\infty]$. We claim that $\lim_{s\to\infty}\sqrt{2}\int_0^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta = a_r\frac{\pi}{\sqrt{2}}$. Let us consider first the case $a_r < +\infty$. Due to $\lim_{x\to+\infty}(g^{-1})'(x) = a_r < +\infty$ there exists $M > a_r$ such that $(g^{-1})'(x) < M$ for all $x \geqslant 0$. Given $\varepsilon > 0$, define $\varepsilon' = \varepsilon/\sqrt{2}$ and let $\bar{x} > 0$ be such that $|(g^{-1})'(x) - a_r| < \frac{\varepsilon'}{\pi}$ for all $x > \bar{x}$. Finally, let s_0 be such that $s_0 \sin\left(\frac{\varepsilon'}{4M}\right) > \bar{x}$. Then if $s > s_0$ we have

$$\begin{aligned}\left|\sqrt{2}\int_0^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta - \frac{a_r\pi}{\sqrt{2}}\right| \leqslant & \left|\sqrt{2}\int_0^{\frac{\varepsilon'}{4M}}\left((g^{-1})'(s\sin\theta) - a_r\right)d\theta\right| \\ & + \left|\sqrt{2}\int_{\frac{\varepsilon'}{4M}}^{\frac{\pi}{2}}\left((g^{-1})'(s\sin\theta) - a_r\right)d\theta\right| \\ \leqslant & \sqrt{2}\left(2M\frac{\varepsilon'}{4M} + \frac{\varepsilon'}{\pi}\frac{\pi}{2}\right) = \sqrt{2}\varepsilon' = \varepsilon.\end{aligned}$$

Let us consider now the case $a_r = +\infty$. Given any $K > 0$, let $\bar{x} > 0$ be such that $(g^{-1})'(x) > K$ for all $x > \bar{x}$. As before, let s_0 be such that $s_0 \sin\left(\frac{\pi}{4}\right) > \bar{x}$. Then, if $s > s_0$ we get that

$$\sqrt{2}\int_0^{\frac{\pi}{2}} (g^{-1})'(s\sin\theta)d\theta \geqslant \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (g^{-1})'(s\sin\theta)d\theta \geqslant K\sqrt{2}\frac{\pi}{4} > K.$$

Thus $\lim_{s\to\infty}\sqrt{2}\int_0^{\frac{\pi}{2}}(g^{-1})'(s\sin\theta)d\theta = +\infty$. Exactly the same way can be proved that

$$\lim_{s\to\infty}\sqrt{2}\int_{-\frac{\pi}{2}}^{0} (g^{-1})'(s\sin\theta)d\theta = a_\ell\frac{\pi}{\sqrt{2}},$$

so the result follows. ■

Lemma 2.3.3. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic systems verifying* **(H)** *and such that $h_0 \equiv +\infty$. Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions which is continuously quantifiable in Λ at $x = x_r(\mu)$ (respectively, $x = x_\ell(\mu)$) by $\alpha(\mu)$ with limit $a(\mu)$. Assume moreover that $\{V_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $x = x_r(\mu)$ (respectively, $x = x_\ell(\mu)$) by $\beta(\mu)$ with limit $b(\mu)$. Then, $\{f_\mu \circ g_\mu^{-1}\}_{\mu\in\Lambda}$ is continuously quantifiable at $+\infty$ (respectively, $-\infty$) by $2(\alpha/\beta)(\mu)$ with limit $\left(ab^{-\alpha/\beta}\right)(\mu)$.*

Proof. We only prove the result for the right hand side of the interval, $x_r(\mu)$. The other case follows similarly. Let $\hat{\mu} \in \Lambda$. Since the family $\{X_\mu\}_{\mu\in\Lambda}$ satisfies hypothesis **(H)**, by Lemma 2.2.8 we have that $\lim_{z\to+\infty} g_\mu^{-1}(z) = x_r(\mu)$ uniformly on a neighbourhood of $\hat{\mu}$. Equivalently, by Lemma 1.3.6 we have that

$$\lim_{(z,\mu)\to(+\infty,\hat{\mu})} g_\mu^{-1}(z) = x_r(\hat{\mu}).$$

Therefore, for any $\hat{\mu} \in \Lambda$, taking into account that $g_\mu^2 = V_\mu$,

$$\begin{aligned}\lim_{(z,\mu)\to(+\infty,\hat{\mu})} \frac{(f_\mu \circ g_\mu^{-1})(z)}{z^{2\frac{\alpha(\mu)}{\beta(\mu)}}} &= \lim_{(x,\mu)\to(x_r(\hat{\mu}),\hat{\mu})} \frac{f_\mu(x)}{V_\mu(x)^{\frac{\alpha(\mu)}{\beta(\mu)}}} \\ &= \lim_{(x,\mu)\to(x_r(\hat{\mu}),\hat{\mu})} \frac{f_\mu(x)(x_r(\mu)-x)^{\alpha(\mu)}}{\left(V_\mu(x)(x_r(\mu)-x)^{\beta(\mu)}\right)^{\frac{\alpha(\mu)}{\beta(\mu)}}} = ab^{-\alpha/\beta}(\mu) \neq 0\end{aligned}$$

as we desired. ■

We can now state our result concerning the criticality at the outer boundary for the case $h_0 \equiv +\infty$. In its statement, and from now on, for a given function $f\colon(-a,a) \longrightarrow \mathbb{R}$, we denote $\mathcal{P}[f](x) := f(x) + f(-x)$. Let us also remark that the assumption requiring the existence of functions $\nu_1, \nu_2, \ldots, \nu_n$ is void in case that $n = 0$.

Theorem C. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic systems verifying* **(H)** *and such that $h_0\equiv+\infty$. Assume that there exist $n\geqslant 0$ continuous functions $\nu_1,\nu_2,\ldots,\nu_n$ in a neighbourhood of some fixed $\hat\mu\in\Lambda$ such that the family $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathcal{P})[z(g_\mu^{-1})''(z)]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $+\infty$ by $\xi(\mu)$ with limit $c(\mu)$. For each $i\in\mathbb{N}$, let $M_i(\mu)$ be the i-th momentum of $(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathcal{P})[z(g_\mu^{-1})''(z)]$, whenever it is well defined. The following assertions hold:*

(a) If $\xi(\hat\mu)>-1$, then $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\leqslant n$.

(b) If $\xi(\hat\mu)<-1$, let $m\in\mathbb{N}$ be such that $\xi(\hat\mu)+2m\in[-1,1)$.

Then $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\leqslant n$ in case that

(b1) either $M_1\equiv M_2\equiv\ldots\equiv M_{j-1}\equiv 0$ and $M_j(\hat\mu)\neq 0$ for some $j\in\{1,2,\ldots,m\}$,

(b2) or $M_1\equiv M_2\equiv\ldots\equiv M_m\equiv 0$ and $\xi(\hat\mu)+2m\notin\{-1,0\}$.

Finally, if the following conditions are verified, then $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathcal{P})[z(g_\mu^{-1})''(z)]\big\}_{\mu\in\Lambda}$ is continuously quantifiable at $+\infty$ by $\xi(\mu)=2\max\big\{\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu),\big(\frac{\alpha_r}{\beta_r}\big)(\mu)\big\}+(n+1)^2-\sum_{i=1}^n\nu_i(\mu)$:

(i) $\{V_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at $x_\ell(\mu)$ by $\beta_\ell(\mu)$ and at $x_r(\mu)$ by $\beta_r(\mu)$ with limits $b_\ell(\mu)$ and $b_r(\mu)$, respectively,

(ii) setting $\mathscr{R}_\mu:=\frac{(V_\mu')^2-2V_\mu V_\mu''}{(V_\mu')^3}$, the function

$$x\longmapsto V_\mu'(x)^{-\frac{n(n+1)}{2}}W\big[V_\mu^{\frac{\nu_1(\mu)-1}{2}},\ldots,V_\mu^{\frac{\nu_n(\mu)-1}{2}},\mathscr{R}_\mu\big](x)$$

is continuously quantifiable at $x_\ell(\mu)$ by $\alpha_\ell(\mu)$ and at $x_r(\mu)$ by $\alpha_r(\mu)$ with limits $a_\ell(\mu)$ and $a_r(\mu)$, respectively,

(iii) and either $\frac{\alpha_\ell}{\beta_\ell}(\mu)\neq\frac{\alpha_r}{\beta_r}(\mu)$ or, otherwise, $\big(a_\ell(b_r)^{\frac{\alpha_r}{\beta_r}}-(-1)^{\frac{n(n+1)}{2}}a_r(b_\ell)^{\frac{\alpha_\ell}{\beta_\ell}}\big)(\mu)\neq 0$.

Proof. Denote $f_\mu(z):=\mathcal{P}[z\big(g_\mu^{-1}\big)''(z)]$ for shortness. Then Lemma 2.2.7 and the hypothesis **(H)** guarantee that $\{f_\mu\}_{\mu\in\Lambda}$ is a continuous family of analytic functions on $[0,+\infty)$. From (2.1) it follows that

$$T_\mu'(h^2)=\frac{1}{\sqrt{2}h^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}z\big(g_\mu^{-1}\big)''(z)\big|_{z=h\sin\theta}d\theta=\frac{1}{\sqrt{2}h^2}\int_0^{\frac{\pi}{2}}f_\mu(h\sin\theta)\,d\theta=\frac{1}{\sqrt{2}h^2}\mathscr{F}[f_\mu](h),$$

so it suffices to prove that there exist $M,\varepsilon>0$ such that $\mathscr{F}[f_\mu](h)$ has at most n isolated zeroes counted with multiplicities for $h>M$ and $\|\mu-\hat\mu\|<\varepsilon$.

Since $\xi(\mu)$ is the quantifier of $\{\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\}_{\mu\in\Lambda}$ in $\hat\mu$ at $+\infty$, by applying Theorem B we can assert that $\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F})[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat\mu$ at $+\infty$ by

$\nu_{n+1}(\mu) := \xi(\mu)$, in cases (a) and $(b2)$, and by $\nu_{n+1}(\mu) := 1-2j$, in case $(b1)$. Then, taking account of the definition of $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}$, see Definition 1.4.4, in these cases we get that

$$\lim_{(h,\mu)\to(+\infty,\hat{\mu})} h^{-\nu_{n+1}(\mu)} \frac{W\left[h^{\nu_1(\mu)}, \ldots, h^{\nu_n(\mu)}, \mathscr{F}[f_\mu](h)\right]}{h^{\sum_{i=1}^{n}(\nu_i(\mu)-i)}} = \ell \neq 0, \tag{2.2}$$

where

$$\ell := \begin{cases} c(\hat{\mu})\mathscr{G}(\xi(\hat{\mu})) & \text{in case } (a), \\ \prod_{i=1}^{j-1}(1-\frac{1}{2i})M_j(\hat{\mu}) & \text{in case } (b1), \\ c(\hat{\mu})\prod_{i=1}^{m}\frac{\xi(\hat{\mu})+2i}{\xi(\hat{\mu})+2i-1}\mathscr{G}(\xi(\hat{\mu})+2m) & \text{in case } (b2). \end{cases}$$

Thus, since $\mathscr{F}[f_\mu](h) = \sqrt{2}h^2T'_\mu(h^2)$, by Lemma 2.3.1 we have $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant n$, as desired. This proves the first part of the result.

Let us turn now to the proof of second part of the result. With this aim in view we note that if ϕ is any analytic function on $(-a,a)$, then

$$\mathscr{L}_{\boldsymbol{\nu}_n}[\mathcal{P}\circ\phi](x) = \frac{W[x^{\nu_1}, x^{\nu_2}, \ldots, x^{\nu_n}, \phi(x)]}{x^{\sum_{i=1}^{n}(\nu_i-i)}} + \frac{W[x^{\nu_1}, x^{\nu_2}, \ldots, x^{\nu_n}, \phi(-x)]}{x^{\sum_{i=1}^{n}(\nu_i-i)}} \tag{2.3}$$

for all $x \in (0,a)$. Let us set $\Delta(\mu) := \sum_{i=1}^{n}(\nu_i(\mu)-i)$ for shortness. Since one can verify that $\big(g^{-1}(z)\big)'' = 2\mathscr{R}\big(g^{-1}(z)\big)$ with $\mathscr{R} = \frac{(V')^2-2VV''}{(V')^3}$ and $V\big(g^{-1}(z)\big) = z^2$, by applying Lemma 1.4.5 some computations show that

$$\frac{W[z^{\nu_1(\mu)}, z^{\nu_2(\mu)}, \ldots, z^{\nu_n(\mu)}, z(g_\mu^{-1})''(z)]}{z^{\Delta(\mu)}} = 2^{1+\frac{n(n+1)}{2}} S_\mu\big(g_\mu^{-1}(z)\big), \text{ for all } z > 0,$$

where

$$S_\mu(x) := \frac{W\left[V_\mu^{\frac{\nu_1(\mu)-1}{2}}, \ldots, V_\mu^{\frac{\nu_n(\mu)-1}{2}}, \mathscr{R}_\mu\right](x)}{V'_\mu(x)^{\frac{n(n+1)}{2}} V_\mu(x)^{\frac{\Delta(\mu)}{2}-\frac{(n+1)(n+2)}{4}}}.$$

Similarly, due to $V(g^{-1}(-z)) = z^2$,

$$\frac{W[z^{\nu_1}, z^{\nu_2}, \ldots, z^{\nu_n}, -z(g_\mu^{-1})''(-z)]}{z^{\Delta(\mu)}} = (-2)^{1+\frac{n(n+1)}{2}} S_\mu\big(g_\mu^{-1}(-z)\big), \text{ for all } z > 0.$$

Accordingly, taking (2.3) with $\phi(z) = z\big(g_\mu^{-1}\big)''(z)$, it turns out that the quantifiers of $S_\mu \circ g_\mu^{-1}$ at $+\infty$ and at $-\infty$ will "generically" determine the quantifier of $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]$ at $+\infty$.

Henceforth, for the sake of shortness, we omit the unessential dependence with respect to μ. On account of (i) and (ii) it follows that $\{S_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at x_ℓ by $\alpha_\ell - \beta_\ell\left(\frac{\Delta}{2} - \frac{(n+1)(n+2)}{4}\right)$ and at x_r by $\alpha_r - \beta_r\left(\frac{\Delta}{2} - \frac{(n+1)(n+2)}{4}\right)$, with limits $a_\ell b_\ell^{\frac{(n+1)(n+2)}{4}-\frac{\Delta}{2}}$ and $a_r b_r^{\frac{(n+1)(n+2)}{4}-\frac{\Delta}{2}}$, respectively. Then, by applying Lemma 2.3.3 and using (i) again, the family $\{S_\mu \circ g_\mu^{-1}\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $-\infty$ by

$2\frac{\alpha_\ell}{\beta_\ell}-\Delta+\frac{(n+1)(n+2)}{2}$ and at $+\infty$ by $2\frac{\alpha_r}{\beta_r}-\Delta+\frac{(n+1)(n+2)}{2}$, with limits $a_\ell(b_\ell)^{-\frac{\alpha_\ell}{\beta_\ell}}$ and $a_r(b_r)^{-\frac{\alpha_r}{\beta_r}}$, respectively. Finally, again from (2.3) with $\phi(z)=z\big(g_\mu^{-1}\big)''(z)$, the assumption (iii) gurarantees that $\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathcal{P})[z(g_\mu^{-1})''(z)]\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $+\infty$ by $\xi=2\max\big\{\big(\frac{\alpha_\ell}{\beta_\ell}\big),\big(\frac{\alpha_r}{\beta_r}\big)\big\}-\Delta+\frac{(n+1)(n+2)}{2}$. This completes the proof of the result because one can easily verify that $\frac{(n+1)(n+2)}{2}-\Delta=(n+1)^2-\sum_{i=1}^n\nu_i$. ∎

Remark 2.3.4. In the case $n=0$, from the expression in (2.2) and the expression of ν and ℓ in the proof of Theorem C we have that $T_\mu'(h)=h^{\alpha_1(\mu)}\big(\Delta_1(\mu)+f_1(h;\mu)\big)$, with $f_1(h;\mu)$ tending to zero as $(h,\mu)\to(+\infty,\hat\mu)$, where

$$\begin{cases}\alpha_1(\mu)=\frac{1}{2}\nu(\mu)-1 \quad\text{and}\quad \Delta_1(\mu)=c(\mu)\mathscr{G}(\xi(\mu)), & \text{in case }(a),\\ \alpha_1(\mu)=-\frac{1}{2}-j \quad\text{and}\quad \Delta_1(\mu)=\prod_{i=1}^{j-1}\big(1-\frac{1}{2i}\big)M_j[\mathcal{P}[z(g_\mu^{-1})''(z)]], & \text{in case }(b1),\\ \alpha_1(\mu)=\frac{1}{2}\nu(\mu)-1 \quad\text{and}\quad \Delta_1(\mu)=c(\mu)\prod_{i=1}^{m}\frac{\xi(\mu)+2i}{\xi(\mu)+2i-1}\mathscr{G}(\xi(\mu)+2m), & \text{in case }(b2).\end{cases}$$

In other words, it gives the quantifier of T_μ' when $\Delta_1(\hat\mu)\neq0$. Moreover, this limit may be useful in order to proof that a fixed parameter is a local bifurcation value at the outer boundary of the period annulus by looking to change of sign of the derivative of the period function. Next result is concerned with this fact and gives a tool in order to prove that a parameter is a local bifurcation value. □

Lemma 2.3.5. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of analytic potential systems such that $h_0\equiv+\infty$ and fix $\hat\mu\in\Lambda$. Suppose that for all $\mu\in\Lambda$ there exist $\alpha_1(\mu)$ and $\Delta_1(\mu)$ such that*

$$\lim_{h\to+\infty}h^{-\alpha_1(\mu)}T_\mu'(h)=\Delta_1(\mu),$$

and that there exist two sequences $\{\mu_n^\pm\}_{n\in\mathbb{N}}$ with $\mu_n^\pm\to\hat\mu$ such that $\Delta_1(\mu_n^+)\Delta_1(\mu_n^-)<0$ for all $n\in\mathbb{N}$. Then $\mathrm{Crit}((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu)\geqslant1$.

Proof. The assumptions on the statement implies that, for all $\delta>0$ and $\bar h>0$, there exist $\mu^\pm\in\Lambda$ and $h^\star>0$ with $\|\mu^\pm-\hat\mu\|<\delta$ and $h^\star>\bar h$ satisfying $T_{\mu^+}'(h)T_{\mu^-}'(h)<0$ for all $h>h^\star$. Then, on account of the continuity of $\mu\mapsto T_\mu'(h^\star)$, there exists $\mu^\star$ in the segment that joins μ^+ and μ^- such that $T_{\mu^\star}'(h^\star)=0$. This shows that $\mathrm{Crit}((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu)\geqslant1$ as we desired. ∎

2.4 Bounding the criticality of potential centers with finite energy level

In this section we shall study the criticality at the outer boundary of the period annulus for families of potential systems such that $h_0(\mu)$ is finite for all $\mu\in\Lambda$.

This section is divided in two parts that can be read independently. The first one, devoted to prove Theorem D, deals with the study of the criticality for analytic potential systems that we call *admissible*. For such systems, Theorem D gives sufficient conditions for a parameter to be a local regular value of the period function at the outer boundary of the period annulus and also to show that at most one critical periodic orbit bifurcates from the outer boundary. The technique used in this first part is the study of the improper integral that gives the expression of the derivative of the period function. On the other hand, the second part is dedicated to prove Theorem E, which is concerned with the upper-bound of the criticality at the outer boundary for systems satisfying hypothesis **(H)**. Here the idea is to proceed similarly as we did for potential systems with infinite energy level. That is, to embed T'_μ into the simplest ECT-system we can consider.

2.4.1 First approach

In this section we shall study the bifurcation of critical periodic orbits in a family of potential systems for which the energy level $h_0(\mu)$ is finite for all $\mu \in \Lambda$.

Lemma 2.4.1. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of analytic potential systems satisfying hypothesis* **(H)** *such that $h_0(\mu)$ is finite and fix $\hat{\mu} \in \Lambda$. Then the following hold:*

(a) Suppose that for all $\mu \in \Lambda$ there exists $\Delta_1(\mu)$ such that

$$\lim_{h\to h_0(\mu)} T'_\mu(h) = \Delta_1(\mu).$$

If there exist two sequences $\{\mu_n^\pm\}_{n\in\mathbb{N}}$ with $\mu_n^\pm \longrightarrow \hat{\mu}$ such that $\Delta_1(\mu_n^+)\Delta_1(\mu_n^-) < 0$ for all $n \in \mathbb{N}$, then $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \geqslant 1$. If the above limit is uniform on Λ and $\Delta_1(\hat{\mu}) \neq 0$, then $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$.

(b) If $\lim_{h\to h_0(\mu)} T'_\mu(h) = \infty$ uniformly on Λ, then $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$ for all $\hat{\mu} \in \Lambda$.

(c) If there exist $\Delta_2(\mu)$ such that the limit

$$\lim_{h\to h_0(\mu)} T''_\mu(h) = \Delta_2(\mu)$$

uniformly on Λ and $\Delta_2(\hat{\mu}) \neq 0$ then $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$.

(d) If $\lim_{h\to h_0(\mu)} T''_\mu(h) = \infty$ uniformly on Λ, then $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$.

Proof. Since $\{X_\mu\}_{\mu\in\Lambda}$ satisfies hypothesis **(H)** and $h_0(\mu)$ is finite, we have that $h_0(\mu)$ is a continuous function and, by a continuous rescaling of variables, we can assume $h_0(\mu) = 1$ for all $\mu \in \Lambda$.

Let us prove the first assertion in (a). The assumption in the statement implies that, for all neighbourhood U of $\hat{\mu}$ and $\bar{h} \in (0,1)$, there exist $\mu^{\pm} \in U$ and $h^{\star} \in (\bar{h},1)$ satisfying $T'_{\mu^+}(h)T'_{\mu^-}(h) < 0$ for all $h \in (h^{\star},1)$. Then, on account of the continuity of $\mu \mapsto T'_{\mu}(h^{\star})$, there exists $\mu^{\star}$ in the segment that joins μ^+ and μ^- such that $T'_{\mu^{\star}}(h^{\star}) = 0$. This shows that $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_{\mu}\big) \geqslant 1$.

Let us turn to the second assertion in (a). Clearly $\{T'_{\mu}\}_{\mu\in\Lambda}$ is a continuous family of continuous functions. On account of Lemma 1.3.6 and since $\Delta_1(\hat{\mu}) \neq 0$ we have that $\lim_{(h,\mu)\to(1,\hat{\mu})} T'_{\mu}(h) \neq 0$. Accordingly there exist a neighbourhood U of $\hat{\mu}$ and $h^{\star} \in (0,1)$ such that $T'_{\mu}(h) \neq 0$ for all $h \in (h^{\star},1)$ and $\mu \in U$. This shows that $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_{\mu}\big) = 0$ and completes the proof of (a).

Let us turn now on the proof of (b). By Lemma 1.3.6, since $\lim_{h\to h_0(\mu)} T'_{\mu}(h) = \infty$ uniformly on Λ, we have that $\lim_{(h,\mu)\to(1,\hat{\mu})} T'_{\mu}(h) \neq 0$. Then the same argument as before proves the result in this case.

Finally let us turn on the proof of (c) and (d). By Lemma 1.3.6 we have that, in both cases, $\lim_{(h,\mu)\to(1,\hat{\mu})} T''_{\mu}(h) \neq 0$. Accordingly there exist a neighbourhood U of $\hat{\mu}$ and $h^{\star} \in (0,1)$ such that $T''_{\mu}(h) \neq 0$ for all $h \in (h^{\star},1)$ and $\mu \in U$. Then $T'_{\mu}(h)$ is monotone for all $h \in (h^{\star},1)$ and $\mu \in U$. Consequently, at most one critical periodic orbit can bifurcate from the outer boundary of the period annulus. ∎

Definition 2.4.2. Let $X = -y\partial_x + V'(x)\partial_y$ be an analytic potential system with a non-degenerated center at the origin and let (x_ℓ, x_r) be the projection on the x-axis of its period annulus. We say that x_ℓ (respectively, x_r) is *regular* if V is analytic at x_ℓ (respectively, x_r) and $V'(x_\ell) \neq 0$ (respectively, $V'(x_r) \neq 0$). Otherwise we say that the endpoint is *non-regular*. Moreover, we say that the potential system is *admissible* if it verifies one of the following conditions:

(a) either x_ℓ or x_r is regular.

(b) $\lim_{x\to x_\ell} V'(x) = \lim_{x\to x_r} V'(x) = 0.$ □

We point out that x_ℓ and x_r cannot be regular simultaneously, otherwise the projection of the period annulus is larger than the interval (x_ℓ, x_r). From now on, when we say admissible we particularly consider, without loss of generality, that x_r is non-regular. Figures 2.3 and 2.4 display the graph of V for all the possible cases giving rise to an admissible potential system under this assumption.

Lemma 2.4.3. *Suppose that $X = -y\partial_x + V'(x)\partial_y$ is an admissible analytic potential system with two non-regular endpoints. The following statements hold:*

(a) *If $(g^{-1})''$ is monotonous near the endpoints of $(-\sqrt{h_0}, \sqrt{h_0})$, then $(g^{-1})''(z)$ tends to $+\infty$ (respectively, $-\infty$) as $z \nearrow \sqrt{h_0}$ (respectively, $z \searrow -\sqrt{h_0}$).*

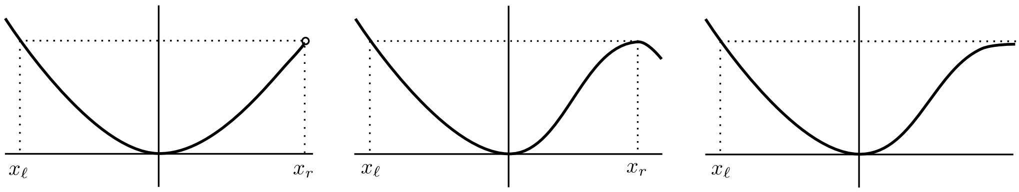

Figure 2.3: Graph of V for admissible potential systems with finite energy and only one non-regular endpoint, cf. (a) in Definition 2.4.2.

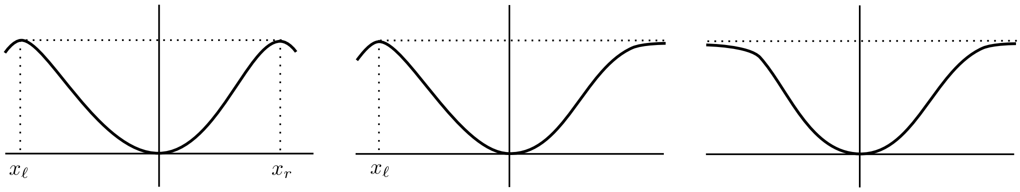

Figure 2.4: Graph of V for admissible potential systems with finite energy and two non-regular endpoints, cf. (b) in Definition 2.4.2.

(b) *If both* $(g^{-1})''$ *and* $(g^{-1})'''$ *are monotonous near the endpoints of* $(-\sqrt{h_0}, \sqrt{h_0})$*, then* $(g^{-1})'''(z)$ *tends to* $+\infty$ *as* $z \to \pm\sqrt{h_0}$*.*

Proof. Let us prove (a). Since the vector field is admissible with two non-regular endpoints, we have that $\lim_{x\to x_\ell} V'(x) = \lim_{x\to x_r} V'(x) = 0$. Since $g(x) = \operatorname{sgn}(x)\sqrt{V(x)}$, this implies that $\lim_{z\to\pm\sqrt{h_0}}(g^{-1})'(z) = +\infty$. Then, due to the fact that the interval $(-\sqrt{h_0}, \sqrt{h_0})$ is bounded, there exist two sequences $a_n \nearrow \sqrt{h_0}$ and $b_n \searrow -\sqrt{h_0}$ such that $(g^{-1})''(a_n)$ and $(g^{-1})''(b_n)$ tend, respectively, to $+\infty$ and $-\infty$ as $n \longrightarrow \infty$. Now the result follows on account of the monotonicity of $(g^{-1})''$ near the endpoints of the interval $(-\sqrt{h_0}, \sqrt{h_0})$. This proves (a). The statement in (b) is proved similarly using the result in (a). ∎

Proposition 2.4.4. *Let* $f : [0, \sigma) \longrightarrow \mathbb{R}$ *be a continuous function that is monotonous near* $x = \sigma$*. Then, for any* $n \in \mathbb{N}$*,*

$$\lim_{s\to 1^-} \int_0^{\frac{\pi}{2}} f(s\sigma \sin\theta) \sin^n \theta d\theta = \int_0^{\frac{\pi}{2}} f(\sigma \sin\theta) \sin^n \theta d\theta,$$

where the improper integral on the right either converges or it tends to infinity.

Proof. Let us prove first the result in case that $L := \int_0^{\frac{\pi}{2}} f(\sigma\sin\theta)\sin^n\theta d\theta$ is a convergent integral. Clearly the limit of $f(z)$ as $z \nearrow \sigma$ exists due to the monotonicity of f near $z = \sigma$. If this limit is finite then the result is straightforward. Hence let us suppose, for instance, that $\lim_{z\to\sigma} f(z) = +\infty$. Thus f is a positive increasing function on $(\sigma - \kappa, \sigma)$ for some $\kappa > 0$. Consider any $\varepsilon > 0$ and let η and δ_1 be small enough positive numbers such that $s\sigma\sin\theta > \sigma - \kappa$ for all $\theta \in (\frac{\pi}{2} - \eta, \frac{\pi}{2})$ and $s \in (1 - \delta_1, 1)$. Then, for these values,

$0 < f(s\sigma\sin\theta) < f(\sigma\sin\theta)$ and consequently

$$0 < \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} f(s\sigma\sin\theta)\sin^n\theta d\theta < \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} f(\sigma\sin\theta)\sin^n\theta d\theta < \frac{\varepsilon}{4}$$

for all $s \in (1-\delta_1, 1)$, where the last inequality follows due to the fact that the integral $\int_0^{\frac{\pi}{2}} f(\sigma\sin\theta)\sin^n\theta d\theta$ is convergent and taking η smaller if necessary. On the other hand, since the function $s \longmapsto \int_0^{\frac{\pi}{2}-\eta} f(s\sigma\sin\theta)\sin^n\theta d\theta$ is continuous at $s=1$, there exists $\delta_2 > 0$ such that

$$\left|\int_0^{\frac{\pi}{2}-\eta}\big(f(\sigma\sin\theta) - f(s\sigma\sin\theta)\big)\sin^n\theta d\theta\right| < \frac{\varepsilon}{2}$$

for all $s \in (1-\delta_2, 1)$. Accordingly if $s \in (1-\delta, 1)$ with $\delta := \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned}\left|L - \int_0^{\frac{\pi}{2}} f(s\sigma\sin\theta)\sin^n\theta d\theta\right| \leqslant{}& \left|\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} f(\sigma\sin\theta)\sin^n\theta d\theta\right| + \left|\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} f(s\sigma\sin\theta)\sin^n\theta d\theta\right| \\ &+ \left|\int_0^{\frac{\pi}{2}-\eta}\big(f(\sigma\sin\theta) - f(s\sigma\sin\theta)\big)\sin^n\theta d\theta\right| < \varepsilon,\end{aligned}$$

and the result follows.

Now let us prove the result in case that $\int_0^{\frac{\pi}{2}} f(\sigma\sin\theta)\sin^n\theta d\theta$ does not converge. This implies, due to the monotonicity of $f(z)$ at $z=\sigma$, that $\int_0^{\frac{\pi}{2}-\eta} f(\sigma\sin\theta)\sin^n\theta d\theta$ tends to infinity as $\eta \searrow 0$. Suppose, for instance, that it tends to $+\infty$. Hence $f(z)$ tends to $+\infty$ as $z \nearrow \sigma$. Take $\bar{z} \in (0,\sigma)$ such that f is positive on $(\bar{z},\sigma)$. Let η_1 and δ_1 be positive numbers such that $s\sigma\sin\theta > \bar{z}$ for all $\theta \in (\frac{\pi}{2}-\eta_1, \frac{\pi}{2})$ and $s \in (1-\delta_1, 1)$. We have

$$\int_0^{\frac{\pi}{2}} f(s\sigma\sin\theta)\sin^n\theta d\theta \geqslant \int_0^{\frac{\pi}{2}-\eta_1} f(s\sigma\sin\theta)\sin^n\theta d\theta \text{ for all } s \in (1-\delta_1, 1). \tag{2.4}$$

Consider at this point any $M > 0$. Then, due to $\int_0^{\frac{\pi}{2}} f(\sigma\sin\theta)\sin^n\theta d\theta = +\infty$, there exists $\eta_2 \in (0,\eta_1)$ small enough such that

$$\int_0^{\frac{\pi}{2}-\eta_2} f(\sigma\sin\theta)\sin^n\theta d\theta > M.$$

Define $S(s) := \int_0^{\frac{\pi}{2}-\eta_2} f(s\sigma\sin\theta)\sin^n\theta d\theta$, which is a continuous function on $[0,1]$. Therefore, on account of $S(1) > M$, there exists $\delta_2 \in (0,\delta_1)$ such that $S(s) > M$ for all $s \in (1-\delta_2, 1)$. Hence, since $f(s\sigma\sin\theta) > 0$ for all $\theta \in (\frac{\pi}{2}-\eta_1, \frac{\pi}{2})$ and $s \in (1-\delta_1, 1)$, from (2.4) we can assert that

$$\int_0^{\frac{\pi}{2}} f(s\sigma\sin\theta)\sin^n\theta d\theta \geqslant \int_0^{\frac{\pi}{2}-\eta_2} f(s\sigma\sin\theta)\sin^n\theta d\theta = S(s) > M$$

for all $s \in (1-\delta_2, 1)$, where in the first inequality we take $0 < \delta_2 < \delta_1$ and $0 < \eta_2 < \eta_1$ also into account. This shows that $\lim_{s\to 1^-}\int_0^{\frac{\pi}{2}} f(s\sigma\sin\theta)\sin^n\theta d\theta = +\infty$, as desired, and completes the proof of the result. ∎

Next result gives the limit value of the period function and its derivative as we approach the outer boundary. Since it is non-parametric, the dependence on μ is omitted for the sake of brevity.

Corollary 2.4.5. *Let X be an admissible analytic potential system with $h_0 < +\infty$ and such that $(g^{-1})''$ is monotonous near the endpoints of the interval $(-\sqrt{h_0}, \sqrt{h_0})$. Then the following hold:*

(a) *either* $\lim_{h\nearrow h_0} T(h) = +\infty$ *or*

$$\lim_{h\nearrow h_0} T(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h_0}\sin\theta)d\theta,$$

and the integral is convergent.

(b) *either* $\lim_{h\nearrow h_0} T'(h) = \pm\infty$ *or*

$$\lim_{h\nearrow h_0} T'(h) = \frac{1}{\sqrt{2h_0}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})''(\sqrt{h_0}\sin\theta)\sin\theta d\theta,$$

and the integral is convergent.

(c) *If we assume additionally that $(g^{-1})'''$ is monotonous near the endpoints of the interval $(-\sqrt{h_0}, \sqrt{h_0})$ and $\lim_{h\nearrow h_0} T'(h) = \Delta_1$ converges, then $\lim_{h\nearrow h_0} T''(h) = \pm\infty$ or*

$$\lim_{h\nearrow h_0} T''(h) = \frac{1}{2\sqrt{2}h_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'''(\sqrt{h_0}\sin\theta)\sin^2\theta d\theta - \frac{\Delta_1}{2h_0}$$

Proof. Clearly the monotonicity assumption on $(g^{-1})''$ implies that $(g^{-1})'$ is monotonous near the endpoints of $(-\sqrt{h_0}, \sqrt{h_0})$ as well. Let us prove the assertion in (a). Denote $f(z) := \sqrt{2}(g^{-1})'(z)$. Then, from (2.1), we can write $T(h_0 s^2) = I_+(s) + I_-(s)$ where

$$I_\pm(s) := \int_0^{\frac{\pi}{2}} f(\pm s\sqrt{h_0}\sin\theta)d\theta.$$

By applying Proposition 2.4.4 we have that $I_\pm(s)$ tends to $I_\pm(1)$ as $s \nearrow 1$, with $I_\pm(1)$ being a positive number or $+\infty$ since $(g^{-1})'$ is a positive function. This proves (a).

Let turn now to the proof of (b). In this case, setting $f(z) := \sqrt{2h_0}(g^{-1})''(z)$, we write $\frac{d}{ds}T(h_0 s^2) = 2h_0 s T'(h_0 s^2) = R_+(s) - R_-(s)$, where

$$R_\pm(s) := \int_0^{\frac{\pi}{2}} f(\pm s\sqrt{h_0}\sin\theta)\sin\theta d\theta.$$

Again, by Proposition 2.4.4, $R_\pm(s)$ tends to $R_\pm(1)$ as $s \nearrow 1$, with $R_\pm(1)$ being a real number or ∞. Accordingly the result follows except in case that $R_-(1)$ and $R_+(1)$ are both ∞. However, due to the admissibility assumption (see Definition 2.4.2), this can only occur if V' tends to zero as we approach to the endpoints of (x_ℓ, x_r). Hence, by Lemma 2.4.3, $f(z)$ tends to $+\infty$ (respectively, $-\infty$) as $z \nearrow \sqrt{h_0}$ (respectively, $z \searrow -\sqrt{h_0}$) and, consequently, $R_-(1)$ and $R_+(1)$ are both $+\infty$. This completes the proof of (b).

Let us finally prove (c). In this case

$$T''(h_0 s^2) = -\frac{1}{2h_0 s^2} T'(h_0 s^2) + \frac{1}{2\sqrt{2} h_0 s^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'''(s\sqrt{h_0}\sin\theta)\sin^2\theta d\theta.$$

The hypothesis of the convergence of T' implies that $\lim_{s\to 1} T'(h_0 s^2)$ converges. Moreover, the value of the limit is the integral in (b). Then the result follows once we prove the limit of the integral in the above expression as s tends to 1. The proof follows similarly as the case (b) so we omit the details here. ■

Once we have established the limit of $T'(h)$ and $T''(h)$ as h tends to h_0, our next goal is to give sufficient conditions to ensure that this limit is uniform with respect to the parameter μ. With this aim in view we prove the following result.

Lemma 2.4.6. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying hypothesis* **(H)** *and such that h_0 and x_ℓ are finite. Assume additionally that $x_\ell(\mu)$ is regular. Then the mappings $(z,\mu) \longmapsto (g_\mu^{-1})''(z)$ and $(z,\mu) \longmapsto (g_\mu^{-1})'''(z)$ are continuous on the set $\big\{(z,\mu) \in \mathbb{R}\times\Lambda : z \in [-\sqrt{h_0(\mu)}, 0]\big\}$.*

Proof. Let us prove the assertion concerning the map $(z,\mu) \longmapsto (g_\mu^{-1})''(z)$. The result for the assertion concerning the map $(z,\mu) \longmapsto (g_\mu^{-1})'''(z)$ follows similarly so we omit the proof here. Since $V_\mu(x) = g_\mu(x)^2$ and $x_\ell(\mu)$ is regular, $g'_\mu\big(x_\ell(\mu)\big) \neq 0$. On the other hand, by implicit derivation, $(g_\mu^{-1})'' = \frac{-g''_\mu}{(g'_\mu)^3} \circ g_\mu^{-1}$. Note also that the map $(x,\mu) \longmapsto \frac{-g''_\mu}{(g'_\mu)^3}(x)$ is continuous on the set $\{(z,\mu) \in \mathbb{R}\times\Lambda : z \in [-\sqrt{h_0(\mu)}, 0]\}$ thanks to hypothesis in **(H)**. By Lemma 2.2.7, $(x,\mu) \longmapsto g_\mu^{-1}(x)$ is continuous on $\{(z,\mu) \in \mathbb{R}\times\Lambda : z \in (-\sqrt{h_0(\mu)}, 0]\}$ and it extends continuously at $(-\sqrt{h_0(\mu)}, \mu)$ by Lemma 2.2.8. The result follows then by composition. ■

Definition 2.4.7. Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of continuous functions defined on $I_\mu = \big(a(\mu), b(\mu)\big)$. Suppose that each endpoint of I_μ is either a continuous function on Λ or identically ∞. We say that the family $\{f_\mu\}_{\mu\in\Lambda}$ is *uniformly monotonous* in $\hat{\mu} \in \Lambda$ at $a(\mu)$ (respectively, at $b(\mu)$) if there exist a neighbourhood U of $\hat{\mu}$ and $\bar{z} \in \mathbb{R}$ such that, for all $\mu \in U$, $\bar{z} \in I_\mu$ and $x \longmapsto f_\mu(x)$ is monotonous on $(a(\mu), \bar{z})$ (respectively, on $(\bar{z}, b(\mu))$). □

For the sake of shortness we collect in the next definition some of the hypothesis we shall use in the statements of the following results.

Definition 2.4.8. Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems. We define the following conditions for a given parameter $\hat{\mu}\in\Lambda$:

(C_1) The family $\left\{\frac{g_\mu''}{(g_\mu')^3}\right\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat{\mu}$ at the non-regular endpoints of $\mathcal{I}_\mu$.

(C_2) The families $\big\{h_0(\mu)-V_\mu\big\}_{\mu\in\Lambda}$, $\{V_\mu'\}_{\mu\in\Lambda}$ and $\{V_\mu''\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat{\mu}$ at the non-regular endpoints of $\mathcal{I}_\mu$.

(C_3) If $\alpha(\mu)$ is the quantifier of $\big\{h_0(\mu)-V_\mu\big\}_{\mu\in\Lambda}$ in a non-regular finite endpoint of $\mathcal{I}_\mu$ then $\alpha(\hat{\mu})\neq -1$.

(C_4) The family $\left\{\frac{3(g_\mu'')^2-g_\mu'''g_\mu'}{(g_\mu')^3}\right\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat{\mu}$ at the non-regular endpoints of $\mathcal{I}_\mu$.

(C_5) The family $\{V_\mu'''\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at the non-regular endpoints of $\mathcal{I}_\mu$.

(C_6) If $\alpha(\mu)$ is the quantifier of $\big\{h_0(\mu)-V_\mu\big\}_{\mu\in\Lambda}$ in a non-regular finite endpoint of $\mathcal{I}_\mu$ then $\alpha(\hat{\mu})\neq -2$.

Let $\alpha_r(\mu)$ be the quantifier of $\big\{h_0(\mu)-V_\mu\big\}_{\mu\in\Lambda}$ at $x_r(\mu)$, which recall that it is non-regular by convention (see Figures 2.3 and 2.4). If $x_\ell(\mu)$ is non-regular too, then we denote the corresponding quantifier at $x_\ell(\mu)$ by $\alpha_\ell(\mu)$. With this notation we define

$$\gamma_M(\mu):=\begin{cases}-\frac{3}{2}\alpha_r(\mu) & \text{if } x_\ell(\mu) \text{ is regular,}\\ -\frac{3}{2}\min\{\alpha_\ell(\mu),\alpha_r(\mu)\} & \text{if } x_\ell(\mu) \text{ is non-regular,}\end{cases}$$

and

$$\gamma_m(\mu):=\begin{cases}-\frac{3}{2}\alpha_r(\mu) & \text{if } x_\ell(\mu) \text{ is regular,}\\ -\frac{3}{2}\max\{\alpha_\ell(\mu),\alpha_r(\mu)\} & \text{if } x_\ell(\mu) \text{ is non-regular.}\end{cases}$$

□

We point out that the functions $\gamma_M(\mu)$ and $\gamma_m(\mu)$ are positive. Indeed, we have that $h_0(\mu)-V_\mu(x)\longrightarrow 0$ as x tends to $x_r(\mu)$. Then it follows that $\alpha_r(\mu)<0$. Exactly the same occurs for $x_\ell(\mu)$.

Lemma 2.4.9. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems. The following statements hold:*

(a) If $\hat{\mu}\in\Lambda$ satisfies (C_2,C_3) then the family $\left\{\frac{g_\mu''g_\mu}{(g_\mu')^2\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x_r(\mu)$ by $-\frac{3}{2}\alpha_r(\mu)$. Moreover, if $x_\ell(\mu)$ is non-regular too, then the family is continuously quantifiable in $\hat{\mu}$ at x_ℓ by $-\frac{3}{2}\alpha_\ell(\mu)$.

(*b*) *If* $\hat\mu \in \Lambda$ *satisfies* **(C$_2$,C$_3$,C$_5$,C$_6$)** *then the family* $\left\{\frac{(3(g_\mu'')^2 - g_\mu' g_\mu''')V_\mu}{(g_\mu')^4\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$ *is continuously quantifiable in* $\hat\mu$ *at* $x_r(\mu)$ *by* $-\frac{5}{2}\alpha_r(\mu)$. *Moreover, if* $x_\ell(\mu)$ *is non-regular too, then the family is continuously quantifiable in* $\hat\mu$ *at* x_ℓ *by* $-\frac{5}{2}\alpha_\ell(\mu)$.

Proof. We prove the first assertion of the result. The second follows similarly. Hypothesis in **(C$_2$)** ensures that $\{h_0(\mu)-g_\mu\}_{\mu\in\Lambda}$, $\{V_\mu'\}_{\mu\in\Lambda}$ and $\{V_\mu''\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat\mu$ at $x_r(\mu)$. Moreover, **(C$_3$)** ensures that $\alpha_r(\hat\mu) \neq -1$ in case that $x_r(\mu)$ is finite. By Hôpital's Rule it is easy to see that in this case, if $x_r(\mu)$ is finite, then the quantifiers of $\{V_\mu'\}_{\mu\in\Lambda}$ and $\{V_\mu''\}_{\mu\in\Lambda}$ are $\alpha_r(\mu)+1$ and $\alpha_r(\mu)+2$ respectively. If $x_r(\mu)$ is infinite then the quantifiers are $\alpha_r(\mu)-1$ and $\alpha_r(\mu)-2$. The result follows then by product of limits and taking into account that $g_\mu^2 = V_\mu$. The proof for the left endpoint follows in the same way. ∎

Proposition 2.4.10. *Let* $\{X_\mu\}_{\mu\in\Lambda}$ *be a family of admissible analytic potential systems satisfying hypothesis* **(H)** *such that* $h_0(\mu)$ *is finite and* $\mathcal{I}_\mu$ *is bounded. The following statements hold:*

(*a*) *Assume that* $\hat\mu \in \Lambda$ *satisfies* **(C$_1$-C$_3$)** *then,*

(*a*1) *If* $\gamma_M(\hat\mu) < 1$ *we have*

$$\lim_{(h,\mu)\to(h_0(\hat\mu),\hat\mu)} T_\mu'(h) = \frac{1}{\sqrt{2h_0(\hat\mu)}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_{\hat\mu}^{-1})''\big(\sqrt{h_0(\hat\mu)}\sin\theta\big)\sin\theta d\theta$$

and the integral is convergent.

(*a*2) *If* $\gamma_M(\hat\mu) > 1$ *and* $\gamma_m(\hat\mu) \neq 1$ *then* $\lim_{(h,\mu)\to(h_0(\hat\mu),\hat\mu)} T_\mu'(h) = \pm\infty$.

(*b*) *Assume that* $\hat\mu \in \Lambda$ *satisfies* **(C$_1$-C$_6$)** *and* $\lim_{(h,\mu)\to(h_0(\hat\mu),\hat\mu)} T_\mu'(h) = \Delta(\hat\mu)$ *converges. Then,*

(*b*1) *If* $\gamma_M(\hat\mu) < \frac{3}{5}$ *we have*

$$\lim_{(h,\mu)\to(h_0(\hat\mu),\hat\mu)} T_\mu''(h) = \frac{1}{2\sqrt{2h_0(\hat\mu)}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_{\hat\mu}^{-1})'''\big(\sqrt{h_0(\hat\mu)}\sin\theta\big)\sin^2\theta d\theta - \frac{\Delta(\hat\mu)}{2h_0(\hat\mu)}.$$

and the integral is convergent.

(*b*2) *If* $\gamma_M(\hat\mu) \in \big(\frac{3}{5},1\big)\setminus\{\frac{3}{4}\}$ *and* $\gamma_m(\hat\mu) \notin \{\frac{3}{5},\frac{3}{4}\}$ *then* $\lim_{(h,\mu)\to(h_0(\hat\mu),\hat\mu)} T_\mu''(h) = \pm\infty$.

Proof. Let us first prove (*a*1). Setting $f_\mu(z) := \sqrt{2h_0(\mu)}(g_\mu^{-1})''(z)$ for the sake of brevity, the derivation of the expression of the period function in (2.1) yields to

$$\frac{d}{ds}T_\mu\big(h_0(\mu)s^2\big) = 2h_0(\mu)sT_\mu'\big(h_0(\mu)s^2\big) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\mu\big(\sqrt{h_0(\mu)}s\sin\theta\big)\sin\theta d\theta. \tag{2.5}$$

We split the interval of integration into $(-\frac{\pi}{2},0)$ and $(0,\frac{\pi}{2})$. We shall prove that

$$\lim_{(s,\mu)\to(1,\hat{\mu})}\int_0^{\frac{\pi}{2}} f_\mu\big(\sqrt{h_0(\mu)}s\sin\theta\big)\sin\theta d\theta=\int_0^{\frac{\pi}{2}} f_{\hat{\mu}}\big(\sqrt{h_0(\hat{\mu})}\sin\theta\big)\sin\theta d\theta=:L \tag{2.6}$$

and that L is a convergent integral. Since the potential systems are admissible, two different situations are considered: either x_ℓ is regular or both x_ℓ and x_r are non-regular. We point out that in the first case the assertion is immediate on $(-\frac{\pi}{2},0)$. Indeed, in this situation the potential family is analytic on $x_\ell(\mu)$ and $V_\mu'(x_\ell(\mu))\neq 0$ for all $\mu\in\Lambda$. Consequently, by Lemma 2.4.6 the function $(z,\mu)\longmapsto f_\mu(z)$ is continuous on the set $\{(x,\mu)\in\mathbb{R}\times\Lambda : x\in[-\sqrt{h_0(\mu)},0]\}$. On the other hand, if both x_ℓ and x_r are non-regular, the proof of the assertion on $(-\frac{\pi}{2},0)$ follows in the same way as the assertion on $(0,\frac{\pi}{2})$. Accordingly the result will follow once we prove (2.6). With this aim in view we claim that, for a given $\varepsilon>0$, there exist positive η, δ and r small enough such that

$$\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}}\Big|f_\mu(s\sqrt{h_0(\mu)}\sin\theta)\Big|\sin\theta d\theta<\varepsilon \tag{2.7}$$

for all $\mu\in B_r(\hat{\mu})$ and $s\in(1-\delta,1)$. Here, and in what follows, we shall denote by $B_r(\hat{\mu})$ the set $\{\mu\in\Lambda : \|\mu-\hat{\mu}\|\leqslant r\}$. To show this let us note first that $\alpha_r(\mu)$ in conditions **(C_1-C_3)** is negative. Indeed, condition **(C_2)** implies that the limit

$$\lim_{(x,\mu)\to(x_r(\hat{\mu}),\hat{\mu})}\big(g_\mu(x)-\sqrt{h_0(\mu)}\big)\big(x_r(\mu)-x\big)^{\alpha_r(\mu)}$$

is finite a different from zero. Due to $g_\mu(x)\nearrow\sqrt{h_0(\mu)}$ as x tends to $x_r(\mu)$ we have then $\alpha_r(\hat{\mu})<0$. The continuity of $\mu\mapsto\alpha_r(\mu)$, see Remark 1.3.3, allows us to suppose $\alpha_r(\mu)<0$ for all $\mu\in B_r(\hat{\mu})$. On the other hand, by condition **(C_2)** and Lemma 2.4.9, the family $\Big\{\frac{g_\mu''g_\mu}{(g_\mu')^2\sqrt{h_0(\mu)-V_\mu}}\Big\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x_r(\mu)$ by $\beta(\mu):=-\frac{3}{2}\alpha_r(\mu)>0$. Moreover, by hypothesis $\gamma_M(\hat{\mu})<1$ so we have $0<\beta(\mu)<1$ for all $\mu\in B_r(\hat{\mu})$ considering r smaller if necessary. Therefore, due to the continuity of $\mu\longmapsto x_r(\mu)$, there exist positive C, ξ and r such that, for all $\mu\in B_r(\hat{\mu})$,

$$\left|\frac{g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2}\right|\frac{1}{\sqrt{h_0(\mu)-V_\mu(x)}}<\frac{C}{(x_r(\mu)-x)^{\beta(\mu)}}$$

for all $x\in\big(x_r(\mu)-\xi,x_r(\mu)\big)$. Therefore

$$\int_{x_r(\mu)-\xi}^{x_r(\mu)}\left|\frac{g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2}\right|\frac{dx}{\sqrt{h_0(\mu)-V_\mu(x)}}<C\frac{\xi^{1-\beta(\mu)}}{1-\beta(\mu)}$$

and so, taking ξ and r smaller if necessary, we can assert that

$$\int_{x_r(\mu)-\xi}^{x_r(\mu)}\left|\frac{g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2}\right|\frac{dx}{\sqrt{h_0(\mu)-V_\mu(x)}}<\varepsilon$$

for all $\mu \in B_r(\hat{\mu})$. If we perform the change of variable $x = (g_\mu^{-1})(\sqrt{h_0(\mu)}\sin\theta)$ in the integral above, the inequality easily implies that

$$\int_{\frac{\pi}{2}-\hat{\eta}}^{\frac{\pi}{2}} \Big| f_\mu\big(\sqrt{h_0(\mu)}\sin\theta\big)\Big| \sin\theta d\theta < \varepsilon \tag{2.8}$$

for all $\mu \in B_r(\hat{\mu})$, where $\hat{\eta} := \frac{\pi}{2} - \max\Big\{\arcsin\Big(\frac{g_\mu(x_r(\mu)-\xi)}{\sqrt{h_0(\mu)}}\Big); \mu \in B_r(\hat{\mu})\Big\} > 0$.

Recall at this point that, by condition **(C$_1$)** and taking $r > 0$ smaller if necessary, there exists $\bar{x} \in \mathbb{R}$ such that, for all $\mu \in B_r(\hat{\mu})$, it holds $\bar{x} \in (x_r(\mu) - \xi, x_r(\mu))$ and $\frac{g''_\mu}{(g'_\mu)^3}$ is monotonous on $(\bar{x}, x_r(\mu))$. Since g_μ is a diffeomorphism from $\big(x_\ell(\mu), x_r(\mu)\big)$ to $\big(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)}\big)$ and $(g_\mu^{-1})'' = \frac{-g''_\mu}{(g'_\mu)^3} \circ g_\mu^{-1}$, if we set $\bar{z} := \max\{g_\mu(\bar{x}); \mu \in B_r(\hat{\mu})\}$, then for all $\mu \in B_r(\hat{\mu})$ the function $(g_\mu^{-1})''$ is monotonous on $(\hat{z}, \sqrt{h_0(\mu)})$. Accordingly, for all $\mu \in B_r(\hat{\mu})$, $z \longmapsto |f_\mu(z)|$ is monotonous on $(\hat{z}, \sqrt{h_0(\mu)})$. Let us take now $\eta \in (0, \hat{\eta})$ and $\delta > 0$ small enough in order that $\sqrt{h_0(\mu)} s \sin\theta > \hat{z}$ for all $s \in (1-\delta, 1)$, $\theta \in (\frac{\pi}{2} - \eta, \frac{\pi}{2})$ and $\mu \in B_r(\hat{\mu})$. If $|f_\mu|$ is increasing then $|f_\mu(s\sqrt{h_0(\mu)}\sin\theta)| < |f_\mu(\sqrt{h_0(\mu)}\sin\theta)|$ for all $s \in (1-\delta, 1)$, $\theta \in (\frac{\pi}{2} - \eta, \frac{\pi}{2})$ and $\mu \in B_r(\hat{\mu})$. Consequently, taking (2.8) into account,

$$\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \Big| f_\mu(s\sqrt{h_0(\mu)}\sin\theta)\Big| \sin\theta d\theta \leqslant \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \Big| f_\mu(\sqrt{h_0(\mu)}\sin\theta)\Big| \sin\theta d\theta < \varepsilon$$

for all $s \in (1-\delta, 1)$ and $\mu \in B_r(\hat{\mu})$. Hence the claim follows in this case. Suppose finally that $|f_\mu|$ is decreasing. Then, for the same values as before,

$$|f_\mu(s\sqrt{h_0(\mu)}\sin\theta)| < |f_\mu\big((1-\delta)\sqrt{h_0(\mu)}\sin\theta\big)|,$$

which yields

$$\int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \Big| f_\mu(s\sqrt{h_0(\mu)}\sin\theta)\Big| \sin\theta d\theta \leqslant \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \Big| f_\mu\big((1-\delta)\sqrt{h_0(\mu)}\sin\theta\big)\Big| \sin\theta d\theta.$$

It is clear that the integral on the right tends to zero as $\eta \longrightarrow 0^+$ uniformly for parameters $\mu \in B_r(\hat{\mu})$ because, on account of Lemma 2.2.7 and hypothesis in **(H)**, the function $(\theta, \mu) \longmapsto |f_\mu\big((1-\delta)\sqrt{h_0(\mu)}\sin\theta\big)|$ is continuous on $[\frac{\pi}{2} - \eta, \frac{\pi}{2}] \times B_r(\hat{\mu})$. Thus the inequality in (2.7) is true for $\eta > 0$ small enough and so the claim follows also in this case.

We are now in position to show (2.6). The fact that L is a convergent integral follows easily by using that, due to the assumption in **(C$_2$)** and Lemma 2.4.9, $\frac{g''_{\hat{\mu}} g_{\hat{\mu}}}{(g'_{\hat{\mu}})^2\sqrt{h_0(\hat{\mu}) - V_{\hat{\mu}}}}$ is quantifiable at $x_r(\hat{\mu})$ by $0 < \beta(\hat{\mu}) < 1$. On the other hand,

$$\begin{aligned}\Big|\int_0^{\frac{\pi}{2}} f_\mu\big(s\sqrt{h_0(\mu)}\sin\theta\big)\sin\theta d\theta - L\Big| \leqslant\ & \Big|\int_0^{\frac{\pi}{2}} \Big(f_\mu\big(s\sqrt{h_0(\mu)}\sin\theta\big) - f_{\hat{\mu}}\big(s\sqrt{h_0(\hat{\mu})}\sin\theta\big)\Big)\sin\theta d\theta\Big| \\ & + \Big|\int_0^{\frac{\pi}{2}} f_{\hat{\mu}}\big(s\sqrt{h_0(\hat{\mu})}\sin\theta\big)\sin\theta d\theta - L\Big|.\end{aligned}$$

Let us denote the first and second summands above by S_1 and S_2, respectively, and consider any $\varepsilon > 0$. Then, by Proposition 2.4.4, there exists $\delta_2 > 0$ such that $S_2 < \varepsilon/2$ for all $s \in (1-\delta_2, 1)$. In addition, taking any $\eta \in (0, \frac{\pi}{2})$, we get

$$S_1 \leqslant \left| \int_0^{\frac{\pi}{2}-\eta} \Big(f_\mu\big(s\sqrt{h_0(\mu)}\sin\theta\big) - f_{\hat\mu}\big(s\sqrt{h_0(\hat\mu)}\sin\theta\big)\Big)\sin\theta d\theta \right| \\ + \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \left| f_\mu\big(s\sqrt{h_0(\mu)}\sin\theta\big)\right| \sin\theta d\theta + \int_{\frac{\pi}{2}-\eta}^{\frac{\pi}{2}} \left| f_{\hat\mu}\big(s\sqrt{h_0(\hat\mu)}\sin\theta\big)\right| \sin\theta d\theta.$$

Let us denote by S_{11}, S_{12} and S_{13} the first, second and third summands above, respectively. By applying the claim in (2.7) twice, there exist positive η, δ_1 and r small enough such that $S_{12}+S_{13} < \varepsilon/4$ for all $\mu \in B_r(\hat\mu)$ and $s \in (1-\delta_1, 1)$. Finally, thanks to Lemma 2.2.7 we have that the function $(\theta, s, \mu) \longmapsto f_\mu\big(s\sqrt{h_0(\mu)}\sin\theta\big)$ is continuous on $[0, \frac{\pi}{2}-\eta] \times [0,1] \times B_r(\hat\mu)$. Then, by making δ_1 and r smaller if necessary, we get that $S_{11} < \varepsilon/4$ for all $\mu \in B_r(\hat\mu)$ and $s \in (1-\delta_1, 1)$. Hence $S_1 + S_2 < \varepsilon$ for all $\mu \in B_r(\hat\mu)$ and $s \in (1-\delta, 1)$ with $\delta := \min\{\delta_1, \delta_2\}$. This shows (2.6) and completes the proof of $(a1)$.

Let us prove $(a2)$. In this case two situations may occur: either $\gamma_m(\hat\mu) < 1 < \gamma_M(\hat\mu)$ or $\gamma_M(\hat\mu) \geqslant \gamma_m(\hat\mu) > 1$. Let us start proving the result in the first situation. In this case x_ℓ and x_r are both non-regular. Let us fix that $\gamma_m(\hat\mu) = -\frac{3}{2}\alpha_\ell(\hat\mu)$ and $\gamma_M(\hat\mu) = -\frac{3}{2}\alpha_r(\hat\mu)$ (the other situation follows exactly in the same way). Lemma 2.4.9 shows that $\gamma_M(\hat\mu)$ and $\gamma_m(\hat\mu)$ are the respective quantifiers of family $\left\{\frac{g_\mu'' g_\mu}{(g_\mu')^2\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$. We split the integration interval of (2.5) into $(-\frac{\pi}{2}, 0)$ and $(0, \frac{\pi}{2})$ giving rise to two integrals that we denote respectively by $L^-(s;\mu)$ and $L^+(s;\mu)$. On account of $\gamma_m(\hat\mu) < 1$ the same proof as in (a) shows that $L^-(s;\mu)$ converges as $(s,\mu) \to (1,\hat\mu)$. We claim at this point that $L^+(s;\mu)$ tends to infinity as $s \nearrow 1$ uniformly in a neighbourhood of $\hat\mu$. Note that once we show this the result will follow taking into account that $h_0(\mu)$ is a continuous function. In order to show the claim we first note that, on account of condition **(C_1)**, g_μ'' is non-vanishing near $x_r(\mu)$. Suppose, for instance, that it is negative. Note that, on account of the assumption in **(C_2)** and Lemma 2.4.9, there exist $\bar x \in \mathbb{R}$ and $\bar r > 0$ verifying

$$\frac{-g_\mu''(x) g_\mu(x)}{(g_\mu'(x))^2\sqrt{h_0(\mu) - V_\mu(x)}} > \frac{C}{(x_r(\mu) - x)^{-\frac{3}{2}\alpha_r(\mu)}}$$

for all $\mu \in B_{\bar r}(\hat\mu)$ and $x \in \big(\bar x, x_r(\mu)\big)$, where we can take $C > 0$ because $g_{\hat\mu}''$ is negative near $x_r(\hat\mu)$. Moreover, on account of Lemma 2.2.8, there exist $\delta > 0$ and $r \in (0, \bar r)$ such that $\bar x < g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big) < x_r(\mu)$ for all $s \in (1-\delta, 1)$ and $\mu \in B_r(\hat\mu)$. Consequently,

$$L_1^+(s;\mu) := \int_{\bar x}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} \frac{-g_\mu''(x) g_\mu(x) dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)s^2 - V_\mu(x)}} \\ > \int_{\bar x}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} \frac{-g_\mu''(x) g_\mu(x) dx}{(g_\mu'(x))^2\sqrt{h_0(\mu) - V_\mu(x)}}$$

$$
\begin{aligned}
&> \int_{\bar{x}}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} C(x_r(\mu)-x)^{\frac{3}{2}\alpha_r(\mu)}dx \\
&= \frac{C}{-(\frac{3}{2}\alpha_r(\mu)+1)}\left((x_r(\mu)-g_\mu^{-1}(\sqrt{h_0(\mu)}s))^{\frac{3}{2}\alpha_r(\mu)+1}-(x_r(\mu)-\bar{x})^{\frac{3}{2}\alpha_r(\mu)+1}\right).
\end{aligned}
$$

Since $\gamma_M(\hat{\mu})>1$ then $\lim_{\mu\to\hat{\mu}}\frac{3}{2}\alpha_r(\mu)+1=\frac{3}{2}\alpha_r(\hat{\mu})+1=1-\gamma_M(\hat{\mu})<0$. Moreover Lemma 2.2.8 shows $g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big)\longrightarrow x_r(\mu)$ as $s\nearrow 1$ uniformly on $B_r(\hat{\mu})$. Therefore the above inequalities show that $L_1^+(s;\mu)$ tends to $+\infty$ as $s\nearrow 1$ uniformly on $B_r(\hat{\mu})$ as desired. This shows the claim and so the result follows in the case $\gamma_M(\hat{\mu})>1>\gamma_m(\hat{\mu})$.

Finally let us consider the case when $\gamma_M(\hat{\mu})\geqslant\gamma_m(\hat{\mu})>1$. If x_ℓ is regular then $L^-(s;\mu)$ converges to a number as $(s,\mu)\to(1,\hat{\mu})$ and the same procedure before shows that $L^+(s;\mu)$ tends to infinity as s tends to 1 uniformly on $B_r(\hat{\mu})$. So the result holds in this case. On the other hand, in case that both x_ℓ and x_r are non-regular, with the same argue we can prove that both $L^-(s;\mu)$ and $L^+(s;\mu)$ tend to infinity as s tends to 1 uniformly on $B_r(\hat{\mu})$. Moreover, on account of Lemma 2.4.3, both integrals tends to $+\infty$. Then, the result follows in this case by additivity. This ends with the proof of $(a2)$.

For the sake of brevity we shall skip the proof of (b) since is essentially the same as (a). We only point out that, since the limit $\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})}T_\mu'(h)=\Delta(\hat{\mu})$ converges, then

$$
\lim_{(s,\mu)\to(1,\hat{\mu})} T_\mu'(h_0(\mu)s^2)=-\frac{\Delta(\hat{\mu})}{2h_0(\hat{\mu})}+\lim_{(s,\mu)\to(1,\hat{\mu})}\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g_\mu^{-1})'''\big(s\sqrt{h_0(\mu)}\sin\theta\big)\sin^2\theta d\theta}{2\sqrt{2}h_0(\mu)s^2}
$$

so the result follows by studying the integral in the above expression. Similar arguments than the ones in (a) proves the result in this case. ■

The next one is the last ingredient for the proof of the main results in the present section.

Proposition 2.4.11. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying hypothesis* **(H)** *such that $h_0(\mu)$ is finite and $\mathcal{I}_\mu$ is unbounded. Consider $\hat{\mu}\in\Lambda$ satisfying condition* **(C_1-C_3)**. *Then $T'(h)$ tends to $\pm\infty$ as $(h,\mu)\longrightarrow(h_0(\hat{\mu}),\hat{\mu})$.*

Proof. The derivative of the expression of the period function in (2.1) gives

$$
\frac{d}{ds}T_\mu\big(h_0(\mu)s^2\big)=2sh_0(\mu)T_\mu'\big(h_0(\mu)s^2\big)=\sqrt{2h_0(\mu)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g_\mu^{-1})''(\sqrt{h_0(\mu)}s\sin\theta)\sin\theta d\theta.
$$

We split the integration interval into $(-\frac{\pi}{2},0)$ and $(0,\frac{\pi}{2})$, namely $L^-(s;\mu)$ and $L^+(s;\mu)$ respectively. Due to the hypothesis of the endpoints of $\mathcal{I}_\mu$ three different cases can be considered: either $x_\ell(\mu)$ is regular and $x_r\equiv+\infty$, or $x_\ell(\mu)\neq-\infty$ non-regular and $x_r\equiv+\infty$, or $x_\ell\equiv-\infty$ and $x_r\equiv+\infty$. Notice that in the three cases $x_r\equiv+\infty$ so the proof for $L^+(s;\mu)$ will be the same.

Let us consider first that $x_\ell(\mu)$ is regular and $x_r \equiv +\infty$. In this case is clear by Lemma 2.4.6 that $L^-(s;\mu)$ tends to a number when $(s,\mu) \longrightarrow (1,\hat{\mu})$. Then let us focus to show that $L^+(s;\mu)$ tends to infinity uniformly on a neighbourhood of $\hat{\mu}$. By making the change of variable $x = g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\sin\theta\big)$, we obtain

$$\begin{aligned} L^+(s;\mu) &= \int_0^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h_0(\mu)}s\sin\theta)\sin\theta d\theta \\ &= \frac{1}{\sqrt{h_0(\mu)}s}\int_0^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} \frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)s^2 - V_\mu(x)}}. \end{aligned}$$

Note that, on account of condition **(C$_1$)**, g_μ'' must be non-vanishing near $x_r(\mu)$. Suppose, for instance, that it is negative. We claim that $L^+(s;\mu)$ tends to $+\infty$ as $s \nearrow 1$ uniformly on some neighbourhood of $\hat{\mu}$ (respectively, if g_μ'' is positive near $x_r(\mu)$ then $L^+(s;\mu)$ tends to $-\infty$ uniformly). It is clear due to the continuity of $h_0(\mu)$ that the result will follow in this case once we prove this. With this aim in view note that, on account of the assumption in **(C$_2$)**, $\alpha_r(\mu)$ is positive. Indeed, we have that the limit

$$\lim_{(x,\mu)\to(+\infty,\hat{\mu})} \frac{g_\mu(x) - \sqrt{h_0(\mu)}}{x^{\alpha_r(\mu)}}$$

is finite and different from zero. Due to $g_\mu(x) \nearrow \sqrt{h_0(\mu)}$ as x tends to $+\infty$ we have then $\alpha_r(\hat{\mu}) < 0$. The continuity of the map $\mu \longmapsto \alpha_r(\mu)$, see Remark 1.3.3, allows us to consider $\alpha_r(\mu) < 0$ for all $\mu \approx \hat{\mu}$. On account of Lemma 2.4.9 we have that the family $\left\{\frac{g_\mu''g_\mu}{(g_\mu')^2\sqrt{h_0(\mu)-V_\mu}}\right\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at infinity by $\beta(\mu) := -\frac{3}{2}\alpha_r(\mu) > 0$. Therefore, since $\lim_{\mu\to\hat{\mu}} x_r(\mu) = +\infty$, there exists $\bar{x} \in \mathbb{R}$ and $\bar{r} > 0$ verifying

$$\frac{-g_\mu''(x)g_\mu(x)}{(g_\mu'(x))^2\sqrt{h_0(\mu) - V_\mu(x)}} > Cx^{\beta(\mu)} \text{ for all } \mu \in B_{\bar{r}}(\hat{\mu}) \text{ and } x \in \big(\bar{x}, x_r(\mu)\big),$$

where we can take $C > 0$ because we assumed $g_{\hat{\mu}}''$ to be negative near $x_r \equiv +\infty$. Moreover, by Lemma 2.2.8, there exists $\delta > 0$ and $r \in (0,\bar{r})$ such that $\bar{x} < g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big) < +\infty$ for all $s \in (1-\delta, 1)$ and $\mu \in B_r(\hat{\mu})$. Then

$$\begin{aligned} L_1^+(s;\mu) := &\int_{\bar{x}}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} \frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu)s^2 - V_\mu(x)}} \\ > &\int_{\bar{x}}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} \frac{-g_\mu''(x)g_\mu(x)dx}{(g_\mu'(x))^2\sqrt{h_0(\mu) - V_\mu(x)}} \\ > &\int_{\bar{x}}^{g_\mu^{-1}(\sqrt{h_0(\mu)}s)} Cx^{\beta(\mu)}dx = \frac{C}{\beta(\mu)+1}\left(g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big)^{\beta(\mu)+1} - \hat{x}^{\beta(\mu)+1}\right) \end{aligned}$$

Since $\beta(\mu) > 0$ and by Lemma 2.2.8 we have $g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big) \longrightarrow +\infty$ as $s \nearrow 1$ uniformly on $B_r(\hat{\mu})$, the above inequalities show that $L_1^+(s;\mu)$ tends to $+\infty$ as $s \nearrow 1$ uniformly on

$B_r(\hat{\mu})$. On the other hand

$$L_2^+(s;\mu) := \int_0^{\bar{x}} \frac{-g''_\mu(x) g_\mu(x) dx}{(g'_\mu(x))^2 \sqrt{h_0(\mu)s^2 - V_\mu(x)}}$$

is continuous on $K := [1-\delta, 1] \times B_r(\hat{\mu})$ because, by construction,

$$\bar{x} < \min\{g_\mu^{-1}\big(\sqrt{h_0(\mu)}s\big), (s,\mu) \in K\}.$$

Accordingly $L_2^+(s;\mu)$ tends to a number as $(s,\mu) \longrightarrow (1,\hat{\mu})$. Therefore, due to

$$L^+(s;\mu) = \frac{L_1^+(s;\mu) + L_2^+(s;\mu)}{\sqrt{h_0(\mu)}s},$$

the claim is true and the result follows in this case.

Now let us consider x_ℓ to be non-regular and finite, and $x_r \equiv +\infty$. In this case $L^-(s;\mu)$ tends to a number as $(s,\mu) \to (1,\hat{\mu})$. We refer the reader to the proof of Proposition 2.4.10 for the details in this case. On the other hand, we have $L^+(s;\mu)$ tends to infinity uniformly on a neighbourhood of $\hat{\mu}$ as we proved before. Consequently $T'_\mu(h)$ tends to infinity as h approach $h_0(\mu)$ uniformly on a neighbourhood of $\hat{\mu}$.

Finally let us consider $x_\ell \equiv -\infty$ and $x_r \equiv +\infty$. The same proof for $x_r \equiv +\infty$ proves that $L^-(s;\mu)$ tends to infinity uniformly on a neighbourhood of $\hat{\mu}$ in case that $x_\ell \equiv -\infty$. Moreover, Lemma 2.4.3 shows that both L^- and L^+ tend to $+\infty$. Then, in this case we have that $T'_\mu(h)$ tends to $+\infty$ as $h \nearrow h_0(\mu)$ uniformly on a neighbourhood of $\hat{\mu}$. This shows the validity of the result in this case and completes the proof. ∎

Now we are in position to prove the main result of this section: a criterion for a parameter to be a local regular value of the period function at the outer boundary of the period annulus and a criterion to ensure that at most one critical periodic orbit bifurcates from the outer boundary of the period annulus.

Theorem D. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of admissible analytic potential systems satisfying hypothesis* **(H)** *such that $h_0(\mu)$ is finite and consider $\hat{\mu} \in \Lambda$. The following statements hold:*

(*a*) *If $\hat{\mu}$ satisfies* **(C_1-C_3)** *then $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$ if one of the following conditions is verified:*

   (*a*1) *$\mathcal{I}_\mu$ is bounded, $\gamma_M(\hat{\mu}) < 1$ and $\Delta_1(\hat{\mu}) := \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''\big(\sqrt{h_0(\mu)}\sin\theta\big)\sin\theta d\theta \neq 0.$*

   (*a*2) *$\mathcal{I}_\mu$ is bounded, $\gamma_M(\hat{\mu}) > 1$ and $\gamma_m(\hat{\mu}) \neq 1$.*

   (*a*3) *$\mathcal{I}_\mu$ is unbounded.*

(b) *If $\hat{\mu}$ satisfies* **(C$_1$-C$_6$)** *then* $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$ *if one of the following conditions is verified:*

(b1) $\mathcal{I}_\mu$ *is bounded,* $\gamma_M(\hat{\mu}) < \frac{3}{5}$, $\Delta_1(\hat{\mu}) = 0$ *and*

$$\Delta_2(\hat{\mu}) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_{\hat{\mu}}^{-1})'''\big(\sqrt{h_0(\hat{\mu})}\sin\theta\big)\sin^2\theta d\theta \neq 0.$$

(b2) $\mathcal{I}_\mu$ *is bounded,* $\gamma_M(\hat{\mu}) \in \big(\frac{3}{5}, 1\big) \setminus \{\frac{3}{4}\}$ *and* $\gamma_m(\hat{\mu}) \notin \{\frac{3}{5}, \frac{3}{4}\}$.

Proof. Let us proof first (a). The assertion in $(a1)$ follows from Proposition 2.4.10, which shows that

$$\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})} T_\mu'(h) = \frac{1}{\sqrt{2h_0(\hat{\mu})}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''\big(\sqrt{h_0(\hat{\mu})}\sin\theta\big)\sin\theta d\theta = \frac{\Delta_1(\hat{\mu})}{\sqrt{2h_0(\hat{\mu})}} \neq 0,$$

and then by applying Lemma 2.4.1. Assertion in $(a2)$ follows also from Proposition 2.4.10. Indeed, this result shows that $\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})} T_\mu'(h) = \pm\infty$. On account of Lemma 1.3.6 we have that $\lim_{h\to h_0(\mu)} T_{\hat{\mu}}'(h) = \pm\infty$ uniformly on compact neighbourhood of $\hat{\mu}$. Then the result follows on account of Lemma 2.4.1. Finally assertion in $(a3)$ follows from Proposition 2.4.11 using again Lemma 2.4.1. This proves (a).

Let us turn now on the proof of $(b1)$. From Proposition 2.4.10 we have

$$\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})} T_\mu'(h) = \frac{\Delta_1(\hat{\mu})}{\sqrt{2h_0(\hat{\mu})}} = 0.$$

and

$$\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})} T_\mu''(h) = \frac{1}{2\sqrt{2}h_0(\hat{\mu})} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_{\hat{\mu}}^{-1})'''\big(\sqrt{h_0(\hat{\mu})}\sin\theta\big)\sin^2\theta d\theta = \frac{\Delta_2(\hat{\mu})}{2\sqrt{2}h_0(\hat{\mu})} \neq 0.$$

Then the result follows by applying Lemma 2.4.1. The assertion in $(b2)$ follows using again Proposition 2.4.10 which shows that $\lim_{(h,\mu)\to(h_0(\hat{\mu}),\hat{\mu})} T_\mu''(h) = \pm\infty$. On account of Lemma 1.3.6 we have that $\lim_{h\to h_0(\mu)} T_{\hat{\mu}}'(h) = \pm\infty$ uniformly on compact neighbourhood of $\hat{\mu}$. Then the result follows on account of Lemma 2.4.1. This proves (b) and end with the proof of the result. ∎

2.4.2 Second approach

In this section we shall study the criticality at the outer boundary of the period annulus for families of potential systems with $h_0(\mu) < +\infty$ for all $\mu \in \Lambda$. If we proceed the same way as for the case $h_0 = +\infty$, we would take an appropriate non-vanishing function f and try to embed fT_μ' into some easy ECT-system. To this end the natural candidate should be $((h_0(\mu)-h)^{\nu_1(\mu)},(h_0(\mu)-h)^{\nu_2(\mu)},\ldots,(h_0(\mu)-h)^{\nu_n(\mu)})$. However we did not succeed with such an approach. Instead we shall take advantage of Theorem B, which is in fact addressed to the case $h_0=+\infty$. This forces us to "translate" the case $h_0<+\infty$ to the case $h_0=+\infty$ and gives rise to some technicalities that make things more complicated than it should be. With this aim in view we define next a differential operator which is conjugated to $\mathscr{L}_{\boldsymbol{\nu}_n}$. The conjugation is precisely the tool that enables us to translate the case $h_0<+\infty$ to the case $h_0=+\infty$ and apply Theorem B.

Definition 2.4.12. Given $\nu_1,\ldots,\nu_n\in\mathbb{R}$, we consider the linear ordinary differential operator $\mathscr{D}_{\boldsymbol{\nu}_n}\colon\mathscr{C}^\omega\big((0,1)\big)\longrightarrow\mathscr{C}^\omega\big((0,1)\big)$ defined by

$$\mathscr{D}_{\boldsymbol{\nu}_n}[f](x):=(x(1-x^2))^{\frac{n(n+1)}{2}}\frac{W\left[\psi_{\nu_1},\ldots,\psi_{\nu_n},f\right](x)}{\prod_{i=1}^n\psi_{\nu_i}(x)},$$

where we use the notation $\boldsymbol{\nu}_n=(\nu_1,\ldots,\nu_n)$ and

$$\psi_\nu(x):=\frac{1}{1-x^2}\left(\frac{x}{\sqrt{1-x^2}}\right)^\nu.$$

In addition, we define $\mathscr{D}_{\boldsymbol{\nu}_0}:=id$. □

Definition 2.4.13. Setting

$$\phi(x):=\frac{x}{\sqrt{1+x^2}},$$

we consider the operator $\mathscr{B}\colon\mathscr{C}^\omega\big([0,1)\big)\longrightarrow\mathscr{C}^\omega\big([0,+\infty)\big)$ defined by

$$\mathscr{B}[f](x):=\big(1-\phi^2(x)\big)\big(f\circ\phi\big)(x)=\frac{1}{1+x^2}\big(f\circ\phi\big)(x).$$

□

We will show next that $\mathscr{B}$ conjugates $\mathscr{D}_{\boldsymbol{\nu}_n}$ and $\mathscr{L}_{\boldsymbol{\nu}_n}$. This fact eventually will enable us to take advantage of Theorem B. Before proving it we introduce the following definition.

Definition 2.4.14. Let f be an analytic function on $[0,1)$. Then, for each $n\in\mathbb{N}$ we call

$$N_n[f]:=\int_0^1\frac{f(x)}{\sqrt{1-x^2}}\left(\frac{x}{\sqrt{1-x^2}}\right)^{2n-2}dx$$

the n-th *momentum* of f, whenever it is well defined. □

Lemma 2.4.15. *Consider $\nu_1,\nu_2,\ldots,\nu_n\in\mathbb{R}$. Then the following hold:*

(a) $\mathscr{B}[\psi_{\nu_i}](x)=x^{\nu_i}$ *for* $i=1,2,\ldots,n$.

(b) $\mathscr{B}\circ\mathscr{D}_{\boldsymbol{\nu}_n}=\mathscr{L}_{\boldsymbol{\nu}_n}\circ\mathscr{B}$.

(c) $\big(\mathscr{F}\circ\mathscr{B}\big)[f](x)=\sqrt{1+x^2}\big(\mathscr{B}\circ\mathscr{F}\big)[f](x)$ *for any* $f\in\mathscr{C}^{\omega}\big((0,1)\big)$.

(d) $N_n=M_n\circ\mathscr{B}$.

Proof. Let us show (b) because (a) follows straightforward. So take $f\in\mathscr{C}^{\omega}\big((0,1)\big)$ and note that

$$(\mathscr{B}\circ\mathscr{D}_{\boldsymbol{\nu}_n})[f](x)=(1-\phi(x)^2)\big(\phi(x)(1-\phi(x)^2)\big)^{\frac{n(n+1)}{2}}\frac{W[\psi_{\nu_1},\dots,\psi_{\nu_n},f](\phi(x))}{\prod_{i=1}^n\psi_{\nu_i}(\phi(x))}$$

by definition. On the other hand, by applying Lemma 1.4.5 we get

$$\begin{aligned}W[\psi_{\nu_1},\dots,\psi_{\nu_n},f](\phi(x))&=(1-\phi(x)^2)^{-n-1}\frac{W\big[\mathscr{B}[\psi_{\nu_1}],\dots,\mathscr{B}[\psi_{\nu_n}],\mathscr{B}[f]\big](x)}{(\phi'(x))^{\frac{n(n+1)}{2}}}\\&=(1-\phi(x)^2)^{-n-1}x^{\frac{n(n+1)}{2}}\frac{W\big[x^{\nu_1},\dots,x^{\nu_n},\mathscr{B}[f](x)\big]}{\big(\phi(x)(1-\phi(x)^2)\big)^{\frac{n(n+1)}{2}}},\end{aligned}$$

where in the second equality we use (a) and that $x\phi'(x)=\phi(x)(1-\phi(x)^2)$. Consequently, since on account of (a) we have $\psi_\nu\big(\phi(x)\big)=\frac{x^\nu}{1-\phi(x)^2}$, the combination of the two previous indented equalities gives

$$(\mathscr{B}\circ\mathscr{D}_{\boldsymbol{\nu}_n})[f](x)=\frac{W[x^{\nu_1},\dots,x^{\nu_n},\mathscr{B}[f](x)]}{x^{\sum_{i=1}^n(\nu_i-i)}}=(\mathscr{L}_{\boldsymbol{\nu}_n}\circ\mathscr{B})[f](x),$$

as desired. Let us turn now to the proof of (c). Take any $s\in(0,1)$ and note that if h analytic on $[0,1)$, then the change of variable $u=s\sin\theta$ gives

$$\mathscr{F}[h](s)=\int_0^{\frac{\pi}{2}}h(s\sin\theta)d\theta=\int_0^s\frac{h(u)}{\sqrt{s^2-u^2}}du. \tag{2.9}$$

If f is any analytic function on $[0,1)$, then performing the change of variable $z=\phi(x)$ it follows that

$$\begin{aligned}\int_0^s\frac{f(z)}{\sqrt{s^2-z^2}}dz&=\frac{1}{\sqrt{1-s^2}}\int_0^{\frac{s}{\sqrt{1-s^2}}}\frac{(f\circ\phi)(x)}{1+x^2}\frac{dx}{\sqrt{\frac{s^2}{1-s^2}-x^2}}\\&=\frac{1}{\sqrt{1-s^2}}\int_0^{\frac{s}{\sqrt{1-s^2}}}\mathscr{B}[f](x)\frac{dx}{\sqrt{\frac{s^2}{1-s^2}-x^2}}.\end{aligned}$$

Then, by applying above the equality in (2.9) with $h=f$ and $h=\mathscr{B}[f]$, we get

$$\mathscr{F}[f](s)=\frac{1}{\sqrt{1-s^2}}(\mathscr{F}\circ\mathscr{B})[f]\left(\frac{s}{\sqrt{1-s^2}}\right).$$

Finally the composition with $\mathscr{B}$ on both sides of this equality and an easy computation yields to

$$\begin{aligned}(\mathscr{B}\circ\mathscr{F})[f](s)&=(1-\phi(s)^2)\frac{1}{\sqrt{1-\phi(s)^2}}(\mathscr{F}\circ\mathscr{B})[f]\left(\frac{\phi(s)}{\sqrt{1-\phi(s)^2}}\right)\\&=(1+s^2)^{-\frac{1}{2}}(\mathscr{F}\circ\mathscr{B})[f](s),\end{aligned}$$

which shows (c). Finally let us prove (d). If f is an analytic function on $[0,1)$, then by means of the change of variable $z=\phi(x)$ once again we get

$$N_n[f]=\int_0^1 \frac{f(z)}{\sqrt{1-z^2}}\left(\frac{z}{\sqrt{1-z^2}}\right)^{2n-2}dz=\int_0^{+\infty}\frac{(f\circ\phi)(x)}{1+x^2}x^{2n-2}dx=M_n\big[\mathscr{B}[f]\big],$$

as desired. This completes the proof of the result. ■

Lemma 2.4.16. *Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions on $[0,1)$. Then $\{f_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $z=1$ by $\alpha(\mu)$ if and only if $\{\mathscr{B}[f_\mu]\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $+\infty$ by $2\alpha(\mu)-2$.*

Proof. By definition, $\mathscr{B}[f_\mu](x)=\frac{1}{1+x^2}f_\mu(\phi(x))$ with $\phi(x)=\frac{x}{\sqrt{1+x^2}}$. Therefore, for a given $\hat\mu\in\Lambda$,

$$\begin{aligned}\lim_{(x,\mu)\to(+\infty,\hat\mu)}\frac{\mathscr{B}[f_\mu](x)}{x^{2\alpha(\mu)-2}}&=\lim_{(x,\mu)\to(+\infty,\hat\mu)}\frac{f_\mu(\phi(x))}{(1+x^2)\,x^{2\alpha(\mu)-2}}=\lim_{(x,\mu)\to(+\infty,\hat\mu)}\frac{f_\mu(\phi(x))}{(1+x^2)^{\alpha(\mu)}}\\&=\lim_{(z,\mu)\to(1,\hat\mu)}\frac{f_\mu(z)}{(1+\phi^{-1}(z)^2)^{\alpha(\mu)}}=\lim_{(z,\mu)\to(1,\hat\mu)}f_\mu(z)(1-z^2)^{\alpha(\mu)},\end{aligned}$$

where we used that $\phi^{-1}(z)=\frac{z}{\sqrt{1-z^2}}$. Since the first limit is different from zero if and only if the last one is different from zero, the result follows. ■

We shall bound the criticality at the outer boundary by means of the following result.

Lemma 2.4.17. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic differential systems such that $\mu\longmapsto h_0(\mu)$ is continuous on Λ. Assume that there exist $n\geqslant 1$ continuous functions $\nu_1,\nu_2\dots,\nu_n$ in a neighbourhood of some fixed $\hat\mu\in\Lambda$ and an analytic non-vanishing function f on $(0,1)$ such that*

$$\lim_{z\longrightarrow 1}(1-z)^{\nu_n(\mu)}W\left[\psi_{\nu_1(\mu)}(z),\dots,\psi_{\nu_{n-1}(\mu)}(z),f(z)T'_\mu(z^2h_0(\mu))\right]=\Delta(\mu),$$

uniformly in $\mu\approx\hat\mu$, and $\Delta(\hat\mu)\neq 0$. Then $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\leqslant n-1$.

Proof. Note that the functions $\nu_1,\nu_2\dots,\nu_{n-1}$ must be pairwise distinct at $\mu=\hat\mu$ because $\Delta(\hat\mu)\neq 0$. Consequently, since $\mathscr{B}[\psi_{\nu_i(\mu)}](x)=x^{\nu_i(\mu)}$ due to (a) in Lemma 2.4.15, by applying Lemmas 1.4.5 and 1.4.7 we can assert that $W[\psi_{\nu_1(\mu)},\dots,\psi_{\nu_k(\mu)}](z)\neq 0$ for all $z\in(0,1)$ and $\mu\approx\hat\mu$, $k=1,2,\dots,n-1$. On the other hand, by the uniformity of the limit as $z\longrightarrow 1$ and the hypothesis $\Delta(\hat\mu)\neq 0$, there exist $\varepsilon>0$ and a neighbourhood U of $\hat\mu$ such that

$$W\left[\psi_{\nu_1(\mu)}(z),\dots,\psi_{\nu_{n-1}(\mu)}(z),f(z)T'_\mu(z^2h_0(\mu))\right]\neq 0 \text{ for all } z\in(1-\varepsilon,1) \text{ and } \mu\in U.$$

Accordingly, by Lemma 1.4.3, $\big(\psi_{\nu_1(\mu)}(z),\ldots,\psi_{\nu_{n-1}(\mu)}(z), f(z)T'_\mu(z^2h_0(\mu))\big)$ is an ECT-system on $(1-\varepsilon,1)$ for all $\mu\in U$. In particular, since f is a unity, there exists $\delta>0$ such that $T'_\mu(h)$ has no more than $n-1$ zeros on $\big(h_0(\mu)-\delta, h_0(\mu)\big)$, counted with multiplicities, for all $\mu\in U$. We claim that this implies $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\leqslant n-1$, see Definition 2.2.2. To show this notice first that, by Remark 2.2.5, the outer boundary of the period annulus varies continuously. Suppose, by contradiction, that there exist n sequences $\{\gamma^k_{\mu_i}\}_{i\in\mathbb{N}}$, $k=1,2,\ldots,n$, where each $\gamma^k_{\mu_i}$ is a critical periodic orbit of X_{μ_i}, such that $\mu_i\to\hat\mu$ and $d_H(\gamma^k_{\mu_i},\Pi_{\hat\mu})\to 0$ as $i\to+\infty$. Then, due to

$$d_H(\gamma^k_{\mu_i},\Pi_{\mu_i})\leqslant d_H(\gamma^k_{\mu_i},\Pi_{\hat\mu})+d_H(\Pi_\mu,\Pi_{\hat\mu}),$$

we have that $d_H(\gamma^k_{\mu_i},\Pi_{\mu_i})$ tends to zero as $i\to+\infty$. This contradicts that, for all $\mu\in U$, $T'_\mu(h)$ has no more than $n-1$ isolated zeroes on $\big(h_0(\mu)-\delta,h_0(\mu)\big)$. So the claim is true and the results follows. ■

Lemma 2.4.18. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of potential analytic systems verifying* **(H)** *such that $h_0(\mu)<+\infty$ for all $\mu\in\Lambda$, and let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions which is continuously quantifiable in Λ at $x=x_r(\mu)$ (respectively, $x=x_\ell(\mu)$) by $\alpha(\mu)$ with limit $a(\mu)$. Assume that the family $\{h_0(\mu)-V_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $x=x_r(\mu)$ (respectively, $x=x_\ell(\mu)$) by $\beta(\mu)$ with limit $b(\mu)$. Then, the family $\big\{(f_\mu\circ g_\mu^{-1})(z\sqrt{h_0(\mu)})\big\}_{\mu\in\Lambda}$ is continuously quantifiable at $z=1$ (respectively, $z=-1$) by $-(\alpha/\beta)(\mu)$ with limit $\big(a\,(2h_0/b)^{\alpha/\beta}\big)(\mu)$.*

Proof. We show the result for $z=1$ (the case $z=-1$ follows exactly the same way). By Lemma 2.2.8, we know that $g_\mu^{-1}(z\sqrt{h_0(\mu)})$ tends to $x_r(\mu)$ uniformly on μ as $z\longrightarrow 1$. Therefore, since $g_\mu^2=V_\mu$,

$$\lim_{z\to 1}\frac{(f_\mu\circ g_\mu^{-1})\big(z\sqrt{h_0(\mu)}\big)}{(1-z^2)^{\left(\frac{\alpha}{\beta}\right)(\mu)}}=\lim_{x\to x_r(\mu)}\frac{h_0(\mu)^{\left(\frac{\alpha}{\beta}\right)(\mu)}f_\mu(x)}{\big(h_0(\mu)-V_\mu(x)\big)^{\left(\frac{\alpha}{\beta}\right)(\mu)}}=\big(a\,(h_0/b)^{\alpha/\beta}\big)(\mu)$$

uniformly on μ. Taking any $\hat\mu\in\Lambda$, this shows that

$$\lim_{(z,\mu)\to(1,\hat\mu)}\frac{(f_\mu\circ g_\mu^{-1})\big(z\sqrt{h_0(\mu)}\big)}{(1-z)^{\left(\frac{\alpha}{\beta}\right)(\mu)}}=\big(a\,(2h_0/b)^{\alpha/\beta}\big)(\hat\mu),$$

and so the result follows. ■

Next one is a non-parametric result and so we omitted the dependence on μ for the sake of shortness.

Lemma 2.4.19. *Let f be an analytic function on $[0,1)$, $\nu_1,\nu_2,\ldots,\nu_n\in\mathbb{R}$ and $\ell\in\mathbb{N}$. Let us assume that $\mathscr{D}_{\boldsymbol{\nu}_{n-1}}[f](z)$ is quantifiable at $z=1$ by ξ. If $\xi<3/2-\ell$, then*

$$N_\ell\big[\mathscr{D}_{\boldsymbol{\nu}_n}[f]\big]=c_n(1-2\ell-\nu_n)N_\ell\big[\mathscr{D}_{\boldsymbol{\nu}_{n-1}}[f]\big],$$

where $c_1:=1$ and $c_n:=\prod_{i=1}^{n-1}(\nu_n-\nu_i)$ for $n\geqslant 2$.

Proof. Due to $f \in \mathscr{C}^\omega([0,1))$, from the definition of $\mathscr{B}$ it follows $\mathscr{B}[f] \in \mathscr{C}^\omega([0,+\infty))$. By Lemma 2.4.16, the function $(\mathscr{B} \circ \mathscr{D}_{\boldsymbol{\nu}_{n-1}})[f]$ is quantifiable at $+\infty$ by $2\xi - 2 < 1 - 2\ell$. Thus, since $\mathscr{B} \circ \mathscr{D}_{\boldsymbol{\nu}_{n-1}} = \mathscr{L}_{\boldsymbol{\nu}_{n-1}} \circ \mathscr{B}$ by (b) in Lemma 2.4.15, by applying Lemma 1.4.9 we can assert that

$$M_\ell\big[(\mathscr{L}_{\boldsymbol{\nu}_n} \circ \mathscr{B})[f]\big] = c_n(1 - 2\ell - \nu_n) M_\ell\big[(\mathscr{L}_{\boldsymbol{\nu}_{n-1}} \circ \mathscr{B})[f]\big].$$

Now the result follows by using (b) and (d) in Lemma 2.4.15. ∎

The following is our main result in order to study the criticality of the outer boundary in case that its energy level is finite. As usual we point out that, in its statement, the assumptions requiring the existence of functions $\nu_1, \nu_2, \ldots, \nu_n$ for $n = 0$ and that $N_1 \equiv N_2 \equiv \ldots \equiv N_{j-1} \equiv 0$ for $j = 1$ are void.

Theorem E. *Let $\{X_\mu\}_{\mu \in \Lambda}$ be a family of potential analytic systems verifying* **(H)** *such that $h_0(\mu) < +\infty$ for all $\mu \in \Lambda$. Assume that there exist $n \geqslant 0$ continuous functions $\nu_1, \nu_2, \ldots, \nu_n$ in a neighbourhood of some fixed $\hat{\mu} \in \Lambda$ such that the family*

$$\big\{(\mathscr{D}_{\boldsymbol{\nu}_n(\mu)} \circ \mathcal{P})\big[z\sqrt{h_0(\mu)}(g_\mu^{-1})''(z\sqrt{h_0(\mu)})\big]\big\}_{\mu \in \Lambda}$$

is continuously quantifiable in Λ at $z = 1$ by $\xi(\mu)$. For each $i \in \mathbb{N}$, let $N_i(\mu)$ be the i-th momentum of $(\mathscr{D}_{\boldsymbol{\nu}_n(\mu)} \circ \mathcal{P})\big[z\sqrt{h_0(\mu)}(g_\mu^{-1})''(z\sqrt{h_0(\mu)})\big]$, whenever it is well defined. The following assertions hold:

(a) If $\xi(\hat{\mu}) > \frac{1}{2}$, then $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant n$.

(b) If $\xi(\hat{\mu}) < \frac{1}{2}$, let $m \in \mathbb{N}$ be such that $\xi(\hat{\mu}) + m \in \big[\frac{1}{2}, \frac{3}{2}\big)$. Then $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant n$ in case that

   (b1) either $N_1 \equiv N_2 \equiv \ldots \equiv N_{j-1} \equiv 0$ and $N_j(\hat{\mu}) \neq 0$ for some $j \in \{1, 2, \ldots, m\}$,

   (b2) or $N_1 \equiv N_2 \equiv \ldots \equiv N_m \equiv 0$ and $\xi(\hat{\mu}) + m \notin \big\{\frac{1}{2}, 1\big\}$.

Finally the family $\big\{(\mathscr{D}_{\boldsymbol{\nu}_n(\mu)} \circ \mathcal{P})\big[z\sqrt{h_0(\mu)}(g_\mu^{-1})''(z\sqrt{h_0(\mu)})\big]\big\}_{\mu \in \Lambda}$ is continuously quantifiable at $z = 1$ by $\xi(\mu) = -\min\big\{\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\mu), \big(\frac{\alpha_r}{\beta_r}\big)(\mu)\big\} - \frac{1}{2}\sum_{i=1}^n \nu_i(\mu) - \frac{n(n+1)}{2} + 1$ if the following conditions are verified:

(i) $\{h_0(\mu) - V_\mu\}_{\mu \in \Lambda}$ is continuously quantifiable at $x_\ell(\mu)$ by $\beta_\ell(\mu)$ and at $x_r(\mu)$ by $\beta_r(\mu)$ with limits $b_\ell(\mu)$ and $b_r(\mu)$, respectively,

(ii) setting $\mathscr{R}_\mu := \frac{(V_\mu')^2 - 2V_\mu V_\mu''}{(V_\mu')^3}$, the function

$$x \longmapsto V'(x)^{-\frac{n(n+1)}{2}} W\left[\left(\frac{V_\mu}{h_0(\mu) - V_\mu}\right)^{\frac{\nu_1(\mu)}{2}}, \ldots, \left(\frac{V_\mu}{h_0(\mu) - V_\mu}\right)^{\frac{\nu_n(\mu)}{2}}, (h_0(\mu) - V_\mu)\, V_\mu^{\frac{1}{2}} \mathscr{R}_\mu\right](x)$$

is continuously quantifiable at $x_\ell(\mu)$ by $\alpha_\ell(\mu)$ and at $x_r(\mu)$ by $\alpha_r(\mu)$ with limits $a_\ell(\mu)$ and $a_r(\mu)$, respectively, (*iii*) *and either* $\frac{\alpha_\ell}{\beta_\ell}(\mu) \neq \frac{\alpha_r}{\beta_r}(\mu)$ *or, otherwise,* $\big(a_r(b_r)^{-\frac{\alpha_r}{\beta_r}} + (-1)^{\frac{n(n+1)}{2}} a_\ell(b_\ell)^{-\frac{\alpha_\ell}{\beta_\ell}}\big)(\mu) \neq 0.$

Proof. Let us set $f_\mu(z) := \mathcal{P}\big[z\sqrt{h_0(\mu)}(g_\mu^{-1})''(z\sqrt{h_0(\mu)})\big]$ for shortness. Then, the hypothesis **(H)** and Lemma 2.2.7 guarantee that $\{f_\mu\}_{\mu\in\Lambda}$ is a continuous family of analytic functions on $[0,1)$. From the expression of the period function in (2.1) it turns out that

$$T'(h) = \frac{1}{\sqrt{2}h}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\big(g^{-1}\big)''(u)\big|_{u=\sqrt{h}\sin\theta}\,d\theta.$$

Thus, on account of the definition of $\mathscr{F}$ in Definition 1.2.1, the obvious rescaling yields to the identity

$$\mathscr{F}[f_\mu](z) = \sqrt{2}z^2h_0(\mu)T_\mu'\big(z^2h_0(\mu)\big), \text{ for all } z\in(0,1). \tag{2.10}$$

So we must show that there exist $\varepsilon > 0$ and a neighbourhood U of $\hat{\mu}$ such that $\mathscr{F}[f_\mu](z)$ has at most n zeros for $z\in(1-\varepsilon,1)$, multiplicities taking into account, for all $\mu\in U$. Recall that, by (b) in Lemma 2.4.15, $\mathscr{B}\circ\mathscr{D}_{\boldsymbol{\nu}_n} = \mathscr{L}_{\boldsymbol{\nu}_n}\circ\mathscr{B}$. This will allow us to transfer the assumptions on the family $\{\mathscr{D}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\}_{\mu\in\Lambda}$, which is defined on $[0,1)$, to another family defined on $[0,+\infty)$ and then apply Theorem B as we did in the proof of Theorem C. With this aim in view we first note that

$$\begin{aligned}(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F}\circ\mathscr{B})[f_\mu](x) &= \mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\big[\sqrt{1+x^2}(\mathscr{B}\circ\mathscr{F})[f_\mu](x)\big]\\ &= (\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{B})\big[(1-x^2)^{-\frac{1}{2}}\mathscr{F}[f_\mu](x)\big]\\ &= (\mathscr{B}\circ\mathscr{D}_{\boldsymbol{\nu}_n(\mu)})\big[(1-x^2)^{-\frac{1}{2}}\mathscr{F}[f_\mu](x)\big],\end{aligned}$$

where in the first equality we use Lemma 2.4.15 (c), in the second equality we use the identity $\sqrt{1+x^2}\mathscr{B}[\phi](x) = \mathscr{B}[(1-x^2)^{-\frac{1}{2}}\phi(x)]$ with $\phi = \mathscr{F}[f_\mu]$, and in the third equality we use Lemma 2.4.15 (b). Note that $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F}\circ\mathscr{B})[f_\mu]\big\}_{\mu\in\Lambda}$ is a continuous family of analytic functions on $[0,+\infty)$.

We claim that if $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F}\circ\mathscr{B})[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable at $+\infty$ in $\hat{\mu}$, then the criticality of X_μ at the outer boundary of the period annulus is at most n. Indeed, to show this suppose that the quantifier is $\eta(\mu)$. Then, on account of the previous equality and Lemma 2.4.16, $\big\{\mathscr{D}_{\boldsymbol{\nu}_n(\mu)}\big[(1-x^2)^{-\frac{1}{2}}\mathscr{F}[f_\mu](x)\big]\big\}_{\mu\in\Lambda}$ is continuously quantifiable at $z=1$ in $\hat{\mu}$ by $\frac{1}{2}\eta(\mu)+1$, i.e.

$$\lim_{(x,\mu)\to(1,\hat{\mu})}(1-x)^{\frac{1}{2}\eta(\mu)+1}\mathscr{D}_{\boldsymbol{\nu}_n(\mu)}\big[(1-x^2)^{-\frac{1}{2}}\mathscr{F}[f_\mu](x)\big] \neq 0.$$

Thus, according to the definition of $\mathscr{D}_{\boldsymbol{\nu}_n}$ in Definition 2.4.12,

$$\lim_{(x,\mu)\to(1,\hat{\mu})}(1-x)^{\frac{\eta(\mu)}{2}+1}\frac{W\big[\psi_{\nu_1(\mu)}(x),\dots,\psi_{\nu_n(\mu)}(x),(1-x^2)^{-\frac{1}{2}}\mathscr{F}[f_\mu](x)\big]}{\big(x(1-x^2)\big)^{-\frac{n(n+1)}{2}}\prod_{i=1}^n\psi_{\nu_i(\mu)}(x)} \neq 0,$$

which, due to $\psi_\nu(x)=\frac{1}{1-x^2}\left(\frac{x}{\sqrt{1-x^2}}\right)^\nu$, easily implies that

$$\lim_{(x,\mu)\to(1,\hat\mu)}(1-x)^{\kappa(\mu)}W\big[\psi_{\nu_1(\mu)}(x),\ldots,\psi_{\nu_n(\mu)}(x),(1-x^2)^{-\frac12}\mathscr{F}[f_\mu](x)\big]\neq 0,$$

where $\kappa(\mu):=\frac12\big(\eta(\mu)+(n+1)(n+2)+\sum_{i=1}^n\nu_i(\mu)\big)$. Now the claim follows by applying Lemma 2.4.17 and taking (2.10) into account.

We are now in position to prove (a) and (b) in the first part of the statement. To this end recall that, by assumption, the family $\big\{\mathscr{D}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $z=1$ by $\xi(\mu)$. On the other hand, by (b) in Lemma 2.4.15, we have that $(\mathscr{B}\circ\mathscr{D}_{\boldsymbol{\nu}_n(\mu)})[f_\mu]=(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{B})[f_\mu]$. Hence by applying Lemma 2.4.16 we can assert that the family $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{B})[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in Λ at $+\infty$ by $2\xi(\mu)-2$. (This is precisely the family defined on $[0,+\infty)$ that in the beginning of the proof we refer to.)

- If $\xi(\hat\mu)>\frac12$, then $2\xi(\hat\mu)-2>-1$ and so Theorem B applied to $\{\mathscr{B}[f_\mu]\}_{\mu\in\Lambda}$ guarantees that $\big\{(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{F}\circ\mathscr{B})[f_\mu]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in a neighbourhood of $\hat\mu$ at $+\infty$ by $2\xi(\mu)-2$. This, thanks to the previous claim, proves that $\operatorname{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\leqslant n$ and hence (a) follows.
- To show (b) we use that, by (b) and (d) in Lemma 2.4.15,

  $$N_\ell\big[\mathscr{D}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big]=M_\ell\big[(\mathscr{B}\circ\mathscr{D}_{\boldsymbol{\nu}_n(\mu)})[f_\mu]\big]=M_\ell\big[(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}\circ\mathscr{B})[f_\mu]\big].$$

  Then the result follows straightforward by applying Theorem B and taking the previous claim into account again.

Let us turn next to the proof of the second part of the result. For the sake of shortness let us denote $\Delta(\mu):=\sum_{i=1}^n(\nu_i(\mu)-i)$. For the same reason, from now on we omit the dependence on μ when it is not essential. That being said note that,

$$\psi_{\nu_i}(z)=\left(\frac{h_0}{h_0-V_\mu}\left(\frac{V_\mu}{h_0-V_\mu}\right)^{\frac{\nu_i}{2}}\right)\big(g_\mu^{-1}(z\sqrt{h_0})\big)\text{ for all }z\in(0,1),$$

due to $V_\mu(g_\mu^{-1}(z))=z^2$. In addition, since $\big(g_\mu^{-1}(z)\big)''=2\mathscr{R}_\mu\big(g_\mu^{-1}(z)\big)$, we have that f_μ is the even part of the function $2(g_\mu\mathscr{R}_\mu)\big(g_\mu^{-1}(z\sqrt{h_0})\big)$. Consequently, taking Lemma 1.4.5 also into account, some computations show that

$$\mathscr{D}_{\boldsymbol{\nu}_n(\mu)}\big[z\sqrt{h_0}(g_\mu^{-1})''(z\sqrt{h_0})\big]=2^{1+\frac{n(n+1)}{2}}h_0^{-\frac{n(n+1)}{2}}S_\mu\big(g_\mu^{-1}(z\sqrt{h_0})\big),\text{ for all }z\in(0,1),$$

where

$$S_\mu(x):=\frac{W\Big[\Big(\frac{V_\mu}{h_0-V_\mu}\Big)^{\frac{\nu_1}{2}},\ldots,\Big(\frac{V_\mu}{h_0-V_\mu}\Big)^{\frac{\nu_n}{2}},(h_0-V_\mu)V_\mu^{\frac12}\mathscr{R}_\mu\Big](x)}{(h_0-V_\mu(x))^{-\frac{\Delta}{2}-\frac{3n(n+1)}{4}+1}V_\mu(x)^{-\frac{n(n+1)}{4}+\frac{\Delta}{2}}(V_\mu'(x))^{\frac{n(n+1)}{2}}}.$$

Similarly, due to $g(z) = -\sqrt{V(z)}$ for $z < 0$, we have that

$$\mathscr{D}_{\boldsymbol{\nu}_n(\mu)}\big[-z\sqrt{h_0}(g_\mu^{-1})''(-z\sqrt{h_0})\big] = -(-2)^{1+\frac{n(n+1)}{2}} h_0^{-\frac{n(n+1)}{2}} S_\mu\big(g_\mu^{-1}(-z\sqrt{h_0})\big),$$

for all $z \in (0,1)$. On account of the assumptions in (i) and (ii), we can assert that $\{S_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at x_ℓ by γ_ℓ and at x_r by γ_r, with limits c_ℓ and c_r, respectively, where

$$\begin{aligned}
\gamma_\ell &:= \alpha_\ell + \beta_\ell\left(\frac{\Delta}{2} + \frac{3n(n+1)}{4} - 1\right)\\
\gamma_r &:= \alpha_r + \beta_r\left(\frac{\Delta}{2} + \frac{3n(n+1)}{4} - 1\right)\\
c_\ell &:= a_\ell b_\ell^{\frac{\Delta}{2}+\frac{3n(n+1)}{4}-1} h_0^{\frac{n(n+1)}{4}-\frac{\Delta}{2}}\\
c_r &:= a_r b_r^{\frac{\Delta}{2}+\frac{3n(n+1)}{4}-1} h_0^{\frac{n(n+1)}{4}-\frac{\Delta}{2}}
\end{aligned}$$

Then, by Lemma 2.4.18, some computations show that $\big\{(S_\mu \circ g_\mu^{-1})(z\sqrt{h_0})\big\}_{\mu\in\Lambda}$ is continuously quantifiable at $z = -1$ by $-\frac{\gamma_\ell}{\beta_\ell}$ and at $z = 1$ by $-\frac{\gamma_r}{\beta_r}$, with limits $c_\ell(\frac{2h_0}{b_\ell})^{\frac{\gamma_\ell}{\beta_\ell}}$ and $c_r(\frac{2h_0}{b_r})^{\frac{\gamma_r}{\beta_r}}$, respectively. Accordingly, by the assumption in (iii), we have that the family $\big\{(\mathscr{D}_{\boldsymbol{\nu}_n(\mu)} \circ \mathcal{P})\big[z\sqrt{h_0}(g_\mu^{-1})''(z\sqrt{h_0})\big]\big\}_{\mu\in\Lambda}$ is continuously quantifiable at $z = 1$ by $\xi = \max\big\{-\frac{\alpha_\ell}{\beta_\ell}, -\frac{\alpha_r}{\beta_r}\big\} - \frac{\Delta}{2} - \frac{3n(n+1)}{4} + 1$. This shows the second assertion and completes the proof. ■

CHAPTER 3

Study of the period function of a two-parameter family of centers

In this chapter we study the period function of the center at the origin of the biparametric family of planar potential systems $\ddot{x} = (1+x)^p - (1+x)^q$, with $p, q \in \mathbb{R}$ and $p > q$. We prove four independent results. The first one establishes some regions in the parameter space where the corresponding center has a monotonous period function. This result extends the previous ones by Miyamoto and Yagasaki for the case $q = 1$. The second result deals with the bifurcation of critical periodic orbits from the center. The third result is addressed to the critical periodic orbits that bifurcate from the period annulus of each one of the three isochronous centers in the family when perturbed by means of a one-parameter deformation. The fourth result is concerned with the bifurcation of critical periodic orbits from the outer boundary of the period annulus. These four results lead us to propose a conjectural bifurcation diagram for the global behaviour of the period function of the family.

3.1 Introduction

In this chapter we consider the two-parameter family of potential differential systems given by

$$X_\mu \quad \begin{cases} \dot{x} = -y, \\ \dot{y} = (1+x)^p - (1+x)^q, \end{cases} \tag{3.1}$$

where $\mu := (q,p)$ with $p, q \in \mathbb{R}$. This is a well defined analytic differential system on the half plane $\{x > -1\}$. The singular point at the origin is a non-degenerated center if $p > q$ and a hyperbolic saddle if $p < q$. Our goal in this chapter is to provide a global study of the qualitative properties of the period function associated to the center, so we will consider X_μ with $\mu \in \Lambda := \{(q,p) \in \mathbb{R}^2 : p > q\}$.

We became interested in this problem because of the previous results by Miyamoto and Yagasaki on the issue. Both authors proved, see [47], that the period function is monotonous when $q = 1$ and $p \in \mathbb{N}$. Later Yagasaki improved the result showing in [65] the monotonicity of the period function for $q = 1$ and any $p \in \mathbb{R}$ with $p > 1$. As it often occurs, they came across the period function when studying the solutions of an elliptic Neumann problem. In [65] the author shows that the monotonicity of the period function of system (3.1) with $q = 1$ is used to determine the global bifurcation diagram of interior single-peak solution in the Neumann boundary problem $\ddot{x} + \lambda(-x + x^p) = 0$ in the interval $(-1, 1)$ with boundary conditions $\dot{x}(\pm 1) = 0$, where λ is considered as a control parameter. This Neumann boundary problem appears in the literature as stationary problem for the shadow system of the Gierer-Meinhardt model on biological pattern formations [23] and the Keller-Segel model on the one-dimensional chemotaxis aggregation [30]. We refer the reader to [32, 49] for details.

Concerning the behaviour of the period function of the family $\{X_\mu\}_{\mu\in\Lambda}$ in (3.1), we propose the following conjectural bifurcation diagram. Figure 3.1 displays the conjecture, where the doted curve is given by the zero set of Δ_1 and the bold curve is Γ_B, defined in (3.4) and (3.7) respectively.

Conjecture 3.1.1. *The bifurcation diagram of the period function of $\{X_\mu\}_{\mu\in\Lambda}$ in (3.1) consists in the union of the curves Δ_1 and Γ_B, that correspond respectively to the local bifurcation values at the inner and outer boundaries of the period annulus. These curves split the parameter space Λ in eight connected components, and the period function of $\{X_\mu\}_{\mu\in\Lambda}$ is either monotonous or has exactly one critical periodic orbit according to Figure 3.1. Moreover, the intersection of the curves Δ_1 and Γ_B correspond to isochronous centers.*

This conjecture claims in particular, and it constitutes the key point, that there are no parameters for which two critical periodic orbits collide disappearing in the interior of the period annulus.

This chapter is devoted to prove some results concerning the previous conjecture. The first one, Theorem F, establishes some regions in the parameter space where the corresponding center has a monotonous period function. This result extends the previous ones by Miyamoto and Yagasaki [47, 65]. Concerning the conjecture, Theorem F covers a big part of the white region in Figure 3.1, where the monotonicity of the period function is conjectured. The second result of this chapter, Theorem G, deals with the bifurcation of critical periodic orbits from the inner boundary of $\mathscr{P}$, i.e., the center. In this result we prove that the curve $\{\Delta_1 = 0\}$ defined in (3.4) consists of local bifurcation values of the period function at the inner boundary of the period annulus, according with Definition 3.3.2, and its criticality is exactly one. The third one, Theorem H, is addressed to the bifurcation of critical periodic orbits from the interior of the period annulus of one of the three isochronous centers of the family under consideration. We prove that at most one critical periodic orbit bifurcates from each of them by means of an analytic one-parametric curve in the parameter space. The fourth result, Theorem I, shows that the parameter in Γ_B defined in (3.7) are local bifurcation values of the period function at the outer boundary according with Definition 2.2.3. Moreover, we prove that the criticality is one for some of the parameters in Γ_B. Finally in Corollary 3.6.1 we show that for parameters belonging to the grey region in Figure 3.1 the period function of X_μ has at least one critical periodic orbit.

The chapter has five additional sections, each one dedicated to state and prove one of the results above. For reader's convenience we advance that the five sections are essentially independent.

3.2 Monotonicity of the period function.

For reader's convenience we recall some notation that we have already introduced in this memoir. For an analytic planar potential system of the form $X = -y\partial_x + V'(x)\partial_y$ with a non-degenerated center at the origin, i.e., $V(0) = V'(0) = 0$ and $V''(0) > 0$, we shall denote the projection of its period annulus $\mathscr{P}$ on the x-axis by $\mathcal{I} = (x_\ell, x_r)$. Thus $x_\ell < 0 < x_r$. The corresponding Hamiltonian function is given by $H(x, y) = \frac{1}{2}y^2 + V(x)$. Then $H(\mathscr{P}) = (0, h_0)$, with $h_0 \in (0, +\infty]$, and in this case we will say that h_0 is the energy level of the outer boundary of $\mathscr{P}$.

Definition 3.2.1. We say that the period function of a center is *monotonous increasing* (respectively, *decreasing*) if there are no critical periodic orbits on $\mathscr{P}$ and, for any two periodic orbits $\gamma_1, \gamma_2 \in \mathscr{P}$ with $\gamma_1 \subset \mathrm{Int}(\gamma_2)$, the period of γ_2 is greater (respectively,

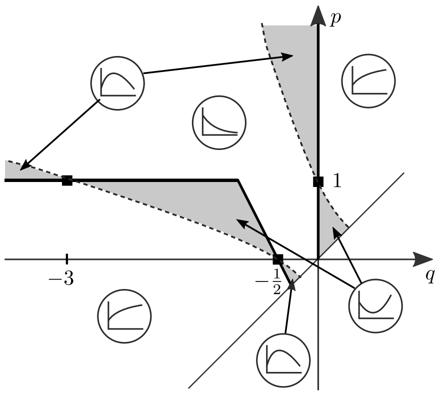

Figure 3.1: Conjectural bifurcation diagram for the period function of the differential system (3.1), where the doted and bold curves consist of local bifurcation values at the inner and outer boundary, respectively. The parameters in the grey region correspond to systems with exactly one critical periodic orbit, whereas parameters in the white region correspond to systems with monotonic period function.

smaller) than the one of γ_1. □

Let us state the main result of this section. To this end we define

$$\Theta(\mu) := 2p^4+p^3(3+4q)+p^2(9q^2+9q-1)+p(4q^3+9q^2+2q-3)+(1+q)^2(2q^2-q-1). \quad (3.2)$$

Then, denoting the light grey region in Figure 3.2 by M_I and the dark grey region by M_D, we will prove the following result:

Theorem F. *The period function of the center at the origin of the potential differential system* (3.1) *is monotonous increasing* (*respectively, decreasing*) *in case that* $\mu \in M_I$ (*respectively,* $\mu \in M_D$)*.*

The proof of Theorem F will be an application of the following monotonicity criterion, see [57], and in its statement we use the notation introduced above.

Theorem 3.2.2 (Schaaf's criterion)**.** *Let* $X = -y\partial_x + V'(x)\partial_y$ *be an analytic potential differential system with a non-degenerated center at the origin and consider its period function* $T(h)$*. Then* $T'(h) > 0$ *for all* $h \in (0, h_0)$ *in case that*

$$(I_1) \qquad 5V'''(x)^2 - 3V''(x)V^{(4)}(x) > 0 \text{ for all } x \in \mathcal{I} \text{ with } V''(x) > 0,$$

and

$$(I_2) \qquad V'(x)V'''(x) < 0 \text{ for all } x \in \mathcal{I} \text{ with } V''(x) = 0.$$

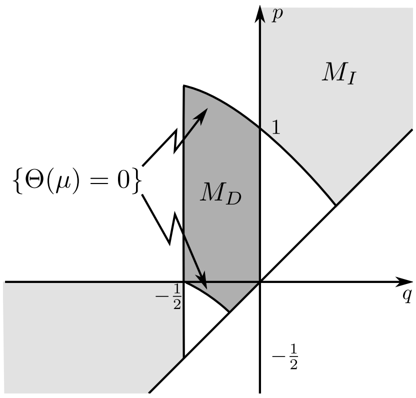

Figure 3.2: Monotonicity regions according to Theorem F.

On the other hand, $T'(h)<0$ for all $h\in(0,h_0)$ in case that

$$(D) \qquad 5V'''(x)^2-3V''(x)V^{(4)}(x)<0 \text{ for all } x\in\mathcal{I} \text{ with } V''(x)\geqslant 0.$$

For the potential differential system (3.1) under consideration we have $\mu=(q,p)$ and

$$V_\mu(x) := \int_1^{x+1}(u^p-u^q)\,du. \tag{3.3}$$

Clearly the origin is a non-degenerated centre for all $\mu\in\Lambda$ because $V''(0)=p-q>0$. Note that $\Lambda=\Lambda_1\cup\Lambda_2\cup\Lambda_3$ with

$$\begin{aligned}\Lambda_1 &:= \{\mu\in\Lambda : -1<q<p\},\\ \Lambda_2 &:= \{\mu\in\Lambda : q\leqslant -1\leqslant p\},\\ \Lambda_3 &:= \{\mu\in\Lambda : q<p<-1\}.\end{aligned}$$

The next result is a straightforward observation and we do not show it for the sake of shortness (see Figure 3.3).

Lemma 3.2.3. *The projection on the x-axis of the period annulus $\mathscr{P}_\mu$ of the center at the origin of (3.1) is $\mathcal{I}_\mu=(-1,\rho(\mu))$ for $\mu\in\Lambda_1$, $\mathcal{I}_\mu=(-1,+\infty)$ for $\mu\in\Lambda_2$ and $\mathcal{I}_\mu=(\rho(\mu),+\infty)$ for $\mu\in\Lambda_3$, where*

$$\rho(\mu) := \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}-1.$$

The key point to apply Schaaf's criterion to the potential differential system X_μ given in (3.1) is that, as one can easily verify, the "test functions" are almost polynomial. Indeed,

$$\begin{aligned}5V_\mu'''(x)^2-3V_\mu''(x)V_\mu^{(4)}(x) &= (1+x)^{2q-4}P_\mu\big((1+x)^{p-q}\big),\\ V_\mu'(x)V_\mu'''(x) &= (1+x)^{2q-2}Q_\mu\big((1+x)^{p-q}\big),\\ V_\mu''(x) &= (1+x)^{q-1}R_\mu\big((1+x)^{p-q}\big),\end{aligned}$$

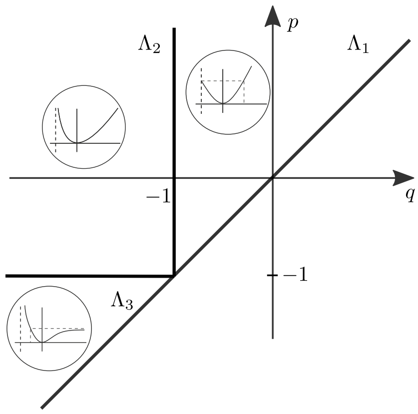

Figure 3.3: Bifurcation diagram of the graph of V_μ.

where

$$P_\mu(z) := (q-1)q^2(1+2q) + pq(3p^2+3q^2-10pq+p+q+2)z + (p-1)p^2(1+2p)z^2,$$
$$Q_\mu(z) := q(q-1) + (p-p^2+q-q^2)z + p(p-1)z^2,$$

and $R_\mu(z) := -q+pz$. Accordingly we get the following result:

Lemma 3.2.4. *The conditions* (I_1), (I_2) *and* (D) *of Schaaf's monotonicity criterion applied to the potential differential system* (3.1) *are equivalent to*

$$(I_1') \qquad P_\mu(z) > 0 \text{ for any } z \in \varphi(\mathcal{I}_\mu) \text{ with } R_\mu(z) > 0,$$
$$(I_2') \qquad Q_\mu(z) < 0 \text{ for any } z \in \varphi(\mathcal{I}_\mu) \text{ with } R_\mu(z) = 0,$$
$$(D') \qquad P_\mu(z) < 0 \text{ for any } z \in \varphi(\mathcal{I}_\mu) \text{ with } R_\mu(z) \geqslant 0,$$

respectively, where $\varphi(x) := (1+x)^{p-q}$.

The next result follows straightforward from Lemma 3.2.3 and so we omit the proof.

Lemma 3.2.5. *Let* $\mu = (q,p) \in \Lambda$ *and define* $L_\mu = \{z \in \varphi(\mathcal{I}_\mu) \colon R_\mu(z) > 0\}$. *Then*

(a) $L_\mu = \left(\frac{q}{p}, \frac{p+1}{q+1}\right)$ *if* $q > 0$,

(b) $L_\mu = \left(0, \frac{p+1}{q+1}\right)$ *if either* $p \geqslant 0$ *and* $-1 < q \leqslant 0$, *or* $p < 0$ *and* $p+q > -1$,

(c) $L_\mu = (0, \infty)$ *if* $p \geqslant 0$ *and* $q < -1$,

(d) $L_\mu = \left(0, \frac{q}{p}\right)$ *if either* $p < 0, q > -1$ *and* $p+q \leqslant -1$, *or* $q < -1$ *and* $-1 < p < 0$,

(e) $L_\mu = \left(\frac{p+1}{q+1}, \frac{q}{p}\right)$ *if* $p < -1$.

In order to be precise in the following results, we introduce at this moment some definitions.

Definition 3.2.6. The *resultant* $R(P,Q)$ of two polynomials $P(x) = a_n x^n + \cdots + a_0$ and $Q(x) = b_m x^m + \cdots + b_0$ is defined by the following determinant

$$R(P,Q) := \begin{vmatrix} a_n & a_{n-1} & \cdots & \cdots & \cdots & a_0 & & & \\ & a_n & a_{n-1} & \cdots & \cdots & \cdots & a_0 & & \\ & & \cdots & & & & & & \\ & & & a_n & a_{n-1} & \cdots & \cdots & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & \cdots & b_0 & & & & \\ & b_m & b_{m-1} & \cdots & \cdots & b_0 & & & \\ & & \cdots & & & & & & \\ & & & \cdots & & & & & \\ & & & & b_m & b_{m-1} & \cdots & \cdots & b_0 \end{vmatrix},$$

where the empty entries of the matrix are filled by zeroes. □

It is well known that the resultant can be also expressed as

$$R(P,Q) := a_n^m b_m^n \prod_{i,j} (\alpha_i - \beta_j)$$

where α_i are the roots of $P(x)$ and β_j the roots of $Q(x)$. We refer the reader to [31] for further details. From the previous expression it is clear that $R(P,Q) = 0$ if and only if P and Q have a common root.

Definition 3.2.7. The *discriminant* $\mathrm{Disc}(P)$ of a polynomial P is defined as

$$\mathrm{Disc}(P) := R(P, P'),$$

where P' denotes the derivative of P. □

Let $\mathcal{Z}(\mu)$ be the number of zeros of a polynomial P_μ in an interval (a_μ, b_μ), where P_μ, a_μ and b_μ depend continuously on the parameter μ. The functions $P_\mu(a_\mu)$, $P_\mu(b_\mu)$ and $\mathrm{Disc}(P_\mu)$ are continuous in μ and the zero level curves of these functions split the parameter space on connected components where $\mathcal{Z}(\mu)$ is constant as the following result establishes.

Proposition 3.2.8. *Let $\mathcal{U} \subset \mathbb{R}^d$ and $\{P_\mu\}_{\mu\in\mathcal{U}}$ be a continuous family of polynomials. Let $\mathcal{Z}(\mu)$ be the number of zeros of P_μ in the interval (a_μ, b_μ), where a_μ and b_μ are continuous functions in $\mathcal{U}$. Let us consider*

$$\mathscr{B} := \{\mu \in \mathcal{U} : P_\mu(a_\mu) P_\mu(b_\mu) \mathrm{Disc}(P_\mu) = 0\}.$$

Then $\mathcal{Z}(\mu)$ is constant in the connected components of $\mathcal{U} \setminus \mathscr{B}$.

Proof. Let Ω be a connected component of $\mathcal{U} \setminus \mathscr{B}$ and fix $\mu_0 \in \Omega$. Let us define the set $R := \{\mu \in \Omega : \mathcal{Z}(\mu) = \mathcal{Z}(\mu_0)\}$. The result will follow once we prove that R is open and closed as a subset of Ω. Then, since Ω is connected and $\mu_0 \in R$, this implies that $R = \Omega$.

Let us prove that R is open in Ω. In this regard we take $\hat{\mu} \in R$ and we denote $B_\delta(\hat{\mu}) := \{\mu \in \mathcal{U} : \|\mu - \hat{\mu}\| < \delta\}$. Since $\hat{\mu} \in \Omega \subset \mathcal{U} \setminus \mathscr{B}$ we have $P_{\hat{\mu}}(a_{\hat{\mu}}) \neq 0$, $P_{\hat{\mu}}(b_{\hat{\mu}}) \neq 0$ and $\mathrm{Disc}(P_{\hat{\mu}}) \neq 0$. Particularly, all the zeros of $P_{\hat{\mu}}$ in $(a_{\hat{\mu}}, b_{\hat{\mu}})$ are simple. By continuity with respect to the parameter, there exists $\delta_1 > 0$ small enough such that for all $\mu \in B_{\delta_1}(\hat{\mu})$ the zeros of P_μ are also simple in (a_μ, b_μ). Moreover, since $P_{\hat{\mu}}(a_{\hat{\mu}}) \neq 0$ and $P_{\hat{\mu}}(b_{\hat{\mu}}) \neq 0$, for a given $\varepsilon > 0$ there exists $\delta_2 > 0$ such that for all $\mu \in B_{\delta_2}(\hat{\mu})$ there are no zeros of the polynomial P_μ in the intervals $[a_\mu - \varepsilon, a_\mu + \varepsilon]$ and $[b_\mu - \varepsilon, b_\mu + \varepsilon]$. Then, taking $\delta := \min\{\delta_1, \delta_2\}$ we have that $\mathcal{Z}(\mu) = \mathcal{Z}(\hat{\mu}) = \mathcal{Z}(\mu_0)$ for all $\mu \in B_\delta(\hat{\mu})$. This proves that R is open. The proof of R is closed follows showing that $\Omega \setminus R$ is open, which can be proved similarly as before. Then R is open and closed and this ends with the proof of the result. ■

The previous proposition deals with a_μ and b_μ continuous functions on the parameters. However, minor modifications show that it is also true when $a_\mu \equiv -\infty$ or $b_\mu \equiv +\infty$. In these cases, $P_\mu(\infty)$ stands for the coefficient of maximum degree of P_μ.

The next result stablish the regions in the parameter space where Schaaf's conditions are verified. The idea is to use the information in Lemma 3.2.5 together with Proposition 3.2.8 and then split the parameter space in connected components where the number of zeros of the polynomials does not change. Then it is enough to check conditions in Lemma 3.2.4 for one parameter in the region under consideration.

Lemma 3.2.9. *The potential differential system* (3.1) *verifies condition* (I_1) *of Schaaf's criterion if* μ *is inside the region* 1, 8, 10, 11, 14 *or* 15 *in Figure 3.4. On the other hand, it verifies condition* (D) *if* μ *is inside the region* 5 *or* 7.

Proof. By applying Lemmas 3.2.4 and 3.2.5, the first assertion is equivalent to require that the quadratic polynomial P_μ is positive on the interval L_μ. For each $\mu \in \Lambda$, let us define $\mathcal{Z}(\mu)$ to be the number of zeros of P_μ inside L_μ counted with multiplicities. The relevant information to study this number is given by the following expressions:

$$
\begin{aligned}
&\mathrm{Disc}(P_\mu) = 3p^2(p-q)^2q^2(3p^2+3q^2-14pq+2p+2q+7),\\
&P_\mu(0) = (q-1)q^2(1+2q),\\
&P_\mu\left(\frac{q}{p}\right) = 5(p-q)^2q^2,\\
&P_\mu(\infty) = (p-1)p^2(1+2p),\\
&P_\mu\left(\frac{p+1}{q+1}\right) = (p-q)^2(1+q)^{-2}\Theta(\mu),
\end{aligned}
$$

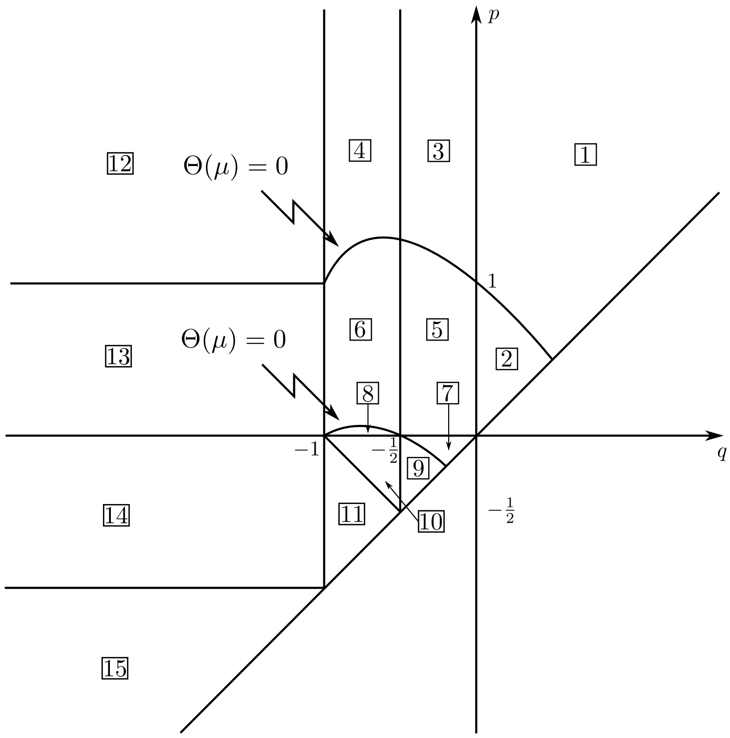

Figure 3.4: Sketch of the regions in the proof of Lemma 3.2.9.

where Θ is defined in (3.2), $\mathrm{Disc}(P_\mu)$ denotes the discriminant of P_μ and $P_\mu(\infty)$ stands for the coefficient of maximum degree of P_μ. The zero level sets of these functions split the parameter space Λ into several connected components. According with Proposition 3.2.8, $\mathscr{Z}(\mu)$ is constant in each connected component. In this regard it is to be pointed out that if $\mathrm{Disc}(P_\mu) = 0$ then one can verify that the corresponding double root is outside L_μ. On account of this, in order to study $\mathscr{Z}(\mu)$ we can rule out the curve $\mathrm{Disc}(P_\mu) = 0$. We obtain in this way 15 connected components, that we display in Figure 3.4. It is clear that if $\mathscr{U}$ is one of these regions and P_μ is positive on L_μ for some $\mu = \hat{\mu} \in \mathscr{U}$, then the same is true for all the parameters $\mu \in \mathscr{U}$. Choosing one parameter inside each one of the 15 regions we prove that this is the case for the regions 1, 8, 10, 11, 14 and 15. This proves the first assertion.

Let us turn now to the second assertion. Thanks to Lemmas 3.2.4 and 3.2.5 again, condition (D) is equivalent to require that P_μ is negative on the interval $\hat{L}_\mu$, where we have $\hat{L}_\mu := L_\mu$ in case that q/p is not an endpoint of L_μ and $\hat{L}_\mu := L_\mu \cup \{q/p\}$ otherwise. (This follows from noting that $R_\mu(z) = 0$ if and only if $z = q/p$.) Arguing as we did with condition (I_1) one can show that this is the case for the parameters inside regions 5 and 7. This proves the result. ∎

Proof of Theorem F. We claim that if $(q, p) \in \Lambda$ verifies $pq > 0$, then the potential differential system under consideration (3.1) satisfies condition (I_2) of Schaaf's criterion.

Indeed, this follows by applying Lemma 3.2.4 and noting that $R_\mu(z) = 0$ if and only if $z = q/p$ and $Q_\mu(q/p) = -q(p-q)^2/p < 0$ when $pq > 0$.

On account of the claim and Lemma 3.2.9, we can assert that the potential differential system (3.1) verifies conditions (I_1) and (I_2) if $\mu = (q,p)$ is inside, see Figure 3.4, the union of the regions 1, 10, 11, 14 and 15, say R_I. Accordingly, by applying Theorem 3.2.2, we can assert that the derivative of the period function is strictly positive in case that $\mu \in R_I$. Note at this point, see also Figure 3.6, that $M_I \setminus R_I$ is the union of three segments, say ℓ_1, ℓ_2 and ℓ_3. Since one can easily verify that (I_1) and (I_2) are fulfilled in $\ell_1 \cup \ell_2 \cup \ell_3$ as well, the result concerning the set M_I follows.

Finally, Lemma 3.2.9 shows that the parameters inside the union of the regions 5 and 7, say R_D, satisfy condition (D) of Schaaf's criterion. Therefore, the derivative of the period function of the center at the origin of system (3.1) is negative for $\mu \in R_D$. Observe that $M_D \setminus R_D$ is the segment $(-1/2, 0) \times \{0\}$. Since one can easily verify that condition (D) also holds for parameters inside this segment, the result concerning M_D follows. This completes the proof of the result. ■

3.3 Criticality at the center.

As we already commented in this memoir, the ultimate goal in the study of the global behaviour of the period function is to decompose the parameter space in such a way the period function is qualitatively the same in each connected component. The main result of this section is addressed to the study of the number of critical periodic orbits of system X_μ in (3.1) that can emerge or disappear from the center itself. In this regard, we introduce precisely the definition of criticality at the inner boundary of the period annulus for a family of centers. In the following definition, d_H stands for the Hausdorff distance between compact sets of $\mathbb{R}^2$ (see Definition 2.2.1).

Definition 3.3.1. Consider a continuous family $\{X_\mu\}_{\mu\in\Lambda}$ of planar analytic vector fields with a center in p_μ and fix some $\hat{\mu} \in \Lambda$. Then, setting

$$N(\delta,\varepsilon) = \sup\{\#\text{ critical periodic orbits } \gamma \text{ of } X_\mu \text{ with } d_H(\gamma, p_{\hat{\mu}}) \leqslant \varepsilon \text{ and } \|\mu - \hat{\mu}\| \leqslant \delta\},$$

we define $\mathrm{Crit}\big((p_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) := \inf_{\delta,\varepsilon} N(\delta,\varepsilon)$ to be the *criticality* of $(p_{\hat{\mu}}, X_{\hat{\mu}})$ with respect to the deformation X_μ. □

Notice that $\mathrm{Crit}\big((p_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big)$ gives the maximal number of critical periodic orbits γ of X_μ that tend to $p_{\hat{\mu}}$ in the Hausdorff sense as $\mu \to \hat{\mu}$.

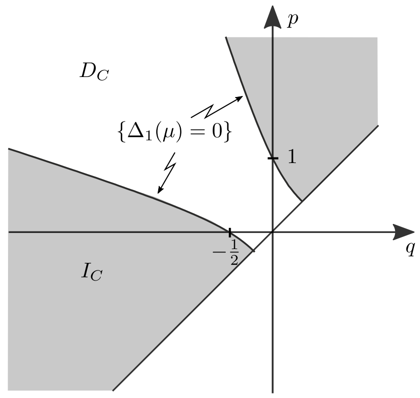

Figure 3.5: Bifurcation diagram of the period function at the inner boundary according to Theorem G.

Definition 3.3.2. We say that $\hat{\mu} \in \Lambda$ is a *local regular value of the period function at the inner boundary* of the period annulus if $\mathrm{Crit}\big((p_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$. Otherwise we say that it is a *local bifurcation value of the period function at the inner boundary.* □

We will prove the following result concerning the bifurcation of critical periodic orbits from the center under consideration, which is depicted in Figure 3.5. In order to state it properly we denote

$$\Delta_1(\mu) = 2p^2 + 2q^2 + 7pq - p - q - 1. \tag{3.4}$$

Theorem G. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of vector fields in (3.1) and consider the period function of the center at the origin. Then the set $\{\mu \in \Lambda : \Delta_1(\mu) \neq 0\}$ consists of local regular values of the period function at the inner boundary of the period annulus. In addition,*

(a) If $\Delta_1(\hat{\mu}) > 0$, then the period function of $X_{\hat{\mu}}$ is increasing near the inner boundary.

(b) If $\Delta_1(\hat{\mu}) < 0$, then the period function of $X_{\hat{\mu}}$ is decreasing near the inner boundary.

Finally if $\Delta_1(\hat{\mu}) = 0$, then $\mathrm{Crit}\big(((0,0), X_{\hat{\mu}}), X_\mu\big) = 1$. In particular, $\hat{\mu}$ is a local bifurcation value of the period function at the inner boundary.

From now on let us consider the set of periodic orbits in $\mathscr{P}_\mu$ parametrized by the energy and let $T_\mu(h)$ the period of the periodic orbit inside the level set $\{\frac{1}{2}y^2 + V_\mu(x) = h\}$. It is well known (see for instance Chicone and Jacobs [11]) that this bifurcation problem can be tackled by studying the solutions of the equation $T_\mu'(h) = 0$ near $h = 0$ as the parameter μ varies. The purpose of this chapter is to study this bifurcation equation for the family under consideration. In the case of a non-degenerated center, the period function $T_\mu(h)$ can be extended analytically to $h = 0$. Then, we can write the Taylor's series of its derivative at $h = 0$ as

$$T'_\mu(h) = a_0(\mu) + a_1(\mu)h + a_2(\mu)h^2 + \cdots,$$

where each function $\mu \mapsto a_k(\mu)$, $k = 1, 2, \ldots$, is analytic and the series converges in some neighbourhood of $h = 0$ uniformly in compact subsets of the parameter space. For a given $\hat{\mu} \in \Lambda$, we want to know the number of isolated zeros of $T'_\mu(h)$ near $h = 0$ for $\mu \approx \hat{\mu}$. There are two different cases to consider. The first one, which is the generic situation, is that for some $n \geqslant 1$,

$$a_0(\hat{\mu}) = a_1(\hat{\mu}) = a_2(\hat{\mu}) = \cdots = a_{n-1}(\hat{\mu}) = 0 \text{ and } a_n(\hat{\mu}) \neq 0.$$

In this case easy considerations show that at most n zeros bifurcate from $h = 0$ as $\mu \approx \hat{\mu}$. The second situation, which correspond to the case when $X_{\hat{\mu}}$ is an isochronous center, is that $a_k(\hat{\mu}) = 0$ for all $k \geqslant 0$. This is a more complicated situation which, in case that the coefficients are polynomial in μ, can be tackled by studying the ideal that generate. To tackle with this bifurcation problem we shall use the techniques developed by Chicone and Jacobs [11] in this regard, which we adapt in Theorem 3.3.7.

The linear part of the system (3.1) under consideration depends on the parameters. This is not convenient since, in this case, the coefficients of the Taylor's development of the period function are not polynomial. For this reason, instead of system X_μ in (3.1) we shall consider the analytic planar potential system

$$\hat{X}_\mu \quad \begin{cases} \dot{u} = -v, \\ \dot{v} = \frac{1}{p-q}\left((1+u)^p - (1+u)^q\right). \end{cases} \tag{3.5}$$

One can verify that if $\mu \in \Lambda$, i.e., if $p - q > 0$, then the coordinate transformation $(x, y) \mapsto (u, v) = (x, \frac{1}{\sqrt{p-q}}y)$ and the constant rescaling of time by $\frac{1}{\sqrt{p-q}}$ brings system (3.1) to system (3.5). This of course guarantees that the properties of the period function that we are interested in do not change at all. Note that the linear part of the differential system (3.5) at the center does not depend on the parameters because, following the obvious notation,

$$\hat{V}_\mu(u) := \frac{1}{p-q}\int_1^{u+1}(s^p - s^q)\,ds = \frac{1}{2}u^2 + \mathrm{o}(u^2).$$

The differential system (3.5) has the additional advantage that it is well-defined for all $(q, p) \in \mathbb{R}^2$, even for the straight line $p = q$, where $\hat{X}_{q,q} = -v\partial_u + (1+u)^q \log(1+u)\partial_v$. Observe in addition the symmetry $\hat{X}_{q,p} = \hat{X}_{p,q}$. Note finally that the projection of the period annulus of the center is the same interval for the differential systems (3.1) and (3.5). Thus, for the sake of simplicity, we shall keep denoting it by $\mathcal{I}_\mu$.

Proposition 3.3.3. *Let $\hat{T}_\mu(h)$ denote the period of the periodic orbit of system* (3.5) *inside the energy level $\{\frac{1}{2}v^2 + \hat{V}_\mu(u) = h\}$. Then, setting $\hat{T}_\mu(0) := 2\pi$, $\hat{T}_\mu(h)$ extends analytically at $h = 0$ and its Taylor development is given by $\hat{T}_\mu(h) = 2\pi + \sum_{i\geqslant 1}\Delta_i(\mu)h^i$ with $\Delta_i \in \mathbb{R}[\mu]$. Moreover,*

$$
\begin{aligned}
\Delta_1(\mu) &= \pi\big(2p^2 + 2q^2 + 7pq - p - q - 1\big),\\
\Delta_2(\mu) &= \frac{5\pi}{24}\big(-23 + 4p^4 - 46q + 21q^2 + 44q^3 + 4q^4 + 4p^3(11 + 43q) +\\
&\quad + 3p^2(7 + 122q + 139q^2) + 2p(-23 + 42q + 183q^2 + 86q^3)\big),\\
\Delta_3(\mu) &= \frac{7\pi}{864}\big(-11237 - 1112p^6 - 33711q - 10641q^2 + 34903q^3 + 22434q^4\\
&\quad - 636q^5 - 1112q^6 + 12p^5(-53 + 803q) + 6p^4(3739 + 25888q + 27289q^2)\\
&\quad + p^3(34903 + 390273q + 734277q^2 + 336347q^3)\\
&\quad + 3p^2(-3547 + 88309q + 284637q^2 + 244759q^3 + 54578q^4)\\
&\quad + 3p(-11237 - 2951q + 88309q^2 + 130091q^3 + 51776q^4 + 3212q^5)\big).
\end{aligned}
$$

Proof. We claim that $\hat{V}_\mu(u) = \sum_{k\geqslant 2}\hat{\alpha}_k(\mu)u^k$ with $\hat{\alpha}_k \in \mathbb{R}[\mu]$ and $\hat{\alpha}_2(\mu) = \frac{1}{2}$. To show this note first of all that $\hat{V}_\mu'(u) = \frac{1}{p-q}\sum_{k\geqslant 1} k!\alpha_k(\mu)u^k$ with

$$\alpha_k(q,p) = p(p-1)\cdots\big(p-(k-1)\big) - q(q-1)\cdots\big(q-(k-1)\big).$$

Since $\alpha_k(q,p) \in \mathbb{R}[q,p]$ and $\alpha_k(q,q) = 0$, we can assert $\hat{\alpha}_{k+1}(p,q) := \frac{k!\alpha_k(q,p)}{(k+1)(p-q)} \in \mathbb{R}[q,p]$. This proves the validity of the claim because the fact that $\hat{\alpha}_2(p,q) = \frac{1}{2}$ is clear.

Let us define $\hat{g}_\mu(x) := \mathrm{sgn}(x)\sqrt{\hat{V}_\mu(x)}$ and suppose that the Taylor development of its inverse at $x = 0$ is given by $\hat{g}_\mu^{-1}(x) = \sum_{k\geqslant 1}\beta_k(\mu)x^k$. Then, see for instance [11], it follows that

$$\hat{T}_\mu(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\hat{g}_\mu^{-1})'(\sqrt{h}\sin\theta)d\theta = \sum_{k\geqslant 0}\Delta_k(\mu)h^k$$

with

$$\Delta_k(\mu) := 2\sqrt{2}(2k+1)\beta_{2k+1}(\mu)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2k}\theta d\theta.$$

Since one can easily verify that $\beta_1(\mu) = \frac{1}{\sqrt{2}}$, the above expression shows in particular that $\hat{T}_\mu(h)$ extends analytically to $h = 0$ setting $\hat{T}_\mu(0) := 2\pi$. It shows moreover that $\Delta_k \in \mathbb{R}[\mu]$ if and only if $\beta_{2k+1} \in \mathbb{R}[\mu]$. Let us show that $\beta_k \in \mathbb{R}[\mu]$ for all $k \in \mathbb{N}$. To this end we note that, by definition, $V_\mu\left(\hat{g}_\mu^{-1}(x)\right) = x^2$ for all $x \in \mathcal{I}_\mu$. Hence

$$\sum_{k=2}^{\infty}\hat{\alpha}_k(\mu)\left(\sum_{i=1}^{\infty}\beta_i(\mu)x^i\right)^k = x^2 \text{ for all } x \approx 0.$$

Using this identity and taking the claim into account, one can prove by induction on k that $\beta_k \in \mathbb{R}[\mu]$ for all $k \in \mathbb{N}$. Therefore $\Delta_k \in \mathbb{R}[\mu]$ for all $k \in \mathbb{N}$. Finally the expression for Δ_1, Δ_2 and Δ_3 that we give in the statement can be easily computed following this approach by using a symbolic manipulator. This concludes the proof of the result. ∎

The coefficient $\Delta_k(\mu)$ is known as the *nth period constant* and play an equivalent role as Lyapunov constants in the framework of limit cycles and the center-focus problem. A necessary and sufficient condition for a point $\hat{\mu} \in \Lambda$ to be an isochronous center is that all $\Delta_k(\hat{\mu}) = 0$ for all $k \geqslant 0$. However, even when the coefficients Δ_k are polynomial, the problem of finding isochronous centers is complicated in general. Next result of Cima, Mañosas and Villadelprat, see [15], provides a useful tool in order to study the isochronicity problem for a center of a potential differential system. It is given in terms of the existence of an *involution*, i.e., a function $\sigma \neq \mathrm{Id}$ such that $\sigma^2 = \mathrm{Id}$.

Proposition 3.3.4. *Let V be an analytic function with $V(0) = 0$ and suppose that the system $X = -y\partial_x + V'(x)\partial_y$ has a center at the origin. Let $\mathcal{I}$ be the projection of its period annulus on the x-axis. Then the origin is an isochronous center of period ω if and only if there exists an analytic involution σ on $\mathcal{I}$ such that $V(x) = \frac{\pi^2}{2\omega^2}\big(x - \sigma(x)\big)^2$.*

For a given ideal $\mathfrak{m}$ over $\mathbb{C}[x]$ we shall denote by $V(\mathfrak{m})$ the *complex variety* of $\mathfrak{m}$. The next result solves in particular the isochronicity problem in the family of centers under consideration. In its statement Δ_1, Δ_2 and Δ_3 are the period constants given in Proposition 3.3.3.

Theorem 3.3.5. *Define $\mu_1 := (-3, 1)$, $\mu_2 := (-1/2, 0)$, $\mu_3 := (0, 1)$, $\mu_4 := (1, -3)$, $\mu_5 := (0, -1/2)$, $\mu_6 := (1, 0)$ and $\mu_7 := (i/\sqrt{3}, -i/\sqrt{3})$. Then the following holds:*

(a) The variety of the ideal $\mathfrak{m}_2 := (\Delta_1, \Delta_2)$ is $V(\mathfrak{m}_2) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \bar{\mu}_7\}$.

(b) The variety of the ideal $\mathfrak{m}_3 := (\Delta_1, \Delta_2, \Delta_3)$ is $V(\mathfrak{m}_3) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$.

Moreover, the center at the origin of the differential system (3.5) is isochronous if and only if $\mu \in V(\mathfrak{m}_3)$.

Proof. The assertions in (a) and (b) can be proved with a symbolic manipulator, for instance using resultants. Let us prove the assertion concerning the isochronous centers of system (3.5). Clearly the necessity follows by definition, so we only need to show the sufficiency, i.e., if $\mu \in V(\mathfrak{m}_3)$ then the center is isochronous. To this end, taking advantage of the symmetry $\hat{X}_{(q,p)} = \hat{X}_{(p,q)}$, it suffices to show that μ_1, μ_2 and μ_3 correspond to isochronous centers. With this end in view, easy computations show that $\sigma_1(x) := -\frac{x}{x+1}$, $\sigma_2(x) := 4 + x - 4\sqrt{x+1}$, and $\sigma_3(x) := -x$ are the involutions associated to $\hat{V}_{\mu_i}$ for $i = 1, 2, 3$, respectively (i.e., such that $\hat{V}_{\mu_i} = \hat{V}_{\mu_i} \circ \sigma_i$). Finally the result follows by Proposition 3.3.4 after verifying that $\hat{V}_{\mu_i}(x) = \frac{1}{8}\left(x - \sigma_i(x)\right)^2$ for $i = 1, 2, 3$. ∎

In the statement of the next result Δ_1 is the first period constant, see Proposition 3.3.3, and μ_i, $i=1,2,\ldots,6$, are the parameters corresponding to isochronous centers of system (3.5), see Theorem 3.3.5.

Proposition 3.3.6. *The following hold:*

(*a*) *If $\Delta_1(\hat{\mu})\neq 0$ with $\hat{\mu}\in\mathbb{R}^2$, then $\mathrm{Crit}\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big)=0$. Moreover, if $\Delta_1(\hat{\mu})$ is positive (respectively, negative), then the period function of $\hat{X}_{\hat{\mu}}$ is increasing (respectively, decreasing) near the center.*

(*b*) *If $\Delta_1(\hat{\mu})=0$ with $\hat{\mu}\in\mathbb{R}^2\setminus\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6\}$, then $\mathrm{Crit}\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big)=1$.*

Proof. Let $\hat{T}_\mu(h)$ denote the period of the periodic orbit of the differential system (3.5) inside the energy level $\frac{1}{2}v^2+\hat{V}_\mu(u)=h$. Then, by Proposition 3.3.3, we have that

$$\hat{T}_\mu'(h)=\Delta_1(\mu)+2\Delta_2(\mu)h+\mathrm{o}(h). \tag{3.6}$$

Clearly, if $\Delta_1(\hat{\mu})\neq 0$, then there exist $\varepsilon>0$ and an open neighbourhood $\mathscr{U}$ of $\hat{\mu}$ such that $\hat{T}_\mu'(h)\neq 0$ for all $h\in(0,\varepsilon)$ and $\mu\in\mathscr{U}$. We claim this implies $\mathrm{Crit}\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big)=0$, see Definition 3.3.1. Suppose, by contradiction, that there exist a sequence $\{\gamma_{\mu_i}\}_{i\in\mathbb{N}}$ where γ_{μ_i} is a critical periodic orbit of X_{μ_i} such that $\mu_i\to\hat{\mu}$ and $d_H(\gamma_{\mu_i},(0,0))\to 0$ as $i\to+\infty$. Then, since the origin is the inner boundary of X_μ for all $\mu\approx\hat{\mu}$, this contradicts that for all $\mu\in\mathscr{U}$, T_μ' has no zeroes in $(0,\varepsilon)$. So the claim is true and the result in (*a*) follows because the assertion concerning the monotonicity of the period function is trivial.

In order to show (*b*) note that if $\Delta_1(\hat{\mu})=0$ with $\hat{\mu}\in\mathbb{R}^2\setminus\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6\}$ then, by Theorem 3.3.5, $\Delta_2(\hat{\mu})\neq 0$. Hence, from (3.6) and by applying the Implicit Function Theorem, a similar argument as before proves that $\mathrm{Crit}\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big)\leqslant 1$. The fact that this upper bound is achieved follows the same way using that the gradient of Δ_1 does not vanish for parameter values with $\Delta_1(\mu)=0$. So the result is proved. ∎

The study of the criticality at the isochronous centers is the last ingredient for the proof of Theorem G. Our approach strongly relies in the following two general results of Chicone and Jacobs [11].

Theorem 3.3.7. *Let $\{Y_\mu\}_{\mu\in\Lambda}$ be an analytic family of analytic Hamiltonian differential systems with a non-degenerate center at the origin. Let H_μ be the Hamiltonian function with $H_\mu(0,0)=0$. Let $T_\mu(h)$ denote the period of the periodic orbit of Y_μ inside the energy level $H_\mu=h$ and let $T_\mu(h)=\sum_{i=0}^{\infty}\Delta_i(\mu)h^i$ be its Taylor development at $h=0$. If the center is isochronous for $\mu=\hat{\mu}$ and if, for all $i\in\mathbb{N}$, Δ_i is inside the ideal $(\Delta_1,\Delta_2,\ldots,\Delta_{k+1})$ over $\mathbb{R}\{\mu\}_{\hat{\mu}}$, the ring of convergent power series at $\hat{\mu}$, then $\mathrm{Crit}\big(((0,0),Y_{\hat{\mu}}),Y_\mu\big)\leqslant k$. Moreover, if the gradients of $\Delta_1,\Delta_2,\ldots,\Delta_{k+1}$ are linearly independent at $\hat{\mu}$, then $\mathrm{Crit}\big(((0,0),Y_{\hat{\mu}}),Y_\mu\big)=k$.*

The previous result is an adaptation of the Isochrone Bifurcation Theorem in [11] to Hamiltonian systems, for which it is more natural to parametrize the periodic orbits with the energy instead of the intersection point with the positive x-axis, and to the definition of criticality that we use in this work. The proof is omitted because it follows verbatim the one Chicone and Jacobs. The next result is a particular case of [11, Theorem A.1].

Proposition 3.3.8. *Suppose that the ideal $\mathfrak{m} = (f_1, \ldots, f_r) \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ satisfies that $V(\mathfrak{m})$ is a finite set and that rank$(\nabla f_1(a), \nabla f_2(a), \ldots, \nabla f_r(a)) = n$ for all $a \in V(\mathfrak{m})$. Then $\mathfrak{m}$ is radical, i.e., $f \in \mathfrak{m}$ if and only if $f\big(V(\mathfrak{m})\big) = 0$.*

Proposition 3.3.9. *Let $\hat{X}_\mu$ be the differential system in (3.5) and let us fix $\mu_i \in \Lambda$, $i = 1, 2, \ldots, 6$, one of the parameters corresponding to the isochronous centers. Then* $\mathrm{Crit}\big(((0,0), \hat{X}_{\mu_i}), \hat{X}_\mu\big) = 1$ *for $i = 1, 2, \ldots, 6$.*

Proof. We apply Proposition 3.3.3 and consider the ideal $\mathfrak{m} := (\Delta_1, \Delta_2)$ over $\mathbb{C}[q,p]$. Then, by Theorem 3.3.5 we have that $V(\mathfrak{m}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \bar{\mu}_7\}$. One can verify in addition that $\nabla\Delta_1$ and $\nabla\Delta_2$ are linearly independent for $\mu \in V(\mathfrak{m})$. Thus, by applying Proposition 3.3.8, $\mathfrak{m}$ is radical over $\mathbb{C}[q,p]$.

We claim that if $f_k(\mu) := (3p^2+3q^2+2)\Delta_k(\mu)$, then $f_k\big(V(\mathfrak{m})\big) = 0$ for all $k \geqslant 3$. Indeed, that $f_k(\mu_i) = 0$ for $i = 1, 2, \ldots, 6$ follows due to the fact that, by Theorem 3.3.5, the center of $\hat{X}_{\mu_i}$ is isochronous. We have on the other hand that $f_k(\mu_7) = 0$ and $f_k(\bar{\mu}_7) = 0$ because these two parameters are the roots of $3p^2 + 3q^2 + 2 = 0$. This proves the claim. Consequently $f_k \in \mathfrak{m}$ for all $k \geqslant 3$. Thus, for each $k \geqslant 3$, there exist $A_k, B_k \in \mathbb{C}[q,p]$ such that $f_k = A_k\Delta_1 + B_k\Delta_2$, so that

$$\Delta_k(q,p) = \frac{A_k(q,p)}{3p^2+3q^2+2}\Delta_1(q,p) + \frac{B_k(q,p)}{3p^2+3q^2+2}\Delta_2(q,p) \text{ for all } k \geqslant 3.$$

Fix any μ_i, $i = 1, 2, \ldots, 6$. Then, since $3p^2 + 3q^2 + 2 \neq 0$ at $\mu = \mu_i$, the above equality shows that $\Delta_k \in \mathfrak{m}$ over the local ring $\mathbb{R}\{\mu\}_{\mu_i}$ for all $k \geqslant 3$. (Here we use that $\Delta_i \in \mathbb{R}[\mu]$.) Hence, by applying Theorem 3.3.7, $\mathrm{Crit}\big(((0,0), \hat{X}_{\mu_i}), \hat{X}_\mu\big) = 1$. So the result is proved. ■

Proof of Theorem G. As we mentioned at the beginning of the present section, any differential system (3.1) with $p > q$ can be brought to (3.5) by means of a conjugation and a constant rescaling of time. On account of this, the result follows by applying Propositions 3.3.6 and 3.3.9. ■

3.4 Criticality at the interior for isochronous centers.

In this section we study the critical periodic orbits that bifurcate from the period annulus of an isochronous center $X_{\hat{\mu}}$ when perturbed by means of a one-parameter deformation inside the family under consideration $\{X_\mu\}_{\mu\in\Lambda}$ in 3.1. We recall that a center is called isochronous if its period function is constant. As we already mentioned in this memoir, we are not in position to give a general definition of local bifurcation value at the interior of the period annulus in terms of the criticality. For this reason, we give the notion of criticality given by Garijo and Villadelprat in [18] for isochronous centers, which is an adaptation of the notion of cyclicity given by Gavrilov in [21, 22].

Definition 3.4.1. Consider a continuous family $\{X_\mu\}_{\mu\in\Lambda}$ of planar analytic vector fields with a center and fix some $\hat{\mu}\in\Lambda$. Assume that $X_{\hat{\mu}}$ is an isochronous center. Then the *criticality* of the pair $(\mathscr{P}_{\hat{\mu}}, X_{\hat{\mu}})$ with respect to the deformation X_μ is

$$\mathrm{Crit}\big((\mathscr{P}_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) := \sup\{\mathcal{N}_K : K\subset\mathscr{P}_{\hat{\mu}}, K \text{ is a compact invariant set of } X_{\hat{\mu}}\}$$

where, for such a K, $\mathcal{N}_K$ is the smallest integer having the property that there exists a neighbourhood V of K and $\delta>0$ such that, for every μ with $\|\mu-\hat{\mu}\|<\delta$, the vector field X_μ has no more than $\mathcal{N}_K$ critical periodic orbits contained in V. □

Although we do not give any definition of local bifurcation value at the interior of the period annulus, it is clear that a parameter corresponding to an isochronous center must be a bifurcation value for any reasonable definition. Now we are in position to state the main result of this section.

Theorem H. *The center at the origin of the differential system X_μ in (3.1) is isochronous if and only if $\mu\in\{(-3,1),(-1/2,0),(0,1)\}$. Moreover, if $\hat{\mu}$ is the parameter value of one of these isochronous centers and $\mu\longmapsto\mu(\varepsilon)$ is any germ of analytic curve in Λ with $\mu(0)=\hat{\mu}$, then $\mathrm{Crit}\big((\mathscr{P}_{\hat{\mu}}, X_{\hat{\mu}}), X_{\mu(\varepsilon)}\big)\leqslant 1$. Finally, for each isochronous center, there exists a germ of analytic curve for which this upper bound is achieved.*

We expect of course $\mathrm{Crit}\big((\mathscr{P}_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big)=1$ for any $\mu\in\{(-3,1),(-1/2,0),(0,1)\}$. In relation to this, but in the context of Hilbert's sixteenth Problem, it is to be quoted the result of L. Gavrilov [22], which shows that the problem of finding the cyclicity of a period annulus with respect to a multi-parameter deformation can be always reduced to the "simpler" problem of finding the cyclicity with respect to a one-parameter deformation. In order to study this problem we recall the following definitions:

Definition 3.4.2. We say that two planar vector fields *commute* on $\mathscr{U}\subset\mathbb{R}^2$ if they are transversal and the Lie bracket $[X,Y]$ vanishes identically on $\mathscr{U}$. □

Definition 3.4.3. Let M and N be two manifolds. Consider X and Y vector fields on M and N, respectively, and let $\varphi\colon M\longrightarrow N$ be a diffeomorphism between M and N. The *pull back* of Y by φ is the vector field φ^*Y on M defined by $(\varphi^*Y)(p) := (D\varphi^{-1})_{\varphi(p)}Y\big(\varphi(p)\big)$ for all $p\in M$. The *push forward* of X by φ is the vector field φ_*X on N defined by $(\varphi_*X)(p) := (D\varphi)_{\varphi^{-1}(p)}X\big(\varphi^{-1}(p)\big)$ for all $p\in N$. □

Suppose that X is an analytic vector field with a center at p. It is well known, see [6] and references there in, that p is an isochronous center if and only if there exists an analytic vector field Y on a neighbourhood $\mathscr{U}$ of p such that X and Y commute on $\mathscr{U}\setminus\{p\}$. In order to prove Theorem H it will be convenient to have a commutator of each X_{μ_i}, $i=1,2,3$, with $\mathscr{U}$ being the whole period annulus $\mathscr{P}_{\mu_i}$. With this end in view we prove the following general result for isochronous centers of potential systems:

Proposition 3.4.4. *Let $X=-y\partial_x+V'(x)\partial_y$ be a potential vector field that has an isochronous center at the origin of period ω and let $\mathscr{P}$ be its period annulus. Define $h(x):=\frac{x-\sigma(x)}{2}$, where σ is the involution such that $V=V\circ\sigma$. Then $(r,\theta)=\varphi(x,y)$, defined by means of $\big\{h(x)=r\cos\theta,\frac{\omega}{2\pi}y=r\sin\theta\big\}$, is a coordinate transformation on $\mathscr{P}$ and the pull-back of*

$$U=r\partial_r-\frac{r\int_0^\theta(h^{-1})''(r\cos s)\cos s ds}{(h^{-1})'(r\cos\theta)}\partial_\theta$$

by φ is an analytic vector field on $\mathscr{P}$ that extends analytically to the origin, and commutes with X on $\mathscr{P}$.

Proof. Without loss of generality we assume that $\omega=2\pi$. Note that h is a diffeomorphism on the projection of the period annulus because $\sigma'(x)=\frac{V'(x)}{V'\big(\sigma(x)\big)}<0$. We claim that $(h^{-1})''$ is an even function and that

$$F(r,\theta):=\int_0^\theta(h^{-1})'(r\cos s)ds$$

is a circle diffeomorphism of degree one, i.e., $F(r,\theta+2\pi)=F(r,\theta)+2\pi$, for each fixed r. To show this observe first that $h\circ\sigma=-h$, so that $\sigma(h^{-1}(u))=h^{-1}(-u)$. Hence

$$u=h\big(h^{-1}(u)\big)=\frac{h^{-1}(u)-\sigma(h^{-1}(u))}{2}=\frac{h^{-1}(u)-h^{-1}(-u)}{2}.$$

Accordingly the odd part of the function h^{-1} is the identity, so $h^{-1}(u)=u+G(u)$, with G being an even function. In particular this shows that $(h^{-1})''$ is an even function. In addition,

$$F(r,\theta+2\pi)-F(r,\theta)=\int_\theta^{\theta+2\pi}\big(h^{-1}\big)'(r\cos s)ds=2\pi+\int_\theta^{\theta+2\pi}G'(r\cos s)ds=2\pi,$$

where the last equality follows by using that G' is odd. This proves the validity of the claim.

Since the origin is an isochronous center, by Proposition 3.3.4 we can write the potential function as $V(x)=\frac{1}{2}h(x)^2$. Therefore, $X=-y\partial_x+h(x)h'(x)\partial_y$ and an easy computation shows that $(u,v)=\varphi_1(x,y):=\big(h(x),y\big)$ brings X to

$$\varphi_{1*}X=\frac{1}{(h^{-1})'(u)}\big(-v\partial_u+u\partial_v\big).$$

Hence, if $(r,\theta)=\varphi_2(u,v)$ denotes the usual polar coordinates $\{u=r\cos\theta, v=r\sin\theta\}$, we get

$$\big(\varphi_2\circ\varphi_1\big)_*X=\varphi_*X=\frac{1}{(h^{-1})'(r\cos\theta)}\partial_\theta.$$

Finally, if $(R,\phi)=\varphi_3(r,\theta):=\big(r,F(r,\theta)\big)$, then we have $\big(\varphi_3\circ\varphi\big)_*X=\partial_\phi$ because $\phi'=\frac{d}{dt}F(r,\theta)=F_\theta(r,\theta)\theta'=1$. (At this point we used that $\theta\longmapsto F(r,\theta)$ is a one-degree circle diffeomorphism.) Clearly, a commutator for ∂_ϕ is given by $\hat{U}:=R\partial_R$, i.e., $[\big(\varphi_3\circ\varphi\big)_*X,\hat{U}]=0$. Then,

$$0=\big(\varphi_3\circ\varphi\big)^*\big[\big(\varphi_3\circ\varphi\big)_*X,\hat{U}\big]=\big[X,\big(\varphi_3\circ\varphi\big)^*\hat{U}\big].$$

The pull-back of $\hat{U}$ by φ_3 is precisely the vector field U given in the statement because $r'=R'=R=r$ and $0=\phi'=F_r(r,\theta)r'+F_\theta(r,\theta)\theta'$. Thus ${\varphi_3}^*\hat{U}=U$ and so the above expression shows that $[X,\varphi^*U]=0$, as desired.

It remains to be shown that φ^*U is an analytic vector field on $\mathscr{P}\cup\{(0,0)\}$. To this end it suffices to prove that $(\varphi_2)^*U$ is an analytic vector field at the origin because $\varphi^*U=\big(\varphi_2\circ\varphi_1\big)^*U=(\varphi_1)^*(\varphi_2)^*U$ and φ_1 is a well-defined analytic diffeomorphism on $\mathscr{P}\cup\{(0,0)\}$. Note that

$$(\varphi_2)^*U=\left(x+y\frac{S(x,y)}{(h^{-1})'(x)}\right)\partial_x+\left(y-x\frac{S(x,y)}{(h^{-1})'(x)}\right)\partial_y,$$

where

$$S(x,y):=r\int_0^\theta (h^{-1})''(r\cos s)\cos s ds\bigg|_{\left\{r=\sqrt{x^2+y^2},\theta=\arctan(y/x)\right\}}.$$

Since $h'(0)\neq 0$, we must show that S is analytic at $(x,y)=(0,0)$. To this end we use that, on account of the claim, $u\longmapsto(h^{-1})''(u)$ is an even function, so we can write $(h^{-1})''(u)=\sum_{i=0}^\infty\beta_i u^{2i}$ for $u\approx 0$. Thus, for $r\approx 0$,

$$r\int_0^\theta (h^{-1})''(r\cos s)\cos s ds=\sum_{i=0}^\infty\beta_i r^{2i+1}\int_0^\theta\cos^{2i+1}s ds=\sum_{i=0}^\infty\beta_i r^{2i+1}\sin\theta P_i(\sin^2\theta),$$

where P_i is a polynomial of degree i. Thanks to the identity $\sin^2\theta+\cos^2\theta=1$, we can write $P_i(\sin^2\theta)=\hat{P}_i(\cos^2\theta,\sin^2\theta)$ with $\hat{P}_i$ being a homogenous polynomial of degree i. Therefore

$$r\int_0^\theta (h^{-1})''(r\cos s)\cos s ds=\sum_{i=0}^\infty\beta_i r\sin\theta\hat{P}_i(r^2\cos^2\theta,r^2\sin^2\theta)$$

and, consequently, $S(x,y)=y\sum_{i=0}^\infty\beta_i\hat{P}_i(x^2,y^2)$ for $(x,y)\approx(0,0)$. This shows the analyticity of S at the origin and completes the proof of the result. ■

Now the desired commutators are given in the following result:

Lemma 3.4.5. *Consider the parameters $\mu_1 = (-3, 1)$, $\mu_2 = (-1/2, 0)$ and $\mu_3 = (0, 1)$ corresponding to the isochronous centers of the family $\{X_\mu\}_{\mu\in\Lambda}$ in (3.1). Define*

$$
\begin{aligned}
U_1 &= \tfrac{S(x,y)}{4(x+1)}\partial_x + \tfrac{y(4+S(x,y))}{4(x+1)^2}\partial_y,\\
U_2 &= (2+2x-2\sqrt{1+x}+y^2)\partial_x + \tfrac{y}{\sqrt{1+x}}\partial_y,\\
U_3 &= x\partial_x + y\partial_y,
\end{aligned}
$$

where $S(x, y) = (x+1)^2y^2 + x(x+2)(x^2+2x+2)$. Then, for $i = 1, 2, 3$, U_i is an analytic vector field on $\mathscr{P}_{\mu_i} \cup \{(0, 0)\}$ that commutes with X_{μ_i} on $\mathscr{P}_{\mu_i}$.

Proof. The commutators follow by applying Proposition 3.4.4 and to this end we need the involutions associated to each potential function. As we already mentioned,

$$\sigma_1(x) = -\frac{x}{x+1},\ \sigma_2(x) = x + 4 - 4\sqrt{x+1} \text{ and } \sigma_3(x) = -x$$

are the involutions for μ_1, μ_2 and μ_3, respectively. By using these functions the result follows after some easy computations which are omitted for the sake of shortness. (Of course, alternatively, the reader may check that $[X_{\mu_i}, U_i] = 0$.) ∎

Fix some $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$ and take a germ of analytic curve $\varepsilon \longmapsto \mu(\varepsilon)$ in the parameter space Λ such that $\mu(0) = \hat{\mu}$. Our first goal is to parametrize the set of periodic orbits of $X_{\mu(\varepsilon)}$ for $\varepsilon \approx 0$. To this end we consider the commutator U of $X_{\hat{\mu}}$ given by Lemma 3.4.5 and we proceed as follows. We choose an arbitrary point $\mathbf{x} \in \mathscr{P}_{\hat{\mu}}$ and we take the solution $\psi(s; \mathbf{x})$ of U such that $\psi(0; \mathbf{x}) = \mathbf{x}$. Then, for some open interval I, $\psi(\,\cdot\,; \mathbf{x})\colon I \longrightarrow \mathbb{R}^2$ is an analytic transverse section to $X_{\hat{\mu}}$ on $\mathscr{P}_{\hat{\mu}}$. By continuity, this will be also the case for $X_{\mu(\varepsilon)}$ with $\varepsilon \approx 0$. Setting $\xi(s) = \psi(s; \mathbf{x})$ for the sake of shortness, we define $T(s; \varepsilon)$ to be the period of the periodic orbit of $X_{\mu(\varepsilon)}$ passing through the point $\xi(s)$. The function $T(s; \varepsilon)$ is analytic for $\varepsilon \approx 0$ and so we can consider its Taylor's series development at $\varepsilon = 0$,

$$T(s; \varepsilon) = \sum_{i=0}^{\infty} T_i(s)\varepsilon^i.$$

Notice that T_0 is constant because $X_{\mu(\varepsilon)}$ is isochronous for $\varepsilon = 0$. Then, if the center is not isochronous for $\varepsilon \neq 0$, there exist $\ell \geqslant 1$ such that

$$T'(s; \varepsilon) = T'_\ell(s)\varepsilon^\ell + o(\varepsilon^\ell),$$

where T'_ℓ is not identically zero and the remainder is uniform in s on each compact subinterval of I. Then, applying the Weierstrass Preparation Theorem, the number of isolated zeros of $T'_\ell(s)$ for $s \in I$, counted with multiplicities, provides an upper-bound for the criticality at the interior of the period annulus of $X_{\hat{\mu}}$. This approach is similar to the use of the so-called *Melnikov functions* for studying the bifurcation of limit cycles arising from the perturbation of an integrable center.

We can now give the fundamental result in order to prove Theorem H. In its statement we use the notation we have just introduced.

Theorem 3.4.6. *Take $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$ and set $\hat{\mu} = (\hat{q}, \hat{p})$. Let U be the commutator of $X_{\hat{\mu}}$ given by Lemma 3.4.5 and take a transverse section $\xi\colon I \longrightarrow \mathbb{R}^2$ to $X_{\hat{\mu}}$ on $\mathscr{P}_{\hat{\mu}}$ given by a solution of U. Then there exist analytic functions A_1 and A_2 on I such that:*

(a) *(A_1, A_2) is an ECT-system on I.*

(b) *If $\varepsilon \longmapsto \mu(\varepsilon)$ with $\mu(\varepsilon) = (\hat{q}+\kappa_1\varepsilon^\ell+\mathrm{o}(\varepsilon^\ell), \hat{p}+\kappa_2\varepsilon^\ell+\mathrm{o}(\varepsilon^\ell))$ is a germ of analytic curve in Λ such that $\kappa_1 \neq 0$ or $\kappa_2 \neq 0$, then the period function $T(s;\varepsilon)$ corresponding to the perturbation $X_{\mu(\varepsilon)}$ verifies $T_0' \equiv T_1' \equiv \cdots \equiv T_{\ell-1}' \equiv 0$ and $T_\ell'(s) = \kappa_1 A_1(s) + \kappa_2 A_2(s)$ for all $s \in I$.*

We remark that, for a given $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$, the functions A_1 and A_2 depend only on the commutator U. In particular, they do not depend on the germ of analytic curve chosen.

To obtain an expression of T_ℓ' we shall apply a result of Grau and Villadelprat that appears in [27]. In order to state it, some additional notation must be introduced. Since $X_{\hat{\mu}}$ and U are transverse on $\mathscr{P}_{\hat{\mu}}$, there exist two analytic functions $\alpha = \alpha(x, y; \varepsilon)$ and $\beta = \beta(x, y; \varepsilon)$ such that

$$X_{\mu(\varepsilon)} = \alpha X_{\hat{\mu}} + \beta U.$$

Note that

$$\alpha = \frac{<X_{\mu(\varepsilon)}, U^\perp>}{<X_{\hat{\mu}}, U^\perp>} \text{ and } \beta = \frac{<X_{\hat{\mu}}^\perp, X_{\mu(\varepsilon)}>}{<X_{\hat{\mu}}^\perp, U>},$$

where $<\ ,\ >$ stands for the scalar product and $X^\perp$ denotes the orthogonal vector field to X. Let us also denote the k-jet of $X_{\mu(\varepsilon)}$ at $\varepsilon = 0$ by $j^k\big(X_{\mu(\varepsilon)}\big)$. With this notation, by applying [27, Theorem 3.2] we get:

Lemma 3.4.7. *Let us assume that, for some $k \in \mathbb{N}$, $j^{k-1}(X_{\mu(\varepsilon)})$ has an isochronous center at the origin for all $\varepsilon \approx 0$. Then $T_0' \equiv T_1' \equiv \cdots \equiv T_{k-1}' \equiv 0$ and*

$$T_k'(s) = -\int_0^{T_0} U(\alpha_k)|_{(x,y)=\varphi(t;s)}\, dt \quad \textit{for all } s \in I,$$

where $\varphi(t; s)$ is the solution of $X_{\hat{\mu}}$ with $\varphi(0; s) = \xi(s)$ and α_k is the kth term of the Taylor development of α at $\varepsilon = 0$.

We point out that the assumption in [27, Theorem 3.2] is that there exists an analytic family of diffeomorphisms $\{\Phi_\varepsilon\}$, defined in a neighbourhood of $(0,0)$, such that Φ_ε linearizes $j^{k-1}(X_{\mu(\varepsilon)})$ for each $\varepsilon \approx 0$. Here we replace it by the assumption that $j^{k-1}(X_{\mu(\varepsilon)})$ has an isochronous center at the origin for all $\varepsilon \approx 0$, which is more easy to verify. The next result shows that both conditions are equivalent:

Lemma 3.4.8. *Let $\Lambda \subset \mathbb{R}^n$ be an open set and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of planar analytic vector fields with center at the origin. Then the center is isochronous for all $\lambda \in \Lambda$ if and only if for each $\lambda_0 \in \Lambda$ there exist a neighbourhood $\mathscr{U}$ of λ_0 and an analytic family of analytic diffeomorphisms $\{\Phi_\lambda\}_{\lambda\in\mathscr{U}}$, defined in a neighbourhood of $(0,0)$, such that Φ_λ linearizes X_λ for each $\lambda \in \mathscr{U}$.*

Proof. First we show that the condition is necessary. Let $\varphi_\lambda(t;p)$ be the solution of X_λ with $\varphi_\lambda(0;p) = p$. In addition, for each $\lambda \in \Lambda$, let $\mathscr{P}_\lambda$ be the period annulus of the center at the origin of X_λ and let T_λ be the period of its periodic orbits. Finally, let $A_\lambda \in M_{2\times 2}$ be the Jacobian matrix of X_λ at the origin. Define

$$\Phi_\lambda(p) := \frac{1}{T_\lambda}\int_0^{T_\lambda} e^{-A_\lambda s}\varphi_\lambda(s;p)ds.$$

One can easily verify that $\Phi_\lambda\big(\varphi_\lambda(t;p)\big) = e^{A_\lambda t}\Phi_\lambda(p)$. Moreover, by applying the variational equations, the linear part of $p \longmapsto \varphi_\lambda(s;p)$ at $p = (0,0)$ is $e^{A_\lambda s}p$. Consequently the Jacobian matrix of Φ_λ at $p = (0,0)$ is the identity. Hence, for each $\lambda \in \Lambda$, the map Φ_λ linearizes X_λ in some neighbourhood U_λ of $(0,0)$. Let us fix $\lambda_0 \in \Lambda$ and take a neighbourhood W of $(0,0)$ inside $\mathscr{P}_{\lambda_0}$. Then there exists a neighbourhood V of λ_0 such that $W \subset \mathscr{P}_\lambda$ for all $\lambda \in V$. Now the result follows by applying the inverse function theorem to the map $\Phi : W\times V \longrightarrow \mathbb{R}^2\times V$ given by $\Phi(p,\lambda) := (\Phi_\lambda(p),\lambda)$, which is analytic, thanks to the analytic dependence of solutions on initial conditions and parameters, and satisfies that its Jacobian matrix at $(p,\lambda) = (0,0,\lambda_0)$ is the identity. This proves the result because the reverse implication is well-known (see [6] for instance). ■

We point out that the diffeomorphism used in the previous proof is a parametric adaptation to the context of isochronus centers of a more general result that appears in the literature (see for instance [48]).

Lemma 3.4.7 constitutes the first ingredient in the proof of Theorem 3.4.6. The second one is a criterion of Grau, Mañosas and Villadelprat [26] that gives a sufficient condition for a collection of Abelian integrals to be an ECT-system. In order to state it precisely some previous definitions must be introduced. Suppose that $H(x,y) = A(x) + B(x)y^{2m}$ is an analytic function in some open subset of the plane that has a local minimum at the origin. Then there exists a punctured neighbourhood $\mathscr{P}$ of the origin foliated by ovals $\gamma_h \subset \{H(x,y)=h\}$. We set $H(0,0)=0$ and then the set of ovals γ_h inside $\mathscr{P}$ is parameterized by the energy levels $h\in(0,h_0)$ for some positive h_0. The projection of $\mathscr{P}$ on the x-axis is an interval (x_ℓ,x_r) with $x_\ell<0<x_r$. Under these assumptions A has a zero of even multiplicity at $x=0$, and it is easy to verify that there exist an analytic involution σ such that

$$A(x)=A\big(\sigma(x)\big) \text{ for all } x\in(x_\ell,x_r).$$

Definition 3.4.9. Given a function κ defined on $(x_\ell,x_r)\setminus\{0\}$, we define its σ-*balance* as $\mathscr{B}_\sigma\big(\kappa\big)(x)=\kappa(x)-\kappa\big(\sigma(x)\big)$. □

In the following result we use the notion of ECT-system introduced in Definition 1.4.1. We also use the notion of CT-system that we introduce now.

Definition 3.4.10. Let $f_0,f_1,\ldots,f_{n-1}$ be analytic function on an open interval $I\subset\mathbb{R}$. The ordered set $(f_0,f_1,\ldots,f_{n-1})$ is a *complete Chebyshev system* (for short, a CT-system) on I if, for all $k=1,2,\ldots,n$, any nontrivial linear combination

$$a_0f_0(x)+a_1f_1(x)+\cdots+a_{k-1}f_{k-1}(x)$$

has at most $k-1$ isolated zeros on I. □

We point out that clearly if $(f_0,f_1,\ldots,f_{n-1})$ is an ECT-system then it is a CT-system. Following this notation we can now state the criterion [26, Theorem B] as follows.

Theorem 3.4.11. *Let $f_0,f_1,\ldots,f_{n-1}$ be analytic functions on (x_ℓ,x_r), and consider the Abelian integrals*

$$I_i(h)=\int_{\gamma_h} f_i(x)y^{2s-1}dx,\ \ i=0,1,\ldots,n-1.$$

Let σ be the involution associated to A and define $\ell_i:=\mathscr{B}_\sigma\Big(\frac{f_i}{A'B^{\frac{2s-1}{2m}}}\Big)$. If $\big(\ell_0,\ell_1,\ldots,\ell_{n-1}\big)$ is a CT-system on $(0,x_r)$ and $s>m(n-2)$, then $(I_0,I_1,\ldots,I_{n-1})$ is an ECT-system on $(0,h_0)$.

The next result can also be found in [26]. It is very useful in order to apply the previous criterion to a collection of Abelian integrals not verifying the condition $s>m(n-2)$.

Lemma 3.4.12. *Let γ_h be an oval inside the level curve $\{A(x)+B(x)y^2=h\}$, and we consider a function F such that F/A' is analytic at $x=0$. Then, for any $k\in\mathbb{N}$,*

$$\int_{\gamma_h}F(x)y^{k-2}dx=\int_{\gamma_h}G(x)y^kdx,$$

where $G(x)=\frac{2}{k}\left(\frac{BF}{A'}\right)'(x)-\left(\frac{B'F}{A'}\right)(x)$.

Proof of Theorem 3.4.6. Fix some $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$ and, setting $\hat{\mu} = (\hat{q}, \hat{p})$, take a germ of analytic curve

$$\mu(\varepsilon) = \big(\hat{q} + \kappa_1 \varepsilon^\ell + \mathrm{o}(\varepsilon^\ell), \hat{p} + \kappa_2 \varepsilon^\ell + \mathrm{o}(\varepsilon^\ell)\big) \text{ with } \kappa_1 \neq 0 \text{ or } \kappa_2 \neq 0.$$

We consider the one-parameter perturbation $X_{\mu(\varepsilon)}$ and an easy computation shows that

$$X_{\mu(\varepsilon)} = X_{\hat{\mu}} + Z\varepsilon^\ell + \mathrm{o}(\varepsilon^\ell) \text{ with } Z := \big(\kappa_2 (x+1)^{\hat{p}} - \kappa_1 (x+1)^{\hat{q}}\big) \log(x+1) \partial_y.$$

Hence $j^{\ell-1}\big(X_{\mu(\varepsilon)}\big)$ is isochronous for all $\varepsilon \approx 0$ and, by applying Lemma 3.4.7, we have $T_0' \equiv T_1' \equiv \cdots \equiv T_{\ell-1}' \equiv 0$ and

$$T_\ell'(s) = -\int_0^{T_0} U(\alpha_\ell)|_{(x,y)=\varphi(t;s)}\, dt \text{ for all } s \in I,$$

where $\varphi(t;s)$ is the solution of $X_{\hat{\mu}}$ with $\varphi(0;s) = \xi(s)$. Note also that, due to

$$\alpha = \frac{< X_{\mu(\varepsilon)}, U^\perp >}{< X_{\hat{\mu}}, U^\perp >} = \sum_{i \geqslant 0} \alpha_i \varepsilon^i,$$

we have $\alpha_\ell = \frac{<Z, U^\perp>}{<X_{\hat{\mu}}, U^\perp>}$.

Recall that $H(x,y) = \frac{1}{2}y^2 + V_{\hat{\mu}}(x)$ is the Hamiltonian associated to $X_{\hat{\mu}}$. Hence, by construction, the solution $\varphi(t;s)$ is inside the energy level $H(x,y) = H(\xi(s)) =: \eta(s)$. Since it is clear that $\eta\colon I \longrightarrow (0, h_0)$ is a diffeomorphism, to prove the result we can consider $J(h) := T_\ell'(\eta^{-1}(h))$ for $h \in (0, h_0)$. More precisely, the result will be proved once we show that there exist analytic functions J_1 and J_2 with (J_1, J_2) being an ECT-system on $(0, h_0)$ and such that $J(h) = \kappa_1 J_1(h) + \kappa_2 J_2(h)$. With this aim in view notice first that

$$J(h) = -\int_{\gamma_h} \frac{U(\alpha_\ell)(x,y)}{y} dx,$$

where as usual γ_h denotes the oval inside the energy level $H(x,y) = h$ for $h \in (0, h_0)$.

We must at this point particularize the proof for each one of the three isochronous centers. We will show in detail the computations for $\hat{\mu} = (-3, 1)$. In this case we have $\alpha_\ell(x,y) = (-\kappa_1 + \kappa_2 (x+1)^4) f(x,y)$ with

$$f(x,y) = \frac{((x+1)^2 y^2 + x(x+2)(x^2+2\,x+2)) \log(x+1)}{((x+1)^2 y^2 + (x^2+2\,x+2)^2 (x+1)^2)(y^2 + x^2(x+2)^2)}.$$

We then compute $U(\alpha_\ell) = \nabla \alpha_\ell \cdot U$, which yields to

$$\begin{aligned} J(h) = {} & \frac{\kappa_1}{16h(h+2)} \int_{\gamma_h} \frac{g_3(x)}{(x+1)^2} y^3 + \frac{g_4(x)}{(1+x)^4} y + \frac{g_5(x)}{(x+1)^6 y} dx \\ & - \frac{\kappa_2}{16h(h+2)} \int_{\gamma_h} (1+x)^2 y^3 + g_1(x) y + \frac{g_2(x)}{(x+1)^2 y} dx, \end{aligned}$$

where

$$\begin{aligned}
g_1(x) &= 2(4x+6x^2+4x^3+x^4)+4\log(x+1),\\
g_2(x) &= (4x+6x^2+4x^3+x^4)(4x+6x^2+4x^3+x^4-4\log(x+1)),\\
g_3(x) &= 4\log(x+1)-1,\\
g_4(x) &= 4(2x^4+8x^3+12x^2+8x-1)\log(x+1)-(4x+6x^2+4x^3+x^4),\\
g_5(x) &= 4(x+1)^4(4x+6x^2+4x^3+x^4)\log(x+1)-(4x+6x^2+4x^3+x^4)^2.
\end{aligned}$$

Here we used that, due to $V_{\mu_1}(x)=\frac{x^2(x+2)^2}{2(x+1)^2}$, we have $y^2+\frac{x^2(x+2)^2}{(x+1)^2}=2h$ for all $(x,y)\in\gamma_h$. Next we apply twice Lemma 3.4.12 to get

$$J(h)=\frac{-1}{12h(h+2)}\left(\kappa_1\int_{\gamma_h}\frac{7y^3dx}{3(x+1)^2}+\kappa_2\int_{\gamma_h}8(x+1)^2y^3dx\right).$$

The projection of the period annulus $\mathscr{P}_{\mu_1}$ on the x-axis is $(-1,+\infty)$ and, according to Lemma 3.4.5, the involution associated to V_{μ_1} is $\sigma(x)=-\frac{x}{x+1}$. Next, setting $f_0(x)=\frac{1}{(x+1)^2}$ and $f_1(x)=(x+1)^2$, we will apply Theorem 3.4.11 with $A=V_{\mu_1}$, $B=\frac{1}{2}$, $m=1$ and $s=n=2$. Following its notation, we obtain

$$\ell_0(x)=\frac{8(x+1)}{x(x+2)}\ \text{ and }\ \ell_1(x)=\frac{8((x+1)^4-2x-x^2)}{x(x+1)(x+2)}.$$

Note that $\ell_0(x)\neq 0$ for all $x\in(0,+\infty)$. One can also verify that the Eronskian of ℓ_0 and ℓ_1 does not vanish on $(0,+\infty)$ neither. Then by Lemma 1.4.3 we can assert that (ℓ_0,ℓ_1) is an ECT-system on $(0,+\infty)$. Therefore, setting

$$J_1(h)=\frac{-7}{36h(h+2)}\int_{\gamma_h}\frac{y^3dx}{(x+1)^2}\ \text{ and }\ J_2(h)=\frac{-2}{3h(h+2)}\int_{\gamma_h}(x+1)^2y^3dx,$$

by applying Theorem 3.4.11 we can conclude that (J_1,J_2) is an ECT-system on $(0,h_0)$. Finally, on account of $T'_\ell(\eta^{-1}(h))=J(h)=\kappa_1J_1(h)+\kappa_2J_2(h)$, the result follows for μ_1 taking $A_i(s)=J_i(\eta(s))$ for $i=1,2$.

Since the proof for $\mu_2=(-1/2,0)$ and $\mu_3=(0,1)$ follows exactly the same way, we omit it here for the sake of brevity. ■

Proof of Theorem H. The fact that the center of the differential system in (3.1) is isochronous if and only if $\mu\in\{\mu_1,\mu_2,\mu_3\}$ follows from Theorem 3.3.5. Let us fix some $\hat\mu\in\{\mu_1,\mu_2,\mu_3\}$ and take a germ of analytic curve $\varepsilon\longmapsto\mu(\varepsilon)$ in Λ with $\mu(0)=\hat\mu$. Let us set $\hat\mu=(\hat q,\hat p)$ and note that there exists $\ell\in\mathbb{N}$ such that

$$\mu(\varepsilon)=\big(\hat q+\kappa_1\varepsilon^\ell+\mathrm{o}(\varepsilon^\ell),\hat p+\kappa_2\varepsilon^\ell+\mathrm{o}(\varepsilon^\ell)\big)\ \text{ with }\kappa_1\neq 0\text{ or }\kappa_2\neq 0.$$

Then, by applying Theorem 3.4.6, the period function $T(s;\varepsilon)$ corresponding to the perturbation $X_{\mu(\varepsilon)}$ verifies $T'_0\equiv T'_1\equiv\cdots\equiv T'_{\ell-1}\equiv 0$ and $T'_\ell(s)=\kappa_1A_1(s)+\kappa_2A_2(s)$ for all

$s \in I$. Moreover (A_1, A_2) is an ECT-system on I. Accordingly, by applying the Implicit Function Theorem, we can assert that, for each $\varepsilon \approx 0$, $T'(s;\varepsilon)$ has at most one zero on I counted with multiplicity. This proves that $\mathrm{Crit}\big((\mathscr{P}_{\hat{\mu}}, X_{\hat{\mu}}), X_{\mu(\varepsilon)}\big) \leqslant 1$.

Finally, in order to show that there exists a perturbation of $X_{\hat{\mu}}$ for which this upper bound is achieved, it suffices to consider

$$\mu(\varepsilon) = \big(\hat{q} + \kappa_1\varepsilon + \mathrm{o}(\varepsilon), \hat{p} + \kappa_2\varepsilon + \mathrm{o}(\varepsilon)\big)$$

taking κ_1 and κ_2 such that $\kappa_1 A_1(\hat{s}) + \kappa_2 A_2(\hat{s}) = 0$ for some $\hat{s} \in I$, i.e., $-\frac{\kappa_1}{\kappa_2} \in \big(\frac{A_2}{A_1}\big)(I)$. Here we use of course that A_1 and A_2 do not depend on the particular curve $\varepsilon \longmapsto \mu(\varepsilon)$ chosen. This proves the result. ∎

3.5 Criticality at the outer boundary.

At this point, for reader's convenience we recall some notation introduced before. For an analytic planar potential differential system $X = -y\partial_x + V'(x)\partial_y$ with a non-degenerated center at the origin, we denote $\mathscr{P}$ its period annulus and $\mathcal{I} = (x_\ell, x_r)$ the projection of $\mathscr{P}$ over the x-axis. The corresponding Hamiltonian function is $H(x,y) = \frac{1}{2}y^2 + V(x)$. Then $H(\mathscr{P}) = (0, h_0)$, where $h_0 \in (0, +\infty]$ is the energy level of the outer boundary of $\mathscr{P}$. We define in addition

$$g(x) := x\sqrt{\frac{V(x)}{x^2}},$$

which is clearly an analytic diffeomorphism from $\mathcal{I}$ to $(-\sqrt{h_0}, \sqrt{h_0})$, and we denote by $T(h)$ the period of the periodic orbit $\gamma_h \subset \{\frac{1}{2}y^2 + V(x) = h\}$. The period function T is analytic on $(0, h_0)$ and it can be extended analytically at $h = 0$. In what follows we shall consider a potential differential system depending on a parameter $\mu \in \Lambda$ and we shall use the previous notations with a subscript μ.

In order to state the main result of this section let us denote

$$\Gamma_B := \{\mu \in \Lambda : q = 0\} \cup \{\mu \in \Lambda : p = 1, q \leqslant -1\} \cup \{\mu \in \Lambda : p + 2q + 1 = 0, q \geqslant -1\} \quad (3.7)$$

and

$$\Gamma_U := \{\mu \in \Lambda : (3q+1)(q+1) = 0\} \cup \{(-1/2, p_0)\},$$

where $p_0 \approx 1.2017$ is the only zero of the function $p \mapsto (2+2p)^{\frac{2+2p}{1+2p}} - 2(1+2p)$ (see Lemma 4.0.6). Here the subscripts B and U stand for bifurcation and unspecified, respectively. The curve Γ_B splits the parameter space Λ into three connected components, see Figure 3.6. We denote by D_B the uncoloured component and by I_B the union of the two other components in dark grey. Therefore, taking into account the definition of criticality at the outer boundary given in Definition 2.2.2, the result concerning the family under consideration is the following:

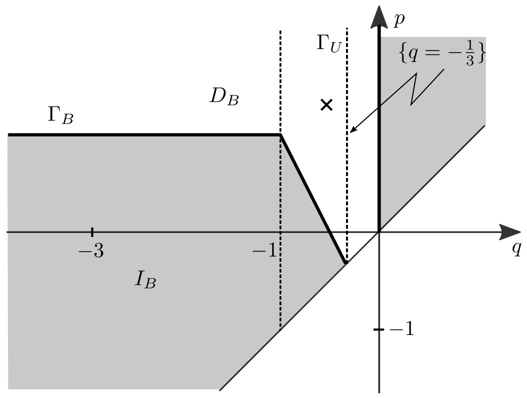

Figure 3.6: Bifurcation diagram of the period function at the outer boundary according to Theorem I.

Theorem I. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of vector fields in (3.1) and consider the period function of the center at the origin. Then the open set $\Lambda\setminus(\Gamma_B\cup\Gamma_U)$ corresponds to local regular values of the period function at the outer boundary of the period annulus. In addition,*

(a) If $\hat\mu\in I_B\setminus\Gamma_U$ then the period function of $X_{\hat\mu}$ is increasing near the outer boundary.

(b) If $\hat\mu\in D_B\setminus\Gamma_U$ then the period function of $X_{\hat\mu}$ is decreasing near the outer boundary.

Moreover the parameters in Γ_B are local bifurcation values of the period function at the outer boundary of the period annulus. Finally, $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)=1$ *for all $\hat\mu=(\hat q,1)$ with $\hat q<-3$, $\hat\mu=(\hat q,-2\hat q-1)$ with $\hat q\in(-\frac{3}{5},-\frac{1}{3})\setminus\{-\frac{1}{2}\}$ and $\hat\mu=(0,\hat p)$ with $\hat p\notin\{\frac{1}{2},1\}$.*

In order to prove the result stated above we shall use the criteria obtained in Chapter 2 concerning the upper-bound of the criticality at the outer boundary for families of potential systems. For this reason, since we have developed different techniques depending on the energy at the outer boundary, we consider the study for each region Λ_i, $i=1,2,3$, separately. The proof of Theorem I concerning parameters in Λ_1 follows from Propositions 3.5.3 and 3.5.6. The proof for the parameters in Λ_2 follows from Proposition 3.5.8 and the proof for Λ_3 is deduced from Proposition 3.5.9. These results are proved in Sections 3.5.1, 3.5.2 and 3.5.3, respectively. For the reader's convenience we point out that the three sections can be read independently.

3.5.1 Parameters in Λ_1

The proof of Theorem I for parameter in Λ_1 follows from Propositions 3.5.3 and 3.5.6. In Proposition 3.5.3 we prove the major part of Theorem I using the techniques in Theorem D. However, these tools do not allow to prove that the parameters $\mu\in\Lambda_1$ with

$q = -1/2$ are local regular values of the period function at the outer boundary. They neither permit to prove that the criticality for parameters with $q = 0$ is one. In order to deal with these two remaining situations, which will be proved in Proposition 3.5.6, we shall use Theorem E.

For the parameter values under consideration, on account of (3.3), we have that

$$V_\mu(x) = \frac{(x+1)^{p+1}}{p+1} - \frac{(x+1)^{q+1}}{q+1} + h_0(\mu), \tag{3.8}$$

where $h_0(\mu) := \frac{p-q}{(p+1)(q+1)}$ is the energy level at the outer boundary of $\mathscr{P}_\mu$. Moreover, on account of Lemma 3.2.3, the projection of $\mathscr{P}_\mu$ on the x-axis is $\mathcal{I}_\mu = (-1, \rho(\mu))$. We point out that the family $\{X_\mu\}_{\mu\in\Lambda_1}$ satisfies hypothesis in **(H)**. Indeed, the map $(x,\mu) \to V_\mu^{(k)}(x)$ is continuous on $\{(x,\mu) \in \mathbb{R}\times\Lambda_1 : x \in I_\mu\}$ for all $k \geqslant 0$ directly by explicit derivation. On the other hand, $x_\ell(\mu) = -1$ and $x_r(\mu) = \rho(\mu)$ which are both continuous functions on Λ_1. Finally, for the parameters under consideration the energy level at the outer boundary $h_0(\mu)$ is a continuous function.

At this point we refer the reader to the definitions of *admissible* analytic potential system and *regular* endpoint of $\mathcal{I}_\mu$ in Definition 2.4.2, and to the definition of hypothesis **(C)** and the function $\gamma_M(\mu)$ in Definition 2.4.8, which are notions used in the following statement.

Lemma 3.5.1. *Let X_μ be the potential vector field defined in* (3.1). *The following statements hold:*

(*a*) *X_μ is admissible for all $\mu \in \Lambda_1$ and $x_r(\mu)$ is regular.*

(*b*) *If $\hat\mu \in \{(q,p) \in \Lambda_1 : q(2q+1) \neq 0\}$ then $\hat\mu$ satisfies condition* **(C_1-C_3)** *and, moreover, $\gamma_M(\mu) = \frac{3}{2}(q+1)$.*

(*c*) *If $\hat\mu \in \{(q,p) \in \Lambda_1 : q(q-1)(2q+1)(3q+2) \neq 0\}$ then $\hat\mu$ satisfies condition* **(C_4-C_6)** *at $x_\ell(\mu)$.*

Proof. For proving the first assertion of the lemma let us show that condition (*a*) of Definition 2.4.2 is satisfied for $\mu \in \Lambda_1$. Indeed, V_μ is analytic at $x_r(\mu) = \rho(\mu)$ and $V_\mu'(x_r(\mu)) = (p-q)(1+p)^{\frac{q}{p-q}}(1+q)^{\frac{p}{q-p}} \neq 0$ so $x_r(\mu)$ is regular.

To prove (*b*) let us fix $\hat\mu = (\hat q, \hat p) \in \Lambda_1$ with $\hat q \neq 0$ and $\hat q \neq -1/2$. We shall prove first condition **(C_1)**. That is, the family $\{g_\mu''/(g_\mu')^3\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat\mu$ at $x_\ell = -1$. With this aim in view we shall show that $(g_\mu''/(g_\mu')^3)' = \frac{g_\mu' g_\mu''' - 3g_\mu''}{(g_\mu')^4}$ does not accumulate zeroes near $x_\ell = -1$ for $\mu \approx \hat\mu$. Since $g_\mu'(x)$ is smooth in $\mathcal{I}_\mu$ it is enough to show that the function $g_\mu' g_\mu''' - 3g_\mu''$ does not accumulate zeroes at $x_\ell = -1$ for $\mu \approx \hat\mu$. By definition,

$$g_\mu' g_\mu''' - 3g_\mu'' = \frac{3V_\mu''(V_\mu')^2 + 6(V_\mu'')^2 V_\mu - 2V_\mu''' V_\mu' V_\mu}{8V_\mu^2}.$$

Again, in this case due to the regularity of V_μ in $\mathcal{I}_\mu$, it is enough to prove that the function on the numerator does not accumulate zeroes. Let us denote by P_μ the numerator of the previous expression. Then some computations show that

$$P_\mu(x-1) = \frac{a_0 + a_1 x^{2(p-q)} + a_2 x^{1+3p-2q} + a_3 x^{p-q} + a_4 x^{1+2p-q} + a_5 x^{1+p} + a_6 x^{1+q}}{x^{2-2q}}$$

where $a_i = a_i(\mu)$ are continuous rational functions on $\mu = (q,p)$ in Λ_1 that we omitted for the sake of shortness. Since $\mu \in \Lambda_1$ we have $p+1 > q+1 > 0$ so all the exponents on the numerator are positive. Notice that the function $(x,\mu) \mapsto x^{2-2q} P_\mu(x-1)$ is continuous at $(0,\hat{\mu})$ with $\hat{\mu} \in \Lambda_1$. Therefore we have that

$$\lim_{(x,\mu)\to(0,\hat{\mu})} x^{2-2q} P_\mu(x-1) = a_0(\hat{q},\hat{p}).$$

An easy computation shows that $a_0(\hat{q},\hat{p}) = \frac{2(\hat{p}-\hat{q})\hat{q}(1+2\hat{q})}{(\hat{p}+1)(\hat{q}+1)}$, which is different from zero in the region under consideration. Consequently the function $P_\mu(x)$ does not vanish near $x_\ell = -1$ for all $\mu \approx \hat{\mu}$ and therefore the family $\{(g_\mu)''/(g_\mu')^3\}$ is uniformly monotonous on $x_\ell = -1$ at $\hat{\mu}$. This proves **(C$_1$)**. Notice that the change of sign in the coefficient $a_0(q,p)$ when $q \approx -\frac{1}{2}$ implies there is no uniformity on the monotonicity in $\hat{q} = -\frac{1}{2}$.

Let us check that $\hat{\mu}$ verifies **(C$_2$)**. On account of the expression in (3.8) we have that

$$\lim_{(x,\mu)\to(-1,\hat{\mu})} (h_0(\mu) - V_\mu(x))(x+1)^{-(q+1)} = \lim_{(x,\mu)\to(-1,\hat{\mu})} \frac{1}{q+1} - \frac{(x+1)^{p-q}}{p+1} = \frac{1}{\hat{q}+1} \neq 0.$$

Then we have that $\{h_0(\mu) - V_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at $\hat{\mu}$ in $x_\ell = -1$ by $\alpha_\ell(\mu) = -(q+1)$. Moreover, on account of expression in (3.8), we can easily see that

$$\lim_{(x,\mu)\to(-1,\hat{\mu})} V_\mu'(x)(x+1)^{-q} = \lim_{(x,\mu)\to(-1,\hat{\mu})} 1 - (x+1)^{p-q} = 1,$$

$$\lim_{(x,\mu)\to(-1,\hat{\mu})} V_\mu''(x)(x+1)^{1-q} = \lim_{(x,\mu)\to(-1,\hat{\mu})} q - p(x+1)^{p-q} = \hat{q} \neq 0.$$

Consequently the families $\{V_\mu'\}_{\mu\in\Lambda}$ and $\{V_\mu''\}_{\mu\in\Lambda}$ are continuously quantifiable in $\hat{\mu}$ at $x_\ell = -1$ by $\alpha = -q$ and $\alpha = 1-q$, respectively. This shows that condition **(C$_2$)** is verified.

Let us show now that $\hat{\mu}$ satisfies **(C$_3$)**. Indeed, the quantifier of $\{h_0(\mu) - V_\mu\}_{\mu\in\Lambda}$ in $\hat{\mu}$ is $\alpha_\ell(\hat{\mu}) = -(\hat{q}+1) \neq -1$ since $\hat{q} \neq 0$. Finally, since x_r is regular, by definition $\gamma_M(\mu) = -\frac{3}{2}\alpha_\ell(\mu) = \frac{3}{2}(q+1)$.

Let us turn to the proof of (c) so let us assume $\hat{\mu} \in \Lambda$ such that $\hat{q} \notin \{-2/3, -1/2, 0, 1\}$. Let us start with proving that $\hat{\mu}$ satisfies **(C$_4$)**. That is, the family $\{\frac{3(g_\mu'')^2 - g_\mu''' g_\mu'}{(g_\mu')^5}\}_{\mu\in\Lambda}$ is uniformly monotonous in $\hat{\mu}$ at $x_\ell(\mu) = -1$. On account that $g_\mu^2 = V_\mu$ we have that

$$\frac{3(g_\mu'')^2 - g_\mu''' g_\mu'}{(g_\mu')^5} = -\frac{4V_\mu^{\frac{1}{2}}(3V_\mu''(V_\mu')^2 - 6(V_\mu'')^2 V_\mu + 2V_\mu''' V_\mu' V_\mu)}{(V_\mu')^5}$$

so we will proof that the derivative of this function does not accumulate zeroes at $x_\ell = -1$. For the sake of simplicity we omit the computations and we have that

$$\left(\frac{3(g_\mu'')^2 - g_\mu''' g_\mu'}{(g_\mu')^5}\right)'(x) = \frac{-2Q_\mu(x)}{\sqrt{V_\mu(x)}V_\mu'(x)^6}$$

where $(x,\mu) \mapsto Q_\mu(x)$ is the sum of 15 monomials of the form $c(\mu)(x+1)^{n_1 p + n_2 q + n_3}$ with $n_i \in \mathbb{Z}$, for $i = 1,2,3$, and c a well defined rational function at $\mu = \hat{\mu}$. Moreover the monomial with the lowest exponent for $\mu \approx \hat{\mu}$ is $(x+1)^{3q-3}$. Consequently,

$$\left(\frac{3(g_\mu'')^2 - g_\mu''' g_\mu'}{(g_\mu')^5}\right)'(x) = \frac{-2(x+1)^{3q-3}}{\sqrt{V_\mu(x)}V_\mu'(x)^6}\left(\frac{-4(p-q)^2 q(2+7q+6q^2)}{(1+p)^2(1+q)^2} + r_\mu(x)\right)$$

with $\lim_{x\to -1} r_\mu(x) = 0$ uniformly for $\mu \approx \hat{\mu}$. Therefore, taking into account the expression of V_μ in (3.8), we have that the previous expression does not accumulate zeroes at $x_\ell = -1$ for $\mu \approx \hat{\mu}$. Consequently, the family is uniformly monotonous in $\hat{\mu}$ at x_ℓ. This proves **(C$_4$)**.

On account of the expression of V_μ we can easily see that $\{V_\mu'''\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at x_ℓ by $2-q$ with limit $\hat{q}(\hat{q}-1)$. This proves that $\hat{\mu}$ satisfies **(C$_5$)** as we desired. Finally, $\alpha_\ell(\hat{\mu}) = -(\hat{q}+1) \neq -2$ since $\hat{q} \neq 1$. This proves **(C$_6$)** and ends with the proof of the lemma. ∎

Proposition 3.5.2. *Consider the period function T_μ of the center at the origin of system* (3.1) *with $(q,p) \in \Lambda_1$. Then the following hold:*

$$(i)\ \lim_{h\to h_0(\mu)} T_\mu(h) = \begin{cases} \sqrt{2\pi}\frac{\sqrt{q+1}}{p-q}\left(\frac{p+1}{q+1}\right)^{\frac{1-q}{2(p-q)}}\frac{\Gamma\left(\frac{1-q}{2(p-q)}\right)}{\Gamma\left(\frac{1+p-q}{2(p-q)}\right)} & \textit{if } -1<q<1,\\ +\infty & \textit{if } q \geqslant 1.\end{cases}$$

$$(ii)\ \lim_{h\to h_0(\mu)} T_\mu'(h) = \begin{cases} -\sqrt{2\pi}\frac{(p+1)^{\frac{3}{2}}(p+2q+1)}{2(p-q)^2\left(\frac{p+1}{q+1}\right)^{\frac{3p+1}{2(p-q)}}}\frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)} & \textit{if } -1<q<-\frac{1}{3},\\ -\infty & \textit{if } -\frac{1}{3}\leqslant q<0,\\ +\infty & \textit{if } q>0.\end{cases}$$

Proof. Since $\mu \in \Lambda_1$ we have that $h_0(\mu)$ is finite. Taking $g_\mu^2 = V_\mu$ into account and the expression of V_μ in (3.3), deriving implicitly it easily follows that $(g_\mu^{-1})'''$ is non-vanishing near the endpoints of $(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)})$. Consequently $(g_\mu^{-1})''$ is monotonous near the endpoints of $(-\sqrt{h_0(\mu)}, \sqrt{h_0(\mu)})$. Since on the other hand X_μ is admissible thanks to Lemma 3.5.1, we can apply Corollary 2.4.5 to conclude that

$$\lim_{h\to h_0(\mu)} T_\mu(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})'(\sqrt{h_0(\mu)}\sin\theta)d\theta$$

and

$$\lim_{h\to h_0(\mu)} T'_\mu(h) = \frac{1}{\sqrt{2h_0(\mu)}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h_0(\mu)}\sin\theta)\sin\theta d\theta.$$

where the previous integrals are considered formally since they may diverge. In the first case, if we perform the change of variable $x = g_\mu^{-1}(\sqrt{h_0(\mu)}\sin\theta)$ we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})'(\sqrt{h_0(\mu)}\sin\theta)d\theta = \int_{-1}^{\rho(\mu)} \frac{dx}{\sqrt{h_0(\mu)-V_\mu(x)}}.$$

Then (i) follows by the first assertion on Lemma 4.0.5 in the Appendix. In the second case, with the same change of variable we have that

$$\frac{1}{\sqrt{2h_0(\mu)}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h_0(\mu)}\sin\theta)\sin\theta d\theta = \frac{\sqrt{2}}{2h_0(\mu)} \int_{-1}^{\rho(\mu)} \frac{-g''_\mu(x)g_\mu(x)}{g'_\mu(x)^2\sqrt{h_0(\mu)-V_\mu(x)}}.$$

Using that $g_\mu^2 = V_\mu$ it follows that

$$\frac{1}{\sqrt{2h_0(\mu)}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h_0(\mu)}\sin\theta)\sin\theta d\theta = \frac{\sqrt{2}}{h_0(\mu)} \int_{-1}^{\rho(\mu)} \frac{\frac{1}{2} - \frac{V_\mu(x)V''_\mu(x)}{V'_\mu(x)^2}}{\sqrt{h_0(\mu)-V_\mu(x)}}.$$

Then (ii) follows by the second assertion on Lemma 4.0.5 in the Appendix. ■

Next proposition is one of the two main results concerning the proof of Theorem I for the parameters in Λ_1.

Proposition 3.5.3. *If $\hat{\mu} = (\hat{q},\hat{p}) \in \Lambda_1$ satisfies $\hat{q}(\hat{p}+2\hat{q}+1)(2\hat{q}+1)(3\hat{q}+1) \neq 0$ then $\hat{\mu}$ is a local regular value of the period function at the outer boundary of system (3.1). Moreover,*

(a) If $\hat{q}(\hat{p}+2\hat{q}+1) > 0$ and $(2\hat{q}+1)(3\hat{q}+1) \neq 0$ then the period function of $X_{\hat{\mu}}$ is increasing near the outer boundary.

(b) If $\hat{q}(\hat{p}+2\hat{q}+1) < 0$ and $(2\hat{q}+1)(3\hat{q}+1) \neq 0$ then the period function of $X_{\hat{\mu}}$ is decreasing near the outer boundary.

On the other hand, if $\hat{q}(\hat{p}+2\hat{q}+1) = 0$ then $\hat{\mu}$ is a local bifurcation value of the period function at the outer boundary of system (3.1). Moreover, $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 1$ *if $\hat{p}+2\hat{q}+1 = 0$ with $\hat{q} \in (-\frac{3}{5},-\frac{1}{3}) \setminus \{-\frac{1}{2}\}$.*

Proof. Consider $\hat{\mu} = (\hat{q},\hat{p}) \in \Lambda_1$ with $\hat{q}(\hat{p}+2\hat{q}+1)(2\hat{q}+1)(3\hat{q}+1) \neq 0$. On account of Lemma 3.5.1 we have that the potential family is admissible and that $\hat{\mu}$ satisfies condition **(C_1-C_3)**. Moreover, $\gamma_M(\hat{\mu}) = \frac{3}{2}(\hat{q}+1)$.

If $\hat{q} > -\frac{1}{3}$ then $\gamma_M(\hat{\mu}) > 1$ and, by applying Theorem D, $\hat{\mu}$ is a local regular value of the period function at the outer boundary. Moreover Proposition 3.5.2 shows that if $\hat{q} < 0$ (respectively, $\hat{q} > 0$) then the period function tends to $-\infty$ (respectively, $+\infty$) as $h \longrightarrow h_0(\mu)$. This proves (a) and (b) for $\hat{q} > -\frac{1}{3}$ and also that, by Lemma 2.4.1, the set of parameters $\{\mu \in \Lambda_1 : q = 0\}$ consists of local bifurcation value of the period function at the outer boundary. On the other hand, if $\hat{q} < -\frac{1}{3}$ then $\gamma_M(\hat{\mu}) < 1$. In addition, Proposition 3.5.2 shows that function $\Delta_1(\mu)$ defined in Theorem D is

$$\Delta_1(\mu) = -\sqrt{2\pi}\frac{(p+1)^{\frac{3}{2}}(p+2q+1)}{2(p-q)^2\left(\frac{p+1}{q+1}\right)^{\frac{3p+1}{2(p-q)}}}\frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)}.$$

Due to $\hat{q}(\hat{p}+2\hat{q}+1)(2\hat{q}+1)(3\hat{q}+1) \neq 0$, we have $\Delta_1(\hat{\mu}) \neq 0$ so Theorem D (a) guarantees that $\hat{\mu}$ is a local regular value of the period function at the outer boundary. This proves the assertion about the regularity. Moreover, if $\hat{p}+2\hat{q}+1 < 0$ and $\hat{q} < -\frac{1}{3}$, then $\Delta_1(\hat{\mu}) > 0$ whereas if $\hat{p}+2\hat{q}+1 > 0$, then $\Delta_1(\hat{\mu}) < 0$. This proves the assertion concerning the monotonicity of the period function near the outer boundary if $\hat{q} < -\frac{1}{3}$. Due to the change of sign of Δ_1, Lemma 2.4.1 shows that if $\hat{p}+2\hat{q}+1 = 0$ then $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \geqslant 1$, so the set of parameters $\{\mu \in \Lambda_1 : p+2q+1 = 0\}$ are local bifurcation values of the period function at the outer boundary.

Finally, if $\hat{p}+2\hat{q}+1 = 0$ and $\hat{q} \in (-\frac{3}{5}, -\frac{1}{3}) \setminus \{-\frac{1}{2}\}$ then $\gamma_M(\hat{\mu}) = \frac{3}{2}(\hat{q}+1) \in (\frac{3}{5}, 1) \setminus \{\frac{3}{4}\}$. Moreover, by Lemma 3.5.1 we have that **(C_1-C_6)** are satisfied so using Theorem D (b) we have $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 1$. ■

As we advanced at the beginning of this section, the previous result does not prove Theorem I in its totality. It remains two issues to be proved: the regularity of the parameter line $\{\mu \in \Lambda_1 : q = -1/2\}$ except the point $(-1/2, p_0)$, and to bound the criticality on the line $\{\mu \in \Lambda_1 : q = 0\}$. In order to prove this, we shall use the techniques developed in Section 2.4.2. Next results deal with these two remaining set of parameters. Before state it, we introduce the notion of *Gaussian Hypergeometric function.*

Definition 3.5.4. The function defined by

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(k)_n := k(k+1)\dots(k+n-1)$ for $n \geqslant 1$ and $(k)_0 := 1$, is the *Gaussian Hypergeometric function.* □

We recall that $\mathscr{D}_{\boldsymbol{\nu}_n}$ and N_n in the following statement refer to the operator and the momentum introduced in Definition 2.4.12 and Definition 2.4.14, respectively.

Lemma 3.5.5. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of potential vector fields in (3.1) and let f_μ be the even part of $z \longmapsto z\sqrt{h_0(\mu)}(g_\mu^{-1})''(z\sqrt{h_0(\mu)})$, where $g_\mu(x) := \operatorname{sgn}(x)\sqrt{V_\mu(x)}$ for $x\in\mathcal{I}_\mu$. Then the following hold:*

(a) If $\mu=(-\frac{1}{2},p)$ with $p\in(-\frac{1}{2},+\infty)\setminus\{0\}$, then $N_1[f_\mu]\neq 0$.

(b) If $\mu=(0,p)$ with $p\in(\frac{1}{2},+\infty)\setminus\{1\}$ and $\nu_1(\mu)=0$, then $N_1\big[\mathscr{D}_{\nu_1(\mu)}[f_\mu]\big]\neq 0$.

Proof. For the sake of shortness we shall omit the non-essential dependence on μ. The change of variable $x=g^{-1}(z\sqrt{h_0})$ gives formally

$$N_1[f]=\int_{-1}^{1}\frac{z\sqrt{h_0}(g^{-1})''(z\sqrt{h_0})}{\sqrt{1-z^2}}dz=\int_{-1}^{\rho}\frac{\big(V'^2-2VV''\big)(x)}{V'(x)^2\sqrt{h_0-V(x)}}dx. \tag{3.9}$$

It is proved in Lemma 4.0.5 that the previous integral is convergent for $p-q>0$ and $q\in(-1,-\frac{1}{3})$. That result provides moreover its precise value in terms of the Gamma function. On account of this we can assert that, for those parameters,

$$N_1[f]=-\sqrt{\pi}\frac{(p+1)^{\frac{1}{2}}(p+2q+1)}{(p-q)(q+1)\big(\frac{p+1}{q+1}\big)^{\frac{3p+1}{2(p-q)}}}\frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)}.$$

Therefore, for $\mu=(-\frac{1}{2},p)$ with $p>-\frac{1}{2}$, $N_1[f]=0$ if, and only if $p=0$. This proves (a).

In order to show (b) let us take $\mu=(0,p)$ with $p>\frac{1}{2}$. We shall take advantage of the second part of Theorem E with $n=0$ to prove that then f is quantifiable at $z=1$ by $\xi<\frac{1}{2}$. Indeed, if $\mu=(0,p)$ then $h_0-V(x)=x+1-\frac{(x+1)^{p+1}}{p+1}$ is quantifiable at $x_\ell=-1$ by $\beta_\ell=-1$ and at $x_r=(p+1)^{\frac{1}{p}}-1$ by $\beta_r=-1$. On the other hand, some computations show that $(h_0-V(x))V(x)^{\frac{1}{2}}\Lambda(x)$ is quantifiable at $x_\ell=-1$ by $\alpha_\ell=-1$ if $p>1$ and by $\alpha_\ell=-p$ if $p<1$, in both cases with limit $a_\ell=\sqrt{h_0}$. Similarly, it is quantifiable at $x_r=(p+1)^{\frac{1}{p}}-1$ by $\alpha_r=-1$ with limit $a_r=\sqrt{h_0}\big(1-2(1+p)^{-\frac{1}{p}}\big)$. According to this and taking $p>\frac{1}{2}$ into account, by applying the second part of Theorem E with $n=0$, it turns out that f is quantifiable at $z=1$ by $\xi<\frac{1}{2}$. Then, by applying Lemma 2.4.19 with $\ell=n=1$ and $\nu_1=0$, $N_1\big[\mathscr{D}_{\nu_1}[f]\big]=-(1+\nu_1)N_1[f]=-N_1[f]$. Consequently, from (3.9),

$$N_1\big[\mathscr{D}_{\nu_1}[f]\big]=-N_1[f]=-\int_{-1}^{\rho}\frac{\big(V'^2-2VV''\big)(x)}{V'(x)^2\sqrt{h_0-V(x)}}dx.$$

Since $V(x)=\frac{(x+1)^{p+1}-1}{p+1}-x$ for $\mu=(0,p)$ with $p\neq-1$, some long but easy computations by applying Lemma 4.0.7 show that

$$G(x):=2\frac{{}_2F_1\big(\frac{1}{2},-\frac{1}{2p},1-\frac{1}{2p},\frac{(x+1)^p}{p+1}\big)-\frac{x}{(x+1)^p-1}\sqrt{1-\frac{(x+1)^p}{1+p}}}{\sqrt{x+1}}$$

is a primitive of $\frac{V'^2-2VV'}{V'^2\sqrt{h_0-V}}$. By [1, 15.1.20], using $p>\frac{1}{2}$, we get

$$\lim_{x\to\rho} G(x)=\frac{2\sqrt{\pi}\Gamma\left(1-\frac{1}{2p}\right)}{(p+1)^{\frac{1}{2p}}\Gamma\left(\frac{p-1}{2p}\right)}.$$

On the other hand, since by definition

$${}_2F_1\left(\frac{1}{2},-\frac{1}{2p},1-\frac{1}{2p},\frac{(x+1)^p}{p+1}\right)=1-\frac{(x+1)^p}{2(2p-1)(p+1)}+\mathrm{o}\big((x+1)^p\big),$$

we can assert, taking $p>\frac{1}{2}$ into account once again, that $\lim_{x\to-1}G(x)=0$. Consequently,

$$N_1\big[\mathscr{D}_{\nu_1}[f]\big]=-N_1[f]=\lim_{x\to-1}G(x)-\lim_{x\to\rho}G(x)=\frac{-2\sqrt{\pi}\Gamma\left(1-\frac{1}{2p}\right)}{(p+1)^{\frac{1}{2p}}\Gamma\left(\frac{p-1}{2p}\right)},$$

which shows that if $p\in(\frac{1}{2},+\infty)\setminus\{1\}$ and $\nu_1=0$, then $N_1\big[\mathscr{D}_{\nu_1}[f]\big]\neq 0$, as desired. ■

The next result, together with Proposition 3.5.3, ends with the proof of Theorem I for parameters in Λ_1.

Proposition 3.5.6. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of potential vector fields in (3.1) and consider the period function of the center at the origin. Then the following hold:*

(*a*) *If $\hat\mu=(-\frac{1}{2},\hat p)$ with $\hat p\in(-\frac{1}{2},+\infty)\setminus\{0,p_0\}$, where $p_0\approx 1.2017$ is the unique zero of the function $p\mapsto(2+2p)^{\frac{2+2p}{1+2p}}-2(1+2p)$ on $(-\frac{1}{2},+\infty)$, then* $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)=0$.

(*b*) *If $\hat\mu=(0,\hat p)$ with $\hat p\in(0,+\infty)\setminus\{\frac{1}{2},1\}$, then* $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)=1$.

Proof. Following the notation in Theorem E, from the expression in (3.8) and on account that $p>q$, it follows easily that $\{h_0(\mu)-V_\mu(x)\}_{\mu\in\Lambda}$ is continuously quantifiable for any $\hat\mu=(\hat q,\hat p)\in\Lambda$ at $x=x_\ell$ by $\beta_\ell(\mu)$ with limit b_ℓ and at $x=x_r$ by $\beta_r(\mu)$ with limit b_r, where

$$\beta_\ell(\mu)=-(q+1),\ \beta_r(\mu)=-1,\ b_\ell=\frac{1}{\hat q+1}\text{ and }b_r=V'_{\hat\mu}(x_r). \tag{3.10}$$

We will prove (a) by applying Theorem E with $n=0$. Let us consider $\hat\mu=(\hat q,\hat p)$ with $\hat q=-\frac{1}{2}$ and $\hat p\in(-\frac{1}{2},+\infty)\setminus\{0,p_0\}$, where p_0 is the unique root of the function $f(p):=(2+2p)^{\frac{2+2p}{1+2p}}-2(1+2p)$ on $(-\frac{1}{2},+\infty)$. We begin by studying the quantifiers of the family $\big\{(h_0-V_\mu)V_\mu{}^{\frac{1}{2}}\mathscr{R}_\mu\big\}_{\mu\in\Lambda}$, where recall that $\mathscr{R}_\mu:=\frac{(V_\mu')^2-2V_\mu V_\mu''}{(V_\mu')^3}$. An easy computation from (3.3) shows that this family is continuously quantifiable in $\hat\mu$ at $x=x_\ell$ by $\alpha_\ell(\mu)=q$ with limit $a_\ell=2\big(2-\frac{1}{\hat p+1}\big)^{\frac{3}{2}}$. Note on the other hand that V_μ is analytic at $x=x_r$ with $V'_{\hat\mu}(x_r)\neq 0$. Consequently the family is continuously quantifiable in $\hat\mu$ at $x=x_r$ by $\alpha_r(\mu)=-1$ with limit

$$a_r=\sqrt{h_0(\hat\mu)}\frac{V'_{\hat\mu}(x_r)^2-2h_0(\hat\mu)V''_{\hat\mu}(x_r)}{V'_{\hat\mu}(x_r)^2},$$

provided that $V_{\hat\mu}'(x_r)^2-2h_0(\hat\mu)V_{\hat\mu}''(x_r)\neq 0$. One can easily check that this inequality is equivalent to require that $f(\hat p)\neq 0$, which is indeed satisfied because $\hat p\neq p_0$ by assumption (we refer (*a*) in Lemma 4.0.6). Accordingly, taking (3.10) also into account, we have that $\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\hat\mu)=\big(\frac{\alpha_r}{\beta_r}\big)(\hat\mu)=1$ and

$$\big(a_r(b_r)^{-\frac{\alpha_r}{\beta_r}}+a_\ell(b_\ell)^{-\frac{\alpha_\ell}{\beta_\ell}}\big)(\hat\mu)=\frac{(2+2\hat p)^{\frac{2+2\hat p}{1+2\hat p}}+4\hat p(1+2\hat p)}{2(1+\hat p)^{\frac{3}{2}}\sqrt{1+2\hat p}}\neq 0.$$

(The fact that this expression is different from zero follows by (*b*) in Lemma 4.0.6.) Accordingly, by the second part of Theorem E, the family $\big\{\mathcal{P}[z\sqrt{h_0(\mu)}(g_\mu^{-1})''(z\sqrt{h_0(\mu)})]\big\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat\mu$ at $z=1$ by $\xi(\mu)=\max\{\frac{q}{q+1},-1\}+1$. We apply next the first part of Theorem E. To this end note that $\xi(\hat\mu)=0$ and that, by (*a*) in Lemma 3.5.5, the first momentum of the even part of $z\longmapsto z\sqrt{h_0(\hat\mu)}(g_{\hat\mu}^{-1})'(z\sqrt{h_0(\hat\mu)})$ does not vanish. Then, the application of (*b*1) in Theorem E with $n=0$ and $j=1$ shows that $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)=0$. This proves the validity of (*a*).

Finally let us turn to the proof of (*b*). So consider now $\hat\mu=(\hat q,\hat p)$ with $\hat q=0$ and $\hat p\in(0,+\infty)\setminus\{\frac{1}{2},1\}$. We note that, by Proposition 3.5.3, $\mathrm{Crit}\big((\Pi_{\hat\mu},X_{\hat\mu}),X_\mu\big)\geqslant 1$, and so the result will follow by applying Theorem E with $n=1$. To this end we need to study the function

$$\Psi_\mu(x):=\frac{1}{V_\mu'(x)}W\left[\left(\frac{V_\mu}{h_0(\mu)-V_\mu}\right)^{\frac{\nu_1(\mu)}{2}},(h_0(\mu)-V_\mu)\,V_\mu^{\frac{1}{2}}\,\mathscr{R}_\mu\right](x),$$

where ν_1 is a continuous function to be determined. Some tedious calculations show that

$$\Psi_\mu=\frac{\psi_\mu}{2V_\mu^{\frac{1}{2}}V_\mu'^5}\left(\frac{V_\mu}{h_0(\mu)-V_\mu}\right)^{\frac{\nu_1(\mu)}{2}},$$

where, omitting the dependence on μ for shortness,

$$\psi:=-(V'^2-2VV'')\big(V'^2(h_0(\nu_1-1)+3V)+6(h_0-V)VV''\big)+4V^2(V-h_0)V'V'''.$$

By means of an algebraic manipulator we can assert that $\psi_\mu(x)$ is the sum of 15 monomials of the form $r(\mu)(x+1)^{n_1p+n_2q+n_3}$ with $n_i\in\mathbb{Z}$, for $i=1,2,3$, and r a well defined rational function at $\mu=\hat\mu$. Moreover the monomial with the lowest exponent for $\mu\approx\hat\mu$ is $(x+1)^{3q-1}$. Consequently,

$$\psi_\mu(x)=(x+1)^{3q-1}\left(\frac{2q(p-q)^2(2q-(q+1)\nu_1)}{(p+1)^2(q+1)^2}+r_\mu(x)\right),$$

with $\lim_{x\to-1}r_\mu(x)=0$ uniformly for $\mu\approx\hat\mu$. This lead us to the choice $\nu_1(\mu)=\frac{2q}{q+1}$, otherwise $\{\Psi_\mu\}_{\mu\in\Lambda}$ would not be continuously quantifiable in $\hat\mu$ at $x=x_\ell$. From now on we set $\nu_1(\mu):=\frac{2q}{q+1}$. Accordingly, the monomial $(x+1)^{3q-1}$ "disappears" and we get that

$$\psi_\mu(x)=\frac{2(p-q)(q-1)}{(p+1)(q+1)^3}(x+1)^{4q}+\frac{4(p-2q)(p-q)^3(1+p-q)}{(p+1)^3(q+1)^3}(x+1)^{p+2q-1}+\hat r_\mu(x)$$

with both, $(x+1)^{-4q}\hat{r}_\mu(x)$ and $(x+1)^{1-p-2q}\hat{r}_\mu(x)$ tending to zero as $x \longrightarrow -1$ uniformly for $\mu \approx \hat{\mu}$. We now consider two cases, $\hat{p} \in (0,1)$ and $\hat{p} > 1$.

- If $\hat{p} \in (0,1)$, then the monomial with lowest exponent in ψ_μ for $\mu \approx \hat{\mu}$ is $(x+1)^{p+2q-1}$. Taking this into account, some computations show that the family $\{\Psi_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x = x_\ell$ by $\alpha_\ell(\mu) = 1-p+4q$.

- If $\hat{p} > 1$, then $(x+1)^{4q}$ is the monomial with lowest exponent in ψ_μ for $\mu \approx \hat{\mu}$ and, similarly as before, the family $\{\Psi_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x = x_\ell$ by $\alpha_\ell(\mu) = 2q$ with limit $a_\ell = \sqrt{\frac{\hat{p}}{\hat{p}+1}}$.

On the other hand, taking advantage of the analyticity of V_μ at $x = x_r$, one can easily check that $\{\Psi_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x = x_r$ by $\alpha_r(\mu) = -\frac{q}{q+1}$ with limit $a_r = \hat{p}^{-\frac{1}{2}}(\hat{p}+1)^{-\frac{\hat{p}+2}{2\hat{p}}}\big(2-(\hat{p}+1)^{\frac{1}{\hat{p}}}\big)$.

We are now in position to conclude the proof. Let us consider the case $\hat{p} \in (0,1)$ first. Then, on account of (3.10) and the values of α_ℓ and α_r obtained above, by applying the second part of Theorem E with $n = 1$ we get $\xi(\hat{\mu}) = -\min\{\hat{p}-1, 0\} = 1-\hat{p}$. Consequently if $\hat{p} \in (0, \frac{1}{2})$, then $\xi(\hat{\mu}) > \frac{1}{2}$ and by (a) in Theorem E we can assert that $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$, as desired. If $\hat{p} \in (\frac{1}{2}, 1)$, then by (b) in Lemma 3.5.5 we have $N_1(\hat{\mu}) \neq 0$ and hence, by $(b1)$ in Theorem E, we get $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$, as well. Let us consider finally the case $\hat{p} > 1$. Then, from (3.10) and the values of α_ℓ and α_r obtained above, $\big(\frac{\alpha_\ell}{\beta_\ell}\big)(\hat{\mu}) = \big(\frac{\alpha_r}{\beta_r}\big)(\hat{\mu}) = 0$ and

$$\big(a_r(b_r)^{-\frac{\alpha_r}{\beta_r}} - a_\ell(b_\ell)^{-\frac{\alpha_\ell}{\beta_\ell}}\big)(\hat{\mu}) = \frac{(\hat{p}+1)^{-\frac{2+\hat{p}}{2\hat{p}}}\big(2-(\hat{p}+1)^{\frac{\hat{p}+1}{\hat{p}}}\big)}{\sqrt{\hat{p}}} \neq 0$$

for all $\hat{p} > 1$. Thus, by the second part of Theorem E, $\xi(\hat{\mu}) = 0$. Since $N_1(\hat{\mu}) \neq 0$ due to (b) in Lemma 3.5.5, by the first part of Theorem E we get $\operatorname{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$ also in this case. This proves the result. ∎

3.5.2 Parameters in Λ_2

The proof of Theorem I for parameters in Λ_2 follows from Proposition 3.5.8.

The energy level at the outer boundary of $\mathscr{P}_\mu$ is $h_0(\mu) = +\infty$ for all $\mu \in \Lambda_2$ and, on account of Lemma 3.2.3, the projection of $\mathscr{P}_\mu$ on the x-axis is $\mathcal{I}_\mu = (x_\ell(\mu), x_r(\mu))$ with $x_\ell(\mu) = -1$ and $x_r(\mu) = +\infty$. We note also that hypothesis in **(H)** are not satisfied for the parameters in the boundary of Δ_2, $(q+1)(p+1) = 0$. Indeed, in every neighbourhood U of $\hat{\mu}$ there exist $\mu_1 \in U \cap \Lambda_1$ and $\mu_2 \in U \cap \Lambda_2$ such that $h_0(\mu_1)$ is finite and $h_0(\mu_2)$ is infinite. Hence the techniques developed in this paper do not apply for these parameters.

We note that if p and q are both different from -1, on account of (3.3) we have that

$$V_\mu(x) = \frac{(x+1)^{p+1}}{p+1} - \frac{(x+1)^{q+1}}{q+1} + \frac{p-q}{(p+1)(q+1)}. \tag{3.11}$$

Lemma 3.5.7. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of potential vector fields in (3.1) and let us denote $f_\mu(z) := z(g_\mu^{-1})''(z) - z(g_\mu^{-1})''(-z)$, where $g_\mu(x) := \mathrm{sgn}(x)\sqrt{V_\mu(x)}$ for $x \in \mathcal{I}_\mu$. If $\mu = (q,p) \in \Lambda_2$ with $q \neq -1$ and $p > 0$, and $\nu_1(\mu) = \frac{1-p}{1+p}$, then $M_1[\mathscr{L}_{\boldsymbol{\nu}_1(\mu)}[f_\mu]] = 0$.*

Proof. For the sake of shortness we shall omit the non-essential dependence on μ. From Definition 1.4.4 we have

$$M_1[\mathscr{L}_{\boldsymbol{\nu}_1}[f]] = \int_{-\infty}^{+\infty} \big((1-\nu_1)z(g^{-1})''(z) + z^2(g^{-1})'''(z)\big)\, dz.$$

On account of $g_\mu^2 = V_\mu$ we have, using the change of variable $z = g(x)$, that

$$M_1[\mathscr{L}_{\boldsymbol{\nu}_1}[f]] = \int_{-1}^{+\infty} \frac{(1-\nu_1)(V')^4 + 2(\nu_1-4)V(V')^2V'' + 12V^2(V'')^2 - 4V^2V'V'''}{(V')^4}(x)dx.$$

Direct derivation shows that

$$G(x) := x(\nu_1+1) - \frac{2\nu_1 V(x)}{V'(x)} - \frac{4V(x)^2V''(x)}{V'(x)^3}$$

is a primitive of the function in the integral above. Taking into account the expression in (3.11) and $\nu_1 = \frac{1-p}{1+p}$, an easy computation shows that $\lim_{x\to -1} G(x) = 0$. On the other hand, the function $G(x)$ can be written as

$$G(x) = \frac{2(p-q)\psi(x)}{(p+1)^2(q+1)^2(x+1)((x+1)^p - (x+1)^q)^3}$$

where $\psi(x)$ is the sum of 8 monomials of the form $r(\mu)(x+1)^{n_1p+n_2q+n_3}$ with $n_i \in \mathbb{Z}$, for $i = 1,2,3$, and r a polynomial function on the parameters. Let us take $\hat\mu \in \Lambda_2$ with $\hat q \neq -1$ and $\hat p > 0$. The monomial with the highest exponent for $\mu \approx \hat\mu$ of $\psi(x)$ is $(x+1)^{1+2p}$. Accordingly, since $p > 0$, we have that $\lim_{x\to+\infty} G(x) = 0$. Consequently, $M_1[\mathscr{L}_{\boldsymbol{\nu}_1(\mu)}[f_\mu]] = 0$ for all $\mu \approx \hat\mu$. ■

Proposition 3.5.8. *Consider $\hat\mu = (\hat q, \hat p) \in \Lambda_2$ with $\hat q \neq -1$. If $\hat p \neq 1$ then $\hat\mu$ is a local regular value of the period function at the outer boundary of system (3.1). Moreover,*

(a) If $\hat p < 1$ then the period function of $X_{\hat\mu}$ is increasing near the outer boundary.

(b) If $\hat p > 1$ then the period function of $X_{\hat\mu}$ is decreasing near the outer boundary.

On the other hand, if $\hat p = 1$ then $\hat\mu$ is a local bifurcation value of the period function at the outer boundary of system (3.1). Moreover, $\mathrm{Crit}\big((\Pi_\mu, X_{\hat\mu}), X_\mu\big) = 1$ *if $\hat p = 1$ and $\hat q < -3$.*

Proof. The proof of the regularity of the parameters $\hat{\mu} \in \Lambda_2$ with $\hat{p} < 0$ follows by the monotonicity result in Theorem F. Then, we shall assume $\hat{p} \geqslant 0$. Following the notation in Theorem C, from the expression in (3.11) and on account that $p > q$, it follows easily that $\{V_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable for any $\hat{\mu} = (\hat{q}, \hat{p}) \in \Lambda$ at $x = x_\ell$ by $\beta_\ell(\mu)$ with limit b_ℓ and at $x = x_r$ by $\beta_r(\mu)$ with limit b_r, where

$$\beta_\ell(\mu) = -(q+1),\ \beta_r(\mu) = p+1,\ b_\ell = -\frac{1}{\hat{q}+1} \text{ and } b_r = \frac{1}{\hat{p}+1}. \tag{3.12}$$

Consider $\hat{\mu} \in \Lambda_2$ with $\hat{q} \neq -1$ and $\hat{p} \in [0, +\infty) \setminus \{1\}$. We will prove that $\hat{\mu}$ is a local regular value of the period function at the outer boundary of the period annulus by applying Theorem C with $n = 0$. This will prove the first assertion of the result. We begin by studying the quantifiers of the family $\{\mathscr{R}_\mu\}_{\mu\in\Lambda}$. Let us recall that $\mathscr{R}_\mu := \frac{(V_\mu')^2 - 2V_\mu V_\mu''}{(V_\mu')^3}$. Easy computations show that $\{\mathscr{R}_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $x = x_\ell$ by $\alpha_\ell(\mu)$ with limit $a_\ell$and at $x = x_r$ by $\alpha_r(\mu)$ with limit a_r, where

$$\alpha_\ell(\mu) = q,\ \alpha_r(\mu) = -p,\ a_\ell = (\hat{q}-1)/(\hat{q}+1), \text{ and } a_r = (1-\hat{p})/(1+\hat{p}).$$

Accordingly, taking (3.12) also into account, we have $\frac{\alpha_\ell}{\beta_\ell}(\mu) = -\frac{q}{q+1}$ and $\frac{\alpha_r}{\beta_r}(\mu) = -\frac{p}{p+1}$ so $\frac{\alpha_\ell}{\beta_\ell}(\hat{\mu}) \neq \frac{\alpha_r}{\beta_r}(\hat{\mu})$ for all $\hat{\mu} \in \{\mu \in \Lambda_2 : (q+1)(p+1) \neq 0\}$. Then, by the second part of Theorem C with $n = 0$, we have that the family $\{\mathcal{P}[z(g_\mu^{-1})''(z)]\}_{\mu\in\Lambda}$ is continuously quantifiable in $\hat{\mu}$ at $z = +\infty$ by $\xi(\mu) = 2\max\{\frac{-q}{q+1}, \frac{-p}{p+1}\} + 1 = 1 - \frac{2p}{p+1}$. We apply next the first part of Theorem C with $n = 0$. To this end note that $\xi(\hat{\mu}) > -1$ for all $\hat{\mu}$ under consideration. Therefore, if $\hat{q} \neq -1$ and $\hat{p} \in [0, +\infty) \setminus \{1\}$, by (a) in Theorem C, we have that $\hat{\mu}$ is a local regular value of the period function at the outer boundary, as we desired. This proves the first assertion of the result.

Let us prove now the assertion concerning the monotonicity of the period function near the outer boundary of the period annulus. Taking into account the previous computations, Remark 2.3.4 shows that in this situation

$$\lim_{(h,\mu)\to(+\infty,\hat{\mu})} h^{1-\frac{\xi(\mu)}{2}} T_\mu'(h) = 2\sqrt{2\pi}(1-\hat{p})(1+\hat{p})^{-\frac{2\hat{p}+1}{\hat{p}+1}} \frac{\Gamma\left(\frac{1}{\hat{p}+1}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{\hat{p}+1}\right)} \neq 0.$$

Notice that the previous limit is positive if $\hat{p} < 1$ and it is negative if $\hat{p} > 1$. This proves the assertion concerning the monotonicity of the period function near the outer boundary. Moreover, Lemma 2.3.5 shows in this case that $\mathrm{Crit}\big((\Pi_\mu, X_{\hat{\mu}}), X_\mu\big) \geqslant 1$ if $\hat{\mu} = (\hat{q}, 1)$, so additionally we have proved that $\hat{\mu}$ is a local bifurcation value of the period annulus at the outer boundary.

Finally let us prove that $\mathrm{Crit}\big((\Pi_\mu, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$ for $\hat{\mu} = (\hat{q}, 1)$ with $\hat{q} < -3$. This will show that the criticality is exactly one for these parameters. The idea is to apply Theorem C with $n = 1$. First we use the second part of Theorem C in order to compute the quantifier ξ at infinity of $\{(\mathscr{L}_{\boldsymbol{\nu}_1(\mu)}\circ\mathcal{P})[z(g_\mu^{-1})''(z)]\}_{\mu\in\Lambda}$. To this end, we need to study the function

$$\Psi_\mu(x):=\frac{1}{V'_\mu}W\left[V_\mu^{\frac{\nu_1(\mu)-1}{2}},\mathscr{R}_\mu\right](x), \tag{3.13}$$

where ν_1 is a continuous function to be determined. Some computations show that

$$\Psi_\mu=\frac{\psi_\mu V_\mu^{\frac{1}{2}(\nu_1(\mu)-3)}}{2(V'_\mu)^5},$$

where, omitting the dependence on μ for shortness,

$$\psi:=(2VV''-(V')^2)((\nu_1-1)(V')^2+6VV'')-4V^2V'V'''.$$

By means of an algebraic manipulator we can assert that $\psi_\mu(x)$ is the sum of 12 monomials of the form $c(\mu)(x+1)^{n_1p+n_2q+n_3}$ with $n_i\in\mathbb{Z}$, for $i=1,2,3$, and c a well defined rational function at $\mu=\hat{\mu}$. Moreover the monomial with the highest exponent for $\mu\approx\hat{\mu}$ is $(x+1)^{4p}$. Consequently,

$$\psi_\mu(x)=(x+1)^{4p}\left(\frac{(p-1)(p-1+(p+1)\nu_1(\mu))}{(p+1)^2}+r_\mu(x)\right)$$

with $\lim_{x\to\infty}r_\mu(x)=0$ uniformly for $\mu\approx\hat{\mu}$. This lead us to the choice $\nu_1(\mu)=\frac{1-p}{1+p}$, otherwise $\{\Psi_\mu\}_{\mu\in\Lambda}$ would not be continuously quantifiable in $\hat{\mu}$ at $x=x_r$. From now on we set $\nu_1(\mu):=\frac{1-p}{1+p}$. Accordingly, the monomial $(x+1)^{4p}$ "disappears" and we get that

$$\psi_\mu(x)=x^{3p-1}\left(\frac{2p(1+3p)(p-q)(q+1)}{(p+1)^2}+\hat{r}_\mu(x)\right)$$

with $\lim_{x\to+\infty}\hat{r}_\mu(x)=0$ uniformly for $\mu\approx\hat{\mu}$. Taking this into account, some computation show that the family $\{\Psi_\mu\}_{\mu\in\Lambda}$ is continuously quantifiable at $x=x_r$ in $\hat{\mu}$ by $\alpha_r=-2-4p$ with limit $a_r=2\sqrt{2}(q^2-1)$.

On the other hand, the monomial with the lowest exponent for $\mu\approx\hat{\mu}$ of ψ_μ is $(x+1)^{4q}$. Consequently,

$$\psi_\mu(x)=(x+1)^{4q}\left(\frac{2(q-1)(q-p)}{(p+1)(q+1)^2}+\bar{r}_\mu(x)\right)$$

with $\lim_{x\to-1}\bar{r}_\mu(x)=0$ uniformly for $\mu\approx\hat{\mu}$. Then, similarly as before, the family $\{\Psi_\mu\}$ under consideration is continuously quantifiable in $\hat{\mu}$ at $x=x_\ell$ by $\alpha_\ell=\frac{1+2p+2q+3pq}{1+p}$ with limit $a_\ell=-\frac{1}{2}(q-1)^2(-q-1)^{-\frac{1}{2}}$.

We are now in position to conclude with the proof of the result. Let us consider $\hat{\mu}\in\{\mu\in\Lambda:p=1,q<-3\}$. On account of the values in (3.12) and the values α_ℓ and α_r obtained above, by applying the second part of Theorem C with $n=1$ we get that the family $\{(\mathscr{L}_{\boldsymbol{\nu}_1(\mu)}\circ\mathcal{P})[z(g_\mu^{-1})''(z)]\}$ is continuously quantifiable at infinity in $\hat{\mu}$ by $\xi(\mu)=\max\{-1+\frac{2}{1+q},-3+\frac{2}{1+p}\}$. Accordingly with the choice of $\hat{\mu}$ above, we have

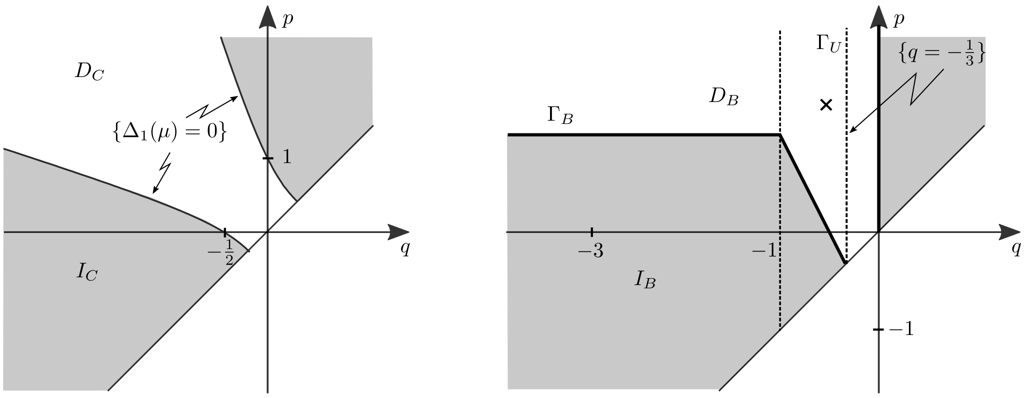

Figure 3.7: The period function associated to the parameters in the grey region (respectively, white region) is increasing (respectively, decreasing) near the inner boundary (left) and the outer boundary (right).

$\xi(\hat{\mu}) = -1 + \frac{2}{1+\hat{q}}$. Consequently, $\xi(\hat{\mu}) \in (-2,-1)$. By Lemma 3.5.7 we have additionally that $M_1[(\mathscr{L}_{\boldsymbol{\nu}_1(\mu)} \circ \mathcal{P})[z(g_\mu^{-1})''(z)]] \equiv 0$ for all $\mu \approx \hat{\mu}$. Thus, by the first part of Theorem C we conclude that $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant 1$ for $\hat{\mu} = (\hat{q}, 1)$ with $\hat{q} < -3$. This proves the result. ■

Theorem C can not be applied to study the criticality of the local bifurcation parameters $\hat{\mu} = (\hat{q}, 1)$ with $\hat{q} \in (-3,-1)$ because $\xi(\hat{\mu}) = -2$ and $M_1[(\mathscr{L}_{\boldsymbol{\nu}_1(\hat{\mu})} \circ \mathcal{P})[z(g_{\hat{\mu}}^{-1})''(z)]] = 0$. In this case the techniques do not apply even in the non-parametric setting, cf. Remark 1.2.11.

3.5.3 Parameters in Λ_3

The proof of Theorem I concerning the parameters in Λ_3 follows from Proposition 3.5.9. We point out that for these parameters the energy level at the outer boundary of $\mathscr{P}_\mu$ is $h_0(\mu) = \frac{p-q}{(p+1)(q+1)}$ and the projection of $\mathscr{P}_\mu$ on the x-axis is $\mathcal{I}_\mu = (\rho(\mu), +\infty)$ with $\rho(\mu) = \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}} - 1$. We also notice that hypothesis **(H)** are satisfied for $\mu \in \Lambda_3$. Therefore, the techniques developed in Section 2.4 could be used in this regard. However, the proof of Proposition 3.5.9 is a corollary of Theorem F, where we proved the global monotonicity of the period function for this parameter region.

Proposition 3.5.9. *If $\hat{\mu} \in \Lambda_3$ then $\hat{\mu}$ is a local regular value of the period function at the outer boundary of system (3.1). Moreover the period function of $X_{\hat{\mu}}$ is increasing near the outer boundary.*

3.6 Lower bound of the number of critical periodic orbits

In this last section we present the corollary we have introduced at the beginning of the chapter. This result deals with the existence of at least one critical periodic orbit inside the regions in the parameter space where we conjecture exactly one critical periodic orbit (see Conjecture 3.1.1). These regions are given by the intersections $D_C \cap I_B$ and $I_C \cap D_B$ (see Figure 3.7).

Corollary 3.6.1. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of vector fields in* (3.1) *and consider the period function of the center at the origin. If $\mu \in (D_C \cap I_B) \cup (I_C \cap D_B)$ then the vector field X_μ has at least one critical periodic orbit.*

Proof. By Theorem G we have that the period function is increasing near the inner boundary in I_C and decreasing in D_C. On the other hand, by Theorem I we have that the period function is increasing near the outer boundary in I_B and decreasing in D_B. Therefore, for parameters $\mu \in D_C \cap I_B$, applying Bolzano's Theorem, we have that X_μ at least one critical periodic orbit. Similarly if $\mu \in I_C \cap D_B$. This proves the result. ∎

CHAPTER 4

Appendix

In this Appendix we show some technical results that are needed in the previous proofs. The first result is a uniform Hôpital's Rule. The authors in [38] give a uniform version of this classical result in case that the function on the denominator tends to infinity. Here we adapt their proof to the case in which the numerator and denominator tend to zero.

Proposition 4.0.2 (Uniform Hôpital's Rule). *Let f_μ and g_μ be two real valued functions defined on an interval (a,b) and depending on a parameter $\mu \in \Lambda \subset \mathbb{R}^d$. Suppose that:*

(a) f_μ and g_μ are differentiable on (a,b),

(b) $g'_\mu(x) \neq 0$ for all $x \in (a,b)$ and $\mu \in \Lambda$,

(c) for all $\mu \in \Lambda$, there exists $L_\mu \in \mathbb{R}$ such that $\lim_{x\to a^+} \frac{f'_\mu(x)}{g'_\mu(x)} = L_\mu$ uniformly on $\mu \in \Lambda$,

(d) $\sup\{|L_\mu|\,;\mu \in \Lambda\} < +\infty$,

(e) there exists $c \in (a,b)$ such that, for each $x \in (a,c)$ we have that

$$\lim_{y\to a^+} \frac{f_\mu(y)}{g_\mu(x)} = 0 \text{ and } \lim_{y\to a^+} \frac{g_\mu(y)}{g_\mu(x)} = 0 \text{ uniformly on } \mu \in \Lambda.$$

Then $\lim_{x\to a^+} \frac{f_\mu(x)}{g_\mu(x)} = L_\mu$ uniformly on $\mu \in \Lambda$.

Proof. Consider a given $\varepsilon > 0$. Setting $M := \sup\{|L_\mu|\,;\mu \in \Lambda\}$, which is well defined by the assumption (d), let us take $\varepsilon_1 := \min\{\frac{\varepsilon}{3+M}, 1\}$. From (c) there exists $\delta > 0$ such that, if $c \in (a, a+\delta)$, then $\left|\frac{f'_\mu(c)}{g'_\mu(c)} - L_\mu\right| < \varepsilon_1$ for all $\mu \in \Lambda$. Let us fix at this point any $x \in (a, a+\delta)$. By the Mean Value Theorem, for each $y \in (a,x)$ there exists $c = c(x,y,\mu) \in (y,x) \subset (a, a+\delta)$ such that $\frac{f_\mu(x)-f_\mu(y)}{g_\mu(x)-g_\mu(y)} = \frac{f'_\mu(c)}{g'_\mu(c)}$. Therefore

$$\left| \frac{\frac{f_\mu(x)}{g_\mu(x)} - \frac{f_\mu(y)}{g_\mu(x)}}{1 - \frac{g_\mu(y)}{g_\mu(x)}} - L_\mu \right| = \left| \frac{f'_\mu(c)}{g'_\mu(c)} - L_\mu \right| < \varepsilon_1. \tag{4.1}$$

On the other hand, the assumption (e) guarantees that there exists $z_x \in (a, x)$ such that

$$\left|\frac{f_\mu(y)}{g_\mu(x)}\right| < \varepsilon_1 \text{ and } \left|\frac{g_\mu(y)}{g_\mu(x)}\right| < \varepsilon_1 \text{ for all } y \in (a, z_x) \text{ and } \mu \in \Lambda. \tag{4.2}$$

Note then that $\left|(L_\mu \pm \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)}\right| < (|L_\mu| + \varepsilon_1)\varepsilon_1$ and, accordingly,

$$-(|L_\mu| + \varepsilon_1)\varepsilon_1 < (L_\mu \pm \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)} < (|L_\mu| + \varepsilon_1)\varepsilon_1. \tag{4.3}$$

The second inequality in (4.2) shows in particular that $1 - \frac{g_\mu(y)}{g_\mu(x)} > 0$ because $\varepsilon_1 < 1$. Hence, from (4.1),

$$(-\varepsilon_1 + L_\mu)\left(1 - \frac{g_\mu(y)}{g_\mu(x)}\right) + \frac{f_\mu(y)}{g_\mu(x)} < \frac{f_\mu(x)}{g_\mu(x)} < (\varepsilon_1 + L_\mu)\left(1 - \frac{g_\mu(y)}{g_\mu(x)}\right) + \frac{f_\mu(y)}{g_\mu(x)}.$$

Therefore,

$$-\varepsilon_1 - (L_\mu - \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)} + \frac{f_\mu(y)}{g_\mu(x)} < \frac{f_\mu(x)}{g_\mu(x)} - L_\mu < \varepsilon_1 - (L_\mu + \varepsilon_1)\frac{g_\mu(y)}{g_\mu(x)} + \frac{f_\mu(y)}{g_\mu(x)}.$$

From this, on account of (4.3) and the first inequality in (4.2), we get that

$$-2\varepsilon_1 - (|L_\mu| + \varepsilon_1)\varepsilon_1 < \frac{f_\mu(x)}{g_\mu(x)} - L_\mu < 2\varepsilon_1 + (|L_\mu| + \varepsilon_1)\varepsilon_1.$$

Accordingly, for all $x \in (a, a + \delta)$ and $\mu \in \Lambda$,

$$\left|\frac{f_\mu(x)}{g_\mu(x)} - L_\mu\right| < \varepsilon_1(2 + |L_\mu| + \varepsilon_1) < \varepsilon_1(3 + |L_\mu|) < \varepsilon_1(3 + M) < \varepsilon,$$

and this proves the result. ■

Next three lemmas deal with the computation of some integrals that appear in the proof of Proposition 3.5.2.

Lemma 4.0.3. *Let α and β be any complex number with strictly positive real part. Then,*

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \int_0^\infty u^{\alpha-1}(1+u)^{-(\alpha+\beta)}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

where Γ denotes the Gamma function.

Proof. See for instance (6.2.1) and (6.2.2) of [1]. ■

Lemma 4.0.4. *Let α and β real numbers such that $\alpha + \beta + 1 \neq 0$. Then,*

$$\int u^\alpha(u+1)^\beta du = \frac{\beta}{\alpha+\beta+1}\int u^\alpha(u+1)^{\beta-1}du + \frac{1}{\alpha+\beta+1}u^{\alpha+1}(1+u)^\beta.$$

Proof. The result follows from

$$\left(\frac{1}{\alpha+\beta+1}u^{\alpha+1}(u+1)^{\beta}\right)' = u^{\alpha}(u+1)^{\beta} - \frac{\beta}{\alpha+\beta+1}u^{\alpha}(u+1)^{\beta-1}.$$

■

Lemma 4.0.5. *Let* $\mu \in \{(q,p) \in \mathbb{R}^2 : p > q > -1\}$ *and let* $\rho(\mu) = \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}} - 1$ *and* $V_\mu(x) = \frac{(x+1)^{p+1}}{p+1} - \frac{(x+1)^{q+1}}{q+1} + \frac{p-q}{(p+1)(q+1)}$. *Then,*

$$(i)\quad \int_{-1}^{\rho(\mu)} \frac{dx}{\sqrt{\frac{p-q}{(p+1)(q+1)} - V_\mu(x)}} = \begin{cases} \sqrt{\pi}\frac{\sqrt{q+1}}{p-q}\left(\frac{p+1}{q+1}\right)^{\frac{1-q}{2(p-q)}} \frac{\Gamma\left(\frac{1-q}{2(p-q)}\right)}{\Gamma\left(\frac{1+p-q}{2(p-q)}\right)} & \text{if } -1 < q < 1, \\ +\infty & \text{if } q \geqslant 1. \end{cases}$$

$$(ii)\quad \int_{-1}^{\rho(\mu)} \frac{\frac{1}{2} - \frac{V_\mu''(x)V_\mu(x)}{V_\mu'(x)^2}}{\sqrt{\frac{p-q}{(p+1)(q+1)} - V_\mu(x)}}dx = \begin{cases} \frac{-\sqrt{\pi}(p+1)^{\frac{1}{2}}(p+2q+1)}{2(p-q)(q+1)(\rho(\mu)+1)^{\frac{3p+1}{2}}} \frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)} & \text{if } -1 < q < -\frac{1}{3}, \\ -\infty & \text{if } -\frac{1}{3} \leqslant q < 0, \\ +\infty & \text{if } q > 0. \end{cases}$$

Proof. Let us prove (i). The improper integral under consideration can be written as

$$\int_{-1}^{\rho(\mu)} \frac{dx}{(x+1)^{\frac{q+1}{2}}\sqrt{\frac{1}{q+1} - \frac{(x+1)^{p-q}}{p+1}}}.$$

In case that $q \geqslant 1$ it is clear that the improper integral is $+\infty$. Let us consider $q < 1$ and perform the change of variable $x = \left(\frac{(p+1)(1-u)}{q+1}\right)^{\frac{1}{p-q}} - 1$. Then the improper integral becomes

$$\frac{\sqrt{q+1}}{p-q}\left(\frac{p+1}{q+1}\right)^{\frac{1-q}{2(p-q)}} \int_0^1 u^{-\frac{1}{2}}(1-u)^{\frac{1-2p+q}{2(p-q)}}du.$$

Notice that, due to $q < 1$, the integral satisfies assumptions in Lemma 4.0.3 and so the result follows immediately by applying this lemma.

For the proof of (ii) let us denote

$$I := \int_{-1}^{\rho(\mu)} \frac{\frac{1}{2} - \frac{V_\mu''(x)V_\mu(x)}{V_\mu'(x)^2}}{\sqrt{\frac{p-q}{(p+1)(q+1)} - V_\mu(x)}}dx.$$

On account of the expression of V_μ and with the help of an algebraic manipulator one can readily see that the improper integral is given by

$$I = \int_{-1}^{\rho(\mu)} \frac{1}{(x+1)^{\frac{3}{2}(q+1)}}\left(\frac{q(p-q)}{2(p+1)\sqrt{q+1}} + G(x;\mu)\right)dx$$

where $G(x;\mu)$ is a continuous function on $\{(x,\mu) : x \in [-1,\rho(\mu)], \mu \in \Lambda_1\}$ and such that $G(-1;\mu) = 0$ for all $\mu \in \Lambda_1$. Consequently, the improper integral is $\pm\infty$ in case that

$q \in (-\frac{1}{3}, +\infty) \setminus \{0\}$ and the sign of the infinity is given by the sign of q, so the result holds in this cases. On the other hand, if $q < -\frac{1}{3}$ then the integral converges. Assume that $-1 < q < -\frac{1}{3}$ in order to compute the integral. Let us denote for the sake of simplicity

$$\begin{aligned}
\Phi(z;\mu) &:= \frac{z^{q+1}}{p+1}\left(\frac{p+1}{q+1} - z^{p-q}\right),\\
l(z;\mu) &:= \frac{1}{z^p - z^q},\\
h(z;\mu) &:= -\Phi(z;\mu)^{\frac{1}{2}} + \frac{p-q}{(p+1)(q+1)}\Phi(z;\mu)^{-\frac{1}{2}}.
\end{aligned}$$

With this notation and considering the expression of V_μ, the improper integral under consideration can be written as

$$I = \lim_{R\to\rho(\mu)+1}\left(\frac{1}{2}\int_0^R \Phi(z;\mu)^{-\frac{1}{2}}dz + \int_0^R l'(z;\mu)h(z;\mu)dz\right).$$

Integrating by parts the second integral it holds that

$$I = \lim_{R\to\rho(\mu)+1}\left(\frac{1}{2}\int_0^R \Phi(z;\mu)^{-\frac{1}{2}}dz + l(z;\mu)h(z;\mu)|_0^R - \int_0^R l(z;\mu)h'(z;\mu)dz\right).$$

Since $l(z;\mu)h'(z;\mu) = \frac{1}{2}\Phi(z;\mu)^{-\frac{1}{2}} + \frac{1}{2}\frac{p-q}{(p+1)(q+1)}\Phi(z;\mu)^{-\frac{3}{2}}$ and $\lim_{z\to 0} l(z;\mu)h(z;\mu) = 0$ we have then

$$I = \lim_{R\to\rho(\mu)+1}\left(l(R;\mu)h(R;\mu) - \frac{1}{2}\frac{p-q}{(p+1)(q+1)}\int_0^R \Phi(z;\mu)^{-\frac{3}{2}}dz\right). \tag{4.4}$$

Moreover, let us perform the change of variable $u = f(z)$ with $f(z) = \frac{z^{p-q}}{\frac{p+1}{q+1} - z^{p-q}}$. We have that

$$\int_0^R \Phi(z;\mu)^{-\frac{3}{2}}dz = \frac{(p+1)^{\frac{3}{2}}}{(p-q)(\rho(\mu)+1)^{\frac{3p+1}{2}}}\int_0^{f(R)} u^{\frac{1}{2}-\lambda}(u+1)^{\lambda-1}du.$$

where $\lambda = \frac{1}{2}\frac{3p+1}{p-q}$. Applying Lemma 4.0.4 to the above integral and taking into account that

$$\lim_{u\to 0} 2u^{\frac{3}{2}-\lambda}(u+1)^{\lambda-1} = 0$$

and that $-1 < q < -\frac{1}{3}$, we have

$$\begin{aligned}
\int_0^R \Phi(z;\mu)^{-\frac{3}{2}}dz = & \frac{2(p+1)^{\frac{3}{2}} f(R)^{\frac{3}{2}-\lambda}(f(R)+1)^{\lambda-1}}{(p-q)(\rho(\mu)+1)^{\frac{3p+1}{2}}}\\
& + \frac{2(\lambda-1)(p+1)^{\frac{3}{2}}}{(p-q)(\rho(\mu)+1)^{\frac{3p+1}{2}}}\int_0^{f(R)} \frac{u^{\frac{1}{2}-\lambda}}{(u+1)^{2-\lambda}}du.
\end{aligned} \tag{4.5}$$

At this point we claim that

$$\lim_{R\to\rho(\mu)+1}\left(l(R;\mu)h(R;\mu) - \frac{f(R)^{-\frac{3q+1}{2(p-q)}}(f(R)+1)^{\frac{p+2q+1}{2(p-q)}}(p+1)^{\frac{1}{2}}}{(q+1)(\rho(\mu)+1)^{\frac{3p+1}{2}}}\right) = 0.$$

Indeed, if we substitute $f(R)=\frac{R^{p-q}}{\frac{p+1}{q+1}-R^{p-q}}$ then we have

$$\frac{f(R)^{-\frac{3q+1}{2(p-q)}}(f(R)+1)^{\frac{p+2q+1}{2(p-q)}}(p+1)^{\frac{1}{2}}}{(q+1)(\rho(\mu)+1)^{\frac{3p+1}{2}}}=(p+1)^{-\frac{1}{2}}R^{-\frac{3q+1}{2}}\left(\frac{p+1}{q+1}-R^{p-q}\right)^{-\frac{1}{2}},$$

and so using the expressions of $l(R;\mu)$ and $h(R;\mu)$ we can obtain that

$$l(R;\mu)h(R;\mu)-(p+1)^{-\frac{1}{2}}R^{-\frac{3q+1}{2}}\left(\frac{p+1}{q+1}-R^{p-q}\right)^{-\frac{1}{2}}=\frac{-\left(\frac{p+1}{q+1}-R^{p-q}\right)^{-\frac{1}{2}}}{\sqrt{p+1}R^{\frac{q+1}{2}}(R^p-R^q)}$$

which clearly tends to 0 as $R\longrightarrow\rho(\mu)+1=\left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}$, so the claim is proved.

Substituting expression in (4.5) into the equality in (4.4) and using the claim we have that

$$I=\lim_{R\to\rho(\mu)+1}\frac{(p+1)^{\frac{1}{2}}(1-\lambda)}{(q+1)(\rho(\mu)+1)^{\frac{3p+1}{2}}}\int_0^{f(R)}\frac{u^{\frac{1}{2}-\lambda}}{(u+1)^{2-\lambda}}du.$$

Finally, since $\lim_{R\to\rho(\mu)+1}f(R)=+\infty$, using Lemma 4.0.3 with $\alpha=-\frac{3q+1}{2(p-q)}>0$ and $\beta=\frac{1}{2}>0$, and substituting the value of λ we have that

$$\lim_{R\to\rho(\mu)+1}(1-\lambda)\int_0^{f(R)}\frac{u^{\frac{1}{2}-\lambda}}{(u+1)^{2-\lambda}}du=\frac{-\sqrt{\pi}(p+2q+1)}{2(p-q)}\frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)}.$$

Consequently we obtain that the value of the improper integral is given by

$$I=\frac{-\sqrt{\pi}(p+1)^{\frac{1}{2}}(p+2q+1)}{2(p-q)(q+1)(\rho(\mu)+1)^{\frac{3p+1}{2}}}\frac{\Gamma\left(-\frac{3q+1}{2(p-q)}\right)}{\Gamma\left(\frac{p-4q-1}{2(p-q)}\right)}$$

as we desired. ∎

Next result deals with some technical details used in Proposition 3.5.6.

Lemma 4.0.6. *The following hold:*

(a) *The function $f_1(x)=(2+2x)^{\frac{2+2x}{1+2x}}-2(1+2x)$ has a unique zero on $(-\frac{1}{2},+\infty)$.*

(b) *The function $f_2(x)=(2+2x)^{\frac{2+2x}{1+2x}}+4x(1+2x)$ is positive on $(-\frac{1}{2},+\infty)$.*

Proof. In order to prove (a) we claim that $g_1(x):=x^{\frac{x}{x-1}}-2x+2$ is monotonous decreasing on $(1,+\infty)$. Note that $g_1(x)=f_1(x/2-1)$ and, consequently, (a) will follow once we prove the claim because one can easily verify that $\lim_{x\to1}g_1(x)=e$ and $\lim_{x\to+\infty}g_1(x)=-\infty$. To show the claim we first note that

$$g_1'(x)=-2+\frac{x^{\frac{x}{x-1}}(x-1-\log(x))}{(x-1)^2}\text{ and }g_1''(x)=\frac{x^{\frac{x}{x-1}}(x\log(x)^2-(x-1)^2)}{(x-1)^4}.$$

Since $\lim_{x\to 1^+} g_1'(x) < 0$, it suffices to show that $g_1''(x) < 0$ for all $x \in (1,+\infty)$, which is equivalent to $\kappa(x) := x\log(x)^2 - (x-1)^2 < 0$. However this is clear because one can verify that $\kappa(1) = \kappa'(1) = \kappa''(1) = 0$ and $\kappa'''(x) = -\frac{2\log(x)}{x^2} < 0$ for all $x > 1$. This shows the validity of (a).

For proving (b) let us denote $g_2(x) := x^{\frac{x}{x-1}} - 1/2$. We point out that it is enough to prove that g_2 is positive on $(1,+\infty)$ due to $f_2(x/2-1) = x^{\frac{x}{x-1}} + 2x^2 - 6x + 4 > g_2(x)$ for all $x \in (1,+\infty)$. To show this notice that

$$g_2'(x) = \frac{x^{\frac{x}{x-1}}(x-1-\log(x))}{(x-1)^2}.$$

Then, since $\lim_{x\to 1} g_2(x) = e - \frac{1}{2} > 0$, in order to verify the result it is enough to see that $g_2' > 0$ on $(1,+\infty)$. That is, we have to prove that $x - 1 - \log(x) > 0$ on $(1,+\infty)$. This follows easily by derivation due to $(x-1-\log(x))' = \frac{x-1}{x} > 0$ for all $x \in (1,+\infty)$ and ends with the proof of the result. ■

Finally, next result is used in Lemma 3.5.5.

Lemma 4.0.7. *If $a, b \in \mathbb{C}$, then $\frac{d}{dz}{}_2F_1(a,b,b+1;z) = \frac{b}{z}\left((1-z)^{-a} - {}_2F_1(a,b,b+1;z)\right)$.*

Proof. This is straightforward by using the formulae in [1]. Indeed, it shows that

$$\frac{d}{dz} z^b {}_2F_1(a,b,b+1;z) = bz^{b-1}{}_2F_1(a,b,b;z) = bz^{b-1}(1-z)^{-a},$$

where the first equality is a particular case of 15.2.4 and the second one follows by applying 15.1.8. Then an easy manipulation yields to the desired equality after deriving the product on the left. ■

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