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A geodesic completeness theorem for locally symmetric Lorentz manifolds

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ABSTRACT. We prove that a locally symmetric and null-complete Lorentz manifold is geodesically complete.

0. INTRODUCTION

The concept of null, timelike and spacelike completeness in Lorentz manifolds are logically inequivalent: Kundt [3] gives an example of a timelike-complete and null-complete spacetime which is not spacelike complete. Geroch [2] shows by an example that a globally hyperbolic manifold may be timelike incomplete but complete in other senses. Finally, an example by Beem [1] together with some modifications to Kundt's example proposed by Geroch [2] show that the other possibilities may actually occur.

The aim of this work is to prove that, for locally symmetric Lorentz manifolds null-completeness implies timelike and spacelike completeness.

A slight modification of the reasoning in 4.1 shows that for these manifolds the three types of completeness are equivalent (see 5.2). The rest of this paper is devoted to prove the following main result:

Theorem 1. *Let $\bar{M}$ and M be locally symmetric, connected, and null-complete Lorentz manifolds. Let $\bar{M}$ be simply connected, and fix $\bar{o}\in\bar{M}$, $o\in M$. Then if $L\colon T_{\bar{o}}\bar{M}\to T_oM$ is a curvature preserving linear isometry, there is a local isometry $\phi\colon\bar{M}\to M$ such that $\phi(\bar{o})=o$ and $d\phi(\bar{o})=L$.*

In particular, if M is locally symmetric and null-complete, its universal covering $\tilde{M}$ will have the same properties. Then for each $\tilde{o}\in\tilde{M}$ the symmetry

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$L:T_{\tilde{o}}\tilde{M} \ni v \to -v \in T_{\tilde{o}}\tilde{M}$ is curvature preserving and, by Theorem 1 there is a local isometry $\phi:\tilde{M}\to\tilde{M}$ such that $\phi(\tilde{o})=\tilde{o}$ and $d\phi(\tilde{o})=L$. Clearly $d\phi^2(\tilde{o})=L^2=id$ and $\phi^2=id$ by 1.1; hence $\tilde{M}$ is symmetric and, in particular, geodesically complete. This completeness is inherited by M and the aim of the paper is achieved:

Corollary 2. *A locally symmetric and null-complete Lorentz manifold is geodesically complete.*

1. PRELIMINARY RESULTS

The following classical results are essential in the proof of theorem 1:

Proposition 1.1 [4] *If $\phi_1, \phi_2:\bar{M}\to M$ are local isometries between connected semi-Riemannian manifolds and $d\phi_1(\bar{p})=d\phi_2(\bar{p})$ for some $\bar{p}\in\bar{M}$, then $\phi_1=\phi_2$.*

Theorem 1.2. (E. Cartan) [4]. *Let $\bar{M},M$ be locally symmetric Lorentz manifolds, $\bar{o}\in\bar{M}$, $o\in M$ and $L:T_{\bar{o}}\bar{M}\to T_oM$ a curvature-preserving linear isometry. Let $\bar{U}_{\bar{o}}$ be a starred neighbourhood of $\bar{o}$ in $T_{\bar{o}}\bar{M}$, such that $exp_{\bar{o}}$ and exp_o are defined on $\bar{U}_{\bar{o}}$ and $U_o=L(\bar{U}_{\bar{o}})$, and $exp_{\bar{o}}$ is a differomorphism from $\bar{U}_{\bar{o}}$ onto its image $\bar{U}$. Define then $\phi=exp_o\cdot L\cdot exp_{\bar{o}}^{-1}:\bar{U}\to exp_o(U_o)=U$.*

Given a geodesic $\bar{\gamma}:[0,1]\to\bar{U}$ such that $\bar{\gamma}(0)=\bar{o}$, $\bar{\gamma}'(0)=\bar{v}\in\bar{U}_{\bar{o}}$, and taking $\gamma(t)=exp_o(tv)$ $t\in[0,1]$, $v=L(\bar{v})$ we get:

$$d\phi(\bar{\gamma}(t))=P_t\cdot L\cdot\bar{P}_t^{-1} \text{ for } t\in[0,1]$$

where P_t and $\bar{P}_t$ are the parallel displacements along γ and $\bar{\gamma}$.

In particular $\phi:\bar{U}\to U$ is a local isometry and in fact the only one such that $d\phi(\bar{o})=L$.

From now onwards, unless the contrary is stated, all the hypotheses of Theorem 1 are assumed.

2. L-PROJECTIONS

The following preliminary result will justify definition 2.2.

Proposition 2.1. *Let $\bar{\gamma}:[0,a]\to\bar{M}$ be a null geodesic such that $\bar{\gamma}(0)=\bar{o}$, $\bar{\gamma}'(0)=\bar{v}$, $L(\bar{v})=v$. Define $\gamma(t)=exp_o(tv)$ for $t\in[0,a]$ (remark that by hypothesis $exp_o(tv)$ is defined for all t). There is a partition $0=t_o<t_1<\ldots<t_r=a$ and a family $\mathcal{U}=\{\bar{U}_i,\phi_i):i=1,\ldots,r\}$ such that:*

P1) $\bar{U}_1$ *is convex and open in* $\bar{M}$, *and* $\bar{\gamma}[t_{i-1}, t_i] \subset \bar{U}_i$ *for* $i=1,\dots,r$.

P2) $\phi_i:\bar{U}_i \to \phi_i(\bar{U}_i)=U_i$ *is an isometry for* $i=1, \dots, r$ *and* $\phi_1(\bar{o})=o$, $d\phi_1(\bar{o})=L$.

P3) $\phi_i(\bar{\gamma}(t))=\gamma(t)$ *for* $t\in[t_{i-1}, t_1]$ $i=1, \dots, r$. *Moreover* $d\phi_i(\bar{\gamma}(t_i))=d\phi_{i+1}(\bar{\gamma}(t_i))$.

Proof: If $P_t, \bar{P}_t$ are as in 1.2 for $t\in[0,a]$ we define

$$L_{\bar{\gamma}}^t = P_t \cdot L \cdot \bar{P}_t^{-1}: T_{\bar{\gamma}(t)}\bar{M} \to T_{\gamma(t)}(M).$$

By local symmetry, P_t and $\bar{P}_t$ are curvature-preserving and so is $L_{\bar{\gamma}}^t$ Using 1.2 we see that for each $t\in[0,a]$ there are $\bar{V}_t$, V_t convex neighbourhoods of $\bar{\gamma}(t)$ and $\gamma(t)$, and an isometry $\psi_t:\bar{V}_t \to V_t$ such that $\psi_t(\bar{\gamma}(t))=\gamma(t)$ and $d\psi_t(\bar{\gamma}(t))=L_{\bar{\gamma}}^t$.

We claim that if $\bar{\gamma}[t_1,t_2] \subset \bar{V}_t$, then for $t\in[t_1,t_2]$ we have $\psi_{t_1}(\bar{\gamma}(t))=\gamma(t)$, and $d\psi_{t_1}(\bar{\gamma}(t))=L_{\bar{\gamma}}^t$.

Hence by 1.1, ψ_{t_1} coincides with ψ_t on the connected component of $\bar{V}_{t_1} \cap \bar{V}_t$ containing $\bar{\gamma}(t)$.

In order to prove the claim note that

$$\psi_{t_1}(\bar{\gamma}(t)) = \psi_{t_1}(\exp_{\bar{\gamma}(t_1)}((t-t_1)\bar{\gamma}'(t_1))) =$$

$$\exp_{\gamma(t_1)}((t-t_1)L_{\bar{\gamma}}^{t_1}(\bar{\gamma}'(t_1))) = \exp_{\gamma(t_1)}((t-t_1)\gamma'(t_1)) = \gamma(t)$$

the second assertion is now an easy consequence of Cartan's theorem.

Using the claim and the compactness of $[0,a]$, we get a partition $0=t_o<t_1<\dots<t_r=a$ and $s_i\in[t_{i-1}, t_i)$ such that if $\bar{U}_i=\bar{V}_{s_i}$ and $\phi_i=\psi_{s_i}$, the family $\mathscr{U} = \{(\bar{U}_i, \phi_i): i=1, \dots, r\}$ verifies $P1$), and automatically $P2$) and $P3$).

We give the following general definition:

Definition 2.2. *Let* $\bar{\gamma}:[0,\alpha]\to\bar{M}$ *be continuous. We say that* $\bar{\gamma}$ *is L-projectable if* $\bar{\gamma}(0)=\bar{o}$ *and we have a continuous curve* $\gamma:[0,a]\to M$, *a partition* $0=t_o<t_1<\dots<t_r=a$ *and a collection* $\mathscr{U}$, *such that* $(P1)$, $(P2)$, $(P3)$ *in proposition* 2.1 *hold.*

We say then that γ *is a L-projection of* $\bar{\gamma}$ *and* (t_i), $\mathscr{U}$ *are the associated partition and covering of the projection.*

The definition is easily adapted for curves defined in [a,b]; this being valid for all results in the section.

Proposition 2.3. *Let* $\bar{\gamma}:[0,a]\to\bar{M}$ *be continuous. Then*

(*i*) *The L-projection* γ, *if it exists, is unique.*

(*ii*) *If* γ *is the L-projection and has a covering* $\mathscr{U}$ *and an associated partition as in 2.2, then the linear isometry*

$$L_{\bar{\gamma}}^{t}=d\phi_{i}(\bar{\gamma}(t)):T_{\bar{\gamma}(t)}\bar{M}\to T_{\gamma(t)}M \text{ for } t\in[t_{i-1},t_{i}] \quad i=1,\dots,r$$

is curvature-preserving and depends only on $\bar{\gamma}$.

(*iii*) *If* $\bar{\gamma}$ *is a null geodesic with* $\bar{\gamma}(0)=\bar{o}$, *then it is L-projectable, and its L-projection* γ *is a null geodesic. Also with the notation in 2.1 we have* $L_{\bar{\gamma}}^{t}=P_{t}\cdot L\cdot\bar{P}_{t}^{-1}$ *for* $t\in[0,a]$.

Proof:

(i) If γ_1 and γ_2 are L-projections we may choose projection coverings $\mathscr{U}^{(\alpha)}=\{(\bar{U}_i^{(\alpha)},\ \phi_i^{(\alpha)}):i=1,\ \dots,\ r\}$ $\alpha=1,2$ with the same associated partition $0=t_o<\dots<t_r=a$. Let $\bar{U}_i$ be the connected component of $\bar{U}_i^{(1)}\cap\bar{U}_i^{(2)}$ containing $\bar{\gamma}[t_{i-1},\ t_i]$. Let us prove that $\phi_i^{(1)}/\bar{U}_i=\phi_i^{(2)}/\bar{U}_i$ $i=1,\ \dots,\ r$. Using condition (P2) and 1.1 the statement is shown to be true, since $d\phi_1^{(1)}(\bar{o})=L=d\phi_1^{(2)}(\bar{o})$. If the statement is true for $i>1$, using (P3) we get:

$$d\phi_{i+1}^{(1)}(\bar{\gamma}(t_i))=d\phi_i^{(1)}(\bar{\gamma}(t_i))=d\phi_i^{(2)}(\bar{\gamma}(t_i))=d\phi_{i+1}^{(2)}(\bar{\gamma}(t_i))$$

and using 1.1 the inductive step can be completed. For each $t\in[t_{i-1},t_i]$ we have $\gamma_1(t)=\phi_i^{(1)}(\bar{\gamma}(t))=\phi_i^{(2)}(\bar{\gamma}(t))=\gamma_2(t)$.

(ii) The preceding argument applied to the projection coverings $\mathscr{U}^{(\alpha)}$ $\alpha=1,2$ of $\bar{\gamma}$ over γ with the same associated partition (t_i) shows that $\phi_i^{(1)}/\bar{U}_i=$ $=\phi_i^{(2)}/\bar{U}_i$, where $\bar{U}_i$ is defined as in (i). In particular $d\phi_i^{(1)}(\bar{\gamma}(t))=d\phi_i^{(2)}(\bar{\gamma}(t))$ for each $t\in[t_{i-1},t_i]$.

(iii) *It follows immediately from* 2.1.

Definition 2.4. *If the hypothesis* 2.3 (*ii*) *holds, the map* $L_{\bar{\gamma}}^{t}:T_{\bar{\gamma}(t)}\bar{M}\to T_{\gamma(t)}M$ *is called the transport of L from* $\bar{\gamma}(0)=\bar{o}$ *to* $\bar{\gamma}(t)$ *along* $\bar{\gamma}$.

We remark that $L_{\bar{\gamma}}^{0}=L$. We will write $L_{\bar{\gamma}}=L_{\bar{\gamma}}^{a}$.

The following statements are elementary and will be used in the future more or less explicitly.

Proposition 2.5.

(*i*) *The L-projectability of $\bar{\gamma}$ depends essentially on im $\bar{\gamma}$ and not on the specific parametrization chosen with compact domain and origin at $\bar{o}$. Analogously the transport $L_{\bar{\gamma}}$ depends only on im$\bar{\gamma}$ and the final end of $\bar{\gamma}$.*

(*ii*) *If $\bar{\gamma}_1$, $\bar{\gamma}_2$ are continuous curves with $\bar{\gamma}_1$ L-projectable and $\bar{\gamma}_2$ $L_{\bar{\gamma}_1}$-projectable, the joint curve $\bar{\gamma}=\bar{\gamma}_1 \vee \bar{\gamma}_2$ is L-projectable and $L_{\bar{\gamma}}=(L_{\bar{\gamma}_1})_{\bar{\gamma}_2}$.*

(*iii*) *Reciprocally, if $\bar{\gamma}$ is continuous and L-projectable in M and we can write $\bar{\gamma}=\bar{\gamma}_1 \vee \bar{\gamma}_2$ we have that $\bar{\gamma}_1$ is L-projectable and $\bar{\gamma}_2$ is $L_{\bar{\gamma}_1}$-projectable.*

Using 2.5 and 2.3 we trivially obtain:

Corollary 2.6. *Any null piecewise geodesic $\bar{\gamma}:[0,a]\to\bar{M}$ with $\bar{\gamma}(0)=\bar{o}$ is L-projectable.*

3. L-PROJECTIONS AND HOMOTOPIES

We will see that in order to show the existence of the local isometry ϕ of theorem 1 it is enough to prove that all continuous curves with origin at $\bar{o}\in\bar{M}$ are L-projectable.

Proposition 3.1. *Let $\bar{\gamma}:[0,1]\times[0,a]\in(s,t)\to\bar{\gamma}_s(t)\in\bar{M}$ be a homotopy with $\bar{\gamma}_s(0)=\bar{o}$, $\bar{\gamma}_s(a)=\bar{p}$ for all $s\in[0,1]$. Suppose that for all $s\in[0,1]$ the curve γ_s is a L-projection of $\bar{\gamma}_s$. Then*

(*i*) *$\gamma:[0,1]\times[0,a]\in(s,t)\to\gamma_s(t)\in M$ is a homotopy and $\gamma_s(a)=p$ does not depend on s.*

(*ii*) *$L_{\bar{\gamma}_s}:T_{\bar{p}}\bar{M}\to T_pM$ does not depend on s.*

Proof: Fix $s_o\in[0,1]$ and set $\bar{\mathscr{U}}=\{(\bar{U}_i,\ \phi_i):i=1,\ ...,\ r\}$ be a projection covering for $\bar{\gamma}_{s_o}$ with associated partition $0=t_o<...<t_r=a$. Note that $\delta>0$ can be chosen such that $\bar{\mathscr{U}}$ is also a projection covering of $\bar{\gamma}_s$ if $|s-s_o|<\delta$, and $\phi_i(\bar{\gamma}_s(t))=\gamma_s(t)$ for $t\in[t_{i-1},\ t_i]$. This proves the continuity of γ. Now for $i=r$ we have $\phi_r(\bar{\gamma}_s(a))=\phi_r(\bar{p})=\gamma_s(a)$ for $|s-s_o|<\delta$. Therefore the map $[0,1]\in s\to\gamma_s(a)\in M$ is locally constant; having connected domain it is constant.

The same sort of argument proves that the map $s\to L_{\bar{\gamma}_s}$, $s\in[0,1]$ is constant.

Corollary 3.2. *If all curves $\bar{\gamma}:[0,a]\to\bar{M}$, $\bar{\gamma}(0)=\bar{o}$, are L-projectable; then there is a local isometry $\phi:\bar{M}\to M$ with $\phi(\bar{o})=0$, $d\phi(\bar{o})=L$.*

Proof.: Given $\bar{p} \in \bar{M}$ there is a curve $\bar{\gamma}:[0,1] \to \bar{M}$ with $\bar{\gamma}(0) = \bar{o}$, $\bar{\gamma}(1) = \bar{p}$. If γ is the L-projection of $\bar{\gamma}$, the point $p = \gamma(1) = \phi(\bar{p})$ and the linear isometry $L_{\bar{p}} = L_{\bar{\gamma}}: T_{\bar{p}}\bar{M} \to T_p M$ are uniquely determined by 3.1 since $\bar{M}$ is simply connected.

If $\bar{\mathscr{U}} = \{(\bar{U}_i, \phi_i): i = 1, \ldots, r\}$ is a projection covering of $\bar{\gamma}$ and (t_i) is the associated partition it is straightforward to conclude that $\phi/\bar{U}_i = \phi_i$ $i = 1, \ldots, r$.

This shows that $L_{\bar{p}} = d\phi(\bar{p})$ and ϕ is a local isometry such that $d\phi(\bar{o}) = L_{\bar{o}} = L$.

We prove the following technical lemma:

Lemma 3.3. *Let $\bar{\gamma}_o:[0,a] \to \bar{M}$ be a continuous curve such that $\bar{\gamma}_o(0) = \bar{o}$. Suppose that for all $a_1 < a$ the curve $\bar{\gamma}_o/[0,a_1]$ is L-projectable and there is a fixed-ends homotopy $\bar{\gamma}:[0,\delta] \times [0,a] \ni (s,t) \to \bar{\gamma}_s(t) \in \bar{M}$ such that*

i) $\bar{\gamma}_s/[0,a-s] = \bar{\gamma}_o/[0,a-s]$;

ii) $\bar{\gamma}_s/[a-s,a]$ is a $L^{a-s}_{\bar{\gamma}_o}$-projectable curve.
Then $\bar{\gamma}_o$ is L-projectable.

Proof.: Since the homotopy $\bar{\gamma}$ keeps ends fixed there is $\bar{p} \in \bar{M}$ such that $\bar{\gamma}_s(a) = \bar{p}$ if $s \in [0,\delta]$. Then if γ_s is the L-projection of $\bar{\gamma}_s$ we get from 3.1 that $p = \gamma_s(a)$ and $L_{\bar{p}} = L_{\bar{\gamma}_s}$ are well determined, independently of s. Moreover $L_{\bar{p}}$ is a linear isometry preserving curvature (2.4 ii). By Cartan's theorem we get convex neighbourhoods $\bar{U}$, U of $\bar{p}$, p and an isometry $\phi:\bar{U} \to U$ such that $d\phi(\bar{p}) = L_{\bar{p}}$. Fix $\delta_1 > 0$ such that $\gamma_s([a-\delta_1, a]) \subset \bar{U}$ if $0 < s \leqslant \delta_1$. We have then $L^t_{\bar{\gamma}_s} = d\phi(\bar{\gamma}_s(t))$ if $t \in [a-\delta_1, a]$, $0 < s \leqslant \delta_1$. Since $\gamma_s(t) = \gamma_o(t)$ for $t \in [0, a-s]$ we get $L^t_{\bar{\gamma}} = d\phi(\bar{\gamma}_o(t))$ for $t \in [a-\delta_1, a]$. Therefore $\bar{\gamma}_o/[a-\delta_1, a]$ is $L^{a-\delta_1}_{\bar{\gamma}_o}$-projectable (its projection is $\gamma_o(t) = \phi(\bar{\gamma}_o(t))$ for $t \in [a-\delta_1, a]$ and $\bar{\gamma}_o/[0, a-\delta_1]$ is L-projectable. Using 2.6 ii) we get that $\bar{\gamma}_o$ is L-projectable.

4. PROOF OF THEOREM 1

By corollary 3.2 we just need to prove that any curve $\bar{\gamma}:[0,a] \to \bar{M}$ is L-projectable. As a first approximation we restrict ourselves to the case where $\bar{\gamma}$ is a non-null geodesic. This requires the following result valid for any Lorentzian manifold.

Theorem 4.1. *Let M be a Lorentz manifold and $\gamma_o:[0,a] \to M$ a non-null geodesic of sign ε. There is a fixed-ends homotopy $\gamma:[0,\delta] \times [0,a] \to M$ such that $\gamma_s/[0,a-s] = \gamma_o/[0,a-s]$, and $\gamma_s/[a-s,a]$ is the join of two null geodesics.*

Proof: We take a convex neighbourhood U of $\gamma_o(a)=p$ and fix the following notation. If $x,y\in U$ we write $xy=exp_x^{-1}(y)$ and $\xi_{xy}(r)=exp_x(rxy)$ for $r\in[0,1]$, is the only geodesic in U defined on $[0,1]$ joining x and y. The function $q(x)=<px,px>$ and the field $P(x)=\xi'_{px}(1)$, $x\in U$ are related by $\text{grad}q=2P$.

If $C_p=\{x\in U: q(x)=0\}$ we have $x\in C_p$ if and only if ξ_{px} is a null geodesic. The point x_s denotes $\gamma_o(a-s)$.

We sketch the proof as follows. Take $\delta_1>0$ such that $\gamma_o(t)\in U$ if $t\in[a-\delta_1,a]$. Let $V(t)$, $t\in[0,a]$ be a null parallel field along γ_o such that

$$\varepsilon<V(a),\ \gamma'_o(a)>>0 \text{ and } \exp_{\gamma_o(t)}V(t)\in U \text{ for } t\in[a-\delta_1,a] \qquad (1)$$

The main idea is to prove that for small s the geodesic $\sigma_s(r)=\exp_{x_s}(rV(a-s))$, $r\in[0,1]$ intersects C_p in a first point p_s.

We construct then the homotopy by a convenient parametrization of the curves $\gamma_s=\gamma_o/[0,a-s]\vee\xi_{x_sp_s}\vee\xi_{p_sp}$.

We go into the details now (see Fig. 1).

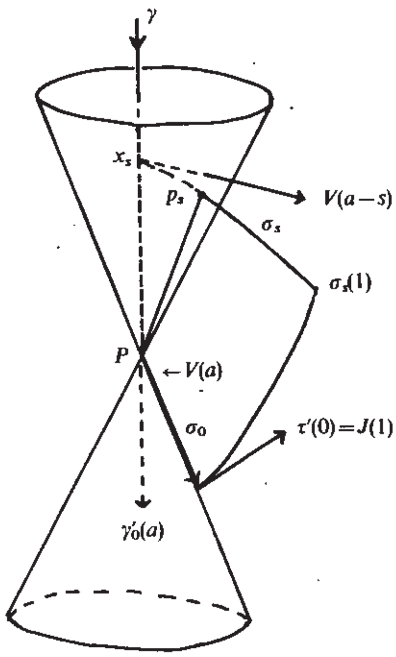

FIG. 1

The variation $\sigma:[0,\delta_1]\times[0,1]\in(s,r)\to\sigma_s(r)=\exp_{x_s}(rV(a-s))\in U$ defines on the geodesic σ_o a Jacobi field $J(r)=\frac{\partial\sigma}{\partial s}/_{(0,r)}$. Consider $\tau:[0,\delta_1]\in s\to\sigma_s(1)\in U$. We remark that $J(0)=-\gamma_o'(a)$, $J(1)=\tau'(0)$, $\sigma_o'(1)=P(\tau(0))$, $\tau(0)=\exp_p(V(a))$.

We get from (1) that $\varepsilon<-\gamma_o'(a), V(a)> =\varepsilon<J(0), \sigma_o'(0)> <0$. Multiplying $V(t)$ if necessary by a suitable constant $\lambda\in[0,1]$ we may assume by continuity that $\varepsilon<J(1), \sigma_o'(1)> <0$ and therefore:

$$0>\varepsilon<J(1), \sigma_o'(1)> =\varepsilon<\tau'(0), P(\tau(0))> =\frac{1}{2}\varepsilon<\tau'(0), \text{grad } q> = \frac{1}{2}\varepsilon(q\cdot\tau)'(0).$$

On the another hand, since $q\cdot\tau(0)=q(\exp_p(V(a))=0$ there is δ such that $0<\delta\leqslant\delta_1$ with $\varepsilon q\tau(s)<0$ for $s\in[0,\delta]$. If $s\in[0,\delta]$ the map $\varepsilon q\cdot\tau_s:[0,1]\to\mathbb{R}$ verifies:

$$\varepsilon q\cdot\sigma_s(0)=\varepsilon q(x_s)>0,\ \varepsilon q\cdot\sigma_s(1)=\varepsilon q\cdot\tau(s)<0.$$

Therefore, there is a first $\theta_s\in(0,1)$ such that $q\cdot\sigma_s(\theta_s)=0$. By the implicit function theorem, the map $(0,\delta]\ni s\to\theta_s\in[0,1]$ is continuous.

Define $p_s=\sigma_s(\theta_s)$. We define (see Fig. 2) the homotopy γ as follows:

$$\gamma_s(t)=\begin{cases}\gamma_o(t) \text{ if } t\in[0,a-s] & \text{(ruled area)}\\ \exp_{x_s}\left(\dfrac{2\theta_s}{s}(t+s-a)V(a-s)\right) \text{ if } t\in\left[a-s,\ a-\dfrac{s}{2}\right] & \text{(dotted area)}\\ \exp_p\left(-\dfrac{2}{s}(t-a)pp_s\right) \text{ if } t\in\left[a-\dfrac{s}{2},\ a\right] & \text{(shaded area)}\end{cases}$$

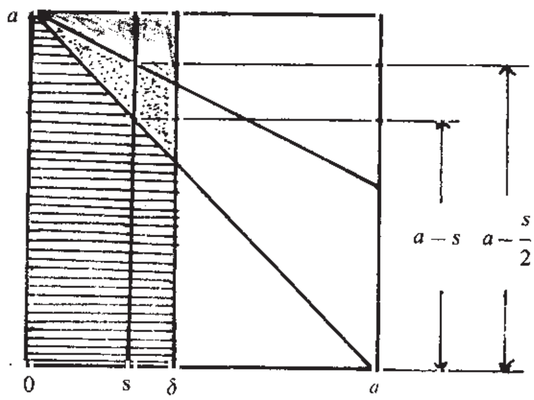

FIG. 2

We check easily the continuity of γ at the points in $[0,\delta]\times[0,a]$ except at $(0,a)$. The continuity at this point is proved using that $p=\lim_{s\to 0} p_s$. Therefore for any convex neighbourhood W of p, there is $\eta>0$ such that $p_s\in W$ for $0<s<\eta$, and ξ_{x,p_s}, $\xi_{p,p}$ are in W.

Corollary 4.2. *Any geodesic $\sigma:[0,1]\to\bar{M}$ with $\bar{\sigma}(0)=\bar{o}$ is L-projectable.*

Proof: Null geodesics are L-projectable by 2.1.
If $\bar{\sigma}$ is a non null geodesic consider $I=\{a\in(0,1]:\bar{\sigma}/[0,a]$ is L-projectable$\}$. By Cartan's Theorem, I is a non empty open subset of $(0,1]$. To see that $(0,1]=I$ we just need to prove the following statement. If $(0,a)\subset I$ for $a\in(0,1)$, then $a\in I$. Let $\bar{\gamma}_o=\bar{\sigma}/[0,a]$. We construct using 4.1 a fixed-ends homotopy $\bar{\gamma}:[0,1]\times[0,a]\to\bar{M}$ of $\bar{\gamma}_o$ such that:

(i) $\bar{\gamma}_s/[0,a-s]=\bar{\gamma}_o/[0,a-s]$ wich is L-projectable by hypothesis.

(ii) $\bar{\gamma}_s/[a-s,a]$ is a piecewise null geodesic, which is $L_{\bar{\gamma}_s}$-projectable by 2.7.

We get from 3.3 that $\bar{\gamma}_o$ is L-projectable and $a\in I$.

Corollary 4.3. *Any curve $\sigma:[0,1]\to\bar{M}$ such that $\bar{\sigma}(0)=\bar{o}$ is L-projectable.*

Proof: As before the set $I=\{a\in(0,1]:\bar{\sigma}/[0,a]$ is L-projectable$\}$ is a non empty open subset of $(0,1]$. We show that $a\in(0,1]$ and $(0,a)\subset I$ imply $a\in I$. Consider $\bar{\gamma}_o=\sigma/[0,a]$. Take $\bar{U}$, convex neighbourhood of $\bar{p}=\bar{\gamma}_o(a)$ and $\delta>0$ such that $\bar{\gamma}_o(t)\in\bar{U}$ if $t\in[a-\delta,a]$. If we define $\bar{p}_s=\bar{\gamma}_o(a-s)$, we can construct a fixed-ends homotopy $\bar{\gamma}:[0,\delta]\times[0,a]\to\bar{M}$ such that:

(i) $\bar{\gamma}_s/[0,a-s]=\bar{\gamma}_o/[0,a-s]$ which is L-projectable by hypothesis.

(ii) $\bar{\gamma}_s(t)=\exp_{\bar{p}_s}\left(\frac{t+s-a}{s}\,\bar{p}_s\bar{p}\right)$, $t\in[a-s,a]$.

Since $\bar{\gamma}_s/[a-s,a]$ is $L^{a-s}_{\bar{\gamma}_s}$-projectable by 4.2 we get from 3.3 the result.

Theorem 1 is now inmediate, since 4.3 and 3.2 give a local isometry $\phi:\bar{M}\to M$ such that $d\phi(\bar{o})=L$.

5. REMARKS

The key point in the proof of theorem 1 is 4.1 which can easily modified in this way.

Theorem 5.1. *Let M be a Lorentz manifold and* γ_o:$[0,a]\rightarrow M$ *a geodesic with sign* ε $(\varepsilon\in\{-1,0,1\})$. *If we fix* $\varepsilon'\in\{-1,0,1\}$, *there is a fixed-ends homotopy of* γ_o, *say* γ:$[0,\delta]\times[0,a]\rightarrow M$ *such that* $\gamma_s/[0,a-s]\neq\gamma_o/[0,a-s]$ *and* $\gamma_s[a-s,a]$ *is a piecewise geodesic with sign* ε'.

Theorem 5.1 allows to modify analogously theorem 1, and, as consequence, also corollary 2, whose new statement becomes.

Corollary 5.2. *The null, timelike and spacelike completeness are equivalent in a locally symmetric Lorentz manifold.*

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