Operadic structure on the Gerstenhaber–Schack complex for prestacks

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Abstract

We introduce an operad Patch which acts on the Gerstenhaber–Schack complex of a prestack as defined by Dinh Van and Lowen, and which in particular allows us to endow this complex with an underlying L_∞-structure. We make use of the operad Quilt which was used by Hawkins in order to solve the presheaf case. Due to the additional difficulty posed by the presence of twists, we have to use Quilt in a fundamentally different way (even for presheaves) in order to allow for an extension to prestacks.

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1 Introduction

The deformation theory of algebras due to Gerstenhaber furnishes the guiding example for algebraic deformation theory. For an algebra A, the Hochschild complex $\mathbf{C}(A)$ is a dg Lie algebra governing the deformation theory of A through the Maurer–Cartan formalism. This dg Lie structure is the shadow of a richer operadic structure, which can be expressed by saying that $\mathbf{C}(A)$ is a homotopy G-algebra [6]. This structure, which captures both the brace operations and the cup product, is a special case of a B_∞-structure [7]. Importantly, this purely algebraic structure constitutes a stepping stone in the proof of the Deligne conjecture, proving $\mathbf{C}(A)$ to be an algebra over the chain little disk operad [10,13].

The deformation theory of algebras was later extended to presheaves of algebras by Gerstenhaber and Schack, who in particular introduced a bicomplex computing the natural bimodule Ext groups [4,5]. However, this GS-complex $\mathbf{C}(\mathcal{A})$ of a presheaf $\mathcal{A}$ does not control deformations of $\mathcal{A}$ as a presheaf, but rather as a *twisted* presheaf, see for instance [2,11]. From this point of view, it is more natural to develop deformation theory at once on the level of twisted presheaves or, more generally, prestacks, that is, pseudofunctors taking values in the 2-category of linear categories (over some fixed commutative ground ring). In [2], Dinh Van and Lowen established a Gerstenhaber–Schack complex for prestacks, involving a differential which features an infinite sequence of higher components in addition to the classical simplicial and Hochshild differentials. Further, for a prestack $\mathcal{A}$, they construct a homotopy equivalence $\mathbf{C}_{GS}(\mathcal{A}) \cong \mathbf{CC}(\mathcal{A}!)$ between the Gerstenhaber–Schack complex $\mathbf{C}_{GS}(\mathcal{A})$ and the Hochschild complex $\mathbf{CC}(\mathcal{A}!)$ of the Grothendieck construction $\mathcal{A}!$ of $\mathcal{A}$. Through homotopy transfer, this endows the GS-complex with an L_∞-structure. This result improves upon the existence of a quasi-isomorphism, which is a consequence of the Cohomology Comparison Theorem due to Gerstenhaber and Schack for presheaves [5] and to Lowen and Van den Bergh for prestacks [12].

Although the GS-complex does not possess a B_∞-structure, its elements - linear maps involving different levels of the prestack - can be composed in an operadic fashion. As such, it makes sense to investigate this higher structure in its own right, and use it directly in order to establish an underlying L_∞-structure. For particular types of presheaves, explicit L_∞-structures on the GS-complex have been established by Frégier et al. [3] and by Barmeier and Frégier [1].

Let Brace be the brace operad and F_2S the homotopy G-operad. In the case of a presheaf $(\mathcal{A}, m, f)$, in [8], Hawkins introduces an operad Quilt $\subseteq F_2S \otimes_H$ Brace which he later extends to an operad mQuilt acting on the GS-complex. These operads are naturally endowed with L_∞-operations as desired. The action of Quilt on the GS-complex considered by Hawkins only involves the restriction functors f of the presheaf, the multiplication m being incorporated later on in mQuilt. Unfortunately, the way in which functoriality of f is built into these actions, does not allow for an extension to twisted presheaves or prestacks.

The goal of this paper is to solve the problem of establishing a natural operadic structure with underlying L_∞-structure on $\mathbf{C}_{GS}(\mathcal{A})$ in the case of a general prestack $(\mathcal{A}, m, f, c)$ with twists c. As part of our solution, we use Quilt in a fundamentally different way in relation to the GS-complex, but still allowing us to make use of the

naturally associated L_∞-structure from [8]. In Sect. 3, we capture the higher structure of $\mathbf{C}_{GS}(\mathcal{A})$ by introducing the new operad Patch $\subseteq$ mNSOp $\otimes_H$ NSOp over which $\mathbf{C}_{GS}(\mathcal{A})$ is shown to be an algebra (see Theorem 3.24). Here, (m)NSOp is the operad of nonsymmetric operads (with multiplication).

In [6], Gerstenhaber and Voronov obtain a brace algebra structure on an operad and a homotopy G-algebra structure on an operad with multiplication. Based upon the expression of these results in terms of the underlying operads NSOp and mNSOp in Sect. 2, we construct a morphism Quilt $\longrightarrow$ Patch$_s$ (see Proposition 3.27) as a restriction of

$$F_2S \otimes_H \text{Brace} \longrightarrow \text{mNSOp}_{st} \otimes_H \text{NSOp}_s,$$

where the operads with subscript denote the (uncolored) graded operads associated to the unsubscripted colored operads. This gives rise to the composition

$$R : \text{Quilt} \longrightarrow \text{Patch}_s \longrightarrow \mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$$

which incorporates the multiplication m and the restrictions f of $\mathcal{A}$.

In Sect. 4, we extend the action R to

$$R_c : \text{Quilt}_b[[c]] \longrightarrow \mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$$

in order to incorporate the twists (see Theorem 4.17). Here, Quilt$_b[[c]]$ is obtained from an operad of formal power series. Further, we establish L_∞-operations on Quilt$_b[[c]]$ extending those on Quilt from [8] (see Theorem 4.10) by adding an infinite series of higher components containing twists. Under the action of Quilt this neatly corresponds to and extends the differential on $\mathbf{C}_{GS}(\mathcal{A})$ obtained in [2]. In the final Sect. 4.5 we briefly discuss the relation of this L_∞-structure with the deformation theory of the prestack $\mathcal{A}$.

The present work naturally grew out of [2], and at the time when [8] appeared large parts of an operadic approach to the GS complex of a prestack had already been developed independently by us. Given the efficient way in which Hawkins' description of Quilt gives rise to an L_∞-structure, we decided it was worthwhile to build on this approach to the presheaf case, albeit in a way which "flips and refines" the action of Quilt in order to make it useful for general prestacks. As a consequence, when we follow through Hawkins' approach, in comparison we manage to incorporate not only the restrictions f, but also the multiplications m in an initial action of Quilt on the GS complex. In analogy with the way in which Hawkins extends his action from Quilt to mQuilt in order to incorporate the multiplications m, we establish an extension from Quilt to Quilt$_b[[c]]$ in order to incorporate the twists c.

The current paper is part of a larger project in which it is our goal to understand the homotopy equivalence $\mathbf{C}_{GS}(\mathcal{A}) \cong \mathbf{CC}(\mathcal{A}!)$ from [2] operadically, showing in particular that the L_∞-structure from [2] and the one established in the present paper actually coincide.

2 Gerstenhaber–Voronov operadically

In the seminal paper [6], Gerstenhaber and Voronov define a brace-algebra structure on the totalisation of a non-symmetric operad $\mathcal{O}$. Moreover, in presence of a multiplication, they define a homotopy G-algebra structure on $\mathcal{O}$ incorporating both the cup product and the Gerstenhaber-bracket.

In this section we describe the morphisms of operads underlying these results. To this end, in Sect. 2.3, we recall the colored operad NSOp encoding non-symmetric operads, and we describe the natural extension mNSOp which adds a multiplication. Let NSOp_s and mNSOp_{st} be their totalised graded (uncolored) operads with suspended, respectively standard degree (see Sects. 2.5 and 2.6). Let Brace be the brace operad (see Sect. 2.1) and F_2S the Gerstenhaber–Voronov operad encoding homotopy G-algebras (see Sect. 2.2). The main goal of this section is the definition of morphisms of dg-operads

$$\phi : \text{Brace} \longrightarrow \text{NSOp}_s$$

and

$$\bar{\phi} : F_2S \longrightarrow \text{mNSOp}_{st}$$

(see Theorems 2.16 and 2.34 respectively). In these definitions, we have to pay particular attention to the choice of signs. For this, we will make use of morphisms of operads (m)NSOp $\longrightarrow$ MultiΔ landing in the multicategory associated to the simplex category Δ (see Proposition 2.11).

For both uncolored as colored operads, we use the term morphism of operads. In case confusion may arise, we add a subscript to differentiate the uncolored operads from their colored counterparts.

2.1 The operad Brace

Throughout, we work over a fixed commutative ground ring k.

The operad Brace encoding brace algebras is defined using trees, that is, planar rooted trees. Following the presentation from [8, §2.2], for a tree T we denote the set of vertices by V_T, the set of edges by E_T, the "vertical" partial order on V_T generated by E_T by $\leq_T$, and the "horizontal" partial order on V_T by $\trianglelefteq_T$. For $(u, v) \in E_T$ we call u the *parent* of v and v a *child* of u.

For $n \in \mathbb{N}$, put $[n] := \{0, \ldots, n\}$ and $\langle n \rangle := \{1, \ldots, n\}$.

Let Tree(n) denote the set of trees with vertex set $\langle n \rangle$ and let Brace(n) be the free k-module on Tree(n) endowed with the $\mathbb{S}_n$-action given by permuting the vertices, i.e., T^σ is the tree defined by replacing vertex i in T by $\sigma^{-1}(i)$. The operadic composition on Brace is based upon substitution of trees, as follows. For trees $T \in \text{Tree}(m)$, $T' \in \text{Tree}(n)$ and $1 \leq i \leq m$, we denote by $Ext(T, T', i) \subseteq \text{Tree}(m + n - 1)$ the set of trees extending T by T' at i (that is, $U \in Ext(T, T', i)$ has T' as a subtree which

upon removal reduces to the vertex i of T). We then define

$$T \circ_i T' := \sum_{U \in Ext(T,T',i)} U.$$

Underlying every such extension lie two maps $\langle n \rangle \overset{\alpha}{\hookrightarrow} \langle n + m - 1 \rangle \overset{\beta}{\twoheadrightarrow} \langle m \rangle$ acting on the vertices, where α embeds n vertices consecutively and β contracts the image of α to the vertex i. We call the pair (α, β) the *extension of m by n at i*. We refer to [8, §2.2] for more details.

2.2 The operad $F_2 S$

The operad $F_2 S$ encodes homotopy G-algebras [6]. Again, we largely follow the exposition from [8, §2.3]. Given a set A, a *word* over A is an element of the free monoid on A. For a word $W = a_1 a_2 \ldots a_k$, correspondiong to the function $W : \langle k \rangle \longrightarrow A : i \longmapsto a_i$, the i-th *letter* of W is the couple (i, a_i). We will often identify a word with its graph $W = \{(i, a_i) \mid i \in \langle k \rangle\} \subseteq \langle k \rangle \times A$, writing $(i, a_i) \in W$.

For $a \in A$, a letter $(i, a) \in W$ is called an *occurrence* of a in W. The letter (i, a) is a *caesura* if there is a later occurrence of a in W, that is, a letter (j, a) with $i < j$. We say that $a \in A$ is *interposed* in W if $W = \cdots ba \ldots b \ldots$. The *length* of $W : \langle k \rangle \longrightarrow A$ is $|W| = k$.

By definition, $F_2 S(n)$ is the free k-module generated by the words W over $\langle n \rangle$ such that:

(1) $W : \langle k \rangle \longrightarrow \langle n \rangle$ is surjective,
(2) $W \neq \cdots uu \ldots$ (nondegeneracy), and
(3) For any $u \neq v \in \langle n \rangle$, $W \neq \cdots u \ldots v \ldots u \ldots v \ldots$ (no interlacing).

The set $F_2 S(n)$ is graded by setting $deg(W) := |W| - n$ and naturally carries a $\mathbb{S}_n$-action by permuting letters, i.e. $W^\sigma = \sigma^{-1} W$.

For a word $W \in F_2 S(n)$ and $u \in \langle n \rangle$, let (i_u, u) be the first occurrence of u in W. Then we obtain a total order $u \downarrow v \iff i_u \leq i_v$ on $\langle n \rangle$.

The operadic composition on $F_2 S$ is based upon merging of words, as follows. For words $W \in F_2 S(m)$, $W' \in F_2 S(n)$ and $1 \leq i \leq m$, we denote by $Ext(W, W', i) \subseteq F_2 S(m + n - 1)$ the set of extensions of W by W' at i (that is, $X \in Ext(W, W', i)$ if up to relabelling and deleting repetitions, W' is a subword of X and upon collapsing the letters from W to i, relabelling and deleting repetitions, we recover W).

In order to define the composition, we need the sign of an extension.

Sign of Extension. Let $W \in F_2 S(m)$ and let $int(W)$ be the set of interposed elements of $\langle m \rangle$ ordered by their first occurrence in W. For $X \in Ext(W, W', i)$ the relabelling gives rise to two maps $\alpha : int(W') \longrightarrow int(X)$ and $\gamma : int(W) \longrightarrow int(X)$ where $\gamma := \beta^{-1}$ except if i is interposed in W, then $\gamma(i) := \alpha(a)$ for $(1, a)$ the first letter of W'.

As $|int(W)| = deg(W)$, an extension X defines a unique $(deg(W), deg(W'))-$ shuffle χ and we define

$$\text{sgn}_{W,W',i}(X) := (-1)^\chi$$

Moreover, it is possible to talk about the boundary of a word, inducing a differential.
Boundary. Given a word $W \in F_2S(n)$ and a letter (i, a) of W for which a is repeated in W, then define $\partial_i W \in F_2S(n)$ as the word obtained by deleting the letter (i, a) from W (and relabelling). If a is not repeated, then set $\partial_i W = 0$.
Sign of Deletion. Given a word $W \in F_2S(n)$ of length k, then we define $\text{sgn}_W :$ $\langle k \rangle \longrightarrow \{-1, 1\}$ by setting $\text{sgn}_W(i) = (-1)^k$ if (i, a_i) is the k-th caesura of W, and otherwise $\text{sgn}_W(i) = (-1)^{k+1}$ if it is the last occurrence, but the previous occurrence is the k-th caesura of W.
The $\mathbb{S}$-module F_2S defines a dg-operad with operadic composition given by

$$W \circ_i W' := \sum_{X \in Ext(W,W',i)} \text{sgn}_{W,W',i}(X)X$$

and boundary given by

$$\partial W := \sum_{i \in \langle |W| \rangle} \text{sgn}_W(i) \partial_i W$$

The following lemma, which we include for the convenience of the reader, shows how F_2S encodes the algebraic operations of a homotopy G-algebra.
Notations. To avoid too large expressions, we leave out certain bracketings by setting as default the bracketing

$$a \circ_i b \circ_j c := (a \circ_i b) \circ_j c$$

Moreover, we compress the following

$$a(\circ_{i_t} a_t)_t := a \circ_{i_1} a_1 \circ_{i_2} \cdots \circ_{i_n} a_n$$

Lemma 2.1 *Let* $M_2 := 12$, $M_{1,0} = 1$ *and* $M_{1,k} := 121 \ldots 1(k+1)1$ *for* $k \geq 1$, *then* F_2S *is generated by these elements and the following holds*

(1) $\partial(M_2) = 0$
(2) $\partial(M_{1,k}) = -(M_2^{(12)} \circ_2 M_{1,k-1}) + \sum_{i=2}^k (-1)^i M_{1,k-1} \circ_i M_2 + (-1)^{k+1} M_2 \circ_1 M_{1,k-1}$

Proof It is a straightforward computation to determine that M_2 and $M_{1,k}$ satisfy these relations.
Let $W \in F_2S(n)$, we then show that it lies in the suboperad generated by $M_2, (M_{1,k})_{k \geq 1}$ using only the above relations. We prove this by induction on n. If

$n = 1$, then $W = 1$. So assume $n > 1$ and apply a permutation such that the first letter of W is 1, then W is of the form

$$W = 1 W_1 1 \ldots 1 W_k 1 W_{k+1}$$

where W_i is the image of a non-empty word $W_i' \in F_2 S(n_i)$ under the map $\gamma_i : \langle n_i \rangle \longrightarrow \langle n \rangle$, except W_{k+1} which is possibly empty. Due to no interlacing we also know that the images $\operatorname{Im}(\gamma_i)$ are pair-wise disjoint. Hence, we can apply a permutation to assume that $\max \operatorname{Im}(\gamma_i) < \min \operatorname{Im}(\gamma_{i'})$ holds for every $i < i'$. In this case, we have that

$$W = M_{1,k} \circ_{k+1} W_k' \circ_k \cdots \circ_2 W_1'$$

if $W_{k+1} = \emptyset$, and

$$W = (M_2 \circ_2 W_{k+1}') \circ_1 (M_{1,k} \circ_{k+1} W_k' \circ_k \ldots \circ_2 W_1')$$

otherwise. By induction, this shows that W is generated by M_2 and $(M_{1,k})_{l \geq 0}$. $\qquad\square$

2.3 The operads NSOp and mNSOp

It is well-known that non-symmetric operads can be encoded using a colored operad NSOp which can be defined using indexed trees, that is, for $q_1, \ldots, q_n \in \mathbb{N}$ and $q' = 1 + \sum_{i=1}^{n}(q_i - 1)$, $\operatorname{NSOp}(q_1, \ldots, q_n; q')$ is the set of pairs (T, I) where $T \in \operatorname{Tree}(n)$ and $I : E_T \longrightarrow \mathbb{N}$ a function such that

- For $(u, v) \in E_T$, $1 \leq I(u, v) \leq q_u$
- $(t, u), (t, v) \in E_T$ and $u \lhd_T v \Longrightarrow I(t, u) < I(t, v)$

We will often write I to denote the indexed tree (T, I). Moreover, NSOp is generated by those trees with a single edge, that is,

$$E_i := \begin{array}{c} 2 \\ \big| \; i \\ 1 \end{array} \;\; \in \operatorname{NSOp}(q_1, q_2; q_1 + q_2 - 1)$$

for every q_1, q_2 and $1 \leq i \leq q_1$, with the following pair of relations

$$\text{(I)} \quad \begin{array}{c} 2 \\ \big| \; i \\ 1 \end{array} \circ_2 \begin{array}{c} 2 \\ \big| \; j \\ 1 \end{array} = \begin{array}{c} 3 \\ \big| \; j \\ 2 \\ \big| \; i \\ 1 \end{array} = \begin{array}{c} 2 \\ \big| \; i-1+j \\ 1 \end{array} \circ_1 \begin{array}{c} 2 \\ \big| \; i \\ 1 \end{array} \quad \text{for } 1 \leq j \leq q_2 \text{ and}$$

$1 \leq i \leq q_1$,

$$\text{(II)} \quad \begin{array}{c} 2 \\ \big| \; i \\ 1 \end{array} \circ_1 \begin{array}{c} 2 \\ \big| \; k \\ 1 \end{array} = \begin{array}{c} 3 \;\; 2 \\ {}_i \diagdown \; \diagup_k \\ 1 \end{array} = \left(\begin{array}{c} 2 \\ \big| \; k-1+q_2 \\ 1 \end{array} \circ_1 \begin{array}{c} 2 \\ \big| \; i \\ 1 \end{array} \right)^{(23)} \quad \text{for}$$

$1 \leq i < k \leq q_1$.

Note that these are the well-known associativity relations for non-symmetric operads.

Definition 2.2 Let mNSOp be the $\mathbb{N}$-colored operad generated by NSOp and an element $m \in \mathrm{mNSOp}(; 2)$ satisfying the relation

$$\left.\begin{array}{c} 2 \\ | \\ 1 \end{array}\right._1 \circ_1 m \circ_1 m = \left.\begin{array}{c} 2 \\ | \\ 1 \end{array}\right._2 \circ_1 m \circ_1 m$$

Remark 2.3 We often write $\left.\begin{array}{c} m \\ | \\ m \end{array}\right._1$ where we have already filled in the plugged in m's.

More explicitly, every representative of an element $X \in \mathrm{mNSOp}(q_1, \ldots, q_n; q)$ is of the form $I \circ_{i_1} m \circ_{i_2} \ldots \circ_{i_k} m$ for $I \in \mathrm{NSOp}$ and appropriate $i_1, \ldots, i_k \in \mathbb{N}$. Due to equivariance, we can always consider a representative of X of the form

$$I \circ_{n+1} m \circ_{n+1} \ldots \circ_{n+1} m$$

for $I \in \mathrm{NSOp}(q_1, \ldots, q_n, 2, \ldots, 2; q)$.

Lemma 2.4 *Let* $X = [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \mathrm{mNSOp}(q_1, \ldots, q_n; q)$, *the partial orders* $<_I$ *and* $\lhd_I$ *on* $\langle n \rangle$ *are independent of the representative of* X. *We denote them by* $<_X$ *and* $\lhd_X$.

Proof We proceed by induction on k the number of m's in X. For $k = 0$ or $k = 1$, there is nothing to show, so assume $k > 1$. It is clear that if the lemma holds for X, then the relations that hold for $<$ and $\lhd$ for trees, also hold for $<_X$ and $\lhd_X$. In particular, if the lemma holds for X and X' and (α, β) is the extension of n by m at i, then for $a, b \notin \mathrm{Im}(\alpha)$ we have

$$a <_{X \circ_i X'} b \iff \beta a <_X \beta b \text{ and } a \lhd_{X \circ_i X'} b \iff \beta a \lhd_X \beta b$$

Now, let $X_{0,i} := X_0 \circ_{n+1} \left.\begin{array}{c} 2 \\ | \\ 1 \end{array}\right._i$ such that $X = X_{0,1} \circ_{n+1} m \circ_{n+1} m = X_{0,2} \circ_{n+1}$ $m \circ_{n+1} m$, then we have by induction that the lemma holds for X_0. Moreover, we have for $a, b \in \langle n \rangle$ that

$$a <_{X_{0,1}} b \iff a <_{X_0} b \iff a <_{X_{0,2}} b \text{ and}$$
$$a \lhd_{X_{0,1}} b \iff a \lhd_{X_0} b \iff a \lhd_{X_{0,2}} b$$

which proves the lemma for X. $\square$

2.4 The morphisms (m)NSOp $\longrightarrow$ MultiΔ

Let $\mathcal{C}$ be a small category. We denote by Multi$\mathcal{C}$ the $\mathrm{Ob}(\mathcal{C})$-colored operad for which Multi$\mathcal{C}(c_1, \ldots, c_n; c)$ is freely generated as a k-module by n-tuples $(\zeta_1, \ldots, \zeta_n)$ of $\mathcal{C}$-morphisms with $\zeta_i : c_i \longrightarrow c$, $\mathbb{S}_n$ acts by permuting labels, and composition is defined in the obvious way.

Let Δ be the simplex category. Next, we construct a morphism of operads

$$\mathrm{NSOp} \longrightarrow \mathrm{Multi}\Delta,$$

by associating to every indexed tree I in $\mathrm{NSOp}(q_1, \ldots, q_n; q)$ a n-tuple ζ_I in Multi$\Delta(q_1, \ldots, q_n; q)$ which assigns to each vertex a, considered as an q_a-corolla, a numbering denoting where its inputs are amongst the inputs of the indexed tree as a whole.

It suffices to define the morphism on the generators $E_i \in \mathrm{NSOp}$ and show that it respects the relations.

Construction 2.5 Let $E_i \in \mathrm{NSOp}(q_1, q_2; q_1 + q_2 - 1)$ for $1 \le i \le q_1$, then we define

$$\zeta_{E_i,1}(t) := \begin{cases} t & t < i \\ t + q_2 - 1 & t \ge i \end{cases} \quad \text{and} \quad \zeta_{E_i,2}(t) := t + i - 1$$

Then, $\zeta_{E_i} \in \mathrm{Multi}\Delta(q_1, q_2; q_1 + q_2 - 1)$, that is, it is a tuple of non-decreasing maps. Moreover, if $q_2 > 0$, then these are strictly increasing.

We will employ it as in the following example.

***Example 2.6*** Let $\mathcal{A}$ be a k-linear category, then its Hochschild complex is defined as

$$\mathbf{C}^n(\mathcal{A}) = \prod_{A_0, \ldots, A_n \in \mathcal{A}} \mathrm{Hom}_k \left(\mathcal{A}(A_1, A_0) \otimes \cdots \otimes \mathcal{A}(A_n, A_{n-1}), \mathcal{A}(A_n, A_0) \right)$$

For a Hochschild cochain $\phi \in \mathbf{C}^n(\mathcal{A})$ and a n-simplex $A_0 \xleftarrow{a_1} A_1 \xleftarrow{a_2} \cdots \xleftarrow{a_{n-1}} A_{n-1} \xleftarrow{a_n} A_n$ in $\mathcal{A}$, we have that $\phi^{A_0, \ldots, A_n}(a_1, \ldots, a_n) \in \mathcal{A}(A_n, A_0)$.

Let $\phi_1 \in \mathbf{C}^{q_1}(\mathcal{A})$ and $\phi_2 \in \mathbf{C}^{q_2}(\mathcal{A})$, then each $E_i \in \mathrm{NSOp}(q_1, q_2; q_1 + q_2 - 1)$ determines a cochain $\phi_1 \circ_i \phi_2 \in \mathbf{C}^{q_1 + q_2 - 1}(\mathcal{A})$ as follows

$$(\phi_1 \circ_i \phi_2)^{A_0, \ldots, A_{q_1+q_2-1}} = \phi_1^{A_{\zeta_{E_i,1}(0)}, \ldots, A_{\zeta_{E_i,1}(q_1)}} \circ_i \phi_2^{A_{\zeta_{E_i,2}(0)}, \ldots, A_{\zeta_{E_i,2}(q_2)}}$$

which we can visualize using n-corollas
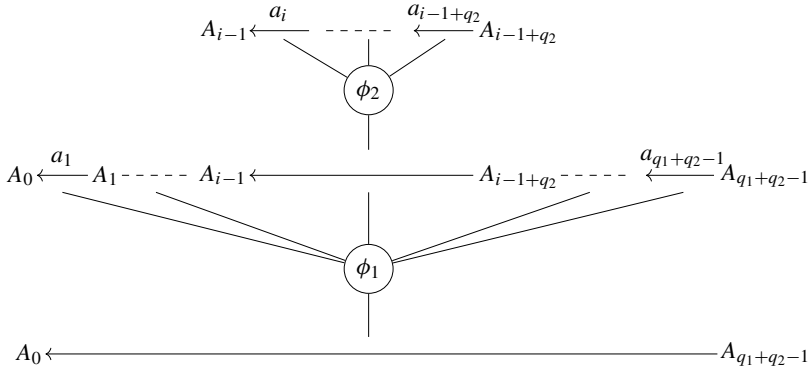
Lemma 2.7 *Construction 2.5 extends to a morphism of operads*

$$\text{NSOp} \longrightarrow \textit{Multi}\Delta : (T, I) \longmapsto \zeta_I$$

Proof It suffices to verify the relations (I) and (II) of NSOp. These are two simple computations and thus we only verify the first relation (I) as an example. Let $\zeta := \zeta_{E_i} \circ_2 \zeta_{E_j}$ denote the left-hand side, then we compute

$$\zeta_1(t) = \zeta_{E_i,1}(t) \qquad = \begin{cases} t & t < i \\ t + q_2 + q_3 - 2 & t \geq i \end{cases}$$

$$\zeta_2(t) = \zeta_{E_i,2} \circ \zeta_{E_j,1}(t) = \begin{cases} t + i - 1 & t < j \\ t + i - 1 + q_3 - 1 & t \geq j \end{cases}$$

$$\zeta_3(t) = \zeta_{E_i,2} \circ \zeta_{E_j,2}(t) = t + i - 1 + j - 1$$

On the other hand, we compute the right-hand side $\zeta' := \zeta_{E_{i-1+j}} \circ_1 \zeta_{E_i}$ and obtain

$$\zeta'_1(t) = \zeta_{E_{i-1+j},1} \circ \zeta_{E_i,1}(t) = \begin{cases} t & t < i \\ t + q_2 - 1 + q_3 - 1 & t \geq i \end{cases}$$

$$\zeta'_2(t) = \zeta_{E_{i-1+j},1} \circ \zeta_{E_i,2}(t) = \begin{cases} t + i - 1 & t < j \\ t + i - 1 + q_3 - 1 & t \geq j \end{cases}$$

$$\zeta'_3(t) = \zeta_{E_{i-1+j},1}(t) \qquad = t + i - 1 + j - 1$$

<div align="right">□</div>

Remark 2.8 In appendix A we have added a generator-free description of this morphism and an alternative proof of Lemma 2.7, which we consider insightful and valuable, especially for concrete computations of signs in later sections.

Example 2.9 Consider the indexed tree

$$
I =
\begin{array}{c}
 \quad 4 \quad\quad 5 \\
\quad {}_1\!\searrow \; \nearrow{}_3 \\
2 \quad\quad 3 \\
{}_1\!\searrow \; \nearrow{}_2 \\
1
\end{array}
\quad \in \mathrm{NSOp}(3, 0, 4, 1, 0; 4)
$$

then we compute ζ_I and obtain

$$
\begin{array}{ccccc}
\zeta_{I,1} & \zeta_{I,2} & \zeta_{I,3} & \zeta_{I,4} & \zeta_{I,5} \\
\end{array}
$$

It is also possible to associate to an element of mNSOp an element of MultiΔ.

Lemma 2.10 *Let* $X = [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \mathrm{mNSOp}(q_1, \ldots, q_n; q)$, *then* $\zeta_{I,t} : [q_t] \longrightarrow [q]$ *for* $t \in \langle n \rangle$ *is independent of the representative* I *of* X.
In this case, we write ζ_X.

Proof We prove the lemma by induction on k the number of occurrences of m. The cases $k = 0$ and $k = 1$ are trivial, so assume $k > 1$. Let $X_{0,i} := X_0 \circ_{n+1} \Big|_{1}^{\;2} i$ such that $X = X_{0,1} \circ_{n+1} m \circ_{n+1} m = X_{0,2} \circ_{n+1} m \circ_{n+1} m$, then by induction and Lemma 2.7 we have for $t \in \langle n \rangle$ that

$$
\zeta_{X_{0,1},t} = \zeta_{X_0,t} = \zeta_{X_{0,2},t}
$$

which proves the lemma. $\qquad\square$

Proposition 2.11 *We have morphisms of operads*

$$
\mathrm{NSOp} \longrightarrow Multi\Delta : I \longmapsto \zeta_I
$$

and its extension

$$
\mathrm{mNSOp} \longrightarrow Multi\Delta : X \longmapsto \zeta_X.
$$

Moreover, this last morphism is surjective, but not an isomorphism. This is due to the existence of vertices with zero inputs which collapse information. We consider a simple example.

Example 2.12 Consider the indexed 2-corolla $I :=$ $\begin{smallmatrix} 2 & & 3 \\ {}_1\diagdown & & \diagup{}_2 \\ & 1 & \end{smallmatrix}$ and its permuted form

$I^{(2,3)} =$ $\begin{smallmatrix} 3 & & 2 \\ {}_1\diagdown & & \diagup{}_2 \\ & 1 & \end{smallmatrix}$ as elements of NSOp$(2, 0, 0; 0)$, then they have the same image in

MultiΔ. Note that this example holds for both NSOp and mNSOp.

Hence, we can consider mNSOp as a finer operad than MultiΔ and thus encoding more information.

2.5 The morphism Brace $\longrightarrow$ NSOp$_s$

In order to define the morphism $\phi :$ Brace $\longrightarrow$ NSOp$_s$ properly we need to compile the colored operad NSOp into a graded non-colored operad

$$\text{NSOp}_s(n) \subseteq \prod_{p_1,\ldots,p_n} \text{NSOp}(p_1, \ldots, p_n; p).$$

where an element $x \in \text{NSOp}(p_1, \ldots, p_n; p)$ is graded as $|x| = \sum_{i=1}^r (p_i - 1) - (p - 1) = 0$ (this is the *suspended* grading, whence the subscript) and NSOp$_s(n)$ is the subspace generated by sequences of elements with constant grading. The composition on NSOp$_s$ is derived from the composition of NSOp where it is set to 0 when the colors do not match. Note in particular that the $\mathbb{S}_n$-action on NSOp$_s(n)$ is affected by this grading: permuting two vertices i and j introduces the signs $(-1)^{(p_i-1)(p_j-1)}$.

Definition 2.13 Let $(T, I) \in \text{NSOp}(p_1, \ldots, p_n; p)$, then (T, I) is a coloring of T and we write $\text{Clr}(T, p_1, \ldots, p_n)$ as the set of all such colorings of T.

   In order to define the sign $\text{sgn}_T(I)$ for $T \in \text{Brace}(n)$, we use the morphism of operads

$$\text{NSOp} \longrightarrow \text{Multi}\Delta : (T, I) \longmapsto \zeta_I$$

and base this definition on the sign $\text{sgn}_Q(\zeta, I)$ from [8, Def. 4.20].

Construction 2.14 We work with the following alphabet

$$1_i, \ldots, (p_i - 1)_i$$

for $i = 1, \ldots, n$ and define the word

$$J^s(p_1, \ldots, p_n) = 1_1 \ldots (p_1 - 1)_1 \ldots 1_n \ldots (p_n - 1)_n$$

We define a second word $J_T^s(I)$ having in the $\zeta_{I,k}(i)$-th position i_k for $1 \leq i \leq q_k - 1$. Note that we start from position 1 for $J_T^s(I)$ (instead of 0).

Definition 2.15 For $I \in \mathrm{NSOp}(p_1, \ldots, p_n; p)$ where we replace those $p_i = 0$ by 2, we define $\mathrm{sgn}_T(I)$ as the sign of the shuffle transforming $J^s(p_1, \ldots, p_n)$ to $J^s_T(I)$.

Theorem 2.16 *We have a morphism of operads*

$$\phi : \mathrm{Brace} \longrightarrow \mathrm{NSOp}_s : T \longmapsto \left(\sum_{I \in \mathrm{Clr}(T, p_1, \ldots, p_n)} \mathrm{sgn}_T(I)(T, I) \right)_{p_1, \ldots, p_n} .$$

Proof Per definition of $\mathrm{sgn}_T(I)$ we see that ϕ is equivariant. Hence, we only need to verify that $\mathrm{sgn}_{T \circ_1 T'}(I \circ_1 I') = \mathrm{sgn}_T(I) \, \mathrm{sgn}_{T'}(I')$ for $T \in \mathrm{Brace}(n)$, $T' \in \mathrm{Brace}(m)$ and $I \in \mathrm{Clr}(T, p_1, \ldots, p_n)$ and $I' \in \mathrm{Clr}(T', p'_1, \ldots, p'_m)$. This equation holds as we can decompose the shuffle $\chi'' : J^s(p'_1, \ldots, p'_m, p_2, \ldots, p_n) \rightsquigarrow J_{T \circ_i T'}(I \circ_i I')$ into two shuffles

$$J^s(p'_1, \ldots, p'_m, p_2, \ldots, p_n) \overset{\chi'}{\rightsquigarrow} J^s_{T'}(I') J^s(p_2, \ldots, p_m) \overset{\chi}{\rightsquigarrow} J^s_{T \circ_1 T'}(I \circ_1 I')$$

where χ and χ' are the corresponding shuffles determining $\mathrm{sgn}_T(I)$ and $\mathrm{sgn}_{T'}(I')$. □

2.6 The morphism $F_2S \longrightarrow \mathrm{mNSOp}_{st}$

In order to define the morphism $\bar{\phi} : F_2S \longrightarrow \mathrm{mNSOp}_{st}$ properly, we again need to compile the colored operad mNSOp to obtain a graded non-colored operad

$$\mathrm{mNSOp}_{st}(n) \subseteq \prod_{q_1, \ldots, q_n, q} \mathrm{mNSOp}(q_1, \ldots, q_n; q)$$

where an element $x \in \mathrm{mNSOp}(q_1, \ldots, q_n; q)$ is graded as $deg(x) = \sum_{i=1}^{r} q_i - q$ (*standard* grading) and $\mathrm{mNSOp}_{st}(n)$ is generated by the sequences of constant grading. The composition on mNSOp_{st} is derived from the composition of mNSOp where it is set to 0 when the colors do not match. Note in particular that the $\mathbb{S}_n$-action on $\mathrm{mNSOp}_{st}(n)$ is affected by this grading: permuting two vertices i and j introduces the sign $(-1)^{q_i q_j}$.

2.6.1 Colorings

Definition 2.17 Let $X := [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \mathrm{mNSOp}(q_1, \ldots, q_n; q)$ for I having $n + k$ vertices, then X is a coloring of $W \in F_2S(n)$ if

- Each vertex $n + 1, \ldots, n + k$ has exactly two children in I
- For $u, v \in \langle n \rangle$ holds

  (1) $u <_X v \iff W = \ldots u \ldots v \ldots u \ldots$
  (2) $u \lhd_X v \iff$ every occurrence of u in W is left of every occurrence of v in W.

We write $\mathrm{Clr}(W, q_1, \ldots, q_n)$ for the set of all such colorings for W.

Remark 2.18 An element $X \in \mathrm{Clr}(W, q_1, \ldots, q_n)$ has $n - deg(W) - 1$ many m's plugged in. Hence, $X \in \mathrm{mNSOp}(q_1, \ldots, q_n; \sum_{i=1}^{n}(q_i - 1) + n - deg(W))$.

We give some examples.

Example 2.19 The following three elements of mNSOp

$$
\begin{array}{c}
1 \quad\quad 2 \\
{}_1\diagdown \quad \diagup{}_2 \\
m
\end{array}
\in \mathrm{mNSOp}(q_1, q_2; q_1{+}q_2), \quad
\begin{array}{c}
2 \quad\quad 3 \\
{}_1\diagdown \quad \diagup{}_2 \\
m \\
\big|\,{}_i \\
1
\end{array}
\in \mathrm{mNSOp}(q_1, q_2, q_3; q_1{+}q_2{+}q_3{-}1)
$$

for $1 \leq i \leq q_1$

and

$$
\begin{array}{c}
\quad\quad 4 \quad\quad 5 \\
\quad\quad {}_1\diagdown \quad \diagup{}_2 \\
3 \quad\quad m \\
{}_1\diagdown \quad \diagup{}_2 \\
2 \quad\quad m \\
{}_j\diagdown \quad \diagup{}_k \\
1
\end{array}
\in \mathrm{mNSOp}(q_1, ..., q_5; \sum_{i=1}^{5} q_i - 2) \quad \text{for } 1 \leq j < k \leq q_1
$$

color respectively the words

$$12 \in \mathrm{F}_2\mathrm{S}(2), \quad 1231 \in \mathrm{F}_2\mathrm{S}(3) \quad \text{and} \quad 1213451 \in \mathrm{F}_2\mathrm{S}(5).$$

Note however that not all elements of mNSOp color a word of $\mathrm{F}_2\mathrm{S}$: the following set of elements

$$
\begin{array}{c}
1 \\
\big|\,{}_r \\
m
\end{array}
\in \mathrm{mNSOp}(q_1; q_1 + 1)
$$

for $r \in \{1, 2\}$, colors no word in $\mathrm{F}_2\mathrm{S}$ because the vertex plugged by m does not have two children.

Lemma 2.20 *Definition 2.17 is well-defined, that is, it is independent of the chosen representative I of X.*

Proof Due to Lemma 2.4, both $<_X$ and $\lhd_X$ are well-defined. We show that the condition stipulating that all vertices of I that are plugged by m's have exactly two children, is independent of the representative of X. Thus, suppose $I = I_0 \circ_k \begin{array}{c} 2 \\ \big|\,{}_1 \\ 1 \end{array}$ for some

$I_0 \in \mathrm{NSOp}$, such that both vertices k and $k + 1$ of I are plugged by m's in X. Due to

the relations in mNSOp, X can equivalently be represented using $I' := I_0 \circ \begin{smallmatrix} 2 \\ | \\ 2 \\ 1 \end{smallmatrix} \, 2$. In this case, we have that vertices k and $k+1$ each have exactly 2 children in I iff vertex k has exactly 3 children in I_0 iff vertices k and $k+1$ each have exactly 2 children in I'. □

We construct a word for every element of mNSOp satisfying the above criteria.

Construction 2.21 Let $X := [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \text{mNSOp}(q_1, \ldots, q_n; q)$ such that each vertex $a > n$ in I has exactly two children, then we construct a word $W_X \in F_2S(n)$ such that $X \in \text{Clr}(W_X, q_1, \ldots, q_n, q)$.

- To every tree T we can associate a word $W_T \in F_2S(n+k)$ (see [8, §2.3]).
- Suppose for $X_0 \in \text{mNSOp}$ such that $X_0 \circ_{n+1} m = X$ we have an associated word $W_{X_0} \in F_2S(n+1)$, then let W_X be the word given by deleting all occurrences of $n+1$. Then $W_X \in F_2S(n)$ because $n+1$ had two children, so no degeneracy can occur.

We consider an example of this procedure.

Example 2.22 We consider the element

$$
\begin{array}{cc}
2 & 3 \\
1 \diagdown \; \diagup 2 \\
m \\
| \; i \\
1
\end{array}
\qquad \in \text{mNSOp}(q_1, q_2, q_3; q_1 + q_2 + q_3 - 1)
$$

for some $1 \le i \le q_1$ from Example 2.19 and show how Construction 2.21 assigns a word. First, we associate to the indexed tree

$$
\begin{array}{cc}
2 & 3 \\
1 \diagdown \; \diagup 2 \\
4 \\
| \; i \\
1
\end{array}
\qquad \in \text{NSOp}(q_1, q_2, q_3, 2; q_1 + q_2 + q_3 - 1)
$$

the word 1424341 and then delete all occurrences of 4 as it is plugged by an instance of m. As a result, we obtain the word 1231.

Lemma 2.23 *For $X \in \text{mNSOp}(q_1, \ldots, q_n; q)$ we have $X \in \text{Clr}(W_X, q_1, \ldots, q_n)$ and if $X \in \text{Clr}(W, q_1, \ldots, q_n)$, then $W = W_X$.*

Proof This clearly holds for $X = I \in \text{NSOp}(q_1, \ldots, q_n; q)$. Assume the lemma holds for $X_0 \in \text{mNSOp}$ and $X = X_0 \circ_{n+1} m$, then $W_{X_0} = W_0(n+1)W_1(n+1)W_2(n+1)W_3$ for W_0 and W_3 possibly empty. In this case, $W_X = W_0 W_1 W_2 W_3$ and it is easy to see that $X \in \text{Clr}(W_X, q_1, \ldots, q_n)$.

Now reversely, if $X \in \text{Clr}(W, q_1, \ldots, q_n)$ and $a \lhd b$ are the two children of $n+1$ in X_0, then $W = W_0 W_1 W_2 W_3$ where $W_1 = a \ldots a$, $W_2 = b \ldots b$ and W_0

and W_3 are possibly empty. In that case, $X_0 \in \text{Clr}(W_0(n+1)W_1(n+1)W_2(n+1)W_3, q_1, \ldots, q_n, 2)$ and thus by induction $W_{X_0} = W_0(n+1)W_1(n+1)W_2(n+1)W_3$. Hence, $W_X = W_0 W_1 W_2 W_3 = W$. $\qquad\square$

Lemma 2.24 *Let* $X \in \text{Clr}(V, q_1, \ldots, q_n, q)$ *and* $Y \in \text{Clr}(W, q_1', \ldots, q_m', q_i)$, *then there exists a unique* $U \in \text{Ext}(V, W, i)$ *such that* $X \circ_i Y \in \text{Clr}(U, q_1, \ldots, q_1', \ldots, q_m', \ldots, q_n, q)$

Proof By Construction 2.21 we obtain a word $U \in F_2S(n+m-1)$ such that $Z := X \circ_i Y \in \text{Clr}(U, \ldots)$. We show that $U \in \text{Ext}(V, W, i)$: let U_α be the word obtained from deleting from U occurrences of vertices not in the image of α and eliminating consecutive repetitions ($uu \mapsto u$). It is easy to check that $W = \cdots u \ldots v \ldots u \ldots$ iff $U_\alpha = \cdots \alpha(u) \ldots \alpha(v) \ldots \alpha(u) \ldots$, and that all occurrences of u are left to those of v in W iff the same holds for $\alpha(u)$ and $\alpha(v)$ in U_α.

Let U_β be the word obtained from U by relabelling by β and eliminating consecutive repetitions. To verify that $U_\beta = V$ is straight forward, except in the following case: $U_\beta = \cdots i \ldots \beta(u) \ldots i \ldots$ where $\beta(u) \neq i$. In this case, there exist v, v' such that $U = \cdots \alpha(v) \ldots u \ldots \alpha(v') \ldots$. Now, we argue that there exists a vertex v'' such that $\alpha(v'') <_Z u$. If neither $\alpha(v)$ nor $\alpha(v')$ do, then U implies that $\alpha(v) \lhd_Z u \lhd_Z \alpha(v')$. As $\alpha(v)$ and $\alpha(v')$ are part of the same subtree $\alpha(Y)$, i.e. the image of Y under α in Z, there must be some vertex a in the tree underlying $\alpha(Y)$ (possibly plugged by m) such that a lies underneath u. As Y is a coloring of a word, the conditions imply that a is not plugged by an instance of m (otherwise it would not have two children in Y). As a result, there is some vertex v'' in Y such that $\alpha(v'') = a <_Z u$. Thus, $i <_X \beta(u)$ which verifies that $V = \cdots i \ldots \beta(u) \ldots i \ldots$. This clearly also holds reversibly. $\qquad\square$

Lemma 2.25 *Let* $U \in \text{Ext}(V, W, i)$ *and* $Z \in \text{Clr}(U, q_1, \ldots, q_1', \ldots, q_m', \ldots, q_n, q)$, *then there exist unique colorings* $X \in \text{Clr}(V, q_1, \ldots, q_n, q)$ *and* $Y \in \text{Clr}(W, q_1', \ldots, q_m', q_i)$ *such that* $Z = X \circ_i Y$.

Proof Let $Z = [I'' \circ_{n+m} m \circ_{n+m} \cdots \circ_{n+m} m]$ with $l + k$ added m's. The word W can be uniquely written as

$$W = W_1 \ldots W_t$$

where two subwords W_j and $W_{j'}$ do not share any occurrence of the same number, and W_j is of the form $a_j \ldots a_j$. As Z is a coloring of U, we have that $\alpha(a_1) \lhd_Z \ldots \lhd_Z \alpha(a_t)$ and no vertex of $\text{Im}(\alpha)$ lies under any a_j. In this case, there exists some vertex $a \in \{n+m, \ldots, n+m+k+l-1\}$ such that $a \leq_{I''} \alpha(a_j)$ which is $\leq_{I''}$-maximal for these conditions (otherwise when applying β to U we will not obtain V).

Let I' be the minimal subtree of I'' on the root a containing $\text{Im}(\alpha)$. By contracting this subtree to a point we obtain a tree I such that, after permutation of some vertices, we obtain $I \circ_i I' = I''$. Consider also the permutation such that $Z = [I'' \circ_{m+i} m \circ_{m+i} \cdots \circ_{m+i} m \circ_{n+m} m \circ_{n+m} \cdots \circ_{n+m} m]$.

It now suffices to show that $X := [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \text{Clr}(V, \ldots)$ and $Y := [I' \circ_{m+1} m \circ_{m+1} \cdots \circ_{m+1} m] \in \text{Clr}(W, \ldots)$, which is a straight forward computation using the facts $X \circ_i Y = Z$, $Z \in \text{Clr}(U, \ldots)$ and $U \in \text{Ext}(V, W, i)$. $\qquad\square$

2.6.2 Signs

In order to define a sign $\mathrm{sgn}_W(X)$ for $W \in \mathrm{F_2S}(n)$ and $X \in \mathrm{Clr}(W, q_1, \ldots, q_n)$ we use the morphism of operads

$$\mathrm{NSOp} \longrightarrow \mathrm{Multi\Delta} : (T, I) \longmapsto \zeta_I$$

which extends to

$$\mathrm{mNSOp} \longrightarrow \mathrm{Multi\Delta} : X \longmapsto \zeta_X$$

We base this definition of the sign on the sign $\mathrm{sgn}_Q(\zeta, I)$ defined in [8, §4.7].

Lemma 2.26 *Let* $X \in \mathrm{Clr}(W, q_1, \ldots, q_n; q)$ *for* $W \in \mathrm{F_2S}(n)$ *and* $q_i > 0$, *then* ζ_X *is a coloring of* W *in the sense of [8, Def. 4.13], that is,*

- $\zeta_X \in \mathrm{Multi\Delta}_+$,
- $\bigcup_{i=1}^{n} \mathrm{Im}(\zeta_{X,i}) = [q]$
- *For each* $a \in \langle n \rangle$ *there exists a function* $\pi_a : [q_a] \longrightarrow W$ *such that*

  (1) The image of π_a *is the set of occurrences of* a *in* W,
  (2) For an $(i, a) \in W$, $\zeta_{X,a}(\pi_a^{-1}(i, a))$ *is an interval*
  (3) If $W = \cdots ab \ldots$, *then*

$$\max\ \zeta_{X,a}(\pi_a^{-1}(i, a)) = \min\ \zeta_{X,b}(\pi_b^{-1}(i+1, b))$$

Proof We first show it holds for $X = (T, I) \in \mathrm{NSOp}$ by induction on the number of vertices: let

$$I = \quad \overset{I^1 \quad \cdots \quad I^k}{\underset{u}{\diagdown_{i_1} \big|\ \diagup^{i_k}}}$$

be its decomposition into its root u with maximal subtrees I^i. In this case, we have $W = uW^1u\ldots uW^ku$ where the subwords W_i represent the subtrees I^i. Let $\gamma_i : \langle k_i \rangle \hookrightarrow \langle n \rangle$ be the maps embedding the tree I_0^i onto I^i in I and $I_0^i \in \mathrm{Clr}(W_0^i, \ldots)$, then they extend to a map $\overline{\gamma_i} : W_0^i \hookrightarrow W$ inserting W_0^i as W^i into W. By induction, the lemma holds for $I_0^i \in \mathrm{Clr}(W_0^i, \ldots)$. Let $(p_1, u), \ldots, (p_{k+1}, u)$ be all the occurrences of u in W, then we define

$$\pi_{\gamma_i(a)} = \overline{\gamma_i} \circ \pi_a^{I_0^i} \text{ and } \pi_u(t) = \begin{cases} (p_1, u) & t < i_1 \\ (p_j, u) & i_{j-1} \le t < i_j \\ (p_{k+1}, u) & i_k \le t \end{cases}$$

then it is easy to verify that these satisfy the above conditions.

Now assume $X = [X_0 \circ_{n+1} m]$ such that the lemma holds for $X_0 \in \mathrm{Clr}(W_{X_0}, \ldots)$, then W_X is obtained from W_{X_0} by deleting all occurrences of $(n+1)$. As the vertex

$n + 1$ has exactly two children in X_0 and $q_{n+1} = 2$, we still have that $\bigcup_{i=1}^{n} \mathrm{Im}(\zeta_{X,i}) = \bigcup_{i=1}^{n+1} \mathrm{Im}(\zeta_{X_0,i}) = [q]$ and that

$$\max \ \zeta_{X_0,a}(\pi_a^{-1}(i,a)) = \min \ \zeta_{X_0,b}(\pi_b^{-1}(i,b))$$

for $W = \cdots ab \ldots$. Hence, ζ_X satisfies the lemma. □

Let us define analogously the sign corresponding to the horizontal part of $\mathrm{sgn}_Q(\zeta_X, I)$.

Construction 2.27 We work with the following alphabet

$$0_i, \ldots, (q_i - 1)_i$$

for $i = 1, \ldots, n$ and define the word

$$J(q_1, \ldots, q_n) = 0_1 \ldots (q_1 - 1)_1 \ldots 0_n \ldots (q_n - 1)_n$$

The second word $J_W(X)$ is the concatenation of two words $J_{0,W}(X)$ and $J_{1,W}(X)$ defined as follows

- $J_{0,W}(X)$ consists of all 0_k for k interposed in W, put in reverse $\downarrow$-order.
- $J_{1,W}(X)$ has in the $\zeta_{X,k}(i)$-th position i_k for $1 \le i \le q_k - 1$ for k interposed, and in the $\zeta_{I,k}(i)$ position i_k for $0 \le i \le q_k - 1$ for k not interposed. Note that we start from position 0.

Definition 2.28 For $X \in \mathrm{mNSOp}(q_1, \ldots, q_n; q)$ where we replace those $q_i = 0$ by 2, we define $\mathrm{sgn}_W(X)$ as the sign of the shuffle transforming $J(q_1, \ldots, q_n)$ to $J_W(X)$.

Example 2.29 Consider the words

$$W = 13121, \quad W' = 1231$$

and colorings

$$X = \ {}_i \diagdown \overset{3 \quad 2}{} \diagup {}_j \ , \qquad X' = \overset{\displaystyle \overset{2 \quad 3}{{}_1 \diagdown \ \diagup {}_2}}{\underset{1}{\left| \ {}_k \right.}} \ \overset{m}{}$$

for which we calculate the words $J_W(X)$ and $J_{W'}(X')$ and their corresponding signs. In the first case, we have

$$J_W(X) = 0_3 0_2 0_1 \ldots (i-1)_1 1_3 \ldots (q_3 - 1)_3 i_1 \ldots (j-1)_1 1_2 \ldots (q_2 - 1)_2 j_1 \ldots (q_1 - 1)_1$$

which corresponds to the sign $\mathrm{sgn}_W(X) = (-1)^{(q_2-1)(q_1-j)+(q_3-1)(q_1-i)+q_3q_2+q_3-1}$. For the second case, we calculate

$$J_{W'}(X') = 0_2 0_1 \ldots (k-1)_1 1_2 \ldots (q_2-1)_2 0_3 \ldots (q_3-1)_3 k_1 \ldots (q_1-1)_1$$

which corresponds to the sign $\mathrm{sgn}_{W'}(X') = (-1)^{(q_2+q_3)(q_1-k)+q_1}$. Note in particular that in $J_{W'}(X')$ the letter 0_3 is not taken to the front of the word as 3 is not interposed in W'.

Lemma 2.30 *Let $X \in \mathrm{Clr}(W, q_1, \ldots, q_n)$ and $X' \in \mathrm{Clr}(W', q'_1, \ldots, q'_m)$, and $W'' \in \mathrm{Ext}(W, W', 1)$, then $\mathrm{sgn}_W(X) \, \mathrm{sgn}_{W'}(X') = \mathrm{sgn}_{W''}(X \circ_1 X') \, \mathrm{sgn}_{W,W',1}(W'')$.*

Proof We can assume that all q_i and q'_j are not zero. We can decompose $\mathrm{sgn}_W(X)$ in three components

- Sign of the shuffle σ shuffling $J_{0,W}(X)$ to $0_{v_1} \ldots 0_{v_k}$ for $v_1 < \ldots < v_k$ the interposed vertices of X,
- Sign of the shuffle τ shuffling $J_{1,W}(X)$ to concatenation of $1_i \ldots (q_i - 1)_i$ for i interposed and $0_i \ldots (q_i - 1)_i$ for i not interposed. We call this latter sequence $J_W^{int}(q_1, \ldots, q_n)$.
- Sign of the shuffle ρ shuffling

$$0_{v_1} \ldots 0_{v_k} J_W^{int}(q_1, \ldots, q_n) \rightsquigarrow J(q_1, \ldots, q_n)$$

We add $'$ and $''$ to denote the correspondings shuffles for X' and $X'' := X \circ_1 X'$.

First we have that $(-1)^{\sigma''} = (-1)^{\sigma+\sigma'} \mathrm{sgn}_{W,W',1}(W'')$ per definition of $\mathrm{sgn}_{W,W',1}(W'')$. Further we clearly have $(-1)^{\tau''} = (-1)^{\tau+\tau'}$ by simply applying them one after the other and renaming using α and β^{-1}

$$J_{1,W''}(X) \rightsquigarrow J_{1,W'} J_W(q_2, \ldots, q_n) \rightsquigarrow J_{W'}(q'_1, \ldots, q'_m) J_W(q_2, \ldots, q_n)$$

as $\zeta_{X''} = \zeta_X \circ_1 \zeta_{X'}$. We also have that $(-1)^{\rho''} = (-1)^{\rho+\rho'}$ because the length of the sequence $J_{W'}^{int}(q'_1, \ldots, q'_m)$ is $q_1 - 1$ if 1 is interposed, and q_1 if 1 is not interposed. □

2.6.3 The morphism

Proposition 2.31 *We have a morphism of graded operads*

$$\bar{\phi} : \mathrm{F}_2\mathrm{S} \longrightarrow \mathrm{mNSOp}_{st} : W \longmapsto \left(\sum_{X \in \mathrm{Clr}(W, q_1, \ldots, q_n)} \mathrm{sgn}_W(X) X \right)_{q_1, \ldots, q_n}$$

Proof By definition of $\mathrm{sgn}_W(X)$ the above linear maps are equivariant. By Lemmas 2.24, 2.25 and 2.30 they define a morphism of graded operads. □

We make mNSOp_{st} into a dg-operad with the hochschild differential, then ϕ will be a morphism of dg-operads.

Definition 2.32 Define for every $q \in \mathbb{N}$ the element

$$D_q = \begin{array}{c} 1 \\ \big| \, 2 \\ m \end{array} + \sum_{i=1}^{q} (-1)^i \begin{array}{c} q \\ \big| \, i \\ 1 \end{array} + (-1)^{q+1} \begin{array}{c} m \\ \big| \, 1 \\ m \end{array} \quad \in \text{mNSOp}(q; q+1) \qquad (1)$$

which compile into an element of degree -1

$$D := (D_q)_{q \geq 0} \in \text{mNSOp}_{st}(1).$$

We consider the associated derivation

$$\partial_D(X) := D \circ_1 X - (-1)^{deg(X)} \sum_{i=1}^{n} X \circ_i D$$

for $X \in \text{mNSOp}_{st}(n)$.

Proposition 2.33 ∂_D *defines a differential making* mNSOp *into a dg-operad, for which holds*

$$\partial_D \left(\bar{\phi}(W) \right) = \bar{\phi} \left(\partial(W) \right)$$

Proof The first part follows directly if $D \circ_1 D = 0$ which is an easy computation (see [6, Prop. 2]).

In order to prove the second part we only need to show this for the generators of W, i.e. 12 and $121 \ldots 1k1$ for $k \geq 1$.

- For $q_1, q_2 \in \mathbb{N}$, it is easy to compute that

$$\partial_D \left(\bar{\phi}(12) \right) = \partial_D \left(\begin{array}{c} 1 \quad\quad 2 \\ \diagdown \quad \diagup \\ 1 \quad\quad 2 \\ m \end{array} \right) = 0$$

in $\text{mNSOp}(q_1, q_2; q_1 + q_2)$.
- For $q_1, \ldots, q_k \in \mathbb{N}$, we also have

$$\bar{\phi}(121 \ldots 1k1) = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq q_1} (-1)^{\epsilon} \begin{array}{c} 2 \quad \cdots \quad k \\ \diagdown \quad \big| \quad \diagup \\ i_1 \quad\quad i_{k-1} \\ 1 \end{array}$$

$$\text{where} \quad \epsilon = \sum_{j=2}^{k} (q_j - 1)(q_1 - i_j + \sum_{l > j} (q_l - 1))$$

and

$$\partial(121 \ldots 1k1) = -2131 \ldots 1 + \sum_{i=2}^{k-1} (-1)^i 1 \ldots 1i(i+1)1 \ldots 1 + (-1)^k 1 \ldots 1k$$

for which it is also a standard computation to see that $\partial_D(\bar{\phi}(121\ldots 1k1)) = \bar{\phi}(\partial(121\ldots 1k1))$ (see [6, Thm 3]). $\qquad\square$

Theorem 2.34 *We have a morphism of dg-operads*

$$\bar{\phi} : F_2S \longrightarrow \mathrm{mNSOp}_{st}$$

Proof This is the direct consequence of Propositions 2.31 and 2.33. $\qquad\square$

3 The Gerstenhaber–Schack complex for prestacks

Let $(\mathcal{A}, m, f, c)$ be a prestack over a small category $\mathcal{U}$ and let $\mathbf{C}_{GS}(\mathcal{A})$ be the associated Gerstenhaber–Schack complex as defined in [2] (see Sect. 3.1). In loc. cit., a homotopy equivalence $\mathbf{C}_{GS}(\mathcal{A}) \cong \mathbf{CC}(\mathcal{A}!)$ is constructed with the Hochschild complex $\mathbf{CC}(\mathcal{A}!)$ of the Grothendieck construction $\mathcal{A}!$ of $\mathcal{A}$. Through homotopy transfer, this allows to endow the GS-complex with an L_∞-structure. However, it is desirable to have a direct description available of this structure, without reference to transfer.

In the case of a *presheaf*, originally considered by Gerstenhaber and Schack, in [8], Hawkins introduces an operad Quilt $\subseteq F_2S \otimes_H$ Brace which he later extends to an operad mQuilt acting on the GS-complex. These operads are naturally endowed with L_∞-operations as desired. The action of Quilt on the GS-complex considered by Hawkins only involves the restriction functors f of the presheaf, the multiplication m being incorporated later on in mQuilt. Unfortunately, the way in which functoriality of f is built into these actions, does not allow for an extension to twisted presheaves or prestacks.

In our solution for the prestack case, we propose to use Quilt in a fundamentally different way in relation to the GS-complex, but still allowing us to make use of the naturally associated L_∞-structure. In this section we capture the higher structure of $\mathbf{C}_{GS}(\mathcal{A})$ by introducing the operad Patch $\subseteq \mathrm{mNSOp} \otimes_H \mathrm{NSOp}$ (see Sect. 3.3) over which the bicomplex $\mathbf{C}^{\bullet,\bullet}(\mathcal{A})$, of which $\mathbf{C}_{GS}(\mathcal{A})$ is the totalisation, is shown to be an algebra (see Theorem 3.24). Next, we construct a morphism Quilt $\longrightarrow$ Patch$_s$ (see Proposition 3.27) as a restriction of

$$\bar{\phi} \otimes_H \phi : F_2S \otimes_H \text{Brace} \longrightarrow \mathrm{mNSOp}_{st} \otimes_H \mathrm{NSOp}_s .$$

This morphism is such that the resulting composition

$$R : \text{Quilt} \longrightarrow \mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$$

incorporates the multiplications m and the restrictions f. Note that in Hawkins' approach to the presheaf case, the initial action of Quilt on $\mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$ only incorporates the restrictions. As far as the structure of both approaches goes, the auxiliary operad Patch we use is the counterpart of the operad ColorQuilt from [8, Def. 4.6].

In Sect. 4, we will further extend the action R in order to incorporate the twists.

3.1 The GS complex

In this section, we recall the notions of prestack and its associated Gerstenhaber–Schack complex, thus fixing terminology and notations. We use the same terminology as in [2,12].

A prestack is a pseudofunctor taking values in k-linear categories. Let $\mathcal{U}$ be a small category.

Definition 3.1 A prestack $\mathcal{A} = (A, m, f, c)$ over $\mathcal{U}$ consists of the following data:

- For every object $U \in \mathcal{U}$, a k-linear category $(\mathcal{A}(U), m^U, 1^U)$ where m^U is the composition of morphisms in $\mathcal{A}(U)$ and 1^U encodes the identity morphisms of $\mathcal{A}(U)$.
- For every morphism $u : V \longrightarrow U$ in $\mathcal{U}$, a k-linear functor $f^u = u^* : \mathcal{A}(U) \longrightarrow \mathcal{A}(V)$. For $u = 1_U$ the identity morphism of U in $\mathcal{U}$, we require that $(1_U)^* = 1_{\mathcal{A}(U)}$.
- For every couple of morphisms $v : W \longrightarrow V, u : V \longrightarrow U$ in $\mathcal{U}$, a natural isomorphism

$$c^{u,v} : v^* u^* \longrightarrow (uv)^*.$$

For $u = 1$ or $v = 1$, we require that $c^{u,v} = 1$. Moreover, the natural isomorphisms have to satisfy the following coherence condition for every triple $w : T \longrightarrow W$, $v : W \longrightarrow V$ and $u : V \longrightarrow U$:

$$c^{u,vw}(c^{v,w} \circ u^*) = c^{uv,w}(w^* \circ c^{u,v}).$$

Given such a prestack $\mathcal{A}$, we have an associated Gerstenhaber–Schack complex $\mathbf{C}_{GS}(\mathcal{A})$. In [2] this is defined as the totalisation of a bicomplex $\mathbf{C}^{\bullet,\bullet}(\mathcal{A})$. We first review some notations.

Notations. Let $\sigma = (U_0 \overset{u_1}{\to} U_1 \to \ldots \overset{u_p}{\to} U_p)$ be a p-simplex in the category $\mathcal{U}$, then we have two functors $\mathcal{A}(U_p) \longrightarrow \mathcal{A}(U_0)$, namely

$$\sigma^{\#} := u_1^* \circ \cdots \circ u_p^* \quad \text{and} \quad \sigma^* := (u_p \circ \cdots \circ u_1)^*$$

For each $1 \leq k \leq p - 1$, denote by $L_k(\sigma)$ and $R_k(\sigma)$ the following simplices

$$L_k(\sigma) = (U_0 \overset{u_1}{\to} U_1 \to \cdots \overset{u_k}{\to} U_k)$$
$$R_k(\sigma) = (U_k \overset{u_{k+1}}{\to} U_{k+1} \to \cdots \overset{u_p}{\to} U_p)$$

and we consider the following natural isomorphisms

$$c^{\sigma,k} = c^{u_k \ldots u_1, u_p \ldots u_{k+1}} : (L_k\sigma)^*(R_k(\sigma))^* \longrightarrow \sigma^*$$
$$\epsilon^{\sigma,k} = u_1^* \ldots u_{k-1}^* c^{u_k, u_{k+1}} u_{k+2}^* \ldots u_p^* : \sigma^{\#} \longrightarrow u_1^* \ldots (u_{k+1}u_k)^* \ldots u_p^*$$

We write $c^{\sigma,k,A} = c^{\sigma,k}(A)$ and $\epsilon^{\sigma,k,A} = \epsilon^{\sigma,k}(A)$ for $A \in \mathcal{A}(U_p)$.

We also define a set $P(\sigma)$ of *formal paths from* $\sigma^{\#}$ *to* σ^{*} inductively. A formal path is finite sequence of couples (τ, i) consisting of a simplex σ and a natural number i. We set

$$P(u_1, u_2) := \{((u_1, u_2), 1)\}$$

and

$$P(\sigma) := \{(r_1, \ldots, r_{p-2}, (\sigma, i)) : 1 \le i \le p - 1 \text{ and } (r_1, \ldots, r_{p-2}) \in P(\partial_i \sigma)\}$$

where ∂_i denotes the ith face-operator of the nerve $N_p(\mathcal{U})$. Given such a formal path $r = (r_1, \ldots, r_{p-1})$ we define its sign

$$(-1)^r = \prod_{i=1}^{p-1} (-1)^{r_i} \text{ where } (-1)^{(\sigma,i)} = (-1)^i.$$

By interpreting the data (σ, i) as the natural isomorphism $\epsilon^{\sigma,i}$, every formal path $r \in P(\sigma)$ induces a sequence of natural isomorphisms $\bar{r} \in N_{p-1}(\text{Fun}(\mathcal{A}(U_p), \mathcal{A}(U_0)))$. Note that $\epsilon^{(u_1,u_2),1} = c^{u_1,u_2}$ and its associated sign is -1.

Let $\mathbb{S}_{t,p-1}$ denote the set of $(t, p - 1)$-shuffles, then given a formal path $r \in P(\sigma)$, a shuffle $\beta \in \mathbb{S}_{t,p-1}$ and a tuple $a = (A_0 \xleftarrow{a_1} A_1 \leftarrow \cdots \xleftarrow{a_t} A_t) \in N_t(\mathcal{A}(U_p))$ let $\beta(a, r) \in N_{p-1+t}(\mathcal{A}(U_0))$ be its shuffle product with respect to evaluation of functors as defined in [2, Ex. 3.2, Ex. 3.4].

Here, we give a more explicit definition of $\beta(a, r)$: first we construct inductively a sequence $(b_1, \ldots, b_{t+p-1})$ which *formally* represents a sequence of morphisms in $A(U_0)$. Every b_i is either of the form (τ, a_i, A_{i-1}) or (r_i, A_j) for τ a simplex, a_i and A_j respectively a morphism and an object occurring in a, and r_i an element of the formal path r. Define

$$b_{t+p-1} = \begin{cases} (\sigma, a_t, A_{t-1}) & \text{if } \beta(t) = t + p - 1 \\ (r_{p-1}, A_t) & \text{if } \beta(t + p - 1) = t + p - 1 \end{cases}$$

then for $1 < i \le t + p - 1$, we have two cases: if $b_{i+1} = (\tau, a_j, A_{j-1})$ for some j, then define

$$b_i = \begin{cases} (\tau, a_{j-1}, A_{j-2}) & \text{if } \beta(j - 1) = i \text{ for } j \le t \\ (r_k, A_{j-1}) & \text{if } \beta(t + k) = i \end{cases}$$

If $b_{i+1} = (r_k, A_j)$ for $r_k = (\tau, l)$, then define

$$b_i = \begin{cases} (\partial_l \tau, a_j, A_{j-1}) & \text{if } \beta(j) = i \text{ for } j < t \\ (r_{k-1}, A_j) & \text{if } \beta(t + k - 1) = i \end{cases}$$

Finally, we define $\beta(a, r)$ as the realization $\overline{b} = (B_0 \overset{\overline{b_1}}{\leftarrow} B_1 \leftarrow \cdots \overset{\overline{b_t}}{\leftarrow} B_{p-1+t})$ of b where $\overline{b_i} = \tau^\# a_j$ if $b_i = (\tau, a_j, A_j)$ and $\overline{b_i} = \overline{r_k}(A_j)$ if $b_i = (r_k, A_j)$.

Definition 3.2 Let $p, q \geq 0$, then define

$$\mathbf{C}^{p,q}(\mathcal{A}) = \prod_{\sigma \in N_p(\mathcal{U})} \prod_{A \in \mathcal{A}(U_p)^{q+1}} \mathrm{Hom}(\bigotimes_{i=1}^{q} \mathcal{A}(U_p)(A_i, A_{i-1}), \mathcal{A}(U_0)(\sigma^\# A_q, \sigma^* A_0))$$

and set

$$\mathbf{C}^n_{GS}(\mathcal{A}) = \bigoplus_{p+q=n} \mathbf{C}^{p,q}(\mathcal{A})$$

The differential d on the GS-complex is defined for $\theta \in \mathbf{C}^{p,q}(\mathcal{A})$ as

$$d(\theta) = \sum_{j=0}^{q+1} d_j(\theta)$$

where $d_j : \mathbf{C}^{p,q}(\mathcal{A}) \longrightarrow \mathbf{C}^{p+j,q+1-j}(\mathcal{A})$ is defined as

-

$$d_0(\theta)^\sigma(A)(a) = m^{U_0}(\sigma^*(a_1), \theta^\sigma(A_1, \ldots, A_{q+1})(a_2, \ldots, a_{q+1}))$$
$$+ \sum_{i=1}^{q}(-1)^i \theta^\sigma(A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{q+1})$$
$$(a_1, \ldots, m^{U_p}(a_i, a_{i+1}), \ldots, a_1)$$
$$+ (-1)^{q+1} m^{U_0}(\theta^\sigma(A_0, \ldots, A_q)(a_1, \ldots, a_q), \sigma^\#(a_{q+1}))$$

-

$$d_1(\theta)^\sigma(A)(a) = (-1)^{p+q+1} m^{U_0}(c^{\sigma,1,A_0}, u_1^*(\theta^{\partial_0\sigma}(A)(a)))$$
$$+ \sum_{i=1}^{p}(-1)^{p+q+1+i} m^{U_0}(\theta^{\partial_i\sigma}(A)(a), \epsilon^{\sigma,i,A_q})$$
$$+ (-1)^q m^{U_0}(c^{\sigma,p,A_0}, \theta^{\partial_{p+1}\sigma}(A)(u_{p+1}^* a_1, \ldots, u_{p+1}^* a_q))$$

-

$$d_j(\theta)^\sigma(A)(a) = \sum_{\substack{r \in P(R_p(\sigma)) \\ \beta \in \mathbb{S}_{q-j+1,j-1}}} (-1)^{r+\beta+q-j+1} m^{U_{p+j}}(c^{\sigma,p,A_0}, \theta^{L_p(\sigma)}(B)(\beta(a, r)))$$

for $\sigma = (u_1, \ldots, u_{p+j}) \in N_{p+j}(\mathcal{U})(U_0, U_{p+j})$, $a = (a_1, \ldots, a_{q-j+1})$ where $a_i \in \mathcal{A}(U_{p+j})(A_i, A_{i-1})$ and such that B is the sequence of objects underlying $\beta(a, r)$.

We will also be interested in the subcomplex $\overline{\mathbf{C}}_{GS}(\mathcal{A}) \subseteq \mathbf{C}_{GS}(\mathcal{A})$ of normalized and reduced cochains which is shown to be quasi-isomorphic to the GS complex (see [2, Prop. 3.16]). A simplex $\sigma = (u_1, \ldots, u_p)$ is *reduced* if $u_i = 1_{U_i}$ for some $1 \le i \le p$. A cochain $\theta = (\theta^\sigma(A))_{\sigma, A} \in \mathbf{C}_{GS}(\mathcal{A})$ is *reduced* if $\theta^\sigma(A) = 0$ for every reduced simplex σ. A simplex $a = (a_1, \ldots, a_q)$ in $\mathcal{A}(U)$ is *normal* if $a_i = 1^U$ for some $1 \le i \le q$. A cochain θ is *normalized* if $\theta^\sigma(A)(a) = 0$ for every normal simplex a in $\mathcal{A}(U_p)$. We come back to this in Sect. 4.

Elements of the GS complex have a neat geometric interpretation as rectangles: for $\theta \in \mathbf{C}^{p,q}(\mathcal{A})$ and the data (σ, A, a) from above, we can represent $\theta^\sigma(A)(a)$ as the rectangle of data

Similarly, we can draw different components of the differential d using rectangles, providing more insight in its rather technical definition. For the hochschild component d_0 we have

$$+ (-1)^{q+1} \quad \sigma^*$$

$$
\begin{array}{c}
a_1 \quad a_2 \quad\quad a_{q+1} \\
\longleftarrow \; \longleftarrow\!-\!-\!-\; \longleftarrow \quad \Big| u_p^* \\
\theta^\sigma(A) \quad\quad \sigma^\# \\
\Big\downarrow \quad\quad \Big| u_1^* \\
\longleftarrow \quad\quad \longleftarrow \\
m^{U_0}
\end{array}
$$

The first component d_1 can similarly be drawn as

$$d_1(\theta)^\sigma(A) = \quad (-1)^{p+q+1} \sigma^* \quad c^{\sigma,1,A_0}$$

$$
\begin{array}{c}
A_0 \quad a_1 \quad\quad a_q \\
\longleftarrow \; \longleftarrow\!-\!-\!-\; \longleftarrow \quad \Big| u_{p+1}^* \\
\theta^{\partial_0\sigma}(A) \\
\Big| u_2^* \\
u_1^* \quad \Big| u_1^* \\
\longleftarrow \\
m^{U_0}
\end{array}
$$

$$+ \quad \sum_{i=1}^{q} (-1)^{p+q+i+1} \quad \sigma^*$$

$$
\begin{array}{c}
a_1 \quad\quad a_q \quad\quad A_q \\
\longleftarrow \;-\!-\!-\; \longleftarrow \quad\quad\quad \Big| u_{p+1}^* \\
R_{i+1}\sigma^\# \\
\theta^{\partial_p\sigma}(A) \quad c^{u_i,u_{i+1}}(A_q) \quad \Big| u_{i+1}^* \\
\Big| u_i^* \\
L_{i-1}\sigma^\# \quad \Big| u_1^* \\
\longleftarrow \\
m^{U_0}
\end{array}
$$

$$+ (-1)^q \quad \sigma^* \quad c^{\sigma,p,A_0}$$

$$
\begin{array}{c}
a_1 \quad\quad a_q \\
\Big| u_{p+1}^* \longleftarrow\!-\!-\!-\; \Big| u_{p+1}^* \Big| u_{p+1}^* \\
\Big| u_p^* \\
\theta^{\partial_0\sigma}(A) \quad \Big| u_1^* \\
\longleftarrow \\
m^{U_0}
\end{array}
$$

Finally, we will draw d_2 as an example from which it is easy to deduce the higher components d_j for $j > 2$. Namely, we have

$$d_2(\theta)^\sigma(A) = \sum_{i=1}^{q} (-1)^{r+\beta+q-2+1}$$

for shuffle $\beta(q) = i$, $\beta(s) = s$ for $s < i$ and $\beta(s) = s+1$ for $s \geq i$, and formal path $r = ((u_p, u_{p+1}), 1)$. Note in particular that we can draw $\beta(a, r)$ as follows

where $b_s = a_{\beta^{-1}(s)}$ for $s \neq i$, and $b_i = c^{u_{p+1}, u_{p+2}}(A_{i-1})$.

We will use this rectangular interpretation as a guide in the next sections.

3.2 Endomorphism operad of a prestack

Although the GS-complex does not have partial compositions $\circ_i$, its elements $\theta = (\theta^\sigma(A))_{(\sigma, A)}$ consist of parts that lie in the endomorphism operad **End**$(\mathcal{A})$.

Definition 3.3 Let $\mathrm{Ob}(\mathcal{U}, \mathcal{A})$ be the set consisting of the triples (U, A, A') for $U \in \mathcal{U}$ and $A, A' \in \mathcal{A}(U)$, then we define the $\mathrm{Ob}(\mathcal{U}, \mathcal{A})$-colored operad **End**$(\mathcal{A})$ as

$$\mathbf{End}(\mathcal{A})((U_1, A_1, A_1'), \ldots, (U_n, A_n, A_n'); (U, A, A'))$$

$$:= \mathrm{Hom}(\bigotimes_{i=1}^{n} \mathcal{A}(U_i)(A_i, A_i'), \mathcal{A}(U)(A, A'))$$

with partial compositions defined by composition of linear maps.

Remark 3.4 Note that $\theta^\sigma(A) \in \mathbf{End}(\mathcal{A})((U_p, A_1, A_0), \ldots, (U_p, A_q, A_{q-1}); (U_0, \sigma^\# A_q, \sigma^* A_0))$.

3.3 The operad Patch

In this section we define an $\mathbb{N} \times \mathbb{N}$-colored operad Patch $\subseteq$ mNSOp $\times$ NSOp. Its elements encode concrete (planar) patchworks of rectangles of size (p_i, q_i) to form a rectangle of size (p, q).

Definition 3.5 Let Patch$((q_1, p_1), \ldots, (q_n, p_n); (q, p))$ consists of the elements $(X, J) \in$ mNSOp$(q_1, \ldots, q_n; q) \times$ NSOp$(p_1, \ldots, p_n; p)$ such that

(1) $a <_J b \Longrightarrow a \lhd_X b$
(2) $a <_X b \Longrightarrow b \lhd_J a$

Remark 3.6 Note that in order for Patch not to be empty, we need to allow a multiplication in one of its coordinates which is not present in the other coordinate.

This has a neat geometric interpretation as well: a (p, q)-rectangle has p inputs on the right-hand side, q inputs on top and a single output on respectively the bottom and the left-hand side
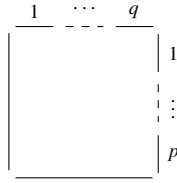
We then interpret a patchwork (X, J) as an ordering of these rectangles: the first coordinate X represent the vertical ordering (from top to bottom) and the second coordinate J the horizontal ordering (from right to left). The multiplications m form a single exception: they appear only vertically, thus we draw them as *flat* rectangles, that is, having no horizontal input and output. From this perspective, the conditions impose planarity on the patchwork such that we have

$$\text{below} <_X \text{above} \quad \text{above} \lhd_J \text{below}$$
$$\text{left} \lhd_X \text{right} \quad \quad \text{left} <_J \text{right}$$

Note that when we write down a patchwork using rectangles, possible 'open spaces' can appear. Moreover, it is possible that multiple rectangles are vertically the 'lowest' elements due to the insertion of multiplication elements m. However, horizontally there can only appear a single most left rectangle which is (horizontally) connected to all other rectangles. We give an example.

Example 3.7 The following pair determines an element in $\mathrm{Patch}((3, 5), (3, 2), (2, 1),$ $(0, 2); (6, 7))$

$$
(X, J) = \begin{pmatrix}
\begin{array}{c}
\overset{3}{\quad}\ \ \overset{4}{\quad} \\
{}^{1}\diagdown\quad\diagup{}^{3} \\
1\qquad 2 \\
{}^{1}\diagdown\quad\diagup{}^{2} \\
m
\end{array}
,\quad
\begin{array}{c}
2\ \ 3\ \overset{1}{\underline{\quad}}\ 4 \\
1\ \diagdown \\
\quad 4\ \diagup\ 2
\end{array}
\end{pmatrix}
$$

which we can draw as the following patchwork of rectangles
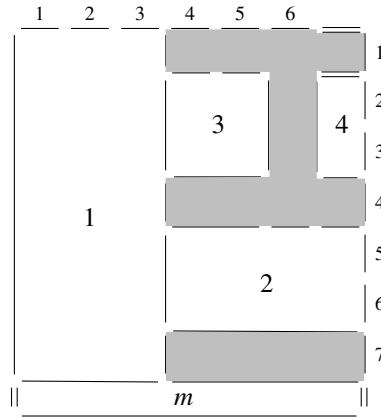
where the grey areas denote the open spaces.

Lemma 3.8 *Patch is a suboperad of* $\mathrm{mNSOp} \otimes_H \mathrm{NSOp}$.

Proof Let $(X, J) \in \mathrm{Patch}((q_1, p_1), \dots, (q_n, p_n); (q, p))$ and $(X', J') \in \mathrm{Patch}((q_1', p_1'), \dots, (q_m', p_m'); (q_i, p_i))$ and we set $X'' := X \circ_i X'$ and $J'' := J \circ_i J'$. Let (α, β) be the extension of n by m at i, then for $a, b \in \langle m \rangle$ we compute

$$
\alpha(a) <_{X''} \alpha(b) \iff a <_{X'} b \implies b \lhd_{J'} a \iff \alpha(b) \lhd_{J''} \alpha(a)
$$

and for $c, d \notin \mathrm{Im}(\alpha)$ we compute

$$
c <_{X''} d \iff \beta c <_X \beta d \implies \beta d \lhd_J \beta c \iff d \lhd_{J''} \beta c
$$

For $c \notin \mathrm{Im}(\alpha)$ and $b \in \langle m \rangle$, we have

$$
c <_{X''} \alpha(b) \implies \beta c <_X i \implies i = \beta\alpha(b) \lhd_J \beta\alpha(b) \implies \alpha(b) \lhd_{J''} c
$$

and the same reasoning shows $\alpha(b) <_{X''} c \implies c \lhd_{J''} \alpha(b)$. Completely symmetrically, this also shows that $c <_{J''} d \implies c \lhd_{X''} d$ for $c, d \in \langle n + m - 1 \rangle$. $\qquad\square$

We again compile the colored operad Patch to obtain a graded non-colored operad

$$\text{Patch}_s(n) \subseteq \prod_{\substack{q_1,\ldots,q_n,q \\ p_1,\ldots,p_n}} \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p))$$

where an element $x \in \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p))$ is graded as

$$|x| = \sum_{i=1}^{n}(q_i + p_i - 1) - (q + p - 1)$$

and $\text{Patch}_s(n)$ is generated as a k-module by the sequences of constant degree. Its composition is derived from Patch where it is set to 0 when the colors do not match. Note in particular that the $\mathbb{S}_n$-action on $\text{Patch}(n)$ is affected by this grading: permuting two vertices i and j introduces a sign $(-1)^{(q_i+p_i-1)(q_j+p_j-1)}$.

Lemma 3.9 Patch_s *is a dg-suboperad of* $(\text{mNSOp}_{st} \otimes_H \text{NSOp}_s, (\partial_D, Id))$.

Proof It suffices to see that the elements $(D_q, 1) \in \text{Patch}((q, p); (q+1, p))$ for every $p, q \in \mathbb{N}$. □

3.4 The morphism Patch$_s$ $\longrightarrow$ End(sC$_{GS}(\mathcal{A})$)

In this section we make the GS-complex $\mathbf{C}_{GS}(\mathcal{A})$ of a prestack $\mathcal{A}$ into a Patch$_s$-algebra. We do so by making its underlying bicomplex $\mathbf{C}^{\bullet,\bullet}(\mathcal{A})$ into a Patch-algebra. We first fix some notations.

Definition 3.10 Let $\sigma = (U_0 \overset{u_1}{\to} U_1 \to \ldots \overset{u_p}{\to} U_p)$ be a p-simplex in the category $\mathcal{U}$ and $\zeta : [p'] \longrightarrow [p]$ a non-decreasing map (or equivalently a non-decreasing sequence), then let $\overline{\zeta}$ be the reflection of ζ, that is,

$$\overline{\zeta}(t) := p - \zeta(p' - t)$$

and define

$$\zeta(\sigma) := N_\bullet(\mathcal{U})(\overline{\zeta})(\sigma)$$

a p'-subsimplex of σ, where $N_\bullet(\mathcal{U})$ denotes the nerve construction on $\mathcal{U}$.

Remark 3.11 Note that we apply the reflection as we count the horizontal inputs of a patchwork from top to bottom (see Example 3.7) instead of bottom to top (see further, Example 3.22).

Given a patchwork $(X, J) \in \text{Patch}$, we now determine which simplices we need to fill in the 'open spaces' in between the rectangles. We first sketch the idea.

Given a simplex σ in $\mathcal{U}$ and a vertex a, we want to determine two sorts of simplices: for every vertical input $i = I(a, b)$ for some vertex b, we want to determine a simplex

$\sigma(a, b)$ that we place between them. For the other vertical inputs $1 \leq i \leq q_a$, we determine a simplex $\sigma_a(i)$ to place on top of a at input i. To do so, we determine the set of left-most vertices which do not "surpass" the ith input and that lie higher than vertex a. In the drawing below, this set consists of the vertices e_1, e_2 and e_3. To calculate $\sigma(a, b)$, we restrict this set to those vertices that still lie below vertex b, in this case, the vertices e_2 and e_3.
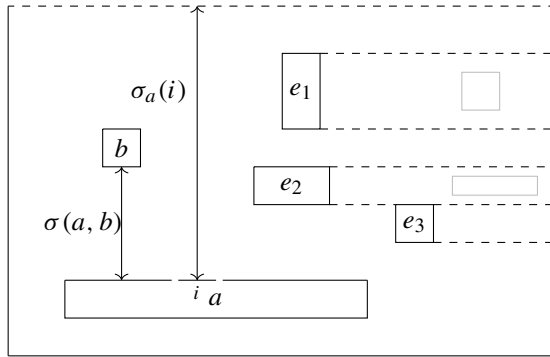
We observe that each element of the GS complex composes in $\mathcal{U}$ the subsimplex corresponding to its horizontal inputs. Hence, using our auxiliary set, we contract the corresponding subsimplices and obtain $\sigma(a, b)$ and $\sigma_a(i)$.

Note that we have not yet treated the multiplications m. In order to do so, we have to add the following complexity. Let $X = [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m]$ where I is an indexed tree with $n + k$ vertices, then we call a vertex a of I *non-plugged* in X if in X it is not inserted by a multiplication element m. We continue with the above chosen representation of X where a non-plugged is equivalent to stating $a \leq n$.

Definition 3.12 We define a function $\downarrow : \langle n + k \rangle \longrightarrow [n]$ on the vertices of I which associates to every vertex a the closest non-plugged vertex in X under or equal to a, or 0 if no such vertex exists. Concretely,

$$\downarrow a := \max_{<_I}\{y \in \langle n \rangle : y \leq_I a\} \text{ and } \downarrow a = 0 \text{ if the set is empty}$$

We also set $a \lhd_J 0$ for every vertex a and define

$$\zeta_{J,a} := \zeta_{J,\downarrow a} \text{ and } \zeta_{J,0} := p$$

where p is the total number of inputs of J.

Remark 3.13 This is clearly independent of the representative I of X. Moreover, $\downarrow$ is for the given representative I the identity on $\langle n \rangle$.

Next, we determine the auxiliary set.

Definition 3.14 Consider a vertex a of I and let $b_1 \lhd_I \ldots \lhd_I b_t$ be the children of a in I lying in $\langle n \rangle$ with $i_s := I(a, b_s)$. We then define

$$
L_a(i) = \begin{cases} \{e \in \langle n \rangle : e \not\lhd_I \downarrow a \text{ and } e \lhd_J \downarrow a\} & i < i_1 \\ \{e \in \langle n \rangle : b_s \lhd_I e \text{ and } e \lhd_J \downarrow a\} & i_s \le i < i_{s+1} \\ \{e \in \langle n \rangle : b_t \lhd_I e \text{ and } e \lhd_J \downarrow a\} & i_t \le i \end{cases}
$$

for $i \in [q_a]$, and

$$
L(a, b_s) := \{e \in \langle n \rangle : b_s \lhd_I e \text{ and } b_s \lhd_J e \lhd_J \downarrow a\}
$$

and let $\min L_a(i)$ and $\min L(a, b_s)$ be respectively the set of $<_J$-minimal elements of $L_a(i)$ or $L(a, b_s)$.

Remark 3.15 Remark that $L_a(i)$, $L(a, b_s) \subseteq \langle n \rangle$ and thus that it contains only vertices which are not plugged by m. By default, we will set the subsimplex underneath the plugged children of a as empty (see Definition 3.18).

Remark 3.16 Note that the condition $e \not\lhd_I \downarrow a$ appearing in the first case becomes superfluous in the others.******

Definition 3.17 Let a be a vertex of I and $\min L_a(i) = \{e_1 \lhd_J \ldots \lhd_J e_l\}$, then we have the sequence of inequalities

$$
0 \le \zeta_{J,e_1}(0) \le \zeta_{J,e_1}(p_{e_1}) \le \cdots \le \zeta_{J,e_l}(p_{e_l}) \le \zeta_{J,a}(0)
$$

and thus the non-decreasing sequence

$$
(0, 1, \ldots, \zeta_{J,e_1}(0), \zeta_{J,e_1}(p_{e_1}), \ldots, \zeta_{J,e_l}(0), \zeta_{J,e_l}(p_{e_l}), \ldots, \zeta_{J,a}(0))
$$

from which we delete $\zeta_{J,e_i}(p_{e_i})$ if $\zeta_{J,e_i}(p_{e_i}) = \zeta_{J,e_{i+1}}(0)$ or $\zeta_{J,e_l}(p_{e_l}) = \zeta_{J,a}(0)$. This defines a subsimplex $\sigma_a(i)$ of σ by Definition 3.10.

Definition 3.18 Let a, b be vertices of I such that b is a child of a, and $\min L(a, b) = \{e_1 \lhd_J \ldots \lhd_J e_l\}$, then we have the sequence of inequalities

- If $i = I(a, b)$ for some vertex $b \in \langle n \rangle$ (non-plugged)

$$
\zeta_{J,b}(p_b) \le \zeta_{J,e_1}(0) \le \zeta_{J,e_1}(p_{e_1}) \le \cdots \le \zeta_{J,e_l}(p_{e_l}) \le \zeta_{J,a}(0)
$$

- If $i = I(a, b)$ for some vertex $b > n$ (plugged)

$$
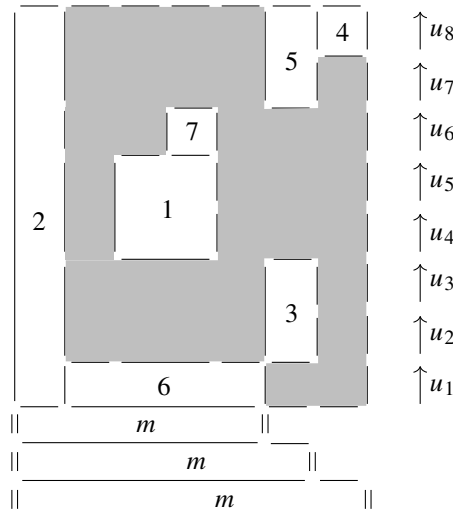\zeta_{J,b}(p_b) = \zeta_{J,a}(0)
$$

and thus the non-decreasing sequence

$$
(\zeta_{J,b}(p_b), \ldots, \zeta_{J,e_1}(0), \zeta_{J,e_1}(p_{e_1}), \ldots, \zeta_{J,e_l}(0), \zeta_{J,e_l}(p_{e_l}), \ldots, \zeta_{J,a}(0))
$$

from which we delete $\zeta_{J,e_i}(p_{e_i})$ if $\zeta_{J,e_i}(p_{e_i}) = \zeta_{J,e_{i+1}}(0)$ or $\zeta_{J,e_l}(p_{e_l}) = \zeta_{J,a}(0)$. We also delete $\zeta_{J,b}(p_b)$ if it equals $\zeta_{J,e_1}(0)$. This defines a subsimplex $\sigma(a, b)$ of σ by Definition 3.10.

We consider an example.

Example 3.19 Given the simplex $\sigma = (u_1, \ldots, u_8)$ and the following patchwork of rectangles

we analyse the case for rectangle 6: we have

$$L_6(0) = L_6(1) = \{1, 3, 4, 5, 7\} \text{ and } \min L_6(0) = \min L_6(1) = \{1, 3, 5, 7\},$$
$$L_6(2) = L_6(3) = \{3, 4, 5\} \text{ and } \min L_6(2) = \min L_6(3) = \{3, 5\}$$
$$L(6, 1) = \{3\} \text{ and } \min L(6, 1) = \{3\}$$

and thus

$$\sigma_6(0) = \sigma_6(1) = (u_3u_2, u_5u_4, u_6, u_8u_7),$$
$$\sigma_6(2) = \sigma_6(3) = (u_3u_2, u_4, u_5, u_6, u_8u_7),$$
$$\sigma(6, 1) = (u_3u_2).$$

Now, we can assemble for every element of Patch a concrete patchwork of elements of $\mathbf{C}_{GS}(\mathcal{A})$ where the first coordinate determines a vertical patching using the operadic structure and the second component determines the horizontal patching to fill in and align the corresponding simplices.

Construction 3.20 Given $(X, J) \in \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p))$ and $\theta_i \in \mathbf{C}^{p_i, q_i}(\mathcal{A})$, then we set $\theta_s = m \in \mathbf{C}^{0,2}(\mathcal{A})$ for $s = n + 1, \ldots, n + k$.

Let σ be a p-simplex in $\mathcal{U}$ and $A = (A_0, \ldots, A_q)$ $(q + 1)$-tuple of objects in $\mathcal{A}(U_p)$, then we define for every vertex a in I

$$\Theta_a := \theta_a^{\zeta_{J,a}(\sigma)} \left(\sigma_a (0)^{\#} A_{\zeta_{I,a}(0)}, \ldots, \sigma_a (q_a)^{\#} A_{\zeta_{I,a}(q_a)} \right)$$

and for every $i \in \langle q_a \rangle$ we make the compositions

- If $i = I(a, b)$ for some vertex b,

$$\Theta_a \circ_i \left(\sigma(a, b)^{\#} \circ_1 \Theta_b \right)$$

- Otherwise,

$$\Theta_a \circ_i \sigma_a (i)^{\#} \left(A_{\zeta_{I,a}(i-1)}, A_{\zeta_{I,a}(i)} \right)$$

All these compositions together define

$$\mathcal{L}(X, J)(\theta_1, \ldots, \theta_n)^{\sigma}(A) \in \mathrm{Hom}\left(\bigotimes_{i=1}^{q} \mathcal{A}(U_p)(A_i, A_{i-1}) ; \mathcal{A}(U_0)\left(\sigma^{\#} A_q, \sigma^* A_0 \right) \right)$$

Lemma 3.21 *Construction 3.20 is independent of the representative I of X.*

Proof It suffices to verify the relation on the formal multiplication elements m in mNSOp. This follows directly from the associativity of the local composition m^U of the category $\mathcal{A}(U)$ for every $U \in \mathcal{U}$. □

Let us work out an example.

Example 3.22 Consider the patching (X, J) from Example 3.7. Let $\theta_1 \in \mathbf{C}^{5,3}(\mathcal{A})$, $\theta_2 \in \mathbf{C}^{2,3}(\mathcal{A})$, $\theta_3 \in \mathbf{C}^{1,2}(\mathcal{A})$ and $\theta_4 \in \mathbf{C}^{2,0}(\mathcal{A})$, then we compute $\mathcal{L}(X, J)(\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbf{C}^{7,6}(\mathcal{A})$. Given the simplex $(u_1, \ldots, u_7) \in N_7(\mathcal{U})(U_0, U_p)$ and the objects $(A_0, \ldots, A_6) \in \mathcal{A}(U_p)$, we first compute

$$\Theta_1 = \theta_1^{(u_1, u_3 u_2, u_4, u_6 u_5, u_7)}(A_0, A_1, A_2, A_3)$$
$$\Theta_2 = \theta_2^{(u_2, u_3)}(u_4^*(u_6 u_5)^* u_7^* A_3, u_4^*(u_6 u_5)^* u_7^* A_5, u_4^*(u_6 u_5)^* u_7^* A_6, u_4^* u_5^* u_6^* u_7^* A_6)$$
$$\Theta_3 = \theta_3^{(u_6 u_5)}(u_7^* A_3, u_7^* A_4, u_7^* A_5)$$
$$\Theta_4 = \theta_4^{(u_5, u_6)}(u_7^* A_6)$$

Then, given $(a_1, \ldots, a_6)$ where $a_i \in \mathcal{A}(U_p)(A_i, A_{i-1})$, we finally compute

$$m^{U_0}(\Theta_1(a_1, a_2, a_3), u_1^* \Theta_2(u_4^* \Theta_3(u_7^*(a_4), u_7^*(a_5)), u_4^*(u_6 u_5)^* u_7^*(a_6), u_4^* \Theta_4)$$

which we can draw as follows
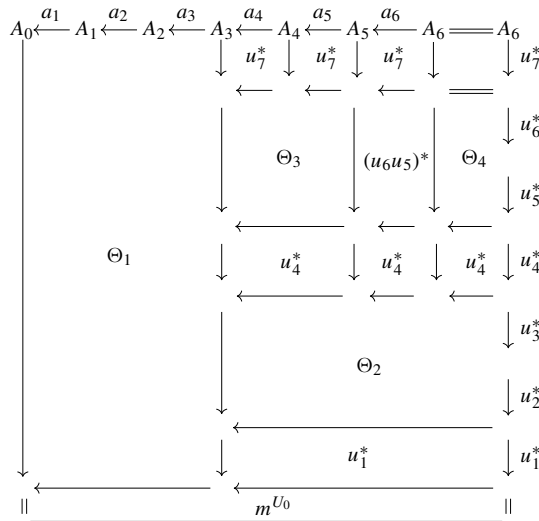
Proposition 3.23

$$\mathcal{L}(X, J) \circ_a \mathcal{L}(X', J') = \mathcal{L}(X \circ_a X', J \circ_a J')$$

Proof We can assume without loss of generality that $a = n$ as $\mathcal{L}$ is clearly equivariant.

Let $(X'', J'') := (X, J) \circ_n (X', J')$, then we add $'$ or $''$ to denote the notions associated to (X', J') or (X'', J''). Let I and I' be the underlying trees representing X and X' having respectively $n + k$ and $m + k'$ vertices, then let (α, β) be the extension of $n + k$ by $m + k'$ at n. Let $(\overline{\alpha}, \overline{\beta})$ be the extension of n by m at n.

We compute

$$\mathcal{L}(X, J)(\theta_1, \ldots, \theta_{n-1}, \mathcal{L}(X', J')(\theta_n, \ldots, \theta_{n+m-1}))^{\sigma}(A) \tag{2}$$

and show that it equals

$$\mathcal{L}(X'', J'')(\theta_1, \ldots, \theta_{n+m-1})^{\sigma}(A) \tag{3}$$

for $\sigma \in N_p(\mathcal{U})$ and $A = (A_0, \ldots, A_q)$ objects in $\mathcal{A}(U_p)$.

It is clear that per construction the blocks involved are composed according to $X'' = X \circ_n X'$. Hence it suffices to verify that they correspond to the blocks Θ''_x in $\mathcal{L}(X'', J'')$ and that the functors used to fill in the open spaces, agree.

First, for x a non plugged vertex of I'' in X'', it is clear that Θ''_x is either $\Theta_{\beta(x)}$, or $\Theta'_{\alpha^{-1}(x)}$ evaluated at $\sigma' = \zeta_{J,n}(\sigma)$. Next, we verify the simplices $\sigma''_i(x)$. For its ith input, we have the following two cases:

- If x does not lie in the image of (X', J'), then $\sigma_{\beta x}(i) = \sigma''_x(i)$ because if $n \in \min L_{\beta x}(i)$ then it is replaced by $\overline{\alpha}(r')$ for r' the root of J' for which holds $\zeta_{J'', \overline{\alpha}(r')} = \zeta_{J, n} \zeta_{J', r'}$.
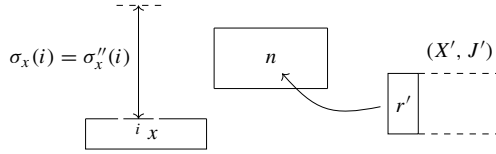
As a result, in both (2) and (3) we have the term $\Theta_{\beta x} \circ_i \sigma_{\beta x}(i)^{\#}$.

- If x is part of (X', J'), i.e. $x = \alpha(x')$ for some vertex x', then $\min L''_{\alpha(x')}(i)$ is the union of $\min L'_{x'}(i)$ and $\min L_n(i')$ for some i'. Hence, we obtain the concatenation of $\sigma'_{x'}(i)$ for $\sigma' = \zeta_{J,n}(\sigma)$ and $\sigma_n(i')$. As $\zeta''_{J''} = \zeta_J \circ_n \zeta_{J'}$, this corresponds exactly to $\sigma''_x(i)$.
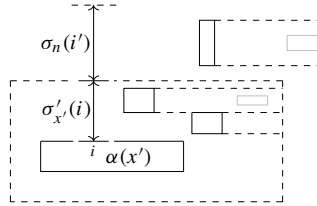
Hence, the corresponding term in both calculations agrees.

Next, we calculate $\sigma''(x, b)$ for b a child of x in (X'', J'') that is not plugged. We again have three cases

- If both x and b lie either outside or inside the image of (X', J'), then clearly $\sigma''(x, b) = \sigma(\beta x, \beta b)$ or $\sigma'(\alpha^{-1} x, \alpha^{-1} b)$ for $\sigma' = \zeta_{J', n}(\sigma)$ due to the previous reasoning and thus the terms agree.
- If b lies in the image of (X', J'), i.e. $b = \alpha(b')$, and x does not, then b' is clearly the root of X'. As a result, $\sigma''(x, b) = \sigma(\beta x, n)$ and thus the terms agree.
- If x lies in the image of (X', J'), i.e. $x = \alpha(x')$, and b does not, then $\min L''_{\alpha(x')}(i)$ is the union of $\min L'_{x'}(i)$ and $\min L_n(i')$ for some i'.
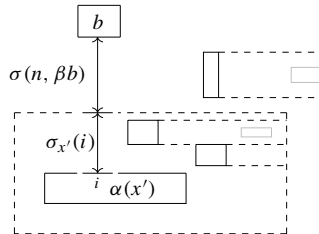
Hence, we obtain in (2) the concatenation of $\sigma(n, \beta b)$ and $\sigma'_{x'}(i)$ for $\sigma' = \zeta_{J,n}(\sigma)$, which corresponds exactly to $\sigma''(x, b)$.

In case either x or b is plugged, we possibly have to apply the functorial property of the restrictions, i.e. $u^* \circ m^U = m^V \circ (u^* \otimes u^*)$ for $u : V \to U$ in $\mathcal{U}$, to pull down $\Theta_{\beta x} = m^{U_{\zeta J, \beta x}(0)}$ or $\Theta'_{\alpha^{-1}x} = m^{U_{\zeta J, n}\zeta_{J', \alpha^{-1}x}(0)}$. Specifically, in the following cases

- Let βx lie on top of n in (X, J) and $\downarrow x = \alpha(y)$ for some vertex y of (X', J').
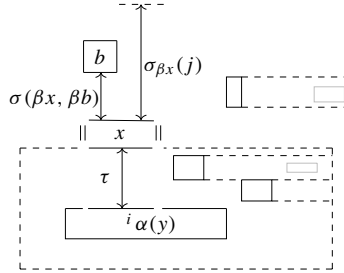
In this case, $\Theta_{\beta x} = m^{U_{\zeta J, n}(0)}$ occurs in (2) and $\Theta''_x = m^{U_{\zeta J, n}\zeta_{J', y}(0)}$ occurs in (3). Using functoriality, in (2) we equivalently have $\tau^\# \circ m^{U_{\zeta J, n}(0)} = m^{U_{\zeta J, n}\zeta_{J', y}(0)} \circ (\tau^\# \otimes \tau^\#)$ for an appropriate simplex τ. As a result they agree.
Next, it is clear from the drawing that $\sigma''_x(j)$ is the concatenation of $\sigma_{\beta x}(j)$ and τ. Moreover, for some vertex b, we have $\sigma''(x, b)$ as the concatenation of $\sigma(\beta x, \beta b)$ and τ, except in the case that b is plugged as well. In the latter case, we can also pull $\Theta_{\beta b}$ in (2) down to $\Theta_{\beta x}$ and obtain $m^{U_{\zeta J, n}\zeta_{J', y}(0)} = \Theta''_x = \Theta''_b$ as in (3).
- Let βx lie on top of n, but $\downarrow x \notin \text{Im}(\alpha)$.
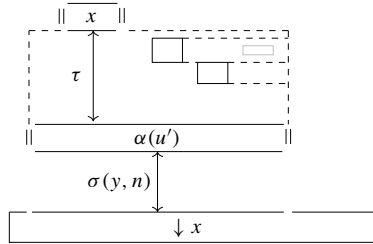
Again, we can pull down $\Theta_{\beta x}$ in (2) past both functors $\tau^\#$ and $\sigma(y, n)^\#$ and obtain $m^{U_{\zeta J, \beta \downarrow y}(0)} = \Theta''_x$. The same reasoning as before also holds for the functors $\sigma''_x(j)$ and $\sigma''(x, b)$ in (3) and its counterparts $\sigma_{\beta x}(j)$ and $\sigma(\beta x, \beta b)$ in (2).
- The case where x lies in the image of (X', J') such that $\downarrow x \notin \text{Im}(\alpha)$, is analogous to the previous one.

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 3.24 *We obtain a morphism of dg-operads*

$$\mathcal{L} : \text{Patch}_s \longrightarrow \textbf{End}(s\textbf{C}_{GS}(\mathcal{A}), d_0).$$

Proof The map $\mathcal{L}$: Patch $\longrightarrow$ **End**$(\mathbf{C}^{\bullet,\bullet}(\mathcal{A}))$ is clearly equivariant and thus it is a morphism of operads due to Proposition 3.23. Hence, the induced map $\mathcal{L}$: Patch$_s$ $\longrightarrow$ **End**$(s\mathbf{C}_{GS}(\mathcal{A}))$ is a morphism of graded operads. Moreover, it is a morphism of dg-operads as $\mathcal{L}(D, 1) = d_0$. □

3.5 The morphism Quilt $\longrightarrow$ Patch$_s$

In [8], Hawkins defines a suboperad Quilt $\subseteq$ F$_2$S $\otimes_H$ Brace for which Quilt(n) is the free k-module generated by $(W, T) \in$ F$_2$S$(n) \times$ Tree(n) such that

(1) $W = \ldots u \ldots v \ldots \Longrightarrow u \not>_T v$;
(2) $W = \ldots u \ldots v \ldots u \ldots \Longrightarrow v \lhd_T u$.

Here, $deg(W, T) := deg(W)$ and the boundary operator is $\partial(W, T) := (\partial W, T)$.

Insightfully, elements of Quilt can also be drawn as a stacking of rectangles in the plane, as extensively explained in [8, §3.2]. We will use Quilt in a fundamentally different way by switching the roles of its first and second component, and thus flipping the rectangles on their side. As such, we also draw the elements of Quilt on their side. We give an example.

Example 3.25 We consider an example from [8, Ex. 3.2] and flip it on its side as follows

Note the double line above rectangle 4: this reflects the fact that 3 is not interposed, otherwise the corresponding word would be 142434.

By definition, we have Patch$_s$ $\subseteq$ mNSOp$_s$ $\otimes_H$ NSOp$_{st}$. In this section, we will construct a morphism of operads Quilt $\longrightarrow$ Patch$_s$ as a restriction of

$$\bar{\phi} \otimes_H \phi : \text{F}_2\text{S} \otimes_H \text{Brace} \longrightarrow \text{mNSOp}_s \otimes_H \text{NSOp}_{st}.$$

Lemma 3.26 *Let* $Q = (W, T) \in$ *Quilt,* $X \in$ Clr$(W, q_1, \ldots, q_n)$ *and* $I \in$ Clr$(T, p_1, \ldots, p_n)$, *then* $(X, I) \in$ *Patch.*

Proof Let $u, v \in \langle n \rangle$, if $u <_I v$, then $u <_T v$ and thus $W \neq \cdots v \ldots u \ldots$ and thus every occurrence of u in W is left of every occurrence of v in W. Hence, $u \lhd_X v$.

The other way around, if $u <_X v$, then $W = \cdots u \ldots v \ldots u \ldots$ and thus $v \lhd_T u$ which is equivalent to $v \lhd_I u$. □

We obtain a morphism of graded operads

$$\bar{\phi} \otimes_H \phi : \text{Quilt} \longrightarrow \text{Patch}_s.$$

defined as

$$(\bar{\phi} \otimes_H \phi)((q_1, p_1), \ldots, (q_n, p_n))(W, T) = \sum_{\substack{X \in \text{Clr}(W, q_1, \ldots, q_n) \\ I \in \text{Clr}(T, p_1, \ldots, p_n)}} \text{sgn}_W(X) \, \text{sgn}_T(I)(-1)^{\sigma}(X, I)$$

where the sign $(-1)^{\sigma}$ is defined as the Koszul sign obtained from switching

$$q_1, \ldots, q_n, p_1 - 1, \ldots, p_n - 1 \rightsquigarrow q_1, p_1 - 1, \ldots, q_n, p_n - 1$$

This is the consequence of the Hadamard product of two graded operads

$$\text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p)) \subseteq \text{mNSOp}(q_1, \ldots, q_n; q) \otimes \text{NSOp}(p_1, \ldots, p_n; p)$$

where we have switched the order of the inputs.

Note in particular that this sign corresponds to the sign defined in [8, §4.7] and that we write $\text{sgn}_Q(X, J) := \text{sgn}_W(X) \, \text{sgn}_T(J)(-1)^{\sigma}$.

As a direct consequence of Lemma 3.9 we have the following.

Proposition 3.27 *We have a morphism of dg-operads*

$$\bar{\phi} \otimes_H \phi : (\text{Quilt}, \partial) \longrightarrow (\text{Patch}_s, \partial_{(D,1)})$$

Corollary 3.28 *We have a morphism of dg-operads*

$$R := \mathcal{L} \circ (\bar{\phi} \otimes_H \phi) : \text{Quilt} \longrightarrow \mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}), d_0)$$

Proof Immediate from Theorem 3.24 and Proposition 3.27.   □

Our action of Quilt on the GS-complex of a prestack is orthogonal to the action constructed in [8, Thm. 4.26] in the case of presheaves, and thus also new for the latter case. This can be interpreted in a geometric sense: our action encodes a quilt $Q = (W, T)$ as a vertical patchwork according to W and a horizontal patchwork according to T. In Hawkins' action their roles are reversed, where the role of the multiplication is filled in by the identity $1^{u,v} : v^*u^* = (uv)^*$. This does not translate to the case of prestacks due to the occurring twists $c^{u,v} : v^*u^* \longrightarrow (uv)^*$.

4 Incorporating twists

The morphism $R : \text{Quilt} \longrightarrow \mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$ from Corollary 3.28 only involves the multiplication m and the functors f of the data of a prestack $(\mathcal{A}, m, f, c)$. In this section, we will incorporate the twists c by adding a formal element with certain relations, resulting in the bounded powerseries operad $\text{Quilt}_b[[c]]$. In Sect. 4.4, we extend R above to a morphism $R_c : \text{Quilt}_b[[c]] \longrightarrow \mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$ (see Theorem 4.17). In Hawkins' approach to the presheaf case, the initial action of Quilt on $\mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$, which only involves the restriction maps f, is later extended in order to incorporate

the multiplications m. As far as the structure of both approaches goes, our operad $\text{Quilt}_b[[c]]$ is the counterpart of the operad mQuilt from [8, Def. 5.2].

In [8, §7.1], Hawkins constructs a morphisms $L_\infty \longrightarrow \text{Quilt}$ (see Sect. 4.2). In Sect. 4.2, we establish a more involved morphism $L_\infty \longrightarrow \text{Quilt}_b[[c]]$ (see Theorem 4.10) by extending to an infinite series of higher components incorporating the element c.

Putting Theorems 4.10 and 4.17 together, we have thus endowed $s\mathbf{C}_{GS}(\mathcal{A})$ with an L_∞-structure. In the case of presheaves, this coincides on reduced and normalised cochains with the L_∞-structure from [8, Thm. 7.13].

In the final Sect. 4.5 we briefly discuss the relation of this structure with the deformation theory of the prestack $\mathcal{A}$.

4.1 Powerseries operads

In order to obtain an L_∞-structure incorporating twists, we will make use of operads of formal power series.

Definition 4.1 Let $\mathcal{O}$ be a graded operad, then define $\mathcal{O}[x]$ as the graded operad generated by $\mathcal{O}$ and an element x of degree t and define the subspaces

$$\mathcal{O}[x](n, r) := \{\gamma \in \mathcal{O}[x](n) : \gamma \text{ has } r \text{ occurrences of } x\} \subseteq \mathcal{O}[x](n)$$

which is well-defined as there are no relations on x. Define

$$\mathcal{O}[[x]](n) := \prod_{r \geq 0} \mathcal{O}[x](n, r)$$

with component-wise $\mathbb{S}_n$-action and write their elements as $\sum_{r \geq 0} Q_r$ for $Q_r \in \mathcal{O}[x](n, r)$. For every $1 \leq k \leq n$ the composition of $\mathcal{O}[x]$ descends to a map

$$\mathcal{O}[x](n, r) \otimes \mathcal{O}[x](n, s) \longrightarrow \mathcal{O}[x](n, r + s)$$

which extends to a composition map

$$\left(\sum_{r \geq 0} Q_r\right) \circ_k \left(\sum_{s \geq 0} P_s\right) := \sum_{t \geq 0} \left(\sum_{i+j=t} Q_i \circ_k P_j\right)$$

We call an element $\sum_{r \geq 0} Q_r \in \mathcal{O}[[x]]$ *bounded* if the set $\{deg(Q_r) : r \geq 0\} \subseteq \mathbb{Z}$ is bounded. Let $\mathcal{O}_b[[x]]$ be the $\mathbb{S}$-submodule of bounded series which is graded by the series with coefficients of constant degree.

Lemma 4.2 *(1)* $\mathcal{O}[[x]]$ *is an operad.*
(2) $\mathcal{O}_b[[x]]$ *is a graded suboperad of* $\mathcal{O}[[x]]$.
(3) *We have a sequence of injective operad morphisms*

$$\mathcal{O} \hookrightarrow \mathcal{O}[x] \hookrightarrow \mathcal{O}_b[[x]] \hookrightarrow \mathcal{O}[[x]]$$

Proof These are straight-forward computations. □

We call $\mathcal{O}[[x]]$ the *operad of powerseries with coefficients in $\mathcal{O}$* and $\mathcal{O}_b[[x]]$ the *operad of bounded powerseries with coefficients in $\mathcal{O}$*.

Definition 4.3 Consider Quilt$[x]$, Quilt$_b[[x]]$ and Quilt$[[x]]$ for x a 0-ary element $x \in$ Quilt(0) of degree -1, then let Quilt$[c]$, Quilt$_b[[c]]$ and Quilt$[[c]]$ be their respective quotients under the following relations

(1) $\partial(c) = 0$
(2) $(12, 1 \text{ —— } 2) \circ_1 c \circ_1 c = 0$
(3) $(W, T) \circ_i c = 0$ if i has more than two children in T or i is repeated in W (that is, it has a child in W).

Remark 4.4 In Definition 4.3, (1) determines that c encodes a natural transformation, (2) embodies the cocycle condition and (3) determines the form of c. The letter c will always stand for the twist subject to its relations, and should not be confused with a free variable.

On inspection of mQuilt from [8, Def. 5.2], we see that our conditions on c are a subset of those imposed on m in mQuilt. Hence, we obtain a morphism Quilt$[c] \longrightarrow$ mQuilt sending c to m.

4.2 The morphism $L_\infty \longrightarrow$ Quilt

In [8, Thm 7.8] a morphism $L_\infty \longrightarrow$ Quilt $: l_n \longmapsto L_n^0$ is defined by setting

$$L_n^0 := \sum_{\substack{Q \in \text{Quilt}(n) \\ deg(Q)=n-2}} \text{sgn}(Q)\, Q$$

In particular, this means that for every $n \geq 2$ the equation

$$0 = \partial L_n^0 + \sum_{\substack{p+q=n+1 \\ p,q \geq 2}} \sum_{\sigma \in Sh_{p-1,q}} (-1)^{(p-1)q}(-1)^\sigma (L_p^0 \circ_p L_q^0)^{\sigma^{-1}} \tag{4}$$

holds.

An important feature which we will need, is that we can write L_n^0 as the antisymmetrization of elements P_n^0. Namely, we set

$$P_n^0 := \sum_{\substack{Q \in \text{Quilt}(n) \\ deg(Q)=n-2 \\ Q \text{ labelled in } \downarrow\text{order}}} (-1)^{1+\frac{n(n-1)}{2}} Q$$

then we have

$$L_n^0 = \sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma (P_n^0)^\sigma.$$

In fact, the L_∞-relations translate to the following

$$\partial P_n^0 + \sum_{\substack{p+q=n+1 \\ p,q\geq 2}} \sum_{j=1}^{p} (-1)^{(p-1)q+(p-j)(q-1)} P_p^0 \circ_j P_q^0 \in \langle Q - Q^\sigma | \sigma \in \mathbb{S}_n \rangle \subseteq \mathrm{Quilt}(n)$$

(5)

where $\langle - \rangle$ denotes 'free k-module generated by$-$'.

4.3 The morphism $L_\infty \longrightarrow \mathrm{Quilt}_b[[c]]$

Next we will define more involved L_∞-operations incorporating c.

Definition 4.5 For $n + r \geq 2$ we define

$$P_n^r := \sum_{1 \leq y_1 < ... < y_r \leq n+r} (-1)^{y_1,...,y_r} P_{n+r}^0 \circ_{y_1} c \circ_{y_2-1} \cdots \circ_{y_r-r+1} c$$

where $(-1)^{y_1,...,y_r}$ denotes the sign of the (r, n)-shuffle defined by $(y_1, \ldots, y_r)$. Using these we set

$$L_n^r := \sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma \left(P_n^r \right)^\sigma$$

Remark 4.6 Note that L_n^r live in $\mathrm{Quilt}[c]$.

Let us compute P_n^r for some low n and r.

Example 4.7 In case no elements c are added, we obtain the original P_n^0

$$P_2^0 = \boxed{\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}}, \quad P_3^0 = \boxed{\begin{array}{|c|} \hline 1 \;\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array} \\ \hline \end{array}},$$

$$P_4^0 = - \; \boxed{1 \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \multicolumn{2}{|c|}{2} \\ \hline\end{array}} \; - \; \boxed{1 \begin{array}{cc} 3 & \\ & 4 \\ & 2 \end{array}} \; - \; \boxed{1 \begin{array}{cc} & 4 \\ 3 & \\ & 2 \end{array}}, \quad \ldots$$

Similar to how we drew elements from mNSOp as trees with vertices plugged by m, elements of $\mathrm{Quilt}[c]$ can be drawn as quilts with rectangles plugged by c. For example, we have

$$P_1^1 = \boxed{\begin{array}{|c|c|}\hline c & 1 \\ \hline\end{array}} \; - \; \boxed{\begin{array}{|c|c|}\hline 1 & c \\ \hline\end{array}}$$

or

$$P_2^2 = \boxed{c \; \begin{array}{|c|c|}\hline c & 2 \\\hline 1 \end{array}} - \boxed{c \; \begin{array}{|c|c|}\hline 2 & c \\\hline 1 \end{array}} - \boxed{1 \; \begin{array}{|c|c|}\hline c & c \\\hline 2 \end{array}} - \boxed{1 \; \begin{array}{c} c \\ c \; \square \\ 2 \end{array}} - \boxed{1 \; \begin{array}{c} c \\ \square \; c \\ 2 \end{array}}$$

Note that depending on where we plug the elements c a sign is added.

This enables us to define the following.

Definition 4.8

$$L_n := \sum_{r \geq 0} L_n^r \text{ for } n \geq 2 \text{ and } L_1 := \sum_{r \geq 1} L_1^r$$

Set

$$\partial' := \partial + \partial_{L_1}$$

where

$$\partial_{L_1}(A) := L_1 \circ_1 A - (-1)^{|A|} \sum_i A \circ_i L_1$$

(a derivation by an element) which will be the new differential.

Remark 4.9 Note that L_n are bounded because their components have constant degree $n - 2$. Hence, L_n live in $\mathrm{Quilt}_b[[c]]$.

The main theorem of this section is the following.

Theorem 4.10 *The map*

$$L_\infty \longrightarrow (\mathrm{Quilt}_b[[c]], \partial') : l_n \longmapsto L_n$$

defines a morphism of dg-operads.

First we need some lemmas.

Lemma 4.11 *For $r \geq 2$ we have $L_0^r = 0$.*

Proof Due to relation (3) in Definition 4.3 of c, there cannot be a vertical composition of c. Hence, $L_0^r = 0$ for $r \geq 3$. $L_0^2 = 0$ due to condition 2 of the definition of c. $\square$

Remark 4.12 Note that we have used 2 out of the three conditions on c to prove this lemma.

The following lemma extends the L_∞-equation of $(L_n^0)_n$ for higher L_n^r.

Lemma 4.13 *For $n \geq 1$ and $r \geq 0$, we have*

$$-\partial(L_n^r) = \sum_{\substack{i+j=r \\ k+l=n+1 \\ k+i\geq 2 \\ l+j\geq 2}} \sum_{\chi\in Sh_{k-1,l}} (-1)^{\chi}(-1)^{(k-1)l} \left(L_k^i \circ_k L_l^j\right)^{\chi^{-1}}.$$

Proof By applying ∂ and using Eq. (5) and $\partial(c) = 0$, we deduce

$$-\partial(L_n^r) = \sum_{\substack{p+q=n+r+1 \\ p,q\geq 2}} \sum_{\sigma\in\mathbb{S}_n} \sum_{\substack{1\leq y_1<...<y_r\leq n+r \\ x=1,...,p}} (-1)^{y_1,...,y_r}(-1)^{\sigma}(-1)^{(p-1)q}(-1)^{(p-x)(q-1)}$$

$$\left(P_p^0 \circ_x P_q^0 \left(\circ_{y_e-e+1}c\right)_e\right)^{\sigma}$$

Given $1 \leq y_1 < \ldots < y_r \leq n + r$ and $1 \leq x \leq p$, we have a subdivision into two groups where $\circ_{y_s-s+1}c$ is inserted into either P_p^0 or P_q^0. Hence, if we are also given a permutation $\sigma \in \mathbb{S}_n$, then we show that there exists unique integers $i + j = r$, $k + l = n + 1$, indices $1 \leq z_1 < \ldots < z_i \leq k + i$ and $1 \leq z_1' < \ldots < z_j' \leq j + l$, permutations $\tau \in \mathbb{S}_{k+i}$, $\tau' \in \mathbb{S}_{j+l}$ and a shuffle $\chi \in Sh_{k-1,l}$ such that

$$(-1)^{y_1,...,y_r}(-1)^{\sigma}(-1)^{(p-1)q}(-1)^{(p-x)(q-1)} \left(P_p^0 \circ_x P_q^0 \left(\circ_{y_e-e+1}c\right)_e\right)^{\sigma}$$

$$= (-1)^{(z_1,...,z_i)+(z_1',...,z_j')}(-1)^{\chi+\tau+\tau'}(-1)^{(k-1)l}$$

$$\left(\left(P_{k+i}^0 \left(\circ_{z_e-e+1}c\right)_e\right)^{\tau} \circ_k \left(P_{l+j}^0(\circ_{z_f'-f+1}c)_f\right)^{\tau'}\right)^{\chi^{-1}} \tag{6}$$

In this case, we obtain

$$-\partial(L_n^r) = \sum_{\substack{i+j=r \\ k+l=n+1 \\ k+i\geq 2 \\ l+j\geq 2}} \sum_{\chi\in Sh_{k-1,l}} (-1)^{\chi}(-1)^{(k-1)l}$$

$$\sum_{\substack{\tau\in\mathbb{S}_k,\tau'\in\mathbb{S}_l}} \sum_{\substack{1\leq z_1<...<z_i\leq k+i \\ 1\leq z_1'<...<z_j'\leq j+l}} (-1)^{\tau+\tau'+(z_1,...,z_i)+(z_1',...,z_j')}$$

$$\left(\left(P_{k+i}^0 \left(\circ_{z_e-e+1}c\right)_e\right)^{\tau} \circ_k \left(P_{l+j}^0(\circ_{z_f'-f+1}c)_f\right)^{\tau'}\right)^{\chi^{-1}} \tag{7}$$

By applying the definition of L_n, this proves the lemma.

We show that Eq. (6) holds: the set $y_1 < \ldots < y_r$ splits into three subsets

- $z_1^1 < \ldots < z_{i_1}^1$ such that $z_t^1 < x$,
- $z_1^2 < \ldots < z_{i_2}^2$ such that $z_t^2 > x + q - 1$,

- $z_1'^0 < \ldots < z_j'^0 = \{y_1 < \ldots < y_r\} \setminus \{z_1^1 < \ldots < z_{i_1}^1, z_1^2 < \ldots < z_{i_2}^2\}.$

from which we define

$$z_t := \begin{cases} z_t^1 & t \le i_1 \\ z_{t-i_1}^2 - q & t > i_1 \end{cases} \qquad\qquad z_t' := z_t'^0 - x + 1$$

$$i := i_1 + i_2 \text{ and } j := r - i, \qquad\qquad k := p - i \text{ and } l := q - j$$

We then compute

$$P_p^0 \circ_x P_q^0 \left(\circ_{y_e - e + 1} c\right)_e = (-1)^{i(l+j)+i_2 j} \left(P_{k+i}^0 \left(\circ_{z_e - e + 1} c\right)_e\right) \circ_{x - i_1} \left(P_{l+j}^0 (\circ_{z_f' - f + 1} c)_f\right) \tag{8}$$

where the sign appears because we move the c's corresponding to $z_t'^0$ past j c's corresponding to $z_t'^0$ and also i c's past P_q^0. Note that if we know x and i_1, then given $i, j, k, l, z_1, \ldots, z_i, z_1', \ldots, z_j'$ we can uniquely determine $p, q, r, n, y_1, \ldots, y_r$.
We also compute the sign of $y_1, \ldots, y_r$: let θ and θ' be the shuffles such that

$$z_1^1, \ldots, z_{i_1}^1 \overset{\theta}{\leadsto} 1, \ldots, i_1$$
$$z_1'^0, \ldots, z_j'^0 \overset{z_1', \ldots, z_j'}{\leadsto} x, \ldots, x + j \overset{j(x-i_1-1)}{\leadsto} i_1 + 1, \ldots, i_1 + j$$
$$z_1^2, \ldots, z_{i_2}^2 \overset{\theta'}{\leadsto} x + q, \ldots, x + q + i_2 - 1 \overset{i_2(l+x-i_1-1)}{\leadsto} i_1 + j + 1, \ldots, r$$

then we obtain that

$$(-1)^{y_1, \ldots, y_r} = (-1)^{\theta + (z_1', \ldots, z_j') + j(x-i_1-1) + \theta' + i_2(l+x-i_1-1)}$$

On the other hand,

$$z_1, \ldots, z_{i_1}, z_{i_1+1}, \ldots, z_i \overset{\theta+\theta'}{\leadsto} 1, \ldots, i_1, x + 1, \ldots, x + i_2 \overset{i_2(x-i_1)}{\leadsto} 1, \ldots, i$$

and thus we have

$$(-1)^{y_1, \ldots, y_r} = (-1)^{(z_1, \ldots, z_i) + (z_1', \ldots, z_j') + j(x-i_1-1) + i_2(l+1)}$$

Given $\sigma \in \mathbb{S}_n$, let $b_1 < \ldots < b_l \subseteq \langle k + l - 1 \rangle$ such that $\sigma(b_t) \in \{x - i_1, \ldots, x - i_1 + l - 1\}$ and let $a_1 < \ldots < a_{k_1} := \langle k + l - 1 \rangle \setminus \{b_1 < \ldots < b_l\}$. We then define the $(k - 1, l)$-shuffle $\chi = (a_1 < \ldots < a_{k-1}, b_1 < \ldots < b_l)$ and

$$\tau(t) = \begin{cases} \sigma\chi(t) & t < k \\ x - i_1 & t = k \end{cases} \text{ and } \tau'(t) = \sigma(b_t) - (x - i_1 - 1)$$

It is then easy to see that

$$\left(P^0_{k+i}\left(\circ_{z_e-e+1}c\right)_e \circ_{x-i_1}\left(P^0_{l+j}(\circ_{z'_f-f+1}c)_f\right)\right)^{\sigma\chi} = \left(P^0_{k+i}\left(\circ_{z_e-e+1}c\right)_e\right)^\tau \circ_k \left(P^0_{l+j}(\circ_{z'_f-f+1}c)_f\right)^{\tau'}$$

Note that τ and $z_1, \ldots, z_i$ determine x and i_1 uniquely. Hence, given $z_1, \ldots, z_i, z'_1, \ldots, z'_j, k, l$ we can uniquely determine $y_1, \ldots, y_r, p, q, n, r$ in the above manner. In order to show that Eq. (6) holds, we only need to verify the corresponding signs: let τ_0 be the permutation such that

$$\sigma\chi(1), \ldots, \sigma\chi(k-1) \overset{\tau_0}{\leadsto} 1, \ldots, x-i_1-1, x-i_1+l, \ldots, k+l-1$$

$$\sigma\chi(k), \ldots, \sigma\chi(k+l-1) \overset{\tau'}{\leadsto} x-i_1, \ldots, x-i_1+l$$

and thus we have that $(-1)^{\sigma+\chi}=(-1)^{\tau_0+\tau'+l(k-x+i_1)}$. On the other hand, we have that τ corresponds to

$$\tau(1), \ldots, \tau(k) \overset{\tau_0}{\leadsto} 1, \ldots, x-i_1-1, x-i_1+1, \ldots, k-1, \tau(k) \overset{k-x+i_1}{\leadsto} 1, \ldots, k-1$$

and thus we have

$$(-1)^{\sigma+\chi} = (-1)^{\tau+\tau'+(l+1)(k-x+i_1)}$$

As such, we can compute

$$(-1)^{(z_1,\ldots,z_i)+(z'_1,\ldots,z'_j)}(-1)^{\chi+\tau+\tau'}(-1)^{(k-1)l}(-1)^{i(l+j)+i_2j}$$
$$= (-1)^{y_1,\ldots,y_r+\sigma}(-1)^{i_2(l+1)+j(x-i_1-1)+(l+1)(k-x+i_1)+(k-1)l+i(l+j)+i_2j}$$

and

$$i_2(l+1) + j(x-i_1-1) + (l+1)(k-x+i_1) + (k-1)l + i(l+j) + i_2j$$
$$= (l+1)(k-x+i) + j(x-i-1) + (k-1)l + i(l+j)$$
$$= (l+1)(p-x) + j(p-x+(k-1)) + (k-1)l + iq$$
$$= (q-1)(p-x) + (k-1)q + iq$$
$$= (q-1)(p-x) + (p-1)q$$

which completes the proof.   □

Lemma 4.14 L_n *are skew symmetric and* ∂' *is a differential making* $\mathrm{Quilt}_\mathrm{b}[[c]]$ *into a dg-operad.*

Proof It is clear from the definition of L^r_n that they are skew symmetric and thus also L_n.

Per definition ∂_{L_1} is a derivation by construction and so is ∂, and thus so is ∂'.

It is clear from the definition of c that $\partial'(c) = 0$. Hence, we only need to show that $\partial'\partial'(Q) = 0$ for every $Q \in$ Quilt. Using Lemmas 4.11 and 4.13 we first prove that $-\partial L_1 = L_1 \circ_1 L_1$. Namely, we compute

$$-\partial(L_1) = \sum_{\substack{r \geq 1}} \sum_{\substack{i+j=r \\ k+l=2 \\ k+i \geq 2 \\ l+j \geq 2}} \sum_{\chi \in Sh_{k-1,l}} (-1)^{\chi}(-1)^{(k-1)l}(L_k^i \circ_k L_l^j)^{\chi^{-1}}$$

$$= \sum_{\substack{i+j=r \\ i,j \geq 1 \\ r \geq 1}} L_1^i \circ_1 L_1^j = L_1 \circ_1 L_1 \tag{9}$$

Now let us compute

$$\partial'\partial'Q = \partial\partial Q + \partial\partial_{L_1}Q + \partial_{L_1}\partial Q + \partial_{L_1}\partial_{L_1}Q$$

we compute separately

$$\partial_{L_1}\partial Q = L_1 \circ_1 \partial Q - (-1)^{|Q|-1}\sum_i \partial Q \circ_i L_1$$

and

$$\partial\partial_{L_1}Q = \partial(L_1 \circ_1 Q) - (-1)^{|Q|}\sum_i \partial(Q \circ_i L_1)$$

$$= \partial L_1 \circ_1 Q + (-1)^{|Q|}L_1 \circ_1 \partial Q - (-1)^{|Q|}\sum_i \partial Q \circ_i L_1 - (-1)^{|Q|-1}\sum_i \partial L_1$$

adding them gives

$$\partial_{L_1}\partial Q + \partial\partial_{L_1}Q = \partial L_1 \circ_1 Q - (-1)^{|Q|-1}\sum_i Q \circ_i \partial L_1$$

Next, we compute $\partial_{L_1}\partial_{L_1}Q$. As L_1 has only a single input and has degree -1, we have for $i \neq j$ that $Q \circ_i L_1 \circ_j L_1 = -Q \circ_j L_1 \circ_i L_1$. Hence, by also using using Eq. (9), we obtain

$$\partial_{L_1}\partial_{L_1}Q = L_1 \circ_1 (L_1 \circ_1 Q) - (-1)^{|Q|}\sum_i L_1 \circ_1 (Q \circ_i L_1)$$

$$+ (-1)^{|Q|}\sum_i (L_1 \circ_1 Q) \circ_i L_1 - \sum_{i,j} Q \circ_i L_1 \circ_j L_1$$

$$= L_1 \circ L_1 \circ_1 Q - \sum_i Q \circ_i (L_1 \circ_1 L_1)$$

$$= \partial_{L_1}\partial Q + \partial\partial_{L_1}Q$$

As $\partial\partial Q = 0$ we thus obtain $\partial'\partial' Q = 0$. □

Proof of Theorem 4.10 We need to show for every $n \geq 2$ that the equation

$$0 = \partial'(L_n) + \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{\sigma \in Sh_{k-1,l}} (-1)^{(k-1)l}(-1)^\sigma (L_k \circ_k L_l)^{\sigma^{-1}}$$

holds, which is equivalent to

$$0 = \partial(L_n) + \sum_{\substack{k+l=n+1 \\ k,l \geq 1}} \sum_{\sigma \in Sh_{k-1,l}} (-1)^{(k-1)l}(-1)^\sigma (L_k \circ_k L_l)^{\sigma^{-1}}$$

This is equivalent to showing for every $r \geq 0$ that the equation

$$0 = \partial(L_n^r) + \sum_{\substack{i+j=r \\ k+l=n+1 \\ k,l \geq 1 \\ (l,j) \neq (1,0) \neq (k,i) \\ (l,j) \neq (0,1) \neq (k,i)}} \sum_{\sigma \in Sh_{k-1,l}} (-1)^{(k-1)l}(-1)^\sigma (L_k^i \circ_k L_l^j)^{\sigma^{-1}} \qquad (10)$$

holds, which follows from Lemma 4.13 and $L_0^i = 0$ for $i \geq 0$ (Lemma 4.11). □

Remark 4.15 Under the natural morphism Quilt$[c] \longrightarrow$ mQuilt sending c to m our L_∞-structure corresponds to the L_∞-structure from [8, Thm. 7.13], that is, we have the commutative diagram

$$
\begin{array}{ccc}
 & \mathrm{Quilt}_b[[c]] & \\
 \nearrow & & \searrow \\
L_\infty \longrightarrow \mathrm{mQuilt} & \longrightarrow & \mathrm{Quilt}_b[[m]]
\end{array}
$$

where Quilt$_b[[m]]$ denotes the quotient of the operad of bounded powerseries by the relations on m in mQuilt.

4.4 The morphism Quilt$_b[[c]] \longrightarrow$ End($s\mathbf{C}_{GS}(\mathcal{A})$)

In this section, we make (the suspension of) $\mathbf{C}_{GS}(\mathcal{A})$ into a Quilt$_b[[c]]$-algebra.

The morphism $R :$ Quilt $\longrightarrow$ **End**($s\mathbf{C}_{GS}(\mathcal{A})$) naturally extends to a morphism of graded operads $R_c :$ Quilt$[c] \longrightarrow$ **End**($s\mathbf{C}_{GS}(\mathcal{A})$) by sending c to $c \in \mathbf{C}_{GS}(\mathcal{A})$ as the axioms of Quilt$[c]$ correspond respectively to c being a natural transformation, the cocycle condition of c and $c \in \mathbf{C}^{2,0}(\mathcal{A})$. Next, we will show that it further extends to the operad of bounded power series.

Lemma 4.16 *Let $\theta_1, \ldots, \theta_n \in \mathbf{C}_{GS}(\mathcal{A})$ where $\theta_i \in C^{p_i,q_i}(\mathcal{A})$ and $Q \in$ Quilt$[c](n,r)$ of degree t, then $R_c(Q)(\theta_1, \ldots, \theta_n) = 0$ if $r > \sum_{i=1}^n q_i - t$.*

Proof This follows from $\mathbf{C}_{GS}(\mathcal{A})$ having only non-negative bidegree. Namely, given $Q = Q' \circ_{i_1} c \circ_{i_2} \ldots \circ_{i_r} c$ for a certain $Q' \in \mathrm{Quilt}(n+r)$, the bidegree of $R_c(Q')$ is $(\sum_{i=1}^{n} p_i + 2r - (n+r-1), \sum_{i=1}^{n} q_i - deg(Q'))$. As $deg(Q') = deg(Q) + r = t + r$, we have that

$$\sum_{i=1}^{n} q_i - deg(Q') < 0 \iff r > \sum_{i=1}^{n} q_i - t$$

proving the lemma. □

Theorem 4.17 *For $\mathcal{A} = (\mathcal{A}, m, f, c)$ a linear prestack over $\mathcal{U}$, the map*

$$R_c : (\mathrm{Quilt}_b[[c]], \partial') \longrightarrow \boldsymbol{End}(s\boldsymbol{C}_{GS}(\mathcal{A}), d)$$

defined as $R_c(Q) = R(Q)$ for $Q \in \mathrm{Quilt}$ and $R_c(c) = c$, is a morphism of dg-operads.

Proof The representation R_c extends to $\mathrm{Quilt}_b[[c]]$ due to Lemma 4.16.

We verify that $R_c \partial' Q = \partial_d R_c Q$ for $Q \in \mathrm{Quilt}$. It suffices to verify that $R_c L_1^j = d_j$ as ∂ corresponds to ∂_{d_0}. Let $\theta \in \mathbf{C}^{p,q}(\mathcal{A})$ and $\sigma \in N_{p+j}(\mathcal{U})$ and note that we write $|\theta|$ for the degree $p + q$ of θ in $\mathbf{C}_{GS}(\mathcal{A})$.
Step 2 $R_c L_1^1 = d_1$

We can write $d_1 = \sum_{s=0}^{p+1} (-1)^{p+q+1+s} d_1^s$ where d_1^s names the sth component of d_1. We write out the left-hand side

$$R_c L_1^1(\theta) = R(P_2^0)(c, \theta) - (-1)^{|\theta|-1} R(P_2^0)(\theta, c) \tag{11}$$

where $P_2^0 = (-1)^{\frac{2 \cdot 1}{2}+1}(12, \ 1 \!\!-\!\!\!- 2 \ )$. There exists two colorings (X_0, J_0) and (X_{p+1}, J_{p+1}) of the quilt $Q = (W, T)$ in $P_2^0(c, \theta)$ given by the patchworks

$$(X_1, J_1) = \qquad \qquad (X_{p+1}, J_{p+1}) =$$

which correspond respectively to d_1^0 and d_1^{p+1}. We verify their signs: the sign $\mathrm{sgn}_Q(X_0, J_0) = \mathrm{sgn}_W(X_0) \, \mathrm{sgn}_T(J_0)(-1)^q$ is determined by two shuffles

$$0_1 1_1 0_2 \ldots (q-1)_2 \rightsquigarrow 0_1 1_1 0_2$$
$$1_2 \ldots (p-1)_2 1_1 \rightsquigarrow 1_1 1_2 \ldots (p-1)_2$$

and thus $\mathrm{sgn}_Q(X_0, J_0) = (-1)^{p-1+q}$. In the second case, we have $\mathrm{sgn}(X_{p+1}, J_{p+1}) = \mathrm{sgn}_W(X_{p+1})\,\mathrm{sgn}_T(J_{p+1})(-1)^q$ determined by the two shuffles

$$0_1 1_1 0_2 \ldots (q-1)_2 \rightsquigarrow 0_1 1_1 0_2$$
$$1_1 1_2 \ldots (p-1)_2 \rightsquigarrow 1_1 1_2 \ldots (p-1)_2$$

Hence, $\mathrm{sgn}_Q(X_{p+1}, J_{p+1}) = (-1)^q$.

There exists p colorings $(X_i, J_i)_{i=1}^p$ of P_2^0 occuring in the second term of (11), given by the patchworks

$$(X_i, J_i) =$$

for $i = 1, \ldots, p$. They correspond to d_1^{p+1-i} and we verify their signs: we have $\mathrm{sgn}_Q(X_i, J_i) = \mathrm{sgn}_W(X_i)\,\mathrm{sgn}_T(J_i)$ determined by the two shuffles

$$0_1 \ldots (q-1)_1 0_2 1_2 \rightsquigarrow 0_1 \ldots (q-1)_1 0_2 1_2$$
$$1_1 \ldots (i-1)_1 1_2 i_1 \ldots (p-1)_1 \rightsquigarrow 1_1 \ldots (p-1)_1 1_2$$

Hence, we obtain the sign

$$-(-1)^{p+q-1}\,\mathrm{sgn}_Q(X_i, J_i) = (-1)^{q-i}$$

Step 2 $R_c L_1^j = d_j$

First, we name the terms of d_j and write

$$d_j(\theta)^\sigma := \sum_{\beta \in \mathbb{S}_{q+1-j}} \sum_{r \in P(R_p(\sigma))} (-1)^{r+\beta+q-j+1} d_j(\theta)^\sigma(\beta, r)$$

such that

$$d_j(\theta)^\sigma(\beta, r) \in \prod_{A \in \mathcal{A}(U_p)^{q-j+2}} \mathrm{Hom}\left(\bigotimes_{i=1}^q \mathcal{A}(U_p)(A_i, A_{i-1}), \mathcal{A}(U_0)(\sigma^\# A_{q-j+2}, \sigma^* A_0)\right)$$

Next, we note that the only non-vanishing term of L_1^j is given by $(-1)^{(j-1)} P_{j+1}^0 \circ_1 c \circ_2 c \circ_2 \ldots \circ_2 c$ as c cannot have any children vertically. Moreover, the only non-

vanishing quilts are of the form

$$Q_T = \left(W', T' \right) = \left(123242\ldots 2\,(j+1)\,2,\ \ 1 \begin{smallmatrix} & T \\ \diagup & \\ \diagdown & \\ & 2 \end{smallmatrix} \right)$$

for any binary tree T on $j-1$ vertices whose numbering is compatible with the word component W of Q_T. Hence, the possible colorings of Q_T are

for $t_1 < \ldots < t_{j-1}$ in $\langle q \rangle$ and J an indexed tree coloring T.

As a result, we have that

$$R_c L_1^j (\theta) = (-1)^{(j-1)+(j-1)(p+q-1)}\, R P_{j+1}^0 (c, \theta, c, \ldots, c)$$

which sums over the terms

$$\mathrm{sgn}_{Q_T} \left(X, J' \right) (-1)^{\frac{(j+1)j}{2}+1}\, \mathcal{L} \left(X, J' \right)$$

We finish the proof by showing that a formal path r and a shuffle $\beta \in \mathbb{S}_{q+1-j,\,j-1}$ correspond uniquely to such a binary tree T and a coloring $\left(X, J' \right)$ of Q_T such that

$$(-1)^{r+\beta+(q+1-j)}\, d_j (\theta)^\sigma (\beta, r)$$
$$= \mathrm{sgn}_Q \left(X, J' \right) (-1)^{\frac{j(j+1)}{2}+1} (-1)^{(j-1)+(j-1)(p+q-1)}\, \mathcal{L} \left(X, J' \right) (c, \theta, c \ldots, c)\, (\theta)^\sigma$$

Given a formal path $r = (r_1, \ldots, r_{j-1}) = \left((\tau_1, i_1), \ldots, (\tau_{j-1}, i_{j-1}) \right)$ we first define T and its coloring J as trees with vertex set $\{3, \ldots, j+1\}$ inductively:

- In the degenerate case $j = 2$, r is uniquely determined and we set T to be the one vertex tree and J the empty function.
- For $j > 2$, let (T_0, J_0) be the indexed tree corresponding to $\left((\tau_1, i_1), \ldots, (\tau_{j-2}, i_{j-2}) \right)$, in order to add vertex $j + 1$ we have three cases

  (1) If $i_{j-1} < i_{j-2}$, then set $j \lhd_J (j+1)$ and start over with (i_{j-1}, i_{j+1})

(2) If $i_{j-1} = i_{j-2}$ or $i_{j-1} = i_{j-2} + 1$, then let $(j, j+1) \in E_T$ and set resp. $J(j, j+1) = 2$ or $= 1$.

(3) If $i_{j-1} > i_{j-2} + 1$, then set $(j+1) \lhd_J j$ and start over with $(i_{j-1}, i_{j+1} - 1)$

which we can draw as follows

(1)   (2)   (3)
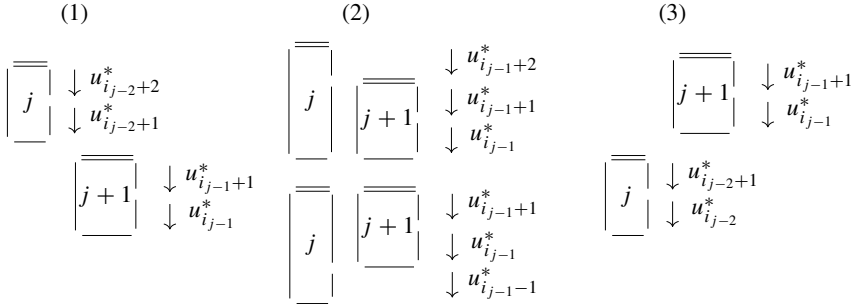
Clearly, this process is reversible: given (T, J) we obtain a unique sequence $r \in P\left(R_p(\sigma)\right)$. By identifying the shuffle $\beta \in Sh_{q+1-j, j-1}$ and X via $t_l = \beta(q - j + 1 + l)$, we clearly obtain that $\mathcal{L}(X, I')(\theta)^{\sigma} = d_j(\theta)^{\sigma}(\beta, r)$.

The remaining work is to verify the signs: $\text{sgn}_{Q_T}(X, J')$ consists of three components

$$\text{sgn}_{W'}(X) \, \text{sgn}_{T'}(J')(-1)^{\sigma}$$

where $(-1)^{\sigma} = (-1)^q$ as it corresponds to the shuffle

$$0, q, 0, \ldots, 0, 1, p - 1, 1, \ldots, 1 \quad \rightsquigarrow \quad 0, 1, q, p - 1, 0, 1, \ldots, 0, 1.$$

The sign $\text{sgn}_{W'}(X)$ corresponds to the shuffle transforming the word

$$\underbrace{0_{(j+1)} \ldots 0_3 0_1 1_1}_{(1)} \underbrace{0_2 \ldots (\beta(t+1) - 1)_2 \, 1_3 \beta(t+1)_2 \ldots 1_{(j+1)} \beta(t + j - 1)_2 \ldots (q-1)_2}_{(2)},$$

for $t = q - j + 1$, into the word

$$0_1 \ldots (q_1 - 1)_1 \ldots 0_{(j+1)} \ldots (q_{j+1} - 1)_{(j+1)}.$$

We observe that shuffling the second part (2)

$$\underbrace{0_2 \ldots (\beta(t+1) - 1)_2}_{} \, 1_3 \, \underbrace{\beta(t+1)_2 \ldots 1_{(j+1)}}_{} \underbrace{\beta(t+j-1)_2 \ldots (q-1)_2}_{} \rightsquigarrow \underbrace{0_2 \ldots (q_2 - 1)_2}_{} \, 1_3 \ldots 1_{(j+1)}$$

almost corresponds to the shuffle β. However, there is in every interval $\beta(s)_2, \ldots, (\beta(s+1) - 1)_2$ exactly one element too many. We remedy this by moving $1_{(j-2)}$ one

place to the right, then $1_{(j-3)}$ two place, and so on. As such, its corresponding sign is $(-1)^{\beta+\sum_{i=1}^{j-2}i} = (-1)^{\beta+\frac{(j-2)(j-1)}{2}}$. Next, we shuffle

$$0_{(j+1)}\ldots 0_3 0_1 1_1 0_2 \ldots (q_2-1)_2 1_3 \ldots 1_{(j+1)}$$
$$\rightsquigarrow 0_1 1_1 0_2 \ldots (q-1)_2 0_3 1_3 \ldots 0_{(j+1)} 1_{(j+1)}$$

whose sign is $(-1)^{(j-1)(q+2)}$. Hence, we obtain that

$$\text{sgn}_{W'}(X) = (-1)^{\beta+\frac{(j-1)(j-2)}{2}+(j-1)q}.$$

Next, we determined $\text{sgn}_{T'}(J')$ as the sign of the shuffle

$$[C_J]1_1 1_2 \ldots (p-1)_2 \quad \rightsquigarrow \quad 1_1 1_2 \ldots (p-1)_2 1_3 \ldots 1_{(j+1)},$$

where $[C_J]$ denotes the word obtained from the indexed tree J. We will show that the sign corresponding to the shuffle $\chi : [C_J] \rightsquigarrow 1_3 \ldots 1_{(j+1)}$ is $(-1)^{r+j-1}$. As a consequence, we obtain that

$$\text{sgn}_{T'}(J') = (-1)^{r+j-1+p(j-1)},$$

and thus that

$$\text{sgn}(X, J') = (-1)^{\beta+r+q+j-1}(-1)^{\frac{(j-1)j}{2}+(j-1)(p+q-1)}.$$

Hence, we have

$$\text{sgn}_Q(X, J')(-1)^{\frac{j(j+1)}{2}+1}(-1)^{(j-1)+(j-1)(p+q-1)} = (-1)^{\beta+r+q-j+1}$$

which completes the proof.

We compute χ inductively: we have that $[C_J] = A1_{(j+1)}B$ for certain words A and B and let χ_0 denote the shuffle $AB \rightsquigarrow 1_3 \ldots 1_j$. By induction we know that $(-1)^{\chi_0} = (-1)^{r_0+j-2}$ for the formal path $r_0 = (r_1, \ldots, r_{j-2})$ and $(-1)^r = (-1)^{r_0+i_{j-1}}$ where $r_{j-1} = (\sigma, i_{j-1})$. Moreover, we have $(-1)^{\chi} = (-1)^{\chi_0+|B|}$ where $|B|$ denotes the length of B. We determine $|B|$. First, observe that the sequence associated to two indexed trees

$$3 \overset{1}{\longrightarrow} 4 \quad \text{and} \quad 3 \overset{2}{\longrightarrow} 4$$

is respectively $1_{h4}1_{h3}$ and $1_{h3}1_{h4}$. Thus, let $3 = v_1 <_T \ldots <_T v_t = j+1$ be the unique chain of vertices from the root of T to $j+1$, then we can define the numbers

- l as the the number of vertices to the right of vertex $j+1$ in T,
- k as the number of vertices in the above chain such that $J(v_g, v_{g+1}) = 1$.

We then easily compute that

- The height i_{j-1} of $r_{j-1} = (\partial_{i_{j-1}}\sigma, i_{j-1})$ is exactly $(1 + l) + k$,
- Length of B is $l + k$

and thus by induction we obtain

$$(-1)^\chi = (-1)^{\chi_0 + |B|} = (-1)^{\chi_0 + i_{j-1} + 1} = (-1)^{r + (j-1)}$$

where the last equality follows from induction. $\square$

Remark 4.18 In the case of presheaves, when looking at the subcomplex $\overline{\mathbf{C}}_{GS}(\mathcal{A})$ of normalised and reduced cochains, the map R_c factorises through $\mathrm{Quilt}_b[[m]]$ as m is sent to the identity $1^{u,v} : u^*v^* = (vu)^*$. Only in the case of normalised and reduced cochains does the identity satisfy all the relations on m.

Note however that our induced morphism mQuilt $\longrightarrow$ $\mathrm{Quilt}_b[[m]]$ $\longrightarrow$ $\mathbf{End}(s\overline{\mathbf{C}}_{GS}(\mathcal{A}))$ does not correspond to the mQuilt-algebra structure [8, Thm. 5.6].

However, the resulting L_∞-algebra structure, in the case of presheaves, does correspond to the one obtained from [8, Thm. 5.6, Thm. 7.13]. This is essentially due to the multiplication of the prestack being unital. A mQuilt-algebra structure also induces a Gerstenhaber-algebra structure on cohomology [8, Thm. 6.11]. Writing down the relevant quilts, it is also easy to see that both mQuilt-algebra structures (ours and [8, Thm. 5.6]) induce the same Gerstenhaber-algebra structure on cohomology.

4.5 Deformations of prestacks

Let k be a field of characteristic 0. Let $(\mathbf{C}, d, (L_n)_{n \geq 2})$ be an L_∞-algebra. By definition, the Maurer–Cartan equation for $\theta \in \mathbf{C}$ is given by

$$MC(\theta) = d(\theta) + \sum_{n=1}^{\infty} \frac{1}{n!} L_n(\theta, \dots, \theta).$$

We consider the set of (degree 1) Maurer–Cartan elements $MC(\mathbf{C}) = \{\theta \in \mathbf{C}^1 \mid MC(\theta) = 0\}$ and for the appropriate notion of gauge equivalence (see [9]), we consider

$$\underline{MC}(\mathbf{C}) = \{\theta \in \mathbf{C}^1 \mid MC(\theta) = 0\}/ \sim .$$

This gives rise to a functor $\underline{MC}_{\mathbf{C}} : \mathsf{Art} \longrightarrow \mathsf{Set} : (R, \mathfrak{m}) \longmapsto \underline{MC}(\mathfrak{m} \otimes \mathbf{C})$ on the category Art of Artin local k-algebras.

Consider the GS-complex $(\mathbf{C}_{GS}(\mathcal{A}), d)$ of a prestack $(\mathcal{A}, (m + f + c))$. In [2, Thm. 3.19] it is shown that normalised reduced 2-cocycles in $\mathbf{C}_{GS}(\mathcal{A})$ correspond to first order deformations of $\mathcal{A}$, that is, deformations in the direction of $R = k[\epsilon]$. More precisely, for $(m', f', c') \in \mathbf{C}_{GS}^2(\mathcal{A})$, we have that $(\mathcal{A}[\epsilon], m + m'\epsilon, f + f'\epsilon, c + c'\epsilon)$ is a first-order deformation of $(\mathcal{A}, m, f, c)$ if and only if $d(m', f', c') = 0$ and (m', f', c') is normalised and reduced. Further, it is shown in loc. cit. that the cohomology of the GS-complex classifies the first-order deformations of $\mathcal{A}$ up to equivalence.

Putting Theorems 4.10 and 4.17 together, $s\mathbf{C}_{GS}(\mathcal{A})$ is endowed with an L_∞-structure which can be used to obtain higher order versions of these results.
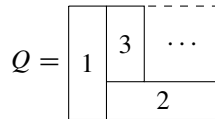
For normalised, reduced cochains, it will be convenient to express the MC-equation in terms of the unsymmetrised components $(P_n)_{n\geq 1}$ from Definition 4.5. The following characteristic-free expression of the MC-equation should be seen as the counterpart of the equation $MC(\theta) = d(\theta) + \theta\{\theta\}$ for the first brace operation (or "dot product") on the Hochschild complex of an algebra. Note that we omit writing R_c and consider everything as elements of $\mathbf{End}(s\mathbf{C}_{GS}(\mathcal{A}))$.

Proposition 4.19 *For a reduced and normalised cochain $\theta = (m', f', c') \in sC_{GS}^1(\mathcal{A})$, we have*

$$MC(\theta) = d(\theta) + P_1(\theta) + P_2(\theta, \theta) + P_3(\theta, \theta, \theta, \theta) + P_4(\theta, \theta, \theta, \theta).$$
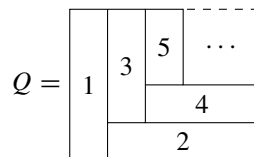
In particular, the MC-equation is quartic.

Proof First note that as $|\theta| = 1$, that $Q^\sigma(\theta, \ldots, \theta) = (-1)^\sigma Q(\theta, \ldots, \theta)$. Moreover, as we can consider $c \in \mathbf{C}^{2,0}(\mathcal{A})$, the MC-equation consists of $d_0(\theta')$ and summations of $Q(\theta_1, \ldots, \theta_n)$ for quilts Q of degree $n - 2$ and $\theta_i \in \mathbf{C}_{GS}^2(\mathcal{A})$. Let $Q = (W, T)$ be such a quilt, part of some P_n^0, then $W = 12 \ldots 2$. For $n \geq 3$, we know that 3 is also a child in T of 1. As 1 can have at most two children in T, for $n \geq 4$, we thus have 4 and 3 are children of 2 in W, i.e. $W = 12324 \ldots 2$.

$$Q = \begin{array}{|c|c|c|}
\hline
 & 3 & \cdots \\
\cline{2-3}
1 & \multicolumn{2}{c|}{} \\
\cline{2-3}
 & \multicolumn{2}{c|}{2} \\
\hline
\end{array}$$

In this case, only the elements c' or c, and m' can be inserted in Q respectively in 1 and 2, with 2 a child of 1 in T. As c' is reduced, this means that $Q(\theta, \ldots, \theta) = 0$ for $n \geq 4$.

As c is not necessarily reduced, more quilts are possible. Note that we write $\theta[i]$ to refer to the element θ inserted in vertex i. The above reasoning still applies and we can once more apply this reasoning to obtain for $n \geq 6$ that $W = 12324546 \ldots 42$ and 1 and 3 have two children in T.

$$Q = \begin{array}{|c|c|c|c|}
\hline
 & & 5 & \cdots \\
 & 3 & \multicolumn{2}{c|}{} \\
1 & & \multicolumn{2}{c|}{4} \\
 & \multicolumn{3}{c|}{2} \\
\hline
\end{array}$$

This means that 2 and 4 need be inserted by m' and hence 3 and 1 by either c' or c. In case either one is c', we already know it is zero as c' is reduced. Thus, consider vertex 1 and c inserted by c, then, as $m'[4]$ is a child of $c[3]$ in T, $c[3]$ is the unit of the corresponding category $\mathcal{A}(U)$ and it is plugged into $m'[2]$ which is normalised, whence we obtain that $Q(\theta_1, \ldots, \theta_n) = 0$ for $n \geq 6$. Hence, $P_n^0(\theta_1, \ldots, \theta_n) = 0$ for $n \geq 6$.

Combining the above reasonings, we have that $P_n(\theta, \ldots, \theta) = 0$ for $n > 5$. In the case $n = 5$ there has to be at least one c present and thus the non-zero terms are contained in P_4. $\qquad\square$

Proposition 4.20 *For a prestack $(\mathcal{A}, m, f, c)$ and $\theta = (m', f', c') \in sC^1_{GS}(\mathcal{A})$, we have that $(\mathcal{A}, m + m', f + f', c + c')$ is a prestack if and only if $MC(\theta) = 0$ and θ is normalised and reduced.*

Proof Using the fact that $\theta = (m', f', c') \in \mathbf{C}^2_{GS}(\mathcal{A})$ is reduced and normalised, we compute $MC(\theta)$ and look at each component $MC(\theta)_{[p,q]} \in \mathbf{C}^{p,q}(\mathcal{A})$ for $p + q = 3$. We will use that $c^{1,-} = 1 = c^{-,1}$, m is unital and that (m, f, c) satisfy the axioms of a prestack. Note that for a cochain $\theta = (\theta^\sigma(A))_{\sigma, A}$, we omit writing the set of objects A explicitly where possible in order to lighten the equations below.

For $(p, q) = (0, 3)$, let $U \in \mathcal{U}$ and $A = (A_0, A_1, A_2, A_3)$ objects in $\mathcal{A}(U)$, then we compute

$$
\begin{aligned}
MC(\theta)^U_{[0,3]}(A) &= d\theta^U_{[0,3]}(A) + P_2(\theta, \theta)^U_{[0,3]}(A) + P_3(\theta, \theta, \theta)^U_{[0,3]}(A) + P_4(\theta, \theta, \theta, \theta)^U_{[0,3]}(A) \\
&= d_0(m') + P_3^0(c, m', m') \\
&= m^U \circ (1^U, m'^U) - m'^U \circ (m^U, 1^U) + m'^U \circ (1^U, m^U) - m^U \circ (m'^U, 1^U) \\
&\quad - m^U \circ (c^{1_U, 1_U}(A_0), m'^U \circ (m'^U, 1^U)) + m^U \circ (c^{1_U, 1_U}(A_0), m'^U \circ (1^U, m'^U)) \\
&= (m^U + m'^U) \circ \left(1^U, (m^U + m'^U)\right) - (m^U + m'^U) \circ \left((m^U + m'^U), 1^U\right)
\end{aligned}
$$

For $(p, q) = (1, 2)$, let $u : U_0 \to U_1$ in $\mathcal{U}$ and $A = (A_0, A_1, A_2)$ objects in $\mathcal{A}(U_1)$, then we compute

$$
\begin{aligned}
d\theta^u_{[1,2]}(A) &= d_0(f')^u(A) + d_1(m')^u(A) \\
&= m^{U_0} \circ (f^u, f'^u) - f'^u \circ m^{U_1} + m^{U_0} \circ (f'^u, f^u) \\
&\quad + m^{U_0} \circ \left(c^{u, 1_{U_0}}(A_0), m'^{U_0} \circ (f^u, f^u)\right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_1}, u}(A_0), f^u \circ m'^{U_1}\right) \\
P_2(\theta, \theta)^u_{[1,2]}(A) &= P_2^0(f', f')^u(A) + P_3^0(c, m', f')^u(A) + P_3^0(c, f', m')^u(A) \\
&= m^{U_0} \circ (f'^u, f'^u) + m^{U_0} \circ \left(c^{u, 1_{U_0}}(A_0), m'^{U_0} \circ (f'^u, f^u)\right) \\
&\quad + m^{U_0} \circ \left(c^{u, 1_{U_0}}(A_0), m'^{U_0}(f^u, f'^u)\right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_1}, u}(A_0), f'^u \circ m'^{U_1}\right) \\
P_3(\theta, \theta, \theta)^u_{[1,2]}(A) &= P_4^0(c, m', f', f')^u(A) \\
&= m^{U_0} \circ \left(c^{u, 1_{U_0}}(A_0), m'^{U_0} \circ (f'^u, f'^u)\right) \\
P_4(\theta, \theta, \theta, \theta)^u_{[1,2]}(A) &= 0
\end{aligned}
$$

and thus

$$MC(\theta)^u_{[1,2]}(A) = -(f^u + f'^u) \circ (m^{U_0} + m'^{U_0}) + (m^{U_0} + m'^{U_0}) \circ \left((f^u + f'^u) \otimes (f^u + f'^u) \right)$$

For $(p, q) = (2, 1)$, let $\sigma = (U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} U_2)$ be a 2-simplex in $\mathcal{U}$ and $A = (A_0, A_1)$ objects in $\mathcal{A}(U_2)$, then we compute

$$
\begin{aligned}
d\theta^\sigma_{[2,1]}(A) &= d_0(c')^\sigma(A) + d_1(f')^\sigma(A) + d_2(m')^\sigma(A) \\
&= m^{U_0} \circ \left(f^{u_2 u_1}, c'^\sigma(A_1) \right) - m^{U_0} \circ \left(c'^\sigma(A_0), f^{u_2} \circ f^{u_2} \right) \\
&\quad - m^{U_0} \circ \left(c^\sigma(A_0), f^{u_1} \circ f'^{u_2} \right) \\
&\quad + m^{U_0} \circ \left(f'^{u_2 u_1}, c^\sigma(A_1) \right) - m^{U_0} \circ \left(c^\sigma(A_0), f'^{u_1} \circ f^{u_2} \right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_0}, u_2 u_1}(A_0), m'^{U_0} \circ \left(c^{u_1, u_2}(A_0), f^{u_2} \circ f^{u_1} \right) \right) \\
&\quad + m^{U_0} \circ \left(c^{1_{U_0}, u_2 u_1}(A_0), m'^{U_0} \circ \left(f^{u_2 u_1}, c^{u_1, u_2}(A_1) \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
P_2(\theta, \theta)^\sigma_{[2,1]}(A) &= P_2^0(f', c')^\sigma(A) + P_2^0(c', f')^\sigma(A) \\
&\quad + P_3^0(c, f', f')^\sigma(A) + P_3^0(c, m', c')^\sigma(A) \\
&\quad + P_4^0(c, m', f', c)^\sigma(A) + P_4^0(c, m', c, f')^\sigma(A) \\
&= m^{U_0} \circ \left(f'^{u_2 u_1}, c'^{u_1, u_2}(A_1) \right) - m^{U_0} \circ \left(c'^{u_1, u_2}(A_0), f'^{u_1} \circ f^{u_2} \right) \\
&\quad - m^{U_0} \circ \left(c'^{u_1, u_2}(A_0), f^{u_1} \circ f'^{u_2} \right) \\
&\quad - m^{U_0} \circ \left(c^{u_1, u_2}(A_0), f'^{u_1} \circ f'^{u_2} \right) \\
&\quad + m^{U_0} \circ \left(c^{1_{U_0}, u_2 u_1}(A_0), m'^{U_0} \circ \left(f^{u_2 u_1}, c'^{u_1, u_2}(A_1) \right) \right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_0}, u_2 u_1}(A_0), m'^{U_0} \circ \left(c'^{u_1, u_2}(A_0), f^{u_1} \circ f^{u_2} \right) \right) \\
&\quad + m^{U_0} \circ \left(c^{1_{U_0}, u_2 u_1}(A_0), m'^{U_0} \circ \left(f'^{u_2 u_1}, c^{u_1, u_2}(A_1) \right) \right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_0}, u_2 u_1}(A_0), m'^{U_0} \circ \left(c^{u_1, u_2}(A_0), f^{u_1} \circ f'^{u_2} \right) \right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_0}, u_2 u_1}(A_0), m'^{U_0} \circ \left(c^{u_1, u_2}(A_0), f'^{u_1} \circ f^{u_2} \right) \right)
\end{aligned}
$$

$$P_3(\theta, \theta, \theta)^\sigma_{[2,1]}(A) = P_3^0(c', f', f')^\sigma(A) + P_4^0(c, m', f', c')^\sigma(A)$$
$$+ P_4^0(c, m', c', f')^\sigma(A) + P_5^0(c, m', c, f', f')^\sigma(A)$$
$$= -m^{U_0} \circ \left(c'^{u_1, u_2}(A_0), f'^{u_1} \circ f'^{u_2} \right)$$
$$+ m^{U_0} \circ \left(c^{1U_0, u_2 u_1}(A_0), m'^{U_0} \circ \left(f'^{u_2 u_1}, c'^{u_1, u_2}(A_1) \right) \right)$$
$$- m^{U_0} \circ \left(c^{1U_0, u_2 u_1}(A_0), m'^{U_0} \circ \left(c'^{u_1, u_2}(A_0), f'^{u_1} \circ f^{u_2} \right) \right)$$
$$- m^{U_0} \circ \left(c^{1U_0, u_2 u_1}(A_0), m'^{U_0} \circ \left(c'^{u_1, u_2}(A_0), f^{u_1} \circ f'^{u_2} \right) \right)$$
$$- m^{U_0} \circ \left(c^{1U_0, u_2 u_1}(A_0), m'^{U_0} \circ \left(c^{u_1, u_2}(A_0), f'^{u_1} \circ f'^{u_2} \right) \right)$$

$$P_4(\theta, \theta, \theta, \theta)^\sigma_{[2,1]}(A) = P_5^0(c, m', c', f', f')^\sigma(A)$$
$$= -m^{U_0} \circ \left(c^{1U_0, u_2 u_1}(A_0), m'^{U_0} \circ \left(c'^{u_1, u_2}(A_0), f'^{u_1} \circ f'^{u_2} \right) \right)$$

and thus

$$MC(\theta)^\sigma_{[2,1]}(A) = \left(m^{U_0} + m'^{U_0} \right) \circ \left(\left(f^{u_2 u_1} + f'^{u_2 u_1} \right), \left(c^{u_1, u_2}(A_0) + c'^{u_1, u_2}(A_0) \right) \right)$$
$$- \left(m^{U_0} + m'^{U_0} \right) \circ \left(\left(c^{u_1, u_2}(A_0) + c'^{u_1, u_2}(A_0) \right), \right.$$
$$\left. \left(f^{u_1} + f'^{u_1} \right) \circ \left(f^{u_2} + f'^{u_2} \right) \right)$$

Lastly, for $(p, q) = (3, 0)$, let $\sigma = (U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} U_2 \xrightarrow{u_3} U_3)$ be a 3-simplex in $\mathcal{U}$ and A_0 an object in $\mathcal{A}(U_2)$, then we compute

$$d\theta^\sigma_{[3,0]}(A_0) = d_3(m')^\sigma(A_0) + d_2(f')^\sigma(A_0) + d_1(c')^\sigma(A_0)$$
$$= m^{U_0} \circ \left(c^{1U_0, u_3 u_2 u_1}(A_0), m'^{U_0} \circ \left(c^{u_2 u_1, u_3}(A_0), c^{u_1, u_2}(u_3^* A_0) \right) \right)$$
$$- m^{U_0} \circ \left(c^{u_1, u_3 u_2}(A_0), f'^{u_1} \circ c^{u_2, u_3}(A_0) \right)$$
$$- m^{U_0} \circ \left(c^{u_1, u_3 u_2}(A_0), f^{u_1} \circ c'^{u_2, u_3}(A_0) \right)$$
$$+ m^{U_0} \circ \left(c^{u_1, u_3 u_2}(A_0), c'^{u_1, u_2}(A_0) \right)$$
$$+ m^{U_0} \circ \left(c'^{u_2 u_1, u_3}(A_0), c^{u_1, u_2}(u_3^* A_0) \right)$$
$$- m^{U_0} \circ \left(c^{u_1, u_3 u_2}(A_0), f^{u_1} \circ c'^{u_2, u_3}(A_0) \right)$$

$$P_2(\theta, \theta)^\sigma_{[3,0]}(A_0) = P_2^0(c', c')^\sigma(A_0) + P_3^0(c, f', c')^\sigma(A_0)$$
$$+ P_3^0(c', f', c)^\sigma(A_0) + P_4^0(c, m', c', c)^\sigma(A_0)$$
$$+ P_4^0(c, m', c, c')^\sigma(A_0) + P_5^0(c, m', c, f', c)^\sigma(A_0)$$
$$= m^{U_0} \circ \left(c'^{u_2 u_1, u_3}(A_0), c'^{u_1, u_2}(u_3^* A_0) \right)$$

$$- m^{U_0} \circ \left(c'^{u_1,u_2}(A_0),\ f^{u_1} \circ c'^{u_2,u_3}(A_0) \right)$$

$$- m^{U_0} \circ \left(c^{u_1,u_2}(A_0),\ f'^{u_1} \circ c'^{u_2,u_3}(A_0) \right)$$

$$- m^{U_0} \circ \left(c'^{u_1,u_2}(A_0),\ f'^{u_1} \circ c^{u_2,u_3}(A_0) \right)$$

$$+ m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c'^{u_2u_1,u_3}(A_0),\ c^{u_1,u_2}(u_3^* A_0) \right) \right)$$

$$- m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c'^{u_1,u_3u_2}(A_0),\ f^{u_1} \circ c^{u_2,u_3}(A_0) \right) \right)$$

$$+ m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c^{u_2u_1,u_3}(A_0),\ c'^{u_1,u_2}(u_3^* A_0) \right) \right)$$

$$- m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c^{u_1,u_3u_2}(A_0),\ f^{u_1} \circ c'^{u_2,u_3}(A_0) \right) \right)$$

$$- m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c^{u_1,u_3u_2}(A_0),\ f^{u_1} \circ c'^{u_2,u_3}(A_0) \right) \right)$$

$$- m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c^{u_1,u_3u_2}(A_0),\ f'^{u_1} \circ c^{u_2,u_3}(A_0) \right) \right)$$

$$
\begin{aligned}
P_3(\theta,\theta,\theta)_{[3,0]} &= P_3^0(c',f',c')^\sigma(A_0) + P_4^0(c,m',c',c')^\sigma(A_0) \\
&\quad + P_5^0(c,m',c',f',c)^\sigma(A_0) + P_5^0(c,m',c,f',c')^\sigma(A_0) \\
&= -m^{U_0} \circ \left(c'^{u_1,u_3u_2}(A_0),\ f'^{u_1} \circ c'^{u_2,u_2}(A_0) \right) \\
&\quad + m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c'^{u_2u_1,u_3}(A_0),\ c'^{u_1,u_2}(u_3^* A_0) \right) \right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c'^{u_1,u_3u_2}(A_0),\ f^{u_1} \circ c'^{u_2,u_3}(A_0) \right) \right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c'^{u_1,u_3u_2}(A_0),\ f'^{u_1} \circ c^{u_2,u_3}(A_0) \right) \right) \\
&\quad - m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c^{u_1,u_3u_2}(A_0),\ f'^{u_1} \circ c'^{u_2,u_3}(A_0) \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
P_4(\theta,\theta,\theta,\theta)_{[3,0]}^\sigma(A_0) &= P_5^0(c,m',c',f',c')^\sigma(A_0) \\
&= -m^{U_0} \circ \left(c^{1_{U_0},u_3u_2u_1}(A_0),\ m'^{U_0} \circ \left(c^{u_1,u_3u_2}(A_0),\ f'^{u_1} \circ c'^{u_2,u_3}(A_0) \right) \right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
MC(\theta)_{[3,0]}^\sigma(A_0) &= \left(m^{U_0} + m'^{U_0} \right) \circ \left(\left(c^{u_2u_1,u_3}(A_0) + c'^{u_2u_1,u_3}(A_0) \right),\right. \\
&\qquad \left(c^{u_1,u_2}(u_3^* A_0) + c'^{u_1,u_2}(u_3^* A_0) \right) \Big) \\
&\quad - \left(m^{U_0} + m'^{U_0} \right) \circ \left(\left(c^{u_1,u_3u_2}(A_0) + c'^{u_1,u_3u_2}(A_0) \right),\right. \\
&\qquad \left(f^{u_1} + f'^{u_2} \right) \circ \left(c^{u_2,u_3}(A_0) + c'^{u_2,u_3}(A_0) \right) \Big)
\end{aligned}
$$

These computations show that $MC(\theta) = 0$ if and only if $(\mathcal{A}, m+m', f+f', c+c')$ is a prestack. $\qquad\square$

Remark 4.21 Note that for the functor condition we only need the cubic part and for the twists the full quartic part of the equation.

Based upon Proposition 4.20, with some more work taking gauge equivalence into account, one may extend [2, Thm. 3.19] to higher order deformations. Recall that for $(R, \mathfrak{m}) \in$ Art an R-deformation of a k-linear prestack $(\mathcal{A}, m, f, c)$ is an R-linear prestack $(R \otimes_k \mathcal{A}, \bar{m}, \bar{f}, \bar{c})$ of which the algebraic structure reduces to that of $\mathcal{A}$ modulo $\mathfrak{m}$, and an equivalence of deformations is an isomorphism between the deformed prestacks which reduces to the identity morphism. Let $\mathrm{Def}_{\mathcal{A}} : \mathrm{Art} \longrightarrow \mathsf{Set}$ be the deformation functor of $\mathcal{A}$ with $\mathrm{Def}_{\mathcal{A}}(R, \mathfrak{m})$ the set of R-deformations of $\mathcal{A}$ up to equivalence of deformations. The following theorem, of which the proof will appear elsewhere, expresses that the deformation theory of $\mathcal{A}$ is controlled by the L_∞-algebra $s\mathbf{C}_{GS}(\mathcal{A})$.

Theorem 4.22 *Let $(\mathcal{A}, m, f, c)$ be a prestack. There is a natural isomorphism of functors* $\mathrm{Art} \longrightarrow \mathsf{Set}$:

$$\mathrm{Def}_{\mathcal{A}} \cong \underline{MC}_{sC_{GS}(\mathcal{A})}.$$

Appendix A: Generator-free description of the morphism NSOp ⟶ MultiΔ

In this appendix, we provide a generator-free description of the morphism NSOp $\longrightarrow$ MultiΔ from Lemma 2.5. Although per construction we have a morphism of operads induced from the generators E_i of NSOp, we consider it valuable in practice to have an explicit definition.

For I an indexed tree in $\mathrm{NSOp}(q_1, \ldots, q_n; \sum_{i=1}^n q_i - n + 1)$, we call $l_I = \sum_{i=1}^n q_i - n + 1$ the *number of leaves* of I. Given a number $m \in \mathbb{N}$, we call the interval

$$[m, m + l_I] = \{m, m + 1, \ldots, m + l_I\}$$

the *numbering set of leaves* of I.

Construction A.1 For I an indexed tree in $\mathrm{NSOp}(q_1, \ldots, q_n; \sum_{i=1}^n q_i - n + 1)$, let I_j be the maximal subtree of I with root j. Let u be the root of I with children

$u_1 \lhd_I \ldots \lhd_I u_k$ which have index $i_j := I(u, u_j)$, then I decomposes as follows

$$I = \begin{array}{ccc} I_{u_1} & \cdots & I_{u_k} \\ {}_{i_1}\diagdown & \big| & \diagup_{i_k} \\ & u & \end{array}$$

Given a number $m \in \mathbb{N}$, we define a non-decreasing map $\zeta(I, m) : [q_u] \longrightarrow [m, m + l_I]$ as follows

$$\zeta(I, m)(t) = \begin{cases} m + t & \text{for } 0 \le t < i_1 \\ m + \sum_{j=1}^{s} l_{I_j} + t & \text{for } i_s \le t < i_{s+1} \\ m + \sum_{j=1}^{k} l_{I_j} + t & \text{for } i_s \le t \le q_u \end{cases}$$

which determines where the leaves of the root of I are placed in its numbering set of leaves $[m, m + l_I]$.

In order to define the tuple $\zeta_I = (\zeta_{I,1}, \ldots, \zeta_{I,n}) \in \mathrm{Multi}\Delta(q_1, \ldots, q_n; \sum_{i=1}^{n} q_i - n + 1)$, we run inductively through the tree I from root u to leaves setting

$$\zeta_{I,u} := \zeta(I, 0) : [q_u] \longrightarrow [l_I]$$

and for vertex a with child b and index $i := I(a, b)$ we set

$$\zeta_{I,b} := \zeta(I_b, \zeta_{I,a}(i-1)) : [q_b] \longrightarrow [\zeta_{I,a}(i-1), \zeta_{I,a}(i-1) + l_{I_b}] \hookrightarrow [l_I]$$

with $l_I = \sum_{i=1}^{n} q_i - n + 1$.

Note that Construction 2.5 corresponds to the above construction applied to the generators E_i. As such, if the generator-free description defines a morphism of operads, they coincide.

Proposition A.2 *Construction A.1 defines a morphism of operads* NSOp $\longrightarrow$ *MultiΔ.*

Proof Let $I \in \mathrm{NSOp}(q_1, \ldots, q_n; q)$ and $I' \in \mathrm{NSOp}(q'_1, \ldots, q'_m; q')$ and consider their composition $I'' := I \circ_i I'$. We will show that

$$\zeta_{I''} = \zeta_I \circ_i \zeta_{I'}$$

As in Construction A.1, let I_j, I'_j and I''_j be the maximal subtrees of respectively I, I' and I'' with root j. Due to equivariance, we can assume that a vertex j belongs to the subtree I_i iff $j \ge i$. We then have three cases to consider which we depict
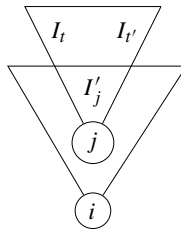
diagrammatically as follows
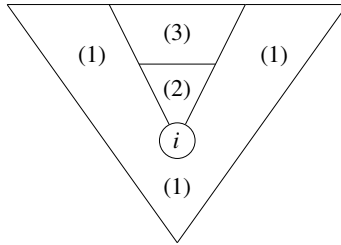
(1) For $j < i$, if we contract the subtree I_i and I_i'' to a single vertex with number of leaves $l_{I_i} = l_{I_i''}$, we obtain the same indexed tree $I \setminus I_i = I'' \setminus I_i''$ in which I_j and I_j'' coincide. Thus, it is easy to see that

$$\zeta_{I'',j} = \zeta_{I,j}$$

(2) If $j \geq i$ and j lies in the image $\mathrm{Im}(I')$ of I' in I'', then I_j'' consists of the subtree I_{j-i+1}' with a sequence of subtrees $I_t, \ldots, I_{t'}$ placed on top.

In this case, $\zeta_{I',j-i+1}$ determines where the leaves of j are placed in I' and $\zeta_{I,i}$ determines where the leaves of i in I are placed. As a result, we see that
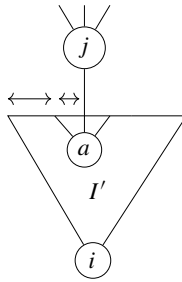
$$\zeta_{I'',j} = \zeta_{I,i} \circ \zeta_{I',j-i+1}$$

which determines where the leaves of j are put in I''.

(3) If $j \geq i$ and j does not lie in the image of I' in I'', then $j - m + 1$ lies in I above i. In this case, the subtrees I_{j-m+1} and I_j'' are equal. If the parent a of j in I'' does not lie in $\mathrm{Im}(I')$, then we clearly have

$$\zeta_{I'',j} = \zeta_{I,j-m+1}.$$

If a lies in $\mathrm{Im}(I')$, then $\zeta_{I'',a} = \zeta_{I,i} \circ \zeta_{I',a-i+1}$ due to the previous case.

Moreover, the index $I''(a, j)$ then equals $I(i, j) - \zeta_{I',a-i+1}(0)$. Hence, we have that

$$\zeta_{I'',j} = \zeta_{I,j-m+1}$$

$\square$

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