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The dimension of the image of the Abel map associated with normal surface singularities

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Abstract

Let (X, o) be a complex normal surface singularity with rational homology sphere link and let $\widetilde{X}$ be one of its good resolutions. Consider an effective cycle Z supported on the exceptional curve and the isomorphism classes $\mathrm{Pic}(Z)$ of line bundles on Z. The set of possible values $h^1(Z, \mathcal{L})$ for $\mathcal{L} \in \mathrm{Pic}(Z)$ can be understood in terms of the dimensions of the images of the Abel maps, as subspaces of $\mathrm{Pic}(Z)$. In this note we present two algorithms, which provide these dimensions. Usually, the dimension of $\mathrm{Pic}(Z)$ and of the dimension of the image of the Abel maps are not topological. However, we provide combinatorial formulae for them in terms of the resolution graph whenever the analytic structure on $\widetilde{X}$ is generic.

Keywords Normal surface singularities · Links of singularities · Plumbing graphs · Rational homology spheres · Abel map · Effective Cartier divisors · Picard group · Brill–Noether theory · Laufer duality · Surgery formulae · Superisolated singularities · Cohomology of line bundles

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1 Introduction

1.1. Fix a complex normal surface singularity (X, o) and let $\widetilde{X}$ be one of its good resolutions. We assume that the link of (X, o) is a rational homology sphere. Before we start a more technical introduction, let us give some motivation for the objectives of the note.

A very important set of analytic invariants of the singularity is provided by the dimensions $h^1(\widetilde{X}, \mathcal{L})$ of the cohomology groups, where $\mathcal{L}$ is a line bundle on $\widetilde{X}$. In general, it is convenient to fix the Chern class l' of the line bundle. Hence, we search for the set of integers $\{h^1(\widetilde{X}, \mathcal{L})\}_{\mathcal{L}\in \mathrm{Pic}^{l'}(\widetilde{X})}$. Note that this problem is the analogue of the famous Brill–Noether problem for curves, which aims to determine the list $\{h^1(C, \mathcal{L})\}$, where C is a fixed smooth curve and $\mathcal{L}$ is a line bundle on C with a given degree. The possible h^1-values provide a stratification of the affine space $\mathrm{Pic}^{l'}(\widetilde{X})$. This provides a very deep information about the analytic type of (X, o), and it is very hard to compute even for very specific analytic structures. In some sense, the final aim of the series of articles of the authors (regarding the Abel maps) is exactly the understanding of this stratification.

In the case of curves, a major tool is the introduction of the effective divisors and the corresponding Abel map, which associates to a divisor the corresponding line bundle. In the present situation, in the case of $\widetilde{X}$, when we run this program we face several obstructions. Firstly, $\widetilde{X}$ is not compact, and the space of divisors (even if we fix their Chern class) is an infinite dimensional space. Therefore, we need to use a 'truncation': we replace $\widetilde{X}$ by an effective cycle Z supported on the exceptional curve, and we consider the space $\mathrm{Eca}^{l'}(Z)$ of effective divisors on the scheme Z and the corresponding Abel map $c^{l'}(Z) : \mathrm{Eca}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$. Note that for $Z \gg 0$ we have an isomorphism $h^1(\widetilde{X}, \mathcal{L}) = h^1(Z, \mathcal{L}|_Z)$ for any $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$, hence we loose no information from the original problem.

A second difficulty appears from the fact that the spaces $\mathrm{Eca}^{l'}(Z)$ and $\mathrm{Pic}^{l'}(Z)$ are not compact and $c^{l'}(Z)$ is not proper.

However, and this is one of the main points of the construction, the fiber structure of the Abel map carries a key information about the needed cohomology ranks: $h^1(Z, \mathcal{L})$ can be related with the dimension of the fiber $(c^{l'}(Z))^{-1}(\mathcal{L})$, and the dimension of the image of $c^{l'}(Z)$ with $h^1(Z, \mathcal{L}^{im}_{gen})$, where $\mathcal{L}^{im}_{gen}$ is a generic element of the image of $c^{l'}(Z)$. Even more, if $\mathcal{L}$ has fixed components (i.e. it is not in the image of the corresponding Abel map), then modifying the Chern class by the cycle of fixed components, $\mathcal{L}$ will be in the image of the new Abel map. Hence analysing all the Abel maps and the dimensions of their images provides a valuable information about the cohomology of line bundles (in particular, by Riemann Roch, about the ranks $h^0(Z, \mathcal{L})$ too, which are important in the study of different linear system and maps).

1.2. Now, we start a more concrete introduction and the presentation of the new results.

Denote by L the lattice $H_2(\widetilde{X}, \mathbb{Z})$ endowed with its negative definite intersection form, by L' its dual lattice $H^2(\widetilde{X}, \mathbb{Z})$ identified with $\{l' \in L \otimes \mathbb{Q} : (l', L) \in \mathbb{Z}\}$, and by $\mathcal{S}' \subset L'$ the Lipman cone of antinef cycles $\{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\}$. The irreducible exceptional curves are denoted by $\{E_v\}_{v\in\mathcal{V}}$, their duals in L' by $\{E^*_v\}_{v\in\mathcal{V}}$, $E := \cup_v E_v$. (For details see Sect. 2).

In [14] for any effective cycle $Z \geq E$ and Chern class $l' \in -\mathcal{S}'$ the authors introduced (based on [4, 6, 7]) and investigated the set of effective Cartier divisors $\mathrm{ECa}^{l'}(Z)$ and the corresponding Abel maps $c^{l'}(Z) : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$. Here $\mathrm{Pic}^{l'}(Z)$ is the affine subspace of the Picard group of line bundles over Z with Chern class l'. The image of the Abel map consists of line bundles without fixed components. [14] and follow-up articles contain several properties of the Abel map, e.g. characterisation when it is dominant, or its relationship with cohomological properties of line bundles. See [15, 16] for the study in the case of generic and elliptic singularities. In all these investigations results regarding the dimension of the image $\mathrm{Im}(c^{l'}(Z))$ have crucial role. The main goal of the present article is the computation of $\dim \mathrm{Im}(c^{l'}(Z))$ in the most general case and the deduction of several new consequences. We consider these as necessary steps towards a long-term final goal: the development of the Brill–Noether theory of normal surface singularities.

Though the dimension (l', Z) (and the homotopy type) of the connected complex manifold $\mathrm{ECa}^{l'}(Z)$ is topological, (i.e. it depends only on the link, or on the lattice L), cf. [14], the dimension $h^1(\mathcal{O}_Z)$ of the target affine space $\mathrm{Pic}^{l'}(Z)$ depends essentially on the analytic structure: if we fix the topological type (and Z), the cohomology group $H^1(\mathcal{O}_Z)$ usually depends on the chosen analytic structure supported by the fixed topological type. The same is true for both $\dim \mathrm{Im}(c^{l'}(Z))$ and $\mathrm{codim}\, \mathrm{Im}(c^{l'}(Z))$. Though (surprisingly) there is a topological characterisation of those cases when $c^{l'}(Z)$ is dominant, oppositely, the cases when (e.g.) $c^{l'}(Z))$ is a point, or it is a hypersurface, have no such topological characterisations. In particular, both integers $\dim \mathrm{Im}(c^{l'}(Z))$ and $\mathrm{codim}\, \mathrm{Im}(c^{l'}(Z))$ are subtle analytic invariants. In fact, it turns out that $\mathrm{codim}\, \mathrm{Im}(c^{l'}(Z))$ equals $h^1(Z, \mathcal{L}^{im}_{gen})$, where $\mathcal{L}^{im}_{gen}$ is a generic line bundle from $\mathrm{Im}(c^{l'}(Z))$. For more about such general facts regarding the Abel maps (and also about several concrete examples) see [14–16].

Maybe it is worth to emphasize that in the case of the Abel map associated with a smooth projective curve the dimension of the image is immediate (for this classical case consult e.g. [1, 3]). This (and almost any other comparison) shows the huge technical differences between the classical smooth curve cases and our situation (which, basically, is the Brill–Noether theory of a non-reduced exceptional curve supported by the exceptional set of a surface singularity resolution).

1.3. The algorithms In the body of the article we present two inductive algorithm for the computation of $d_Z(l') := \dim \mathrm{Im}(c^{l'}(Z))$. The induction follows a sequential blow up procedure starting from the resolution $\widetilde{X}$. Write $-l' = \sum_{v \in \mathcal{V}} a_v E^*_v \in \mathcal{S}' \setminus \{0\}$ (hence each $a_v \in \mathbb{Z}_{\geq 0}$). Then, for every $v \in \mathcal{V}$ with $a_v > 0$ we fix a_v generic points on E_v, say p_{v,k_v}, $1 \leq k_v \leq a_v$. Starting from each p_{v,k_v} we consider a sequence of blowing ups: first we blow up p_{v,k_v} and we create the exceptional curve $F_{v,k_v,1}$, then we blow up a generic point of $F_{v,k_v,1}$ and we create $F_{v,k_v,2}$, and we do this, say, $\mathbf{s}_{v,k_v}$ times (an exact bound is given in Sect. 3). We proceed in this way with all points p_{v,k_v}, hence we get $\sum_v a_v$ chains of modifications. Hence, a set of integers $\mathbf{s} = \{\mathbf{s}_{v,k_v}\}_{v \in \mathcal{V},\ 1 \leq k_v \leq a_v}$ provides a modification $\pi_{\mathbf{s}} : \widetilde{X}_{\mathbf{s}} \to \widetilde{X}$. In $\widetilde{X}_{\mathbf{s}}$ we find the exceptional curves

$$\Big(\bigcup_{v\in\mathcal{V}} E_v\Big)\cup\bigcup_{v,k_v}\ \bigcup_{1\le t\le \mathbf{s}_{v,k_v}} F_{v,k_v,t}.$$

At each level $\mathbf{s}$ we set $Z_{\mathbf{s}} := \pi^*_{\mathbf{s}}(Z)$ and $-l'_{\mathbf{s}} := \sum_{v,k_v} F^*_{v,k_v,\mathbf{s}_{v,k_v}}$ (in $L'(\widetilde{X}_{\mathbf{s}})$, where $F_{v,k_v,0} = E_v$). We also write $d_{\mathbf{s}} := \dim \mathrm{Im}(c^{l'_{\mathbf{s}}}(Z_{\mathbf{s}}))$. Note that $d_{\mathbf{0}} = d_Z(l')$, and it turns out that $d_{\mathbf{s}} = 0$ whenever the entries of $\mathbf{s}$ are large enough. (Sometimes we abridge the pair (v, k_v) by (v, k).)

In order to run an induction, for any $\mathbf{s}$ and (v, k) let $\mathbf{s}^{v,k}$ denote that tuple which is obtained from $\mathbf{s}$ by increasing $\mathbf{s}_{v,k}$ by one. The inductive algorithm compares $d_{\mathbf{s}}$ with all possible $d_{\mathbf{s}^{v,k}}$.

Using the fact (cf. the proof of Theorem 8.2.1) that $\mathrm{ECa}^{l'_{\mathbf{s}^{v,k}}}(Z_{\mathbf{s}^{v,k}})$ is birational with a codimension one subspace of $\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$, we obtain

$$d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0, 1\}. \tag{1.3.1}$$

A very subtle part of the theory is to identify all those pairs $(\mathbf{s}, \mathbf{s}^{v,k})$, where the gaps/jumps occur (that is, when the difference in (1.3.1) is 0 or 1). The identification of such places carries a deep analytic content (and even if in some cases it can be characterised topologically—e.g., in the case of a generic analytic structure—it might be guided by rather complicated combinatorial patterns).

Example 1.3.2 To create a good intuition for such a phenomenon, let us recall the classical case of Weierstrass points. Let C be a smooth projective complex curve of genus g and let us fix a point $p \in C$. For any $s \in \mathbb{Z}_{\ge 0}$ consider $\ell(s) := h^0(C, \mathcal{O}_C(sp))$. Then $\ell(0) = 1$ and $\ell(2g-1+k) = g+k$ for $k \ge 0$. Moreover, $\ell(s)-\ell(s-1) \in \{0, 1\}$ for any $s \ge 0$. Those s values when this difference is 0 are called the gaps, there are g of them. For a generic point the gaps are $\{1, 2, \ldots, g\}$, otherwise p is called a Weierstrass point. For Weierstrass point the set of gaps might depend on the choice of p and on the analytic structure of C. The characterization of all possible gap-sets is still unsettled.

In order to characterize completely our gaps/jump places, we will use *test functions*. For such a test function, say $\tau_{\mathbf{s}}$, we will require the following properties. Firstly, it is a function $\mathbf{s} \mapsto \tau_{\mathbf{s}} \in \mathbb{Z}_{\ge 0}$, such that $d_{\mathbf{s}} \le \tau_{\mathbf{s}}$ for any $\mathbf{s}$. Usually, $\tau_{\mathbf{s}}$ is defined by a weaker (more robust) geometric construction, which approximates/bounds $\mathrm{Im}(c^{l'}(Z))$, and which hopefully is easier to compute. Secondly, $\tau_{\mathbf{s}}$ satisfies the following remarkable *testing property* formulated by the next pattern theorem.

Pattern Theorem. *The sequence of integers $d_{\mathbf{s}}$ are determined inductively as follows:*

(1) $d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0, 1\}$ *(cf.* (1.3.1)),
(2) *if for some fixed* $\mathbf{s}$ *the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathbf{s}} = \max_{v,k}\{d_{\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value* $d_{\mathbf{s}^{v,k}}$ *equals* $\tau_{\mathbf{s}}$, *then* $d_{\mathbf{s}} = \tau_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$; *otherwise* $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}} + 1$.

More precisely, we wish to determine from the collection $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ the term $d_{\mathbf{s}}$ (as a decreasing induction). Using *(1)* this is ambiguous only if all this numbers are the same, say d. In this case $d_{\mathbf{s}}$ can be d or $d+1$. Well, if the inequality (†) $d_{\mathbf{s}} \le \tau_{\mathbf{s}}$ is not obstructed by the choice of $d_{\mathbf{s}} = d + 1$, then this value is taken. Otherwise it is d. That is, $d_{\mathbf{s}}$ is as large as it can be, modulo *(1)* and (†).

This can be an interesting procedure even if s is a 1-entry parameter. E.g., in the case of classical Weierstrass points, the inequality $\ell(s) \leq 1 + \lfloor s/2 \rfloor$ (valid for $s \leq 2g-1$), given by Clifford's theorem, by this 'maximal-testing procedure' gives the sequence $\{1, 1, 2, 2, \dots\}$ for $s \geq 0$, with gaps $\{1, 3, \dots, 2g-1\}$. In fact, in the case of hyperelliptic curves the Weierstrass points are the branch points of the hyperelliptic projection and their gap-set is uniformly $\{1, 3, 5, \dots, 2g-1\}$. (However, for non-hyperelliptic curves we are not aware of the existence of a non-trivial test function.)

If the Pattern Theorem from above holds, then it turns out (see e.g. Corollary 3.2.4) that $d_{\mathbf{s}} = \min_{\mathbf{s} \leq \widetilde{\mathbf{s}}} \{|\widetilde{\mathbf{s}} - \mathbf{s}| + \tau_{\widetilde{\mathbf{s}}}\}$ for any $\mathbf{s}$. (Here $|\mathbf{s}| = \sum_{v,k} s_{v,k_v}$.) In particular,

$$d_Z(l') = d_{\mathbf{0}} = \min_{\mathbf{0} \leq \mathbf{s}} \{|\mathbf{s}| + \tau_{\mathbf{s}}\}. \tag{1.3.3}$$

Such type of formulas already appeared in the computation of $d_Z(l')$ for weighted homogeneous singularities (and specific l') in [14], case which lead us to the present general case. (The type of formula, and also the conceptual approach behind, can also be compared e.g. with Pflueger's formula regarding the dimension of the Brill–Noether varieties of a generic smooth projective curve C with fixed gonality, cf. [5, 24].) Nevertheless, the approach of the testing function (and the corresponding min-type close formulae) is the novelty of the present manuscript.

1.4. The testing functions for $\mathbf{d_s}$ Obviously, the above theorem is valuable only if $\tau_{\mathbf{s}}$ is essentially different than $d_{\mathbf{s}}$ and also if it is computable from other different geometrical data. It is also clear that not any upper bound $d_{\mathbf{s}} \leq \tau_{\mathbf{s}}$ satisfies the testing property *(2)*: this is satisfied only for bounds $\tau(\mathbf{s})$ with very structural relationship, symbiosis with the original $d_{\mathbf{s}}$. Hence it is not easy to find testing functions, they must 'testify' about some deep geometric property: even the existence of computable testing function(s) is really remarkable.

Our first test function is defined as follows. Consider again $Z \geq E$, $l' \in -\mathcal{S}'$ associated with a resolution $\widetilde{X}$, as above. Then, besides the Abel map $c^{l'}(Z)$ one can consider its 'multiples' $\{c^{nl'}(Z)\}_{n\geq 1}$. It turns out that $n \mapsto \dim \mathrm{Im}(c^{nl'}(Z))$ is a non-decreasing sequence, $\mathrm{Im}(c^{nl'}(Z))$ is an affine subspace for $n \gg 1$, whose dimension $e_Z(l')$ is independent of $n \gg 0$, and essentially it depends only on the E^*-support of l' (i.e., on $I \subset \mathcal{V}$, where $-l' = \sum_{v\in I} a_v E^*_v$ with all $\{a_v\}_{v\in I}$ nonzero). From construction $d_Z(l') \leq e_Z(l')$, however they usually are not the same. Furthermore, $e_Z(l') = e_Z(I)$ plays a crucial role in different analytic properties of $\widetilde{X}$ (surgery formula, $h^1(\mathcal{L})$-computations, base point freeness properties). For details see [14] or Sects. 2 and 2 here, especially Definition 3.1.1 and Theorem 2.2.5 (and also the proof of Theorem 3.2.2). Now, at any step of the tower $\widetilde{X}_{\mathbf{s}}$ one can consider this invariant $e_{Z_{\mathbf{s}}}(l'_{\mathbf{s}})$, an integer denoted by $e_{\mathbf{s}}$.

Theorem 3.2.2 (the 'first algorithm') guarantees that $e_{\mathbf{s}}$ is a testing function for $d_{\mathbf{s}}$.

The invariants $\{e_{\mathbf{s}}\}_{\mathbf{s}}$ are still hard to compute (cf. 4). However, the first algorithm is a necessary intermediate step for the second algorithm, valid for another testing function.

The advantage of the second testing function is that it is defined at the level of $\widetilde{X}$ only. It is based on Laufer's perfect pairing $H^1(\mathcal{O}_Z)\otimes\mathcal{G}_Z\to\mathbb{C}$, where $\mathcal{G}_Z$ denoted the space of classes of forms $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$. The vector space $\mathcal{G}_Z$ has a natural divisorial filtration $\{\mathcal{G}_l\}_{0\leq l\leq Z}$, where $\mathcal{G}_l$ is generated by forms with pole $\leq l$. Its dimension (via Laufer duality) is $h^1(\mathcal{O}_l)$. (For more see [14] and 2 here.) Next, for any $\mathbf{s}$ define the cycle $l_{\mathbf{s}}\in L$ of $\widetilde{X}$ by

$$l_{\mathbf{s}}:=\min\Big\{\sum_{v\in\mathcal{V}}\min_{1\leq k_v\leq a_v}\{\mathbf{s}_{v,k_v}\}E_v,\,Z\Big\}\in L.$$

Set also $g_{\mathbf{s}}:=\dim\mathcal{G}_{l_{\mathbf{s}}}$ as well. It turns out (see 4) that $d_{\mathbf{s}}\leq e_{\mathbf{s}}\leq h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$. Usually, the equality $e_{\mathbf{s}}=h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$ rarely happens, however, it happens whenever the testing property requires it! Theorem 4.1.2 (the 'second algorithm') says that $h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$ is a testing function for $d_{\mathbf{s}}$ indeed.

The cases of superisolated singularities is exemplified.

The second algorithm has several consequences. E.g., a 'numerical' one, cf. (4.1.6):

$$d_Z(l')=\min_{0\leq Z_1\leq Z}\{\,(l',Z_1)+h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})\,\},\ \text{or},$$
$$\operatorname{codim}\operatorname{Im}(c^{l'}(Z))=\max_{0\leq Z_1\leq Z}\{\,h^1(\mathcal{O}_{Z_1})-(l',Z_1)\,\}.$$

The cycles Z_1 for which the above minimum is realized have several additional geometric properties (cf. Lemma 4.1.14 and 4). In particular, such a Z_1 imposes the following conceptual consequence:

Structure Theorem for the image of the Abel map. *Fix a resolution $\widetilde{X}$, a cycle $Z\geq E$ and a Chern class $l'\in-\mathcal{S}'$ as above. Then there exists an effective cycle $Z_1\leq Z$, such that: (i) the map* $\mathrm{ECa}^{l'}(Z_1)\to H^1(\mathcal{O}_{Z_1})$ *is birational onto its image, and (ii) the generic fibres of the restriction of r, r^{im} :* $\operatorname{Im}(c^{l'}(Z))\to\operatorname{Im}(c^{l'}(Z_1))$, *have dimension $h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})$. In particular, for any such Z_1, the space* $\operatorname{Im}(c^{l'}(Z))$ *is birationally equivalent with an affine fibration over* $\mathrm{ECa}^{l'}(Z_1)$ *with affine fibers of dimension $h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})$.*

1.5. The case of generic analytic structure In Sect. 5 we prove that if $\widetilde{X}$ has a generic analytic structure (in the sense of [9, 15]), and $Z\geq E$ and $l'\in-\mathcal{S}'$ then both $\dim\operatorname{Im}(c^{l'}(Z))$ and $\operatorname{codim}\operatorname{Im}(c^{l'}(Z))$ are topological. E.g., we have (where χ is the usual Riemann–Roch expression):

$$\operatorname{codim}\operatorname{Im}(c^{l'}(Z))=\max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}.\qquad(1.5.1)$$

The maximum at the right hand side is realized e.g. for the cohomology cycle of $\mathcal{L}^{im}_{gen}\in\operatorname{Im}(c^{l'}(Z))\subset\operatorname{Pic}^{l'}(Z)$. Furthermore,

$$h^1(Z,\mathcal{L})\geq\max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}$$

for any $\mathcal{L}\in \mathrm{Im}(c^{l'}(Z))$ and equality holds for generic $\mathcal{L}^{im}_{gen}\in \mathrm{Im}(c^{l'}(Z))$.

The identity (1.5.1), valid for a generic analytic structure of $\widetilde{X}$, extends to an optimal inequality valid for *any analytic structure*.

Theorem 1.5.2 *Consider an* arbitrary *normal surface singularity* (X,o)*, its resolution* $\widetilde{X}$*,* $Z\geq E$ *and* $l'\in -\mathcal{S}'$*. Then* $\mathrm{codim}\,\mathrm{Im}(c^{l'}(Z))=h^1(Z,\mathcal{L}^{im}_{gen})$ *satisfies*

$$\mathrm{codim}\,\mathrm{Im}(c^{l'}(Z))\geq \max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}. \tag{1.5.3}$$

In particular, for any $\mathcal{L}\in \mathrm{Im}(c^{l'}(Z))$ *one also has*

$$\begin{aligned} &h^1(Z,\mathcal{L})\geq h^1(Z,\mathcal{L}^{im}_{gen})\\ &=\mathrm{codim}\,\mathrm{Im}(c^{l'}(Z))\geq \max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}.\end{aligned}$$

The right hand side of (1.5.3) is a sharp topological lower bound for $\mathrm{codim}\,\mathrm{Im}(c^{l'}(Z))$. The inequality (1.5.3) can also be interpreted as the semi-continuity statement

$$\mathrm{codim}\,\mathrm{Im}(c^{l'}(Z))(\text{arbitrary analytic structure})\geq \mathrm{codim}\,\mathrm{Im}(c^{l'}(Z))(\text{generic analytic structure}).$$

1.6. Generalization Sections 7 and 8 target generalizations of the previous parts, valid for $\{h^1(Z,\mathcal{L})\}_{\mathcal{L}\in \mathrm{Im} c^{l'}(Z)}$, to the shifted case, valid for $\{h^1(Z,\mathcal{L}_0\otimes\mathcal{L})\}_{\mathcal{L}\in \mathrm{Im} c^{l'}(Z)}$, where $\mathcal{L}_0\in \mathrm{Pic}^{l'_0}(Z)$ is a fixed bundle without fixed components. In order to run a parallel theory based on Abel maps, we have to create the *new Abel map* $c^{l'}_{\mathcal{L}_0}(Z):\mathrm{ECa}^{l'}(Z)\to \mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z)$, where $\mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z)$ is an affine space associated with the vector space $\mathrm{Pic}^{0}_{\mathcal{L}_0}(Z)\simeq H^1(Z,\mathcal{L}_0)$. ($\mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z)$ appears also as an affine quotient of the classical $\mathrm{Pic}^{l'}(Z)$ as well.) Section 7 contains the definitions and the needed exact sequences. Section 8 contains the extension of the two algorithms to this situation.

2 Preliminaries

2.1. Notations regarding a good resolution [14, 19–21] Let (X,o) be the germ of a complex analytic normal surface singularity, and let us fix a good resolution $\phi:\widetilde{X}\to X$ of (X,o). Let E be the exceptional curve $\phi^{-1}(0)$ and $\cup_{v\in\mathcal{V}}E_v$ be its irreducible decomposition. Define $E_I:=\sum_{v\in I}E_v$ for any subset $I\subset\mathcal{V}$.

We will assume that each E_v is rational, and the dual graph is a tree. This happens exactly when the link M of (X,o) is a rational homology sphere.

Define the lattice L as $H_2(\widetilde{X},\mathbb{Z})$, it is endowed with a negative definite intersection form $(\,,\,)$. It is freely generated by the classes of $\{E_v\}_{v\in\mathcal{V}}$. The dual lattice is $L'=\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})=\{l'\in L\otimes\mathbb{Q}\,:\,(l',L)\in\mathbb{Z}\}$. It is generated by the (anti)dual classes $\{E^*_v\}_{v\in\mathcal{V}}$ defined by $(E^*_v,E_w)=-\delta_{vw}$ (where δ_{vw} stays for the Kronecker symbol). It is also identified with $H^2(\widetilde{X},\mathbb{Z})$, where the first Chern classes live.

All the E_v-coordinates of any E^*_u are strict positive. We define the Lipman cone as $\mathcal{S}' := \{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\}$. As a monoid it is generated over $\mathbb{Z}_{\geq 0}$ by $\{E^*_v\}_v$.

L embeds into L' with $L'/L \simeq H_1(M, \mathbb{Z})$, abridged by H. Each class $h \in H = L'/L$ has a unique representative $r_h \in L'$ in the semi-open cube $\{\sum_v r_v E_v \in L' : r_v \in \mathbb{Q} \cap [0, 1)\}$, such that its class $[r_h]$ is h.

There is a natural (partial) ordering of L' and L: we write $l'_1 \geq l'_2$ if $l'_1 - l'_2 = \sum_v r_v E_v$ with all $r_v \geq 0$. We set $L_{\geq 0} = \{l \in L : l \geq 0\}$ and $L_{>0} = L_{\geq 0} \setminus \{0\}$.

The support of a cycle $l = \sum n_v E_v$ is defined as $|l| = \cup_{n_v \neq 0} E_v$.

The *(anti)canonical cycle* $Z_K \in L'$ is defined by the *adjunction formulae* $(Z_K, E_v) = (E_v, E_v) + 2$ for all $v \in \mathcal{V}$. We write $\chi : L' \to \mathbb{Q}$ for the (Riemann–Roch) expression $\chi(l') := -(l', l' - Z_K)/2$.

2.1.1 Natural line bundles. Let $\phi : (\widetilde{X}, E) \to (X, o)$ be as above. Consider the 'exponential' cohomology exact sequence (with $H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}}) = \mathrm{Pic}(\widetilde{X})$, the group of isomorphic classes of holomorphic line bundles on $\widetilde{X}$, and $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \mathrm{Pic}^0(\widetilde{X})$)

$$0 \to \mathrm{Pic}^0(\widetilde{X}) \longrightarrow \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} H^2(\widetilde{X}, \mathbb{Z}) \to 0. \tag{2.1.2}$$

Here $c_1(\mathcal{L}) \in H^2(\widetilde{X}, \mathbb{Z}) = L'$ is the first Chern class of $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$. Since the link M is a rational homology sphere, $\mathrm{Pic}^0(\widetilde{X}) \simeq H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \simeq \mathbb{C}^{p_g}$, where p_g is the geometric genus. Write also $\mathrm{Pic}^{l'}(\widetilde{X}) = c_1^{-1}(l')$. Furthermore, see e.g. [20, 23], there exists a unique homomorphism (split) $s_1 : L' \to \mathrm{Pic}(\widetilde{X})$ of c_1, that is $c_1 \circ s_1 = id$, such that s_1 restricted to L is $l \mapsto \mathcal{O}_{\widetilde{X}}(l)$. The line bundles $s_1(l')$ are called *natural line bundles* of $\widetilde{X}$. For several definitions of them see [20]. E.g., $\mathcal{L}$ is natural if and only if one of its power has the form $\mathcal{O}_{\widetilde{X}}(l)$ for some *integral* cycle $l \in L$ supported on E. In order to have a uniform notation we write $\mathcal{O}_{\widetilde{X}}(l')$ for $s_1(l')$ for any $l' \in L'$.

For any $Z \geq E$ let $\mathcal{O}_Z(l')$ be the restriction of the natural line bundle $\mathcal{O}_{\widetilde{X}}(l')$ to Z. In fact, $\mathcal{O}_Z(l')$ can be defined in an identical way as $\mathcal{O}_{\widetilde{X}}(l')$ starting from the exponential cohomological sequence $0 \to \mathrm{Pic}^0(Z) \to \mathrm{Pic}(Z) \to H^2(\widetilde{X}, \mathbb{Z}) \to 0$ as well. Set also $\mathrm{Pic}^{l'}(Z) = c_{1,Z}^{-1}(l')$.

2.2. The Abel map [14] For any $Z \geq E$ let $\mathrm{ECa}(Z)$ be the space of (analytic) effective Cartier divisors on Z. Their supports are zero-dimensional in E. Taking the line bundle of a Cartier divisor provides the *Abel map* $c = c(Z) : \mathrm{ECa}(Z) \to \mathrm{Pic}(Z)$. Let $\mathrm{ECa}^{l'}(Z)$ be the set of effective Cartier divisors with Chern class $l' \in L'$, i.e. $\mathrm{ECa}^{l'}(Z) := c^{-1}(\mathrm{Pic}^{l'}(Z))$. The restriction of c is denoted by $c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$.

A line bundle $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ is in the image $\mathrm{im}(c^{l'})$ if and only if it has a section without fixed components, that is, if $H^0(Z, \mathcal{L})_{reg} \neq \emptyset$, where $H^0(Z, \mathcal{L})_{reg} := H^0(Z, \mathcal{L}) \setminus \cup_v H^0(Z - E_v, \mathcal{L}(-E_v))$. By this definition (see (3.1.5) of [14]) $\mathrm{ECa}^{l'}(Z) \neq \emptyset$ if and only if $-l' \in \mathcal{S}' \setminus \{0\}$. It is advantageous to have a similar statement for $l' = 0$ too, hence we redefine $\mathrm{ECa}^0(Z)$ as $\{\emptyset\}$, a set/space with one element (the empty divisor), and $c^0 : \mathrm{ECa}^0(Z) \to \mathrm{Pic}^0(Z)$ by $c^0(\emptyset) = \mathcal{O}_Z$. In particular,

$$H^0(Z, \mathcal{L})_{reg} \neq \emptyset \ \Leftrightarrow\ \mathcal{L} = \mathcal{O}_Z \ \Leftrightarrow\ \mathcal{L} \in \mathrm{im}(c^0) \quad \text{whenever } c_1(\mathcal{L}) = 0. \tag{2.2.1}$$

Hence, the extended statement valid for any l' is:

$$\mathrm{ECa}^{l'}(Z) \neq \emptyset \ \Leftrightarrow\ -l' \in \mathcal{S}'. \tag{2.2.2}$$

Sometimes even for $\mathcal{L} \in \mathrm{Pic}^{l'}(\widetilde{X})$ we write $\mathcal{L} \in \mathrm{Im}(c^{l'})$ whenever $\mathcal{L}|_Z \in \mathrm{Im}(c^{l'}(Z))$ for some $Z \gg 0$. This happens if and only if $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ has no fixed components.

It turns out that $\mathrm{ECa}^{l'}(Z)$ $(-l' \in \mathcal{S}')$ is a smooth complex algebraic variety of dimension (l', Z) and the Abel map is an algebraic regular map. For more properties and applications see [14, 15].

2.2.3. The modified Abel map. Multiplication by $\mathcal{O}_Z(-l')$ gives an isomorphism of the affine spaces $\mathrm{Pic}^{l'}(Z) \to \mathrm{Pic}^0(Z)$. Furthermore, we identify (via the exponential exact sequence) $\mathrm{Pic}^0(Z)$ with the vector space $H^1(Z, \mathcal{O}_Z)$.

It is convenient to replace the Abel map $c^{l'}$ with the composition

$$\widetilde{c}^{l'} : \mathrm{ECa}^{l'}(Z) \xrightarrow{c^{l'}} \mathrm{Pic}^{l'}(Z) \xrightarrow{\mathcal{O}_Z(-l')} \mathrm{Pic}^0(Z) \xrightarrow{\simeq} H^1(\mathcal{O}_Z).$$

The advantage of this new set of maps is that all the images sit in the same vector space $H^1(\mathcal{O}_Z)$.

Consider the natural additive structure $s^{l'_1,l'_2}(Z) : \mathrm{ECa}^{l'_1}(Z) \times \mathrm{ECa}^{l'_2}(Z) \to \mathrm{ECa}^{l'_1+l'_2}(Z)$ $(l'_1, l'_2 \in -\mathcal{S}')$ provided by the sum of the divisors. One verifies (see e.g. [14, Lemma 6.1.1]) that $s^{l'_1,l'_2}(Z)$ is dominant and quasi-finite. There is a parallel multiplication $\mathrm{Pic}^{l'_1}(Z) \times \mathrm{Pic}^{l'_2}(Z) \to \mathrm{Pic}^{l'_1+l'_2}(Z)$, $(\mathcal{L}_1, \mathcal{L}_2) \mapsto \mathcal{L}_1 \otimes \mathcal{L}_2$, which satisfies $c^{l'_1+l'_2} \circ s^{l'_1,l'_2} = c^{l'_1} \otimes c^{l'_2}$ in $\mathrm{Pic}^{l'_1+l'_2}$. This, in the modified case, using $\mathcal{O}_Z(l'_1 + l'_2) = \mathcal{O}_Z(l'_1) \otimes \mathcal{O}_Z(l'_2)$, reads as $\widetilde{c}^{l'_1+l'_2} \circ s^{l'_1,l'_2} = \widetilde{c}^{l'_1} + \widetilde{c}^{l'_2}$ in $H^1(\mathcal{O}_Z)$.

Definition 2.2.4 For any $l' \in -\mathcal{S}'$ let $A_Z(l')$ be the smallest dimensional affine subspace of $H^1(\mathcal{O}_Z)$ which contains $\mathrm{Im}(\widetilde{c}^{l'})$. Let $V_Z(l')$, be the parallel vector subspace of $H^1(\mathcal{O}_Z)$, the translation of $A_Z(l')$ to the origin.

For any $I \subset \mathcal{V}$, $I \neq \emptyset$, let (X_I, o_I) be the multigerm $\widetilde{X}/_{\cup_{v\in I} E_v}$ at its distinguished points, obtained by contracting the connected components of $\cup_{v\in I} E_v$ in $\widetilde{X}$. If $I = \emptyset$ then by convention (X_I, o_I) is a smooth germ.

Theorem 2.2.5 [14, Prop. 5.6.1, Lemma 6.1.6 and Th. 6.1.9] *Assume that* $Z \geq E$.

(a) *For any* $-l' = \sum_v a_v E^*_v \in \mathcal{S}'$ *let the* E^**-support of* l' *be* $I(l') := \{v : a_v \neq 0\}$. *Then* $V_Z(l')$ *depends only on* $I(l')$. *(This motivates to write* $V_Z(l')$ *as* $V_Z(I)$ *where* $I = I(l')$.*)*
(b) $V_Z(I_1 \cup I_2) = V_Z(I_1) + V_Z(I_2)$ *and* $A_Z(l'_1 + l'_2) = A_Z(l'_1) + A_Z(l'_2) = \{a_1 + a_2 : a_i \in A_Z(l'_i) \subset H^1(\mathcal{O}_Z),\ i = 1, 2\}$ *(and* $a_1 + a_2$ *is the sum in* $H^1(\mathcal{O}_Z)$*).*
(c) $\dim V_Z(I) = h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$.

(d) *If* $\mathcal{L}^{im}_{gen}$ *is a generic bundle of* $\mathrm{Im}(c^{l'})$ *then* $h^1(Z,\mathcal{L}^{im}_{gen})=h^1(\mathcal{O}_Z)-\dim(\mathrm{Im}(c^{l'}))$.

(e) *For* $n\gg 1$ *one has* $\mathrm{Im}(\widetilde{c}^{nl'})=A_Z(nl')$*, and* $h^1(Z,\mathcal{L})=h^1(\mathcal{O}_Z)-\dim V_Z(I)=h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$ *for any* $\mathcal{L}\in \mathrm{Im}(c^{nl'})$.

For different geometric reinterpretations of $\dim V_Z(I)$ see also [14, §9].

2.3 Theorem 4.1.1 of [14] says that $c^{l'}(Z)$ is dominant if and only if $\chi(-l')<\chi(-l'+l)$ for any $0<l\leq Z$. In particular, the dominance of $c^{l'}(Z)$ is a topological property. If $c^{l'}(Z)$ is dominant then $c^{l'}(Z')$ is dominant for any $0<Z'\leq Z$.

2.4. Review of Laufer duality [8], [10, p. 1281] Following Laufer, we identify the dual space $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})^*$ with the space of global holomorphic 2-forms on $\widetilde{X}\setminus E$ up to the subspace of those forms which can be extended holomorphically over $\widetilde{X}$.

For this, use first Serre duality $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})^*\simeq H^1_c(\widetilde{X},\Omega^2_{\widetilde{X}})$. Then, in the next exact sequence

$$0\to H^0_c(\widetilde{X},\Omega^2_{\widetilde{X}})\to H^0(\widetilde{X},\Omega^2_{\widetilde{X}})\to H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})\to H^1_c(\widetilde{X},\Omega^2_{\widetilde{X}})\to H^1(\widetilde{X},\Omega^2_{\widetilde{X}})$$

$H^0_c(\widetilde{X},\Omega^2_{\widetilde{X}})=H^2(\widetilde{X},\mathcal{O}_{\widetilde{X}})^*=0$ by dimension argument, while $H^1(\widetilde{X},\Omega^2_{\widetilde{X}})=0$ by the Grauert–Riemenschneider vanishing. Hence,

$$H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})^*\simeq H^1_c(\widetilde{X},\Omega^2_{\widetilde{X}})\simeq H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})/H^0(\widetilde{X},\Omega^2_{\widetilde{X}}). \tag{2.4.1}$$

2.4.2. Above $H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})$ can be replaced by $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))$ for a large cycle Z (e.g. for $Z\geq\lfloor Z_K\rfloor$). Indeed, for any cycle $Z>0$ from the exacts sequence of sheaves $0\to\Omega^2_{\widetilde{X}}\to\Omega^2_{\widetilde{X}}(Z)\to\mathcal{O}_Z(Z+K_{\widetilde{X}})\to 0$ and from the vanishing $h^1(\Omega^2_{\widetilde{X}})=0$ and Serre duality one has

$$H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})=H^0(\mathcal{O}_Z(Z+K_{\widetilde{X}}))\simeq H^1(\mathcal{O}_Z)^*. \tag{2.4.3}$$

Since $H^1(\mathcal{O}_Z)\simeq H^1(\mathcal{O}_{\widetilde{X}})$ for $Z\geq\lfloor Z_K\rfloor$, the natural inclusion

$$H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})\hookrightarrow H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}}) \tag{2.4.4}$$

is an isomorphism.

This pairing reduces to a perfect pairing at the level of an arbitrary $Z>0$, cf. [14, 7.4]. Indeed, consider the above perfect pairing $\langle\cdot,\cdot\rangle : H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})\otimes H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})\to\mathbb{C}$ given via integration of class representatives. In $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ let A be the image of $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-Z))$, hence $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})/A=H^1(\mathcal{O}_Z)$. On the other hand, in $H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})$ consider the subspace $B:=H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ of dimension $h^1(\mathcal{O}_Z)$ (cf. (2.4.3)). Since $\langle A,B\rangle=0$, the pairing factorizes to a perfect pairing $H^1(\mathcal{O}_Z)\otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})\to\mathbb{C}$. It can be described by the very same integral form of the corresponding class representatives.

2.4.5. The linear subspace arrangement $\{V_Z(I)\}_I\subset H^1(\mathcal{O}_Z)$ **and differential forms.** The arrangement $\{V_Z(I)\}_I$ transforms into a linear subspace arrangement of $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ via the (Laufer) non-degenerate pairing $H^1(\mathcal{O}_Z)\otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})\to\mathbb{C}$ as follows. Let $\Omega_Z(I)$ be the subspace $H^0(\Omega^2_{\widetilde{X}}(Z|_{\mathcal{V}\setminus I}))/H^0(\Omega^2_{\widetilde{X}})$ in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, that is, the subspace generated by those forms which have no poles along generic points of any E_v, $v\in I$. .

Proposition 2.4.6 [14, 8.3] *Via Laufer duality* $V_Z(I)=\Omega_Z(I)^{\perp}=\{x:\langle x,\Omega_Z(I)\rangle=0\}$ *for* $Z\geq E$.

2.4.7. Furthermore, for any $l'\in -\mathcal{S}'\setminus\{0\}$ consider a divisor $D\in \mathrm{ECa}^{l'}(Z)$, which is a union of (l',E) disjoint divisors $\{D_i\}_i$, each of them $\mathcal{O}_Z$-reduction of reduced divisors $\{\widetilde{D}_i\}_i$ of $\widetilde{X}$ intersecting E transversally. Set $\widetilde{D}=\cup_i\widetilde{D}_i$ and $\mathcal{L}:=\widetilde{c}^{l'}(D)\in H^1(\mathcal{O}_Z)$. Write also $Z=\sum_{v\in\mathcal{V}}r_vE_v$.

We introduce a subsheaf $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$ of $\Omega^2_{\widetilde{X}}(Z)$ consisting of those forms ω which have the property that the residue $\mathrm{Res}_{\widetilde{D}_i}(\omega)$ has no poles along $\widetilde{D}_i$ for all i. This means that the restrictions of $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$ and $\Omega^2_{\widetilde{X}}(Z)$ on the complement of the support of $\widetilde{D}$ coincide, however along $\widetilde{D}$ one has the following local picture. Introduce near $p=E\cap\widetilde{D}_i=E_{v_i}\cap\widetilde{D}_i$ local coordinates (u,v) such that $\{u=0\}=E$ and $\widetilde{D}_i$ has local equation v. Then a local section of $\Omega^2_{\widetilde{X}}(Z)$ in this system has the form $\omega=\sum_{k\geq -r_{v_i},j\geq 0}a_{k,j}u^kv^j du\wedge dv$. Then, by definition, the residue $\mathrm{Res}_{\widetilde{D}_i}(\omega)$ is $(\omega/dv)|_{v=0}=\sum_k a_{k,0}u^kdu$, hence the pole-vanishing reads as $a_{k,0}=0$ for all $k<0$. Note that $\Omega^2_{\widetilde{X}}(Z-\widetilde{D})$ and the sheaf of regular forms $\Omega^2_{\widetilde{X}}$ are subsheaves of $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$.

Set $\Omega_Z(D):=H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$. This can be regarded as a subspace of $H^1(\mathcal{O}_Z)^*=H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$.

Theorem 2.4.8 [14, Th. 10.1.1] *In the above situation one has the following facts.*

(a) *The sheaves* $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}/\Omega^2_{\widetilde{X}}$ *and* $\mathcal{O}_Z(K_{\widetilde{X}}+Z-D)$ *are isomorphic.*
(b) $H^1(Z,\mathcal{L})^*\simeq\Omega_Z(D)$.
(c) *The image* $(T_D\widetilde{c})(T_D\mathrm{ECa}^{l'}(Z))$ *of the tangent map at* D *of* $\widetilde{c}:\mathrm{ECa}^{l'}(Z)\to H^1(\mathcal{O}_Z)$ *is the intersection of kernels of linear maps* $T_{\mathcal{L}}\omega: T_{\mathcal{L}}H^1(\mathcal{O}_Z)\to\mathbb{C}$, *where* $\omega\in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})$.

If I is the E^*-support of l' (that is, $\widetilde{D}$ intersects E exactly along $\cup_{v\in I}E_v$), then $\Omega_Z(I)\subset\Omega_Z(D)\subset H^1(\mathcal{O}_Z)^*$. Dually, via Proposition 2.4.6 and Theorem 2.4.8*(c)* (and up to a linear translation of $\mathrm{Im}(T_D\widetilde{c})$)

$$(T_D\widetilde{c})(T_D\mathrm{ECa}^{l'}(Z))=\Omega_Z(D)^{\perp}\subset\Omega_Z(I)^{\perp}=V_Z(I)\subset H^1(\mathcal{O}_Z). \quad (2.4.9)$$

Let us fix a point $p\in E$ and a local coordinate system (u,v) around p such that $E=\{u=0\}$, cf. 2. Fix also some $\omega\in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))$ which has pole of order $o>0$ at the exceptional divisor in E containing p. We say that (the divisor of) ω has no support point at p if it can be represented locally as $(\varphi(u,v)/u^o)du\wedge dv$ with φ holomorphic and $\varphi(0,0)\neq 0$. The other points are the support points denoted by $\mathrm{supp}(\omega)$.

Lemma 2.4.10 *Fix $\omega \in H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))$ such that there exists a point $p \in E_v$, a local divisor $\widetilde{D}_1$ in $\widetilde{X}$ with the following properties:*

(a) *$\widetilde{D}_1$ is part of certain $\widetilde{D} = \widetilde{D}_1 + \widetilde{D}_2$, such that $\widetilde{D}_1 \cap E = \widetilde{D}_1 \cap E_v = p \notin \widetilde{D}_2 \cup \mathrm{supp}(\omega)$, and*

(b) *$\widetilde{D}$ is a lift of $D \in \mathrm{ECa}^{l'}(Z)$, and the class of ω in $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ restricted on $\mathrm{Im} T_D\widetilde{c}^{l'}(Z)$ is zero.*

Then ω has no pole along E_v.

Proof Assume that ω has a pole of order $o > 0$ along E_v. Fix some local coordinated (u, v) at $p := \widetilde{D}_1 \cap E_v$ such that ω locally is $du \wedge dv/u^o$ and $\widetilde{D}_1$ is $\{g(u, v) = 0\}$. A deformation $g_t(u, v)$ of g produces a tangent vector in $T_D\mathrm{ECa}^{l'}(Z)$ and the action of ω on it is given by (for details see [14, 7.2])

$$\frac{d}{dt}\Big|_{t=0} \int_{|u|=\epsilon,\ |v|=\epsilon} \log \frac{g_t(u, v)}{g(u, v)} \cdot \frac{du \wedge dv}{u^o}. \tag{2.4.11}$$

Hence if we realize a deformation g_t for which the expression from (2.4.11) is non-zero, we get a contradiction. Note that g necessarily has the form $cv^k + \sum_{n>k} c_n v^n + uh(u, v) = cv^k + h'$ for some $k \geq 1, c_n \in \mathbb{C}$ and $c \in \mathbb{C}^*$. Then set $g_t = c(v - tu^{o-1})^k + h'$. Then the t-coefficient of the integrant is $\frac{kdu \wedge dv}{uv} \cdot (1 - \frac{h'}{cv^k} + (\frac{h'}{cv^k})^2 - \cdots)$, hence (2.4.11) is non-zero. □

Definition 2.4.12 Additionally to the linear subspace arrangement $\{\Omega_Z(I)\}_I \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \simeq H^1(\mathcal{O}_Z)^*$ we consider a more subtle object, a filtration indexed by $l \in L$, $0 \leq l \leq Z$ as well, called the *multivariable divisorial filtration of forms*. Indeed, for any such l we define $\mathcal{G}_l := H^0(\Omega^2_{\widetilde{X}}(l))/H^0(\Omega^2_{\widetilde{X}}) \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, equivalent to $H^1(\mathcal{O}_l)^* \hookrightarrow H^1(\mathcal{O}_Z)^*$, dual to the natural epimorphisms $H^1(\mathcal{O}_Z) \twoheadrightarrow H^1(\mathcal{O}_l)$. In particular, $\mathcal{G}_l \simeq H^1(\mathcal{O}_l)^*$. The subspace $\mathcal{G}_l$ is generated by forms with pole $\leq l$. In particular, $\mathcal{G}_0 = 0$, $\mathcal{G}_Z$ is the total vector space, $\mathcal{G}_{l_1} \subset \mathcal{G}_{l_2}$ whenever $l_1 \leq l_2$, and $\mathcal{G}_{l_1} \cap \mathcal{G}_{l_2} = \mathcal{G}_{\min\{l_1, l_2\}}$.

Note that if $l = \sum_{v \notin I} r_v E_v$ and all $r_v \gg 0$ then $\mathcal{G}_{\min(l,Z)} = \Omega_Z(I)$.

3 The first algorithm for the computation of dim Im($c^{l'}(Z)$)

3.1. We fix $Z \geq E$ and $l' \in -\mathcal{S}'$ as above.

Definition 3.1.1 For any $l' \in -\mathcal{S}'$ with E^*-support I ($\emptyset \subset I \subset \mathcal{V}$) we set the following notations: $e_Z(l') = e_Z(I) := \dim V_Z(l') = \dim V_Z(I)$ and $d_Z(l') := \dim \mathrm{Im}(c^{l'}(Z))$.

From definitions and Theorem 2.2.5 and Proposition 2.4.6 (see also 2.4.9) we obtain

$$\begin{aligned} &(i)\ \ d_Z(l') \leq e_Z(l') \\ &(ii)\ \ e_Z(I) = h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}}) = h^1(\mathcal{O}_Z) - \dim \Omega_Z(I). \end{aligned} \tag{3.1.2}$$

Usually $d_Z(l') \neq e_Z(l')$. The next lemma provides a criterion for the validity of the equality $d_Z(l') = e_Z(l')$.

Lemma 3.1.3 *Let $l' \in -S'$ with E^*-support I and $Z \geq E$. Assume that $\mathcal{L}$ is a regular value of $\widetilde{c}^{l'}$ in $\mathrm{Im}(\widetilde{c}^{l'})$ such that for any $\omega \in H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))$ there exists a section $s \in H^0(\mathcal{L})_{reg}$ such that $\mathrm{div}(s) \cap \mathrm{supp}(\omega) = \emptyset$. (This is guaranteed e.g. if the bundle $\mathcal{L}$ has no base points.) Then $T_{\mathcal{L}}(\mathrm{Im}\widetilde{c}^{l'}) = A_Z(l')$, hence $d_Z(l') = e_Z(l')$.*

Proof Since $\mathcal{L}$ is a regular value, $\mathcal{L}$ is a smooth point of $\mathrm{Im}(\widetilde{c}^{l'})$ and $T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'}) = \mathrm{Im}(T_D\widetilde{c}^{l'})$ for *any* $D \in (\widetilde{c}^{l'})^{-1}(\mathcal{L})$ (cf. [14, 3.3.2]). We have to prove that $T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'}) = A_Z(l')$ (as affine subspaces); we prove the dual identity in the space of forms, namely, $(T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'})^{\perp} = \Omega_Z(I)$ (up to a linear translation, see (2.4.9)).

Assume the contrary, that is, $(T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'}))^{\perp} \neq \Omega_Z(I)$. Since $\Omega_Z(I) \subset (T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'}))^{\perp}$ (the duality integral on $\Omega_Z(I) \times T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'})$ is zero, cf. [14, 7.2] or (2.4.9)) we get, that there is a form $\omega \in (T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'}))^{\perp} \setminus \Omega_Z(I)$.

Next choose $D \in (\widetilde{c}^{l'})^{-1}(\mathcal{L})$ such that its lift $\widetilde{D}$ satisfies $\widetilde{D} \cap \mathrm{supp}(\omega) = \emptyset$. But $\omega \in (T_{\mathcal{L}}\mathrm{Im}(\widetilde{c}^{l'}))^{\perp} = (\mathrm{Im}(T_D\widetilde{c}^{l'}))^{\perp}$ and $\omega \notin \Omega_Z(I)$ contradict Lemma 2.4.10. □

In this section we provide an algorithm, valid for any analytic structure, which determines $d_Z(l')$ in terms of a finite collection of invariants of type $e_Z(l')$, associated with a finite sequence of resolutions obtained via certain extra blowing ups from $\widetilde{X}$.

3.2. Preparation for the algorithm Fix some resolution $\widetilde{X}$ of (X, o) and $-l' = \sum_{v\in\mathcal{V}} a_v E^*_v \in \mathcal{S}'\setminus\{0\}$ (hence each $a_v \in \mathbb{Z}_{\geq 0}$). In the next construction we will consider a finite sequence of blowing ups starting from $\widetilde{X}$. In order to find a bound for the number of blowing ups recall that for any representative ω in $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ the order of pole of ω along some E_v is less than or equal to the E_v-multiplicity m_v of $\max\{0, \lfloor Z_K \rfloor\}$ (see e.g. [14, 7.1.3] or 2 here). Then, for every $v \in \mathcal{V}$ with $a_v > 0$ we fix a_v generic points on E_v, say p_{v,k_v}, $1 \leq k_v \leq a_v$. Starting from each p_{v,k_v} we consider a sequence of blowing ups of length m_v: first we blow up p_{v,k_v} and we create the exceptional curve $F_{v,k_v,1}$, then we blow up a generic point of $F_{v,k_v,1}$ and we create $F_{v,k_v,2}$, and we do this all together m_v times. We proceed in this way with all points p_{v,k_v}, hence we get $\sum_v a_v$ chains of modifications. If $a_v m_v = 0$ we do no modification along E_v. A set of integers $\mathbf{s} = \{\mathbf{s}_{v,k_v}\}_{v\in\mathcal{V},\ 1\leq k_v\leq a_v}$ with $0 \leq \mathbf{s}_{v,k} \leq m_v$ provides an intermediate step of the tower: in the (v, k_v) tower we do exactly $\mathbf{s}_{v,k_v}$ blowing ups; $\mathbf{s}_{v,k_v} = 0$ means that we do not blow up p_{v,k_v} at all. (In the sequel, in order to avoid aggregation of indices, we simplify k_v into k.) Let us denote this modification by $\pi_{\mathbf{s}} : \widetilde{X}_{\mathbf{s}} \to \widetilde{X}$. In $\widetilde{X}_{\mathbf{s}}$ we find the exceptional curves

$$\Big(\bigcup_{v\in\mathcal{V}} E_v\Big) \cup \bigcup_{v,k}\ \bigcup_{1\leq t\leq \mathbf{s}_{v,k}} F_{v,k,t}.$$

We index the set of vertices as

$$\mathcal{V}_{\mathbf{s}} := \mathcal{V} \cup \bigcup_{v,k}\ \bigcup_{1\leq t\leq \mathbf{s}_{v,k}} \{w_{v,k,t}\}.$$

At each level $\mathbf{s}$ we set the next objects: $Z_{\mathbf{s}} := \pi^*_{\mathbf{s}}(Z)$, $I_{\mathbf{s}} := \cup_{v,k}\{w_{v,k,\mathbf{s}_{v,k}}\}$, $-l'_{\mathbf{s}} := \sum_{v,k} F^*_{v,k,\mathbf{s}_{v,k}}$ (in $L'_{\mathbf{s}}$, where $F_{v,k,0} = E_v$), $d_{\mathbf{s}} := \dim \mathrm{Im} c^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$

and $e_{\mathbf{s}} := e_{Z_{\mathbf{z}}}(I_{\mathbf{s}})$ (both considered in $\widetilde{X}_{\mathbf{s}}$).

By similar argument as in (3.1.2) one has again $d_{\mathbf{s}} \leq e_{\mathbf{s}}$ for any $\mathbf{s}$.

From definitions, for $\mathbf{s} = \mathbf{0}$ one has $I_{\mathbf{0}} = |l'|$, $e_{\mathbf{0}} = e_Z(l')$ and $d_{\mathbf{0}} = d_Z(l')$.

There is a natural partial ordering on the set of $\mathbf{s}$-tuples. Some of the above invariants are constant with respect to $\mathbf{s}$, some of them are only monotonous. E.g., by Leray spectral sequence one has $h^1(\mathcal{O}_{Z_{\mathbf{s}}}) = h^1(\mathcal{O}_Z)$ for all $\mathbf{s}$. One the other hand,

$$\text{if } \mathbf{s}_1 \leq \mathbf{s}_2 \text{ then } e_{\mathbf{s}_1} = h^1(\mathcal{O}_{Z_{\mathbf{s}_1}}) - \dim \Omega_{Z_{\mathbf{s}_1}}(I_{\mathbf{s}_1}) \geq h^1(\mathcal{O}_{Z_{\mathbf{s}_2}}) - \dim \Omega_{Z_{\mathbf{s}_2}}(I_{\mathbf{s}_2}) = e_{\mathbf{s}_2} \tag{3.2.1}$$

because $\Omega_{Z_{\mathbf{s}_1}}(I_{\mathbf{s}_1}) \subset \Omega_{Z_{\mathbf{s}_2}}(I_{\mathbf{s}_2})$. In fact, for any ω, the pole-order along $F_{v,k,\mathbf{s}_{v,k}+1}$ of its pullback is one less than the pole-order of ω along $F_{v,k,\mathbf{s}_{v,k}}$. Hence, for $\mathbf{s} = \mathbf{m}$ (that is, when $\mathbf{s}_{v,k} = m_v$ for all v and k, hence all the possible pole-orders along $I_{\mathbf{m}}$ automatically vanish) one has $\Omega_{Z_{\mathbf{m}}}(I_{\mathbf{m}}) = H^0(\widetilde{X}_{\mathbf{m}}, \Omega^2_{\widetilde{X}_{\mathbf{m}}}(Z_{\mathbf{m}}))/H^0(\Omega^2_{\widetilde{X}_{\mathbf{m}}})$. Hence $e_{\mathbf{m}} = 0$. In particular, necessarily $d_{\mathbf{m}} = 0$ too.

More generally, for any $\mathbf{s}$ and (v, k) let $\mathbf{s}^{v,k}$ denote the tuple which is obtained from $\mathbf{s}$ by increasing $\mathbf{s}_{v,k}$ by one. By the above discussion if no form has pole along $F_{v,k,\mathbf{s}}$ then $\Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}}) = \Omega_{Z_{\mathbf{s}^{v,k}}}(I_{\mathbf{s}^{v,k}})$, hence $e_{\mathbf{s}} = e_{\mathbf{s}^{v,k}}$. Furthermore, by Laufer duality (or, integral presentation of the Abel map as in [14, §7]), under such condition $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$ as well.

Therefore, we can redefine $e_{\mathbf{s}}$ and $d_{\mathbf{s}}$ for tuples $\mathbf{s} = \{\mathbf{s}_{v,k}\}_{v,k}$ even for arbitrary $\mathbf{s}_{v,k} \geq 0$: $e_{\mathbf{s}} = e_{\min\{\mathbf{s},\mathbf{m}\}}$ and $d_{\mathbf{s}} = d_{\min\{\mathbf{s},\mathbf{m}\}}$ (and these values agree with the ones which might be obtained by the first original construction applied for larger chains of blow ups).

The next theorem relates the invariants $\{d_{\mathbf{s}}\}_{\mathbf{s}}$ and $\{e_{\mathbf{s}}\}_{\mathbf{s}}$.

Theorem 3.2.2 (First algorithm) *With the above notations the following facts hold.*

(1) $d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0, 1\}$.
(2) *If for some fixed* $\mathbf{s}$ *the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathbf{s}} = \max_{v,k}\{d_{\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value* $d_{\mathbf{s}^{v,k}}$ *equals* $e_{\mathbf{s}}$, *then* $d_{\mathbf{s}} = e_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$; *otherwise* $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}} + 1$.

The proof of Theorem 3.2.2 together with the proof of Theorem 4.1.2 (the 'Second algorithm') from the next section will be given in a more general context in section 8.

3.2.3 Theorem 3.2.2 is suitable to run a decreasing induction over the entries of $\mathbf{s}$ in order to determine $\{d_{\mathbf{s}}\}_{\mathbf{s}}$ from $\{e_{\mathbf{s}}\}_{\mathbf{s}}$. In fact we can obtain even a closed-form expression.

Corollary 3.2.4 *With the notations of Theorem 3.2.2 one has* $d_{\mathbf{s}} = \min_{\mathbf{s}\leq\widetilde{\mathbf{s}}\leq\mathbf{m}}\{|\widetilde{\mathbf{s}} - \mathbf{s}| + e_{\widetilde{\mathbf{s}}}\}$ *for any* $\mathbf{0} \leq \mathbf{s} \leq \mathbf{m}$. *(Here* $|\mathbf{s}| = \sum_{v,k} s_{v,k_v}$*.) In particular,*

$$d_Z(l') = d_{\mathbf{0}} = \min_{\mathbf{0}\leq\mathbf{s}\leq\mathbf{m}}\{|\mathbf{s}| + e_{\mathbf{s}}\}.$$

(By the end of 3.2 one also has $\min_{\mathbf{s}\leq\widetilde{\mathbf{s}}\leq\mathbf{m}}\{|\widetilde{\mathbf{s}} - \mathbf{s}| + e_{\widetilde{\mathbf{s}}}\} = \min_{\mathbf{s}\leq\widetilde{\mathbf{s}}}\{|\widetilde{\mathbf{s}} - \mathbf{s}| + e_{\widetilde{\mathbf{s}}}\}$ *and* $\min_{\mathbf{0}\leq\mathbf{s}\leq\mathbf{m}}\{|\mathbf{s}| + e_{\mathbf{s}}\} = \min_{\mathbf{0}\leq\mathbf{s}}\{|\mathbf{s}| + e_{\mathbf{s}}\}$.*)*

Proof By Theorem 3.2.2*(1)* for any $\widetilde{\mathbf{s}} \geq \mathbf{s}$ one has $d_{\mathbf{s}} - d_{\widetilde{\mathbf{s}}} \leq |\widetilde{\mathbf{s}} - \mathbf{s}|$, and by (3.1.2) $d_{\widetilde{\mathbf{s}}} \leq e_{\widetilde{\mathbf{s}}}$. These two imply $d_{\mathbf{s}} \leq |\widetilde{\mathbf{s}} - \mathbf{s}| + e_{\widetilde{\mathbf{s}}}$, hence $d_{\mathbf{s}} \leq \min_{\mathbf{s} \leq \widetilde{\mathbf{s}} \leq \mathbf{m}} \{|\widetilde{\mathbf{s}} - \mathbf{s}| + e_{\widetilde{\mathbf{s}}}\}$. Next we show that $d_{\mathbf{s}}$ in fact equals $|\widetilde{\mathbf{s}} - \mathbf{s}| + e_{\widetilde{\mathbf{s}}}$ for some $\widetilde{\mathbf{s}}$. The wished $\widetilde{\mathbf{s}}$ is the last term of the sequence $\{\mathbf{s}_i\}_{i=0}^t$ constructed as follows. Set $\mathbf{s}_0 := \mathbf{s}$. Then, assume that $\mathbf{s}_i$ is already constructed, and that there exists (v, k) such that $d_{\mathbf{s}_i} = d_{(\mathbf{s}_i)^{v,k}} + 1$. Then set $\mathbf{s}_{i+1} := (\mathbf{s}_i)^{v,k}$ (for one of the choices of such possible (v, k)). This inductive construction will stop after finitely many steps (since each $d_{\mathbf{s}} \geq 0$). But if $d_{\mathbf{s}_t} = d_{(\mathbf{s}_t)^{v,k}}$ for all (v, k), then by 3.2.2*(2)* $d_{\mathbf{s}_t} = e_{\mathbf{s}_t}$. Hence $e_{\mathbf{s}_t} = d_{\mathbf{s}_t} = d_{\mathbf{s}} - |\mathbf{s}_t - \mathbf{s}|$. □

4 The second algorithm for the computation of dim Im$(c^{l'}(Z))$

4.1. Preparation The algorithm from the previous section determines the dimensions of the Abel maps $d_Z(l')$ in terms of a finite collection of invariants of type $e_Z(l')$ associated with a finite sequence of resolutions obtained via certain extra blowing ups from $\widetilde{X}$. Though, in principle, $e_Z(l')$ is much simpler than $d_Z(l')$ (it is the 'stabilizer' of $d_Z(l')$), the algorithm is still slightly cumbersome, it is more theoretical, it is not easy to apply in concrete examples: one needs to know all the integers $\{e_{\mathbf{s}}\}_{\mathbf{s}}$, that is, cf. Proposition 2.2.5, all the integers $\{h^1(\mathcal{O}_{Z_{\mathbf{s}}|\mathcal{V}_{\mathbf{s}} \setminus I_{\mathbf{s}}}\}_{\mathbf{s}}$ associated with the tower of blowing ups. (However, it is a necessary intermediate step in the proof of the new algorithm).

The new algorithm is considerably simpler, e.g. it can be formulated in terms of the resolution $\widetilde{X}$ (see also the comments below). It provides $d_Z(l')$ in terms of the filtration $\{\mathcal{G}_l\}_l$ of 2-forms.

As a starting point, consider the construction from 3. For any $\mathbf{s}$ define the cycle $l_{\mathbf{s}} \in L$ of $\widetilde{X}$ by

$$l_{\mathbf{s}} := \min \left\{ \sum_{v \in \mathcal{V}} \min_{1 \leq k_v \leq a_v} \{\mathbf{s}_{v,k_v}\} E_v,\, Z \right\} \in L.$$

Set $\mathcal{G}_{\mathbf{s}} := \mathcal{G}_{l_{\mathbf{s}}}$ and $g_{\mathbf{s}} := \dim \mathcal{G}_{\mathbf{s}}$ as well. Note that (via pullback) there is an inclusion $\mathcal{G}_{\mathbf{s}} \subset \Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}})$. Indeed, if the pole order of certain ω along E_v is $\leq \mathbf{s}_{v,k_v}$ then its pullback along $F_{v,k_v,\mathbf{s}_{v,k_v}}$ has no pole. Hence $g_{\mathbf{s}} \leq \dim \Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}}) = h^1(\mathcal{O}_Z) - e_{\mathbf{s}}$ too (cf. (3.1.2)). In particular,

$$d_{\mathbf{s}} \leq e_{\mathbf{s}} \leq h^1(\mathcal{O}_Z) - g_{\mathbf{s}}. \tag{4.1.1}$$

However, in principle it can happen that for a certain ω with even higher pole than $l_{\mathbf{s}}$ its pullback is in $\Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}})$. E.g., if ω in some local coordinates (u, v) of an open set U is $v du \wedge dv/u^o$ (and $U \cap E = \{u = 0\}$) then its pullback via blowing up (once) at $u = v = 0$ has pole order $o - 2$. This phenomenon can happen even if we blow up a generic point: imagine a family of forms ω_t with 'moving divisor', parametrized by t given by $(v - t) du \wedge dv/u^o$. Then, even if we blow up E at a generic point $u = v - t_0 = 0$, in the family $\{\omega_t\}_t$ there is a form ω_{t_0} whose pole along E_v is o while its pullback has pole $o - 2$. Hence the equality of subspaces $\mathcal{G}_{\mathbf{s}} \subset \Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}})$, or the equality $e_{\mathbf{s}} = h^1(\mathcal{O}_Z) - g_{\mathbf{s}}$ is subtle and it is hard to test.

Note also that the invariant $h^1(\mathcal{O}_Z) - g_{\mathbf{s}}$ conceptually (and technically) is much simpler than $e_{\mathbf{s}}$. E.g., it depends only on $v \mapsto \min_{k_v \le a_v}\{\mathbf{s}_{v,k_v}\}$, and it can be described via a cycle of $\widetilde{X}$ (namely $l_{\mathbf{s}}$) instead of the geometry of the tower $\widetilde{X}_{\mathbf{s}}$. Nevertheless, via the next theorem, it still contains sufficient information to determine $d_{\mathbf{s}}$, in particular $d_Z(l')$. In order to emphasize the parallelism between the two algorithms we formulate them in a completely symmetric way (in particular, the first parts are completely identical).

Theorem 4.1.2 (Second algorithm) *With the above notations the following facts hold.*

(1) $d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0, 1\}$.
(2) *If for some fixed* $\mathbf{s}$ *the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathbf{s}} = \max_{v,k}\{ d_{\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value* $d_{\mathbf{s}^{v,k}}$ *equals* $h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$, *then* $d_{\mathbf{s}} = h^1(\mathcal{O}_Z)-g_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$; *otherwise* $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}+1$.

For the proof see Sect. 8.

Corollary 4.1.3 *With the notations of* 4 *and of Theorem* 4.1.2, *for* $l' \in -S'$ *and* $Z \ge E$ *one has*

$$d_Z(l') = \min_{\mathbf{s}}\{ |\mathbf{s}| + h^1(\mathcal{O}_Z) - g_{\mathbf{s}} \}. \tag{4.1.4}$$

The proof runs similarly as the proof of Corollary 3.2.4.

The formula (4.1.4) can be rewritten in a different flavour.

Corollary 4.1.5 *For* $l' \in -S'$ *and* $Z \ge E$ *one has*

$$d_Z(l') = \min_{0\le Z_1\le Z}\{ (l', Z_1) + h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1}) \}. \tag{4.1.6}$$

Proof From 2.4.12 $g_{\mathbf{s}} = \dim \mathcal{G}_{\mathbf{s}} = h^1(\mathcal{O}_{l_{\mathbf{s}}})$ and also $|\mathbf{s}| \ge \sum_v a_v(l_{\mathbf{s}})_v = (l', l_{\mathbf{s}})$, and $0 \le l_{\mathbf{s}} \le Z$, hence $\min_{\mathbf{s}}\{ |\mathbf{s}| + h^1(\mathcal{O}_Z) - g_{\mathbf{s}} \} \ge \min_{0\le Z_1\le Z}\{ (l', Z_1) + h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1}) \}$. The opposite inequality is also true since any such Z_1 can be represented as a certain $l_{\mathbf{s}}$ with $|\mathbf{s}| = (l', l_{\mathbf{s}})$. □

Example 4.1.7 (1) ($c^{l'}(Z)$ **constant)** For any $0 \le Z_1 \le Z$ one has $(l', Z_1) \ge 0$ and $h^1(\mathcal{O}_Z) \ge h^1(\mathcal{O}_{Z_1})$, hence $d_Z(l') = 0$ happens exactly when there exists Z_1 with $(l', Z_1) + h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1}) = 0$, or, $(l', Z_1) = 0$ and $h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_{Z_1})$. This means that $Z_1 \le Z|_{\mathcal{V}\setminus I}$, where I is the E^*-support of l', a fact which (together with $h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_{Z_1})$) implies $h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$ too. Hence, $d_Z(l') = 0$ if and only if $h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$. This is exactly the statement of [14, 6.3(v)].

(2) $c^{l'}(Z)$ **is dominant** if and only if $d_Z(l') = h^1(\mathcal{O}_Z)$, hence, via (4.1.6), if and only if $h^1(\mathcal{O}_{Z_1}) \le (l', Z_1)$ for any $0 \le Z_1 \le Z$. This can be seen in a different way as follows. First, if $c^{l'}(Z)$ is dominant, then, for any $0 < Z_1 \le Z$, $c^{l'}(Z_1)$ is dominant too, hence $(l', Z_1) = \dim(\mathrm{ECa}^{l'}(Z_1)) \ge \dim(H^1(\mathcal{O}_{Z_1}))$. Conversely, if $(l', Z_1) \ge h^1(\mathcal{O}_{Z_1})$ and $Z_1 > 0$ then $(l', Z_1) - h^1(\mathcal{O}_{Z_1}) > -h^0(\mathcal{O}_{Z_1})$, that is, $\chi(-l') < \chi(-l' + Z_1)$, hence $c^{l'}(Z)$ is dominant by [14, Thm. 4.1.1], cf. 2 here. Note that the characterization 2 for dominant property is topological.

(3) By (4.1.6) $\mathrm{Im}(c^{l'}(Z))$ **is a hypersurface** if and only if $\min_{0\leq Z_1\leq Z}\{(l', Z_1)-h^1(\mathcal{O}_{Z_1})\}=-1$. Since $h^0(\mathcal{O}_{Z_1})\geq 1$, this implies that $\chi(-l')=\min_{0\leq l\leq Z}\chi(-l'+l)$.

The converse statement is not true: take e.g. a Gorenstein elliptic singularity with length of elliptic sequence $m+1$. (For elliptic singularities consult [16–18]. For more on the Abel map of elliptic singularities see [16].) Set $Z\gg 0$ and $-l'=Z_{min}$, the fundamental (minimal) cycle. Then $\mathrm{Im}(c^{l'}(Z))=1$ and $h^1(Z)=p_g=m+1$. However, $\chi(Z_{min})=\min_{0\leq l\leq Z}\chi(Z_{min}+l)=0$. Therefore, if $m=1$ then $\mathrm{Im}(c^{l'})$ is a hypersurface, but for $m\geq 2$ it is not. It is instructive to consider with the same topological data (elliptic numerically Gorenstein singularity with $m\geq 1$, $Z\gg 0$, $-l'=Z_{min}$) the generic analytic structure. Then $p_g=1$ (cf. [10, 15]) but $\mathrm{Im}(c^{l'}(Z))$ is a point (this follows from part *(1)* too). Hence $\mathrm{Im}(c^{l'}(Z))$ is a hypersurface for any $m\geq 1$. In particular, the property that $\mathrm{Im}(c^{l'}(Z))$ is a hypersurface is not a topological property.

Example 4.1.8 (*Superisolated singularities*) Assume that (X,o) is a hypersurface superisolated singularity associated with an irreducible projective curve whose link is a rational homology sphere. More precisely, $(X,o)=\{F(x_1,x_2,x_3)=0\}$, where the homogeneous terms F_i of F are as follows: $\{F_d=0\}$ defines an irreducible rational cuspidal curve in $\mathbb{CP}^2$ and $\{F_{d+1}=0\}\cap \mathrm{Sing}\{F_d=0\}$ is empty in $\mathbb{CP}^2$. (For details see [11, 12, 14].) Consider the minimal good resolution and let E_0 be the irreducible exceptional curve corresponding to C (the exceptional curve of the first blow up of the maximal ideal). Assume that $l'=-kE_0^*$ for some $k\geq 1$ and $Z\geq Z_K$. For any $\mathbf{m}=(m_1,m_2,m_3)\in\mathbb{Z}^3_{\geq 0}$ write $|\mathbf{m}|=\sum_i m_i$. Then by the discussion from [14, 11.2] one has the following facts: $p_g=d(d-1)(d-2)/6=\#\{\mathbf{m}:|\mathbf{m}|\leq d-3\}$, this is exactly the cardinality of the set of forms of type $\mathbf{x}^{\mathbf{m}}\omega$, where ω is the Gorenstein form. The pole order of ω along E_0 is $d-2$, and the vanishing order of $\mathbf{x}^{\mathbf{m}}$ along E_0 is $|\mathbf{m}|$. The classes of the forms $\{\mathbf{x}^{\mathbf{m}}\omega\}_{\mathbf{m}}$ constitute a basis in $H^0(\Omega^2_{\tilde{X}}(Z))/H^0(\Omega^2_{\tilde{X}})$. Hence, for $0\leq s\leq d-2$ one has $g_s=\dim\mathcal{G}_{sE_0}=\#\{\mathbf{m}:d-2-s\leq|\mathbf{m}|\leq d-3\}$ and $h^1(\mathcal{O}_Z)-g_s=\binom{d-s}{3}$. In particular,

$$d_Z(-kE_0^*)=\min_{0\leq s\leq d-2}\{ks+\tbinom{d-s}{3}\}.$$

In [14, 11.2] $d_Z(-kE_0^*)$ was computed in a different way as $\sum_{j=0}^{d-3}\min\{k,\binom{j+2}{2}\}$. The identification of the two numerical answers is left to the reader. (Use $\sum_{j=0}^{t}\binom{j+2}{2}=\binom{t+3}{3}$.)

Example 4.1.9 For **weighted homogeneous germs** (and $l'=-kE_0^*$, where E_0 is the central vertex of the star shaped graph) $d_Z(l')$ was computed by a similar method in [14, §12].

Remark 4.1.10 (1) In Theorems 3.2.2 and 4.1.2 (and Corollaries 3.2.4 and 4.1.3 as well) the functions $\mathbf{s}\mapsto e_{\mathbf{s}}$ and $\mathbf{s}\mapsto h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$ serve as ‘test-functions’: “if this common value $d_{\mathbf{s}^{v,k}}$ equals the test value, then $d_{\mathbf{s}}=d_{\mathbf{s}^{v,k}}$, otherwise $d_{\mathbf{s}}=d_{\mathbf{s}^{v,k}}+1$”. Via this fact in mind, the second algorithm is rather surprising: the test function for each fixed v depends only on $\mathbf{s}\mapsto\min_{0\leq k_v\leq a_v}s_{v,k_v}=(l_{\mathbf{s}})_v$, hence does not depend on the number of integers $\{s_{v,k_v}\}_{0\leq k_v\leq a_v}$, or, on a_v. However, the final output, namely $d_{\mathbf{s}}$ (and the right hand side of (4.1.4) and the algorithm itself) do depend on l'. We encourage the reader to work out the algorithm for an example when $a_v\geq 2$ (say, for $-l'=2E_v^*$).

(2) Notice that the formulas $\min_{\mathbf{s}}(|\mathbf{s}|+h^1(Z)-g_{\mathbf{s}})$ and $\min_{\mathbf{s}}(|\mathbf{s}|+e_{\mathbf{s}})$ can be defined without any restriction on the numbers $g_{\mathbf{s}}$ and $e_{\mathbf{s}}$, however in our case these numbers are restricted. For example we have $\min_{\mathbf{s}\geq\mathbf{s}_1}(|\mathbf{s}|-|\mathbf{s}_1|+h^1(Z)-g_{\mathbf{s}})-\min_{\mathbf{s}\geq\mathbf{s}_1^{v,k}}(|\mathbf{s}\geq\mathbf{s}_1^{v,k}|+h^1(Z)-g_{\mathbf{s}})\in\{0,1\}$ for all $v,k,\mathbf{s}_1$. Or, $g_{\mathbf{s}}\leq|\mathbf{s}|$ for all $\mathbf{s}$ if and only if $\chi(-l')<\chi(-l'+l)$ for all $Z\geq l>0$ (cf. Example 4.1.7(2)).

(3) **(Bounds for** $\operatorname{codim}\operatorname{Im}c^{l'}(Z)$**)** In some expression the codimension of $\operatorname{Im}(c^{l'}(Z))$ appears more naturally. E.g., we have the following two general statements from [14, Prop. 5.6.1] (under the conditions of Corollary 4.1.5):

(a) $h^1(Z,\mathcal{L})\geq\operatorname{codim}\operatorname{Im}(c^{l'}(Z))$ for any $\mathcal{L}\in\operatorname{Im}(c^{l'}(Z))$. Equality holds whenever $\mathcal{L}$ is generic in $\operatorname{Im}(c^{l'}(Z))$.

(b) $\operatorname{codim}\operatorname{Im}c^{l'}(Z)\geq\chi(-l')-\min_{0\leq l\leq Z}\chi(-l'+l)$, and this inequality is strict whenever $c^{l'}(Z)$ is not dominant. (This can be compared with the discussion from Example 4.1.7(3).)

Note that Corollary 4.1.5 reads as:

$$\operatorname{codim}\operatorname{Im}(c^{l'}(Z))=\max_{0\leq Z_1\leq Z}\{h^1(\mathcal{O}_{Z_1})-(l',Z_1)\}. \tag{4.1.11}$$

4.1.12. Before we state the next theorem let us emphasise the obvious fact that for any $0\leq Z_1\leq Z$ the natural restriction (linear projection) $r:H^1(\mathcal{O}_Z)\to H^1(\mathcal{O}_{Z_1})$ is surjective, hence for any irreducible constructible subset $C_1\subset H^1(\mathcal{O}_{Z_1})$ one has $\dim r^{-1}(C_1)-\dim C_1=h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})$.

However, though the restriction of r to $\operatorname{Im}(c^{l'}(Z))\to\operatorname{Im}(c^{l'}(Z_1))$ is dominant, in general $\dim\operatorname{Im}(c^{l'}(Z))$ can be smaller than $\dim r^{-1}(\operatorname{Im}(c^{l'}(Z_1)))$.

4.1.13. It is instructive to see that certain extremal geometric phenomenons (indexed by effective cycles) are realized by the very same set of cycles.

Lemma 4.1.14 *The following three sets of cycles coincide (for fixed $Z\geq E$ and $l'\in-\mathcal{S}'$ as above):*

(I) the set of cycles Z_1 with $0\leq Z_1\leq Z$ realizing the minimality in (4.1.6), that is: $d_Z(l')=(l',Z_1)+h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})$.

(II) the set of cycles Z_1 with $0\leq Z_1\leq Z$ such that (i) the map $\mathrm{ECa}^{l'}(Z)\to H^1(\mathcal{O}_{Z_1})$ is birational onto its image, and (ii) the generic fibres of the restriction of r, $r^{im}:\operatorname{Im}(c^{l'}(Z))\to\operatorname{Im}(c^{l'}(Z_1))$, have dimension $h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})$. (That is, the fibers of r^{im} have maximal possible dimension.)

(III) the set of cycles Z_1 with $0\leq Z_1\leq Z$ such that for the generic element $\mathcal{L}^{im}_{gen}\in\operatorname{Im}(c^{l'}(Z))$ and arbitrary section $s\in H^0(Z_1,\mathcal{L}^{im}_{gen})_{reg}$ with divisor D (i) in the (analogue of the Mittag–Lefler sequence associated with the exact sequence

$$0 \to \mathcal{O}_{Z_1} \xrightarrow{\times s} \mathcal{L}^{im}_{gen} \to \mathcal{O}_D \to 0, \textit{ cf. } [14,\ 3.2]),$$

$$0 \to H^0(\mathcal{O}_{Z_1}) \xrightarrow{\times s} H^0(Z_1, \mathcal{L}^{im}_{gen}) \to \mathbb{C}^{(Z_1,l')} \xrightarrow{\delta} H^1(\mathcal{O}_{Z_1}) \to h^1(Z_1, \mathcal{L}^{im}_{gen}) \to 0$$

δ is injective, and (ii) $h^1(Z, \mathcal{L}^{im}_{gen}) = h^1(Z_1, \mathcal{L}^{im}_{gen})$.

Proof For (I)⇒(II) use the following. First recall that $\dim \mathrm{ECa}^{l'}(Z') = (l', Z')$ for any effective cycle Z'. Next, from (4.1.6), there exists an effective cycle $Z_1 \leq Z$, such that $\dim \mathrm{Im}(c^{l'}(Z)) = (l', Z_1) + h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$. But $\dim(\mathrm{Im}(c^{l'}(Z_1))) \leq \dim \mathrm{ECa}^{l'}(Z_1) = (l', Z_1)$ (cf. 2) and $\dim(\mathrm{Im}(c^{l'}(Z))) - \dim(\mathrm{Im}(c^{l'}(Z_1))) \leq h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$. Hence, necessarily we have equalities in both these inequalities. (I)⇐(II) is similar.

For (II)(i)⇔(III)(i) use the fact that δ is the tangent application $T_D \mathrm{Im} c^{l'}(Z_1)$ at D, cf. [14, 3.2], and for (II)(ii)⇔(III)(ii) use Remark 4.1.10(3)(a). □

4.2. Structure theorem for the Abel map The geometric interpretation from Lemma 4.1.14(II) has the following consequence.

Theorem 4.2.1 (Structure theorem) *Fix a resolution $\widetilde{X}$, a cycle $Z \geq E$ and a Chern class $l' \in -\mathcal{S}'$ as above.*

(a) *There exists an effective cycle $Z_1 \leq Z$, such that: (i) the map* $\mathrm{ECa}^{l'}(Z_1) \to H^1(Z_1)$ *is birational onto its image, and (ii) the generic fibres of the restriction of r, $r^{im} : \mathrm{Im}(c^{l'}(Z)) \to \mathrm{Im}(c^{l'}(Z_1))$, have dimension $h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$. (Cf. Lemma* 4.1.14(II).*)*

(b) *In particular, for any such Z_1, the space* $\mathrm{Im}(c^{l'}(Z))$ *is birationally equivalent with an affine fibration with affine fibers of dimension $h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$ over* $\mathrm{ECa}^{l'}(Z_1)$.

(c) *The set of effective cycles Z_1 with property as in (a) has a unique minimal and a unique maximal element denoted by $C_{min}(Z, l')$ and $C_{max}(Z, l')$. Furthermore, $C_{min}(Z, l')$ coincides with the cohomology cycle of the pair $(Z, \mathcal{L}^{im}_{gen})$ (the unique minimal element of the set $\{0 \leq Z_1 \leq Z : h^1(Z, \mathcal{L}^{im}_{gen}) = h^1(Z_1, \mathcal{L}^{im}_{gen})\}$) for the generic $\mathcal{L}^{im}_{gen} \in \mathrm{Im}(c^{l'}(Z))$.*

Proof *(a)* Use Lemma 4.1.14.

(c) Assume that two cycles Z_1 and Z_2 satisfy *(a)*. We claim that $Z' := \max\{Z_1, Z_2\}$ satisfies too.

First, for any cycle Z'' with $Z_1 \leq Z'' \leq Z$, if Z_1 satisfies *(a)(ii)*

then Z'' satisfies too. This applies for Z' too. To prove *(a)(i)* for Z', let us denote by $\mathrm{ECa}^{l'}(Z'')_0 \subset \mathrm{ECa}^{l'}(Z'')$ the set of divisors whose support is disjoint from the singular points of E. If $l' = \sum_v a_v E^*_v$ then $\mathrm{ECa}^{l'}(Z)_0 = \prod_v \mathrm{ECa}^{a_v E^*_v}(Z)_0$. Using this fact one shows that the product $\mathrm{ECa}^{l'}(Z') \to \mathrm{ECa}^{l'}(Z_1) \times \mathrm{ECa}^{l'}(Z_2)$ of the two restrictions $\mathrm{ECa}^{l'}(Z') \to \mathrm{ECa}^{l'}(Z_j)$ $(j = 1, 2)$ is birational onto its image (BioIm). This composed with the product of the maps $\mathrm{ECa}^{l'}(Z_1) \to H^1(Z_1)$ and $\mathrm{ECa}^{l'}(Z_2) \to H^1(Z_2)$ (both BioIm) guarantees that $\mathrm{ECa}^{l'}(Z') \to H^1(Z_1) \times H^1(Z_2)$ is BioIm too. This map writes as the composition $\mathrm{ECa}^{l'}(Z') \to H^1(Z') \to H^1(Z_1) \times H^1(Z_2)$, hence the first term $\mathrm{ECa}^{l'}(Z') \to H^1(Z')$ should be BioIm. Hence the claim and the existence of $C_{max}(Z, l')$ follows.

In order to prove the existence of $C_{min}(Z, l')$, first we claim that the set of cycles Z^{ii}, which satisfy *(a)(ii)* has a unique minimal element Z^{ii}_{min}. This fact via Remark 4.1.10(3)(a) is equivalent with the existence of the (unique) cohomological cycle for the pair $(Z, \mathcal{L}^{im}_{gen})$. This was proved in [14, 5.5], see also [25, 4.8]. Next, we claim that the map $\mathrm{ECa}^{l'}(Z^{ii}_{min}) \to H^1(Z^{ii}_{min})$ is BioIm as well. From the existence of the cycle $C_{max}(\cdot, l')$ (already proved above), applied for Z^{ii}_{min}, there exists a cycle $C_{max}(Z^{ii}_{min}, l') \le Z^{ii}_{min}$, which satisfies *(a)*. In particular, *(a)(ii)* is valid for the pair $C_{max}(Z^{ii}_{min}, l') \le Z^{ii}_{min}$. By the definition of Z^{ii}_{min} the condition *(a)(ii)* is valid for the pair $Z^{ii}_{min} \le Z$ too. Hence, *(a)(ii)* is valid for the pair $C_{max}(Z^{ii}_{min}, l') \le Z$ as well. Therefore, by the definition of Z^{ii}_{min} necessarily $C_{max}(Z^{ii}_{min}, l') = Z^{ii}_{min}$, hence Z^{ii}_{min} satisfies *(a)*. □

5 Example: The case of generic analytic structure

5.1. Let us fix the topological type of a good resolution of a normal surface singularity, and we assume that the analytic type on $\widetilde{X}$ is generic (in the sense of [15], see [9] as well). Recall that in such a situation, if $Z' = \sum n_v E_v$ is a non-zero effective cycle, whose support $|Z'| = \cup_{n_v \neq 0} E_v$ is connected, then by [15, Corollary 6.1.7] one has

$$h^1(\mathcal{O}_{Z'}) = 1 - \min_{|Z'| \le l \le Z',\ l \in L} \{\chi(l)\}.$$

Corollary 5.1.1 *Assume that $\widetilde{X}$ has a generic analytic type, $Z \ge E$ an integral cycle and $l' \in -S'$. For any $0 \le Z_1 \le Z$ write $E_{|Z_1|}$ for $\sum_{E_v \subset |Z_1|} E_v$. Then*

$$d_Z(l') = 1 - \min_{E \le l \le Z} \{\chi(l)\} + \min_{0 \le Z_1 \le Z} \Big\{ (l', Z_1) + \min_{E_{|Z_1|} \le l \le Z_1} \{\chi(l)\} - \chi(E_{|Z_1|}) \Big\}. \tag{5.1.2}$$

In particular, $d_Z(l') = \dim(\mathrm{Im} c^{l'}(Z))$ is topological.

Let us concentrate again on the *codimension* $h^1(\mathcal{O}_Z) - d_Z(l')$ of $\mathrm{Im}(c^{l'}(Z)) \subset \mathrm{Pic}^{l'}(Z)$ instead of the dimension. Then, (5.1.2) reads as

$$\mathrm{codim}\, \mathrm{Im}(c^{l'}(Z)) = \max_{0 \le Z_1 \le Z} \Big\{ -(l', Z_1) - \min_{E_{|Z_1|} \le l \le Z_1} \{\chi(l)\} + \chi(E_{|Z_1|}) \Big\}. \tag{5.1.3}$$

This is a rather complicated combinatorial expression in terms of the intersection lattice L. The next lemma aims to simplify it.

Proposition 5.1.4 *Consider the assumptions of Corollary 5.1.1. Let Z_1 be minimal such that the maximum in (5.1.3) is realized for it. Then* $\min_{E_{|Z_1|} \le l \le Z_1} \{\chi(l)\} = \chi(Z_1)$.

In particular,

$$\operatorname{codim}\operatorname{Im}(c^{l'}(Z)) = \max_{0\le Z_1\le Z}\big\{ -(l',Z_1) - \chi(Z_1) + \chi(E_{|Z_1|})\big\}. \tag{5.1.5}$$

The maximum at the right hand side is realized e.g. for the cohomology cycle of $\mathcal{L}^{im}_{gen} \in \operatorname{Im}(c^{l'}(Z)) \subset \operatorname{Pic}^{l'}(Z)$. *Furthermore,*

$$h^1(Z,\mathcal{L}) \ge \max_{0\le Z_1\le Z}\big\{ -(l',Z_1) - \chi(Z_1) + \chi(E_{|Z_1|})\big\} \tag{5.1.6}$$

for any $\mathcal{L} \in \operatorname{Im}(c^{l'}(Z))$ *and equality holds for generic* $\mathcal{L}^{im}_{gen} \in \operatorname{Im}(c^{l'}(Z))$.

Proof Assume that the minimum $\min_{E_{|Z_1|}\le l\le Z_1}\{\chi(l)\} = \chi(Z_1)$ is realized by some l_1. Then $(l',Z_1) \ge (l',l_1)$ (since $l' \in -\mathcal{S}'$), $\min_{E_{|Z_1|}\le l\le Z_1}\{\chi(l)\} = \min_{E_{|l_1|}\le l\le l_1}\{\chi(l)\}$ and $\chi(E_{|Z_1|}) = \chi(E_{|l_1|})$ hence $-(l',Z_1) - \min_{E_{|Z_1|}\le l\le Z_1}\{\chi(l)\} + \chi(E_{|Z_1|}) \le -(l',l_1) - \min_{E_{|l_1|}\le l\le l_1}\{\chi(l)\} + \chi(E_{|l_1|})$. Since the maximality in (5.1.3) is realized by Z_1, which is minimal with this property, necessarily $Z_1 = l_1$. Next,

$$\begin{aligned}&\max_{0\le Z_1\le Z}\big\{ -(l',Z_1) - \min_{E_{|Z_1|}\le l\le Z_1}\{\chi(l)\} + \chi(E_{|Z_1|})\big\} \ge\\ &\max_{0\le Z_1\le Z}\big\{ -(l',Z_1) - \chi(Z_1) + \chi(E_{|Z_1|})\big\}.\end{aligned}$$

But the maximum at the left hand side is realized by a term from the right.

For the last statement use again Remark 4.1.10(3)(a). □

5.2. The identity (5.1.5), valid for a generic analytic structure of $\widetilde{X}$, extends to an optimal inequality valid for *any analytic structure*.

Theorem 5.2.1 *Consider an* arbitrary *normal surface singularity* (X,o)*, its resolution* $\widetilde{X}$*,* $Z \ge E$ *and* $l' \in -\mathcal{S}'$. *Then* $\operatorname{codim}\operatorname{Im}(c^{l'}(Z)) = h^1(Z,\mathcal{L}^{im}_{gen})$ *(cf. Remark* 4.1.10(3)(a)*) satisfies*

$$\operatorname{codim}\operatorname{Im}(c^{l'}(Z)) \ge \max_{0\le Z_1\le Z}\big\{ -(l',Z_1) - \chi(Z_1) + \chi(E_{|Z_1|})\big\}. \tag{5.2.2}$$

In particular, for any $\mathcal{L} \in \operatorname{Im}(c^{l'}(Z))$ *one also has (everything computed in* $\widetilde{X}$*)*

$$\begin{aligned}&h^1(Z,\mathcal{L}) \ge h^1(Z,\mathcal{L}^{im}_{gen})\\ &= \operatorname{codim}\operatorname{Im}(c^{l'}(Z)) \ge \max_{0\le Z_1\le Z}\big\{ -(l',Z_1) - \chi(Z_1) + \chi(E_{|Z_1|})\big\}.\end{aligned} \tag{5.2.3}$$

Note that the right hand side of (5.2.2) is a sharp topological lower bound for $\operatorname{codim}\operatorname{Im}(c^{l'}(Z))$. The inequality (5.2.2) can also be interpreted as the semi-continuity statement

$$\begin{aligned}&\operatorname{codim}\operatorname{Im}(c^{l'}(Z))(\text{arbitrary analytic structure})\\ &\ge \operatorname{codim}\operatorname{Im}(c^{l'}(Z))(\text{generic analytic structure}).\end{aligned}$$

Proof Consider the identity (4.1.11) applied for an arbitrary $\widetilde{X}$ and for the generic $\widetilde{X}$, denoted by $\widetilde{X}_{gen}$. Then, by semi-continuity of $h^1(\mathcal{O}_{Z_1})$ with respect to the analytic structure as parameter space (see e.g. [15, 3.6]), for any fixed effective cycle $Z_1 > 0$, $h^1(\mathcal{O}_{Z_1})$ computed in $\widetilde{X}$ is greater than or equal to $h^1(\mathcal{O}_{Z_1})$ computed in $\widetilde{X}_{gen}$. Therefore, by (4.1.11) one has codim $\mathrm{Im}(c^{l'}(Z))(\text{in } \widetilde{X}) \geq$ codim $\mathrm{Im}(c^{l'}(Z))(\text{in } \widetilde{X}_{gen})$. Then for $\widetilde{X}_{gen}$ apply (5.1.5). □

Remark 5.2.4 Certain upper bounds for $\{h^1(Z,\mathcal{L})\}_{\mathcal{L}\in \mathrm{Pic}^{l'}(Z)}$, valid for any analytic structure, were established in [14, Prop. 5.7.1] (see alo Remark 5.3.3). However, an optimal upper bound is not known (see [22] for a particular case). Large h^1-values are realized by special strata, their existence is extremely hard to detect.

5.3. The cohomology of $\mathcal{L}^{im}_{gen}(l)$ Assume that $Z \geq E$, $l' \in -\mathcal{S}'$ and let $\mathcal{L}^{im}_{gen}$ be a generic element of $\mathrm{Im}(c^{l'}(Z))$. If the analytic structure of (X, o) is generic, then by Proposition 5.1.4 $h^1(Z, \mathcal{L}^{im}_{gen}) = t_Z(l')$, where $t_Z(l')$ is the topological expression from the right hand side of (5.1.5).

Our goal is to give a topological lower bound for $h^1(Z,\mathcal{L})$, where $\mathcal{L} := \mathcal{L}^{im}_{gen}(l) = \mathcal{L}^{im}_{gen} \otimes \mathcal{O}(l) \in \mathrm{Pic}^{l'+l}(Z)$ whenever $l \in L_{>0}$. In this way we will control the generic element of the 'new' strata $\mathcal{O}(l) \otimes (\mathrm{Im}(c^{l'}(Z)))$ of $\mathrm{Pic}^{l'+l}(Z)$, unreachable directly by the previous result. Our hidden goal is to construct in this way line bundles with 'high' h^1.

For simplicity we will assume that all the coefficients of Z are sufficiently large (even compared with l, hence the coefficients of $Z-l$ are large as well). The monomorphism of sheaves $\mathcal{L}^{im}_{gen}|_{Z-l} \hookrightarrow \mathcal{L}^{im}_{gen}(l)$ gives $h^0(Z-l, \mathcal{L}^{im}_{gen}) \leq h^0(Z, \mathcal{L}^{im}_{gen}(l))$, hence

$$h^1(Z-l, \mathcal{L}^{im}_{gen}) + \chi(Z-l, \mathcal{L}^{im}_{gen}) \leq h^1(Z, \mathcal{L}^{im}_{gen}(l)) + \chi(Z, \mathcal{L}^{im}_{gen}(l)).$$

By a computation regarding χ this transforms into

$$h^1(Z, \mathcal{L}^{im}_{gen}(l)) \geq h^1(Z-l, \mathcal{L}^{im}_{gen}) + \chi(-l'-l) - \chi(-l').$$

If $\widetilde{X}$ is generic and $Z, Z-l \gg 0$ then $h^1(Z-l, \mathcal{L}^{im}_{gen}) = t_{Z-l}(l') = t_Z(l')$, hence

$$h^1(Z, \mathcal{L}^{im}_{gen}(l)) \geq t_Z(l') - \chi(-l') + \chi(-l'-l). \tag{5.3.1}$$

E.g., with the choice $l = -l' \in \mathcal{S}' \cap L_{>0}$ we get that $\mathcal{L}^{im}_{gen}(-l') \in \mathrm{Pic}^0(Z)$ and

$$h^1(Z, \mathcal{L}^{im}_{gen}(-l')) \geq t_Z(l') - \chi(-l'). \tag{5.3.2}$$

Remark 5.3.3 By [14, Prop. 5.7.1] for $Z \gg 0$, $\mathcal{L} \in \mathrm{Pic}(Z)$ with $c_1(\mathcal{L}) \in -\mathcal{S}'$ one has $h^1(Z,\mathcal{L}) \leq p_g$ whenever either $H^0(Z,\mathcal{L}) = 0$ or $\mathcal{L} \in \mathrm{Im}(c^{l'}(Z))$. For other line bundles a weaker bound is established (see [loc. cit.]), which does not guarantee $h^1(\mathcal{L}) \leq p_g$. However, it is not so easy to find singularities and bundles with $h^1(\mathcal{L}) >$

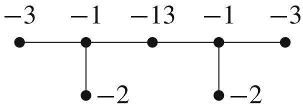
p_g in order to show that such cases indeed might appear. In the next 5.3.4 we provide such an examples (with a recipe to find many others as well) based partly on (5.3.2).

Example 5.3.4 Assume that we can construct a nonrational resolution graph (that is a graph, which does not satisfy Artin's criterion of rationality) which satisfies the following (combinatorial) properties, valid for certain $Z \gg 0$ and $l' \in -\mathcal{S}' \cap L$:

$$\begin{aligned} &(a) \quad t_Z(l') \geq \chi(-l') - \min_{l\geq 0} \chi(-l'+l) + 2, \text{ and} \\ &(b) \quad -l' \leq \max \mathcal{M}, \text{ where } \mathcal{M} := \{l \in L_{>0} : \chi(l) = \min \chi\}. \end{aligned} \tag{5.3.5}$$

Now, if we consider the generic analytic structure supported on this topological type, then $\min_{l\geq 0} \chi(-l'+l) \overset{(b)}{=} \min \chi = 1 - p_g$ (for the second identity use [15, Cor. 5.2.1]), hence $t_Z(l') - \chi(-l') \overset{(a)}{\geq} -1 + p_g + 2 = p_g + 1$. This combined with (5.3.2) gives $h^1(Z, \mathcal{L}^{im}_{gen}(-l')) > p_g$.

Next we show that (5.3.5) can be realized. Consider two copies Γ_1 and Γ_2 of the following graph



The wished graph Γ consists of Γ_1, Γ_2 and a new vertex v, which has two adjacent edges connecting v to the (-13)-vertices of Γ_1 and Γ_2. Let the decoration of v be $-b_v$ where $b_v \gg 0$. One verifies that the minimal cycle is $Z_{min} = (b_v - 2)E^*_v$, whose E_v-multiplicity is 1. We set $-l' := Z_{min}$. Since $\max \mathcal{M} \in \mathcal{S}_{an} \subset \mathcal{S}' \cap L$ (cf. [15, 5.7]) we get that $-l' = Z_{min} \leq \max \mathcal{M}$. One verifies that $\chi(Z_{min}) = -3$ (e.g. by Laufer's criterion), and also that $\min \chi = -5$ (realized e.g. for $2Z_{min} - E_v$). Therefore $\chi(-l') - \min_{l\geq 0} \chi(-l'+l) + 2 = -3 + 5 + 2 = 4$. On the other hand, the expression (under max) in (5.1.5) for $Z_1 = Z_{min}(\Gamma_1) + Z_{min}(\Gamma_2)$ supported on $\Gamma \setminus v$ is 4, hence $t_Z(l') \geq 4$.

6 Geometrical aspects behind the lower bound Theorem 5.2.1

6.1. Let us discuss with more details the geometry behind the inequality (5.2.2). Along the discussion we will provide a second independent proof of it and we also provide several examples, which show its sharpness/weakness in several situations. Similar construction (with similar philosophy) will appear in forthcoming manuscripts on the subject as well. The construction of the present section shows also in a conceptual way how one can produce different sharp lower bounds for sheaf cohomologies (for another case see e.g. Sect. 7).

We provide the new proof in several steps. First, we define a topological lower bound for $\operatorname{codim} \operatorname{Im}(c^{l'}(Z))$, which (a priori) will have a more elaborated form then the right hand side $t_Z(l')$ of (5.2.2). Then via several steps we will simplify it and we show that in fact it is exactly $t_Z(l')$.

Definition 6.1.1 For any $Z>0$ with $|Z|$ connected we define $D(Z,l')$ as 0 if $c^{l'}(Z)$ is dominant and 1 otherwise. (For a criterion see 2.) Furthermore, set

$$T(Z,l') := \chi(-l') - \min_{0\leq l\leq Z, l\in L} \chi(-l'+l) + D(Z,l'). \tag{6.1.2}$$

By [14, Theorem 5.3.1] for any singularity (X,o), any resolution $\widetilde{X}$, any $Z>0$ and $l'\in L'$, and for $\mathcal{L}_{gen}$ generic in $\mathrm{Pic}^{l'}(Z)$ one has

$$h^1(Z,\mathcal{L}_{gen}) = \chi(-l') - \min_{0\leq l\leq Z, l\in L} \chi(-l'+l). \tag{6.1.3}$$

By [14, Prop. 5.6.1], see also 4.1.10(3), for any $Z\geq E$ and for any $l'\in -\mathcal{S}'$, if $\mathcal{L}^{im}_{gen}$ is a generic element of $\mathrm{Im}(c^{l'}(Z))$, then $h^1(Z,\mathcal{L}^{im}_{gen}) = \mathrm{codim}\,\mathrm{Im}(c^{l'}(Z))$ satisfies (the semicontinuity)

$$\begin{aligned} h^1(Z,\mathcal{L}^{im}_{gen}) &\geq \chi(-l') - \min_{0\leq l\leq Z, l\in L} \chi(-l'+l) + D(Z,l') \\ &= h^1(Z,\mathcal{L}_{gen}) + D(Z,l') = T(Z,l'). \end{aligned} \tag{6.1.4}$$

Remark 6.1.5 Assume that $Z>0$ is a nonzero cycle with connected support $|Z|$, but with $Z\not\geq E$. Then the statements from (6.1.4) remain valid for such Z once we replace l' by its restriction $R(l')$, where $R: L'\to L'(|Z|)$ is the natural cohomological operator dual to the natural homological inclusion $L(|Z|)\hookrightarrow L$. (For this apply the statement for the singularity supported on $|Z|$.) On the other hand, for $l\in L(|Z|)$ one has $\chi(-R(l'))-\chi(-R(l')+l) = -\chi(l) - (R(l'),l)_{L(|Z|)} = -\chi(l) - (l',l) = \chi(-l')-\chi(-l'+l)$. Hence, in fact, (6.1.4) remains valid in its original form for any such $Z>0$ with $|Z|$ connected.

Example 6.1.6 The difference $h^1(Z,\mathcal{L}^{im}_{gen}) - h^1(Z,\mathcal{L}_{gen})$ can be arbitrary large. Indeed, let us start with a singularity with an arbitrary analytic structure, we fix a resolution $\widetilde{X}$ with dual graph Γ, and we distinguish a vertex, say v_0, associated with the irreducible divisor E_0. Let k ($k>0$) be the number of connected components of $\Gamma\setminus v_0$, and we assume that each of them is non-rational. Furthermore, we choose $Z\gg 0$, hence $h^1(\mathcal{O}_Z)=p_g$. Let $\widetilde{X}|_{\mathcal{V}\setminus v_0}$ be a small neighbourhood of $\cup_{v\neq v_0}E_v$, let $\{\widetilde{X}_i\}_{i=1}^k$ be its connected components, and set $p_{g,i}=h^1(\mathcal{O}_{\widetilde{X}_i})$ for the geometric genus of the singularities obtained from $\widetilde{X}_i$ by collapsing its exceptional curves. Write also $\Gamma\setminus v_0=\cup_i\Gamma_i$. We also assume that $-l'=nE_0^*$ with $n\gg 0$.

Since n is large, $\mathrm{Im}(\widetilde{c}^{l'}(Z)) = A_Z(l')$, hence $d_Z(l')=e_Z(l')=p_g-\sum_i p_{g,i}$, cf. [14, Th. 6.1.9] or Theorem 2.2.5 here. Hence, cf. (6.1.4), $\mathrm{codim}(\mathrm{Im}\widetilde{c}^{l'}(Z)) = h^1(\mathcal{O}_Z)-d_Z(l') = h^1(Z,\mathcal{L}^{im}_{gen}) = \sum_i p_{g,i}$ (in particular, $\widetilde{c}^{l'}$ is not dominant).

Next we compute $h^1(Z,\mathcal{L}_{gen}) = \chi(nE_0^*) - \min_{l\geq 0}\chi(nE_0^*+l)$. Write l as $l_0E_0+\widetilde{l}$, where $\widetilde{l}$ is supported on $\cup_{v\neq v_0}E_v$. Then $\chi(nE_0^*)-\chi(nE_0^*+l) = -\chi(l)-nl_0$. If $l_0=0$ then $-\chi(l)=-\chi(\widetilde{l})$, and its maximal value is $M:=\sum_i(-\min\chi(\Gamma_i))$. On the other hand, if $l_0 > 0$ then for $n > -M - \min\chi$ one has $-\chi(l) - l_0 n < M$. Hence $h^1(Z, \mathcal{L}_{gen}) = \chi(nE_0^*) - \min_{l\geq 0}\chi(nE_0^*+l) = \sum_i(-\min\chi(\Gamma_i))$.

Now, $p_{g,i} \geq 1-\min\chi(\Gamma_i)$ (cf. [26] or [15]), hence $h^1(Z, \mathcal{L}^{im}_{gen}) - h^1(Z, \mathcal{L}_{gen}) \geq k$.

6.1.7 We wish to estimate $h^1(Z, \mathcal{L}^{im}_{gen})$. Note that the estimate given by (6.1.4), that is, $h^1(Z, \mathcal{L}^{im}_{gen}) \geq T(Z, l')$, sometimes is week, see the previous example. However, surprisingly, if we replace Z by a smaller cycle $Z' \leq Z$, then we might get a better bound. More precisely, first note that if $\mathcal{L}^{im}_{gen}$ is a generic element of $\mathrm{Im}(c^{l'}(Z))$, and $0 < Z' \leq Z$, then its restriction $r(\mathcal{L}^{im}_{gen})$ (via $r : \mathrm{Pic}^{l'}(Z) \to \mathrm{Pic}^{R(l')}(Z')$) is a generic element of $\mathrm{Im}(c^{l'}(Z'))$. If Z' has more connected components, $Z' = \sum_i Z'_i$ (where each $|Z'_i|$ is connected and $|Z'_i| \cap |Z'_j| = \emptyset$ for $i \neq j$), then for each Z'_i we can apply (6.1.4). Therefore, we get

$$h^1(Z, \mathcal{L}^{im}_{gen}) \geq h^1(Z', r(\mathcal{L}^{im}_{gen})) = \sum_i h^1(Z'_i, r(\mathcal{L}^{im}_{gen})) \geq \sum_i T(Z'_i, l'). \quad (6.1.8)$$

Define

$$t(Z, l') := \max_{0<Z'\leq Z} \sum_i T(Z'_i, l') = \max_{0<Z'\leq Z} \Big(\sum_i (\chi(-l') - \min_{0\leq l_i\leq Z'_i} \chi(-l'+l_i) + D(Z'_i, l')) \Big). \quad (6.1.9)$$

(Here there is no need to restrict l', cf. Remark 6.1.5.) Hence (6.1.8) reads as

$$h^1(Z, \mathcal{L}^{im}_{gen}) \geq t(Z, l'). \quad (6.1.10)$$

In this estimate the point is the following: though $\sum_i(\chi(-l') - \min_{0\leq l_i\leq Z'_i}\chi(-l'+l_i)) = \chi(-l') - \min_{0\leq l\leq Z'}\chi(-l'+l)$ is definitely not larger than $\chi(-l') - \min_{0\leq l\leq Z}\chi(-l'+l)$, the number of components of Z' might be large, and the sum of the 'non-dominant' contribution terms $\sum_i D(Z'_i, l')$ might increase the right hand side of (6.1.10)—compared with $T(Z, l')$—drastically.

Example 6.1.11 **(Continuation of Example** 6.1.6) The last computation of Example 6.1.6 shows that the maximum of $\chi(nE_0^*) - \min_{l\geq 0}\chi(nE_0^*+l)$ is obtained for $l_0 = 0$ and $T(Z, l') = 1 + \sum_i(-\min\chi(\Gamma_i))$. Hence, taking $Z' = \sum_i Z'_i$, each Z'_i supported on Γ_i and large, we get that the restriction of l' is zero and $\sum_i T(Z'_i, l') = \sum_i(1-\min\chi(\Gamma_i)) = T(Z, l') + k - 1$.

Summarized (also from Example 6.1.6), for any analytic type one has $\sum_i p_{g,i} = h^1(Z, \mathcal{L}^{im}_{gen}) \geq t(Z, l') \geq \sum_i T(Z'_i, l') = \sum_i(1-\min\chi(\Gamma_i))$. However, if $\widetilde{X}$ is generic then $p_{g,i} = 1-\min\chi(\Gamma_i)$ (cf. [15]), hence, all the inequalities transform into equalities. Hence, for generic analytic structure $h^1(Z, \mathcal{L}^{im}_{gen}) = t(Z, l')$, that is, (6.1.10) provides the optimal sharp topological lower bound.

Note also that both $t(Z, l')$ and $\sum_i(1-\min\chi(\Gamma_i))$ are topological, hence if they agree for $\widetilde{X}$ generic, then they are in fact equal. Since $p_{g,i} - 1 + \min\chi(\Gamma_i)$ for arbitrary analytic type can be considerably large, for *arbitrary* analytic types the inequality (6.1.10) can be rather week.

6.2 Our goal is to simplify the expression (6.1.9) of $t(Z, l')$.

First we analyse the set of cycles Z' for which the maximum in the right hand side of (6.1.9) can be realized. E.g., if $c^{l'}(Z)$ is dominant (equivalently, $t(Z, l') = 0$, cf. 2) then any $0 \leq Z' \leq Z$ realizes the maximum 0 (with all $l_i = 0$). (Indeed, use the fact that $D(Z_2, l') \geq D(Z_1, l')$ for $Z_2 \geq Z_1$ and $|Z_i|$ connected.)

In the next Lemmas 6.2.1 and 6.2.4 we will assume that $c^{l'}(Z)$ is not dominant.

Lemma 6.2.1 (a) *Assume that Z' is a minimal cycle (or a cycle with minimal number of connected components) among those cycles which realize the maximum in the right hand side of* (6.1.9). *Then $D(Z'_i, l') = 1$ for all i.*

(b) *If $D(Z'_i, l') = 1$ then the minimal value $\min_{0 \leq l_i \leq Z'_i} \chi(-l' + l_i)$ can be realized by $l_i > 0$.*

Proof *(a)* Otherwise, $c^{l'}(Z'_i)$ is dominant, and by 2 $\chi(-l') - \min_{0 \leq l_i \leq Z'_i} \chi(-l' + l_i) = 0$ (realized for $l_i = 0$). Hence $T(Z'_i, l') = 0$, that is, the right hand side of (6.1.9) is realized by $Z' - Z'_i$ too, contradicting the minimality of Z'. *(b)* If the wished minimum is realized by $l_i = 0$, and *only* by $l_i = 0$, then by 2 $c^{l'}(Z'_i)$ is dominant, contradicting $D(Z'_i, l') = 1$. □

Example 6.2.2 Though in Example 6.1.6 we have shown that $h^1(Z, \mathcal{L}^{im}_{gen}) = t(Z, l')$ can be much larger than $T(Z, l')$ (that is, the maximizing Z' usually should be necessarily strict smaller than Z), in some cases $Z' = Z$ still works. Indeed, we claim that

$$\text{if the } E^*\text{–support } I \text{ of } l' \text{ is included in the set of end vertices of } \Gamma, \quad \text{then } t(Z, l') = T(Z, l').$$

Let Z' be a cycle for minimal number n of connected components $\{Z'_i\}_{i=1}^n$ for which the right hand side of (6.1.9) is realized. We claim that $n = 1$. Indeed, by Lemma 6.2.1, each $D(Z'_i, l') = 1$. Let l_i be a cycle which realizes $\chi(-l') - \min_{0 \leq l \leq Z'_i} \chi(-l' + l)$. By Lemma 6.2.1 we can assume $l_i \neq 0$.

If $n > 1$ then let Z_1 and Z_2 be two adjacent component, which means, that there is a vertex $u \in |Z'_1|$ and $v \in |Z'_2|$ and a (minimal) path $u_1 = u, u_2, \ldots, u_t = v$, such that $u_2, \ldots, u_{t-1} \notin |Z'|$ and u_k and u_{k+1} are neighbours in the resolution graph. Moreover, define a new cycle by $Z'_{1,new} = Z'_1 + Z'_2 + \sum_{2 \leq k \leq t-1} E_{u_k}$ and $Z'_{new} = Z'_{1,new} + \sum_{3 \leq i \leq n} Z'_i$. Similarly, let us have a minimal path between $|l_1|$ and $|l_2|$: vertices $w_1, \ldots, w_l$, such that $w_1 \in |l_1|$ and $w_l \in |l_2|$, $w_2, \ldots, w_{l-1} \notin |l_1| \cup |l_2|$ and w_k, w_{k+1} are neighbours in the resolution graph. Then define $l_{1,new} = l_1 + l_2 + \sum_{2 \leq k \leq l-1} E_{w_k}$. The point is that the vertices $w_2, \ldots, w_{l-1}$ are not end vertices, in particular $(l', \sum_{2 \leq k \leq l-1} E_{w_k}) = 0$.

Note also that $D(Z'_{1,new}, l') = 1$. Then a computation gives that

$$\chi(-l') - \chi(-l' + l_{1,new}) + D(Z'_{1,new}, l') \geq T(Z_1, l') + T(Z_2, l'), \quad (6.2.3)$$

or, $T(Z_{1,new}, l') \geq T(Z_1, l') + T(Z_2, l')$, contradicting the minimality of Z'. Hence necessarily $n = 1$.

On the other hand, if Z' is connected, then $T(Z', l') \le T(Z, l')$, hence the maximal value in the right hand side of (6.1.10) is realized for Z as well (and maybe by several other smaller cycles too; here we minimalized $\#|Z'|$ by increasing Z').

The present example together with Examples 6.1.6 and 6.1.11 show that the structure of possible cycles Z' for which the maximality in (6.1.9) realizes can be rather subtle.

Lemma 6.2.4 *Assume that Z' is a minimal cycle among those cycle which realizes the maximum in the right hand side of* (6.1.9). *Then the following facts hold:*

(a) $\min_{0\le l_i\le Z'_i} \chi(-l'+l_i)$ *is realized by* $l_i = Z'_i$.

(b) $\min_{0\le l_i\le Z'_i} \chi(l)$ *is realized by* $l_i = Z'_i$.

(c) $t(Z', l') = t(Z, l') = \sum_i \big(-(Z'_i, l') - \chi(Z'_i) + 1 \big)$.

Proof (a) For each Z'_i let l_i be minimal non-zero cycle (cf. Lemma 6.2.1) such that $M_i := \chi(-l') - \min_{0\le l\le Z'_i} \chi(-l'+l)$ is realized by l_i. Let $l_i = \cup_k l_{i,k}$ be its decomposition into cycles with $|l_{i,k}|$ connected and disjoint. Since $M_i = -\chi(l_i) - (l', l_i) \ge 0$, there exists k such that $\chi(-l') - \chi(-l'+l_{i,k}) = -\chi(l_{i,k}) - (l', l_{i,k}) \ge 0$, hence by the criterion from 2 the Abel map $c^{l'}(l_{i,k})$ must be non-dominant. Thus (using also $D(Z'_i, l') = 1$ from Lemma 6.2.1(a))

$$\sum_k T(l_{i,k}, l') \ge \chi(-l') - \chi(-l'+l_i) + 1 = T(Z'_i, l'). \tag{6.2.5}$$

In particular, by the minimality of Z'_i, $Z'_i = l_i$.

(b) By part *(a)* $\chi(Z'_i) + (Z'_i, l') \le \chi(l_i) + (l_i, l')$ for any $0 \le l_i \le Z'_i$. But, since $l' \in -\mathcal{S}'$, $(Z'_i, l') \ge (l_i, l')$, hence $\chi(Z'_i) \le \chi(l_i)$ for any $0 \le l_i \le Z'_i$. Part *(c)* follows from (6.1.9) and *(a)*. $\square$

Recall that in 5 we defined $t_Z(l') := \max_{0\le Z'\le Z} \big\{ -(l', Z') - \chi(Z') + \chi(E_{|Z'|}) \big\}$.

Corollary 6.2.6 $t(Z, l') = t_Z(l')$.

Proof If $c^{l'}(Z)$ is dominant then both sides are zero. Otherwise, by Lemma 6.2.4(c) (with its notations) $t(Z, l') = \sum_i \big(-(Z'_i, l') - \chi(Z'_i) + 1 \big) \le t_Z(l')$. On the other hand, let us fix some $Z' = \cup_i Z'_i$ for which the maximum in $t_Z(l')$ is realized. Then we can assume that each $c^{l'}(Z'_i)$ is not dominant. Then $-(Z'_i, l') - \chi(Z'_i) + 1 = \chi(-l') - \chi(-l' + Z'_i) + 1 \le \chi(-l') - \min_{0\le l_i\le Z'_i} \chi(-l'+l_i) + D(Z'_i, l')$. Hence $t_Z(l') \le t(Z, l')$ too. $\square$

Remark 6.2.7 The second proof of Theorem 5.2.1 follows from (6.1.10) and Corollary 6.2.6.

7 The $\mathcal{L}_0$-projected Abel map

In this section we introduce a new object, a modification of the Picard group $\mathrm{Pic}(Z)$, which will play a key role in the cohomology computation of the shifted line bundles of type $\{\mathcal{L}_0 \otimes \mathcal{L}\}_{\mathcal{L}\in \mathrm{Im}(c^{l'}(Z))}$.

7.1. The $\mathcal{L}_0$-projected Picard group Let (X, o) be a normal surface singularity. For simplicity we assume (as always in this manuscript) that the link is a rational homology sphere. Let $\widetilde{X}$ be one of its good resolutions and $Z \geq E$ an effective cycle. Fix also $\mathcal{L}_0 \in \mathrm{Pic}(Z)$ such that $H^0(Z, \mathcal{L}_0)_{reg} \neq \emptyset$ (cf. 2). Choose $s_0 \in H^0(Z, \mathcal{L}_0)_{reg}$ arbitrarily, and write $\mathrm{div}(s_0) = D_0 \in \mathrm{ECa}^{l'_0}(Z)$, where $l'_0 = c_1(\mathcal{L}_0) \in -\mathcal{S}'$. Motivated by the exponential exact sequence of sheaves $0 \to \mathbb{Z}_Z \stackrel{i}{\to} \mathcal{O}_Z \to \mathcal{O}_Z^* \to 0$, we define $\mathcal{L}_0^*$ as the cokernel of the composed map $\mathbb{Z}_Z \stackrel{i}{\to} \mathcal{O}_Z \xrightarrow{s_0} \mathcal{L}_0$, where the second morphism is the multiplication by (restrictions of) s_0. Then we have the following commutative diagram of sheaves:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_Z & \stackrel{i}{\longrightarrow} & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_Z^* & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow s_0 & & \downarrow s_0^* & & \\
0 & \longrightarrow & \mathbb{Z}_Z & \longrightarrow & \mathcal{L}_0 & \longrightarrow & \mathcal{L}_0^* & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{D_0} & = & \mathcal{O}_{D_0} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where s_0^* is induced by s_0. At cohomological level we get the (identical/renamed) diagrams

$$\begin{array}{ccccccc}
 & H^0(\mathcal{O}_{D_0}) & = & H^0(\mathcal{O}_{D_0}) & & & \\
 & \downarrow \delta^0 & & \downarrow \delta & & & \\
0 \to & H^1(\mathcal{O}_Z) & \to & H^1(\mathcal{O}_Z^*) & \stackrel{c_1}{\to} & L' \to 0 \\
 & \downarrow s^0 & & \downarrow s & & \downarrow = \\
0 \to & H^1(\mathcal{L}_0) & \to & H^1(\mathcal{L}_0^*) & \stackrel{c_1}{\to} & L' \to 0 \\
 & \downarrow & & \downarrow & & \\
 & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & H^0(\mathcal{O}_{D_0}) & = & H^0(\mathcal{O}_{D_0}) & & & \\
 & \downarrow \delta^0 & & \downarrow \delta & & & \\
0 \to & \mathrm{Pic}^0(Z) & \to & \mathrm{Pic}(Z) & \stackrel{c_1}{\to} & L' \to 0 \\
 & \downarrow s^0 & & \downarrow s & & \downarrow = \\
0 \to & \mathrm{Pic}^0_{\mathcal{L}_0}(Z) & \to & \mathrm{Pic}_{\mathcal{L}_0}(Z) & \stackrel{c_1}{\to} & L' \to 0 \\
 & \downarrow & & \downarrow & & \\
 & 0 & & 0 & &
\end{array}$$

where we use the notation $\mathrm{Pic}_{\mathcal{L}_0}(Z) := H^1(Z, \mathcal{L}_0^*)$—and call it *the $\mathcal{L}_0$–projected Picard group*— and (its linearization) $\mathrm{Pic}^0_{\mathcal{L}_0}(Z) := H^1(Z, \mathcal{L}_0)$. Note that the classical first Chern class map c_1 factorizes to a well-defined map $c_1 : \mathrm{Pic}_{\mathcal{L}_0}(Z) \to L'$. Set also $\mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z) := c_1^{-1}(l')$ for any $l' \in L'$; it is an affine space isomorphic to $\mathrm{Pic}^{l'}(Z)/\mathrm{Im}(\delta)$ associated with the vector space $\mathrm{Pic}^0_{\mathcal{L}_0}(Z) = H^1(Z, \mathcal{L}_0) = H^1(\mathcal{O}_Z)/\mathrm{Im}(\delta^0)$.

The corresponding vector spaces appear in the following exact sequences as well. Let us take another line bundle $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ without fixed components, $s \in H^0(Z, \mathcal{L})_{reg}$ and $D := \mathrm{div}(s)$. Then one can take the exact sequences $0 \to \mathcal{O}_Z \stackrel{s}{\to} \mathcal{L} \to \mathcal{O}_D \to 0$ and $0 \to \mathcal{L}_0 \stackrel{s}{\to} \mathcal{L}_0 \otimes \mathcal{L} \to \mathcal{O}_D \to 0$. They induce (at cohomology, or 'tangent' vector space level) the following commutative diagram

$$\begin{array}{ccccccc}
 & & H^0(\mathcal{O}_{D_0}) & = & H^0(\mathcal{O}_{D_0}) & & \\
 & & \downarrow \delta^0 & & \downarrow & & \\
H^0(\mathcal{O}_D) & \stackrel{\delta^0_{\mathcal{L}}}{\to} & H^1(\mathcal{O}_Z) & \stackrel{s}{\to} & H^1(\mathcal{L}) & \to 0 & \\
\big\downarrow = & & \big\downarrow s^0_{\mathcal{L}_0} & & \big\downarrow & & \\
H^0(\mathcal{O}_D) & \stackrel{\bar{\delta}^0_{\mathcal{L}}}{\to} & H^1(\mathcal{L}_0) & \stackrel{s}{\to} & H^1(\mathcal{L}_0\otimes\mathcal{L}) & \to 0 & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

This is related with the Abel map $c^{l'}(Z) : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ as follows. Recall from [14, 3.2.2] that the tangent linear map $T_D\, c^{l'}(Z) : T_D\, \mathrm{ECa}^{l'}(Z) \to T_{\mathcal{L}}\, \mathrm{Pic}^{l'}(Z)$ can be identified with $\delta^0_{\mathcal{L}} : H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_Z)$. Therefore, if $\mathcal{L} = \mathcal{L}^{im}_{gen}$ is a generic element of $\mathrm{Im}(c^{l'}(Z))$ then $\mathrm{codim}\, \mathrm{Im}(c^{l'}(Z)) = \dim\, H^1(\mathcal{O}_Z)/\mathrm{Im}(\delta^0_{\mathcal{L}}) = h^1(Z, \mathcal{L})$. Similarly, consider the composition

$$c^{l'}_{\mathcal{L}_0}(Z) : \mathrm{ECa}^{l'}(Z) \xrightarrow{c^{l'}(Z)} \mathrm{Pic}^{l'}(Z) \xrightarrow{s^0_{\mathcal{L}_0}} \mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z).$$

We call it *the $\mathcal{L}_0$-projection of the Abel map* $c^{l'}(Z)$. Using the previous paragraph we obtain that the tangent linear map $T_D\, c^{l'}_{\mathcal{L}_0}(Z) : T_D\, \mathrm{ECa}^{l'}(Z) \to T_{\mathcal{L}}\, \mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z)$ can be identified with $\bar{\delta}^0_{\mathcal{L}} = s^0_{\mathcal{L}_0} \circ \delta^0_{\mathcal{L}} : H^0(\mathcal{O}_D) \to H^1(\mathcal{L}_0)$. Therefore, if $\mathcal{L}$ is a generic element of $\mathrm{Im}(c^{l'}_{\mathcal{L}_0}(Z))$ (or, it is the image by $s_{\mathcal{L}_0}$ of a generic element $\mathcal{L}^{im}_{gen}$ of $\mathrm{Im}(c^{l'}(Z))$) then

$$\mathrm{codim}\, \mathrm{Im}(c^{l'}_{\mathcal{L}_0}(Z)) = \dim\, H^1(\mathcal{L}_0)/\mathrm{Im}(\bar{\delta}^0_{\mathcal{L}}) = h^1(Z, \mathcal{L}_0 \otimes \mathcal{L}). \qquad (7.1.1)$$

This fact fully motivates the next point of view: if one wishes to study $h^1(Z, \mathcal{L}_0 \otimes \mathcal{L})$ with $\mathcal{L}_0$ fixed and $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ then—as a tool—the right Abel map is the $\mathcal{L}_0$-projected $c^{l'}_{\mathcal{L}_0}(Z)$.

7.2 The cohomology $h^1(Z, \mathcal{L}_0 \otimes \mathcal{L})$. Using the exact sequence $H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_Z) \stackrel{s}{\to} H^1(Z, \mathcal{L}) \to 0$ and $h^0(\mathcal{O}_D) = (l', Z)$ we obtain the inequality $h^1(Z, \mathcal{L}) \geq h^1(\mathcal{O}_Z) - (l', Z)$. Usually it is not sharp, since $\delta^0_{\mathcal{L}}$ might not be injective. However, as in the prototype construction from Sect. 6 (and even in its preceding sections), if we consider any $Z_1 \leq Z$ then we also have $h^1(Z, \mathcal{L}) \geq h^1(Z_1, \mathcal{L}) \geq h^1(\mathcal{O}_{Z_1}) - (l', Z_1)$, hence $h^1(Z, \mathcal{L}) \geq \max_{Z_1 \leq Z}\{h^1(\mathcal{O}_{Z_1}) - (l', Z_1)\}$, and, remarkably, this for the generic $\mathcal{L}^{im}_{gen} \in \mathrm{Im}(c^{l'}(Z))$ is an equality (cf. (4.1.11)).

Similarly, using the exact sequence $H^0(\mathcal{O}_D) \to H^1(Z, \mathcal{L}_0) \stackrel{s}{\to} H^1(Z, \mathcal{L}_0 \otimes \mathcal{L}) \to 0$ we obtain $h^1(Z, \mathcal{L}_0 \otimes \mathcal{L}) \geq h^1(\mathcal{L}_0) - (l', Z)$. Again, this usually is not sharp.

However, by the same procedure,

$$h^1(Z, \mathcal{L}_0 \otimes \mathcal{L}) \geq \max_{0 \leq Z_1 \leq Z} \{h^1(Z_1, \mathcal{L}_0) - (l', Z_1)\}. \tag{7.2.1}$$

In the next section (cf. Corollary 8.3.4) we will prove that this is again an equality for the generic $\mathcal{L} = \mathcal{L}^{im}_{gen} \in \mathrm{Im}(c^{l'}_{\mathcal{L}_0}(Z))$. (The above inequality (7.2.1) can be compared with (5.3.1) as well.)

7.3. Compatibility with Laufer duality and differential forms Consider the perfect pairing $\langle\, ,\, \rangle : H^1(\mathcal{O}_Z) \otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$ from 2, see alo [14]. Once we fix $D_0 = \mathrm{div}(s_0)$ of certain $s_0 \in H^0(Z, \mathcal{L}_0)_{reg}$, we can define $\Omega_Z(D_0) := (\mathrm{Im}(\delta^0_{\mathcal{L}_0}))^\perp \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$. It is generated by forms which vanish on the image of the tangent map $T_{D_0} c^{l'_0}(Z)$, identified with $\delta^0_{\mathcal{L}_0}$, cf. 2 and (2.4.9). The pairing $\langle\, ,\, \rangle$ induces a perfect pairing $\langle\, ,\, \rangle_{\mathcal{L}_0} : H^1(Z, \mathcal{L}_0) \otimes \Omega_Z(D_0) \to \mathbb{C}$, see also Theorem 2.4.8.

7.3. The $\mathcal{G}$-filtration of $\Omega_Z(D_0) = H^1(\mathcal{L}_0)^*$ Consider the situation and notations of Definition 2.4.12; in particular, $\mathcal{G}_l = H^0(\Omega^2_{\widetilde{X}}(l))/H^0(\Omega^2_{\widetilde{X}})$ for any $0 < l \leq Z$. In the presence of $\mathcal{L}_0 = \mathcal{O}_Z(D_0)$ as above, we have the subspace $\Omega_Z(D_0) = (\mathrm{Im}\delta^0)^\perp \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, and the induced perfect pairing $\langle\, ,\, \rangle_{\mathcal{L}_0} : H^1(Z, \mathcal{L}_0) \otimes \Omega_Z(D_0) \to \mathbb{C}$. Similarly, for any $0 < l \leq Z$, we have the analogous data $\Omega_l(D_0) = (\mathrm{Im}(\delta^0|_l))^\perp \subset H^0(\Omega^2_{\widetilde{X}}(l))/H^0(\Omega^2_{\widetilde{X}})$, and the induced perfect pairing $\langle\, ,\, \rangle_{\mathcal{L}_0|_l} : H^1(l, \mathcal{L}_0) \otimes \Omega_l(D_0) \to \mathbb{C}$. One has the following inclusions inside $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$

$$\begin{array}{ccc} \Omega_l(D_0) & \longrightarrow & \Omega_Z(D_0) \\ \downarrow & & \downarrow \\ \mathcal{G}_l & \longrightarrow & H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \end{array}$$

and, in fact, $\Omega_l(D_0) = \Omega_Z(D_0) \cap \mathcal{G}_l$. Hence $\{\Omega_l(D_0)\}_l = \{\Omega_Z(D_0) \cap \mathcal{G}_l\}_l$ filters $\Omega_Z(D_0)$. Moreover, by $\langle\, ,\, \rangle_{\mathcal{L}_0|_l}$, one has $\dim \Omega_Z(D_0) \cap \mathcal{G}_l = \dim \Omega_l(D_0) = h^1(l, \mathcal{L}_0)$.

8 $\mathcal{L}_0$-projected versions of the algorithms

8.1. Preliminary notations Let us keep all the notations of the previous section.

Let us denote the dimension of $\mathrm{Im}(c^{l'}_{\mathcal{L}_0}(Z))$ by $d_{\mathcal{L}_0,Z}(l')$.

If $A_Z(l')$ is the smallest affine space which contains $\mathrm{Im}(c^{l'}(Z))$ in $\mathrm{Pic}^{l'}(Z)$, then $s_{\mathcal{L}_0}(A_Z(l'))$ is the smallest affine space which contains $\mathrm{Im}(c^{l'}_{\mathcal{L}_0}(Z))$. We denote it by $A_{\mathcal{L}_0,Z}(l')$ and its dimension by $e_{\mathcal{L}_0,Z}(l')$. From definitions $d_{\mathcal{L}_0,Z}(l') \leq e_{\mathcal{L}_0,Z}(l')$.

In this section we provide two algorithms for the computation of $d_{\mathcal{L}_0,Z}(l')$, the analogues of the algorithms from Theorems 3.2.2 and 4.1.2 .

8.2. The setup Let us fix (X, o), a good resolution $\widetilde{X}$, $Z \geq E$ and $l' \in -\mathcal{S}'$. We also fix a line bundle $\mathcal{L}_0$ as in Sect. 7, whose notations we will adopt. In order to estimate

$d_{\mathcal{L}_0,Z}(l')$ we proceed as in Sects. 3 and 4 . In particular, we perform the modificatiosn $\pi_{\mathbf{s}}:\widetilde{X}_{\mathbf{s}}\to\widetilde{X}$, and we adopt the notations of 3 as well. By the generic choice of the centers of blow ups we can assume that they differ from the support of D_0. Notice that we have a natural identification between $H^1(\mathcal{O}_Z)$ and $H^1(\mathcal{O}_{Z_{\mathbf{s}}})$, and also between $H^1(\mathcal{O}_Z^*)$ and $H^1(\mathcal{O}_{Z_{\mathbf{s}}}^*)$. Furthermore, we denote the divisor $\pi_{\mathbf{s}}^{-1}(D_0)$ on $\widetilde{X}_{\mathbf{s}}$ still by D_0 (basically unmodified), and the line bundle $\mathcal{O}_{Z_{\mathbf{s}}}(D_0)$ still by $\mathcal{L}_0$. Then we have the identification of $H^0(Z,\mathcal{O}_D)$ with $H^0(Z_{\mathbf{s}},\mathcal{O}_D)$, and also $H^1(Z,\mathcal{L}_0)\simeq H^1(Z_{\mathbf{s}},\mathcal{L}_0)$ and $H^1(Z,\mathcal{L}_0^*)\simeq H^1(Z_{\mathbf{s}},\mathcal{L}_0^*)$ (hence identifications of the corresponding commutative diagrams from 7 as well). The subspace $\Omega_{Z_{\mathbf{s}}}(D_0)$ in $H^1(\mathcal{O}_{Z_{\mathbf{s}}})^*=H^1(\mathcal{O}_Z)^*$ is also 'stable' of dimension $h^1(Z,\mathcal{L}_0)$.

Write $d_{\mathcal{L}_0,\mathbf{s}}$ and $e_{\mathcal{L}_0,\mathbf{s}}$ the corresponding dimensions associated with $\widetilde{X}_{\mathbf{s}}$ defined as in 8. Then $d_{\mathcal{L}_0,\mathbf{s}}\leq e_{\mathcal{L}_0,\mathbf{s}}$. If $\mathbf{s}=\mathbf{0}$ then $d_{\mathcal{L}_0,\mathbf{0}}=d_{\mathcal{L}_0,Z}(l')$ and $e_{\mathcal{L}_0,\mathbf{0}}=e_{\mathcal{L}_0,Z}(l')$.

Theorem 8.2.1 (1) $d_{\mathcal{L}_0,\mathbf{s}}-d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\in\{0,1\}$. *Moreover,* $d_{\mathcal{L}_0,\mathbf{s}}=d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ *if and only if for a generic point* $\bar{\mathcal{L}}\in\mathrm{Im}(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))$ *the set of divisors in* $(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))^{-1}(\bar{\mathcal{L}})$ *do not have a base point on* $F_{v,k,\mathbf{s}_{v,k}}$.

(2) *If for some fixed* $\mathbf{s}$ *the numbers* $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathcal{L}_0,\mathbf{s}}=\max_{v,k}\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value* $d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ *equals* $e_{\mathcal{L}_0,\mathbf{s}}$, *then* $d_{\mathcal{L}_0,\mathbf{s}}=e_{\mathcal{L}_0,\mathbf{s}}=d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$*; otherwise* $d_{\mathcal{L}_0,\mathbf{s}}=d_{\mathcal{L}_0,\mathbf{s}^{v,k}}+1$.

Proof (1) Assume first that either $s_{v,k}\geq 1$ or $a_v=1$. Then divisors from $\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$ intersect $F_{v,k,\mathbf{s}_{v,k}}$ by multiplicity one, hence the intersection (supporting) point gives a map $q:\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})\to F_{v,k,\mathbf{s}_{v,k}}$, which is dominant. Moreover, $\mathrm{ECa}^{l'_{\mathbf{s}^{v,k}}}(Z_{\mathbf{s}^{v,k}})$ is birational with a generic fiber of q (the fiber over the point which was blown up), hence the first statement follows. Note also that $d_{\mathcal{L}_0,\mathbf{s}}=d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ if and only if the generic fiber of the $\mathcal{L}_0$-projected Abel map $c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}$ is not included in a q-fiber. This implies the second part of *(1)*.

If $\mathbf{s}_{v,k}=0$ and $a_v>1$ then write $l'_-:=l'_{\mathbf{s}}-E^*_v$ and consider the 'addition map' $s:\mathrm{ECa}^{E^*_v}(Z_{\mathbf{s}})\times\mathrm{ECa}^{l'_-}(Z_{\mathbf{s}})\to\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$, which is dominant and quasifinite (cf. [14, Lemma 6.1.1]). Let $q:\mathrm{ECa}^{E^*_v}(Z_{\mathbf{s}})\to E_v$ be given by the supporting point as before. Then if $q^{-1}(gen)$ is a generic fiber of q (above the point which was blown up), then the restriction of s to $q^{-1}(gen)\times\mathrm{ECa}^{l'_-}(Z_{\mathbf{s}})$ with target $\mathrm{ECa}^{l'_{\mathbf{s}^{v,k}}}(Z_{\mathbf{s}^{v,k}})$ is dominant and quasifinite. Hence the arguments can be repeated.

(2) First notice that if the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are not the same then from *(1)* we have $d_{\mathcal{L}_0,\mathbf{s}}\leq\min_{v,k}d_{\mathcal{L}_0,\mathbf{s}^{v,k}}+1\leq\max_{v,k}d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\leq d_{\mathcal{L}_0,\mathbf{s}}$, hence $d_{\mathcal{L}_0,\mathbf{s}}=\max_{v,k}d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$.

Next, assume that the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are the same, say d.

If $d_{\mathcal{L}_0,\mathbf{s}}=d$ then part *(1)* reads as follows: $d_{\mathcal{L}_0,\mathbf{s}}=d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ for all v and k if and only if for a generic $\bar{\mathcal{L}}\in\mathrm{Im}(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))$ the set of divisors in $(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))^{-1}(\bar{\mathcal{L}})$ do not have a base point on any of the curves $\{F_{v,k,\mathbf{s}_{v,k}}\}_{v,k}$.

Let us choose a generic element $\bar{\mathcal{L}}\in\mathrm{Im}(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))$, which is in particular a regular value of $c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}})$ and the generic divisors in $\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$ mapped to $\bar{\mathcal{L}}$ are in fact generic divisors of $\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$ itself.

Next, take an element in $\Omega_{Z_{\mathbf{s}}}(D_0)$ (for details see 7) represented by a form ω, such that the class of ω vanishes on $T_{\bar{\mathcal{L}}}\mathrm{Im}(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))$.

Then choose a generic D from $\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$, which is mapped to $\bar{\mathcal{L}}$ and which has no common points with the support of ω (we can even assume additionally that it is transversal and reduced). Then we apply the previous statements for $\bar{\mathcal{L}} := c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}})(D)$.

In particular, the class of ω vanishes on $\mathrm{Im}(T_D c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))$ so ω cannot have pole along any of the curves $\{F_{v,k,\mathbf{s}_{v,k}}\}_{v,k}$, that is, it belongs to $\Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}})$, cf. Theorem 2.4.8 and Lemma 2.4.10. Hence $d_{\mathcal{L}_0,\mathbf{s}} = e_{\mathcal{L}_0,\mathbf{s}}$, cf. Lemma 3.1.3, and also $d = e_{\mathcal{L}_0,\mathbf{s}}$ too.

On the other hand if $d = e_{\mathcal{L}_0,\mathbf{s}}$, then from $d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \le d_{\mathcal{L}_0,\mathbf{s}} \le e_{\mathcal{L}_0,\mathbf{s}}$ we get $d = d_{\mathcal{L}_0,\mathbf{s}}$. Hence $d_{\mathcal{L}_0,\mathbf{s}} = d$ if and only if $d = e_{\mathcal{L}_0,\mathbf{s}}$. Otherwise $d_{\mathcal{L}_0,\mathbf{s}}$ should be $d+1$ by *(1)*. □

8.3. Notations for the second algorithm Consider the setup of 4 and combine it with the one from 8, where $\mathcal{L}_0$ enters in the picture. Accordingly, we have the following subspaces (inclusions):

$$\begin{array}{ccccccc}
\Omega_{Z_{\mathbf{s}}}(D_0)\cap\mathcal{G}_{l_{\mathbf{s}}} & \to & \Omega_{Z_{\mathbf{s}}}(D_0)\cap\Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}}) & \stackrel{j}{\longrightarrow} & \Omega_{Z_{\mathbf{s}}}(D_0) & = & H^1(Z,\mathcal{L}_0)^* \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{G}_{l_{\mathbf{s}}} & \to & \Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}}) & \stackrel{i}{\longrightarrow} & H^0(\Omega^2_{\widetilde{X}_{\mathbf{s}}}(Z_{\mathbf{s}}))/H^0(\Omega^2_{\widetilde{X}_{\mathbf{s}}}) & = & H^1(\mathcal{O}_Z)^*
\end{array}$$

The codimension of the inclusion i is $e_{\mathbf{s}}$ and the dimension of $\mathcal{G}_{\mathbf{s}}$ is $g_{\mathbf{s}}$ providing the inequality $e_{\mathbf{s}} \le h^1(\mathcal{O}_Z) - g_{\mathbf{s}}$. Similarly, the codimension of j is $e_{\mathcal{L}_0,\mathbf{s}}$ and the dimension of $\Omega_{Z_{\mathbf{s}}}(D_0)\cap\mathcal{G}_{l_{\mathbf{s}}}$ will be denoted by $g_{\mathcal{L}_0,\mathbf{s}}$ providing the inequality $e_{\mathcal{L}_0,\mathbf{s}} \le h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$. Hence

$$d_{\mathcal{L}_0,\mathbf{s}} \le e_{\mathcal{L}_0,\mathbf{s}} \le h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}. \tag{8.3.1}$$

It is conveninent to lift the $\mathbf{s}$-independent subspace $\Omega_{Z_{\mathbf{s}}}(D_0) = \Omega_Z(D_0)$ of $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ as $\Omega_{\widetilde{X}}(D_0) := \pi^{-1}(\Omega_Z(D_0))$ by the projection $\pi : H^0(\Omega^2_{\widetilde{X}}(Z)) \to H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$.

Theorem 8.3.2 (1) $d_{\mathcal{L}_0,\mathbf{s}} - d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \in \{0,1\}$.

(2) *If for some fixed* $\mathbf{s}$ *the numbers* $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathcal{L}_0,\mathbf{s}} = \max_{v,k}\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value* $d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ *equals* $h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$, *then* $d_{\mathcal{L}_0,\mathbf{s}} = h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$; *otherwise* $d_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}} + 1$.

Proof Part (1) was already proved in Theorem 8.2.1. Regarding part *(2)*, if the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are not the same then we argue again as in the proof of Theorem 8.2.1.

Next, assume that the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are the same, say d. Via (8.3.1) and the first algorithm Theorem 8.2.1 we need to show that if $d = e_{\mathcal{L}_0,\mathbf{s}}$ then necessarily $d = h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$ as well. However, if $d = e_{\mathcal{L}_0,\mathbf{s}}$ then we have $e_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ for all (v,k), hence by (8.3.1) we get $e_{\mathcal{L}_0,\mathbf{s}} = d = d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \le e_{\mathcal{L}_0,\mathbf{s}^{v,k}}$. But $e_{\mathcal{L}_0,\mathbf{s}} \ge e_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ by the combination of the argument from (3.2.1) and the diagram from 8. Hence, $d_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}}$ for all k and v implies $e_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}}$ for all v and k.

In particular, it is enough to verify the (stronger statement):

$$\text{if } e_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}} \text{ for all } v \text{ and } k \text{ then } e_{\mathcal{L}_0,\mathbf{s}} = h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}} \text{ as well.} \tag{8.3.3}$$

Assume that (8.3.3) is not true, that is, $e_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}}$ for all v and k, but $e_{\mathcal{L}_0,\mathbf{s}} < h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$. The last inequality via the diagram from 8 says that the inclusion $\Omega_{Z_\mathbf{s}}(D_0) \cap \mathcal{G}_{l_\mathbf{s}} \subset \Omega_{Z_\mathbf{s}}(D_0) \cap \Omega_{Z_\mathbf{s}}(I_\mathbf{s})$ is strict. This means, that there is a differential form $\omega \in \Omega_{\widetilde{X}}(D_0)$, with class $[\omega]$ in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \subset H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$, such that ω does not have a pole along the exceptional divisor $F_{v,k,\mathbf{s}_{v,k}}$, however $[\omega] \notin \mathcal{G}_\mathbf{s}$. In particular, there exists a vertex $v \in |l'|$, such that the pole order of ω along E_v is larger than $(l_\mathbf{s})_v$. Notice that this also means $(l_\mathbf{s})_v = \min_{1\le i\le a_v} \mathbf{s}_{v,i} < Z_v$.

Let $1 \le i \le a_v$ be an integer such that $\mathbf{s}_{v,i} = (l_\mathbf{s})_v$ (abridged in the sequel by t) and we denote the order of vanishing of ω on an arbitrary exceptional divisor E_u by b_u, where u is an arbitrary vertex along the blowing up procedure. Next we focus on the string between v and $w_{v,i,\mathbf{s}_{v,i}}$ and we denote them by $v_0 = v, \dots, v_t = w_{v,i,\mathbf{s}_{v,i}}$. Set $r := \min\{0 \le s \le t \, : \, b_{v_s} + t - s \ge 0\}$. Since for $s = t$ one has $b_{v_t} \ge 0$ (since ω has no pole along $F_{v,i,\mathbf{s}_{v,i}}$) r is well-defined. On the other hand we have $r \ge 1$. Indeed, $b_{v_0} + t < 0$, since pole order of ω along E_v is higher than $(l_\mathbf{s})_v = t$. Note that $b_{v_{r-1}} + t - r + 1 < 0$ and $b_{v_r} + t - r \ge 0$ imply $b_{v_r} - b_{v_{r-1}} \ge 2$ (†).

Let $\widetilde{X}'$ be that resolution obtained from $\widetilde{X}$, as an intermediate step of the tower between $\widetilde{X}$ and $\widetilde{X}_\mathbf{s}$, when in the (v,i) sequence of blow ups we do not proceed all $\mathbf{s}_{v,i}$ of them, but we create only the divisors $\{F_{v,i,k}\}_{k\le r-1}$. Let $\mathcal{V}'$ be its vertex set and $\{E_u\}_{u\in\mathcal{V}'}$ its exceptional divisors. On $\widetilde{X}'$ consider the line bundle $\mathcal{L} := \Omega^2_{\widetilde{X}'}(-\sum_{u\in\mathcal{V}'} b_u E_u)$. Since F_{v,i,v_r} was created by blowing up a *generic point* p of $E_{v_{r-1}} = F_{v,i,v_{r-1}}$, the existence of ω guarantees the existence of a section $s \in H^0(\widetilde{X}', \mathcal{L})$, which does not vanish along $E_{v_{r-1}}$ and it has multiplicity $m := b_{v_r} - b_{v_{r-1}} - 1$ at the generic point $p \in E_{v_{r1}}$. By (†) $m \ge 1$. By construction, ω (or s) belongs also to the subvectorspace $\Omega_{\widetilde{X}}(D_0)$ after certain identifications.

Now by the technical Lemma 9.1.1 (valid for general line bundles, and separated in Sect. 9) for any $0 \le k < m$ and a generic point $p \in E_{v_{r-1}}$ there exists a section $s' \in H^0(\widetilde{X}', \mathcal{L})$, which does not vanish along the exceptional divisor $E_{v_{r-1}}$, and the divisor of s' has multiplicity k at p. We apply for $k = -(b_{v_{r-1}} + t - r + 1) - 1$. (Note that $0 \le k < m$.) The section s' gives a differential form $\omega' \in \Omega_{\widetilde{X}}(D_0)$, such that if we blow up $E_{v_{r-1}}$ in the generic point p and we denote the new exceptional divisor by $E_{v_{r,new}}$, then ω' has wanishing order $-(t - r + 1)$ on $E_{v_{r,new}}$. This means, that if we blow up it in generic points $t - r + 1$ times, then ω' has a pole on $E_{v_{t,new}}$, but has no pole on $E_{v_{t+1,new}}$. This means that $e_{\mathcal{L}_0,\mathbf{s}^{v,i}} \ne e_{\mathcal{L}_0,\mathbf{s}}$, which is a contradiction. □

The analogues of Corollaries 4.1.3 and 4.1.5 (with similar proofs) are:

Corollary 8.3.4 *For any $l' \in -\mathcal{S}'$, $Z \geq E$ and $\mathcal{L}_0$ with $H^0(Z, \mathcal{L}_0)_{reg} \neq \emptyset$ one has*

$$\begin{aligned} d_{\mathcal{L}_0, Z}(l') &= \min_{\mathbf{s}} \{ |s| + h^1(Z, \mathcal{L}_0) - g_{\mathcal{L}_0, \mathbf{s}} \} \\ &= \min_{0 \leq Z_1 \leq Z} \{ (l', Z_1) + h^1(Z, \mathcal{L}_0) - h^1(Z_1, \mathcal{L}_0) \}. \end{aligned}$$

This combined with (7.1.1) gives for a generic $\mathcal{L}^{im}_{gen} \in \mathrm{Im}(c^{l'}(Z))$:

$$h^1(Z, \mathcal{L}_0 \otimes \mathcal{L}^{im}_{gen}) = \max_{0 \leq Z_1 \leq Z} \{ h^1(Z_1, \mathcal{L}_0) - (l', Z_1) \}.$$

Example 8.3.5 This is a continuation of Example 4.1.8 (based on [14, §11]), whose notations and statements we will use. Assume that $Z \gg 0$ and $l' = -kE^*_0$ as in 4.1.8. Additionally we take a generic line bundle $\mathcal{L}_0$ with $c_1(\mathcal{L}_0) = l'_0 = -k_0 E^*_0$, $k_0 \geq 0$, (hence $\widetilde{D}_0$ consists of k_0 generic irreducible cuts of E_0). Recall that $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ admits a basis consisting of elements of type $\mathbf{x}^{\mathbf{m}}\omega$, where ω is the Gorenstein form and $0 \leq |\mathbf{m}| \leq d-3$. Each 'block' $\{|\mathbf{m}| = j\}$ $(0 \leq j \leq d-3)$ (which can be identified with $H^0(\mathbb{P}^2, \mathcal{O}(j))$) contributes with $\binom{j+2}{2}$ monomials. The k_0 generic divisors impose $\min\{k_0, \binom{j+2}{2}\}$ independent conditions (see [14, 11.2] for the explication), hence the block $\{|\mathbf{m}| = j\}$ $(0 \leq j \leq d-3)$ contributes into $\dim \Omega_Z(D_0) = h^1(\mathcal{L}_0)$ with $\binom{j+2}{2} - \min\{k_0, \binom{j+2}{2}\} = \max\{0, \binom{j+2}{2} - k_0\}$. In particular, $h^1(\mathcal{L}_0) = \sum_{j=0}^{d-3} \max\{0, \binom{j+2}{2} - k_0\}$ and $h^1(\mathcal{L}_0) - g_{\mathcal{L}_0, s} = \sum_{j=0}^{d-3-s} \max\{0, \binom{j+2}{2} - k_0\}$ $(0 \leq s \leq d-2)$. Therefore,

$$d_{\mathcal{L}_0, Z}(-kE^*_0) = \min_{0 \leq s \leq d-2} \left\{ ks + \sum_{j=0}^{d-3-s} \max\left\{0, \tbinom{j+2}{2} - k_0\right\} \right\}.$$

However, if $\mathcal{L}_0 = \mathcal{O}_Z(D_0)$ is not generic, then the points D_0 might fail to impose independent conditions on the corresponding linear systems, and the determination of the dimension of $\Omega_Z(D_0)$ can be harder. See [14, 11.3] for discussion, examples and connection with the Cayley–Bacharach type theorems (cf. [2]). Those discussions with combined with the present section produces further examples for $d_{\mathcal{L}_0, Z}(l')$ whenever D_0 is special (and (X, o) is superisolated).

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9 Appendix: A technical lemma

9.1. The next lemma is used in the body of the article, however, it might have also an independent general interest.

Lemma 9.1.1 *Let $\widetilde{X}$ be an arbitrary resolution of a normal surface singularity $(X, 0)$. Let us fix an arbitrary line bundle $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ with $c_1(\mathcal{L}) = l' \in -S'$, an irreducible exceptional curve E_v, and an integer $m > 0$.*

Assume that there exists a linear subspace $V \subset H^0(\widetilde{X}, \mathcal{L})$ with the following property: for a generic point $p \in E_v$ there exists a section $s \in V$ such that s does not vanish along E_v and the multiplicity of the divisor of s at $p \in E_v$ is m. Then for any number $0 \leq k \leq m$ and a generic point $p \in E_v$ there exists a section $s \in V$ such that s does not vanish along E_v and the multiplicity of the divisor of s at $p \in E_v$ is k.

Proof By induction we need to prove the statement only for $k = m - 1$.

First we fix a very large integer $N \gg m$, and consider the restriction $r : H^0(\widetilde{X}, \mathcal{L}) \to H^0(NE_v, \mathcal{L})$. Then r induces a map from $H^0(\widetilde{X}, \mathcal{L})_{reg} := H^0(\widetilde{X}, \mathcal{L}) \setminus H^0(\widetilde{X}, \mathcal{L}(-E_v))$ to $H^0(NE_v, \mathcal{L})_{reg} := H^0(NE_v, \mathcal{L}) \setminus H^0((N-1)E_v, \mathcal{L}(-E_v))$. Denote its restriction $H^0(\widetilde{X}, \mathcal{L})_{reg} \cap V \to H^0(NE_v, \mathcal{L})_{reg} \cap r(V)$ by r_V. Consider also the natural map $\mathrm{div} : H^0(NE_v, \mathcal{L})_{reg} \to \mathrm{ECa}^{l'}(NE_v)$, and the composition map $\mathrm{div} \circ r_V = g : H^0(\widetilde{X}, \mathcal{L})_{reg} \cap V \to \mathrm{ECa}^{l'}(NE_v)$, which sends a section to its divisor restricted to the cycle NE_v.

Next, for any $p \in E_v^0 := E_v \setminus \cup_{u \neq v} E_u$ set $D_{m,p} \subset \mathrm{ECa}^{l'}(NE_v)$, the set of divisors with multiplicity m at p. (Since $N \gg m$ this notion is well-defined). Set also $D_m := \cup_p D_{m,p}$.

By the assumption, the image of g intersects $D_{m,p}$ for any generic p. Since D_m is constructible subvariety of $\mathrm{ECa}^{l'}(NE_v)$, $g^{-1}(D_m)$ is a nonempty constructible subset of $H^0(\widetilde{X}, \mathcal{L})_{reg} \cap V$. Define an analytic curve $h_0 : (-\epsilon, \epsilon) \to g^{-1}(D_m)$ such that its image is not a subset of some $g^{-1}(D_{m,p})$. Let us denote the zeros of the section $h_0(0)$ along E_v^0 by $\{p_1, \ldots, p_r\}$. Then there exists a small neighborhood U of one of the points p_i and a restriction of h_0 to some smaller $(-\epsilon', \epsilon')$, such that for any $t \in (-\epsilon', \epsilon')$ the restriction of $h_0(t)$ to U has a unique zero, say $p(t)$, and its multiplicity is m. Furthermore, $t \mapsto p(t)$, $(-\epsilon', \epsilon') \to U \cap E_v^0$ is not constant, hence taking further restrictions to some interval we can assume that $t \mapsto p(t)$ is locally invertible. Reparametrising h_0 by the inverse of this map, we obtain an analytic map $U \cap E_v^0 \to g^{-1}(D_m)$, $t \mapsto h(t)$ such that the restriction of the section $h(t)$ to some local chart U has only one zero, namely t, and the multiplicity of the section at t is m. In some local coordinates (x, y) of U (with $U \cap E_v = \{y = 0\}$) the equation of $h(t)$ has the form (modulo y^N)

$$h(t) = \sum_{j \geq 0, i \geq 0} (x - t)^j y^i c_{j,i}(t), \tag{9.1.2}$$

where by the multiplicity condition $c_{j,i} \equiv 0$, if $j + i < m$ and, there is a pair (j, i), such that $j + i = m$ and $c_{j,i}(t) \not\equiv 0$. Moreover, by the non-vanishing condition $y \nmid h(t)$, or, $c_{j,0}(t) \not\equiv 0$ for some j.

We claim that there is a generic choice of $t_1, \dots, t_r$ (for some large r) of t-values, and a convenient choice of the coefficients $\{\alpha_l\}_{l=1}^r$ such that $s := \sum_{l=1}^r \alpha_l h(t_l)$ satisfies the requirements. Indeed, first we consider the Taylor expansion of $h(t)$ in variables (x, y) at a point $(x, y) = (q, 0)$ with q generic (and modulo y^N as usual):

$$\sum_{j,i}(x-q+q-t)^j y^i c_{j,i}(t) = \sum_{j,i}\sum_{k=0}^{j}(x-q)^k y^i \binom{j}{k}(q-t)^{j-k}c_{j,i}(t).$$

The fact that s at $(q, 0)$ has multiplicity $\geq m-1$ transforms into a linear system

$$\sum_{l=1}^{r}\alpha_l\Big(\sum_{j\geq k}\binom{j}{k}(q-t_l)^{j-k}c_{j,i}(t_l)\Big)=0$$

for any (k, i) with $k, i \geq 0$ and $k + i \leq m - 2$.

This linear system $LS(r, m-2)$ with unknowns $\{\alpha_l\}_{l=1}^r$ has matrix $M(r, m-2)$ of size $r \times m(m-1)/2$. If $r \gg m(m-1)/2$ then the system has a nontrivial solution. We need to show that for a generic choice of the solutions $\{\alpha_l\}_l$ the section s has multiplicity $m-1$ at q. Assume that this is not the case. Then the generic solution of the system $LS(r, m-2)$ is automatically solution of $LS(r, m-1)$ too (the last one defined similarly). This means that $\mathrm{rank} M(r, m-2) = \mathrm{rank} M(r, m-1)$ (†) for generic $\{t_l\}_l$.

The matrix $M(r, m-1)$ has m additional rows corresponding to the indexes (k, i) with $k, i \geq 0$ and $k+i = m-1$. Let us fix one of them, corresponding to the following choice.

Now let d be the minimal number, such that there exists j, i such that $i \leq m-1$, $j+i = d$ and $c_{j,i}(t)$ is not identically 0. Since by assumption (by non-vanishing of $h(t)$ along E_v) there exists certain $j \geq m$ with $c_{j,0} \not\equiv 0$, such a d exists. Fix i_0 such that $i_0 \leq m-1$, $j_0 + i_0 = d$ and $c_{j_0,i_0}(t) \not\equiv 0$.

Then, from the additional rows of $M(r, m-1)$ we chose the one indexed by $(m-1-i_0, i_0)$.

Consider the minor of $M(r, m-1)$ of size $m(m-1)/2+1$, whose last row is the row corresponding to $(m-1-i_0, i_0)$, and the other rows belong to $M(r, m-2)$, while the last column corresponds to the generic $t_r = t$. Then its determinant should be zero by (†). Expanded it by the last column gives

$$\begin{aligned}&\sum_{j\geq m-1-i_0}\binom{j}{m-1-i_0}(q-t)^{j-m+1+i_0}c_{j,i_0}(t)\\ &\quad=\sum_{k,i\geq 0;k+i\leq m-2}\beta_{k,i}(q)\cdot\sum_{j\geq k}\binom{j}{k}(q-t)^{j-k}c_{j,i}(t)\end{aligned}$$

for some holomorphic functions $\beta_{k,i}(q)$. But such an identity cannot exist. Indeed, since $c_{j_0,i_0} \not\equiv 0$, but $c_{j,i_0} \equiv 0$ for any $j < j_0$, the vanishing order of $q-t$ at the left hand side is exactly $d-m+1$, while on the right hand side—since $j \geq d-i$

(otherwise $c_{j,i} \equiv 0$) and $k \leq m-2-i$ implies $j-k \geq d-m+2$—we get vanishing order $\geq d-m+2$.

Finally we need to show that this generic s does not vanish along E_v. This follows from a similar argument as above, or one can proceed as follows. For any generic q consider a section s which has multiplicity $m-1$ at $(q, 0)$. If it vanishes along E_v then $s + h(q)$ does not vanish along E_v and it has multiplicity $m-1$ at $(q, 0)$. □

Remark 9.1.3 We claim that under the assumptions of Lemma 9.1.1 the following property also holds: *For any finite set $F \subset E_v$ there exists a section $s \in V$ such that s does not vanish along E_v*, $\mathrm{div}(s) \cap F = \emptyset$,*and at each $p \in \mathrm{div}(s) \cap E_v$ the intersection of* $\mathrm{div}(s)$ *with E_v is transversal.* Indeed, we can use first Lemma 9.1.1 for $k=1$ and then show that a generic combination of 'moving' sections of multiplicity one works.

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