



On the p -adic Beilinson conjecture and the equivariant Tamagawa number conjecture

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Abstract

Let E/K be a finite Galois extension of totally real number fields with Galois group G . Let p be an odd prime and let $r > 1$ be an odd integer. The p -adic Beilinson conjecture relates the values at $s = r$ of p -adic Artin L -functions attached to the irreducible characters of G to those of corresponding complex Artin L -functions. We show that this conjecture, the equivariant Iwasawa main conjecture and a conjecture of Schneider imply the ‘ p -part’ of the equivariant Tamagawa number conjecture for the pair $(h^0(\text{Spec}(E))(r), \mathbb{Z}[G])$. If $r > 1$ is even we obtain a similar result for Galois CM-extensions after restriction to ‘minus parts’.

Keywords Beilinson conjecture · Equivariant Tamagawa number conjecture · Iwasawa theory · Regulator maps

Mathematics Subject Classification 19F27 · 11R23 · 11R42 · 11R70

1 Introduction

Let E/K be a finite Galois extension of number fields with Galois group G and let r be an integer. The equivariant Tamagawa number conjecture (ETNC) for the pair $(h^0(\text{Spec}(E))(r), \mathbb{Z}[G])$ as formulated by Burns and Flach [17] asserts that a certain canonical element $T\Omega(E/K, r)$ in the relative algebraic K -group $K_0(\mathbb{Z}[G], \mathbb{R})$ vanishes. This element relates the leading terms at $s = r$ of Artin L -functions to natural arithmetic invariants.

If $r = 0$ this might be seen as a vast generalization of the analytic class number formula for number fields, and refines Stark’s conjecture for E/K as discussed by Tate in [75] and the ‘Strong Stark conjecture’ of Chinburg [25, Conjecture 2.2]. It is known

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to imply a whole bunch of conjectures such as Chinburg's ' Ω_3 -conjecture' [25,26], the Rubin–Stark conjecture [68], Brumer's conjecture, the Brumer–Stark conjecture (see [75, Chapitre IV, Sect. 6]) and generalizations thereof due to Burns [13] and the author [57]. If r is a negative integer, the ETNC refines a conjecture of Gross [42] and implies (generalizations of) the Coates–Sinnott conjecture [28] and a conjecture of Snaith [71] on annihilators of the higher K -theory of rings of integers (see [56]). If $r > 1$ the ETNC likewise predicts constraints on the Galois module structure of p -adic wild kernels [62].

The functional equation of Artin L -functions suggests that the ETNC at r and $1 - r$ are equivalent. This is not known in general, but leads to a further conjecture which is sometimes referred to as the local ETNC. Except for the validity of the local ETNC it therefore suffices to consider the (global) ETNC for either odd or even integers r . Note that the local ETNC is widely believed to be easier to settle. For instance, the 'global epsilon constant conjecture' of Bley and Burns [6] measures the compatibility of the closely related 'leading term conjectures' at $s = 0$ [12] and $s = 1$ [10] and is known to hold for arbitrary tamely ramified extensions [6, Corollary 7.7] and also for certain weakly ramified extensions [7].

Now suppose that E/K is a Galois extension of totally real number fields and let p be an odd prime. If $r < 0$ is odd Burns [14] and the author [60] independently have shown that the ' p -part' of the ETNC for the pair $(h^0(\text{Spec}(E))(r), \mathbb{Z}[G])$ holds provided that a certain Iwasawa μ_p -invariant vanishes (which conjecturally is always true). The latter condition is mainly present because the equivariant Iwasawa main conjecture (EIMC) for totally real fields then holds by independent work of Ritter and Weiss [67] and of Kakde [51].

The case $r \geq 0$ is more subtle. Burns and Venjakob [22,23] (see also [14, Corollary 2.8]) proposed a strategy for proving the p -part of the ETNC for the pair $(h^0(\text{Spec}(E))(1), \mathbb{Z}[G])$. More precisely, this special case of the ETNC is implied by the vanishing of the relevant μ_p -invariant, Leopoldt's conjecture for E at p and the ' p -adic Stark conjecture at $s = 1$ '. The latter conjecture relates the leading terms at $s = 1$ of the complex and p -adic Artin L -functions attached to characters of G by certain comparison periods. Note that Burns and Venjakob actually assume these conjectures for all odd primes p and then deduce the ETNC for the pair $(h^0(\text{Spec}(E))(1), \mathbb{Z}[\frac{1}{2}][G])$, but their approach has recently been refined by Johnston and the author [49] so that one can indeed work prime-by-prime.

There are similar results on minus parts if L/K is a Galois CM-extension with Galois group G , i.e. K is totally real and L is a totally complex quadratic extension of a totally real field L^+ . Namely, if $r < 0$ is even and μ_p vanishes, then the minus p -part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), \mathbb{Z}[G])$ holds [14,60]. Burns [15] recently proposed a strategy for proving the minus p -part of the ETNC in the case $r = 0$. In comparison with the strategy in the case $r = 1$, Leopoldt's conjecture is replaced with the conjectural non-vanishing of Gross's regulator [41], and the p -adic Stark conjecture is replaced with the 'weak p -adic Gross–Stark conjecture' [41, Conjecture 2.12b] (now a theorem for linear characters by work of Dasgupta, Kakde and Ventullo [32]). For an approach that only relies upon the validity of the EIMC we refer the reader to [58,61].

The aim of this article is to propose a similar strategy in the remaining cases, i.e. we will consider the ETNC for Tate motives $h^0(\mathrm{Spec}(L))(r)$ where $r > 1$ and L is a CM-field. Note that we can treat all integers $r > 1$ simultaneously as the ‘plus p -part’ of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}[G])$ naturally identifies with the corresponding conjecture for the extension L^+/K of totally real fields. We show that the p -adic Beilinson conjecture at $s = r$, a conjecture of Schneider [69] and the EIMC imply the plus (resp. minus) p -part of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}[G])$ if r is odd (resp. even).

We follow the formulation of the p -adic Beilinson conjecture in [5]. It relates the values at $s = r$ of the complex and p -adic Artin L -functions by certain comparison periods involving Besser’s syntomic regulator [4]. For absolutely abelian extensions variants of the p -adic Beilinson conjecture have been formulated and proved by Coleman [29], Gros [39,40] and Kolster and Nguyen Quang Do [52]. Thus the p -adic Beilinson conjecture holds for absolutely abelian characters (see Sect. 3.13 for a precise statement).

Let us compare our approach to the earlier work mentioned above. The formulation of both the p -adic Beilinson conjecture and the p -adic Stark conjecture involves the choice of a field isomorphism $j : \mathbb{C} \simeq \mathbb{C}_p$. We show in Sect. 3.12 that the p -adic Beilinson conjecture does not depend upon this choice if and only if a conjecture of Gross [42] holds. The latter is revisited in Sect. 3.8 and might be seen as a higher analogue of Stark’s conjecture; a similar result in the case $r = 1$ has recently been established by Johnston and the author in [49]. In both cases the independence of j is therefore equivalent to the rationality part of the appropriate special case of the ETNC. This eventually allows us to establish a prime-by-prime descent result analogous to [49, Theorem 8.1].

In a little more detail, we formulate conjectural ‘higher refined p -adic class number formulae’ analogous to [15, Conjecture 3.5] (where $r = 0$), and show that these follow from the EIMC and Schneider’s conjecture in Sect. 4.6. Here, as will be shown in Sect. 4.5, the latter conjecture ensures that the relevant complexes are semisimple at all Artin characters as Leopoldt’s conjecture does in the case $r = 1$ and the non-vanishing of Gross’s regulator does in the case $r = 0$. This is a necessary condition in order to apply the descent formalism of Burns and Venjakob [23]. A second condition is the vanishing of the aforementioned Iwasawa μ_p -invariant, but given recent progress of Johnston and the author [47,50] on the EIMC without assuming $\mu_p = 0$, we wish to circumvent this hypothesis. For this purpose, we develop a different descent argument that makes no use of this assumption, but requires a more delicate analysis of the relevant complexes. The higher refined p -adic class number formula at $s = r$ may then be combined with the p -adic Beilinson conjecture at $s = r$ to deduce the plus, respectively minus, p -part of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}[G])$ in Sect. 4.7. For this, it is crucial to relate Besser’s syntomic regulators to Soulé’s p -adic Chern class maps [72] and the Bloch–Kato exponential maps [8] that appear in the formulation of the ETNC. This is carried out in Sect. 3.9 (in particular see Proposition 3.16). The formulation of the ETNC that is most suitable for our purposes is a reformulation due to the author [62]. This has primarily been introduced in order to construct (conjectural) annihilators of p -adic wild kernels.

Our prime example are totally real Galois extensions E/\mathbb{Q} with Galois group isomorphic to $\text{Aff}(q)$, where $q = \ell^n$ is a prime power and $\text{Aff}(q)$ denotes the group of affine transformations on the finite field \mathbb{F}_q with q elements. We show that Gross's conjecture holds in this case (Theorem 3.14 (iv)). Moreover, the relevant cases of the EIMC hold unconditionally by recent work of Johnston and the author [47] (see also [50]) and the p -adic Beilinson conjecture reduces to the case of the trivial extension E^H/E^H where H denotes the subgroup $\text{GL}_1(\mathbb{F}_q)$ of $\text{Aff}(q)$. See Example 4.24 for more details.

Finally, we note that the ETNC for the pair $(h^0(\text{Spec}(L))(r), \mathbb{Z}[G])$ has been verified for any integer r whenever L is abelian over the rationals by work of Burns, Greither and Flach [19,20,35]. However, if $r > 1$ and L is not absolutely abelian, then we are not aware of any previous (conditional) results that establish the (p -part of the) ETNC for the pair $(h^0(\text{Spec}(L))(r), \mathbb{Z}[G])$.

Notation and conventions

All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. Unadorned tensor products will always denote tensor products over \mathbb{Z} . For a ring Λ we write $\zeta(\Lambda)$ for its center and Λ^\times for the group of units in Λ . For every field F we fix a separable closure F^c of F and write $G_F := \text{Gal}(F^c/F)$ for its absolute Galois group. If $n > 0$ is an integer coprime to the characteristic of F , we let ζ_n denote a primitive n th root of unity in F^c .

A finite Galois extension of totally real number fields will usually be denoted by E/K , whereas L/K denotes an arbitrary Galois extension of number fields. Galois CM-extensions will usually be denoted by L/K as well.

2 Algebraic preliminaries

2.1 Derived categories and Galois cohomology

Let Λ be a noetherian ring and let $\text{PMod}(\Lambda)$ be the category of all finitely generated projective Λ -modules. We write $\mathcal{D}(\Lambda)$ for the derived category of Λ -modules and $\mathcal{C}^b(\text{PMod}(\Lambda))$ for the category of bounded complexes of finitely generated projective Λ -modules. Recall that a complex of Λ -modules is called perfect if it is isomorphic in $\mathcal{D}(\Lambda)$ to an element of $\mathcal{C}^b(\text{PMod}(\Lambda))$. We denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$ comprising perfect complexes by $\mathcal{D}^{\text{perf}}(\Lambda)$.

If M is a Λ -module and n is an integer, we write $M[n]$ for the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

where M is placed in degree $-n$. Note that this is compatible with the usual shift operator on cochain complexes.

Let L be an algebraic extension of the number field K . For a finite set S of places of K containing the set S_∞ of all archimedean places we let $G_{L,S}$ be the Galois group

over L of the maximal extension of L that is unramified outside $S(L)$; here we write $S(L)$ for the set of places of L lying above those in S . We let $\mathcal{O}_{L,S}$ be the ring of $S(L)$ -integers in L . For any topological $G_{L,S}$ -module M we write $R\Gamma(\mathcal{O}_{L,S}, M)$ for the complex of continuous cochains of $G_{L,S}$ with coefficients in M . If F is a field and M is a topological G_F -module, we likewise define $R\Gamma(F, M)$ to be the complex of continuous cochains of G_F with coefficients in M .

If F is a global or a local field of characteristic zero, and M is a discrete or a compact G_F -module, then for $r \in \mathbb{Z}$ we denote the r th Tate twist of M by $M(r)$. Now fix a prime p and suppose that S also contains all p -adic places of K . Then for each integer i the cohomology group in degree i of $R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ naturally identifies with $H_{\text{ét}}^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$, the i th étale cohomology group of the affine scheme $\text{Spec}(\mathcal{O}_{L,S})$ with coefficients in the étale p -adic sheaf $\mathbb{Z}_p(r)$. We set $H_{\text{ét}}^i(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_{\text{ét}}^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$.

2.2 Representations and characters of finite groups

Let G be a finite group and let F be a field of characteristic zero. We write $R_F^+(G)$ for the set of characters attached to finite-dimensional F -valued representations of G , and $R_F(G)$ for the ring of virtual characters generated by $R_F^+(G)$. Moreover, we denote the subset of irreducible characters in $R_F^+(G)$ and the ring of F -valued virtual characters of G by $\text{Irr}_F(G)$ and $\text{Char}_F(G)$, respectively.

For a subgroup H of G and $\psi \in R_F^+(H)$ we write $\text{ind}_H^G \psi \in R_F^+(G)$ for the induced character; for a normal subgroup N of G and $\chi \in R_F^+(G/N)$ we write $\text{infl}_{G/N}^G \chi \in R_F^+(G)$ for the inflated character. For $\sigma \in \text{Aut}(F)$ and $\chi \in \text{Char}_F(G)$ we set $\chi^\sigma := \sigma \circ \chi$ and note that this defines a group action from the left even though we write exponents on the right of χ . We denote the trivial character of G by $\mathbb{1}_G$.

2.3 χ -twists

Let G be a finite group and let F be a field of characteristic zero. If M is a $\mathbb{Z}[G]$ -module we let M^G be the maximal submodule of M upon which G acts trivially. Likewise we write M_G for the maximal quotient module with trivial G -action. For any $\chi \in R_F^+(G)$ we fix a (left) $F[G]$ -module V_χ with character χ . For any $F[G]$ -module M and any $\alpha \in \text{End}_{F[G]}(M)$ we write M^χ for the F -vector space

$$\text{Hom}_{F[G]}(V_\chi, M) \simeq \text{Hom}_F(V_\chi, M)^G$$

and α^χ for the induced map $(f \mapsto \alpha \circ f) \in \text{End}_F(M^\chi)$. We note that $\det_F(\alpha^\chi)$ is independent of the choice of V_χ . The following is [49, Lemma 2.1] and very similar to [75, Chapitre I, 6.4].

Lemma 2.1 *Let M be an $F[G]$ -module and let $\alpha \in \text{End}_{F[G]}(M)$. Let H be a subgroup of G and let $M|_H$ denote M considered as an $F[H]$ -module. Let N be a normal subgroup of G and view M^N as an $F[G/N]$ -module in the obvious way.*

- (i) *For $\chi_1, \chi_2 \in R_F^+(G)$ one has $\det_F(\alpha^{\chi_1 + \chi_2}) = \det_F(\alpha^{\chi_1}) \det_F(\alpha^{\chi_2})$.*

- (ii) For $\chi \in R_F^+(H)$ there is an isomorphism $M^{\text{ind}_H^G \chi} \simeq (M|_H)^\chi$ and an equality $\det_F(\alpha^{\text{ind}_H^G \chi}) = \det_F(\alpha^\chi)$.
- (iii) For $\chi \in R_F^+(G/N)$ there is an isomorphism $M^{\text{infl}_{G/N}^G \chi} \simeq (M^N)^\chi$ and an equality $\det_F(\alpha^{\text{infl}_{G/N}^G \chi}) = \det_F((\alpha|_{M^N})^\chi)$.

Let p be a prime. For each $\chi \in \text{Irr}_{\mathbb{C}_p}(G)$ we fix a subfield F_χ of \mathbb{C}_p which is both Galois and of finite degree over \mathbb{Q}_p and such that χ can be realized over F_χ . We write $e_\chi := |G|^{-1} \chi(1) \sum_{g \in G} \chi(g) g^{-1}$ for the associated primitive central idempotent in $\mathbb{C}_p[G]$ and choose an indecomposable idempotent f_χ of $F_\chi[G]e_\chi$. Let \mathcal{O}_χ be the ring of integers in F_χ and choose a maximal \mathcal{O}_χ -order \mathcal{M}_χ in $F_\chi[G]$ containing f_χ . Then $T_\chi := f_\chi \mathcal{M}_\chi$ is an \mathcal{O}_χ -free right $\mathcal{O}_\chi[G]$ -module.

For any (left) $\mathbb{Z}_p[G]$ -module M we define a (left) $\mathcal{O}_\chi[G]$ -module $M[\chi] := T_\chi \otimes_{\mathbb{Z}_p} M$, where $g \in G$ acts upon $t \otimes m$ by the rule $g(t \otimes m) = tg^{-1} \otimes gm$ for all $t \in T_\chi$ and $m \in M$. We define \mathcal{O}_χ -modules $M^{(\chi)} := M[\chi]^G$ and $M_{(\chi)} := M[\chi]_G \simeq T_\chi \otimes_{\mathbb{Z}_p[G]} M$. We thereby obtain left, respectively right exact functors $M \mapsto M^{(\chi)}$ and $M \mapsto M_{(\chi)}$ from the category of $\mathbb{Z}_p[G]$ -modules to the category of \mathcal{O}_χ -modules. Note that there is an isomorphism $F_\chi \otimes_{\mathcal{O}_\chi} M^{(\chi)} \simeq (F_\chi \otimes_{\mathbb{Z}_p} M)^\chi$ for every finitely generated $\mathbb{Z}_p[G]$ -module M .

Since multiplication by the trace $\text{Tr}_G := \sum_{g \in G} g$ gives rise to an isomorphism $P_{(\chi)} \simeq P^{(\chi)}$ for projective $\mathbb{Z}_p[G]$ -module P (in fact for each cohomologically trivial G -module P), these functors extend to naturally isomorphic exact functors $\mathcal{D}^{\text{perf}}(\mathbb{Z}_p[G]) \rightarrow \mathcal{D}^{\text{perf}}(\mathcal{O}_\chi)$ (and $\mathcal{D}^{\text{perf}}(\mathbb{Q}_p[G]) \rightarrow \mathcal{D}^{\text{perf}}(F_\chi)$).

Lemma 2.2 *Let $\chi \in \text{Irr}_{\mathbb{C}_p}(G)$ and let $a \leq b$ be integers. If $C^\bullet \in \mathcal{D}^{\text{perf}}(\mathbb{Z}_p[G])$ is acyclic outside $[a, b]$, then $C_{(\chi)}^\bullet$ is also acyclic outside $[a, b]$ and there are natural isomorphisms of \mathcal{O}_χ -modules*

$$H^a(C_{(\chi)}^\bullet) \simeq H^a(C^\bullet)^{(\chi)} \quad \text{and} \quad H^b(C_{(\chi)}^\bullet) \simeq H^b(C^\bullet)_{(\chi)}.$$

For $C^\bullet \in \mathcal{D}^{\text{perf}}(\mathbb{Q}_p[G])$ we have isomorphisms $H^i(C_{(\chi)}^\bullet) \simeq H^i(C^\bullet)_{(\chi)} \simeq H^i(C^\bullet)^{(\chi)}$ for every $i \in \mathbb{Z}$.

Proof Since (finitely generated) $\mathbb{Q}_p[G]$ -modules are cohomologically trivial, the functors $M \mapsto M^{(\chi)}$ and $M \mapsto M_{(\chi)}$ are naturally isomorphic exact functors on the category of finitely generated $\mathbb{Q}_p[G]$ -modules. The final assertion of the lemma is therefore clear.

Now suppose that $C^\bullet \in \mathcal{D}^{\text{perf}}(\mathbb{Z}_p[G])$ is acyclic outside $[a, b]$. If $b - a \leq 1$ the claim is [15, Lemma 5.1]. We repeat the short argument for convenience. Choose a complex $A \rightarrow B$ of cohomologically trivial $\mathbb{Z}_p[G]$ -modules that is isomorphic to C^\bullet in $\mathcal{D}(\mathbb{Z}_p[G])$. Here A and B are placed in degrees a and $a + 1$, respectively. Then we obtain a commutative diagram of \mathcal{O}_χ -modules

$$\begin{array}{ccccccc} A_{(\chi)} & \longrightarrow & B_{(\chi)} & \longrightarrow & H^{a+1}(C^\bullet)_{(\chi)} & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & H^a(C^\bullet)^{(\chi)} & \longrightarrow & A^{(\chi)} & \longrightarrow & B^{(\chi)} \end{array}$$

which implies the claim. If $b - a \geq 2$ we choose a complex $P^\bullet \in \mathcal{C}^b(\text{PMod}(\mathbb{Z}_p[G]))$ that is isomorphic to C^\bullet in $\mathcal{D}(\mathbb{Z}_p[G])$ and consider the exact sequence of perfect complexes

$$0 \longrightarrow \tau_{\geq b-1} P^\bullet \longrightarrow P^\bullet \longrightarrow \tau_{\leq b-2} P^\bullet \longrightarrow 0,$$

where $\tau_{\geq b-1}$ and $\tau_{\leq b-2}$ denote naive truncation. Note that the complexes $\tau_{\geq b-1} P^\bullet$ and $\tau_{\leq b-2} P^\bullet$ are acyclic outside $[b-1, b]$ and $[a, b-2]$, respectively. It follows by induction that $C_{(\chi)}^\bullet$ is acyclic outside $[a, b]$ and, since $H^b(C^\bullet) = H^b(\tau_{\geq b-1} P^\bullet)$, that we have an isomorphism $H^b(C_{(\chi)}^\bullet) \simeq H^b(C^\bullet)_{(\chi)}$. If $b - a \geq 3$ then we likewise have that $H^a(C^\bullet) = H^a(\tau_{\leq b-2} P^\bullet)$ and we may again conclude by induction that we have an isomorphism $H^a(C_{(\chi)}^\bullet) \simeq H^a(C^\bullet)_{(\chi)}$. If $b - a = 2$ we may alternatively consider the exact sequence of perfect complexes

$$0 \longrightarrow \tau_{\geq b} P^\bullet \longrightarrow P^\bullet \longrightarrow \tau_{\leq b-1} P^\bullet \longrightarrow 0$$

and deduce as above. \square

3 The p -adic Beilinson conjecture

3.1 Setup and notation

Let L/K be a finite Galois extension of number fields with Galois group G . For any place v of K we choose a place w of L above v and write G_w and I_w for the decomposition group and inertia subgroup of L/K at w , respectively. We denote the completions of L and K at w and v by L_w and K_v , respectively, and identify the Galois group of the extension L_w/K_v with G_w . For each non-archimedean place w we let \mathcal{O}_w be the ring of integers in L_w . We identify $\overline{G_w} := G_w/I_w$ with the Galois group of the corresponding residue field extension which we denote by $L(w)/K(v)$. Finally, we let $\phi_w \in \overline{G_w}$ be the Frobenius automorphism, and we denote the cardinality of $K(v)$ by $N(v)$. We let S be a finite set of places of K containing the set S_∞ of archimedean places. If a prime p is fixed, we will usually assume that the set S_p of all p -adic places is also contained in S .

By a Galois CM-extension of number fields we shall mean a finite Galois extension L/K such that K is totally real and L is a CM-field. Thus complex conjugation induces a unique automorphism τ in the center of G and we denote the maximal totally real subfield of L by L^+ . Then L^+/K is also Galois with group $G^+ := G/\langle \tau \rangle$.

3.2 Higher K -theory

For an integer $n \geq 0$ and a ring R we write $K_n(R)$ for the Quillen K -theory of R . In the cases $R = \mathcal{O}_{L,S}$ and $R = L$ the groups $K_n(\mathcal{O}_{L,S})$ and $K_n(L)$ are equipped with a natural G -action and for every integer $r > 1$ the inclusion $\mathcal{O}_{L,S} \subseteq L$ induces an isomorphism of $\mathbb{Z}[G]$ -modules

$$K_{2r-1}(\mathcal{O}_{L,S}) \simeq K_{2r-1}(L). \quad (3.1)$$

Moreover, if S' is a second finite set of places of K containing S , then for every $r > 1$ there is a natural exact sequence of $\mathbb{Z}[G]$ -modules

$$0 \longrightarrow K_{2r}(\mathcal{O}_{L,S}) \longrightarrow K_{2r}(\mathcal{O}_{L,S'}) \longrightarrow \bigoplus_{w \in S'(L) \setminus S(L)} K_{2r-1}(L(w)) \longrightarrow 0. \quad (3.2)$$

Both results (3.1) and (3.2) are due to Soulé [72]; see [77, Chapter V, Theorem 6.8]. We also note that sequence (3.2) remains left-exact in the case $r = 1$. The structure of the finite $\mathbb{Z}[\overline{G_w}]$ -modules $K_{2r-1}(L(w))$ has been determined by Quillen [63] (see also [77, Chapter IV, Theorem 1.12 and Corollary 1.13]) to be

$$K_{2r-1}(L(w)) \simeq \mathbb{Z}[\overline{G_w}]/(\phi_w - N(v)^r). \quad (3.3)$$

3.3 The regulators of Borel and Beilinson

Let $\Sigma(L)$ be the set of embeddings of L into the complex numbers; we then have $|\Sigma(L)| = r_1 + 2r_2$, where r_1 and r_2 are the number of real embeddings and the number of pairs of complex embeddings of L , respectively. For an integer $k \in \mathbb{Z}$ we define a finitely generated \mathbb{Z} -module

$$H_k(L) := \bigoplus_{\Sigma(L)} (2\pi i)^{-k} \mathbb{Z}$$

which is endowed with a natural $\text{Gal}(\mathbb{C}/\mathbb{R})$ -action, diagonally on $\Sigma(L)$ and on $(2\pi i)^{-k}$. The invariants of $H_k(L)$ under this action will be denoted by $H_k^+(L)$, and it is easily seen that we have

$$d_k := \text{rank}_{\mathbb{Z}}(H_{1-k}^+(L)) = \begin{cases} r_1 + r_2 & \text{if } 2 \nmid k \\ r_2 & \text{if } 2 \mid k. \end{cases} \quad (3.4)$$

The action of G on $\Sigma(L)$ endows $H_k^+(L)$ with a natural G -module structure.

Let $r > 1$ be an integer. Borel [9] has proved that the even K -groups $K_{2r-2}(\mathcal{O}_L)$ (and thus $K_{2r-2}(\mathcal{O}_{L,S})$ for any S as above by (3.2) and (3.3)) are finite, and that the odd K -groups $K_{2r-1}(\mathcal{O}_L)$ are finitely generated abelian groups of rank d_r . More precisely, for each $r > 1$ Borel constructed an equivariant regulator map

$$\rho_r^{\text{Bor}} : K_{2r-1}(\mathcal{O}_L) \longrightarrow H_{1-r}^+(L) \otimes \mathbb{R} \quad (3.5)$$

with finite kernel. Its image is a full lattice in $H_{1-r}^+(L) \otimes \mathbb{R}$. The covolume of this lattice is called the Borel regulator and will be denoted by $R_r^{\text{Bor}}(L)$. Moreover, Borel showed that

$$\frac{\zeta_L^*(1-r)}{R_r^{\text{Bor}}(L)} \in \mathbb{Q}^\times, \quad (3.6)$$

where $\zeta_L^*(1-r)$ denotes the leading term at $s = 1-r$ of the Dedekind zeta function $\zeta_L(s)$ attached to the number field L .

In the context of the ETNC, however, it is more natural to work with Beilinson's regulator map [3]. By a result of Burgos Gil [11] Borel's regulator map is twice the regulator map of Beilinson. Hence we will work with $\rho_r := \frac{1}{2}\rho_r^{\text{Bor}}$ in the following.

Remark 3.1 We will sometimes refer to [62] where we have worked with Borel's regulator map. However, if we are interested in rationality questions or in verifying the p -part of the ETNC for an odd prime p , the factor 2 essentially plays no role. In contrast, the p -adic Beilinson conjecture below predicts an equality of two numbers in \mathbb{C}_p so that this factor indeed matters.

3.4 The Quillen–Lichtenbaum conjecture

Fix an odd prime p and assume that S contains S_∞ and the set S_p of all p -adic places of K . Then for any integer $r > 1$ and $i = 1, 2$ Soulé [72] has constructed canonical G -equivariant p -adic Chern class maps

$$ch_{r,i}^{(p)} : K_{2r-i}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p \longrightarrow H_{\text{ét}}^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)).$$

We need the following deep result.

Theorem 3.2 (Quillen–Lichtenbaum Conjecture) *Let p be an odd prime. Then for every integer $r > 1$ and $i = 1, 2$ the p -adic Chern class maps $ch_{r,i}^{(p)}$ are isomorphisms.*

Proof Soulé [72] proved surjectivity. Building on work of Rost and Voevodsky, Weibel [76] completed the proof of the Quillen–Lichtenbaum Conjecture. \square

Let p be a prime. For an integer $n \geq 0$ and a ring R we write $K_n(R; \mathbb{Z}_p)$ for the K -theory of R with coefficients in \mathbb{Z}_p . The following result is due to Hesselholt and Madsen [43].

Theorem 3.3 *Let p be an odd prime and let w be a finite place of L . Then for every integer $r > 1$ and $i = 1, 2$ there are canonical isomorphisms of $\mathbb{Z}_p[G_w]$ -modules*

$$K_{2r-i}(\mathcal{O}_w; \mathbb{Z}_p) \simeq H_{\text{ét}}^i(L_w, \mathbb{Z}_p(r)).$$

3.5 Local Galois cohomology

We keep the notation of Sect. 3.1. In particular, L/K is a Galois extension of number fields with Galois group G . Let p be an odd prime. We denote the (finite) set of places of K that ramify in L/K by S_{ram} and let S be a finite set of places of K containing S_{ram} and all archimedean and p -adic places (i.e. $S_\infty \cup S_p \cup S_{\text{ram}} \subseteq S$).

Let M be a topological $G_{L,S}$ -module. Then M becomes a topological G_{L_w} -module for every $w \in S(L)$ by restriction. For any $i \in \mathbb{Z}$ we put

$$P^i(\mathcal{O}_{L,S}, M) := \bigoplus_{w \in S(L)} H_{\text{ét}}^i(L_w, M).$$

For any integers r and i we define $P^i(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))$ to be $P^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Lemma 3.4 *Let $r > 1$ be an integer. Then we have isomorphisms of $\mathbb{Q}_p[G]$ -modules*

$$P^i(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \simeq \begin{cases} H_{-r}^+(L) \otimes \mathbb{Q}_p & \text{if } i = 0 \\ L \otimes_{\mathbb{Q}} \mathbb{Q}_p & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof This is [62, Lemma 3.3] (see also [1, Lemma 5.2.4]). The case $i = 1$ will be crucial in the following so that we briefly recall its proof. Let $w \in S_p(L)$ and put $D_{dR}^{L_w}(\mathbb{Q}_p(r)) := H^0(L_w, B_{dR} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r))$, where B_{dR} denotes Fontaine's de Rham period ring. Then the Bloch–Kato exponential map

$$\exp_r^{BK} : L_w = D_{dR}^{L_w}(\mathbb{Q}_p(r)) \longrightarrow H_{\text{ét}}^1(L_w, \mathbb{Q}_p(r)) \quad (3.7)$$

is an isomorphism for every $w \in S_p(L)$ as follows from [8, Corollary 3.8.4 and Example 3.9]. Since the groups $H_{\text{ét}}^1(L_w, \mathbb{Z}_p(r))$ are finite for $w \notin S_p(L)$, we obtain isomorphisms of $\mathbb{Q}_p[G]$ -modules

$$P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \simeq \bigoplus_{w \in S_p(L)} H_{\text{ét}}^1(L_w, \mathbb{Q}_p(r)) \simeq \bigoplus_{w \in S_p(L)} L_w \simeq L \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

□

3.6 Schneider's conjecture

Let M be a topological $G_{L,S}$ -module. For any integer i we denote the kernel of the natural localization map

$$H_{\text{ét}}^i(\mathcal{O}_{L,S}, M) \longrightarrow P^i(\mathcal{O}_{L,S}, M)$$

by $\text{III}^i(\mathcal{O}_{L,S}, M)$. We call $\text{III}^i(\mathcal{O}_{L,S}, M)$ the *Tate–Shafarevich group* of M in degree i . We recall the following conjecture of Schneider [69, p. 192].

Conjecture 3.5 ($\text{Sch}(L, p, r)$) *Let $r \neq 0$ be an integer. Then the Tate–Shafarevich group $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ vanishes.*

Remark 3.6 It is not hard to show that Conjecture 3.5 does not depend on the choice of the set S .

Remark 3.7 Schneider originally conjectured that $H_{\text{ét}}^2(\mathcal{O}_{L,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ vanishes. Both conjectures are in fact equivalent (see [62, Proposition 3.8 (ii)]).

Remark 3.8 It can be shown that Schneider's conjecture for $r = 1$ is equivalent to Leopoldt's conjecture (see [54, Chapter X, Sect. 3]).

Remark 3.9 For a given number field L and a fixed prime p , Schneider's conjecture holds for almost all r . This follows from [69, Sect. 5, Corollar 4] and [69, Sect. 6, Satz 3].

Remark 3.10 Schneider's conjecture $\text{Sch}(L, p, r)$ holds whenever $r < 0$ by work of Soulé [72]; see also [54, Theorem 10.3.27].

For an integer r we let $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$.

Lemma 3.11 *Let $r \neq 0$ be an integer and let p be an odd prime. Then the Tate–Shafarevich group $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is torsion-free. In particular, Schneider's conjecture $\text{Sch}(L, p, r)$ holds if and only if $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))$ vanishes.*

Proof The first claim is [62, Proposition 3.8 (i)]. The second claim is immediate. \square

3.7 Artin L -series

Let L/K be a finite Galois extension of number fields with Galois group G and let S be a finite set of places of K containing all archimedean places. For any irreducible complex-valued character χ of G we denote the S -truncated Artin L -series by $L_S(s, \chi)$, and the leading coefficient of $L_S(s, \chi)$ at an integer r by $L_S^*(r, \chi)$. We shall sometimes use this notion even if $L_S^*(r, \chi) = L_S(r, \chi)$ (which will happen frequently in the following).

Recall that there is a canonical isomorphism $\zeta(\mathbb{C}[G]) \simeq \prod_{\chi \in \text{Irr}_{\mathbb{C}}(G)} \mathbb{C}$. We define the equivariant S -truncated Artin L -series to be the meromorphic $\zeta(\mathbb{C}[G])$ -valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \text{Irr}_{\mathbb{C}}(G)}.$$

For any $r \in \mathbb{Z}$ we also put

$$L_S^*(r) := (L_S^*(r, \chi))_{\chi \in \text{Irr}_{\mathbb{C}}(G)} \in \zeta(\mathbb{R}[G])^{\times}.$$

3.8 A conjecture of Gross

Let $r > 1$ be an integer. Since Borel's regulator map (3.5) induces an isomorphism of $\mathbb{R}[G]$ -modules, the Noether–Deuring Theorem (see [54, Lemma 8.7.1] for instance) implies the existence of $\mathbb{Q}[G]$ -isomorphisms

$$\phi_{1-r} : H_{1-r}^+(L) \otimes \mathbb{Q} \xrightarrow{\sim} K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{Q}. \quad (3.8)$$

Let χ be a complex character of G and let V_{χ} be a $\mathbb{C}[G]$ -module with character χ . Composition with $\rho_r \circ \phi_{1-r}$ induces an automorphism of $\text{Hom}_G(V_{\chi}, H_{1-r}^+(L) \otimes \mathbb{C})$. Let $R_{\phi_{1-r}}(\chi) = \det_{\mathbb{C}}((\rho_r \circ \phi_{1-r})^{\check{\chi}}) \in \mathbb{C}^{\times}$ be its determinant. If χ' is a second character, then $R_{\phi_{1-r}}(\chi + \chi') = R_{\phi_{1-r}}(\chi) \cdot R_{\phi_{1-r}}(\chi')$ by Lemma 2.1 so that we obtain a map

$$R_{\phi_{1-r}} : R(G) \longrightarrow \mathbb{C}^\times$$

$$\chi \mapsto \det_{\mathbb{C}}(\rho_r \circ \phi_{1-r} \mid \operatorname{Hom}_G(V_{\check{\chi}}, H_{1-r}^+(L) \otimes \mathbb{C})),$$

where $R(G) := R_{\mathbb{C}}(G)$ denotes the ring of virtual complex characters of G . We likewise define

$$A_{\phi_{1-r}}^S : R(G) \longrightarrow \mathbb{C}^\times$$

$$\chi \mapsto R_{\phi_{1-r}}(\chi) / L_S^*(1-r, \chi).$$

Gross [42, Conjecture 3.11] conjectured the following higher analogue of Stark's conjecture.

Conjecture 3.12 (Gross) *We have $A_{\phi_{1-r}}^S(\chi^\sigma) = A_{\phi_{1-r}}^S(\chi)^\sigma$ for all $\sigma \in \operatorname{Aut}(\mathbb{C})$.*

Remark 3.13 It is not hard to see that Gross's conjecture does not depend on S and the choice of ϕ_{1-r} (see also [56, Remark 6]). A straightforward substitution shows that if it is true for χ then it is true for χ^τ for every choice of $\tau \in \operatorname{Aut}(\mathbb{C})$.

We record some cases where Gross's conjecture is known and deduce a few new cases. If $q = \ell^n$ is a prime power, we let $\operatorname{Aff}(q)$ be the group of affine transformations on \mathbb{F}_q . Thus we may write $\operatorname{Aff}(q)$ as a semi-direct product $N \rtimes H$, where $H = \{x \mapsto ax \mid a \in \mathbb{F}_q^\times\} \simeq \mathbb{F}_q^\times$ acts on $N = \{x \mapsto x + b \mid b \in \mathbb{F}_q\} \simeq \mathbb{F}_q$ in the natural way. Note that N is the commutator subgroup of $\operatorname{Aff}(q)$.

Theorem 3.14 *Let L/K be a finite Galois extension of number fields with Galois group G and let $\chi \in R(G)$ be a virtual character. Let $r > 1$ be an integer. Then Gross's conjecture (Conjecture 3.12) holds in each of the following cases.*

- (i) χ is absolutely abelian, i.e. there is a normal subgroup N of G such that χ factors through $G/N \simeq \operatorname{Gal}(L^N/K)$ and L^N/\mathbb{Q} is abelian;
- (ii) $\chi = \mathbb{1}_G$ is the trivial character;
- (iii) χ is a virtual permutation character, i.e. a \mathbb{Z} -linear combination of characters of the form $\operatorname{ind}_H^G \mathbb{1}_H$ where H ranges over subgroups of G ;
- (iv) $G \simeq \operatorname{Aff}(q) = N \rtimes H$ and L^N/\mathbb{Q} is abelian;
- (v) $L^{\ker(\chi)}$ is totally real and r is even;
- (vi) $L^{\ker(\chi)}/K$ is a CM-extension, χ is an odd character and r is odd.

Proof We first note that (ii) is Borel's result (3.6) above. Since Gross's conjecture is invariant under induction and respects addition of characters, (ii) implies (iii). For (i), (v) and (vi) we refer the reader to [62, Theorem 5.2] and the references given therein. We now prove (iv). It suffices to show that Gross's conjecture holds for every $\chi \in \operatorname{Irr}_{\mathbb{C}}(G)$. If χ is linear, it factors through G/N so that χ is indeed absolutely linear. Thus Gross's conjecture holds by (i). It has been shown in the proof of [49, Theorem 10.5] that there is a unique non-linear irreducible character χ_{nl} of G and that this character can be expressed as a \mathbb{Z} -linear combination of $\operatorname{ind}_H^G \mathbb{1}_H$ and linear characters in $\operatorname{Irr}_{\mathbb{C}}(G)$. As Gross's conjecture holds for the linear characters and for $\operatorname{ind}_H^G \mathbb{1}_H$ by (iii), it also holds for χ_{nl} . \square

For any integer k we write

$$\iota_k : L \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \bigoplus_{\Sigma(L)} \mathbb{C} = (H_{1-k}^+(L) \oplus H_{-k}^+(L)) \otimes \mathbb{C} \quad (3.9)$$

for the canonical $\mathbb{C}[G \times \text{Gal}(\mathbb{C}/\mathbb{R})]$ -equivariant isomorphism which is induced by mapping $l \otimes z$ to $(\sigma(l)z)_{\sigma \in \Sigma(L)}$ for $l \in L$ and $z \in \mathbb{C}$. Now fix an integer $r > 1$. We define a $\mathbb{C}[G]$ -isomorphism

$$\begin{aligned} \lambda_r : (K_{2r-1}(\mathcal{O}_L) \oplus H_{-r}^+(L)) \otimes \mathbb{C} &\simeq (H_{1-r}^+(L) \oplus H_{-r}^+(L)) \otimes \mathbb{C} \\ &\simeq L \otimes_{\mathbb{Q}} \mathbb{C}. \end{aligned} \quad (3.10)$$

Here, the first isomorphism is induced by $\rho_r \oplus \text{id}_{H_{-r}^+(L)}$, whereas the second isomorphism is ι_r^{-1} . As above, there exist $\mathbb{Q}[G]$ -isomorphisms

$$\phi_r : L \xrightarrow{\sim} (K_{2r-1}(\mathcal{O}_L) \oplus H_{-r}^+(L)) \otimes \mathbb{Q}. \quad (3.11)$$

We now define maps

$$\begin{aligned} R_{\phi_r} : R(G) &\longrightarrow \mathbb{C}^\times \\ \chi &\mapsto \det_{\mathbb{C}}(\lambda_r \circ \phi_r \mid \text{Hom}_G(V_{\check{\chi}}, L \otimes_{\mathbb{Q}} \mathbb{C})) \end{aligned}$$

and

$$\begin{aligned} A_{\phi_r}^S : R(G) &\longrightarrow \mathbb{C}^\times \\ \chi &\mapsto R_{\phi_r}(\chi) / L_S(r, \check{\chi}). \end{aligned}$$

Proposition 3.15 *Fix an integer $r > 1$ and a character χ . Then Gross's conjecture 3.12 holds if and only if we have $A_{\phi_r}^S(\chi^\sigma) = A_{\phi_r}^S(\chi)^\sigma$ for all $\sigma \in \text{Aut}(\mathbb{C})$.*

Proof This is [62, Proposition 5.5]. \square

3.9 The comparison period

We henceforth assume that p is an odd prime and that L/K is a Galois CM-extension. Recall that $\tau \in G$ is the unique automorphism induced by complex conjugation. For each $n \in \mathbb{Z}$ we define a central idempotent $e_n := \frac{1-(-1)^n\tau}{2} \in \mathbb{Z}[\frac{1}{2}][G]$. Now let $r > 1$ be an integer. Since L is CM, the idempotent e_r acts trivially on $H_{1-r}^+(L) \otimes \mathbb{C}$, whereas $e_r(H_{-r}^+(L) \otimes \mathbb{C})$ vanishes. Thus (3.10) induces a $\mathbb{C}[G]$ -isomorphism

$$\mu_\infty(r) : K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{C} \xrightarrow{\sim} e_r(L \otimes_{\mathbb{Q}} \mathbb{C}).$$

We likewise define a $\mathbb{C}_p[G]$ -homomorphism

$$\mu_p(r) : K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{C}_p \longrightarrow e_r(L \otimes_{\mathbb{Q}} \mathbb{C}_p)$$

as follows. For each $w \in S_p(L)$ we let $H_{\text{syn}}^i(\mathcal{O}_w, r)$ be the i th syntomic cohomology group as considered by Besser [4]. We let $\text{reg}_r^w : K_{2r-1}(\mathcal{O}_w) \rightarrow H_{\text{syn}}^1(\mathcal{O}_w, r)$ be the syntomic regulator [4, Theorem 7.5]. By [5, Lemma 2.15] (which heavily relies on [4, Proposition 8.6]) we have canonical isomorphisms $H_{\text{syn}}^1(\mathcal{O}_w, r) \simeq L_w$ for each $w \in S_p(L)$. The map $\mu_p(r)$ is induced by the following chain of homomorphisms

$$\begin{aligned}
 K_{2r-1}(\mathcal{O}_L) &\longrightarrow \bigoplus_{w \in S_p(L)} K_{2r-1}(\mathcal{O}_w) \\
 &\longrightarrow \bigoplus_{w \in S_p(L)} H_{\text{syn}}^1(\mathcal{O}_w, r) \\
 &\simeq \bigoplus_{w \in S_p(L)} L_w \\
 &\simeq L \otimes_{\mathbb{Q}} \mathbb{Q}_p.
 \end{aligned} \tag{3.12}$$

The map $\mu_p(r)$ shows up in the formulation of the p -adic Beilinson conjecture. However, the following map will be more suitable for the relation to the ETNC. We define a $\mathbb{C}_p[G]$ -homomorphism

$$\begin{aligned}
 \tilde{\mu}_p(r) : K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{C}_p &= K_{2r-1}(\mathcal{O}_{L,S}) \otimes \mathbb{C}_p \\
 &\simeq H_{\text{ét}}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \\
 &\longrightarrow e_r(P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p) \\
 &\simeq e_r(L \otimes_{\mathbb{Q}} \mathbb{C}_p).
 \end{aligned}$$

Here, the first map is induced by the p -adic Chern class map $ch_{r,1}^{(p)}$ which is an isomorphism by Theorem 3.2; the arrow is the natural localization map, and the last isomorphism is induced by the Bloch–Kato exponential maps (see Lemma 3.4).

The following result will be crucial for relating the p -adic Beilinson conjecture to the ETNC.

Proposition 3.16 *For each $r > 1$ we have $\mu_p(r) = \tilde{\mu}_p(r)$.*

Proof For any abelian group A we write \widehat{A} for its p -completion, that is $\widehat{A} := \varprojlim_n A/p^n A$. The localization maps (3.12) induce a map

$$K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{Z}_p \longrightarrow \bigoplus_{w \in S_p(L)} \widehat{K_{2r-1}(\mathcal{O}_w)}.$$

For each $w \in S_p(L)$ the Universal Coefficient Theorem [77, Chapter IV, Theorem 2.5] implies that there is a natural (injective) map

$$\widehat{K_{2r-1}(\mathcal{O}_w)} \longrightarrow K_{2r-1}(\mathcal{O}_w; \mathbb{Z}_p).$$

By [4, Corollary 9.10] there is a natural map $H_{\text{syn}}^1(\mathcal{O}_w, r) \rightarrow H_{\text{ét}}^1(L_w, \mathbb{Q}_p(r))$ such that the diagram

$$\begin{array}{ccc} \widehat{K_{2r-1}(\mathcal{O}_w)} & \longrightarrow & K_{2r-1}(\mathcal{O}_w; \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ H_{\text{syn}}^1(\mathcal{O}_w, r) & \longrightarrow & H_{\text{ét}}^1(L_w, \mathbb{Q}_p(r)) \end{array}$$

commutes. Here, the left-hand vertical arrow is induced by the syntomic regulator, and the map on the right by the isomorphism in Theorem 3.3. Moreover, the composite map

$$L_w \simeq H_{\text{syn}}^1(\mathcal{O}_w, r) \longrightarrow H_{\text{ét}}^1(L_w, \mathbb{Q}_p(r))$$

is the Bloch–Kato exponential map (3.7) by [4, Proposition 9.11]. Unravelling the definitions we now see that the maps $\mu_p(r)$ and $\tilde{\mu}_p(r)$ coincide. \square

We will henceforth often not distinguish between the maps $\mu_p(r)$ and $\tilde{\mu}_p(r)$. Since the Tate–Shafarevich group $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is torsion-free by Lemma 3.11, the following result is now immediate.

Lemma 3.17 *The map $\mu_p(r)$ is a $\mathbb{C}_p[G]$ -isomorphism if and only if $\text{Sch}(L, p, r)$ holds.*

Definition 3.18 Let $j : \mathbb{C} \simeq \mathbb{C}_p$ be a field isomorphism and let $\rho \in R_{\mathbb{C}_p}^+(G)$. Let $r > 1$ be an integer. We define the comparison period attached to j, ρ and r to be

$$\Omega_j(r, \rho) := \det_{\mathbb{C}_p}(\mu_p(r) \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_\infty(r))^{-1})^\rho \in \mathbb{C}_p.$$

We record some basic properties of $\Omega_j(r, -)$.

Lemma 3.19 *Let H, N be subgroups of G with N normal in G .*

- (i) *Let $\rho_1, \rho_2 \in R_{\mathbb{C}_p}^+(G)$. Then $\Omega_j(r, \rho_1 + \rho_2) = \Omega_j(r, \rho_1)\Omega_j(r, \rho_2)$.*
- (ii) *Let $\rho \in R_{\mathbb{C}_p}^+(H)$. Then $\Omega_j(r, \text{ind}_H^G \rho) = \Omega_j(r, \rho)$.*
- (iii) *Let $\rho \in R_{\mathbb{C}_p}^+(G/N)$. Then $\Omega_j(r, \text{infl}_{G/N}^G \rho) = \Omega_j(r, \rho)$.*

Proof Each part follows from the corresponding part of Lemma 2.1. \square

Remark 3.20 Since $\mu_\infty(r)$ is an isomorphism, for any two choices of field isomorphism $j, j' : \mathbb{C} \simeq \mathbb{C}_p$ we have that $\Omega_j(r, \rho) = 0$ if and only if $\Omega_{j'}(r, \rho) = 0$.

Remark 3.21 For any fixed choice of field isomorphism $j : \mathbb{C} \simeq \mathbb{C}_p$ we have

$$\begin{aligned} \text{Sch}(L, p, r) \text{ holds} &\iff \mu_p(r) \text{ is an isomorphism} \\ &\iff \Omega_j(r, \rho) \neq 0 \quad \forall \rho \in \text{Irr}_{\mathbb{C}_p}(G), \end{aligned}$$

where the first equivalence is Lemma 3.17. Thus the non-vanishing of $\Omega_j(r, \rho)$ can be thought of as the ‘ ρ -part’ of $\text{Sch}(L, p, r)$. Moreover, if $\Omega_j(r, \rho) \neq 0$ then we may set $\Omega_j(r, -\rho) := \Omega_j(r, \rho)^{-1}$ and so if we assume $\text{Sch}(L, p, r)$ then Lemma 3.19 (i) shows that the definition of $\Omega_j(r, \rho)$ naturally extends to any virtual character $\rho \in R_{\mathbb{C}_p}(G)$.

Remark 3.22 Assume that G is abelian. For each integer r , Burns, Kurihara and Sano [21, Sect. 2.2] define canonical period-regulator isomorphisms

$$\varepsilon_r \bigwedge_{\mathbb{C}_p[G]}^{r_j^\varepsilon} H_{\text{ét}}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \rightarrow \varepsilon_r \bigwedge_{\mathbb{C}_p[G]}^{r_j^\varepsilon} P^0(\mathcal{O}_{L,S}, \mathbb{Z}_p(-r)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p. \quad (3.13)$$

Here $\varepsilon_r \in \mathbb{Z}_p[G]$ are certain idempotents such that the ε_r -parts of both $\mathbb{C}_p[G]$ -modules in the exterior products are free of the same rank r_j^ε . If $r > 1$ and $\text{Sch}(L, p, r)$ holds, then one may take $\varepsilon_r = e_r$ and $r_j^\varepsilon = 0$. In this case the diagram [21, p. 125] gives an exact sequence of $\mathbb{C}_p[G]$ -modules

$$0 = e_r(H_{\text{ét}}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p) \rightarrow e_r(L \otimes_{\mathbb{Q}} \mathbb{C}_p)^* \rightarrow e_r(H_{1-r}^+(L) \otimes \mathbb{C}_p)^* \rightarrow 0,$$

where $(-)^*$ denotes \mathbb{C}_p -linear duals (note also that their $H_L(r-1)^+$ is $H_{1-r}^+(L)$ in our notation). The non-trivial map is (up to sign) the dual of $\mu_p(r) \circ \rho_r^{-1}$. Hence the exterior product on the left of (3.13) canonically identifies with

$$e_r \det_{\mathbb{C}_p[G]}((L \otimes_{\mathbb{Q}} \mathbb{C}_p)^* \otimes_{\mathbb{C}_p[G]} \det_{\mathbb{C}_p[G]}^{-1}((H_{1-r}^+(L) \otimes \mathbb{C}_p)^*))$$

and the isomorphisms $\mu_p(r) \circ \rho_r^{-1}$ and ι_r induce a map to $\det_{\mathbb{C}_p[G]}(H_{-r}^+(L) \otimes \mathbb{C}_p)$ which can be identified with the exterior power on the right by a variant of Lemma 3.4. For more details we refer the interested reader to [21, Sect. 2.2.4].

The authors then use the isomorphism (3.13) to define generalized Stark elements and to state [21, Conjecture 3.6] which might be seen as an analogue and refinement of a conjecture of Rubin [68] in the case $r = 0$. It is then shown in [21, Sect. 4] that their conjecture is implied by the appropriate special case of the ETNC. As the formulation of the latter involves the Bloch–Kato exponential map rather than the syntomic regulator, a variant of Proposition 3.16 is already implicit in their work (for instance, see [21, Remark 2.7])

3.10 p -adic Artin L -functions

Let E/K be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/K)$. Let p be an odd prime and let S be a finite set of places of K containing $S_p \cup S_\infty$. For each $\rho \in R_{\mathbb{C}_p}(G)$ the S -truncated p -adic Artin L -function attached to ρ is the unique p -adic meromorphic function $L_{p,S}(\rho, s) : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ with the property

that for each strictly negative integer n and each field isomorphism $j : \mathbb{C} \simeq \mathbb{C}_p$ we have

$$L_{p,S}(n, \rho) = j \left(L_S(n, (\rho \otimes \omega^{n-1})^{j^{-1}}) \right),$$

where $\omega : G_K \rightarrow \mathbb{Z}_p^\times$ is the Teichmüller character and we view $\rho \otimes \omega^{n-1}$ as a character of $\text{Gal}(E(\zeta_p)/K)$. By a result of Siegel [70] the right-hand side does indeed not depend on the choice of j . In the case that ρ is linear, $L_{p,S}(s, \rho)$ was constructed independently by Deligne and Ribet [33], Barsky [2] and Cassou-Nogués [24]. Greenberg [37] then extended the construction to the general case using Brauer induction.

3.11 Statement of the p -adic Beilinson conjecture

We now formulate our variant of the p -adic Beilinson conjecture.

Conjecture 3.23 (The p -adic Beilinson conjecture) *Let E/K be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/K)$. Let p be an odd prime and let S be a finite set of places of K containing $S_p \cup S_\infty$. Let $\rho \in R_{\mathbb{C}_p}^+(G)$ and let $r > 1$ be an integer. Then for every choice of field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ we have*

$$L_{p,S}(r, \rho) = \Omega_j(r, \rho \otimes \omega^{r-1}) \cdot j \left(L_S(r, (\rho \otimes \omega^{r-1})^{j^{-1}}) \right). \quad (3.14)$$

Remark 3.24 It is straightforward to show that Conjecture 3.23 does not depend on the choice of S .

Remark 3.25 One can show (see Theorem 4.12 below) that $L_{p,S}(r, \rho) \neq 0$ if and only if $\Omega_j(r, \rho \otimes \omega^{r-1}) \neq 0$. In this case (and thus in particular if $\text{Sch}(E(\zeta_p), p, r)$ holds) the statement of Conjecture 3.23 naturally extends to all virtual characters $\rho \in R_{\mathbb{C}_p}(G)$.

Remark 3.26 It is clear from the definitions that Conjecture 3.23 is compatible with the p -adic Beilinson conjecture as considered by Besser, Buckingham, de Jeu and Roblot [5, Conjecture 3.18]. More concretely, the equality (3.14) is equivalent to the appropriate special case of [5, Conjecture 3.18 (i)–(iii)], whereas [5, Conjecture 3.18 (iv)] then is equivalent to the non-vanishing of $\Omega_j(r, \rho \otimes \omega^{r-1})$ as follows from Remark 3.25.

Remark 3.27 Since both complex and p -adic Artin L -functions satisfy properties analogous to those of $\Omega_j(r, -)$ given in Lemma 3.19, the truth of Conjecture 3.23 is invariant under induction and inflation; moreover, if it holds for $\rho_1, \rho_2 \in R_{\mathbb{C}_p}(G)$ then it holds for $\rho_1 + \rho_2$.

3.12 The relation to Gross's conjecture

The following results are the analogues of [49, Theorem 4.16 and Corollary 4.18], respectively.

Theorem 3.28 *Let E/K be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/K)$. Let p be an odd prime and let S be a finite set of places of K containing $S_p \cup S_\infty$. Let $\rho \in R_{\mathbb{C}_p}^+(G)$ and let $r > 1$ be an integer. We put $\psi := \rho \otimes \omega^{r-1}$. If $\Omega_j(r, \psi) \neq 0$ for some (and hence every) choice of field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ then the following statements are equivalent.*

- (i) $\Omega_j(r, \psi) \cdot j(L_S(r, \psi^{j^{-1}}))$ is independent of the choice of $j : \mathbb{C} \cong \mathbb{C}_p$.
- (ii) Gross's conjecture at $s = 1 - r$ holds for $\check{\psi}^{j^{-1}} \in R_{\mathbb{C}}^+(\text{Gal}(E(\zeta_p)/K))$ and some (and hence every) choice of $j : \mathbb{C} \cong \mathbb{C}_p$.

Proof The first and second occurrence of ‘and hence every’ in the statement of the theorem follow from Remarks 3.13 and 3.20, respectively.

Let $j, j' : \mathbb{C} \cong \mathbb{C}_p$ be field isomorphisms and let $\chi := \psi^{j^{-1}}$. Then $j = j' \circ \sigma$ for some $\sigma \in \text{Aut}(\mathbb{C})$ and so $\psi^{j'^{-1}} = \chi^\sigma$. For every $\mathbb{Q}[G]$ -isomorphism ϕ_r as in (3.11) we have

$$\Omega_j(r, \psi) \cdot j(R_{\phi_r}(\check{\chi})) = \det_{\mathbb{C}_p}(\mu_p(r) \circ (\mathbb{C}_p \otimes_{\mathbb{Q}} \phi_r))^\psi,$$

which does not depend on j . Hence we have

$$\begin{aligned} \frac{\Omega_j(r, \psi) \cdot j(L_S(r, \psi^{j^{-1}}))}{\Omega_{j'}(r, \psi) \cdot j'(L_S(r, \psi^{(j')^{-1}}))} &= \frac{j'(R_{\phi_r}(\check{\chi})) \cdot j(L_S(r, \chi))}{j(R_{\phi_r}(\check{\chi})) \cdot j'(L_S(r, \chi^\sigma))} \\ &= j' \left(\frac{\sigma(L_S(r, \chi)) \cdot R_{\phi_r}(\check{\chi})}{\sigma(R_{\phi_r}(\check{\chi})) \cdot L_S(r, \chi^\sigma)} \right). \end{aligned}$$

By Proposition 3.15 the last expression is equal to 1 if and only if Gross's conjecture at $s = 1 - r$ (Conjecture 3.12) holds for the character $\check{\chi}$. \square

Corollary 3.29 *Let E/K be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/K)$. Fix a prime p and let $r > 1$ be an integer. Assume that $\text{Sch}(E(\zeta_p), p, r)$ holds. If the p -adic Beilinson conjecture at $s = r$ holds for all $\rho \in R_{\mathbb{C}_p}^+(G)$ then Gross's conjecture at $s = 1 - r$ holds for $\chi \otimes \omega^{r-1}$ for all $\chi \in R_{\mathbb{C}}^+(G)$.*

3.13 Absolutely abelian characters

Since our conjecture is compatible with that of [5] by Remark 3.26 and invariant under induction and inflation of characters by Remark 3.27, we deduce the following result from work of Coleman [29] (see [5, Proposition 4.17]).

Theorem 3.30 *Let E/K be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/K)$. Let p be a prime and let $r > 1$ be an integer. Suppose that $\rho \in R_{\mathbb{C}_p}^+(G)$ is an absolutely abelian character, i.e., there exists a normal subgroup N of G such that ρ factors through $G/N \cong \text{Gal}(E^N/K)$ and E^N/\mathbb{Q} is abelian. Then the p -adic Beilinson conjecture (Conjecture 3.23) holds for ρ .*

4 Equivariant Iwasawa theory

4.1 Bockstein homomorphisms

We recall some background material regarding Bockstein homomorphisms. The reader may also consult [22, Sect. 3.1–Sect. 3.2].

Let \mathcal{G} be a compact p -adic Lie group that contains a closed normal subgroup H such that $\Gamma := \mathcal{G}/H$ is isomorphic to \mathbb{Z}_p . We fix a topological generator γ of Γ . The Iwasawa algebra of \mathcal{G} is

$$\Lambda(\mathcal{G}) := \mathbb{Z}_p[[\mathcal{G}]] = \varprojlim \mathbb{Z}_p[\mathcal{G}/\mathcal{N}],$$

where the inverse limit is taken over all open normal subgroups \mathcal{N} of \mathcal{G} . If F is a finite field extension of \mathbb{Q}_p with ring of integers $\mathcal{O} = \mathcal{O}_F$, we put $\Lambda^{\mathcal{O}}(\mathcal{G}) := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) = \mathcal{O}[[\mathcal{G}]]$. We consider continuous homomorphisms

$$\pi : \mathcal{G} \longrightarrow \text{Aut}_{\mathcal{O}}(T_{\pi}), \quad (4.1)$$

where T_{π} is a finitely generated free \mathcal{O} -module. For $g \in \mathcal{G}$ we denote its image in Γ under the canonical projection by \bar{g} . We view $\Lambda^{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} T_{\pi}$ as a $(\Lambda^{\mathcal{O}}(\Gamma), \Lambda(\mathcal{G}))$ -bimodule, where $\Lambda^{\mathcal{O}}(\Gamma)$ acts by left multiplication and $\Lambda(\mathcal{G})$ acts on the right via

$$(\lambda \otimes_{\mathcal{O}} t)g := \lambda \bar{g} \otimes_{\mathcal{O}} g^{-1}t$$

for $\lambda \in \Lambda^{\mathcal{O}}(\Gamma)$, $t \in T_{\pi}$ and $g \in \mathcal{G}$. For each complex $C^{\bullet} \in \mathcal{D}^{\text{perf}}(\Lambda(\mathcal{G}))$ we define a complex $C_{\pi}^{\bullet} \in \mathcal{D}^{\text{perf}}(\Lambda^{\mathcal{O}}(\Gamma))$ by

$$C_{\pi}^{\bullet} := (\Lambda^{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} T_{\pi}) \otimes_{\Lambda(\mathcal{G})}^{\mathbb{L}} C^{\bullet}.$$

Given an open normal subgroup U of \mathcal{G} we set $C_U^{\bullet} := \mathbb{Z}_p[\mathcal{G}/U] \otimes_{\Lambda(\mathcal{G})}^{\mathbb{L}} C^{\bullet}$ and, if U is contained in the kernel of π , we furthermore obtain a complex

$$C_{U,(\pi)}^{\bullet} = T_{\pi} \otimes_{\mathbb{Z}_p[\mathcal{G}/U]}^{\mathbb{L}} C_U^{\bullet} = T_{\pi} \otimes_{\Lambda(\mathcal{G})}^{\mathbb{L}} C^{\bullet} \in \mathcal{D}^{\text{perf}}(\mathcal{O})$$

which does actually not depend on U . The natural exact triangles

$$C_{\pi}^{\bullet} \xrightarrow{\gamma-1} C_{\pi}^{\bullet} \longrightarrow C_{U,(\pi)}^{\bullet}$$

in $\mathcal{D}(\mathcal{O})$ induce short exact sequences of \mathcal{O} -modules

$$0 \longrightarrow H^i(C_{\pi}^{\bullet})_{\Gamma} \xrightarrow{\alpha^i} H^i(C_{U,(\pi)}^{\bullet}) \xrightarrow{\tilde{\alpha}^i} H^{i+1}(C_{\pi}^{\bullet})_{\Gamma} \longrightarrow 0 \quad (4.2)$$

for each $i \in \mathbb{Z}$. The *Bockstein homomorphism* in degree i is defined to be the composite homomorphism

$$\beta_{C^\bullet, \pi}^i : H^i(C_{U, (\pi)}^\bullet) \xrightarrow{\tilde{\alpha}^i} H^{i+1}(C_\pi^\bullet)^\Gamma \longrightarrow H^{i+1}(C_\pi^\bullet)_\Gamma \xrightarrow{\alpha^{i+1}} H^{i+1}(C_{U, (\pi)}^\bullet),$$

where the middle arrow is the tautological map. We obtain a bounded complex of \mathcal{O} -modules

$$\Delta_{C^\bullet, \pi} : \dots \xrightarrow{\beta_{C^\bullet, \pi}^{i-1}} H^i(C_{U, (\pi)}^\bullet) \xrightarrow{\beta_{C^\bullet, \pi}^i} H^{i+1}(C_{U, (\pi)}^\bullet) \xrightarrow{\beta_{C^\bullet, \pi}^{i+1}} \dots$$

where the term $H^i(C_{U, (\pi)}^\bullet)$ is placed in degree i . Note that the Bockstein homomorphisms and the complex $\Delta_{C^\bullet, \pi}$ actually depend on the choice of γ , though our notation does not reflect this. The complex C^\bullet is called *semisimple at π* if $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Delta_{C^\bullet, \pi}$ is acyclic (for any and hence every choice of γ).

For any $\Lambda(\mathcal{G})$ -module M we put $M_\pi := (\Lambda^\mathcal{O}(\Gamma) \otimes_{\mathcal{O}} T_\pi) \otimes_{\Lambda(\mathcal{G})} M$.

Lemma 4.1 *Assume that \mathcal{G} is a compact p -adic Lie group of dimension 1 and let $C^\bullet \in \mathcal{D}^{\text{perf}}(\Lambda(\mathcal{G}))$ be acyclic outside degree a for some $a \in \mathbb{Z}$. Further assume that the $\Lambda(\mathcal{G})$ -module $H^a(C^\bullet)$ has projective dimension at most 1. Then for every continuous homomorphism $\pi : \mathcal{G} \rightarrow \text{Aut}_{\mathcal{O}}(T_\pi)$ the complex C_π^\bullet is acyclic outside degree a and there is a canonical isomorphism of $\Lambda^\mathcal{O}(\Gamma)$ -modules $H^a(C_\pi^\bullet) \simeq H^a(C^\bullet)_\pi$.*

Proof This has been shown in the course of the proof of [15, Lemma 5.6]. We repeat the short argument for convenience.

We may assume that $a = 0$. Then C^\bullet may be represented by a complex $P^{-1} \xrightarrow{d} P^0$, where P^{-1} and P^0 are projective $\Lambda(\mathcal{G})$ -modules placed in degrees -1 and 0 , respectively, and the homomorphism d is injective. Then C_π^\bullet is represented by

$$(T_\pi \otimes_{\mathbb{Z}_p} P^{-1})^H \xrightarrow{d'} (T_\pi \otimes_{\mathbb{Z}_p} P^0)^H,$$

where d' is injective since d is. The result follows. \square

4.2 Algebraic K -theory

Let R be a noetherian integral domain with field of fractions E . Let A be a finite-dimensional semisimple E -algebra and let \mathfrak{A} be an R -order in A . For any field extension F of E we set $A_F := F \otimes_E A$. Let $K_0(\mathfrak{A}, F) = K_0(\mathfrak{A}, A_F)$ denote the relative algebraic K -group associated to the ring homomorphism $\mathfrak{A} \hookrightarrow A_F$. We recall that $K_0(\mathfrak{A}, A_F)$ is an abelian group with generators $[X, g, Y]$ where X and Y are finitely generated projective \mathfrak{A} -modules and $g : F \otimes_R X \rightarrow F \otimes_R Y$ is an isomorphism of A_F -modules; for a full description in terms of generators and relations, we refer the reader to [74, p. 215]. Moreover, there is a long exact sequence of relative K -theory (see [74, Chapter 15])

$$K_1(\mathfrak{A}) \longrightarrow K_1(A_F) \xrightarrow{\partial} K_0(\mathfrak{A}, A_F) \longrightarrow K_0(\mathfrak{A}) \longrightarrow K_0(A_F). \quad (4.3)$$

The reduced norm map $\text{Nrd} = \text{Nrd}_{A_F} : A_F \rightarrow \zeta(A_F)$ is defined componentwise on the Wedderburn decomposition of A_F and extends to matrix rings over A_F (see [31, Sect. 7D]); thus it induces a map $K_1(A_F) \rightarrow \zeta(A_F)^\times$, which we also denote by Nrd .

In the case $E = F$ the relative K -group $K_0(\mathfrak{A}, A)$ identifies with the Grothendieck group whose generators are $[C^\bullet]$, where C^\bullet is an object of the category $\mathcal{C}_{\text{tors}}^b(\text{PMod}(\mathfrak{A}))$ of bounded complexes of finitely generated projective \mathfrak{A} -modules whose cohomology modules are R -torsion, and the relations are as follows: $[C^\bullet] = 0$ if C^\bullet is acyclic, and $[C_2^\bullet] = [C_1^\bullet] + [C_3^\bullet]$ for every short exact sequence

$$0 \rightarrow C_1^\bullet \rightarrow C_2^\bullet \rightarrow C_3^\bullet \rightarrow 0$$

in $\mathcal{C}_{\text{tors}}^b(\text{PMod}(\mathfrak{A}))$ (see [77, Chapter 2] or [73, Sect. 2], for example).

We denote the full triangulated subcategory of $\mathcal{D}_{\text{tors}}^{\text{perf}}(\mathfrak{A})$ comprising perfect complexes whose cohomology modules are R -torsion by $\mathcal{D}_{\text{tors}}^{\text{perf}}(\mathfrak{A})$. Then every object C^\bullet of $\mathcal{D}_{\text{tors}}^{\text{perf}}(\mathfrak{A})$ defines a class $[C^\bullet]$ in $K_0(\mathfrak{A}, A)$.

Let p be an odd prime and let \mathcal{G} be a one-dimensional compact p -adic Lie group that surjects onto \mathbb{Z}_p . Then \mathcal{G} may be written as $\mathcal{G} = H \rtimes \Gamma$ with a finite normal subgroup H of \mathcal{G} and a subgroup $\Gamma \simeq \mathbb{Z}_p$. Let \mathcal{O} be the ring of integers in some finite extension F of \mathbb{Q}_p . We consider $\mathfrak{A} = \Lambda^{\mathcal{O}}(\mathcal{G}) = \mathcal{O}[[\mathcal{G}]]$ as an order over $R := \Lambda^{\mathcal{O}}(\Gamma_0)$, where Γ_0 is an open subgroup of Γ that is central in \mathcal{G} . We denote the fraction field of $\Lambda^{\mathcal{O}}(\Gamma_0)$ by $\mathcal{Q}^F(\Gamma_0)$ and let $A = \mathcal{Q}^F(\mathcal{G}) := \mathcal{Q}^F(\Gamma_0) \otimes_R \Lambda^{\mathcal{O}}(\mathcal{G})$ be the total ring of fractions of $\Lambda^{\mathcal{O}}(\mathcal{G})$. Then [78, Corollary 3.8] shows that the map ∂ in (4.3) is surjective; thus the sequence

$$K_1(\Lambda^{\mathcal{O}}(\mathcal{G})) \rightarrow K_1(\mathcal{Q}^F(\mathcal{G})) \xrightarrow{\partial} K_0(\Lambda^{\mathcal{O}}(\mathcal{G}), \mathcal{Q}^F(\mathcal{G})) \rightarrow 0 \quad (4.4)$$

is exact. If $\xi \in K_1(\mathcal{Q}^F(\mathcal{G}))$ is a pre-image of some $x \in K_0(\Lambda^{\mathcal{O}}(\mathcal{G}), \mathcal{Q}^F(\mathcal{G}))$, we say that ξ is a *characteristic element* for x . We also set $\mathcal{Q}(\mathcal{G}) := \mathcal{Q}^{\mathbb{Q}_p}(\mathcal{G})$.

We include the following consequence of (4.4) for later use.

Lemma 4.2 *Let M be a finitely generated $\Lambda^{\mathcal{O}}(\mathcal{G})$ -module of projective dimension at most one. Assume that M is torsion as an R -module. Then M admits a free resolution of the form*

$$0 \rightarrow \Lambda^{\mathcal{O}}(\mathcal{G})^m \rightarrow \Lambda^{\mathcal{O}}(\mathcal{G})^m \rightarrow M \rightarrow 0 \quad (4.5)$$

for some positive integer m .

Let $\Gamma' \simeq \mathbb{Z}_p$ be an open normal subgroup of \mathcal{G} and set $G := \mathcal{G}/\Gamma'$. If in addition $M_{\Gamma'}$ is finite, then $M^{\Gamma'}$ vanishes and (4.5) induces a short exact sequence of $\mathbb{Z}_p[G]$ -modules

$$0 \rightarrow \mathbb{Z}_p[G]^m \rightarrow \mathbb{Z}_p[G]^m \rightarrow M_{\Gamma'} \rightarrow 0.$$

Proof Choose a surjective $\Lambda^{\mathcal{O}}(\mathcal{G})$ -homomorphism $\Lambda^{\mathcal{O}}(\mathcal{G})^m \rightarrow M$. Its kernel is a projective $\Lambda^{\mathcal{O}}(\mathcal{G})$ -module P by assumption. Since M is R -torsion, we see that the classes of P and $\Lambda^{\mathcal{O}}(\mathcal{G})^m$ in $K_0(\Lambda^{\mathcal{O}}(\mathcal{G}))$ have the same image in $K_0(\mathcal{Q}^F(\mathcal{G}))$. Hence they coincide by (4.4). In other words, P and $\Lambda^{\mathcal{O}}(\mathcal{G})^m$ are stably isomorphic. By enlarging m if necessary, we may assume that P is free of rank m and we have established the existence of (4.5). By [54, Lemma 5.3.11] this sequence induces an exact sequence of $\mathbb{Z}_p[G]$ -modules

$$0 \rightarrow M^{\Gamma'} \rightarrow \mathbb{Z}_p[G]^m \rightarrow \mathbb{Z}_p[G]^m \rightarrow M_{\Gamma'} \rightarrow 0.$$

It follows that $M^{\Gamma'}$ is a free \mathbb{Z}_p -module of the same rank as $M_{\Gamma'}$. This proves the remaining claims. \square

Now let $\pi : \mathcal{G} \rightarrow \text{Aut}_{\mathcal{O}}(T_{\pi})$ be a continuous homomorphism as in (4.1) and set $n := \text{rank}_{\mathcal{O}}(T_{\pi})$. There is a ring homomorphism $\Phi_{\pi} : \Lambda(\mathcal{G}) \rightarrow M_{n \times n}(\Lambda^{\mathcal{O}}(\Gamma))$ induced by the continuous group homomorphism

$$\begin{aligned} \mathcal{G} &\longrightarrow (M_{n \times n}(\mathcal{O}) \otimes_{\mathbb{Z}_p} \Lambda(\Gamma))^{\times} = \text{GL}_n(\Lambda^{\mathcal{O}}(\Gamma)) \\ g &\mapsto \pi(g) \otimes \bar{g}, \end{aligned}$$

where \bar{g} denotes the image of g in $\mathcal{G}/H \simeq \Gamma$. By [27, Lemma 3.3] this homomorphism extends to a ring homomorphism $\mathcal{Q}(\mathcal{G}) \rightarrow M_{n \times n}(\mathcal{Q}^F(\Gamma))$ and this in turn induces a homomorphism

$$\Phi_{\pi} : K_1(\mathcal{Q}(\mathcal{G})) \longrightarrow K_1(M_{n \times n}(\mathcal{Q}^F(\Gamma))) \simeq \mathcal{Q}^F(\Gamma)^{\times}.$$

For $\xi \in K_1(\mathcal{Q}(\mathcal{G}))$ we set $\xi(\pi) := \Phi_{\pi}(\xi)$. If $\pi = \pi_{\rho}$ is an Artin representation with character ρ , we also write Φ_{ρ} and $\xi(\rho)$ for $\Phi_{\pi_{\rho}}$ and $\xi(\pi_{\rho})$, respectively, and let

$$j_{\rho} : \zeta(\mathcal{Q}(\mathcal{G}))^{\times} \longrightarrow \mathcal{Q}^F(\Gamma)^{\times}$$

be the map defined by Ritter and Weiss in [66]. By [60, Lemma 2.3] (choose $r = 0$) we have a commutative triangle

$$\begin{array}{ccc} K_1(\mathcal{Q}(\mathcal{G})) & & \\ \downarrow \text{Nrd} & \searrow \Phi_{\rho} & \\ \zeta(\mathcal{Q}(\mathcal{G}))^{\times} & \xrightarrow{j_{\rho}} & \mathcal{Q}^F(\Gamma)^{\times} \end{array}$$

We shall also write $\xi^*(\rho)$ for the leading term at $T = 0$ of the power series $\Phi_{\rho}(\xi)$.

We choose a maximal $\Lambda(\Gamma_0)$ -order $\mathfrak{M}(\mathcal{G})$ in $\mathcal{Q}(\mathcal{G})$ such that $\Lambda(\mathcal{G})$ is contained in $\mathfrak{M}(\mathcal{G})$.

Lemma 4.3 *Let $C^{\bullet} \in \mathcal{D}_{\text{tors}}^{\text{perf}}(\Lambda(\mathcal{G}))$ be a complex and let ξ be a characteristic element for C^{\bullet} . Let π_{ρ} be an Artin representation with character ρ . Then $\xi(\rho) \cdot j_{\rho}(x)$ is a characteristic element for C^{\bullet}_{ρ} for every $x \in \zeta(\mathfrak{M}(\mathcal{G}))^{\times}$.*

Proof In the case that $x = 1$ this is [15, Lemma 5.4 (vii)] (and follows from the naturality of connecting homomorphisms). By [66, Remark H] the image of $\zeta(\mathfrak{M}(\mathcal{G}))^\times$ under j_ρ is contained in $\Lambda^\mathcal{O}(\Gamma)^\times$. This proves the claim. \square

4.3 Cohomology with compact support

Let p be an odd prime. We denote the cyclotomic \mathbb{Z}_p -extension of a number field K by K_∞ and set $\Gamma_K := \text{Gal}(K_\infty/K)$. Let S be a finite set of places of K containing the set $S_\infty \cup S_p$. Let M be a topological $G_{K,S}$ -module. Following Burns and Flach [17] we define the compactly supported cohomology complex to be

$$R\Gamma_c(\mathcal{O}_{K,S}, M) := \text{cone} \left(R\Gamma(\mathcal{O}_{K,S}, M) \longrightarrow \bigoplus_{v \in S} R\Gamma(K_v, M) \right) [-1],$$

where the arrow is induced by the natural restriction maps. For any integers i and r we abbreviate $H^i R\Gamma_c(\mathcal{O}_{K,S}, M)$ to $H_c^i(\mathcal{O}_{K,S}, M)$ and set $H_c^i(\mathcal{O}_{K,S}, \mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^i(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$.

Let E/K be a finite Galois extension of totally real fields and set $L := E(\zeta_p)$. Then L is a CM-field and we denote its maximal totally real subfield by L^+ as in Sect. 3.9. Set $\mathcal{G} := \text{Gal}(L_\infty/K)$ and let

$$\chi_{\text{cyc}} : \mathcal{G} \longrightarrow \mathbb{Z}_p^\times,$$

be the p -adic cyclotomic character defined by $\sigma(\zeta) = \zeta^{\chi_{\text{cyc}}(\sigma)}$ for any $\sigma \in \mathcal{G}$ and any p -power root of unity ζ . The composition of χ_{cyc} with the projections onto the first and second factors of the canonical decomposition $\mathbb{Z}_p^\times = \langle \zeta_{p-1} \rangle \times (1 + p\mathbb{Z}_p)$ are given by the Teichmüller character ω and a map that we denote by κ .

Assume in addition that S contains all places that ramify in L_∞/K . For each integer r we define a complex of $\Lambda(\mathcal{G})$ -modules

$$C_{r,S}^\bullet := R\Gamma_c(\mathcal{O}_{K,S}, e_r \Lambda(\mathcal{G})^\sharp(r)),$$

where $\Lambda(\mathcal{G})^\sharp(r)$ denotes the $\Lambda(\mathcal{G})$ -module $\Lambda(\mathcal{G})$ upon which $\sigma \in G_K$ acts on the right via multiplication by the element $\chi_{\text{cyc}}^r(\sigma) \bar{\sigma}^{-1}$; here $\bar{\sigma}$ denotes the image of σ in \mathcal{G} . Note that the complexes $C_{r,S}^\bullet$ are perfect by [36, Proposition 1.6.5] and we have natural isomorphisms

$$C_{r,S}^\bullet \simeq C_{1,S}^\bullet \otimes_{\mathbb{Z}_p}^{\mathbb{L}} \mathbb{Z}_p(r-1) \quad (4.6)$$

for every integer r .

Each $\Lambda(\mathcal{G})$ -module M naturally decomposes as a direct sum $M^+ \oplus M^-$ with $M^\pm = \frac{1 \pm \tau}{2} M$. Similarly, each complex C^\bullet of $\mathcal{D}^{\text{perf}}(\Lambda(\mathcal{G}))$ gives rise to subcomplexes $(C^\bullet)^+$ and $(C^\bullet)^-$. Moreover, we let $(C^\bullet)^\vee := R\text{Hom}(C^\bullet, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontryagin dual of C^\bullet . By a Shapiro Lemma argument and Artin–Verdier duality we then have isomorphisms

$$C_{r,S}^\bullet \simeq \begin{cases} R\Gamma(\mathcal{O}_{L_\infty^+,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee[-3] & \text{if } 2 \nmid r \\ (R\Gamma(\mathcal{O}_{L_\infty^+,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee)^-[-3] & \text{if } 2 \mid r. \end{cases} \quad (4.7)$$

in $\mathcal{D}(e_r \Lambda(\mathcal{G}))$. We let M_S be the maximal abelian pro- p -extension of L_∞^+ unramified outside S . Then $X_S := \text{Gal}(M_S/L_\infty^+)$ is a finitely generated $\Lambda(\mathcal{G}^+)$ -module, where we put $\mathcal{G}^+ := \mathcal{G}/\langle \tau \rangle = \text{Gal}(L_\infty^+/K)$. Iwasawa [45] has shown that X_S is in fact torsion as a $\Lambda(\Gamma_{L^+})$ -module. We let $\mu_p(L^+)$ denote the Iwasawa μ -invariant of X_S and note that this does not depend on the choice of S (see [54, Corollary 11.3.6]). Hence $\mu_p(L^+)$ vanishes if and only if X_S is finitely generated as a \mathbb{Z}_p -module. It is conjectured that we always have $\mu_p(L^+) = 0$ and as explained in [47, Remark 4.3], this is closely related to the classical Iwasawa ‘ $\mu = 0$ ’ conjecture for L at p . Thus a result of Ferrero and Washington [34] on this latter conjecture implies that $\mu_p(L^+) = 0$ whenever E/\mathbb{Q} and thus L/\mathbb{Q} is abelian.

The only non-trivial cohomology groups of $R\Gamma(\mathcal{O}_{L_\infty^+,S}, \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ occur in degrees -1 and 0 and canonically identify with X_S and \mathbb{Z}_p , respectively. Hence (4.7) with $r = 1$ and (4.6) imply that for each integer r the cohomology of $C_{r,S}^\bullet$ is concentrated in degrees 2 and 3 and we have

$$H^2(C_{r,S}^\bullet) \simeq X_S(r-1), \quad H^3(C_{r,S}^\bullet) \simeq \mathbb{Z}_p(r-1).$$

4.4 The main conjecture

The following is an obvious reformulation of the equivariant Iwasawa main conjecture for the extension L_∞^+/K (without its uniqueness statement).

Conjecture 4.4 (equivariant Iwasawa main conjecture) *Let L/K be a Galois CM-extension such that L contains a primitive p th root of unity. Let S be a finite set of places of K containing S_∞ and all places that ramify in L_∞^+/K . Then there exists an element $\zeta_S \in K_1(\mathcal{Q}(\mathcal{G}^+))$ such that $\partial(\zeta_S) = [C_{1,S}^\bullet]$ and for every irreducible Artin representation π_ρ of \mathcal{G}^+ with character ρ and for each integer $n \geq 1$ divisible by $p-1$ we have*

$$\zeta_S^*(\rho\kappa^n) = L_{p,S}(1-n, \rho) = j \left(L_S(1-n, \rho^{j^{-1}}) \right) \quad (4.8)$$

for every field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$.

Part (i) of the following theorem has been shown by Ritter and Weiss [67] and by Kakde [51] independently. Part (ii) is due to Johnston and the present author [50].

Theorem 4.5 *Conjecture 4.4 holds for L_∞^+/K in each of the following cases.*

- (i) *The μ -invariant $\mu_p(L^+)$ vanishes.*
- (ii) *The Galois group \mathcal{G}^+ has an abelian Sylow p -subgroup.*

By starting out from the work of Deligne and Ribet [33], Greenberg [37] has shown that for each topological generator γ of Γ there is a unique element $f_{\rho,S}(T)$ in the

quotient field of $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[T]]$ (which we will identify with $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)$ via the usual map that sends γ to $1 + T$) such that

$$L_{p,S}(1 - s, \rho) = f_{\rho,S}(u^s - 1),$$

where $u := \kappa(\gamma)$. For each integer a we let $x \mapsto t_{\text{cyc}}^a(x)$ be the automorphisms on $\mathcal{Q}^c(\mathcal{G})$ induced by $g \mapsto \chi_{\text{cyc}}(g)^a g$ for $g \in \mathcal{G}$. We use the same notation for the induced group homomorphisms on $K_1(\mathcal{Q}(\mathcal{G}))$ and $K_1(e_r \mathcal{Q}(\mathcal{G}))$, $r \in \mathbb{Z}$.

Proposition 4.6 *Suppose that Conjecture 4.4 holds for L_∞^+/K . Then for each $r \in \mathbb{Z}$ there exists an element $\zeta_{r,S} \in K_1(e_r \mathcal{Q}(\mathcal{G}))$ such that $\partial(\zeta_{r,S}) = [C_{r,S}^\bullet]$ and for every irreducible Artin representation π_ρ of \mathcal{G}^+ with character ρ we have*

$$\zeta_{r,S}(\rho \otimes \omega^{r-1}) = f_{\rho,S}(u^{1-r}(1 + T) - 1). \quad (4.9)$$

Proof When $r = 1$ we may take $\zeta_{1,S} = \zeta_S$ by [49, Proposition 7.5]. Then $\zeta_{r,S} := t_{\text{cyc}}^{1-r}(\zeta_{1,S})$ is a characteristic element for $C_{r,S}^\bullet$ by (4.6) and (4.9) follows from [15, Lemma 5.4 (v)] (see also [14, Lemma 9.5]). \square

Corollary 4.7 *Let $r \in \mathbb{Z}$ be an integer and let π_ρ be an irreducible Artin representation of \mathcal{G}^+ with character ρ . Then $f_{\rho,S}(u^{1-r}(1 + T) - 1)$ is a characteristic element of $C_{r,\rho \otimes \omega^{r-1}}^\bullet$.*

Proof Since the main conjecture (Conjecture 4.4) holds ‘over the maximal order’ by [47, Theorem 4.9] (this result is essentially due to Ritter and Weiss [66]), the equality (4.9) holds unconditionally up to a factor $j_\rho(x)$ for some $x \in \zeta(\mathfrak{M}(\mathcal{G}))^\times$. Thus the claim follows from Lemma 4.3. \square

4.5 Schneider’s conjecture and semisimplicity

We recall the following result from [62, Propositions 3.11 and 3.12]. Part (i) is a special case of [16, Proposition 1.20] and of [36, Proposition 1.6.5].

Proposition 4.8 *Let L/K be a Galois extension of number fields with Galois group G . Let $r > 1$ be an integer and let p be an odd prime. Then the following hold.*

- (i) *The complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ belongs to $\mathcal{D}^{\text{perf}}(\mathbb{Z}_p[G])$ and is acyclic outside degrees 1, 2 and 3.*
- (ii) *We have an exact sequence of $\mathbb{Z}_p[G]$ -modules*

$$0 \longrightarrow H_{-r}^+(L) \otimes \mathbb{Z}_p \longrightarrow H_c^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \longrightarrow \text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \longrightarrow 0.$$

- (iii) *We have an isomorphism of $\mathbb{Z}_p[G]$ -modules*

$$H_c^3(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq \mathbb{Z}_p(r-1)_{G_L}.$$

- (iv) *The $\mathbb{Z}_p[G]$ -module $\text{III}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is finite.*

(v) The \mathbb{Z}_p -rank of $H_c^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ equals d_{r+1} if and only if Schneider's conjecture $\text{Sch}(L, p, r)$ holds.

We now return to the situation considered in Sect. 4.3. Before proving the next result we recall that for a finitely generated $\Lambda^{\mathcal{O}}(\Gamma)$ -torsion module M the following are equivalent: (i) M^{Γ} is finite; (ii) M_{Γ} is finite; (iii) $f_M(0) \neq 0$, where f_M denotes a characteristic element for M . Moreover, by [54, Proposition 5.3.19] we have that $M = 0$ if and only if $M_{\Gamma} = M^{\Gamma} = 0$.

Proposition 4.9 *Let $r > 1$ be an integer and let ρ be an irreducible Artin character of \mathcal{G}^+ such that $\ker(\rho)$ contains Γ_{L^+} . Set $\psi := \rho \otimes \omega^{r-1}$. Then we have that $H^i(C_{r,S,\psi}^{\bullet}) = 0$ for $i \neq 2, 3$ and natural isomorphisms of \mathcal{O}_{ψ} -modules*

$$H^2(C_{r,S,\psi}^{\bullet})^{\Gamma} \simeq H_c^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)} \simeq \text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)} \quad (4.10)$$

and

$$H^3(C_{r,S,\psi}^{\bullet})_{\Gamma} \simeq H_c^3(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{(\psi)} \simeq (\mathbb{Z}_p(r-1)_{G_L})_{(\psi)} \quad (4.11)$$

and a short exact sequence of \mathcal{O}_{ψ} -modules

$$0 \longrightarrow H^2(C_{r,S,\psi}^{\bullet})_{\Gamma} \longrightarrow H^2(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{(\psi)}) \longrightarrow H^3(C_{r,S,\psi}^{\bullet})^{\Gamma} \longrightarrow 0. \quad (4.12)$$

In particular, the \mathcal{O}_{ψ} -modules $H^3(C_{r,S,\psi}^{\bullet})_{\Gamma}$ and $H^3(C_{r,S}^{\bullet})^{\Gamma}$ are finite.

Proof We put $\mathcal{O} := \mathcal{O}_{\psi}$ for brevity. Since the complex $C_{r,S}^{\bullet}$ is acyclic outside degrees 2 and 3 and the functor $M \mapsto M_{\psi} = (\Lambda^{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} T_{\psi}) \otimes_{\Lambda(\mathcal{G})} M$ is right exact, it is clear that $H^i(C_{r,S,\psi}^{\bullet})$ vanishes for $i \geq 4$. We let $U = \Gamma_L = \text{Gal}(L_{\infty}/L)$. Then U is contained in the kernel of ψ , and by [36, Proposition 1.6.5] we have an isomorphism

$$C_{r,S,U}^{\bullet} \simeq e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) := e_r \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G]}^{\mathbb{L}} R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$$

in $\mathcal{D}(e_r \mathbb{Z}_p[G])$. We now consider the exact sequence (4.2) for the case at hand and various integers i . We will repeatedly apply Lemma 2.2. In particular, the complex $C_{r,S,U,(\psi)}^{\bullet}$ is acyclic outside degrees 1, 2 and 3 by Proposition 4.8 (i). For $i \leq 0$ we find that $H^i(C_{r,S,\psi}^{\bullet})_{\Gamma}$ and $H^{i+1}(C_{r,S,\psi}^{\bullet})^{\Gamma}$ vanish. Thus $H^i(C_{r,S,\psi}^{\bullet})$ vanishes for $i \leq 0$ and even for $i = 1$ once we show that the \mathcal{O} -module $H^1(C_{r,S,\psi}^{\bullet})_{\Gamma}$ is trivial. We already know that it is finite. Sequence (4.2) in the case $i = 1$ and Lemma 2.2 now give rise to a short exact sequence

$$0 \longrightarrow H^1(C_{r,S,\psi}^{\bullet})_{\Gamma} \longrightarrow H_c^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)} \longrightarrow H^2(C_{r,S,\psi}^{\bullet})^{\Gamma} \longrightarrow 0.$$

Since the central idempotent e_r annihilates $H_{-r}^+(L) \otimes \mathbb{Z}_p$ and $e_r e_{\psi} = e_{\psi}$ we have an isomorphism

$$H_c^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)} \simeq \text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)}$$

by Proposition 4.8 (ii). Since the Tate–Shafarevich group $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is torsion-free by Lemma 3.11, the \mathcal{O} -module $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)}$ is free. Therefore $H^1(C_{r,S,\psi}^\bullet)_\Gamma$ vanishes as desired and we have established (4.10). Proposition 4.8 (iii), Lemma 2.2 and the case $i = 3$ of sequence (4.2) imply (4.11). Finally, sequence (4.12) is the case $i = 2$ of sequence (4.2). \square

Lemma 4.10 *Let r be an arbitrary integer. Then there are finitely generated $e_r \Lambda(\mathcal{G})$ -modules $Y_{r,S}$ and Z_r with all of the following properties:*

- (i) *The projective dimension of both $Y_{r,S}$ and Z_r is at most 1;*
- (ii) *both $Y_{r,S}$ and Z_r are torsion as R -modules;*
- (iii) *there is an exact triangle*

$$Z_r[-3] \longrightarrow C_{r,S}^\bullet \longrightarrow Y_{r,S}[-2]$$

in $\mathcal{D}(e_r \Lambda(\mathcal{G}))$;

- (iv) *we have that $Y_{r,S} = Y_{0,S}(r)$ and $Z_r = Z_0(r)$;*
- (v) *the coinvariants $(Z_r)_{\Gamma_L}$ are finite if $r \neq 1$.*

Proof We first consider the case $r = 0$. It is shown in [48, Proposition 8.5] that the complex $C_{0,S}^\bullet$ is isomorphic in $\mathcal{D}(e_0 \Lambda(\mathcal{G}))$ to a complex

$$\dots \longrightarrow 0 \longrightarrow Y_{0,S} \longrightarrow Z_0 \longrightarrow 0 \longrightarrow \dots$$

where $Y_{0,S}$ is placed in degree 2. More precisely, in the notation of [48] we have

$$C_{0,S}^\bullet = C_S^\bullet(L_\infty^+/K)(-1)[-3], \quad Y_{0,S} = Y_S^T(-1), \quad Z_0 = I_T = \left(\bigoplus_{v \in T} \text{ind}_{\mathcal{G}_{w_\infty}}^{\mathcal{G}} \mathbb{Z}_p(-1) \right)^-$$

where T is a finite set of places of K disjoint from S with certain properties, and \mathcal{G}_{w_∞} denotes the decomposition group at a chosen place w_∞ of L_∞ above v for each $v \in T$; moreover, we write $\text{ind}_{\mathcal{U}}^{\mathcal{G}} M := \Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{U})} M$ for any open subgroup \mathcal{U} of \mathcal{G} and any $\Lambda(\mathcal{U})$ -module M . By [48, Lemmas 8.4 and 8.5] the modules Z_0 and $Y_{0,S}$ are R -torsion and of projective dimension at most 1. Thus (i) and (ii) also hold for $Y_{r,S} := Y_{0,S}(r) = Y_S^T(r-1)$ and $Z_r := Z_0(r) = e_r \bigoplus_{v \in T} \text{ind}_{\mathcal{G}_{w_\infty}}^{\mathcal{G}} \mathbb{Z}_p(r-1)$. It is now clear that (iv) holds and that $C_{r,S}^\bullet$ is isomorphic to the complex

$$\dots \longrightarrow 0 \longrightarrow Y_{r,S} \longrightarrow Z_r \longrightarrow 0 \longrightarrow \dots$$

in $\mathcal{D}(e_r \Lambda(\mathcal{G}))$. Hence (iii) also holds. Moreover, the coinvariants $(Z_r)_{\Gamma_L}$ are clearly finite for $r \neq 1$. \square

Remark 4.11 We point out that similar constructions repeatedly appear in the literature. In fact, by [60, Theorem 2.4] the complex $C_S^\bullet(L_\infty^+/K)$ naturally identifies with the complex constructed by Ritter and Weiss [65]. Choosing their maps Ψ and $\tilde{\psi}$ in [66,

p.562 f.] suitably one can take $Y_{1,S} = \text{coker}(\Psi)$ and $Z_1 = \text{coker}(\tilde{\psi})$. Moreover, Burns [15, Sect. 5.3.1] constructed an exact triangle in $\mathcal{D}(e_0\Lambda(\mathcal{G}))$ of the form

$$R\Gamma_T(\mathcal{O}_{K,S}, e_0\Lambda(\mathcal{G})^\sharp(1)) \longrightarrow R\Gamma(\mathcal{O}_{K,S}, e_0\Lambda(\mathcal{G})^\sharp(1)) \longrightarrow \bigoplus_{v \in T} R\Gamma(K(v), e_0\Lambda(\mathcal{G})^\sharp(1))$$

where the set T is as in the proof of Lemma 4.10. For each $v \in T$ the complex $R\Gamma(K(v), e_0\Lambda(\mathcal{G})^\sharp(1))$ is acyclic outside degree 1 and we have an $e_0\Lambda(\mathcal{G})$ -isomorphism

$$H^1(K(v), e_0\Lambda(\mathcal{G})^\sharp(1)) \simeq e_0 \text{ind}_{\mathcal{G}_{w_\infty}}^{\mathcal{G}} \mathbb{Z}_p(1).$$

For each $\Lambda(\mathcal{G})$ -module M and $i \in \mathbb{Z}$ we set $E^i(M) := \text{Ext}_{\Lambda(\mathcal{G})}^i(M, \Lambda(\mathcal{G}))$. We then have an isomorphism of $e_0\Lambda(\mathcal{G})$ -modules

$$\bigoplus_{v \in T} H^1(K(v), e_0\Lambda(\mathcal{G})^\sharp(1)) \simeq E^1(I_T).$$

If C^\bullet is a complex in $\mathcal{D}(e_0\Lambda(\mathcal{G}))$, we denote the complex $R\text{Hom}_{e_0\Lambda(\mathcal{G})}(C^\bullet, e_0\Lambda(\mathcal{G}))$ by $(C^\bullet)^*$. If M is an R -torsion $e_0\Lambda(\mathcal{G})$ -module of projective dimension at most 1, then we have isomorphisms $M[-n]^* \simeq E^1(M)[n-1]$ in $\mathcal{D}(e_0\Lambda(\mathcal{G}))$ for every $n \in \mathbb{Z}$. This yields

$$\bigoplus_{v \in T} R\Gamma(K(v), e_0\Lambda(\mathcal{G})^\sharp(1)) \simeq I_T^*.$$

The complex $R\Gamma_T(\mathcal{O}_{K,S}, e_0\Lambda(\mathcal{G})^\sharp(1))$ is acyclic outside degree 2 and its second cohomology group $H_T^2(\mathcal{O}_{K,S}, e_0\Lambda(\mathcal{G})^\sharp(1))$ is of projective dimension at most 1 by [15, Proposition 5.5]. Since we have an isomorphism

$$R\Gamma(\mathcal{O}_{K,S}, e_0\Lambda(\mathcal{G})^\sharp(1)) \simeq (C_{0,S}^\bullet)^*[-3]$$

by Artin–Verdier duality, it follows that we have an isomorphism

$$R\Gamma_T(\mathcal{O}_{K,S}, e_0\Lambda(\mathcal{G})^\sharp(1)) \simeq Y_S^T(-1)^*[-1]$$

in $\mathcal{D}(e_0\Lambda(\mathcal{G}))$ and an isomorphism of $e_0\Lambda(\mathcal{G})$ -modules

$$H_T^2(\mathcal{O}_{K,S}, e_0\Lambda(\mathcal{G})^\sharp(1)) \simeq E^1(Y_S^T(-1)).$$

Finally, the minus Tate module $T_p(\mathcal{M}_{S,T}^{\mathcal{K}})^-$ of the Iwasawa-theoretic 1-motive $\mathcal{M}_{S,T}^{\mathcal{K}}$ constructed by Greither and Popescu [38] plays the role of $E^1(Y_S^T(-1))$; the $e_0\Lambda(\mathcal{G})$ -module $T_p(\Delta_{\mathcal{K},T})^-$ is our $E^1(I_T)$. See in particular [38, Remark 3.10].

Theorem 4.12 *Let $r > 1$ be an integer and let ρ be an irreducible Artin character of \mathcal{G}^+ such that $\ker(\rho)$ contains Γ_{L^+} . Set $\psi := \rho \otimes \omega^{r-1}$. Then the following conditions are equivalent.*

- (i) *We have that $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)}$ vanishes.*
- (ii) *We have that $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))^{(\psi)}$ vanishes.*
- (iii) *We have that $H_c^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))^{(\psi)}$ vanishes.*
- (iv) *The period $\Omega_j(r, \psi)$ is non-zero for any (and hence every) choice of $j : \mathbb{C} \simeq \mathbb{C}_p$.*
- (v) *We have that $L_{p,S}(r, \rho) \neq 0$.*

If these equivalent conditions hold (in particular if $\text{Sch}(L, p, r)$ holds), then the complex $C_{r,S}^\bullet$ is semisimple at ψ .

Proof We again set $\mathcal{O} := \mathcal{O}_\psi$ for brevity. We have already observed in the proof of Proposition 4.9 that $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)}$ is a free \mathcal{O} -module. The equivalence of (i) and (ii) is therefore clear. We have an exact sequence of $\mathbb{Z}_p[G]$ -modules

$$\begin{aligned} 0 \longrightarrow \text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) &\longrightarrow H_{\text{ét}}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \longrightarrow P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \\ &\longrightarrow H_c^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \longrightarrow \text{III}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \longrightarrow 0. \end{aligned}$$

Since $\text{III}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is finite by Proposition 4.8 (iv) and the $e_r \mathbb{Q}_p[G]$ -modules $e_r H_{\text{ét}}^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \simeq e_r K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{Q}_p$ and $e_r P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \simeq e_r (L \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ are (non-canonically) isomorphic, there is a (non-canonical) isomorphism of $e_r \mathbb{Q}_p[G]$ -modules

$$e_r \text{III}^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \simeq e_r H_c^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)).$$

Thus also $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))^{(\psi)}$ and $H_c^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))^{(\psi)}$ are (non-canonically) isomorphic and so (ii) and (iii) are indeed equivalent. The equivalence of (ii) and (iv) is easy (see Remark 3.21).

We next establish the equivalence of (i) and (v). By Lemma 4.1 the triangle of Lemma 4.10 (iii) induces an exact triangle

$$Z_{r,\psi}[-3] \longrightarrow C_{r,S,\psi}^\bullet \longrightarrow Y_{r,S,\psi}[-2] \quad (4.13)$$

in $\mathcal{D}(\Lambda^{\mathcal{O}}(\Gamma))$. Let $h_{r,\psi}(T)$ and $g_{r,S,\psi}(T)$ be characteristic elements of $Z_{r,\psi}$ and $Y_{r,S,\psi}$, respectively. Corollary 4.7 implies that we may assume that

$$g_{r,S,\psi}(T) = h_{r,\psi}(T) \cdot f_{\rho,S}(u^{1-r}(1+T) - 1).$$

Since $(Z_{r,\psi})_\Gamma$ is finite by Lemma 4.10 (v) we have $h_{r,\psi}(0) \neq 0$. Thus $L_{p,S}(r, \rho) = f_{\rho,S}(u^{1-r} - 1)$ is non-zero if and only if $g_{r,S,\psi}(0) \neq 0$ if and only if $Y_{r,S,\psi}^\Gamma$ vanishes, where the latter equivalence uses Lemma 4.2 (with $M = Y_{r,S,\psi}$ and $\mathcal{G} = \Gamma$) and Lemma 4.10 (i) and (ii). By (4.13) we have an exact sequence of $\Lambda^{\mathcal{O}}(\Gamma)$ -modules

$$0 \longrightarrow H^2(C_{r,S,\psi}^\bullet) \longrightarrow Y_{r,S,\psi} \longrightarrow Z_{r,\psi} \longrightarrow H^3(C_{r,S,\psi}^\bullet) \longrightarrow 0.$$

Since taking Γ -invariants is left exact and $Z_{r,\psi}^\Gamma$ vanishes by another application of Lemma 4.2, it follows that there is an isomorphism $Y_{r,S,\psi}^\Gamma \simeq H^2(C_{r,S,\psi}^\bullet)^\Gamma$. The latter identifies with $\text{III}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^{(\psi)}$ by Proposition 4.9 which proves the claim.

Finally, if these equivalent conditions all hold, then Proposition 4.9 implies that the \mathcal{O} -modules $H^i(C_{r,S,\psi}^\bullet)^\Gamma$ and $H^i(C_{r,S,\psi}^\bullet)_\Gamma$ are finite for all $i \in \mathbb{Z}$. It follows from the short exact sequences (4.2) that $H^i(C_{r,S,U}^\bullet)$ is finite for all $i \in \mathbb{Z}$, where we set $U = \Gamma_L$ as before. Thus the whole complex $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Delta_{C_{r,S},\psi}^\bullet$ vanishes. In particular, the complex $C_{r,S}^\bullet$ is semisimple at ψ . \square

Remark 4.13 By specializing $K = L^+$ in the above argument, we see that Schneider's conjecture $\text{Sch}(L, p, r)$ implies that $Y_{r,S}^{\Gamma_L}$ vanishes and that $(Y_{r,S})_{\Gamma_L}$ is finite.

4.6 Higher refined p -adic class number formulae

We keep the notation of Sect. 4.3. We let $\partial_p : \zeta(\mathbb{C}_p[G])^\times \simeq K_1(\mathbb{C}_p[G]) \rightarrow K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ be the composition of the inverse of the reduced norm and the connecting homomorphism to relative K -theory. By abuse of notation we shall use the same symbol for the induced maps on ' e_r -parts'. We recall that G denotes $\text{Gal}(L/K)$ and that $G^+ = G/\langle \tau \rangle = \text{Gal}(L^+/K)$. We define

$$L_{p,S}(r) := \sum_{\rho \in \text{Irr}_{\mathbb{C}_p}(G^+)} L_{p,S}(r, \rho) e_{\rho \otimes \omega^{r-1}} \in \zeta(e_r \mathbb{Q}_p[G]).$$

Let us assume that Schneider's conjecture $\text{Sch}(L, p, r)$ holds. Then we actually have that $L_{p,S}(r) \in \zeta(e_r \mathbb{Q}_p[G])^\times$ and the cohomology groups of the complex $e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ are finite by Proposition 4.8 and Theorem 4.12. This complex therefore is an object in $\mathcal{D}_{\text{tors}}^{\text{perf}}(e_r \mathbb{Z}_p[G])$. We now state our conjectural higher refined p -adic class number formula.

Conjecture 4.14 *Let $r > 1$ be an integer and assume that $\text{Sch}(L, p, r)$ holds. Then in $K_0(e_r \mathbb{Z}_p[G], \mathbb{Q}_p)$ one has*

$$\partial_p(L_{p,S}(r)) = [e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))].$$

Remark 4.15 In the case $r = 1$ Burns [15, Conjecture 3.5] has formulated a conjectural refined p -adic class number formula. Conjecture 4.14 might be seen as a higher analogue of his conjecture. Accordingly, Theorem 4.17 below is the higher analogue of [15, Theorem 3.6].

Lemma 4.16 *Let $r > 1$ be an integer and assume that $\text{Sch}(L, p, r)$ holds. Then*

$$HR_p(L/K, r) := \partial_p(L_{p,S}(r)) - [e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))]$$

does not depend on the set S .

Proof Let S' be a second sufficiently large finite set of places of K . By embedding S and S' into the union $S \cup S'$ we may and do assume that $S \subseteq S'$. By induction we may additionally assume that $S' = S \cup \{v\}$, where v is not in S . In particular, v is unramified in L/K and $v \nmid p$. By [17, (30)] we have an exact triangle

$$\bigoplus_{w|v} R\Gamma_f(L_w, \mathbb{Z}_p(r))[-1] \longrightarrow R\Gamma_c(\mathcal{O}_{L,S'}, \mathbb{Z}_p(r)) \longrightarrow R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)), \quad (4.14)$$

where $R\Gamma_f(L_w, \mathbb{Z}_p(r))$ is a perfect complex of $\mathbb{Z}_p[G_w]$ -modules which is naturally quasi-isomorphic to

$$\mathbb{Z}_p[G_w] \xrightarrow{1 - \phi_w N(v)^{-r}} \mathbb{Z}_p[G_w] \quad (4.15)$$

with terms in degree 0 and 1. We set

$$\varepsilon_v(r) := \left(\det_{\mathbb{C}_p} (1 - \phi_w N(v)^{-r} \mid V_\chi)^{-1} \right)_{\chi \in \text{Irr}_{\mathbb{C}_p}(G)} \in \zeta(\mathbb{Q}_p[G])^\times.$$

We compute

$$\begin{aligned} [e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))] - [e_r R\Gamma_c(\mathcal{O}_{L,S'}, \mathbb{Z}_p(r))] &= [e_r \bigoplus_{w|v} R\Gamma_f(L_w, \mathbb{Z}_p(r))] \\ &= \partial_p(e_r \varepsilon_v(r)) \\ &= \partial_p(L_{p,S}(r)) - \partial_p(L_{p,S'}(r)), \end{aligned}$$

where the first and second equality follow from (4.14) and (4.15), respectively. This implies the claim. \square

Our main evidence for Conjecture 4.14 is provided by the following result which, crucially, does not depend upon the vanishing of $\mu_p(L^+)$.

Theorem 4.17 *Let $r > 1$ be an integer and assume that $\text{Sch}(L, p, r)$ holds. If the equivariant Iwasawa main conjecture (Conjecture 4.4) holds for L_∞^+/K (and so in particular if $\mu_p(L^+) = 0$ or if \mathcal{G}^+ has an abelian Sylow p -subgroup), then Conjecture 4.14 holds.*

Proof We first observe that the complex $C_{r,S}^\bullet$ is semisimple at $\rho \otimes \omega^{r-1}$ for all $\rho \in \text{Irr}_{\mathbb{C}_p}(G^+)$ by Theorem 4.12. Moreover, if we put $U = \Gamma_L$ as above, then we have an isomorphism

$$C_{r,S,U}^\bullet \simeq e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \quad (4.16)$$

in $\mathcal{D}(e_r \mathbb{Z}_p[G])$. If Conjecture 4.4 holds, then by Proposition 4.6 there is a characteristic element $\zeta_{r,S}$ of $[C_{r,S}^\bullet]$ such that $\zeta_{r,S}(\rho \otimes \omega^{r-1}) = f_{\rho,S}(u^{1-r}(1+T) - 1)$. Since

$f_{\rho,S}(u^{1-r} - 1) = L_{p,S}(r, \rho) \neq 0$ we have that $\zeta_{r,S}^*(\rho \otimes \omega^{r-1}) = L_{p,S}(r, \rho)$. If $\mu_p(L^+)$ vanishes or p does not divide the cardinality of G^+ , then [23, Theorem 2.2] implies the claim (as noted above the complexes $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Delta_{C_{r,S}^\bullet, \rho \otimes \omega^{r-1}}$ all vanish).

In order to avoid these assumptions, we proceed as follows. Observe that both $(Z_r)_{\Gamma_L}$ and $(Y_{r,S})_{\Gamma_L}$ are finite by Lemma 4.10 (v) and Theorem 4.12 (or rather Remark 4.13), respectively, since Schneider's conjecture holds by assumption. Recall from Lemma 4.10 (iii) that we have an exact triangle

$$Z_r[-3] \longrightarrow C_{r,S}^\bullet \longrightarrow Y_{r,S}[-2]$$

in $\mathcal{D}(e_r \Lambda(\mathcal{G}))$. It now follows from (4.16) and Lemma 4.2 for both Z_r and $Y_{r,S}$ that we likewise have an exact triangle

$$(Z_r)_{\Gamma_L}[-3] \longrightarrow e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \longrightarrow (Y_{r,S})_{\Gamma_L}[-2]$$

in $\mathcal{D}(e_r \mathbb{Z}_p[G])$. Let H_r and $G_{r,S}$ in $\zeta(e_r \mathcal{Q}(\mathcal{G}))^\times$ be the reduced norms of characteristic elements of Z_r and $Y_{r,S}$, respectively. Note that both H_r and $G_{r,S}$ are actually reduced norms of matrices with coefficients in $e_r \Lambda(\mathcal{G})$. Since Conjecture 4.4 holds by assumption, we may assume that $\text{Nrd}(\zeta_{r,S}) = G_{r,S}/H_r$, where $\zeta_{r,S}$ is the characteristic element of $[C_{r,S}^\bullet]$ that occurs in Proposition 4.6. Now by (the proof of) [55, Theorem 6.4] one has

$$\partial_p(\overline{G_{r,S}}) = [(Y_{r,S})_{\Gamma_L}], \quad \partial_p(\overline{H_r}) = [(Z_r)_{\Gamma_L}],$$

where

$$\overline{G_{r,S}} := \sum_{\rho \in \text{Irr}_{\mathbb{C}_p}(G^+)} \text{aug}_\Gamma(j_{\rho \otimes \omega^{r-1}}(G_{r,S})) e_{\rho \otimes \omega^{r-1}} \in \zeta(e_r \mathbb{Q}_p[G])^\times$$

and $\overline{H_r}$ is defined similarly. Hence we obtain

$$\begin{aligned} & [e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))] \\ &= [(Y_{r,S})_{\Gamma_L}] - [(Z_r)_{\Gamma_L}] \\ &= \partial_p(\overline{G_{r,S}}/\overline{H_r}) \\ &= \partial_p \left(\sum_{\rho \in \text{Irr}_{\mathbb{C}_p}(G^+)} \text{aug}_\Gamma(j_{\rho \otimes \omega^{r-1}}(\text{Nrd}(\zeta_{r,S}))) e_{\rho \otimes \omega^{r-1}} \right) \\ &\stackrel{(*)}{=} \partial_p(L_{p,S}(r)). \end{aligned}$$

It remains to justify the last equality (*). For this we compute

$$\begin{aligned} \text{aug}_\Gamma(j_{\rho \otimes \omega^{r-1}}(\text{Nrd}(\zeta_{r,S}))) &= \text{aug}_\Gamma(\zeta_{r,S}(\rho \otimes \omega^{r-1})) \\ &= f_{\rho,S}(u^{1-r} - 1) = L_{p,S}(r, \rho). \end{aligned}$$

Here the first and second equality follow from [59, Lemma 2.3] and (4.9), respectively. This finishes the proof of (*). \square

Let us write $K_0(e_r \mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$ for the torsion subgroup of $K_0(e_r \mathbb{Z}_p[G], \mathbb{Q}_p)$.

Corollary 4.18 *Let $r > 1$ be an integer and assume that $\text{Sch}(L, p, r)$ holds. Then we have that $HR_p(L/K, r) \in K_0(e_r \mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$.*

Proof Let $K' \subseteq L^+$ be a totally real field containing K . Denote the Galois group $\text{Gal}(L/K')$ by G' . Then $HR_p(L/K, r)$ maps to $HR_p(L/K', r)$ under the natural restriction map $K_0(e_r \mathbb{Z}_p[G], \mathbb{Q}_p) \rightarrow K_0(e_r \mathbb{Z}_p[G'], \mathbb{Q}_p)$. If $L' \subseteq L$ is a Galois CM-extension of K , then likewise $HR_p(L/K, r)$ maps to $HR_p(L'/K, r)$ under the natural quotient map $K_0(e_r \mathbb{Z}_p[G], \mathbb{Q}_p) \rightarrow K_0(e_r \mathbb{Z}_p[\text{Gal}(L'/K)], \mathbb{Q}_p)$. Since $HR(L'/K', r)$ vanishes for each intermediate Galois CM-extension of degree prime to p by Theorem 4.17, we deduce the result by a slight modification of the argument given in [64, Proof of Proposition 11] (also see [58, Proof of Corollary 2]). \square

4.7 An application to the equivariant Tamagawa number conjecture

Let r be an integer and let L/K be a finite Galois extension of number fields with Galois group G . We set $(\mathbb{Q}(r))_L := h^0(\text{Spec}(L))(r)$ which we regard as a motive defined over K and with coefficients in the semisimple algebra $\mathbb{Q}[G]$. The ETNC [17, Conjecture 4 (iv)] for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ asserts that a certain canonical element $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ in $K_0(\mathbb{Z}[G], \mathbb{R})$ vanishes. Note that in this case the element $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ is indeed well-defined as observed in [18, Sect. 1]. If $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ is rational, i.e. belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$, then by means of the canonical isomorphism

$$K_0(\mathbb{Z}[G], \mathbb{Q}) \simeq \bigoplus_p K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$$

we obtain elements $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])_p$ in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$.

If $r > 1$ is an integer and L/K is a Galois CM-extension, the following result provides a strategy for proving the ETNC for the pair $(\mathbb{Q}(r)_L, e_r \mathbb{Z}[\frac{1}{2}][G])$. We let $T\Omega(\mathbb{Q}(r)_L, e_r \mathbb{Z}[\frac{1}{2}][G])$ be the image of $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ under the canonical maps

$$K_0(\mathbb{Z}[G], \mathbb{R}) \longrightarrow K_0(\mathbb{Z}[\frac{1}{2}][G], \mathbb{R}) \longrightarrow K_0(e_r \mathbb{Z}[\frac{1}{2}][G], \mathbb{R})$$

induced by extension of scalars. We define $T\Omega(\mathbb{Q}(r)_L, e_r \mathbb{Z}[\frac{1}{2}][G])_p$ similarly.

Theorem 4.19 *Let $r > 1$ be an integer and let p be an odd prime. Let L/K be a Galois CM-extension with Galois group G and set $\tilde{L} := L(\zeta_p)$. Assume that both Schneider's conjecture $\text{Sch}(\tilde{L}, p, r)$ and the p -adic Beilinson conjecture (Conjecture 3.23) for \tilde{L}^+/K hold. Then $T\Omega(\mathbb{Q}(r)_L, e_r \mathbb{Z}[\frac{1}{2}][G])$ is rational and we have that*

$$T\Omega(\mathbb{Q}(r)_L, e_r \mathbb{Z}[\frac{1}{2}][G])_p \in K_0(e_r \mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}.$$

If we assume in addition that the equivariant Iwasawa main conjecture (Conjecture 4.4) holds for \tilde{L}_∞^+/K , then the p -part of the ETNC for the pair $(\mathbb{Q}(r)_L, e_r\mathbb{Z}[\frac{1}{2}][G])$ holds, i.e. the element $T\Omega(\mathbb{Q}(r)_L, e_r\mathbb{Z}[\frac{1}{2}][G])_p$ vanishes.

Proof Since $T\Omega(\mathbb{Q}(r)_{\tilde{L}}, e_r\mathbb{Z}[\frac{1}{2}][\text{Gal}(\tilde{L}/K)])$ maps to $T\Omega(\mathbb{Q}(r)_L, e_r\mathbb{Z}[\frac{1}{2}][G])$ under the canonical quotient map $K_0(e_r\mathbb{Z}[\frac{1}{2}][\text{Gal}(\tilde{L}/K)], \mathbb{R}) \rightarrow K_0(e_r\mathbb{Z}[\frac{1}{2}][G], \mathbb{R})$ by [17, Theorem 4.1], we may and do assume that $\tilde{L} = L$. Since both Schneider's conjecture and the p -adic Beilinson conjecture hold, Corollary 3.29 implies that Gross's conjecture at $s = 1 - r$ holds for all even (odd) irreducible characters of G if r is even (odd). By [62, Proposition 5.5 and Theorem 6.5] (or rather the ' e_r -parts' of these results) this is indeed equivalent to the rationality of $T\Omega(\mathbb{Q}(r)_L, e_r\mathbb{Z}[\frac{1}{2}][G])$.

Let us define

$$\Omega_j(r) := \sum_{\rho \in \text{Irr}_{\mathbb{C}_p}(G^+)} \Omega_j(r, \rho \otimes \omega^{r-1}) e_{\rho \otimes \omega^{r-1}} \in \zeta(e_r\mathbb{C}_p[G])^\times.$$

By the validity of the p -adic Beilinson conjecture we have that

$$L_{p,S}(r) = j(e_r L_S^*(r)) \Omega_j(r). \quad (4.17)$$

We clearly have that $\Omega_j(r) = \text{Nrd}([e_r(L \otimes_{\mathbb{Q}} \mathbb{C}_p) \mid \mu_p(r) \circ (\mathbb{C}_p \otimes_{\mathbb{C},j} \mu_\infty(r))^{-1}])$. Moreover, by Proposition 3.16 the automorphism

$$t(r, S, j) := \iota_r \circ (\mathbb{C}_p \otimes_{\mathbb{C},j} \mu_\infty(r)) \circ \mu_p(r)^{-1} \circ \iota_r^{-1} \in \text{Aut}_{e_r\mathbb{C}_p[G]}(H_{1-r}^+ \otimes_{\mathbb{Z}} \mathbb{C}_p)$$

coincides with the ' e_r -part' of the trivialization of the same name in [62, Sect. 6.2] (up to an insignificant factor 2; cf. Remark 3.1). Since we have that

$$\text{Nrd}([H_{1-r}^+ \otimes_{\mathbb{Z}} \mathbb{C}_p \mid t(r, S, j)]) = \Omega_j(r)^{-1},$$

the object

$$\Psi_{r,S}^j := [e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))] - \partial_p(\Omega_j(r)) \in K_0(e_r\mathbb{Z}_p[G], \mathbb{C}_p) \quad (4.18)$$

is equal to the e_r -part of the object denoted by $\Omega_{r,S}^j$ in [62, Sect. 6.2]. We now compute

$$\begin{aligned} HR_p(L/K, r) &= \partial_p(L_{p,S}(r)) - [e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))] \\ &= \partial_p(j(e_r L_S^*(r))) + \partial_p(\Omega_j(r)) - [e_r R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))] \\ &= \partial_p(j(e_r L_S^*(r))) - \Psi_{r,S}^j \\ &= T\Omega(\mathbb{Q}(r)_L, e_r\mathbb{Z}[\frac{1}{2}][G])_p. \end{aligned}$$

Here, the first equality holds by definition of $HR_p(L/K, r)$, the second and third equality follow from (4.17) (essentially the p -adic Beilinson conjecture) and (4.18), respectively, and the last equality follows from [62, Proposition 6.4 and Theorem 6.5]. Now Corollary 4.18 and Theorem 4.17 give the result. \square

Remark 4.20 Fix an integer $r > 1$. We have assumed throughout that \tilde{L} contains a p th root of unity. What was actually needed in the above considerations, however, is that ω^{r-1} restricted to $G_{\tilde{L}}$ is trivial. Let $E \subseteq \tilde{L}_r \subseteq \tilde{L}$ be the smallest intermediate field such that ω^{r-1} restricted to $G_{\tilde{L}_r}$ is trivial. Then we can replace \tilde{L} by \tilde{L}_r throughout. Note that \tilde{L}_r is totally real and thus $\tilde{L}_r^+ = \tilde{L}_r$ whenever r is odd, whereas we have $\tilde{L}_r = \tilde{L}$ otherwise. In particular, we can replace \tilde{L} by E whenever $r \equiv 1 \pmod{p-1}$.

Corollary 4.21 *Let p be an odd prime and let $r > 1$ be an integer such that $r \equiv 1 \pmod{p-1}$. Let E/K be a Galois extension of totally real fields with Galois group G . Assume that Schneider's conjecture $\text{Sch}(E, p, r)$, the p -adic Beilinson conjecture (Conjecture 3.23) for E/K and the equivariant Iwasawa main conjecture (Conjecture 4.4) for E_{∞}/K all hold. Then $T\Omega(\mathbb{Q}(r)_E, \mathbb{Z}[\frac{1}{2}][G])$ is rational and the p -part of the ETNC for the pair $(\mathbb{Q}(r)_E, \mathbb{Z}[\frac{1}{2}][G])$ holds.*

Proof This follows from Theorem 4.19 and Remark 4.20. \square

Corollary 4.22 *Let L/K be a Galois extension of number fields with Galois group G and let p be an odd prime. Assume in addition that L/\mathbb{Q} is abelian. Then $T\Omega(\mathbb{Q}(r)_L, e_r \mathbb{Z}[\frac{1}{2}][G])$ is rational and the p -part of the ETNC for the pair $(\mathbb{Q}(r)_L, e_r \mathbb{Z}[\frac{1}{2}][G])$ holds for all but finitely many $r > 1$.*

Proof We may assume that $K = \mathbb{Q}$ by functoriality. As $L(\zeta_p)$ is abelian over the rationals, the relevant Iwasawa invariant $\mu_p(L(\zeta_p)^+)$ vanishes by the aforementioned result of Ferrero and Washington [34] (see the discussion following (4.7)). Hence Conjecture 4.4 holds for $L(\zeta_p)_{\infty}^+/\mathbb{Q}$ by either part of Theorem 4.5 (but note that in this case a variant of the equivariant Iwasawa main conjecture can be deduced from work of Mazur and Wiles [53] as in [65, Theorem 8] for example). The p -adic Beilinson conjecture holds for $L(\zeta_p)^+/\mathbb{Q}$ by Theorem 3.30. Finally, Schneider's conjecture $\text{Sch}(L(\zeta_p), p, r)$ holds for all but finitely many r by Remark 3.9. Thus the result follows from Theorem 4.19. \square

Remark 4.23 Of course, the result of Corollary 4.22 is not new. In fact, the ETNC for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ holds for every integer r whenever L/\mathbb{Q} is abelian. If $r \leq 0$ this is the main result of Burns and Greither in [20] (important difficulties with the prime 2 have subsequently been resolved by Flach [35]). The case $r > 0$ is due to Burns and Flach [19]. A slightly weaker variant of the ETNC, where the integral group ring $\mathbb{Z}[G]$ is essentially replaced by a maximal order containing it, has been studied earlier by Huber and Kings [44].

Example 4.24 Let E/\mathbb{Q} be a Galois extension of totally real fields with Galois group $G \simeq \text{Aff}(q)$, where $q = \ell^n$ is a prime power. Let $r > 1$ be an integer. Since Gross's conjecture holds for all $\chi \in R(G)$ by Theorem 3.14 (iv), we have that $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[\frac{1}{2}][G])$ is rational by [62, Theorem 6.5 (i)]. Let us write $\text{Aff}(q) \simeq N \rtimes H$, where N denotes the commutator subgroup of $\text{Aff}(q)$. Then the p -adic group ring $\mathbb{Z}_p[\text{Aff}(q)]$ is ' N -hybrid' in the sense of [46, Definition 2.5] by [46, Example 2.16] for every prime $p \neq \ell$. Since every p -adic group ring is $\{1\}$ -hybrid, we deduce from [48, Theorem 10.2] (or just as well from Theorem 4.5 (ii)) that the

equivariant Iwasawa main conjecture holds unconditionally for $E(\zeta_p)_{\infty}^+/\mathbb{Q}$ for every odd prime p . Thus Conjecture 4.14 holds by Theorem 4.17 whenever $\text{Sch}(E(\zeta_p), p, r)$ holds. In particular, this conjecture holds for almost all $r > 1$ for a fixed prime p .

We now assume for simplicity that $r \equiv 1 \pmod{p-1}$. The p -adic Beilinson conjecture holds for all linear characters of G by Theorem 3.30. As we have already observed in the proof of Theorem 3.14 (iv) there is only one non-linear character χ_{nl} of $\text{Aff}(q)$ which is a \mathbb{Z} -linear combination of linear characters and of $\text{ind}_H^G \mathbb{1}_H$. Hence it suffices to show Conjecture 3.23 for the trivial character $\mathbb{1}_H$, i.e. for the trivial extension E^H/E^H . Assuming this we can apply Corollary 4.21 to deduce that the p -part of the ETNC for the pair $(\mathbb{Q}(r)_E, e_r \mathbb{Z}[\frac{1}{2}][G])$ holds. So what is missing here (apart from Schneider's conjecture) is a higher analogue of Colmez's p -adic analytic class number formula [30] and its complex analytic counterpart. (A closer analysis of the proof of Theorem 4.19 shows that similar observations indeed hold for arbitrary $r > 1$.)

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