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Real Hypersurfaces of Complex Quadric in Terms of Star-Ricci Tensor

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Abstract. In this article, we introduce the notion of star-Ricci tensors in the real hypersurfaces of complex quadric Q^m. It is proved that there exist no Hopf hypersurfaces in Q^m, $m \geq 3$, with commuting star-Ricci tensor or parallel star-Ricci tensor. As a generalization of star-Einstein metric, star-Ricci solitons on M are considered. In this case we show that M is an open part of a tube around a totally geodesic $\mathbb{C}P^{\frac{m}{2}} \subset Q^m$, $m \geq 4$.

1. Introduction

The complex quadric Q^m is a Hermitian symmetric space $\mathrm{SO}_{m+2}/\mathrm{SO}_m\mathrm{SO}_2$ with rank two in the class of compact type. It can be regarded as a complex hypersurface of complex projective space $\mathbb{C}P^{m+1}$. Also, the complex quadric Q^m can be regarded as a kind of real Grassmannian manifolds of compact type with rank two. In the complex quadric Q^m there are two important geometric structures, a complex conjugation structure A and Kähler structure J, with each other being anti-commuting, that is, $AJ = -JA$. Another distinguished geometric structure in Q^m is a parallel rank two vector bundle $\mathfrak{U}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. Here the parallel vector bundle $\mathfrak{U}$ means that $(\widetilde{\nabla}_X A)Y = q(X)AY$ for all $X, Y \in T_zQ^m$, $z \in Q^m$, where $\widetilde{\nabla}$ and q denote a connection and a certain 1-form on T_zQ^m, respectively.

Recall that a nonzero tangent vector $W \in T_zQ^m$, $z \in Q^m$, is called *singular* if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{U}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{U}$-principal.
2. If there exist a conjugation $A \in \mathfrak{U}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{U}$-isotropic.

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Let M be a real hypersurface of Q^m. The Kähler structure J on Q^m induces a structure vector field ξ called *Reeb vector field* on M by $\xi := -JN$, where N is a local unit normal vector field of M in Q^m. It is well-known that there is an almost contact structure (ϕ, η, ξ, g) on M induced from complex quadric. Moreover, if the Reeb vector field ξ is invariant under the shape operator S, i.e. $S\xi = \alpha\xi$, where $\alpha = g(S\xi, \xi)$ is a smooth function, then M is said to be a *Hopf hypersurface*. For the real Hopf hypersurfaces of complex quadric many characterizations were obtained by Suh (see [9, 10, 11, 12, 13] etc.). In particular, we note that Suh in [9] introduced parallel Ricci tensor, i.e. $\nabla \mathrm{Ric} = 0$, for the real hypersurfaces in Q^m and gave a complete classification for this case. In addition, if the real hypersurface M admits commuting Ricci tensor, i.e. $\mathrm{Ric} \circ \phi = \phi \circ \mathrm{Ric}$, Suh also proved the followings:

THEOREM 1 ([13]). *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$, with commuting Ricci tensor. Then the unit normal vector field N of M is either $\mathfrak{U}$-principal or $\mathfrak{U}$-isotropic.*

THEOREM 2 ([13]). *There exist no Hopf real hypersurfaces in the complex quadric Q^m, $m \geq 4$, with commuting and parallel Ricci tensor.*

Since the Ricci tensor of an Einstein hypersurface in the complex quadric Q^m is a constant multiple of g, it satisfies naturally commuting and parallelism. Thus we have the following.

COROLLARY 1 ([13]). *There exist no Hopf Einstein real hypersurfaces in the complex quadric Q^m, $m \geq 4$.*

As a generalization of Einstein metrics, recently Suh in [14] has shown a complete classification of Hopf hypersurfaces with a Ricci soliton, which is given by

$$\frac{1}{2}(\mathfrak{L}_W g)(X, Y) + \mathrm{Ric}(X, Y) = \lambda g(X, Y) .$$

Here λ is a constant and W is a vector field on M, which are said to be *Ricci soliton constant* and *potential vector field*, respectively, and $\mathfrak{L}_W$ denotes the Lie derivative along the direction of the vector field W.

Notice that, as the corresponding of Ricci tensor, Tachibana [15] introduced the idea of star-Ricci tensor. These ideas apply to almost contact metric manifolds, and in particular, to real hypersurfaces in complex space forms by Hamada in [3]. The star-Ricci tensor Ric^* is defined by

$$\mathrm{Ric}^*(X, Y) = \frac{1}{2}\mathrm{trace}\{\phi \circ R(X, \phi Y)\} , \quad \text{for all } X,\ Y \in TM . \tag{1}$$

If the star-Ricci tensor is a constant multiple of $g(X, Y)$ for all X, Y orthogonal to ξ, then M is said to be a *star-Einstein manifold*. Hamada gave a classification of star-Einstein hypersurfaces of $\mathbb{C}P^n$ and $\mathbb{C}H^n$, and further Ivey and Ryan updated and refined the work of Hamada in 2011 ([4]).

Motivated by the present work, in this paper we introduce the notion of star-Ricci tensor in the real hypersurfaces of complex quadric Q^m and study the characterizations of a real Hopf hypersurface whose star-Ricci tensor satifies certain conditions.

First we consider the real hypersurface with commuting star-Ricci tensor, i.e. $\phi \circ \mathrm{Ric}^* = \mathrm{Ric}^* \circ \phi$. We assert the following:

THEOREM 3. *There exist no Hopf hypersurfaces of Q^m, $m \geq 3$, with commuting star-Ricci tensor.*

For the Hopf hypersurfaces of Q^m, $m \geq 3$, with parallel star-Ricci tensor, we also prove the following non-existence.

THEOREM 4. *There exist no Hopf hypersurfaces of Q^m, $m \geq 3$, with parallel star-Ricci tensor.*

As the generalization of star-Einstein metric Kaimakamis and Panagiotidou [5] introduced a so-called *star-Ricci soliton*, that is, a Riemannian metric g on M satisfying

$$\frac{1}{2}\mathcal{L}_W g + \mathrm{Ric}^* = \lambda g \,. \tag{2}$$

In this case we obtain the following characterization:

THEOREM 5. *Let M be a real hypersurface in Q^m, $m \geq 4$, admitting a star-Ricci soliton with potential vector field ξ, then M is an open part of a tube around a totally geodesic $\mathbb{C}P^{\frac{m}{2}} \subset Q^m$.*

This paper is organized as follows. In Sections 2 and 3, some basic concepts and formulas for real hypersurfaces in complex quadric are presented. In Section 4 we consider Hopf hypersurfaces with commuting star-Ricci tensor and give the proof of Theorem 3. In Section 5 we will prove Theorem 4. At last we assume that a Hopf hypersurface admits star-Ricci soliton and give the proof of Theorem 5 as Section 6.

2. The complex quadric

In this section we will summarize some basic notations and formulas about the complex quadric Q^m. For more detail see [1, 2, 7, 6]. The complex quadric Q^m is the hypersurface of complex projective space $\mathbb{C}P^{m+1}$, which is defined by $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. In the complex quadric it is equipped with a Riemannian metric $\widetilde{g}$ induced from the Fubini-Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. Also the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure $(J, \widetilde{g})$ on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, i.e. $\mathbb{C}P^{m+1} = \mathrm{SU}_{m+2}/S(U_1U_{m+1})$. The special orthogonal group $\mathrm{SO}_{m+2} \subset \mathrm{SU}_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$, where $o = [0, \ldots, 1] \in \mathbb{C}P^{m+1}$ is the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. We can identify Q^m with a homogeneous space $\mathrm{SO}(m+2)/\mathrm{SO}_2\mathrm{SO}_m$, which is the second singular orbit of this action. Such a homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. From now on we always assume $m \geq 3$ because it is well known that Q^1 is isometric to a sphere S^2 with constant curvature and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature.

For a unit normal vector ρ of Q^m at a point $z \in Q^m$ we denote by $A = A_\rho$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to ρ, which is an involution on the tangent space $T_z Q^m$, and the tangent space can be decomposed as

$$T_z Q^m = V(A_\rho) \oplus JV(A_\rho)\,,$$

where $V(A_\rho)$ is the $(+1)$-eigenspace and $JV(A_\rho)$ is the (-1)-eigenspace of A_ρ. This means that the shape operator A defines a real structure on $T_z Q^m$, equivalently, A is a complex conjugation. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{U}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations. Notice that J and each complex conjugation $A \in \mathfrak{U}$ anti-commute, i.e. $AJ = -JA$.

3. Real hypersurface of complex quadric and its star-Ricci tensor

Let M be an immersed real hypersurface of Q^m with induced metric g. There exists a local defined unit normal vector field N on M and we write $\xi := -JN$ by the structure vector field of M. An induced one-form η is defined by $\eta(\cdot) = \widetilde{g}(J\cdot, N)$, which is dual to ξ. For any vector field X on M the tangent part of JX is denoted by $\phi X = JX - \eta(X)N$. Moreover, the following identities hold:

$$\phi^2 = -Id + \eta \otimes \xi\,, \quad \eta \circ \phi = 0\,, \quad \phi \circ \xi = 0\,, \quad \eta(\xi) = 1\,, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \quad g(X, \xi) = \eta(X)\,, \tag{4}$$

where $X, Y \in \mathfrak{X}(M)$. By these formulas, we know that (ϕ, η, ξ, g) is an almost contact metric structure on M. The tangent bundle TM can be decomposed as $TM = \mathcal{C} \otimes \mathbb{R}\xi$, where $\mathcal{C} = \ker \eta$ is the maximal complex subbundle of TM. Denote by ∇, S the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(SX, Y)N\,, \quad \widetilde{\nabla}_X N = -SX\,, \tag{5}$$

where $\widetilde{\nabla}$ is the connection on Q^m with respect to $\widetilde{g}$. Also, we have

$$(\nabla_X \phi)Y = \eta(Y)SX - g(SX, Y)\xi\,, \quad \nabla_X \xi = \phi SX\,. \tag{6}$$

The curvature tensor R and Codazzi equation of M are given respectively as follows (see [9]):

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY \\ &\quad + g(SY,Z)SX - g(SX,Z)SY\,, \end{aligned} \tag{7}$$

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= \eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z) - 2\eta(Z)g(\phi X,Y) \\ &\quad + g(X,AN)g(AY,Z) - g(Y,AN)g(AX,Z) \\ &\quad + g(X,A\xi)g(JAY,Z) - g(Y,A\xi)g(JAX,Z) \end{aligned} \tag{8}$$

for any vector fields X, Y, Z on M.

At each point $z \in M$ we denote

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{U}_z\}$$

by a maximal $\mathfrak{U}$-invariant subspace of T_zM. For the subspace the following lemma was proved.

LEMMA 1 (see [10]). *For each $z \in M$ we have*

- *If N_z is $\mathfrak{U}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
- *If N_z is not $\mathfrak{U}$-principal, there exist a conjugation $A \in \mathfrak{U}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \frac{\pi}{4}]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX+Y)$.*

For each point $z \in M$ we choose $A \in \mathfrak{U}_z$, then there exist two orthonormal vectors $Z_1, Z_2 \in V(A)$ such that

$$\begin{cases} N &= \cos(t)Z_1 + \sin(t)JZ_2\,, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2\,, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1\,, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1 \end{cases} \tag{9}$$

for $0 \le t \le \frac{\pi}{4}$ (see [8, Proposition 3]). From this we get $g(AN,\xi) = 0$.

In the real hypersurface M we introduce the star-Ricci tensor Ric^* defined by

$$\mathrm{Ric}^*(X,Y) = \frac{1}{2}\mathrm{trace}\{\phi \circ R(X,\phi Y)\}\,, \quad \text{for all } X,\ Y \in TM\,.$$

Taking a local frame $\{e_i\}$ of M such that $e_1 = \xi$ and using (4), we derive from (7)

$$\begin{aligned} &\sum_{i=1}^{2m-1} g(\phi \circ R(X,\phi Y)e_i, e_i) \\ &= g(\phi Y,\phi X) - g(X,\phi^2 Y) + g(\phi^2 Y,\phi^2 X) - g(\phi X,\phi^3 Y) + 2(2m-2)g(\phi X,\phi Y) \end{aligned}$$

$$
\begin{aligned}
&+ g(A\phi Y, \phi AX) - g(AX, \phi A\phi Y) + g(JA\phi Y, \phi JAX) - g(JAX, \phi JA\phi Y)\\
&+ g(S\phi Y, \phi SX) - g(SX, \phi S\phi Y)\\
= {}& 4mg(\phi X, \phi Y) - 2g(AX, \phi A\phi Y) + 2g(JA\phi Y, \phi JAX) - 2g(SX, \phi S\phi Y)\,.
\end{aligned}
$$

In view of (1), the star-Ricci tensor is given by

$$
\begin{aligned}
\mathrm{Ric}^*(X, Y) = {}& 2mg(\phi X, \phi Y) - g(AX, \phi A\phi Y)\\
&+ g(JA\phi Y, \phi JAX) - g(SX, \phi S\phi Y)\,.
\end{aligned}
\tag{10}
$$

Since $AJ = -JA$ and $\xi = -JN$, we have

$$
\begin{aligned}
JA\phi Y &= -AJ\phi Y = AY - \eta(Y)A\xi\,,\\
\phi JAX &= J(JAX) - \eta(JAX)N = -AX + g(N, AX)N\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
g(JA\phi Y, \phi JAX) &= -g(AX, AY) + \eta(Y)\eta(X) + g(N, AX)g(AY, N)\\
&= g(\phi^2 X, Y) + g(N, AX)g(AY, N)\,.
\end{aligned}
\tag{11}
$$

Because

$$
\begin{aligned}
JA\phi Y &= \phi A\phi Y + \eta(A\phi Y)N\\
&= \phi A\phi Y + g(\xi, AJY - \eta(Y)AN)N\\
&= \phi A\phi Y + g(J\xi, AY)N\\
&= \phi A\phi Y + g(N, AY)N\,,
\end{aligned}
$$

we have

$$
\begin{aligned}
g(AX, \phi A\phi Y) &= g(AX, JA\phi Y - g(N, AY)N)\\
&= g(AX, JA\phi Y) - g(N, AY)g(AX, N)\\
&= -g(\phi^2 X, Y) - g(N, AY)g(AX, N)\,.
\end{aligned}
\tag{12}
$$

Thus substituting (11) and (12) into (10) implies

$$
\mathrm{Ric}^*(X, Y) = -\,2(m-1)g(\phi^2 X, Y) - 2g(N, AX)g(AY, N) - g((\phi S)^2 X, Y) \tag{13}
$$

for all $X, Y \in TM$.

In the following we always assume that M is a Hopf hypersurface in Q^m, i.e. $S\xi = \alpha\xi$ for a smooth function $\alpha = g(S\xi, \xi)$. As in [9], since $g(AN, \xi) = 0$, by taking $Z = \xi$ in the Codazzi equation (8), we have

$$
\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&= -\,2g(\phi X, Y) + 2g(X, AN)g(AY, \xi) - 2g(Y, AN)g(AX, \xi)\,.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\
&= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\
&= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y)\,.
\end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ gives

$$Y\alpha = (\xi\alpha)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\,. \tag{14}$$

Moreover, we have the following.

LEMMA 2 ([10, Lemma 4.2]). *Let M be a Hopf hypersurface in Q^m with (local) unit normal vector field N. For each point in $z \in M$ we choose $A \in \mathfrak{U}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$. Then*

$$\begin{aligned}
0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\
& + 2g(X, AN)g(Y, A\xi) - 2g(Y, AN)g(X, A\xi) \\
& + 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\}
\end{aligned} \tag{15}$$

holds for all vector fields X, Y on M.

From this lemma we can prove the following.

LEMMA 3. *Let M be a Hopf hypersurface in complex quadric Q^m, then*

$$(\phi S)^2 = (S\phi)^2\,. \tag{16}$$

PROOF. From the equation (15) we assert the followings:

$$\begin{aligned}
g((S\phi)^2X, Y) = {} & \frac{1}{2}\alpha g((\phi S + S\phi)\phi X, Y) + g(\phi^2 X, Y) - g(\phi X, AN)g(Y, A\xi) \\
& + g(\phi X, A\xi)g(Y, AN) + g(\xi, A\xi)g(\phi X, AN)\eta(Y)\,, \\
g((\phi S)^2X, Y) = {} & \frac{1}{2}\alpha g(\phi(\phi S + S\phi)X, Y) + g(\phi^2 X, Y) - g(X, AN)g(\phi A\xi, Y) \\
& + g(X, A\xi)g(\phi AN, Y) - g(\xi, A\xi)\eta(X)g(\phi AN, Y)\,.
\end{aligned} \tag{17}$$

Thus we obtain

$$\begin{aligned}
g((S\phi)^2X - (\phi S)^2X, Y) = {} & -g(\phi X, AN)g(Y, A\xi) + g(\phi X, A\xi)g(Y, AN) \\
& + g(\xi, A\xi)g(\phi X, AN)\eta(Y) + g(X, AN)g(\phi A\xi, Y) \\
& - g(X, A\xi)g(\phi AN, Y) + g(\xi, A\xi)\eta(X)g(\phi AN, Y) \\
= {} & \eta(X)g(AN, N)g(Y, A\xi) - g(\xi, A\xi)g(X, A\xi)\eta(Y) \\
& - g(X, A\xi)\eta(Y)g(AN, N) + g(\xi, A\xi)\eta(X)g(Y, A\xi)
\end{aligned}$$

$$= \Big(\eta(X)g(A\xi, Y) - g(X, A\xi)\eta(Y)\Big)\Big(g(AN, N) + g(\xi, A\xi)\Big).$$

Here we have used the following relations:

$$g(A\xi, \phi X) = g(A\xi, JX - \eta(X)N) = g(AN, X), \tag{18}$$

$$g(A\phi X, N) = g(AJX - \eta(X)AN, N) = -g(X, A\xi) - \eta(X)g(AN, N). \tag{19}$$

From (9), we get $g(AN, N) + g(\xi, A\xi) = 0$, which yields (16). □

4. Proof of Theorem 3

In this section we suppose that M is a real Hopf hypersurface with commuting star-Ricci tensor, that is, $\phi \circ \mathrm{Ric}^* = \mathrm{Ric}^* \circ \phi$. Making use of (13), a straightforward computation gives

$$\begin{aligned} 0 &= g((\phi \circ \mathrm{Ric}^* - \mathrm{Ric}^* \circ \phi)X, Y) \\ &= -\mathrm{Ric}^*(X, \phi Y) - \mathrm{Ric}^*(\phi X, Y) \\ &= 2g(N, AX)g(A\phi Y, N) + 2g(N, A\phi X)g(AY, N) \\ &\quad + g(\phi[(S\phi)^2 - (\phi S)^2]X, Y). \end{aligned}$$

Thus Lemma 3 implies

$$g(N, AX)g(A\phi Y, N) + g(N, A\phi X)g(AY, N) = 0.$$

Replacing X and Y by ϕX and ϕY respectively gives

$$g(N, A\phi X)g(Y, AN) + g(X, AN)g(A\phi Y, N) = 0.$$

Now, if $X = Y$, we find $g(AN, \phi X)g(AN, X) = 0$ for all vector field X on M, which means $AN = N$. Therefore we prove the following.

LEMMA 4. *Let M be a Hopf hypersurface of complex quadric Q^m, $m \geq 3$, with commuting star-Ricci tensor. Then the unit normal vector field N is $\mathfrak{U}$-principal.*

In terms of (17), the star-Ricci tensor (13) becomes

$$\mathrm{Ric}^*(X, Y) = (-2m + 1)g(\phi^2 X, Y) - \frac{1}{2}\alpha g(\phi(\phi S + S\phi)X, Y).$$

Moreover, from (15) we obtain

$$\begin{aligned} \mathrm{Ric}^*(X) &= (-2m + 1)\phi^2 X - \frac{1}{2}\alpha\phi(\phi S + S\phi)X \\ &= (-2m + 1)\phi^2 X - \frac{1}{2}\alpha\phi^2 SX - \frac{1}{4}\alpha^2(\phi S + S\phi)X - \frac{1}{2}\alpha\phi X. \end{aligned}$$

By virtue of [9, Lemma 4.3] and Lemma 4, it implies that α is constant. If $\alpha \neq 0$, making use of the previous formula, we conclude that

$$0 = \phi\mathrm{Ric}^*(X) - \mathrm{Ric}^*(\phi X) = \frac{1}{2}\alpha(\phi SX - S\phi X)$$

for all $X \in TM$. That means that the Reeb flow is isometric. In view of [2, Proposition 6.1], the normal vector field N is isotropic everywhere, which is contradictory with Lemma 4. Hence $\alpha = 0$ and the star-Ricci tensor becomes

$$\mathrm{Ric}^*(X, Y) = (-2m + 1)g(\phi^2 X, Y)\,. \tag{20}$$

Now replacing X and Y by ϕX and ϕY respectively in (13) and using (20), we get

$$(2m-1)(\phi X, \phi Y) = 2(m-1)g(X, \phi Y) - 2g(N, A\phi X)g(A\phi Y, N) - g((S\phi)^2 X, Y)\,.$$

Interchanging X and Y and applying the resulting equation to subtract the previous equation, we obtain

$$g((S\phi)^2 X - (\phi S)^2 X, Y) = 4(m-1)g(X, \phi Y)\,.$$

So from Lemma 3, we conclude that

$$4(m-1)g(X, \phi Y) = 0\,,$$

which is impossible since $m \geq 3$. We finish the proof of Theorem 3. $\square$

REMARK 1. Formula (20) with $X, Y \in \mathcal{C}$, we have $\mathrm{Ric}^*(X, Y) = (2m-1)g(X, Y)$, namely M is star-Einstein, thus we have proved that there exist no star-Einstein Hopf hypersurfaces in complex quadric Q^m, $m \geq 3$, which is analogous to the conclusion of Corollary 1 in the introduction.

5. Proofs of Theorem 4

In this section we assume M is a Hopf hypersurface of Q^m, $m \geq 3$, with parallel star-Ricci tensor. In order to prove Theorem 4, we first prove the following lemma.

LEMMA 5. *Let M be a Hopf hypersurface of Q^m, $m \geq 3$, with parallel star-Ricci tensor. Then the unit normal vector N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

PROOF. Since $\nabla \mathrm{Ric}^* = 0$, differentiating equation (13) covariantly along vector field Z gives

$$\begin{aligned}
0 = {} & 2(m-1)g((\nabla_Z\phi)\phi X + \phi(\nabla_Z\phi)X, Y) \\
& + 2g(\widetilde{\nabla}_Z N, AX)g(AY, N) + 2g(N, (\widetilde{\nabla}_Z A)X)g(AY, N) \\
& + 2g(\widetilde{\nabla}_Z N, AY)g(AX, N) + 2g(N, (\widetilde{\nabla}_Z A)Y)g(AX, N) \\
& + g((\nabla_Z\phi)S\phi SX, Y) + g(\phi(\nabla_Z S)\phi SX, Y) \\
& + g(\phi S(\nabla_Z\phi)SX, Y) + g(\phi S\phi(\nabla_Z S)X, Y)\,.
\end{aligned}$$

Here we have used $(\widetilde{\nabla}_Z A)X = q(Z)AX$ for a certain 1-form q as in the introduction. Moreover, by (5) we have

$$0 = -2(m-1)g(SZ, \phi X)\eta(Y) + 2(m-1)\eta(X)g(\phi SZ, Y)$$

$$- 2g(SZ, AX)g(AY, N) + 4q(Z)g(N, AX)g(AY, N)$$
$$- 2g(SZ, AY)g(AX, N) - g(SZ, S\phi SX)\eta(Y) + g(\phi(\nabla_Z S)\phi SX, Y)$$
$$+ \eta(SX)g(\phi S^2 Z, Y) + g(\phi S\phi(\nabla_Z S)X, Y)\,. \tag{21}$$

Since $S\xi = \alpha\xi$, letting $X = \xi$ we get

$$\begin{aligned} 0 =& 2(m-1)g(\phi SZ, Y) - 2g(SZ, A\xi)g(AY, N) \\ &+ \alpha g(\phi S^2 Z, Y) + g((\nabla_Z S)\xi, \phi S\phi Y) \\ =& 2(m-1)g(\phi SZ, Y) - 2g(SZ, A\xi)g(AY, N) \\ &+ \alpha g(\phi S^2 Z, Y) + g(\alpha\phi SZ - S\phi SZ, \phi S\phi Y)\,. \end{aligned}$$

Moreover, if $Z = \xi$ then we get $\alpha g(A\xi, \xi)g(AY, N) = 0$. If $\alpha \neq 0$ then $\cos(2t)g(AY, N) = 0$ by (9). That means that $t = \frac{\pi}{4}$ or $AY \in TM$, that is, the unit normal vector N is $\mathfrak{U}$-principal or $\mathfrak{U}$-isotropic. If $\alpha = 0$ then $g(Y, AN)g(\xi, A\xi) = 0$ for any $Y \in TM$ by (14), thus we have same conclusion. The proof is complete. □

We first assume that the unit normal vector field N is $\mathfrak{U}$-isotropic. In this case these expressions in (9) become

$$\begin{cases} N &= \frac{1}{\sqrt{2}}(Z_1 + JZ_2)\,, \\ AN &= \frac{1}{\sqrt{2}}(Z_1 - JZ_2)\,, \\ \xi &= \frac{1}{\sqrt{2}}(Z_2 - JZ_1)\,, \\ A\xi &= \frac{1}{\sqrt{2}}(Z_2 + JZ_1)\,. \end{cases}$$

Thus

$$g(A\xi, \xi) = g(AN, N) = 0\,. \tag{22}$$

So (15) becomes

$$\begin{aligned} S\phi SX =& \frac{1}{2}\alpha(\phi S + S\phi)X + \phi X \\ &- g(X, AN)A\xi + g(X, A\xi)AN\,. \end{aligned} \tag{23}$$

The formula (21) with $Z = \xi$ implies

$$\begin{aligned} 0 =& -2g(S\xi, AX)g(AY, N) + 4q(\xi)g(N, AX)g(AY, N) \\ &-2g(S\xi, AY)g(AX, N) - g((\nabla_\xi S)\phi SX, \phi Y) \\ &+ g((\nabla_\xi S)X, \phi S\phi Y)\,. \end{aligned} \tag{24}$$

By Codazzi equation (8), we get

$$(\nabla_\xi S)Y = \alpha\phi SY - S\phi SY + \phi Y - g(Y, AN)A\xi$$

$$+ g(Y, A\xi)AN$$
$$= \frac{1}{2}\alpha(\phi S - S\phi)Y .$$

Thus substituting this into (24) gives

$$0 = -2\alpha g(\xi, AX)g(AY, N) + 4q(\xi)g(N, AX)g(AY, N)$$
$$- 2\alpha g(\xi, AY)g(AX, N) - \frac{1}{2}\alpha g(S\phi SX + \phi S\phi S\phi X, Y) . \tag{25}$$

Moreover, by (23) we have $S\phi SX + \phi S\phi S\phi X = 0$, thus taking $X = A\xi$ in (25) yields

$$\alpha g(AY, N) = 0 .$$

Here we have used $g(A\xi, A\xi) = 1$ and $g(AN, A\xi) = 0$. From this we derive $\alpha = 0$ since N is $\mathfrak{U}$-isotropic.

On the other hand, we put $Y = \xi$ in (21) and get

$$0 = 2(m-1)g(SZ, \phi X) + 2g(SZ, A\xi)g(AX, N) + g(SZ, S\phi SX) .$$

Applying (23) in the above formula, we have

$$0 = (2m-1)g(SZ, \phi X) + g(SZ, A\xi)g(AX, N) + g(SZ, AN)g(X, A\xi) .$$

That is,

$$0 = (2m-1)S\phi X + g(AX, N)SA\xi + g(X, A\xi)SAN . \tag{26}$$

When $X = AN$, it comes to

$$0 = (2m-1)S\phi AN + SA\xi .$$

Then $A\xi = \phi AN$ implies $SA\xi = 0$. Similarly, $SAN = 0$. Therefore from (26) we obtain $S\phi X = 0$ for all $X \in TM$. As $S\xi = 0$ we know $SX = 0$ for all $X \in TM$, thus $\nabla_\xi S = 0$, that means that the hypersurface M admits parallel shape operator. But Suh [10] has showed the non-existence of this type hypersurfaces.

In the following if N is $\mathfrak{U}$-principal, that is, $AN = N$, then (13) becomes

$$\mathrm{Ric}^*(X, Y) = -2(m-1)g(\phi^2 X, Y) - g((\phi S)^2 X, Y) .$$

In this case we see that the star-Ricci tensor is commuting by Lemma 3. Thus we see $\alpha = 0$ from the proof of Theorem 3. In this case, the formulas (21) with $Y = \xi$ and (15) respectively become $2(m-1)g(SZ, \phi X) + g(SZ, S\phi SX) = 0$ and $S\phi SX = \phi X$, respectively. From these two equations we obtain $g(SZ, \phi X) = 0$, that is, $\phi SZ = 0$. This implies $SZ = \alpha\eta(Z)\xi = 0$. As before, this is impossible.

Summing up the above discussion, we complete the proof of Theorem 4.

6. Proof of Theorem 5

In order to prove our theorem, we first give the following property.

PROPOSITION 1. *Let M be a real hypersurface in Q^m, $m \geq 3$, admitting a star-Ricci soliton with potential vector field ξ, then M must be Hopf.*

PROOF. Since $\mathcal{L}_W g$ and g are symmetry, the *-Ricci soliton equation (2) implies the star-Ricci tensor is also symmetry, i.e. $\mathrm{Ric}^*(X, Y) = \mathrm{Ric}^*(Y, X)$ for any vector fields X, Y on M. It yields from (13) that

$$(\phi S)^2 X = (S\phi)^2 X \tag{27}$$

for all $X \in TM$.

On the other hand, from the star-Ricci soliton equation (2) it follows

$$\mathrm{Ric}^*(X, Y) = \lambda g(X, Y) + \frac{1}{2} g((S\phi - \phi S)X, Y)\,. \tag{28}$$

By (13), we have

$$\begin{aligned} &-2(m-1)g(\phi^2 X, Y) - 2g(N, AX)g(AY, N) - g((\phi S)^2 X, Y) \\ &= \lambda g(X, Y) + \frac{1}{2} g((S\phi - \phi S)X, Y)\,. \end{aligned} \tag{29}$$

Putting $X = Y = \xi$ gives $\lambda = 0$ since $g(AN, \xi) = 0$. Therefore the previous formula with $X = \xi$ yields

$$(\phi S)^2 \xi = \frac{1}{2} \phi S \xi\,.$$

Using (27) we get $\phi S \xi = 0$, which shows $S\xi = \alpha \xi$ with $\alpha = g(S\xi, \xi)$. □

Moreover, by (28) we have

$$\mathrm{Ric}^*(X) = \frac{1}{2}(S\phi - \phi S)X\,. \tag{30}$$

Thus by a straightforward computation we find $\phi \circ \mathrm{Ric}^* + \mathrm{Ric}^* \circ \phi = 0$ since the relation $\phi^2 S = S\phi^2$ holds by Proposition 1. Namely the following result holds.

PROPOSITION 2. *Let M be a real hypersurface in Q^m, $m \geq 3$, admitting a star-Ricci soliton with potential vector field ξ, then the star-Ricci tensor is anti-commuting.*

Next we will compute the convariant derivative of $\phi \circ \mathrm{Ric}^* + \mathrm{Ric}^* \circ \phi = 0$. First of all, by (30) and (6), we compute

$$\begin{aligned} (\nabla_X \mathrm{Ric}^*)(Y) &= \frac{1}{2}\Big\{(\nabla_X S)\phi Y + S(\nabla_X \phi)Y - (\nabla_X \phi)SY - \phi(\nabla_X S)Y\Big\} \\ &= \frac{1}{2}\Big\{(\nabla_X S)\phi Y + \eta(Y)S^2 X - \alpha g(SX, Y)\xi \end{aligned}$$

$$- \alpha\eta(Y)SX + g(SX, SY)\xi - \phi(\nabla_X S)Y\Big\} . \tag{31}$$

Now differentiating $\phi \circ \mathrm{Ric}^* + \mathrm{Ric}^* \circ \phi = 0$ convariantly gives

$$\begin{aligned} 0 &= (\nabla_X\phi)\mathrm{Ric}^*(Y) + \phi(\nabla_X\mathrm{Ric}^*)Y + (\nabla_X\mathrm{Ric}^*)\phi Y + \mathrm{Ric}^*(\nabla_X\phi)Y \\ &= -g(SX, \mathrm{Ric}^*(Y))\xi + \phi(\nabla_X\mathrm{Ric}^*)Y + (\nabla_X\mathrm{Ric}^*)\phi Y + \eta(Y)\mathrm{Ric}^*(SX) \\ &= -\frac{1}{2}g(SX, S\phi Y - \phi SY)\xi + \phi(\nabla_X\mathrm{Ric}^*)Y + (\nabla_X\mathrm{Ric}^*)\phi Y \\ &\quad + \frac{1}{2}\eta(Y)(S\phi SX - \phi S^2X) . \end{aligned}$$

Applying (31) in the above formula, we get

$$\begin{aligned} 0 &= g(SX, \phi SY)\xi + \Big\{ - \alpha\eta(Y)\phi SX + g((\nabla_X S)Y, \xi)\xi\Big\} \\ &\quad + \Big\{\eta(Y)(\nabla_X S)\xi - \alpha g(SX, \phi Y)\xi\Big\} + \eta(Y)S\phi SX \\ &= g(SX, \phi SY)\xi - \alpha\eta(Y)\phi SX + \Big\{g((Y, X(\alpha)\xi + \alpha\phi SX - S\phi SX)\Big\}\xi \\ &\quad + \eta(Y)\Big\{X(\alpha)\xi + \alpha\phi SX - S\phi SX\Big\} - \alpha g(SX, \phi Y)\xi + \eta(Y)S\phi SX \\ &= 2g(SX, \phi SY)\xi + 2\eta(Y)X(\alpha)\xi - 2\alpha g(SX, \phi Y)\xi , \end{aligned}$$

i.e.

$$g(SX, \phi SY) + \eta(Y)X(\alpha) - \alpha g(SX, \phi Y) = 0 . \tag{32}$$

From this we know $X(\alpha) = 0$ by taking $Y = \xi$, i.e. α is constant. Hence formula (32) becomes

$$g(SX, \phi SY) = \alpha g(SX, \phi Y) .$$

Now interchanging X and Y and comparing the resulting equation with the previous equation, we have $\alpha(\phi S - \phi S)X = 0$, which shows that either $\alpha = 0$ or $\phi S = S\phi$. Namely the following lemma has been proved.

LEMMA 6. *Let M be a real hypersurface in Q^m, $m \geq 3$, admitting a star-Ricci soliton with potential vector field ξ, then either the Reeb flow is isometric, or $\alpha = 0$.*

If the Reeb flow of M is isometric, Berndt and Suh proved the following conclusion:

THEOREM 6 ([2]). *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

In the following we set $\alpha = 0$, it follows from (32) that

$$S\phi SX = 0 , \quad \text{for all } X \in TM . \tag{33}$$

And it is easy to show that the normal vector N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic from (14). In the following let us consider these two cases.

Case I: N is $\mathfrak{A}$-principal, that is, $AN = N$. We follow from (15) that

$$S\phi SX = \phi X .$$

By comparing with (33) we find $\phi X = 0$, which is impossible.

Case II: N is $\mathfrak{A}$-isotropic. Using (33), we derive from (15)

$$g(\phi X, Y) = g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) . \tag{34}$$

Using (33) again, we learn (29) becomes

$$\begin{aligned} &-2(m-1)g(\phi^2 X, Y) - 2g(N, AX)g(AY, N) \\ &= \frac{1}{2}g((S\phi - \phi S)X, Y) . \end{aligned}$$

Moreover, replacing Y by ϕY gives

$$\begin{aligned} &-2(m-1)g(\phi X, Y) + 2g(N, AX)g(Y, A\xi) \\ &= \frac{1}{2}g((S\phi - \phi S)X, \phi Y) . \end{aligned} \tag{35}$$

Here we have used $g(A\phi Y, N) = -g(Y, A\xi)$, which follows from (19) and (22).

By interchanging Y and X in the formula (35) and applying the resulting equation to subtract this equation, we get

$$\begin{aligned} &2g(N, AX)g(Y, A\xi) - 2g(N, AY)g(X, A\xi) \\ &= \frac{1}{2}g((S\phi - \phi S)X, \phi Y) + 2(m-1)g(\phi X, Y) \\ &\quad - \frac{1}{2}g((S\phi - \phi S)Y, \phi X) - 2(m-1)g(\phi Y, X) \\ &= 4(m-1)g(\phi X, Y) . \end{aligned}$$

Combining this with (34) we get $(m-3)\phi X = 0$, which is a contradiction if $m \geq 4$. Hence we complete the proof of Theorem 5.

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