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Parabolic Flows on Almost Complex Manifolds

Masaya KAWAMURA

Tokyo Metropolitan University

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Abstract. We define two parabolic flows on almost complex manifolds, which coincide with the pluriclosed flow and the Hermitian curvature flow respectively on complex manifolds. We study the relationship between these parabolic evolution equations on a compact almost Hermitian manifold.

1. Introduction

In [4] and [6], Streets and Tian introduced a parabolic evolution equation of pluriclosed metrics with a pluriclosed initial metric on a Hermitian manifold, which is called the pluriclosed flow. In this paper, we would like to show that some of their results hold for almost Hermitian cases as well. Let (M, J) be a compact almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1, 0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1, 1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1} g_{r\bar{k}} \zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric. We would like to define a parabolic flow of almost Hermitian metrics with an almost Hermitian initial metric ω_0 on (M, J). We will call it the almost Hermitian flow (AHF):

$$\text{(AHF)} \quad \begin{cases} \frac{\partial}{\partial t}\omega(t) = \partial \partial^*_{g(t)}\omega(t) + \bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - P(\omega(t)) =: -\Phi(\omega(t))\,, \\ \omega(0) = \omega_0\,, \end{cases}$$

where $\partial^*_{g(t)}$ and $\bar{\partial}^*_{g(t)}$ are the L^2-adjoint operators with respect to metrics $g(t)$, and $P(\omega)$ is one of the Ricci-type curvatures of the Chern curvature. One has with an arbitrary $(1, 0)$-frame $\{Z_r\}$ with respect to g, $P_{i\bar{j}} = g^{k\bar{l}} \Omega_{i\bar{j}k\bar{l}} = -g^{k\bar{l}} Z_{\bar{j}} Z_i(g_{k\bar{l}}) + \mathcal{O}(Z(g)) + \mathcal{O}(B)$, where Ω is the curvature of the Chern connection ∇ on (M, g, J).

The first goal of this paper is to prove that the operator $\omega \mapsto \Phi(\omega)$ is a strictly elliptic operator for an almost Hermitian metric ω, which means that the equation (AHF) with an almost Hermitian initial metric is a strictly parabolic equation. Hence the short-time existence and the uniqueness of the solution (AHF) follows from the standard parabolic theory since the

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manifold is supposed to be compact. This flow (AHF) coincides with the pluriclosed flow if J is integrable and also the initial metric is pluriclosed (cf. [4], [6]).

THEOREM 1.1. *Given a compact almost Hermitian manifold* (M, ω_0, J), *there exists a unique solution to* (*AHF*) *with initial condition* ω_0 *on* $[0, \varepsilon)$ *for some* $\varepsilon > 0$.

We denote by S one of the Ricci-type curvatures of the Chern curvature, which is locally given by $S_{i\bar{j}} = g^{k\bar{l}}\Omega_{k\bar{l}i\bar{j}}$. The second goal of this paper is to prove that a solution of the almost Hermitian flow with initial condition ω_0 is equivalent to a solution of the following parabolic flow on a compact almost complex manifold with an almost Hermitian metric, we will call it the almost Hermitian curvature flow (AHCF):

$$\text{(AHCF)} \quad \begin{cases} \frac{\partial}{\partial t}\omega(t) = -S(\omega(t)) - Q^7(\omega(t)) - Q^8(\omega(t)) + BT'(\omega(t)) + \bar{Z}(T')(\omega(t)) \,, \\ \omega(0) = \omega_0 \,, \end{cases}$$

where $w_i := T_{ir\bar{r}}$,

$$BT'_{i\bar{j}} := B^{j}_{\bar{r}p}T_{ir\bar{p}} + B^{r}_{\bar{p}i}T_{pr\bar{j}} + B^{p}_{\bar{r}r}T_{pi\bar{j}} + B^{r}_{\bar{j}i}w_r \,,$$

and

$$\bar{Z}(T')_{i\bar{j}} := -Z_{\bar{r}}(T^{s}_{ri})g_{s\bar{j}} - Z_{\bar{j}}(w_i) - g^{p\bar{q}}T^{r}_{pi}Z_{\bar{j}}(g_{r\bar{q}}) \,.$$

These components are defined using an arbitrary unitary frame. In all this paper, we assume the Einstein convention omitting the symbol of sum over repeated indexes.

Concerning the difference between the flow (AHCF) and Vezzoni's flow in [7], Vezzoni studied the parabolic flow on a compact almost Hermitian manifold (M^{2n}, ω_0, J) such that

$$\frac{\partial}{\partial t}\omega = -S + \frac{1}{2}Q^1 - \frac{1}{4}Q^2 - \frac{1}{2}Q^3 + Q^4 - Q^7 - Q^8 =: K \,,$$

where Q^1, Q^2, Q^3, Q^4, Q^7, Q^8 are quadratics in the torsion of the Chern connection (cf. [7, pg. 712])

$$Q^1_{i\bar{j}} := T_{ik\bar{r}}T_{\bar{j}\bar{k}r}, \quad Q^2_{i\bar{j}} := T_{kri}T_{\bar{k}\bar{r}\bar{j}}, \quad Q^3_{i\bar{j}} := T_{ik\bar{k}}T_{\bar{j}\bar{r}r} \,,$$

and

$$Q^4_{i\bar{j}} := \frac{1}{2}(T_{rk\bar{k}}T_{\bar{r}\bar{j}i} + T_{\bar{r}\bar{k}k}T_{ri\bar{j}}) \,, \quad Q^7_{i\bar{j}} := T_{irk}T_{\bar{r}\bar{k}\bar{j}} \,, \quad Q^8_{i\bar{j}} := T_{irk}T_{\bar{j}\bar{k}\bar{r}} \,.$$

These components are defined using an arbitrary unitary frame. Also, Vezzoni considered a functional

$$\mathbf{F}(g) = \mathrm{Vol}(M)^{\frac{1-n}{n}} \int_M k dV_g \,,$$

where $k := g^{i\bar{j}}K_{i\bar{j}}$, and showed in [7, Theorem 6.5] that a metric g is a critical point of $\mathbf{F}$ if and only if k is constant and

$$K - \frac{k}{n} g = 0 .$$

The view point of (AHCF) is that we would like to generalize Streets-Tian identifiability theorem in [4].

Note that we have

$$P = S + \operatorname{div}^{\nabla} T' - \nabla \bar{w} + Q^7 + Q^8$$

for any almost Hermitian metric g (cf. [7, Lemma 3.5]), where T' is the torsion of the Chern connection ∇ associated to g, $(\operatorname{div}^{\nabla} T')_{i\bar{j}} = g^{k\bar{l}} \nabla_{\bar{l}} T_{ki\bar{j}}$, $(\nabla \bar{w})_{i\bar{j}} = g^{k\bar{l}} \nabla_i T_{\bar{j}\bar{l}k}$. This flow (AHCF) coincides with the flow called the Hermitian curvature flow (HCF):

$$\text{(HCF)} \qquad \begin{cases} \frac{\partial}{\partial t}\omega(t) = -S(\omega(t)) + Q^1(\omega(t)) , \\ \omega(0) = \omega_0 , \end{cases}$$

starting at a pluriclosed metric ω_0 if J is integrable.

PROPOSITION 1.1. *The parabolic flow* (*AHCF*) *coincides with the flow* (*HCF*) *starting at a pluriclosed metric if J is integrable.*

PROOF. Under our assumption that J is integrable, we have $Q^7 = Q^8 = BT' = 0$. Also, since we have $\partial_{\bar{r}} T_{ri\bar{j}} = \partial_{\bar{j}} T_{ri\bar{r}}$ for a pluriclosed metric on a Hermitian manifold (one can check that (HCF) preserves the pluriclosedness (cf. [4, Theorem 3.4])) and then we may choose a frame $Z_r = \frac{\partial}{\partial z_r}$ for some complex local coordinate $\{z_1, \dots, z_n\}$, we obtain

$$\begin{aligned} \bar{Z}(T')_{i\bar{j}} &= -\partial_{\bar{r}}(T^s_{ri}) g_{s\bar{j}} - \partial_{\bar{j}}(w_i) - g^{p\bar{q}} T^r_{pi} \partial_{\bar{j}} g_{r\bar{q}} \\ &= -\partial_{\bar{r}} T_{ri\bar{j}} + T^s_{ri} \partial_{\bar{r}} g_{s\bar{j}} - \partial_{\bar{j}} T_{ir\bar{r}} - g^{p\bar{q}} T^r_{pi} \Gamma^{\bar{k}}_{\bar{j}\bar{q}} g_{r\bar{k}} \\ &= -\partial_{\bar{j}} T_{ri\bar{r}} - \partial_{\bar{j}} T_{ir\bar{r}} + T^s_{ri} \Gamma^{\bar{k}}_{\bar{r}\bar{j}} g_{s\bar{k}} - T^s_{ri} \Gamma^{\bar{k}}_{\bar{j}\bar{r}} g_{s\bar{k}} \\ &= T_{ir\bar{s}} T_{\bar{j}\bar{r}s} \\ &= Q^1_{i\bar{j}} , \end{aligned}$$

where we are writing ∂_r, $\partial_{\bar{j}}$ for $\frac{\partial}{\partial z_r}$, $\frac{\partial}{\partial z_{\bar{j}}}$ respectively. Combining these yields the result. □

Streets and Tian asked whether or not it is possible to prove classification results in higher dimensions for complex non-Kähler manifolds using geometric evolution equations as in the case that the Ricci flow was used for proving uniformization of Riemann surfaces. They tried to have a parabolic flow such that it preserves Hermitianness and as much additional structure as possible and also is as close to the Kähler-Ricci flow as possible. Since a pluriclosed form ω is locally given by $\omega = \partial \eta + \bar{\partial} \bar{\eta}$ for some $\eta \in \Lambda^{0,1}$ (cf. [3, Lemma 3.9]), they concluded that it is natural to define a flow of pluriclosed metrics using a second order closed $(1, 1)$-form (the Chern curvature form) and a first-order $(0, 1)$-form. From this point of view, they defined the pluriclosed flow, whose RHS is the same as (AHF), starting at a pluriclosed metric. They showed that the solution of the pluriclosed flow coincides with the solution of (HCF) (cf. [4]). Our approach is to try to generalize their flow to almost Hermitian cases and to expect to obtain similar results as in the complex cases. The result in Proposition 1.1 tells us that our flow (AHCF) can be considered as a generalized flow of the pluriclosed flow and (HCF). The following result indicates that the parabolic flow (AHCF) could play the same role as the flow (HCF) on complex manifolds. We may expect to have some other similar results as in [4], [5] and [6] for (AHCF).

THEOREM 1.2. *Given a compact almost Hermitian manifold* (M, ω_0, J), *if* $(M, g(t), J)$ *is a solution to* (*AHF*) *starting at* ω_0, *then it coincides with the solution to* (*AHCF*).

As the third goal of this paper, we prove that the equation (AHCF) is a strictly parabolic equation. Therefore, the short-time existence and the uniqueness of the solution to (AHCF) with initial condition ω_0 follow from the standard parabolic theory since the manifold is supposed to be compact. Since the solution to (AHCF) are unique, the solution to (AHCF) exactly coincides with the solution to (AHF).

THEOREM 1.3. *Given a compact almost Hermitian manifold* (M, ω_0, J), *there exists a unique solution to* (*AHCF*) *with initial condition* ω_0 *on* $[0, \varepsilon)$ *for some* $\varepsilon > 0$.

2. Preliminaries

Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

which is the Nijenhuis tensor of J. An almost complex structure J is called integrable if $N = 0$ on M. Giving a complex structure to a differentiable manifold M is equivalent to giving an integrable almost complex structure to M. Let (M, J) be an almost complex manifold. A Riemannian metric g on M is called J-invariant if J is compatible with g, i.e., for any $X, Y \in \Gamma(TM)$,

$$g(X, Y) = g(JX, JY).$$

In this case, the pair (g, J) is called an almost Hermitian structure. The fundamental 2-form ω associated to a J-invariant Riemannian metric g, i.e., an almost Hermitian metric, is determined by, for $X, Y \in \Gamma(TM)$,

$$\omega(X, Y) = g(JX, Y).$$

Indeed we have, for any $X, Y \in \Gamma(TM)$,

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y)$$

and $\omega \in \Gamma(\bigwedge^2 T^*M)$. We will also refer to the associated real fundamental $(1,1)$-form ω as an almost Hermitian metric. The form ω is related to the volume form dV_g by $\omega^n = \frac{1}{n!}dV_g$. Let a local $(1,0)$-frame $\{Z_r\}$ on (M, J) with an almost Hermitian metric g. Since g is almost Hermitian, its components satisfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$.

We write $T^{\mathbf{R}}M$ for the real tangent space of M. Then its complexified tangent space is given by

$$T^{\mathbf{C}}M = T^{\mathbf{R}}M \otimes_{\mathbf{R}} \mathbf{C}\,.$$

By extending J $\mathbf{C}$-linearly and g, ω $\mathbf{C}$-bilinearly to $T^{\mathbf{C}}M$, they are also defined on $T^{\mathbf{C}}M$ and we observe that the complexified tangent space $T^{\mathbf{C}}M$ can be decomposed as

$$T^{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M\,,$$

where $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let $\Lambda^r(M) = \bigoplus_{p+q=r} \Lambda^{p,q}(M)$ for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p, q)-forms. The de Rham operator splits in $d = N + \partial + \bar{\partial} + \bar{N}$.

We define the adjoint operator ∂_g^* of ∂ with respect to an almost Hermitian metric g by

$$(\partial_g^*\gamma, \eta)_g = (\gamma, \partial\eta)_g$$

for $\gamma \in \Lambda^{p+1,q}(M)$ and $\eta \in \Lambda^{p,q}(M)$, where $(\cdot,\cdot)_g$ is the L^2-inner product with respect to g. For instance, $(\alpha, \beta)_g = \int_M g^{\bar{j}i}\alpha_{\bar{j}}\overline{\beta_{\bar{i}}}dV_g$ for $(0,1)$-forms α, β. Analogously, we define the adjoint operator $\bar{\partial}_g^*$ of $\bar{\partial}$.

Let (M, g, J) be an almost Hermitian manifold. There exists a unique affine connection ∇ preserving g and J on M whose torsion has vanishing $(1,1)$-part (cf. [3],[4]), which is called the Chern connection. Now let ∇ be the Chern connection on M. We can write

$$[Z_i, Z_j] = B^r_{ij}Z_r + B^{\bar{r}}_{ij}Z_{\bar{r}}, \quad [Z_i, Z_{\bar{j}}] = B^r_{i\bar{j}}Z_r + B^{\bar{r}}_{i\bar{j}}Z_{\bar{r}}\,.$$

Notice that J is integrable if and only if the $B^{\bar{r}}_{ij}$'s vanish. Since the Chern connection ∇ preserves J, we have

$$\nabla_i Z_j = \Gamma^r_{ij}Z_r\,, \quad \nabla_i Z_{\bar{j}} = \Gamma^{\bar{r}}_{i\bar{j}}Z_{\bar{r}}\,,$$

where

$$\Gamma^r_{ij} = g^{r\bar{s}}Z_i(g_{j\bar{s}}) - g^{r\bar{s}}g_{j\bar{l}}B^{\bar{l}}_{i\bar{s}}\,,$$
$$\Gamma^{\bar{r}}_{i\bar{j}} = B^{\bar{r}}_{i\bar{j}}\,.$$

For any (0, 1)-form β, for a coframe $\{\zeta^r\}$ associated to a local (1, 0)-frame $\{Z_r\}$ with respect to g around a fixed point in M, we have $\beta = \beta_{\bar{j}}\zeta^{\bar{j}}$,

$$\nabla_k \beta_{\bar{j}} = Z_k(\beta_{\bar{j}}) - \Gamma^{\bar{l}}_{k\bar{j}}\beta_{\bar{l}} = Z_k(\beta_{\bar{j}}) - B^{\bar{l}}_{k\bar{j}}\beta_{\bar{l}}\,,$$

and

$$\partial\beta = \nabla_k \beta_{\bar{j}}\zeta^k \wedge \zeta^{\bar{j}}$$

and so $(\partial\beta)_{k\bar{j}} = \nabla_k \beta_{\bar{j}}$.

Note that the mixed derivatives $\nabla_i Z_{\bar{j}}$ do not depend on g. The torsion T of ∇ has no (1, 1)-part and the only vanishing components are as follows:

$$T^r_{ij} = \Gamma^r_{ij} - \Gamma^r_{ji} - B^r_{ij}\,, \quad T^{\bar{r}}_{ij} = -B^{\bar{r}}_{ij}\,,$$

which tells us that T splits in $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}(M) \otimes T^{1,0}M)$ and $T'' \in \Gamma(\Lambda^{2,0}(M) \otimes T^{0,1}M)$. Note that T'' depends only on J and it can be regarded as the Nijenhuis tensor of J, that is, J is integrable if and only if T'' vanishes. We denote by Ω the curvature of the Chern connection ∇. We can regard Ω as a section of $\Lambda^2(M) \otimes TM$, $\Omega \in \Gamma(\Lambda^2(M) \otimes TM)$ and Ω splits in $\Omega = H + R + \bar{H}$, where $R \in \Gamma(\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M))$, $H \in \Gamma(\Lambda^{2,0}(M) \otimes \Lambda^{1,1}(M))$. In terms of Z_r's, we have

$$R^r_{i\bar{j}k} = Z_i(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^l_{\bar{j}k}\Gamma^r_{il} - \Gamma^l_{ik}\Gamma^r_{\bar{j}l} - B^l_{i\bar{j}}\Gamma^r_{lk} - B^{\bar{l}}_{i\bar{j}}\Gamma^r_{\bar{l}k}\,,$$
$$H^r_{ijk} = Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^l_{jk}\Gamma^r_{il} - \Gamma^l_{ik}\Gamma^r_{jl} - B^l_{ij}\Gamma^r_{lk} - B^{\bar{l}}_{ij}\Gamma^r_{\bar{l}k}\,.$$

3. Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

Let (M^{2n}, g, J) be a compact almost Hermitian manifold. Let $\{Z_r\}$ be a local unitary (1, 0)-frame with respect to g around a fixed point $p \in M$. Note that unitary frames always exist locally since we can take any frame and apply the Gram-Schmidt process. Then with respect to such a frame, we have $g_{i\bar{j}} = \delta_{ij}$, $Z_k(g_{i\bar{j}}) = 0$ and the Christoffel symbols satisfy

$$\Gamma^k_{ij} = -\Gamma^{\bar{j}}_{i\bar{k}} = -B^{\bar{k}}_{i\bar{j}}.$$

With respect to such a frame, the components of the torsion can be written as

$$T^k_{ij} = -B^{\bar{j}}_{i\bar{k}} + B^{\bar{i}}_{j\bar{k}} - B^k_{ij}$$

and the components of w can be written as

$$w_j = -B^r_{jr} - B^{\bar{r}}_{j\bar{r}} + B^{\bar{j}}_{r\bar{r}}\,.$$

We need the following lemma.

LEMMA 3.1 ([7, Lemma 4.1]). *Let α be a $(0,1)$-form and $\{Z_j\}$ be an arbitrary unitary $(1,0)$-frame. Then we have*

$$\int_M Z_j(\alpha_{\bar{j}})dV_g = \int_M \alpha_{\bar{j}}(B^r_{jr} + B^{\bar{r}}_{j\bar{r}})dV_g \, .$$

Let $\{Z_r\}$ be an arbitrary local unitary $(1,0)$-frame with respect to g and let $\{\zeta^r\}$ be the associated coframe. Then ω takes the expression $\omega = \sqrt{-1}\delta_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$ and we can write $\alpha = \alpha_{\bar{r}}\zeta^{\bar{r}}$. Hence we have (cf. [7, Lemma 4.1])

$$\begin{aligned}\partial\alpha &= \nabla_r\alpha_{\bar{k}}\zeta^r \wedge \zeta^{\bar{k}} \\ &= (Z_r(\alpha_{\bar{k}}) - \Gamma^{\bar{m}}_{r\bar{k}}\alpha_{\bar{m}})\zeta^r \wedge \zeta^{\bar{k}} \\ &= (Z_r(\alpha_{\bar{k}}) - B^{\bar{m}}_{r\bar{k}}\alpha_{\bar{m}})\zeta^r \wedge \zeta^{\bar{k}},\end{aligned}$$

that is, we have

$$(\partial\alpha)_{i\bar{j}} = \nabla_i\alpha_{\bar{j}}.$$

Applying the result of Lemma 3.1, we obtain the following formulae.

LEMMA 3.2. *Given an almost Hermitian metric ω, we have*

$$(\partial^*_g\omega)_{\bar{j}} = -w_{\bar{j}}\, , \quad (\bar{\partial}^*_g\omega)_k = -w_k\, .$$

PROOF. Let $\{Z_r\}$ be an arbitrary local g-unitary frame. For any $\alpha \in \Lambda^{0,1}$, we compute

$$\begin{aligned}(\partial^*_g\omega, \alpha)_g &= (\omega, \partial\alpha)_g \\ &= \sqrt{-1}\int_M \overline{(\partial\alpha)_{j\bar{j}}}dV_g \\ &= \sqrt{-1}\int_M \overline{\nabla_j\alpha_{\bar{j}}}dV_g \\ &= \sqrt{-1}\int_M \overline{(Z_j(\alpha_{\bar{j}}) - B^{\bar{r}}_{j\bar{j}}\alpha_{\bar{r}})}dV_g \\ &= \sqrt{-1}\int_M \overline{(B^r_{jr} + B^{\bar{r}}_{j\bar{r}} - B^{\bar{j}}_{r\bar{r}})}\overline{\alpha_{\bar{j}}}dV_g \\ &= -\sqrt{-1}\int_M w_{\bar{j}}\overline{\alpha_{\bar{j}}}dV_g\, ,\end{aligned}$$

where we used the result in Lemma 3.1 in the fourth line. This gives the first formula, and the second one follows analogously. □

LEMMA 3.3. *Given an almost Hermitian metric ω, we have*

$$(\partial\partial^*_g\omega)_{i\bar{j}} = -\nabla_i w_{\bar{j}}\, , \quad (\bar{\partial}\bar{\partial}^*_g\omega)_{k\bar{l}} = -\nabla_{\bar{l}}w_k\, .$$

PROOF. In general, we have for any $\alpha \in \Lambda^{0,1}$, $(\partial\alpha)_{i\bar{j}} = \nabla_i \alpha_{\bar{j}}$. Hence we have

$$\begin{aligned}(\partial\partial_g^*\omega)_{i\bar{j}} &= -(\partial w)_{i\bar{j}}\\ &= -\nabla_i w_{\bar{j}}\,.\end{aligned}$$

The second follows analogously. $\square$

PROPOSITION 3.1. *The operator $\omega \mapsto \Phi(\omega)$ is a non-linear second-order elliptic operator for an almost Hermitian metric ω.*

PROOF. Let $\{Z_r\}$ be an arbitrary $(1,0)$-frame with respect to J. We compute

$$\begin{aligned}Z_{\bar{j}}(w_i) = {}& g^{s\bar{l}} Z_{\bar{j}} Z_i(g_{r\bar{l}}) g_{s\bar{r}} - g^{s\bar{l}} g_{r\bar{q}} Z_{\bar{j}}(B^{\bar{q}}_{i\bar{l}}) g_{s\bar{r}} - g^{s\bar{l}} Z_{\bar{j}} Z_r(g_{i\bar{l}}) g_{s\bar{r}} + g^{s\bar{l}} g_{i\bar{q}} Z_{\bar{j}}(B^{\bar{q}}_{r\bar{l}}) g_{s\bar{r}}\\ &- Z_{\bar{j}}(B^s_{ir}) g_{s\bar{r}} + \mathcal{O}(Z(g))\,.\end{aligned}$$

Then we compute with using the formula above,

$$\begin{aligned}&(\partial\partial_g^*\omega)_{i\bar{j}} + (\bar{\partial}\bar{\partial}_g^*\omega)_{i\bar{j}}\\ &\quad= -\nabla_i w_{\bar{j}} - \nabla_{\bar{j}} w_i\\ &\quad= -(Z_i(w_{\bar{j}}) - \Gamma^{\bar{r}}_{i\bar{j}} w_{\bar{r}}) - (Z_{\bar{j}}(w_i) - \Gamma^r_{\bar{j}i} w_r)\\ &\quad= -Z_i(w_{\bar{j}}) + B^{\bar{r}}_{i\bar{j}} w_{\bar{r}} - Z_{\bar{j}}(w_i) + B^r_{\bar{j}i} w_r\\ &\quad= -Z_i(\Gamma^{\bar{s}}_{\bar{j}\bar{r}} - \Gamma^{\bar{s}}_{\bar{r}\bar{j}} - B^{\bar{s}}_{\bar{j}\bar{r}}) g_{r\bar{s}} + \mathcal{O}(Z(g)) + B^{\bar{r}}_{i\bar{j}} w_{\bar{r}}\\ &\qquad - Z_{\bar{j}}(\Gamma^s_{ir} - \Gamma^s_{ri} - B^s_{ir}) g_{s\bar{r}} + \mathcal{O}(Z(g)) + B^r_{\bar{j}i} w_r\\ &\quad= -g^{k\bar{s}} Z_i Z_{\bar{j}}(g_{k\bar{r}}) g_{r\bar{s}} + g^{k\bar{s}} g_{l\bar{r}} Z_i(B^l_{\bar{j}k}) g_{r\bar{s}} + g^{k\bar{s}} Z_i Z_{\bar{r}}(g_{k\bar{j}}) g_{r\bar{s}} - g^{k\bar{s}} g_{l\bar{j}} Z_i(B^l_{\bar{r}k}) g_{r\bar{s}}\\ &\qquad + Z_i(B^{\bar{s}}_{\bar{j}\bar{r}}) g_{r\bar{s}} + \mathcal{O}(Z(g)) + B^{\bar{r}}_{i\bar{j}} w_{\bar{r}}\\ &\qquad - g^{s\bar{l}} Z_{\bar{j}} Z_i(g_{r\bar{l}}) g_{s\bar{r}} + g^{s\bar{l}} g_{r\bar{q}} Z_{\bar{j}}(B^{\bar{q}}_{i\bar{l}}) g_{s\bar{r}} + g^{s\bar{l}} Z_{\bar{j}} Z_r(g_{i\bar{l}}) g_{s\bar{r}} - g^{s\bar{l}} g_{i\bar{q}} Z_{\bar{j}}(B^{\bar{q}}_{r\bar{l}}) g_{s\bar{r}}\\ &\qquad + Z_{\bar{j}}(B^s_{ir}) g_{s\bar{r}} + \mathcal{O}(Z(g)) + B^r_{\bar{j}i} w_r\,,\end{aligned}$$

where $\mathcal{O}(Z(g))$ denotes the set of all terms including $Z(g)$. Here note that $B^{\bar{q}}_{j\bar{b}}$, $B^q_{\bar{j}b}$'s do not depend on g, which depend only on J since the mixed derivatives $\nabla_j Z_{\bar{b}}$ do not depend on g.

We also compute with using that $[Z_i, Z_r] = B^s_{ir} Z_s + B^{\bar{s}}_{ir} Z_{\bar{s}}$,

$$\begin{aligned}Z_{\bar{j}}(B^s_{ir}) g_{s\bar{r}} &= g(\nabla_{\bar{j}}(B^s_{ir}) Z_s, Z_{\bar{r}})\\ &= g(\nabla_{\bar{j}}(B^s_{ir} Z_s) - B^s_{ir} \nabla_{\bar{j}} Z_s, Z_{\bar{r}})\\ &= Z_{\bar{j}} g(B^s_{ir} Z_s, Z_{\bar{r}}) + \mathcal{O}(Z(g))\\ &= Z_{\bar{j}} g([Z_i, Z_r] - B^{\bar{s}}_{ir} Z_{\bar{s}}, Z_{\bar{r}}) + \mathcal{O}(Z(g))\\ &= Z_{\bar{j}} g((\Gamma^s_{ir} - \Gamma^s_{ri}) Z_s, Z_{\bar{r}}) + \mathcal{O}(Z(g))\end{aligned}$$

$$= Z_{\bar{j}}(\Gamma^s_{ir})g_{s\bar{r}} - Z_{\bar{j}}(\Gamma^s_{ri})g_{s\bar{r}} + \mathcal{O}(Z(g))\,,$$

where we recall that the mixed derivatives $\nabla_{\bar{j}} Z_s$ and $B^{\bar{s}}_{ir}$ do not depend on g. We calculate

$$\begin{aligned} Z_{\bar{j}}(\Gamma^s_{ir})g_{s\bar{r}} &= Z_{\bar{j}}(g^{s\bar{l}} Z_i(g_{r\bar{l}}) - g^{s\bar{l}} g_{r\bar{q}} B^{\bar{q}}_{i\bar{l}})g_{s\bar{r}} \\ &= g^{s\bar{l}} Z_{\bar{j}} Z_i(g_{r\bar{l}})g_{s\bar{r}} + \mathcal{O}(Z(g))\,, \end{aligned}$$

and also,

$$\begin{aligned} Z_{\bar{j}}(\Gamma^s_{ri})g_{s\bar{r}} &= Z_{\bar{j}}(g^{s\bar{l}} Z_r(g_{i\bar{l}}) - g^{s\bar{l}} g_{i\bar{q}} B^{\bar{q}}_{r\bar{l}})g_{s\bar{r}} \\ &= g^{s\bar{l}} Z_{\bar{j}} Z_r(g_{i\bar{l}})g_{s\bar{r}} + \mathcal{O}(Z(g))\,. \end{aligned}$$

Likewise, we calculate

$$\begin{aligned} Z_i(B^{\bar{s}}_{\bar{j}\bar{r}})g_{r\bar{s}} &= g(Z_r, \nabla_i(B^{\bar{s}}_{\bar{j}\bar{r}})Z_{\bar{s}}) \\ &= g(Z_r, \nabla_i(B^{\bar{s}}_{\bar{j}\bar{r}}Z_{\bar{s}}) - B^{\bar{s}}_{\bar{j}\bar{r}}\nabla_i Z_{\bar{s}}) \\ &= Z_i\, g(Z_r, B^{\bar{s}}_{\bar{j}\bar{r}}Z_{\bar{s}}) + \mathcal{O}(Z(g)) \\ &= Z_i\, g(Z_r, [Z_{\bar{j}}, Z_{\bar{r}}] - B^{s}_{\bar{j}\bar{r}}Z_s) + \mathcal{O}(Z(g)) \\ &= Z_i\, g(Z_r, (\Gamma^{\bar{s}}_{\bar{j}\bar{r}} - \Gamma^{\bar{s}}_{\bar{r}\bar{j}})Z_{\bar{s}}) + \mathcal{O}(Z(g)) \\ &= Z_i(\Gamma^{\bar{s}}_{\bar{j}\bar{r}})g_{r\bar{s}} - Z_i(\Gamma^{\bar{s}}_{\bar{r}\bar{j}})g_{r\bar{s}} + \mathcal{O}(Z(g))\,, \end{aligned}$$

$$\begin{aligned} Z_i(\Gamma^{\bar{s}}_{\bar{j}\bar{r}})g_{r\bar{s}} &= Z_i(g^{l\bar{s}} Z_{\bar{j}}(g_{l\bar{r}}) - g^{l\bar{s}} g_{p\bar{r}} B^{p}_{\bar{j}l})g_{r\bar{s}} \\ &= g^{l\bar{s}} Z_i Z_{\bar{j}}(g_{l\bar{r}})g_{r\bar{s}} + \mathcal{O}(Z(g))\,, \end{aligned}$$

and also,

$$\begin{aligned} Z_i(\Gamma^{\bar{s}}_{\bar{r}\bar{j}})g_{r\bar{s}} &= Z_i(g^{l\bar{s}} Z_{\bar{r}}(g_{l\bar{j}}) - g^{l\bar{s}} g_{p\bar{j}} B^{p}_{\bar{r}l})g_{r\bar{s}} \\ &= g^{l\bar{s}} Z_i Z_{\bar{r}}(g_{l\bar{j}})g_{r\bar{s}} + \mathcal{O}(Z(g))\,. \end{aligned}$$

Hence we obtain

$$\begin{aligned} (\partial\partial^*_g\omega)_{i\bar{j}} + (\bar\partial\bar\partial^*_g\omega)_{i\bar{j}} &= Z_i(B^{l}_{\bar{j}r})g_{l\bar{r}} - Z_i(B^{l}_{\bar{r}r})g_{l\bar{j}} + Z_{\bar{j}}(B^{\bar{q}}_{i\bar{r}})g_{r\bar{q}} - Z_{\bar{j}}(B^{\bar{q}}_{r\bar{r}})g_{i\bar{q}} \\ &\quad + B^{\bar{r}}_{i\bar{j}}w_{\bar{r}} + B^{r}_{\bar{j}i}w_r + \mathcal{O}(Z(g))\,. \end{aligned}$$

.

Therefore, $(\partial\partial^*_g\omega)_{i\bar{j}} + (\bar\partial\bar\partial^*_g\omega)_{i\bar{j}}$ involves only the first derivatives of g, and the principal part of $\Phi(\omega)$ is determined by $P(\omega)$.

We also have

$$P_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -g^{k\bar{l}} Z_{\bar{j}} Z_i (g_{\bar{k}\bar{l}}) + \mathcal{O}(Z(g)) + \mathcal{O}(B)\,,$$

which implies that the operator Φ is elliptic. $\square$

Since the operator $\omega \mapsto \Phi(\omega)$ is strictly elliptic, the equation (AHF) is strictly parabolic and the standard parabolic theory implies that (AHF) has a unique solution for t sufficiently small. Hence we have proved the first main result.

LEMMA 3.4. *The following formula holds for any almost Hermitian metric*

$$div^{\nabla} T'_{i\bar{j}} = -(\bar{\nabla} w)_{i\bar{j}} - BT'_{i\bar{j}} - \bar{Z}(T')_{i\bar{j}}\,,$$

where T' is the torsion given by $T = T' + T''$ (T *is the torsion of* ∇), *which is a section of* $\Lambda^{2,0}M \otimes T^{1,0}M$.

PROOF. Let $\{Z_r\}$ be a local unitary $(1,0)$-frame with respect to g. Using the equality $Z_{\bar{q}}(g_{r\bar{j}}) = 0 = Z_{\bar{j}}(g_{r\bar{q}})$ with respect to such a frame in the fourth line below, we compute

$$\begin{aligned}
\mathrm{div}^{\nabla} T'_{i\bar{j}} &= g^{p\bar{q}} \nabla_{\bar{q}} T_{pi\bar{j}} \\
&= g^{p\bar{q}} (Z_{\bar{q}}(T_{pi\bar{j}}) - \Gamma^r_{\bar{q}p} T_{ri\bar{j}} - \Gamma^r_{\bar{q}i} T_{pr\bar{j}} - \Gamma^{\bar{r}}_{\bar{q}\bar{j}} T_{pi\bar{r}}) \\
&= g^{p\bar{q}} (Z_{\bar{q}}(T^r_{pi}) g_{r\bar{j}} + T^r_{pi} Z_{\bar{q}}(g_{r\bar{j}}) - \Gamma^r_{\bar{q}p} T_{ri\bar{j}} - \Gamma^r_{\bar{q}i} T_{pr\bar{j}} - \Gamma^{\bar{r}}_{\bar{q}\bar{j}} T_{pi\bar{r}}) \\
&= g^{p\bar{q}} (Z_{\bar{q}}(T^r_{pi}) g_{r\bar{j}} + T^r_{pi} Z_{\bar{j}}(g_{r\bar{q}}) - \Gamma^r_{\bar{q}p} T_{ri\bar{j}} - \Gamma^r_{\bar{q}i} T_{pr\bar{j}} - \Gamma^{\bar{r}}_{\bar{q}\bar{j}} T_{pi\bar{r}}) \\
&= g^{p\bar{q}} (Z_{\bar{q}}(T^r_{pi}) g_{r\bar{j}} + Z_{\bar{j}}(T^r_{pi} g_{r\bar{q}}) - Z_{\bar{j}}(T^r_{pi}) g_{r\bar{q}} - \Gamma^r_{\bar{q}p} T_{ri\bar{j}} - \Gamma^r_{\bar{q}i} T_{pr\bar{j}} - \Gamma^{\bar{r}}_{\bar{q}\bar{j}} T_{pi\bar{r}}) \\
&= g^{p\bar{q}} (Z_{\bar{q}}(T^r_{pi}) g_{r\bar{j}} - \nabla_{\bar{j}} T_{ip\bar{q}} + \Gamma^r_{\bar{j}p} T_{ri\bar{q}} + \Gamma^r_{\bar{j}i} T_{pr\bar{q}} + \Gamma^{\bar{r}}_{\bar{j}\bar{q}} T_{pi\bar{r}} - Z_{\bar{j}}(T^r_{pi}) g_{r\bar{q}} \\
&\quad - \Gamma^r_{\bar{q}p} T_{ri\bar{j}} - \Gamma^r_{\bar{q}i} T_{pr\bar{j}} - \Gamma^{\bar{r}}_{\bar{q}\bar{j}} T_{pi\bar{r}}) \\
&= g^{p\bar{q}} (Z_{\bar{q}}(T^r_{pi}) g_{r\bar{j}} - \nabla_{\bar{j}} T_{ip\bar{q}} + \Gamma^r_{\bar{j}i} T_{pr\bar{q}} - Z_{\bar{j}}(T^r_{pi} g_{r\bar{q}}) + T^r_{pi} Z_{\bar{j}}(g_{r\bar{q}}) \\
&\quad - \Gamma^r_{\bar{q}p} T_{ri\bar{j}} - \Gamma^r_{\bar{q}i} T_{pr\bar{j}} + (\Gamma^{\bar{r}}_{\bar{j}\bar{q}} - \Gamma^{\bar{r}}_{\bar{q}\bar{j}} - B^{\bar{r}}_{\bar{j}\bar{q}}) T_{pi\bar{r}} + \Gamma^r_{\bar{j}p} T_{ri\bar{q}} + B^{\bar{r}}_{\bar{j}\bar{q}} T_{pi\bar{r}}) \\
&= -\nabla_{\bar{j}} w_i - B^j_{\bar{r}p} T_{ir\bar{p}} - B^r_{\bar{p}i} T_{pr\bar{j}} - B^p_{\bar{r}r} T_{pi\bar{j}} - B^r_{\bar{j}i} w_r \\
&\quad + Z_{\bar{r}}(T^s_{ri}) g_{s\bar{j}} + Z_{\bar{j}}(w_i) + g^{p\bar{q}} T^r_{pi} Z_{\bar{j}}(g_{r\bar{q}}) \\
&= -(\bar{\nabla} w)_{i\bar{j}} - BT'_{i\bar{j}} - \bar{Z}(T')_{i\bar{j}}\,,
\end{aligned}$$

where we used the following in the eighth line

$$\begin{aligned}
&g^{p\bar{q}} \big((\Gamma^{\bar{r}}_{\bar{j}\bar{q}} - \Gamma^{\bar{r}}_{\bar{q}\bar{j}} - B^{\bar{r}}_{\bar{j}\bar{q}}) T_{pi\bar{r}} + \Gamma^r_{\bar{j}p} T_{ri\bar{q}} + B^{\bar{r}}_{\bar{j}\bar{q}} T_{pi\bar{r}} \big) \\
&\qquad = T^{\bar{r}}_{\bar{j}\bar{p}} T_{pi\bar{r}} - (B^r_{\bar{j}p} + B^{\bar{p}}_{\bar{j}\bar{r}}) T_{ir\bar{p}}
\end{aligned}$$

$$
\begin{aligned}
&= -T_{\bar{j}\bar{p}r}T_{ip\bar{r}} - \overline{(B^{\bar{r}}_{j\bar{p}} + B^{p}_{jr} - B^{\bar{j}}_{r\bar{p}})}T_{ir\bar{p}} - B^{j}_{\bar{r}p}T_{ir\bar{p}} \\
&= -Q^1_{i\bar{j}} + \overline{(\Gamma^{p}_{jr} - \Gamma^{p}_{rj} - B^{p}_{jr})}T_{ir\bar{p}} - B^{j}_{\bar{r}p}T_{ir\bar{p}} \\
&= -Q^1_{i\bar{j}} + Q^1_{i\bar{j}} - B^{j}_{\bar{r}p}T_{ir\bar{p}} \\
&= -B^{j}_{\bar{r}p}T_{ir\bar{p}} \,.
\end{aligned}
$$

Here we used $B^{\bar{r}}_{j\bar{p}} = \Gamma^{\bar{r}}_{j\bar{p}} = -\Gamma^{p}_{jr}$ with respect to a unitary frame in the third line above. □

We note the following equality.

LEMMA 3.5 ([7, Lemma 3.5]). *The following formula holds for any almost Hermitian metric*

$$
P - S = div^{\nabla} T' - \nabla \bar{w} + Q^7 + Q^8 \,.
$$

Combining these formulae, we obtain the following.

PROPOSITION 3.2. *Let $(M, \omega(t), J)$ be a solution to (AHF) starting at the initial almost Hermitian metric ω_0. Then we have*

$$
\frac{\partial}{\partial t}\omega(t) = -S - Q^7 - Q^8 + BT' + \bar{Z}(T') \,.
$$

PROOF. Taking into account Lemma 3.3, the flow (AHF) takes the expression

$$
\begin{aligned}
\frac{\partial}{\partial t}\omega &= \partial\partial^*_g\omega + \bar{\partial}\bar{\partial}^*_g\omega - P \\
&= -\nabla\bar{w} - \bar{\nabla}w - P \,.
\end{aligned}
$$

From the result of Lemma 3.5, we have

$$
P = \mathrm{div}^{\nabla} T' - \nabla\bar{w} + S + Q^7 + Q^8 \,.
$$

Applying the result of Lemma 3.4 yields

$$
P = -\bar{\nabla}w - BT' - \bar{Z}(T') - \nabla\bar{w} + S + Q^7 + Q^8 \,.
$$

Hence, we conclude

$$
\frac{\partial}{\partial t}\omega = -S - Q^7 - Q^8 + BT' + \bar{Z}(T') \,.
$$

□

Hence we have proved the second main result.

Consider the operator

$$
\Psi(\omega) := (S + Q^7 + Q^8 - BT' - \bar{Z}(T'))(\omega) \,.
$$

PROPOSITION 3.3. *The operator $\omega \mapsto \Psi(\omega)$ is a non-linear second-order elliptic operator for an almost Hermitian metric ω.*

PROOF. Since $(Q^7+Q^8-BT')(\omega)$ involves only the first derivatives of g, the principal part of $\Psi(\omega)$ is determined by $(S-\bar{Z}(T'))(\omega)$. Let $\{Z_r\}$ be an arbitrary $(1,0)$-frame with respect to J. We compute

$$\begin{aligned}
-\bar{Z}(T')_{i\bar{j}} &= Z_{\bar{r}}(T^s_{ri})g_{s\bar{j}} + Z_{\bar{j}}(w_i) + g^{p\bar{q}}T^r_{pi}Z_{\bar{j}}(g_{r\bar{q}}) \\
&= Z_{\bar{r}}(\Gamma^s_{ri}-\Gamma^s_{ir}-B^s_{ri})g_{s\bar{j}} + Z_{\bar{j}}(\Gamma^s_{ir}-\Gamma^s_{ri}-B^s_{ir})g_{s\bar{r}} + \mathcal{O}(Z(g)) + g^{p\bar{q}}T^r_{pi}Z_{\bar{j}}(g_{r\bar{q}}) \\
&= Z_{\bar{r}}(g^{s\bar{l}}Z_r(g_{i\bar{l}}) - g^{s\bar{l}}g_{i\bar{q}}B^{\bar{q}}_{r\bar{l}})g_{s\bar{j}} - Z_{\bar{r}}(g^{s\bar{l}}Z_i(g_{r\bar{l}}) - g^{s\bar{l}}g_{r\bar{q}}B^{\bar{q}}_{i\bar{l}})g_{s\bar{j}} - Z_{\bar{r}}(B^s_{ri})g_{s\bar{j}} \\
&\quad + g^{s\bar{l}}Z_{\bar{j}}Z_i(g_{r\bar{l}})g_{s\bar{r}} - g^{s\bar{l}}g_{r\bar{q}}Z_{\bar{j}}(B^{\bar{q}}_{i\bar{l}})g_{s\bar{r}} - g^{s\bar{l}}Z_{\bar{j}}Z_r(g_{i\bar{l}})g_{s\bar{r}} + g^{s\bar{l}}g_{i\bar{q}}Z_{\bar{j}}(B^{\bar{q}}_{r\bar{l}})g_{s\bar{r}} \\
&\quad - Z_{\bar{j}}(B^s_{ir})g_{s\bar{r}} + \mathcal{O}(Z(g)) \\
&= Z_{\bar{r}}Z_r(g_{i\bar{j}}) - g^{s\bar{l}}g_{i\bar{q}}Z_{\bar{r}}(B^{\bar{q}}_{r\bar{l}})g_{s\bar{j}} - Z_{\bar{r}}Z_i(g_{r\bar{j}}) + g^{s\bar{l}}g_{r\bar{q}}Z_{\bar{r}}(B^{\bar{q}}_{i\bar{l}})g_{s\bar{j}} - Z_{\bar{r}}(B^s_{ri})g_{s\bar{j}} \\
&\quad + Z_{\bar{j}}Z_i(g_{r\bar{r}}) - Z_{\bar{j}}(B^{\bar{q}}_{i\bar{r}})g_{r\bar{q}} - Z_{\bar{j}}Z_r(g_{i\bar{r}}) + Z_{\bar{j}}(B^{\bar{q}}_{r\bar{r}})g_{i\bar{q}} - Z_{\bar{j}}(B^s_{ir})g_{s\bar{r}} + \mathcal{O}(Z(g)),
\end{aligned}$$

where we used the formula for $Z_{\bar{j}}(w_i)$ as in Proposition 3.1.

We also compute with using that $[Z_r, Z_i] = B^s_{ri}Z_s + B^{\bar{s}}_{ri}Z_{\bar{s}}$,

$$\begin{aligned}
Z_{\bar{r}}(B^s_{ri})g_{s\bar{j}} &= g(\nabla_{\bar{r}}(B^s_{ri})Z_s, Z_{\bar{j}}) \\
&= g(\nabla_{\bar{r}}(B^s_{ri}Z_s) - B^s_{ri}\nabla_{\bar{r}}Z_s, Z_{\bar{j}}) \\
&= Z_{\bar{r}}g(B^s_{ri}Z_s, Z_{\bar{j}}) + \mathcal{O}(Z(g)) \\
&= Z_{\bar{r}}g([Z_r, Z_i] - B^{\bar{s}}_{ri}Z_{\bar{s}}, Z_{\bar{j}}) + \mathcal{O}(Z(g)) \\
&= Z_{\bar{r}}g((\Gamma^s_{ri} - \Gamma^s_{ir})Z_s, Z_{\bar{j}}) + \mathcal{O}(Z(g)) \\
&= Z_{\bar{r}}(\Gamma^s_{ri})g_{s\bar{j}} - Z_{\bar{r}}(\Gamma^s_{ir})g_{s\bar{j}} + \mathcal{O}(Z(g)),
\end{aligned}$$

where we recall that the mixed derivatives $\nabla_{\bar{r}}Z_s$ and $B^{\bar{s}}_{ri}$ do not depend on g. We calculate

$$\begin{aligned}
Z_{\bar{r}}(\Gamma^s_{ri})g_{s\bar{j}} &= Z_{\bar{r}}(g^{s\bar{l}}Z_r(g_{i\bar{l}}) - g^{s\bar{l}}g_{i\bar{q}}B^{\bar{q}}_{r\bar{l}})g_{s\bar{j}} \\
&= g^{s\bar{l}}Z_{\bar{r}}Z_r(g_{i\bar{l}})g_{s\bar{j}} + \mathcal{O}(Z(g)),
\end{aligned}$$

and also,

$$\begin{aligned}
Z_{\bar{r}}(\Gamma^s_{ir})g_{s\bar{j}} &= Z_{\bar{r}}(g^{s\bar{l}}Z_i(g_{r\bar{l}}) - g^{s\bar{l}}g_{r\bar{q}}B^{\bar{q}}_{i\bar{l}})g_{s\bar{j}} \\
&= g^{s\bar{l}}Z_{\bar{r}}Z_i(g_{r\bar{l}})g_{s\bar{j}} + \mathcal{O}(Z(g)).
\end{aligned}$$

Likewise, we compute

$$Z_{\bar{j}}(B^s_{ir})g_{s\bar{r}} = g(\nabla_{\bar{j}}(B^s_{ir})Z_s, Z_{\bar{r}})$$

$$\begin{aligned}
&= g(\nabla_{\bar{j}}(B^s_{ir} Z_s) - B^s_{ir}\nabla_{\bar{j}} Z_s, Z_{\bar{r}}) \\
&= Z_{\bar{j}} g(B^s_{ir} Z_s, Z_{\bar{r}}) + \mathcal{O}(Z(g)) \\
&= Z_{\bar{j}} g([Z_i, Z_r] - B^{\bar{s}}_{ir} Z_{\bar{s}}, Z_{\bar{j}}) + \mathcal{O}(Z(g)) \\
&= Z_{\bar{j}} g((\Gamma^s_{ir} - \Gamma^s_{ri}) Z_s, Z_{\bar{j}}) + \mathcal{O}(Z(g)) \\
&= Z_{\bar{j}}(\Gamma^s_{ir}) g_{s\bar{j}} - Z_{\bar{j}}(\Gamma^s_{ri}) g_{s\bar{j}} + \mathcal{O}(Z(g))\,,
\end{aligned}$$

$$\begin{aligned}
Z_{\bar{j}}(\Gamma^s_{ir}) g_{s\bar{j}} &= Z_{\bar{j}}(g^{s\bar{l}} Z_i(g_{r\bar{l}}) - g^{s\bar{l}} g_{r\bar{q}} B^{\bar{q}}_{i\bar{l}}) g_{s\bar{j}} \\
&= g^{s\bar{l}} Z_{\bar{j}} Z_i(g_{r\bar{l}}) g_{s\bar{j}} + \mathcal{O}(Z(g))\,,
\end{aligned}$$

and also,

$$\begin{aligned}
Z_{\bar{j}}(\Gamma^s_{ri}) g_{s\bar{j}} &= Z_{\bar{j}}(g^{s\bar{l}} Z_r(g_{i\bar{l}}) - g^{s\bar{l}} g_{i\bar{q}} B^{\bar{q}}_{r\bar{l}}) g_{s\bar{j}} \\
&= g^{s\bar{l}} Z_{\bar{j}} Z_r(g_{i\bar{l}}) g_{s\bar{j}} + \mathcal{O}(Z(g))\,.
\end{aligned}$$

Combining these, we obtain

$$-\bar{Z}(T')_{i\bar{j}} = -Z_{\bar{r}}(B^{\bar{q}}_{r\bar{j}}) g_{i\bar{q}} + Z_{\bar{r}}(B^{\bar{q}}_{i\bar{j}}) g_{r\bar{q}} - Z_{\bar{j}}(B^{\bar{q}}_{i\bar{r}}) g_{r\bar{q}} + Z_{\bar{j}}(B^{\bar{q}}_{r\bar{r}}) g_{i\bar{q}} + \mathcal{O}(Z(g)).$$

Here note that $B^{\bar{q}}_{j\bar{b}}$, $B^{q}_{\bar{j}b}$'s do not depend on g, which depend only on J since the mixed derivatives $\nabla_j Z_{\bar{b}}$ do not depend on g.

Therefore, we conclude that $-\bar{Z}(T')$ involves only the first derivatives of g and the principal part of $\Psi(\omega)$ is determined by $S(\omega)$. We also have

$$S_{i\bar{j}} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}} = -g^{k\bar{l}} Z_{\bar{l}} Z_k(g_{i\bar{j}}) + \mathcal{O}(Z(g)) + \mathcal{O}(B)\,,$$

which implies that the operator Ψ is elliptic. □

Since the operator $\omega \mapsto \Psi(\omega)$ is strictly elliptic, the equation (AHCF) is strictly parabolic and the standard parabolic theory implies that (AHCF) has a unique solution for t sufficiently small. Hence we have proved the third main result.

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Present Address:
TOKYO METROPOLITAN UNIVERSITY,
1–1 MINAMI-OSAWA, HACHIOJI-SHI, TOKYO 192–0397, JAPAN.
e-mail: wander276lust@yahoo.co.jp