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Truncated Bernoulli-Carlitz and Truncated Cauchy-Carlitz Numbers

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Abstract. In this paper, we define the truncated Bernoulli-Carlitz numbers and the truncated Cauchy-Carlitz numbers as analogues of hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers, and as extensions of Bernoulli-Carlitz numbers and the Cauchy-Carlitz numbers. These numbers can be expressed explicitly in terms of incomplete Stirling-Carlitz numbers.

1. Introduction

For $N \ge 1$, hypergeometric Bernoulli numbers $B_{N,n}$ ([10, 11, 13]) are defined by the generating function

$$\frac{1}{{}_1F_1(1; N+1; x)} = \frac{x^N/N!}{e^x - \sum_{n=0}^{N-1} x^n/n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^n}{n!}, \tag{1}$$

where

$$ {}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!}$$

is the confluent hypergeometric function with $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \ge 1)$ and $(x)^{(0)} = 1$. When $N = 1$, $B_n = B_{1,n}$ are classical Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

In addition, hypergeometric Cauchy numbers $c_{N,n}$ (see [16]) are defined by

$$\frac{1}{{}_2F_1(1, N; N+1; -x)} = \frac{(-1)^{N-1} x^N/N}{\log(1+x) - \sum_{n=1}^{N-1} (-1)^{n-1} x^n/n} = \sum_{n=0}^{\infty} c_{N,n} \frac{x^n}{n!}, \tag{2}$$

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where

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^\infty \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!} $$

is the Gauss hypergeometric function. When $N=1$, $c_n=c_{1,n}$ are classical Cauchy numbers defined by

$$ \frac{x}{\log(1+x)}=\sum_{n=0}^\infty c_n\frac{x^n}{n!}\,. $$

On the other hand, L. Carlitz [1] introduced analogues of Bernoulli numbers for the rational function (finite) field $K=\mathbf{F}_r(T)$, which are called Bernoulli-Carlitz numbers now. Bernoulli-Carlitz numbers have been studied since then (e.g., see [2, 3, 5, 12, 21]). According to the notations by Goss [6], Bernoulli-Carlitz numbers are defined by

$$ \frac{x}{e_C(x)}=\sum_{n=0}^\infty \frac{BC_n}{\Pi(n)}x^n\,. \tag{3} $$

Here, $e_C(x)$ are the Carlitz exponential defined by

$$ e_C(x)=\sum_{i=0}^\infty \frac{x^{r^i}}{D_i}\,, \tag{4} $$

where $D_i=[i][i-1]^r\cdots[1]^{r^{i-1}}$ $(i\ge 1)$ with $D_0=1$, and $[i]=T^{r^i}-T$. The Carlitz factorial $\Pi(i)$ is defined by

$$ \Pi(i)=\prod_{j=0}^m D_j^{c_j} \tag{5} $$

for a non-negative integer i with r-ary expansion:

$$ i=\sum_{j=0}^m c_j r^j \quad (0\le c_j<r)\,. \tag{6} $$

As analogues of the classical Cauchy numbers c_n, Cauchy-Carlitz numbers CC_n ([14]) are introduced as

$$ \frac{x}{\log_C(x)}=\sum_{n=0}^\infty \frac{CC_n}{\Pi(n)}x^n\,. \tag{7} $$

Here, $\log_C(x)$ is the Carlitz logarithm defined by

$$\log_C(x) = \sum_{i=0}^{\infty}(-1)^i \frac{x^{r^i}}{L_i}\,, \tag{8}$$

where $L_i = [i][i-1]\cdots[1]$ $(i \ge 1)$ with $L_0 = 1$.

In [14], Bernoulli-Carlitz numbers and Cauchy-Carlitz numbers are expressed explicitly by using the Stirling-Carlitz numbers of the second kind and of the first kind, respectively. These properties are the extensions that Bernoulli numbers and Cauchy numbers are expressed explicitly by using the Stirling numbers of the second kind and of the first kind, respectively.

In this paper, we define the truncated Bernoulli-Carlitz numbers and the truncated Cauchy-Carlitz numbers as analogues of hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers, and as extensions of Bernoulli-Carlitz numbers and the Cauchy-Carlitz numbers. These numbers can be expressed explicitly in terms of incomplete Stirling-Carlitz numbers.

2. Preliminaries

For $N \ge 0$, define the truncated Bernoulli-Carlitz numbers $BC_{N,n}$ and the truncated Cauchy-Carlitz numbers $CC_{N,n}$ by

$$\frac{x^{r^N}/D_N}{e_C(x) - \sum_{i=0}^{N-1} x^{r^i}/D_i} = \sum_{n=0}^{\infty} \frac{BC_{N,n}}{\Pi(n)} x^n \tag{9}$$

and

$$\frac{(-1)^N x^{r^N}/L_N}{\log_C(x) - \sum_{i=0}^{N-1}(-1)^i x^{r^i}/L_i} = \sum_{n=0}^{\infty} \frac{CC_{N,n}}{\Pi(n)} x^n\,, \tag{10}$$

respectively. When $N = 0$, $BC_n = BC_{0,n}$ and $CC_n = CC_{0,n}$ are the original Bernoulli-Carlitz numbers and Cauchy-Carlitz numbers, respectively. As the concept of these definitions in (9) and (10) in function fields are the same as (1) and (2) in complex numbers, the numbers $BC_{N,n}$ and $CC_{N,n}$ could be called the hypergeometric Bernoulli-Carlitz numbers and the hypergeometric Cauchy-Carlitz numbers, respectively. However, the generating functions of (9) and (10) are not related to the existing Carlitz hypergeometric functions (e.g., see [15, 24]). In [20], the *truncated Euler polynomials* are introduced and studied in complex numbers.

3. Hasse-Teichmüller derivatives

Let $\mathbf{F}$ be a field (of any characteristic), $\mathbf{F}((z))$ the field of Laurent series in z, and $\mathbf{F}[[z]]$ the ring of formal power series. The Hasse-Teichmüller derivative $H^{(n)}$ of order n is defined by

$$H^{(n)}\left(\sum_{m=R}^{\infty} a_m z^m\right) = \sum_{m=R}^{\infty} a_m \binom{m}{n} z^{m-n}$$

for $\sum_{m=R}^{\infty} a_m z^m \in \mathbf{F}((z))$, where R is an integer and $a_m \in \mathbf{F}$ for any $m \geq R$.

The Hasse-Teichmüller derivatives satisfy the product rule [23], the quotient rule [7] and the chain rule [9]. One of the product rules can be described as follows.

LEMMA 1. *For $f_i \in \mathbf{F}[[z]]$ $(i = 1, \ldots, k)$ with $k \geq 2$ and for $n \geq 1$, we have*

$$H^{(n)}(f_1 \cdots f_k) = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} H^{(i_1)}(f_1) \cdots H^{(i_k)}(f_k)\,.$$

The quotient rules can be described as follows.

LEMMA 2. *For $f \in \mathbf{F}[[z]]\backslash\{0\}$ and $n \geq 1$, we have*

$$H^{(n)}\left(\frac{1}{f}\right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k = n}} H^{(i_1)}(f) \cdots H^{(i_k)}(f) \tag{11}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} H^{(i_1)}(f) \cdots H^{(i_k)}(f)\,. \tag{12}$$

By using the Hasse-Teichmüller derivative of order n, we shall obtain some explicit expressions of the hypergeometric Bernoulli-Carlitz numbers $BC_{N,n}$ and hypergeometric Cauchy numbers $CC_{N,n}$, respectively.

THEOREM 1. *For $n \geq 1$,*

$$BC_{N,n} = \Pi(n) \sum_{k=1}^{n} (-D_N)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ r^{N+i_1}+\cdots+r^{N+i_k} = n+kr^N}} \frac{1}{D_{N+i_1} \cdots D_{N+i_k}}\,.$$

REMARK 1. For $N \geq 1$, it is clear that $BC_{N,n} = 0$ if $r \nmid n$ or $r^N(r-1) > n$. When $N = 0$, we have

$$BC_n = \Pi(n) \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ r^{i_1}+\cdots+r^{i_k} = n+k}} \frac{1}{D_{i_1} \cdots D_{i_k}}\,,$$

which is Theorem 4.2 in [12]. It is known that $BC_n = 0$ for $n \not\equiv 0 \pmod{r-1}$ (see [12, Corollary 4.3]).

PROOF OF THEOREM 1. Put

$$h := \frac{\sum_{i=N}^{\infty} \dfrac{x^{r^i}}{D_i}}{\dfrac{x^{r^N}}{D_N}} = \sum_{j=0}^{\infty} \frac{D_N}{D_{N+j}} x^{r^{N+j}-r^N} .$$

Note that

$$\begin{aligned} H^{(e)}(h)\Big|_{x=0} &= \sum_{j=0}^{\infty} \frac{D_N}{D_{N+j}} \binom{r^{N+j}-r^N}{e} x^{r^{N+j}-r^N-e}\Bigg|_{x=0} \\ &= \begin{cases} \dfrac{D_N}{D_{N+i}} & \text{if } e = r^{N+i} - r^N , \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, by using Lemma 2 (11), we have

$$\begin{aligned} \frac{BC_{N,n}}{\Pi(n)} &= H^{(n)}\left(\frac{1}{h}\right)\Bigg|_{x=0} \\ &= \sum_{k=1}^{n} \frac{(-1)^k}{h^{k+1}}\Bigg|_{x=0} \sum_{\substack{e_1,\dots,e_k \ge 1 \\ e_1+\dots+e_k=n}} H^{(e_1)}(h)\Big|_{x=0} \cdots H^{(e_k)}(h)\Big|_{x=0} \\ &= \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\dots,i_k \ge 1 \\ r^{N+i_1}+\dots+r^{N+i_k}=n+kr^N}} \frac{D_N}{D_{N+i_1}} \cdots \frac{D_N}{D_{N+i_k}} \\ &= \sum_{k=1}^{n} (-D_N)^k \sum_{\substack{i_1,\dots,i_k \ge 1 \\ r^{N+i_1}+\dots+r^{N+i_k}=n+kr^N}} \frac{1}{D_{N+i_1}\cdots D_{N+i_k}} . \end{aligned}$$

□

EXAMPLES. Let $r = 3$ and $N = 2$. Then $BC_{2,n} = 0$ if $18 \nmid n$. When $n = 18$, consider the set

$$S_k = \left\{(i_1, \dots, i_k) \mid i_1, \dots, i_k \ge 1,\ 3^{i_1+2} + \dots + 3^{i_k+2} = 18 + 9k\right\} .$$

Then $S_1 = \{(1)\}$, and S_k is empty when $k \ge 2$ because $3^{i_1+2} + 3^{i_2+2} \ge 54 > 36$. Hence, we obtain

$$BC_{2,18} = \Pi(18)(-D_2)\frac{1}{D_3} .$$

When $n = 36$, consider the set

$$S_k = \left\{(i_1, \dots, i_k) \mid i_1, \dots, i_k \ge 1,\ 3^{i_1+2} + \dots + 3^{i_k+2} = 36 + 9k\right\} .$$

Then, S_k ($k=1, k\ge 3$) are empty because $3^{i_1+2}+3^{i_2+2}+3^{i_3+2}\ge 81>63$. By $S_2=\{(1,1)\}$, we have

$$BC_{2,36}=\Pi(36)(-D_2)^2\frac{1}{D_3D_3}=\Pi(36)\frac{D_2^2}{D_3^2}\,.$$

When $n=72$, consider the set

$$S_k=\left\{(i_1,\dots,i_k)\mid i_1,\dots,i_k\ge 1,\ 3^{i_1+2}+\cdots+3^{i_k+2}=72+9k\right\}.$$

Since $S_1=\{(2)\}$, $S_4=\{(1,1,1,1)\}$ and S_k is empty for $k=2,3$ and $k\ge 5$, we have

$$BC_{2,72}=\Pi(72)\left(\frac{-D_2}{D_4}+\frac{D_2^4}{D_3D_3D_3D_3}\right).$$

In fact,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{BC_{2,n}}{\Pi(n)}x^n&=\frac{\dfrac{x^9}{D_2}}{\sum_{i=2}^{\infty}\dfrac{x^{3^i}}{D_i}}\\&=1-\frac{D_2}{D_3}x^{18}+\frac{D_2^2}{D_3^2}x^{36}-\frac{D_2^3}{D_3^3}x^{54}\\&\quad+\left(\frac{D_2^4}{D_3^4}-\frac{D_2}{D_4}\right)x^{72}-\left(\frac{D_2^5}{D_3^5}-\frac{2D_2^2}{D_3D_4}\right)x^{72}+\cdots.\end{aligned}$$

We can express the hypergeometric Bernoulli-Carlitz numbers in terms of the binomial coefficients too. By using Lemma 2 (12) instead of Lemma 2 (11) in the proof of Theorem 1, we obtain the following:

PROPOSITION 1. *For $n\ge 1$,*

$$BC_{N,n}=\Pi(n)\sum_{k=1}^{n}\binom{n+1}{k+1}(-D_N)^k\sum_{\substack{i_1,\dots,i_k\ge 0\\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^N}}\frac{1}{D_{N+i_1}\cdots D_{N+i_k}}\,.$$

REMARK 2. When $N=0$, we have

$$BC_n=\Pi(n)\sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^k\sum_{\substack{i_1,\dots,i_k\ge 0\\ r^{i_1}+\cdots+r^{i_k}=n+k}}\frac{1}{D_{i_1}\cdots D_{i_k}}\,,$$

which is Proposition 4.4 in [12].

EXAMPLE. Let $r = 3$ and $N = 2$. When $n = 18$, consider the set

$$S_k = \left\{(i_1, \dots, i_k) \mid i_1, \dots, i_k \geq 0,\ 3^{i_1+2} + \cdots + 3^{i_k+2} = 18 + 9k\right\}.$$

Since $S_1 = \{(1)\}$, $S_2 = \{(0, 1), (1, 0)\}$, $S_3 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$, $\dots$, $S_{18} = \{(\underbrace{0, \dots, 0}_{17}, 1), \dots, (1, \underbrace{0, \dots, 0}_{17})\}$, we have

$$\begin{aligned}
\frac{BC_{2,18}}{\Pi(18)} &= -\binom{19}{2}\frac{D_2}{D_3} + \binom{19}{3}\frac{2D_2^2}{D_2 D_3} - \binom{19}{4}\frac{3D_2^3}{D_2^2 D_3} + \cdots + \binom{19}{19}\frac{18D_2^{18}}{D_2^{17}D_3} \\
&= \frac{D_2}{D_3}\sum_{k=1}^{18}(-1)^k\binom{19}{k+1}k \\
&= \frac{D_2}{D_3}\left(\sum_{k=0}^{19}(-1)^{k-1}\binom{19}{k}(k-1) - 1\right) \\
&= \frac{D_2}{D_3}\left(\sum_{k=0}^{19}(-1)^{k-1}\binom{19}{k}k + \sum_{k=0}^{19}(-1)^k\binom{19}{k} - 1\right) \\
&= -\frac{D_2}{D_3}.
\end{aligned}$$

Next, we shall give an explicit formula for hypergeometric Cauchy-Carlitz numbers.

THEOREM 2. *For $n \geq 1$,*

$$CC_{N,n} = \Pi(n)\sum_{k=1}^{n}(-L_N)^k \sum_{\substack{i_1,\dots,i_k \geq 1 \\ r^{N+i_1}+\cdots+r^{N+i_k} = n + kr^N}} \frac{(-1)^{i_1+\cdots+i_k}}{L_{N+i_1}\cdots L_{N+i_k}}.$$

REMARK 3. It is clear that $CC_{N,n} = 0$ if $r \nmid n$ or $r^N(r-1) > n$. When $N = 1$, we have

$$CC_n = \Pi(n)\sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1,\dots,i_k \geq 1 \\ r^{i_1}+\cdots+r^{i_k} = n + k}} \frac{(-1)^{i_1+\cdots+i_k}}{L_{i_1}\cdots L_{i_k}},$$

which is Theorem 3 in [14].

PROOF OF THEOREM 2. Put

$$h := \frac{\sum_{i=N}^{\infty}(-1)^i \dfrac{x^{r^i}}{L_i}}{(-1)^N \dfrac{x^{r^N}}{L_N}} = \sum_{j=0}^{\infty}(-1)^j \frac{L_N}{L_{N+j}} x^{r^{N+j} - r^N}.$$

Note that

$$
\begin{aligned}
H^{(e)}(h)\Big|_{x=0} &= \sum_{j=0}^{\infty}(-1)^j \frac{L_N}{L_{N+j}}\binom{r^{N+j}-r^N}{e}x^{r^{N+j}-r^N-e}\Bigg|_{x=0} \\
&= \begin{cases} \dfrac{(-1)^i L_N}{L_{N+i}} & \text{if } e = r^{N+i}-r^N\,, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Hence, by using Lemma 2 (11), we have

$$
\begin{aligned}
\frac{CC_{N,n}}{\Pi(n)} &= H^{(n)}\left(\frac{1}{h}\right)\Bigg|_{x=0} \\
&= \sum_{k=1}^{n}\frac{(-1)^k}{h^{k+1}}\Bigg|_{x=0}\sum_{\substack{e_1,\dots,e_k\ge 1\\ e_1+\dots+e_k=n}} H^{(e_1)}(h)\Big|_{x=0}\cdots H^{(e_k)}(h)\Big|_{x=0} \\
&= \sum_{k=1}^{n}(-1)^k\sum_{\substack{i_1,\dots,i_k\ge 1\\ r^{N+i_1}+\dots+r^{N+i_k}=n+kr^N}}\frac{(-1)^{i_1}L_N}{L_{N+i_1}}\cdots\frac{(-1)^{i_k}L_N}{L_{N+i_k}} \\
&= \sum_{k=1}^{n}(-L_N)^k\sum_{\substack{i_1,\dots,i_k\ge 1\\ r^{N+i_1}+\dots+r^{N+i_k}=n+kr^N}}\frac{(-1)^{i_1+\dots+i_k}}{L_{N+i_1}\cdots L_{N+i_k}}\,.
\end{aligned}
$$

□

EXAMPLE. Let $r=3$ and $N=3$. Then $CC_{3,n}=0$ if $54\nmid n$. When $n=270$, consider the set

$$
S_k=\left\{(i_1,\dots,i_k)\mid i_1,\dots,i_k\ge 1,\ 3^{i_1+3}+\dots+3^{i_k+3}=270+27k\right\}.
$$

Then $S_2=\{(1,2),\ (2,1)\}$, $S_5=\{(1,1,1,1,1)\}$ and S_k is empty when $k=1,3,4$ and $k\ge 6$. Hence, we obtain

$$
\begin{aligned}
\frac{CC_{3,270}}{\Pi(270)} &= \left((-L_3)^2\frac{(-1)^3\cdot 2}{L_4L_5}+(-L_3)^5\frac{(-1)^5}{L_4^5}\right) \\
&= \frac{L_3^5}{L_4^5}-\frac{2L_3^2}{L_4L_5}\,.
\end{aligned}
$$

In fact,

$$
\sum_{n=0}^{\infty}\frac{CC_{3,n}}{\Pi(n)}x^n=\frac{-\dfrac{x^{27}}{L_3}}{\sum_{i=3}^{\infty}(-1)^i\dfrac{x^{3^i}}{L_i}}
$$

$$
\begin{aligned}
&= 1 + \frac{L_3}{L_4}x^{54} + \frac{L_3^2}{L_4^2}x^{108} + \frac{L_3^3}{L_4^3}x^{162} \\
&\quad + \left(\frac{L_3^5}{L_4^5} - \frac{2L_3^2}{L_4L_5}\right)x^{270} + \left(\frac{L_3^6}{L_4^6} - \frac{3L_3^3}{L_4^2L_5}\right)x^{324} + \cdots .
\end{aligned}
$$

We can express the hypergeometric Cauchy numbers in terms of the binomial coefficients too. In fact, by using Lemma 2 (12) instead of Lemma 2 (11) in the proof of Theorem 2, we obtain the following:

PROPOSITION 2. *For $n \ge 1$,*

$$
CC_{N,n} = \Pi(n)\sum_{k=1}^{n}\binom{n+1}{k+1}(-L_N)^k \sum_{\substack{i_1,\dots,i_k\ge 0\\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^N}} \frac{(-1)^{i_1+\cdots+i_k}}{L_{N+i_1}\cdots L_{N+i_k}} .
$$

4. Incomplete Stirling-Carlitz numbers

In [14], as analogues of the Stirling numbers of the first kind $\left[{n \atop k}\right]$ defined by

$$
\frac{\big(-\log(1-t)\big)^k}{k!} = \sum_{n=0}^{\infty}\left[{n \atop k}\right]\frac{t^n}{n!}, \tag{13}
$$

the Stirling-Carlitz numbers of the first kind $\left[{n \atop k}\right]_C$ were introduced by

$$
\frac{\big(\log_C(z)\big)^k}{\Pi(k)} = \sum_{n=0}^{\infty}\left[{n \atop k}\right]_C\frac{z^n}{\Pi(n)} . \tag{14}
$$

As analogues of the Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ defined by

$$
\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty}\left\{{n \atop k}\right\}\frac{t^n}{n!},
$$

the Stirling-Carlitz numbers of the second kind $\left\{{n \atop k}\right\}_C$ were introduced by

$$
\frac{\big(e_C(z)\big)^k}{\Pi(k)} = \sum_{n=0}^{\infty}\left\{{n \atop k}\right\}_C\frac{z^n}{\Pi(n)} . \tag{15}
$$

By the definition (14), we have

$$
\left[{n \atop 0}\right]_C = 0 \quad (n\ge 1), \qquad \left[{n \atop m}\right]_C = 0 \quad (n<m) \quad \text{and} \quad \left[{n \atop n}\right]_C = 1 \quad (n \ge 0) \tag{16}
$$

and

$$\left\{ {n \atop 0} \right\}_C = 0 \quad (n \geq 1), \quad \left\{ {n \atop m} \right\}_C = 0 \quad (n < m) \quad \text{and} \quad \left\{ {n \atop n} \right\}_C = 1 \quad (n \geq 0)\,. \tag{17}$$

On the other hand, in [4, 17, 18, 19], so-called incomplete Stirling numbers of the first kind and of the second kind were introduced as some generalizations of the classical Stirling numbers of the first kind and of the second kind. One of the incomplete Stirling numbers is *restricted Stirling number*, and another is *associated Stirling number*. Associated Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_{\geq m}$ are given by

$$\frac{\left(e^x - E_{m-1}(x)\right)^k}{k!} = \sum_{n=0}^\infty \left\{ {n \atop k} \right\}_{\geq m} \frac{x^n}{n!} \quad (m \geq 1)\,, \tag{18}$$

where

$$E_m(x) = \sum_{n=0}^m \frac{x^n}{n!}\,.$$

When $m = 1$, $\left\{ {n \atop k} \right\} = \left\{ {n \atop k} \right\}_{\geq 1}$ is the classical Stirling numbers of the second kind. Restricted Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_{\leq m}$ are given by

$$\frac{\left(E_m(x) - 1\right)^k}{k!} = \sum_{n=0}^\infty \left\{ {n \atop k} \right\}_{\leq m} \frac{x^n}{n!} \quad (m \geq 1)\,. \tag{19}$$

When $m \to \infty$, $\left\{ {n \atop k} \right\} = \left\{ {n \atop k} \right\}_{\leq \infty}$ is the classical Stirling numbers of the second kind.

Associated Stirling numbers of the first kind $\left[{n \atop k} \right]_{\geq m}$ are given by

$$\frac{\left(-\log(1-x) + F_{m-1}(-x)\right)^k}{k!} = \sum_{n=0}^\infty \left[{n \atop k} \right]_{\geq m} \frac{x^n}{n!} \quad (m \geq 1)\,, \tag{20}$$

where

$$F_m(t) = \sum_{k=1}^m (-1)^{k+1} \frac{t^k}{k}\,.$$

When $m = 1$, $\left[{n \atop k} \right] = \left[{n \atop k} \right]_{\geq 1}$ is the classical Stirling numbers of the first kind. Restricted Stirling numbers of the first kind $\left[{n \atop k} \right]_{\leq m}$ are given by

$$\frac{\left(-F_m(-x)\right)^k}{k!} = \sum_{n=0}^\infty \left[{n \atop k} \right]_{\leq m} \frac{x^n}{n!} \quad (m \geq 1)\,. \tag{21}$$

When $m \to \infty$, $\left[{n \atop k} \right] = \left[{n \atop k} \right]_{\leq \infty}$ is the classical Stirling numbers of the first kind.

Now, we introduce *associated Stirling-Carlitz numbers* and *restricted Stirling-Carlitz numbers*. The partial sum of the Carlitz exponential is denoted by

$$\mathcal{E}_m(x)=\sum_{i=0}^{m}\frac{x^{r^i}}{D_i}\,.$$

The associated Stirling-Carlitz numbers of the second kind $\left\{{n \atop k}\right\}_{C,\ge m}$ are defined by

$$\frac{\bigl(e_C(z)-\mathcal{E}_{m-1}(z)\bigr)^k}{\Pi(k)}=\sum_{n=0}^{\infty}\left\{{n \atop k}\right\}_{C,\ge m}\frac{z^n}{\Pi(n)}\,. \tag{22}$$

The restricted Stirling-Carlitz numbers of the second kind $\left\{{n \atop k}\right\}_{C,\le m}$ are defined by

$$\frac{\bigl(\mathcal{E}_m(z)\bigr)^k}{\Pi(k)}=\sum_{n=0}^{\infty}\left\{{n \atop k}\right\}_{C,\le m}\frac{z^n}{\Pi(n)}\,. \tag{23}$$

When $m=0$ in (22) or $m\to\infty$ in (23), $\left\{{n \atop k}\right\}_{C}=\left\{{n \atop k}\right\}_{C,\ge 0}=\left\{{n \atop k}\right\}_{C,\le\infty}$ is the original Stirling-Carlitz number of the second kind. The partial sum of the Carlitz logarithm is denoted by

$$\mathcal{F}_m(x)=\sum_{i=0}^{m}(-1)^i\frac{x^{r^i}}{L_i}\,.$$

The associated Stirling-Carlitz numbers of the first kind $\left[{n \atop k}\right]_{C,\ge m}$ are defined by

$$\frac{\bigl(\log_C(z)-\mathcal{F}_{m-1}(z)\bigr)^k}{\Pi(k)}=\sum_{n=0}^{\infty}\left[{n \atop k}\right]_{C,\ge m}\frac{z^n}{\Pi(n)}\,. \tag{24}$$

The restricted Stirling-Carlitz numbers of the first kind $\left[{n \atop k}\right]_{C,\le m}$ are defined by

$$\frac{\bigl(\mathcal{F}_m(z)\bigr)^k}{\Pi(k)}=\sum_{n=0}^{\infty}\left[{n \atop k}\right]_{C,\le m}\frac{z^n}{\Pi(n)}\,. \tag{25}$$

When $m=0$ in (24) or $m\to\infty$ in (25), $\left[{n \atop k}\right]_{C}=\left[{n \atop k}\right]_{C,\ge 0}=\left[{n \atop k}\right]_{C,\le\infty}$ is the original Stirling-Carlitz number of the first kind.

Due to associated Stirling-Carlitz numbers of the second kind in (22), we can obtain a more explicit expression of hypergeometric Bernoulli-Carlitz numbers, expressed in Theorem 1 or Proposition 1.

THEOREM 3. *For $N\ge 1$ and $n\ge 1$, we have*

$$BC_{N,n}=\Pi(n)\sum_{k=1}^{n}\binom{n+1}{k+1}\frac{(-D_N)^k\Pi(k)}{\Pi(n+kr^N)}\left\{{n+kr^N \atop k}\right\}_{C,\ge N}\,.$$

PROOF. From (22), we have

$$\left(\sum_{j=0}^\infty \frac{x^{r^{N+j}-r^N}}{D_{N+j}}\right)^k = \left(\frac{e_C(x)-\mathcal{E}_{N-1}(x)}{x^{r^N}}\right)^k$$
$$= \sum_{n=k}^\infty \frac{\Pi(k)}{\Pi(n)} \begin{Bmatrix} n \\ k \end{Bmatrix}_{C,\ge N} x^{n-kr^N}$$
$$= \sum_{n=-(r^N-1)k}^\infty \frac{\Pi(k)}{\Pi(n+kr^N)} \begin{Bmatrix} n+kr^N \\ k \end{Bmatrix}_{C,\ge N} x^n .$$

Notice that

$$H^{(e)}\left(\frac{e_C(x)-\mathcal{E}_{N-1}(x)}{x^{r^N}}\right)\Bigg|_{x=0} = \sum_{j=0}^\infty \frac{1}{D_{N+j}}\binom{r^{N+j}-r^N-e}{e} x^{r^{N+j}-r^N-e}\Bigg|_{x=0}$$
$$= \begin{cases} \frac{1}{D_{N+i}} & \text{if } r^{N+i}-r^N=e\,, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 1 with

$$f_1(t)=\cdots=f_k(t)=\frac{e_C(x)-\mathcal{E}_{N-1}(x)}{x^{r^N}}\,,$$

we get

$$\frac{\Pi(k)}{\Pi(n+kr^N)} \begin{Bmatrix} n+kr^N \\ k \end{Bmatrix}_{C,\ge N} = \sum_{\substack{i_1,\dots,i_k\ge 0 \\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^N}} \frac{1}{D_{N+i_1}\cdots D_{N+i_k}}\,. \tag{26}$$

Together with Proposition 1, we can get the desired result. □

EXAMPLE. Let $r=3$, $N=2$ and $n=18$. Comparing the coefficient of x^n on both sides of

$$\sum_{n=0}^\infty \frac{\Pi(k)}{\Pi(n)} \begin{Bmatrix} n \\ k \end{Bmatrix}_{C,\ge 2} x^n = \left(\frac{x^9}{D_2}+\frac{x^{27}}{D_3}+\frac{x^{81}}{D_4}+\cdots\right)^k ,$$

for $k=1,2,\dots,18$, we have

$$\frac{\Pi(k)}{\Pi(18+9k)} \begin{Bmatrix} 18+9k \\ k \end{Bmatrix}_{C,\ge 2} = \frac{k}{D_2^{k-1}D_3}\,.$$

Hence,

$$\begin{aligned}\frac{BC_{2,18}}{\Pi(18)} &= \sum_{k=1}^{18}\binom{19}{k+1}(-D_2)^k\frac{k}{D_2^{k-1}D_3}\\ &= \frac{D_2}{D_3}\sum_{k=1}^{18}(-1)^k\binom{19}{k+1}k = -\frac{D_2}{D_3}.\end{aligned}$$

Bernoulli-Carlitz numbers can be expressed in term of the Stirling-Carlitz numbers of the second kind:

$$BC_n = \sum_{j=0}^{\infty}\frac{(-1)^j D_j}{L_j^2}\left\{ {n \atop r^j - 1} \right\}_C$$

([14, Theorem 2]). When $N = 0$, Theorem 3 is reduced to a different expression of Bernoulli-Carlitz numbers in terms of the Stirling-Carlitz numbers of the second kind.

COROLLARY 1. *For $n \geq 1$, we have*

$$BC_n = \Pi(n)\sum_{k=1}^{n}\binom{n+1}{k+1}\frac{(-1)^k\Pi(k)}{\Pi(n+k)}\left\{ {n+k \atop k} \right\}_C .$$

REMARK 4. This is an analogue of

$$B_n = \sum_{k=1}^{n}\frac{(-1)^k\binom{n+1}{k+1}}{\binom{n+k}{k}}\left\{ {n+k \atop k} \right\},$$

which is a simple formula appeared in [8, 22].

Similarly, due to associated Stirling-Carlitz numbers of the first kind in (24), we can obtain a more explicit expression of hypergeometric Cauchy-Carlitz numbers, expressed in Theorem 2 or Proposition 2.

THEOREM 4. *For $N \geq 1$ and $n \geq 1$, we have*

$$CC_{N,n} = \Pi(n)\sum_{k=1}^{n}\binom{n+1}{k+1}\frac{(-1)^{Nk}(-L_N)^k\Pi(k)}{\Pi(n+kr^N)}\left[{n+kr^N \atop k} \right]_{C,\geq N} .$$

PROOF. From (24), we have

$$\begin{aligned}\left(\sum_{j=0}^{\infty}\frac{(-1)^{N+j}x^{r^{N+j}-r^N}}{L_{N+j}}\right)^k &= \left(\frac{\log_C(x)-\mathcal{F}_{N-1}(x)}{x^{r^N}}\right)^k\\ &= \sum_{n=k}^{\infty}\frac{\Pi(k)}{\Pi(n)}\left[{n \atop k} \right]_{C,\geq N} x^{n-kr^N}\end{aligned}$$

$$= \sum_{n=-(r^N-1)k}^{\infty} \frac{\Pi(k)}{\Pi(n+kr^N)} \begin{bmatrix} n+kr^N \\ k \end{bmatrix}_{C,\ge N} x^n .$$

Notice that

$$H^{(e)}\left(\frac{\log_C(x)-\mathcal{F}_{N-1}(x)}{x^{r^N}}\right)\Bigg|_{x=0} = \sum_{j=0}^{\infty} \frac{(-1)^{N+j}}{L_{N+j}} \binom{r^{N+j}-r^N-e}{e} x^{r^{N+j}-r^N-e}\Bigg|_{x=0}$$
$$= \begin{cases} \frac{(-1)^{N+i}}{L_{N+i}} & \text{if } r^{N+i}-r^N=e\,, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 1 with

$$f_1(t)=\cdots=f_k(t)=\frac{\log_C(x)-\mathcal{F}_{N-1}(x)}{x^{r^N}}\,,$$

we get

$$\frac{\Pi(k)}{\Pi(n+kr^N)} \begin{bmatrix} n+kr^N \\ k \end{bmatrix}_{C,\ge N} = \sum_{\substack{i_1,\dots,i_k\ge 0 \\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^N}} \frac{(-1)^{Nk+i_1+\cdots+i_k}}{L_{N+i_1}\cdots L_{N+i_k}}\,. \tag{27}$$

Together with Proposition 2, we can get the desired result. $\square$

EXAMPLE. Let $r=3$, $N=3$ and $n=270$. Comparing the coefficient of x^n on both sides of

$$\sum_{n=0}^{\infty} \frac{\Pi(k)}{\Pi(n)} \begin{bmatrix} n \\ k \end{bmatrix}_{C,\ge 3} x^n = \left(-\frac{x^{27}}{L_3}+\frac{x^{81}}{L_4}-\frac{x^{243}}{L_5}+\frac{x^{729}}{L_6}-\cdots\right)^k ,$$

for $k=1,2,3,4$, we have

$$\frac{\Pi(k)}{\Pi(270+27k)} \begin{bmatrix} 270+27k \\ k \end{bmatrix}_{C,\ge 3} = (-1)^{k-1}\frac{k(k-1)}{L_3^{k-2}L_4L_5}\,.$$

and for $k=5,6,\dots,270$, we have

$$\frac{\Pi(k)}{\Pi(270+27k)} \begin{bmatrix} 270+27k \\ k \end{bmatrix}_{C,\ge 3} = (-1)^{k-1}\frac{\binom{k}{5}}{L_3^{k-5}L_4^5} + (-1)^{k-1}\frac{k(k-1)}{L_3^{k-2}L_4L_5}\,.$$

Therefore,

$$\frac{CC_{3,270}}{\Pi(270)} = \sum_{k=1}^{270} \binom{271}{k+1} (-1)^{3k}(-L_3)^k \frac{(-1)^{k-1}k(k-1)}{L_3^{k-2}L_4L_5}$$
$$+ \sum_{k=5}^{270} \binom{271}{k+1} (-1)^{3k}(-L_3)^k \frac{(-1)^{k-1}\binom{k}{5}}{L_3^{k-5}L_4^5}$$

$$
\begin{aligned}
&= \frac{L_3^2}{L_4 L_5} \sum_{k=1}^{270} (-1)^{k-1} k(k-1) \binom{271}{k+1} + \frac{L_3^5}{L_4^5} \sum_{k=5}^{270} (-1)^{k-1} \binom{271}{k+1} \binom{k}{5} \\
&= -\frac{2L_3^2}{L_4 L_5} + \frac{L_3^5}{L_4^5} .
\end{aligned}
$$

Cauchy-Carlitz numbers can be expressed in term of the Stirling-Carlitz numbers of the first kind:

$$
CC_n = \sum_{j=0}^{\infty} \frac{1}{L_j} \begin{bmatrix} n \\ r^j - 1 \end{bmatrix}_C
$$

([14, Theorem 1]). When $N = 0$, Theorem 4 is reduced to a different expression of Cauchy-Carlitz numbers in terms of the Stirling-Carlitz numbers of the first kind.

COROLLARY 2. *For $n \geq 1$, we have*

$$
CC_n = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k \Pi(k)}{\Pi(n+k)} \begin{bmatrix} n+k \\ k \end{bmatrix}_C .
$$

REMARK 5. This is an analogue of

$$
c_n = \sum_{k=1}^{n} \frac{(-1)^{n-k} \binom{n+1}{k+1}}{\binom{n+k}{k}} \begin{bmatrix} n+k \\ k \end{bmatrix} ,
$$

which is Proposition 2 in [14].

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