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Complex Interpolation of Certain Closed Subspaces of Generalized Morrey Spaces

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Abstract. In this paper, we give a description about the first and second complex interpolation between L^∞ and the generalized Morrey spaces. Our result can be viewed as a supplement of the complex interpolation of generalized Morrey spaces, discussed in [9]. We also give an explicit description of some closed subspaces of generalized Morrey spaces and their complex interpolation spaces.

1. Introduction

Based on the study of the solution of certain elliptic partial differential equations by C.B. Morrey in [15], many researchers studied the Morrey spaces. For $0 < q \le p < \infty$, the Morrey space $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^q_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}^p_q} := \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B(x,r))} < \infty \,. \tag{1.1}$$

Here, $B(x,r)$ denotes the ball centered at $x \in \mathbb{R}^n$ with radius r. Remark that, for $p = q$, we have the Morrey space $\mathcal{M}^p_q$ is equal to the Lebesgue space L^p. The function $r \in (0,\infty) \mapsto r^{n/p}$ in (1.1) can be generalized to a suitable function $\varphi : (0,\infty) \to (0,\infty)$ to define the generalized Morrey space $\mathcal{M}^\varphi_q = \mathcal{M}^\varphi_q(\mathbb{R}^n)$ whose norm is given by

$$\|f\|_{\mathcal{M}^\varphi_q} := \sup_{x\in\mathbb{R}^n, r>0} \frac{\varphi(r)}{|B(x,r)|^{1/q}} \|f\|_{L^q(B(x,r))} < \infty \,. \tag{1.2}$$

The space $\mathcal{M}^\varphi_q$ was introduced by Nakai in [16]. Here, we may assume that $\varphi \in \mathcal{G}_q$, that is, φ is increasing and $r \mapsto r^{-n/q}\varphi(r)$ is decreasing (see [16]). Remark that, when $\varphi(r) = r^{\frac{n}{p}}$ and $\psi(r) = 1$, we have $\mathcal{M}^\varphi_q = \mathcal{M}^p_q$ and $\mathcal{M}^\psi_q = L^\infty$ (see [17, Proposition 3.3]), respectively.

It is known that Morrey spaces do not have an interpolation property in general, as indicated in [3, 19]. However, there has been some progress in the interpolation theory of Morrey

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spaces. Let $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$ be the first and second Calderón complex interpolation spaces, respectively (see [1, 5]). It was shown by Cobos et al. in [7] that

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$$

whenever $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $1 \le q \le p < \infty$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.3}$$

Moreover, by adding the assumption $\frac{p_0}{q_0} = \frac{p_1}{q_1}$, Lu et al. [14, Theorem 1.2] showed the following description:

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}. \tag{1.4}$$

For the second complex interpolation space, Lemarié-Rieusset [13] proved that

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q \tag{1.5}$$

when one assumes (1.3) and $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Furthermore, the results in [13] and [14, Theorem 1.2] were extended to generalized Morrey spaces in [9]. Meanwhile, Burenkov and Nursultanov [4] obtained a real interpolation method for local Morrey spaces, and their results were extended to B^w_u setting by Nakai and Sobukawa [18]. The interpolation of Morrey-Campanato spaces and smoothness spaces by the complex method, the Peetre-Gagliardo method, and the $\pm$ method can be found in [26].

Recall the complex interpolation of Lebesgue spaces:

$$[L^{p_0}, L^{p_1}]_\theta = [L^{p_0}, L^{p_1}]^\theta = L^p \tag{1.6}$$

where $1 \le p_0 \le \infty$, $1 \le p_1 \le \infty$, and $1 \le p \le \infty$ satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Remark that, when p_0 and p_1 are finite, (1.6) is a special case of (1.4) and (1.5). Our aim is to give a supplement of (1.4), (1.5), and [9] which recovers (1.6) for the case $p_0 = \infty$. More precisely, one of our main results is stated as follows:

THEOREM 1.1. *Let $\theta \in (0,1)$, $1 \le q < \infty$, and $\varphi \in \mathcal{G}_q$. Then we have*

$$[L^\infty, \mathcal{M}^\varphi_q]_\theta = \left\{ f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \lim_{N\to\infty} \|f \chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0 \right\} \tag{1.7}$$

and

$$[L^\infty, \mathcal{M}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}. \tag{1.8}$$

When $1 \le q < p < \infty$, we know that $f(x) := |x|^{-n/p} \in \mathcal{M}^p_q$, and for any $R > 0$, we have

$$\|f - \chi_{B(0,R)} f\|_{\mathcal{M}^p_q} = \|f \chi_{B(0,R)}\|_{\mathcal{M}^p_q} = \|f\|_{\mathcal{M}^p_q}. \tag{1.9}$$

This shows the difficulty of approximating functions in the Morrey space $\mathcal{M}^p_q$ by compactly supported functions. Recently, the description of the closure in $\mathcal{M}^p_q$ of L^∞_c was given in [9, Lemma 7]. For the next discussion, we use the following notation:

DEFINITION 1.2. Let $1 \le q < \infty$, $\varphi \in \mathcal{G}_q$, and L^0_{c} be the set of compactly supported functions. The spaces $\widetilde{\mathcal{M}^\varphi_q}$, $\overset{*}{\mathcal{M}}{}^\varphi_q$, and $\overline{\mathcal{M}^\varphi_q}$ denote the closure in $\mathcal{M}^\varphi_q$ of L^∞_{c}, $L^0_{\mathrm{c}} \cap \mathcal{M}^\varphi_q$, and $L^\infty \cap \mathcal{M}^\varphi_q$, respectively. We also write $\widetilde{L^\infty}$ for the closure of L^∞_c in L^∞. If $\varphi(t) := t^{n/p}$, then we write $\widetilde{\mathcal{M}^p_q}$, $\overset{*}{\mathcal{M}}{}^p_q$, and $\overline{\mathcal{M}^p_q}$ for the corresponding closed subspaces of Morrey spaces.

Our results on the explicit description of $\widetilde{\mathcal{M}^\varphi_q}$ and $\overset{*}{\mathcal{M}}{}^\varphi_q$ are given as follows:

THEOREM 1.3. *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

$$\widetilde{\mathcal{M}^\varphi_q} = \left\{ f \in \mathcal{M}^\varphi_q : \lim_{R \to \infty} \|\chi_{\{|f|>R\} \cup (\mathbb{R}^n \setminus B(0,R))} f\|_{\mathcal{M}^\varphi_q} = 0 \right\} \tag{1.10}$$

and

$$\overset{*}{\mathcal{M}}{}^\varphi_q = \left\{ f \in \mathcal{M}^\varphi_q : \lim_{R \to \infty} \|\chi_{\mathbb{R}^n \setminus B(0,R)} f\|_{\mathcal{M}^\varphi_q} = 0 \right\}. \tag{1.11}$$

Note that, the identity (1.10) for the case $\inf \varphi = 0$ can be seen in [9, Lemma 15]. We also remark that another characterization of $\overset{*}{\mathcal{M}}{}^p_q$ was given by Yuan et al. in [26, Lemma 2.33]. Meanwhile, the description of the space $\overline{\mathcal{M}^p_q}$ and $\overline{\mathcal{M}^\varphi_q}$ was given in [6, Lemma 3.1] and [10, Lemma 2.6], respectively. We mention that we do not discuss the closed subspaces of Morrey spaces having smoothness property. Morrey spaces, which date back to [15], have significant progress from the point of smoothness. From the point of the function spaces having smoothness, it turned out that Morrey spaces can be realized with function spaces related to the Carleson measure; see [24] for an exhaustive account. We refer to [22, 23] for more recent surveys. We refer to [8, 11, 12, 25] for some recent approaches in this direction.

Related to the complex interpolation of closed subspaces of Morrey spaces, we state one of our main theorems:

THEOREM 1.4. *Suppose that $\theta \in (0,1)$, $q_0, q_1, q \in [1, \infty)$, $\varphi_0 \in \mathcal{G}_{q_0}$, $\varphi_1 \in \mathcal{G}_{q_1}$, and $\varphi \in \mathcal{G}_q$ satisfy*

$$q_0 \neq q_1, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \varphi = \varphi_0^{1-\theta} \varphi_1^\theta, \quad \text{and} \quad \varphi_0^{q_0} = \varphi_1^{q_1}. \tag{1.12}$$

Then we have the following characterizations:

$$[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta = [\mathcal{M}^{\varphi_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}]^\theta = \{ f \in \mathcal{M}^\varphi_q : \chi_{\{a \le |f| \le b\}} f \in \overset{*}{\mathcal{M}}{}^\varphi_q \text{ for all } 0 < a < b < \infty \}, \tag{1.13}$$

$$[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta = [\mathcal{M}^{\varphi_0}_{q_0}, \widetilde{\mathcal{M}^{\varphi_1}_{q_1}}]^\theta = \{ f \in \mathcal{M}^\varphi_q : \chi_{\{a \le |f| \le b\}} f \in \overset{*}{\mathcal{M}}{}^\varphi_q \text{ for all } 0 < a < b < \infty \}, \tag{1.14}$$

and

$$[\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^{\theta} = [\mathcal{M}^{\varphi_0}_{q_0}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}]^{\theta} = \mathcal{M}^{\varphi}_{q} . \tag{1.15}$$

Theorem 1.4 can be seen as a refinement of our previous results, namely Theorems 3 and 4 in [9] and the following theorem:

THEOREM 1.5 ([10, Corollary 1.10]). *Suppose that $\theta \in (0,1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define $\varphi := \varphi_0^{1-\theta}\varphi_1^{\theta}$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then*

$$[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \widetilde{\mathcal{M}^{\varphi_1}_{q_1}}]^{\theta} = [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}]^{\theta} = \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^{\varphi}_{q} : \chi_{\{a \le |f| \le b\}} f \in \widetilde{\mathcal{M}^{\varphi}_{q}}\} \tag{1.16}$$

and

$$[\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}]^{\theta} = \mathcal{M}^{\varphi}_{q} . \tag{1.17}$$

We also consider the complex interpolation between L^{∞} and each of the spaces $\widetilde{\mathcal{M}^{\varphi}_{q}}$, $\overset{*}{\mathcal{M}}{}^{\varphi}_{q}$, and $\overline{\mathcal{M}^{\varphi}_{q}}$. One of our results is the following theorem:

THEOREM 1.6. *Let $\theta \in (0,1)$, $1 \le q < \infty$, and $\varphi \in \mathcal{G}_q$. Then we have*

$$[L^{\infty}, \widetilde{\mathcal{M}^{\varphi}_{q}}]_{\theta} = [L^{\infty}, \overset{*}{\mathcal{M}}{}^{\varphi}_{q}]_{\theta} = \widetilde{\mathcal{M}^{\varphi^{\theta}}_{q/\theta}} . \tag{1.18}$$

Now, we outline the rest of this paper. We shall recall some notation and definition related to the complex interpolation spaces in Section 2. We also recall some known results about the interpolation of generalized Morrey spaces. The proof of Theorem 1.1 is given in Section 3. In Section 4, we give the proof of Theorem 1.3. The proof of Theorem 1.4 is given in Section 5. In the last section, we prove Theorem 1.6 and we also present other complex interpolation spaces between L^{∞} and each of the spaces $\widetilde{\mathcal{M}^{\varphi}_{q}}$, $\overset{*}{\mathcal{M}}{}^{\varphi}_{q}$, and $\overline{\mathcal{M}^{\varphi}_{q}}$.

2. Preliminaries

In this section, we recall the definition of the first and second complex interpolation methods and we also state some previous results which will be used in the proof of our main theorem. We begin with the first complex interpolation method.

DEFINITION 2.1 (Calderón's first complex interpolation space). Let $\overline{X} = (X_0, X_1)$ be a compatible couple of Banach spaces. Let $\overline{S} := \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$ and S be its interior.

1. Define $\mathcal{F}(X_0, X_1)$ as the set of all functions $F : \overline{S} \to X_0 + X_1$ such that
   (a) F is continuous on $\overline{S}$ and $\sup_{z \in \overline{S}} \|F(z)\|_{X_0+X_1} < \infty$,
   (b) F is holomorphic in S,

(c) the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

The space $\mathcal{F}(X_0, X_1)$ is equipped with the norm

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1}\right\}.$$

2. Let $\theta \in (0, 1)$. Define the complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) to be the set of all functions $f \in X_0 + X_1$ such that $f = F(\theta)$ for some $F \in \mathcal{F}(X_0, X_1)$. The norm on $[X_0, X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

The fact that $[X_0, X_1]_\theta$ is a Banach space can be seen in [5] and [1, Theorem 4.1.2]. We invoke the following useful lemmas:

LEMMA 2.2 ([5], [1, Theorem 4.2.2]). *Let $\theta \in (0, 1)$ and (X_0, X_1) be a compatible couple of Banach spaces. Then we have $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$.*

LEMMA 2.3 ([1, Lemma 4.3.2]). *Let $\theta \in (0, 1)$ and $F \in \mathcal{F}(X_0, X_1)$. Then we have*

$$\|F(\theta)\|_{[X_0,X_1]_\theta} \leq \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|F(it)\|_{X_0} P_0(\theta, t)\, dt\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \|F(1+it)\|_{X_1} P_1(\theta, t)\, dt\right)^{\theta} \tag{2.1}$$

where $P_0(\theta, t)$ and $P_1(\theta, t)$ are defined by

$$P_0(\theta, t) := \frac{\sin(\pi\theta)}{2(\cosh(\pi t) - \cos(\pi\theta))} \text{ and } P_1(\theta, t) := \frac{\sin(\pi\theta)}{2(\cosh(\pi t) + \cos(\pi\theta))}.$$

In order to obtain the explicit description of the first complex interpolation spaces, sometimes it is easier to calculate the Calderón product and then use the result of Sestakov in [21]. The definition of Calderón product and Sestakov's lemma are given as follows:

DEFINITION 2.4. Let $\overline{X} = (X_0, X_1)$ be a compatible couple of Banach lattices on $\mathbb{R}$ and $\theta \in (0, 1)$. The Calderón product ${X_0}^{1-\theta}{X_1}^{\theta}$ of X_0 and X_1 is defined by

$${X_0}^{1-\theta}{X_1}^{\theta} := \bigcup_{f_0\in X_0,\, f_1\in X_1} \left\{f : \mathbb{R}^n \to \mathbb{C} : |f(x)| \leq |f_0(x)|^{1-\theta}|f_1(x)|^{\theta}\right\}.$$

For $f \in {X_0}^{1-\theta}{X_1}^{\theta}$, define the norm

$$\|f\|_{{X_0}^{1-\theta}{X_1}^{\theta}} := \inf\left\{\|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^{\theta} : f_0 \in X_0,\, f_1 \in X_1,\, |f| \leq |f_0|^{1-\theta}|f_1|^{\theta}\right\}.$$

LEMMA 2.5. *For every $\theta \in (0, 1)$, we have $[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^{\theta}}$.*

Now, we recall the definition of the second complex interpolation method.

DEFINITION 2.6. Let X be a Banach space. Denote by $\mathrm{Lip}(\mathbb{R}, X)$ the set of all functions $f : \mathbb{R} \to X$ such that

$$\|f\|_{\mathrm{Lip}(\mathbb{R},X)} := \sup_{-\infty<s<t<\infty} \frac{\|f(t) - f(s)\|_X}{|t - s|}$$

is finite.

DEFINITION 2.7 ([1, 5](Calderón's second complex interpolation space)). Let (X_0, X_1) be a compatible couple of Banach spaces.

1. Denote by $\mathcal{G}(X_0, X_1)$ the set of all functions $G : \bar{S} \to X_0 + X_1$ such that:
   (a) G is continuous on $\bar{S}$ and $\sup_{z\in\bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$,
   (b) G is holomorphic in S,
   (c) the functions
   $$t \in \mathbb{R} \mapsto G(j + it) - G(j) \in X_j$$
   are Lipschitz continuous on $\mathbb{R}$ for $j = 0, 1$.

   The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm
   $$\|G\|_{\mathcal{G}(X_0,X_1)} := \max\left\{\|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \|G(1 + i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\right\}. \tag{2.2}$$
2. Let $\theta \in (0, 1)$. The second complex interpolation space $[X_0, X_1]^\theta$ with respect to (X_0, X_1) is defined to be the set of all $f \in X_0 + X_1$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(X_0, X_1)$. The norm on $[X_0, X_1]^\theta$ is defined by
   $$\|f\|_{[X_0,X_1]^\theta} := \inf\{\|G\|_{\mathcal{G}(X_0,X_1)} : f = G'(\theta) \text{ for some } G \in \mathcal{G}(X_0, X_1)\}.$$

Relation between the inclusion and the complex interpolation is given as follows:

LEMMA 2.8. *If $X_0 \subseteq Y_0$ and $X_1 \subseteq Y_1$ with continuous embedding, then*

$$[X_0, X_1]^\theta \subseteq [Y_0, Y_1]^\theta .$$

PROOF. Let $f \in [X_0, X_1]^\theta$. Then there exists $G \in \mathcal{G}(X_0, X_1)$ such that $f = G'(\theta)$. By using the following inequalities

$$\|x_0\|_{Y_0} \lesssim \|x_0\|_{X_0}, \quad \|x_1\|_{Y_1} \lesssim \|x_1\|_{X_1}, \quad \text{and} \quad \|x\|_{Y_0+Y_1} \lesssim \|x\|_{X_0+X_1},$$

for every $x_0 \in X_0$, $x_1 \in X_1$, and $x \in X_0 + X_1$, we can verify that $G \in \mathcal{G}(Y_0, Y_1)$. Hence, $f \in [Y_0, Y_1]^\theta$. □

The relation between the first and second complex interpolation functors is given in the following lemma:

LEMMA 2.9 ([10, Lemma 2.4]). *For* $G \in \mathcal{G}(X_0, X_1)$, $z \in \overline{S}$, *and* $k \in \mathbb{N}$, *define*

$$h_k(z) := \frac{G(z + 2^{-k}i) - G(z)}{2^{-k}i} .$$

Then we have $h_k(\theta) \in [X_0, X_1]_\theta$.

PROOF. We give a simplified proof of [10, Lemma 2.4]. The continuity and holomorphicity of h_k follows from the corresponding property of G. Since $t \in \mathbb{R} \mapsto G(j + it) \in X_j$ are Lipschitz-continuous for $j = 0, 1$, we see that $t \in \mathbb{R} \mapsto h_k(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$. Moreover,

$$\begin{aligned}
\|h_k(\theta)\|_{[X_0,X_1]_\theta} &\le \|h_k\|_{\mathcal{F}(X_0,X_1)} \\
&= \max_{j=0,1} \sup_{t\in\mathbb{R}} \left\| \frac{G(j + i(t + 2^{-k})) - G(j + it)}{2^{-k}i} \right\|_{X_j} \\
&\le \|G\|_{\mathcal{G}(X_0,X_1)} < \infty ,
\end{aligned}$$

as desired. □

We use the following density result in addition to Lemma 2.5:

LEMMA 2.10 ([2]). *Let* (X_0, X_1) *be a compatible couple and* $\theta \in (0, 1)$. *Then we have*

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{[X_0,X_1]^\theta} . \tag{2.3}$$

We invoke the previous result about the second complex interpolation of generalized Morrey spaces.

THEOREM 2.11 ([9, Lemmas 4 and 12 and Theorem 2]). *Suppose that* $\theta, q_0, q_1, q, \varphi_0, \varphi_1$, *and* φ *are defined in Theorem* 1.4. *For* $f \in \mathcal{M}^\varphi_q$ *and* $z \in \overline{S}$, *define*

$$F(z) := \operatorname{sgn}(f)|f|^{q\left(\frac{1-z}{q_0}+\frac{z}{q_1}\right)} \quad \text{and} \quad G(z) := (z-\theta)\int_0^1 F(\theta + (z-\theta)t)\, dt . \tag{2.4}$$

Then we have:

1. $|G(z)| \le (1 + |z|)\left(|f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}}\right)$ *for every* $z \in \overline{S}$;
2. $G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$;
3. $[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta = \mathcal{M}^\varphi_q$.

3. The proof of Theorem 1.1

To prove (1.7), we combine the Calderón product of L^∞ and $\mathcal{M}^\varphi_q$ with Lemma 2.5. Our description of the Calderón product of L^∞ and $\mathcal{M}^\varphi_q$ is given as follows:

LEMMA 3.1. *Let* $\theta \in (0,1)$, $1 \le q < \infty$, *and* $\varphi \in \mathcal{G}_q$. *Then we have*

$$(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta = \mathcal{M}_{q/\theta}^{\varphi^\theta}. \tag{3.1}$$

PROOF. For $f \in \mathcal{M}_{q/\theta}^{\varphi^\theta}$, define $f_0 := 1$ and $f_1 := |f|^{1/\theta}$. Since

$$|f_0|^{1-\theta}|f_1|^\theta = |f|, \quad \|f_0\|_{L^\infty} = 1, \quad \text{and} \quad \|f_1\|_{\mathcal{M}_q^\varphi} = \|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}}^{1/\theta} < \infty,$$

we have $f \in (L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta$ and $\|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta} \le \|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. Therefore, $\mathcal{M}_{q/\theta}^{\varphi^\theta} \subseteq (L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta$ with embedding constant 1.

Conversely, for $f \in (L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta$ and $\varepsilon > 0$, choose $f_0 \in L^\infty$ and $f_1 \in \mathcal{M}_q^\varphi$ such that

$$|f| \le |f_0|^{1-\theta}|f_1|^\theta \quad \text{and} \quad \|f_0\|_{L^\infty}^{1-\theta}\|f_1\|_{\mathcal{M}_q^\varphi}^\theta \le (1+\varepsilon)\|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta}. \tag{3.2}$$

Let $x \in \mathbb{R}^n$ and $r > 0$. As a consequence of (3.2), we have

$$\begin{aligned}\frac{\varphi(r)^\theta}{|B(x,r)|^{\frac{\theta}{q}}}\left(\int_{B(x,r)}|f(y)|^{q/\theta}\,dy\right)^{\frac{\theta}{q}} &\le \frac{\varphi(r)^\theta}{|B(x,r)|^{\frac{\theta}{q}}}\left(\int_{B(x,r)}|f_0(y)|^{\frac{q(1-\theta)}{\theta}}|f_1(y)|^q\,dy\right)^{\frac{\theta}{q}} \\ &\le \|f_0\|_{L^\infty}^{1-\theta}\|f_1\|_{\mathcal{M}_q^\varphi}^\theta \\ &\le (1+\varepsilon)\|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta},\end{aligned}$$

and hence, $f \in \mathcal{M}_{q/\theta}^{\varphi^\theta}$ with $\|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \le \|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta}$. Therefore, $(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta \subseteq \mathcal{M}_{q/\theta}^{\varphi^\theta}$. Thus, (3.1) holds. □

The proof of the first complex interpolation of L^∞ and $\mathcal{M}_q^\varphi$ is given as follows:

PROOF OF (1.7). We combine Lemmas 2.5 and 3.1 to obtain

$$[L^\infty, \mathcal{M}_q^\varphi]_\theta = \overline{L^\infty \cap \mathcal{M}_q^\varphi}^{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta} = \overline{L^\infty \cap \mathcal{M}_q^\varphi}^{\mathcal{M}_{q/\theta}^{\varphi^\theta}}. \tag{3.3}$$

Let $f \in \mathcal{M}_{q/\theta}^{\varphi^\theta}$ be such that

$$\lim_{N\to\infty}\|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} = 0. \tag{3.4}$$

For each $N \in \mathbb{N}$, define $f_N := f\chi_{\{\frac{1}{N}\le|f|\le N\}}$. Since $f_N \in L^\infty$,

$$\|f_N\|_{\mathcal{M}_q^\varphi} \le (1/N)^{1-\frac{1}{\theta}}\||f|^{1/\theta}\|_{\mathcal{M}_q^\varphi} = N^{\frac{1}{\theta}-1}\|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}}^{1/\theta} < \infty,$$

and

$$\|f - f_N\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = \|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $N \to \infty$, we see that $f \in \overline{L^\infty \cap \mathcal{M}^\varphi_q}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = [L^\infty, \mathcal{M}^\varphi_q]_\theta$.

Conversely, let $f \in [L^\infty, \mathcal{M}^\varphi_q]_\theta$. As a consequence of (3.3), for each $\varepsilon > 0$, there exists $g = g_\varepsilon \in L^\infty \cap \mathcal{M}^\varphi_q$ such that

$$\|f - g\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} < \frac{\varepsilon}{6}. \tag{3.5}$$

For each $N \in \mathbb{N}$, we have

$$\begin{aligned}\left|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\right| &\le \left|f\chi_{\{|f|<\frac{1}{N}\}}\right| + |f\chi_{\{|f|>N\}}| \\ &\le 2|f-g| + \left|g\chi_{\{|f|<\frac{1}{N}\}\cap\{|g|>\frac{2}{N}\}}\right| + \left|g\chi_{\{|g|\le\frac{2}{N}\}}\right| \\ &\quad + \left|f\chi_{\{|f|>N\}\cap\{|g|\le\frac{N}{2}\}}\right| + \left|g\chi_{\{|g|>\frac{N}{2}\}}\right|.\end{aligned} \tag{3.6}$$

Observe that, on the set $\{|f| < \frac{1}{N}\} \cap \{|g| > \frac{2}{N}\}$, we have

$$|g| \le |f-g| + |f| < |f-g| + \frac{1}{N} < |f-g| + \frac{|g|}{2}.$$

Therefore,

$$|g\chi_{\{|f|<\frac{1}{N}\}\cap\{|g|>\frac{2}{N}\}}| \le 2|f-g|. \tag{3.7}$$

Meanwhile, on the set $\{|f| > N\} \cap \{|g| \le \frac{N}{2}\}$, we have

$$|f| \le |f-g| + |g| \le |f-g| + \frac{N}{2} < |f-g| + \frac{|f|}{2},$$

and hence,

$$\left|f\chi_{\{|f|>N\}\cap\{|g|\le\frac{N}{2}\}}\right| \le 2|f-g|. \tag{3.8}$$

By combining (3.6)–(3.8), for

$$N > 2\max\left\{\|g\|_{L^\infty}, \left(\frac{1+\|g\|^\theta_{\mathcal{M}^\varphi_q}}{\varepsilon}\right)^{\frac{1}{1-\theta}}\right\}, \tag{3.9}$$

we have

$$\begin{aligned}|f\chi_{\{|f|<1/N\}\cup\{|f|>N\}}| &\le 6|f-g| + \left|g\chi_{\{|g|\le\frac{2}{N}\}}\right| + \left|g\chi_{\{|g|>\frac{N}{2}\}}\right| \\ &\le 6|f-g| + \left(\frac{2}{N}\right)^{1-\theta}|g|^\theta + \left|g\chi_{\{|g|>\frac{N}{2}\}}\right|\end{aligned}$$

$$\le 6|f-g| + \frac{\varepsilon}{1+\|g\|_{\mathcal{M}_q^{\varphi}}^{\theta}}|g|^{\theta}\,.$$

We combine the last inequality and (3.5) to obtain

$$\begin{aligned}\left\|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\right\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} &\le 6\|f-g\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} + \frac{\varepsilon}{1+\|g\|_{\mathcal{M}_q^{\varphi}}^{\theta}}\||g|^{\theta}\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} \\ &< \varepsilon + \frac{\varepsilon}{1+\|g\|_{\mathcal{M}_q^{\varphi}}^{\theta}}\|g\|_{\mathcal{M}_q^{\varphi}}^{\theta} < 2\varepsilon\,.\end{aligned}$$

This shows that $\lim_{N\to\infty}\|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} = 0.$ □

Our proof of (1.8) combines (1.7) and Lemma 2.9. We give the proof of the second complex interpolation of L^{∞} and $\mathcal{M}_q^{\varphi}$ as follows:

PROOF OF (1.8). Let $f \in [L^{\infty}, \mathcal{M}_q^{\varphi}]^{\theta}$ and $\varepsilon > 0$. Then there exists $G \in \mathcal{G}(L^{\infty}, \mathcal{M}_q^{\varphi})$ such that

$$G'(\theta) = f \quad \text{and} \quad \|G\|_{\mathcal{G}(L^{\infty},\mathcal{M}_q^{\varphi})} \le (1+\varepsilon)\|f\|_{[L^{\infty},\mathcal{M}_q^{\varphi}]^{\theta}}\,.$$

For every $z \in \overline{S}$ and $k \in \mathbb{N}$, define

$$H_k(z) := \frac{G(z+2^{-k}i) - G(z)}{2^{-k}i}\,.$$

By Lemma 2.9, we have $H_k(\theta) \in [L^{\infty}, \mathcal{M}_q^{\varphi}]_{\theta}$ with

$$\|H_k(\theta)\|_{[L^{\infty},\mathcal{M}_q^{\varphi}]_{\theta}} \le (1+\varepsilon)\|f\|_{[L^{\infty},\mathcal{M}_q^{\varphi}]^{\theta}}\,. \tag{3.10}$$

By combining (3.10) and (1.7), we have

$$\|H_k(\theta)\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} \le (1+\varepsilon)\|f\|_{[L^{\infty},\mathcal{M}_q^{\varphi}]^{\theta}}\,. \tag{3.11}$$

Since $\lim_{k\to\infty} H_k(\theta) = G'(\theta) = f$ in $L^{\infty} + \mathcal{M}_q^{\varphi}$, we can find a subsequence $\{H_{k_j}(\theta)\}_{j=1}^{\infty} \subseteq \{H_k(\theta)\}_{k=1}^{\infty}$ such that

$$\lim_{j\to\infty} H_{k_j}(\theta)(x) = f(x) \quad \text{a.e.}$$

By Fatou's lemma and (3.11), we get

$$\|f\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} \le \liminf_{j\to\infty}\|H_{k_j}(\theta)\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} \le (1+\varepsilon)\|f\|_{[L^{\infty},\mathcal{M}_q^{\varphi}]^{\theta}}\,.$$

Since $\varepsilon > 0$ is arbitrary, we have $\|f\|_{\mathcal{M}_{q/\theta}^{\varphi^{\theta}}} \le \|f\|_{[L^{\infty},\mathcal{M}_q^{\varphi}]^{\theta}}$.

Conversely, let us suppose that $f \in \mathcal{M}_{q/\theta}^{\varphi^{\theta}}$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}} \quad \text{and} \quad G(z) := (z-\theta)\int_0^1 F(\theta + (z-\theta)t)\,dt\,. \tag{3.12}$$

Let $F_0(z) := \chi_{\{|f|\le 1\}}F(z)$, $F_1(z) := F(z) - F_0(z)$, $G_0(z) := \chi_{\{|f|\le 1\}}G(z)$, and $G_1(z) := G(z) - G_0(z)$. Let $u \in \overline{S}$. Since $\mathrm{Re}(u) \in [0, 1]$, we have

$$|F_0(u)| = \chi_{\{|f|\le 1\}}|f|^{\frac{\mathrm{Re}(u)}{\theta}} \le 1 \quad \text{and} \quad |F_1(u)| = \chi_{\{|f|>1\}}|f|^{\frac{\mathrm{Re}(u)}{\theta}} \le |f|^{\frac{1}{\theta}}.$$

Consequently,

$$\|F(u)\|_{L^\infty+\mathcal{M}^\varphi_q} \le \|F_0(u)\|_{L^\infty} + \|F_1(u)\|_{\mathcal{M}^\varphi_q} \le 1 + \||f|^{1/\theta}\|_{\mathcal{M}^\varphi_q} = 1 + \|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}, \quad (3.13)$$

$$\|G_0(z)\|_{L^\infty} = \left\|\int_\theta^z F_0(u)\,du\right\|_{L^\infty} \le |z-\theta| \le (1+|z|) \quad (3.14)$$

and

$$\|G_1(z)\|_{\mathcal{M}^\varphi_q} = \left\|\int_\theta^z F_1(u)\,du\right\|_{\mathcal{M}^\varphi_q} \le |z-\theta|\||f|^{\frac{1}{\theta}}\|_{\mathcal{M}^\varphi_q} \le (1+|z|)\,\|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} < \infty. \quad (3.15)$$

This implies $G(z) \in L^\infty + \mathcal{M}^\varphi_q$ and

$$\sup_{z\in\overline{S}} \left\|\frac{G(z)}{1+|z|}\right\|_{L^\infty+\mathcal{M}^\varphi_q} \le 1 + \|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} < \infty. \quad (3.16)$$

Fix $0 < \varepsilon \ll 1$. Let $z \in S$ with $\varepsilon < \mathrm{Re}(z) < 1-\varepsilon$ and $w \in \mathbb{C}$ with $|w| < \frac{\varepsilon}{2}$. Since

$$G(z+w) - G(z) = \int_z^{z+w} F(u)\,du = \frac{F(z+w)-F(z)}{\log(|f|^{1/\theta})},$$

we have

$$\begin{aligned}
\left|\frac{G_0(z+w)-G_0(z)}{w} - F_0(z)\right| &= |F_0(z)|\left|\frac{\exp(w\log(|f|^{1/\theta}))-1}{w\log(|f|^{1/\theta})} - 1\right| \\
&= \chi_{\{|f|\le 1\}}|f|^{\frac{\mathrm{Re}(z)}{\theta}}\left|\frac{\exp(w\log(|f|^{1/\theta}))-1}{w\log(|f|^{1/\theta})} - 1\right| \\
&\le \chi_{\{|f|\le 1\}}|f|^{\frac{\varepsilon}{\theta}}\left|\frac{\exp(w\log(|f|^{1/\theta}))-1}{w\log(|f|^{1/\theta})} - 1\right| \\
&\le \sup_{0<t\le 1} t^\varepsilon\left|\frac{\exp(w\log t)-1}{w\log t} - 1\right|.
\end{aligned}$$

Observe that, for every $t \in (0, 1)$, we have

$$\begin{aligned}
t^\varepsilon\left|\frac{\exp(w\log t)-1}{w\log t} - 1\right| &= t^\varepsilon\left|\sum_{k=2}^\infty \frac{(w\log t)^{k-1}}{k!}\right| \\
&\le -t^\varepsilon|w|(\log t)\sum_{k=2}^\infty \frac{(-|w|\log t)^{k-2}}{(k-2)!}
\end{aligned}$$

$$
\begin{aligned}
&\le -t^{\varepsilon}|w|(\log t)\exp(-|w|\log t)\\
&\le -t^{\varepsilon}|w|(\log t)\exp\Big(-\frac{\varepsilon}{2}\log t\Big)\\
&= -t^{\frac{\varepsilon}{2}}(\log t)|w| \le \frac{2}{\varepsilon e}|w|\,.
\end{aligned}
$$

Consequently,

$$
\left\|\frac{G_0(z+w)-G_0(z)}{w}-F_0(z)\right\|_{L^\infty} \le \frac{2}{\varepsilon e}|w|\,. \tag{3.17}
$$

By a similar argument, we also have

$$
\left\|\frac{G_1(z+w)-G_1(z)}{w}-F_1(z)\right\|_{\mathcal{M}^\varphi_q} \le \frac{2}{\varepsilon e}|w|\||f|^{\frac{1}{\theta}}\|_{\mathcal{M}^\varphi_q} = \frac{2}{\varepsilon e}|w|\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}\,. \tag{3.18}
$$

We combine (3.17) and (3.18) to obtain

$$
\left\|\frac{G(z+w)-G(z)}{w}-F(z)\right\|_{L^\infty+\mathcal{M}^\varphi_q} \le \frac{2}{\varepsilon e}\left(1+\|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}\right)|w| \to 0 \tag{3.19}
$$

as $w \to 0$. According to (3.13) and (3.19), we have $G : S_\varepsilon \to L^\infty + \mathcal{M}^\varphi_q$ is a holomorphic function. Since ε is arbitrary, we conclude that $G : S \to L^\infty + \mathcal{M}^\varphi_q$ is holomorphic.

Note that, for $j = 0, 1$ and $t_1, t_2 \in \mathbb{R}$, we have

$$
G(j+it_2) - G(j+it_1) = i\int_{t_1}^{t_2} F(j+it)\,dt\,. \tag{3.20}
$$

By combining (3.20), $|F(it)| = 1$, and $|F(1+it)| = |f|^{\frac{1}{\theta}}$, we have

$$
\|G(it_2) - G(it_1)\|_{L^\infty} \le |t_2 - t_1|
$$

and

$$
\|G(1+it_2) - G(1+it_1)\|_{\mathcal{M}^\varphi_q} \le |t_2 - t_1|\|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}\,,
$$

which verify Lipschitz-continuity of the functions $t \in \mathbb{R} \mapsto G(it) - G(0) \in L^\infty$ and $t \in \mathbb{R} \mapsto G(1+it)-G(1) \in \mathcal{M}^\varphi_q$. In total, we have $G \in \mathcal{G}(L^\infty, \mathcal{M}^\varphi_q)$. Since $f = F(\theta) = G'(\theta)$, we conclude that $f \in [L^\infty, \mathcal{M}^\varphi_q]^\theta$ as desired. □

4. Description of the spaces $\widetilde{\mathcal{M}^\varphi_q}$, $\overset{*}{\mathcal{M}}{}^\varphi_q$, and $\overline{\mathcal{M}^\varphi_q}$

In this section, we give the proof of Theorem 1.3 and we also explain the relation between $\widetilde{\mathcal{M}^\varphi_q}$, $\overset{*}{\mathcal{M}}{}^\varphi_q$, and $\overline{\mathcal{M}^\varphi_q}$. Our proof utilizes the information about the level sets of the function $f \in \widetilde{\mathcal{M}^\varphi_q}$ and the function $g \in L^\infty_{\rm c} \cap \mathcal{M}^\varphi_q$ which approximate f. The proof of (1.10) is given as follows:

PROOF OF THEOREM 1.3 (1.10). Let $f \in \mathcal{M}^{\varphi}_q$ be such that

$$\lim_{R\to\infty} \|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}^{\varphi}_q} = 0\,.$$

For each $R > 0$, define $f_R := \chi_{\{|f|\le R\}\cap B(0,R)} f$. Since $|f_R| \le R$ and $\operatorname{supp}(f_R) \subseteq B(0,R)$, we have $f_R \in L^{\infty}_{\mathrm{c}}$. Since

$$\lim_{R\to\infty} \|f - f_R\|_{\mathcal{M}^{\varphi}_q} = \lim_{R\to\infty} \|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}^{\varphi}_q} = 0$$

and $f_R \in L^{\infty}_{\mathrm{c}}$, we see that $f \in \widetilde{\mathcal{M}^{\varphi}_q}$.

Conversely, let $f \in \widetilde{\mathcal{M}^{\varphi}_q}$ and $\varepsilon > 0$. Choose $g \in L^{\infty}_{\mathrm{c}}$ such that

$$\|f - g\|_{\mathcal{M}^{\varphi}_q} < \frac{\varepsilon}{2}\,.$$

Choose $R_\varepsilon > 0$ such that $R_\varepsilon \ge 2\|g\|_{L^\infty}$ and $\operatorname{supp}(g) \subseteq B(0,R_\varepsilon)$. For every $R > R_\varepsilon$, we have

$$\begin{aligned}
|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f| &\le |f-g| + |\chi_{\{|f|>R\}} g| + |\chi_{\mathbb{R}^n\setminus B(0,R)} g| \\
&\le |f-g| + \chi_{\{|f|>R\}} \frac{R}{2} \qquad (4.1)\\
&\le |f-g| + \chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} \frac{|f|}{2}\,. \qquad (4.2)
\end{aligned}$$

Therefore, for every $R > R_\varepsilon$, we have

$$|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f| \le 2|f-g|\,,$$

and hence

$$\|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}^{\varphi}_q} \le 2\|f-g\|_{\mathcal{M}^{\varphi}_q} < \varepsilon\,.$$

This shows that $\lim_{R\to\infty} \|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}^{\varphi}_q} = 0$. □

Now, we give the proof of Theorem 1.3 (1.11):

PROOF OF THEOREM 1.3 (1.11). Assume that $f \in \mathcal{M}^{\varphi}_q$ and that $\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}^{\varphi}_q} = 0$. For every $R > 0$, define $f_R := \chi_{B(0,R)} f$. Then $f_R \in L^0_{\mathrm{c}} \cap \mathcal{M}^{\varphi}_q$, and it follows that

$$\lim_{R\to\infty} \|f - f_R\|_{\mathcal{M}^{\varphi}_q} = 0\,,$$

so then $f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q$. Conversely, let $f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q$. Given $\varepsilon > 0$, there exists $g_\varepsilon \in L^0_{\mathrm{c}} \cap \mathcal{M}^{\varphi}_q$ such that

$$\|f - g_\varepsilon\|_{\mathcal{M}^{\varphi}_q} < \varepsilon\,. \qquad (4.3)$$

For any $R > 0$, we have

$$|\chi_{\mathbb{R}^n\setminus B(0,R)} f| \le |\chi_{\mathbb{R}^n\setminus B(0,R)} g_\varepsilon| + |\chi_{\mathbb{R}^n\setminus B(0,R)} (f - g_\varepsilon)| \le |\chi_{\mathbb{R}^n\setminus B(0,R)} g_\varepsilon| + |f - g_\varepsilon|\,.$$

Choose $R_\varepsilon > 0$ such that $\operatorname{supp}(g_\varepsilon) \subset B(0, R_\varepsilon)$. Then, for all $R > R_\varepsilon$, we have

$$|\chi_{\mathbb{R}^n \setminus B(0,R)} f| \le |f - g_\varepsilon| .$$

Consequently, for all $R > R_\varepsilon$, we have

$$\|\chi_{\mathbb{R}^n \setminus B(0,R)} f\|_{\mathcal{M}^\varphi_q} \le \|f - g_\varepsilon\|_{\mathcal{M}^\varphi_q} < \varepsilon .$$

This shows that $\lim_{R\to\infty} \|\chi_{\mathbb{R}^n \setminus B(0,R)} f\|_{\mathcal{M}^\varphi_q} = 0$. □

Next, we move on to the description of $\overline{\mathcal{M}^\varphi_q}$ given in [10, Lemma 2.6]. For the sake of completeness, we also give the proof here.

LEMMA 4.1 ([10, Lemma 2.6]). *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then*

$$\overline{\mathcal{M}^\varphi_q} = \left\{ f \in \mathcal{M}^\varphi_q : \lim_{R\to\infty} \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^\varphi_q} = 0 \right\}. \tag{4.4}$$

PROOF. Let $f \in \mathcal{M}^\varphi_q$ be such that $\lim_{R\to\infty} \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^\varphi_q} = 0$. Define $f_R := \chi_{\{|f|\le R\}} f$. Since $f_R \in L^\infty \cap \mathcal{M}^\varphi_q$ and

$$\|f - f_R\|_{\mathcal{M}^\varphi_q} = \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^\varphi_q} \to 0$$

as $R \to \infty$, we see that $f \in \overline{\mathcal{M}^\varphi_q}$.

Conversely, let $f \in \overline{\mathcal{M}^\varphi_q}$ and $\varepsilon > 0$. Choose $g \in L^\infty \cap \mathcal{M}^\varphi_q$ be such that

$$\|f - g\|_{\mathcal{M}^\varphi_q} < \frac{\varepsilon}{2} .$$

Let $R_\varepsilon := 2\|g\|_{L^\infty}$. Then, for every $R > R_\varepsilon$, we have

$$\begin{aligned} |\chi_{\{|f|>R\}} f| &\le |\chi_{\{|f|>R\}} (f - g)| + |\chi_{\{|f|>R\}} g| \\ &\le |f - g| + \chi_{\{|f|>R\}} \frac{R}{2} \\ &\le |f - g| + \chi_{\{|f|>R\}} \frac{|f|}{2} , \end{aligned}$$

so $|\chi_{\{|f|>R\}} f| \le 2|f - g|$. Therefore, for every $R > R_\varepsilon$, we have

$$\|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^\varphi_q} \le 2\|f - g\|_{\mathcal{M}^\varphi_q} < \varepsilon .$$

This shows that $\lim_{R\to\infty} \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^\varphi_q} = 0$, as desired. □

Now, we compare the space $\widetilde{\mathcal{M}^\varphi_q}$, $\overset{*}{\mathcal{M}}{}^\varphi_q$, and $\overline{\mathcal{M}^\varphi_q}$ when $\varphi(t) = t^{n/p}$ and $p > q$. By using (1.10), (1.11), and (4.4), we can verify that $\widetilde{\mathcal{M}^p_q} \subsetneq \overset{*}{\mathcal{M}}{}^p_q \subsetneq \mathcal{M}^p_q$ and $\overline{\mathcal{M}^p_q} \subsetneq \mathcal{M}^p_q$ as follows:

EXAMPLE 4.2. Let $1 \le q < p < \infty$. Define $f(x) := |x|^{-n/p}$, $g(x) := f(x)\chi_{\mathbb{R}^n \setminus B(0,1)}(x)$, and $h(x) := f(x)\chi_{B(0,1)}(x)$. Then $f \in \mathcal{M}^p_q \setminus (\overline{\mathcal{M}^p_q} \cup \overset{*}{\mathcal{M}}{}^p_q)$, $g \in \overline{\mathcal{M}^p_q} \setminus \overset{*}{\mathcal{M}}{}^p_q$, and $h \in \overset{*}{\mathcal{M}}{}^p_q \setminus \widetilde{\mathcal{M}^p_q}$.

Finally, we give a corollary of Theorem 1.3 and Lemma 4.1, that is, $\widetilde{\mathcal{M}_q^\varphi}$ is the intersection of $\overset{*}{\mathcal{M}}{}_q^\varphi$ and $\overline{\mathcal{M}_q^\varphi}$:

THEOREM 4.3. *Let* $1 \le q < \infty$ *and* $\varphi \in \mathcal{G}_q$. *Then,* $\widetilde{\mathcal{M}_q^\varphi} = \overset{*}{\mathcal{M}}{}_q^\varphi \cap \overline{\mathcal{M}_q^\varphi}$.

PROOF. Since $\widetilde{\mathcal{M}_q^\varphi} \subseteq \overset{*}{\mathcal{M}}{}_q^\varphi$ and $\widetilde{\mathcal{M}_q^\varphi} \subseteq \overline{\mathcal{M}_q^\varphi}$, we have $\widetilde{\mathcal{M}_q^\varphi} \subseteq \overset{*}{\mathcal{M}}{}_q^\varphi \cap \overline{\mathcal{M}_q^\varphi}$. Conversely, let $f \in \overset{*}{\mathcal{M}}{}_q^\varphi \cap \overline{\mathcal{M}_q^\varphi}$. Define $A_R := \{|f| > R\} \cup (\mathbb{R}^n \setminus B(0,R))$. Then

$$\|\chi_{A_R} f\|_{\mathcal{M}_q^\varphi} \le \|\chi_{\mathbb{R}^n \setminus B(0,R)} f\|_{\mathcal{M}_q^\varphi} + \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}_q^\varphi} . \tag{4.5}$$

Since $f \in \overset{*}{\mathcal{M}}{}_q^\varphi$ and $f \in \overline{\mathcal{M}_q^\varphi}$, by combining Theorem 1.3, Lemma 4.1, and (4.5), we have

$$\lim_{R\to\infty} \|\chi_{A_R} f\|_{\mathcal{M}_q^\varphi} = 0 , \tag{4.6}$$

and hence, $f \in \widetilde{\mathcal{M}_q^\varphi}$. This shows that $\overset{*}{\mathcal{M}}{}_q^\varphi \cap \overline{\mathcal{M}_q^\varphi} \subseteq \widetilde{\mathcal{M}_q^\varphi}$. □

5. The proof of Theorem 1.4

In this section, we prove Theorem 1.4. Our proof uses Lemma 2.3 and also the description of the space $\overset{*}{\mathcal{M}}{}_q^\varphi$. We begin with the lattice property of $\overset{*}{\mathcal{M}}{}_q^\varphi$:

LEMMA 5.1. *Let f and g belong to $\mathcal{M}_q^\varphi$ with $|f| \le |g|$. If $g \in \overset{*}{\mathcal{M}}{}_q^\varphi$, then $f \in \overset{*}{\mathcal{M}}{}_q^\varphi$.*

PROOF. Given $\varepsilon > 0$, there exists $g_\varepsilon \in L^0_{\rm c} \cap \mathcal{M}_q^\varphi$ such that $\|g - g_\varepsilon\|_{\mathcal{M}_q^\varphi} < \varepsilon$. Define $f_\varepsilon := \frac{f}{g} g_\varepsilon \chi_{\{g \ne 0\}}$. Then $f_\varepsilon \in L^0_{\rm c} \cap \mathcal{M}_q^\varphi$ and

$$\|f - f_\varepsilon\|_{\mathcal{M}_q^\varphi} \le \|g - g_\varepsilon\|_{\mathcal{M}_q^\varphi} < \varepsilon .$$

This shows that $f \in \overset{*}{\mathcal{M}}{}_q^\varphi$. □

REMARK 5.2. By the same argument, we also have the lattice property of $\widetilde{\mathcal{M}_q^\varphi}$.

Here and below, we use the same notation and assumptions as in Theorem 1.4. We use the following auxiliary lemma which is a special case of [10, Lemma 3.5]. For convenience, we supply its proof.

LEMMA 5.3 ([10, Lemma 3.5]). *Let* $0 < a < b < \infty$. *Then*

$$\overline{\overset{*}{\mathcal{M}}{}_q^\varphi}^{\overset{*}{\mathcal{M}}{}_{q_0}^{\varphi_0} + \mathcal{M}_{q_1}^{\varphi_1}} \cap \mathcal{M}_q^\varphi \subseteq \left\{ f \in \mathcal{M}_q^\varphi : \chi_{\{a \le |f| \le b\}} f \in \overset{*}{\mathcal{M}}{}_q^\varphi \right\} . \tag{5.1}$$

PROOF. Without loss of generality, we assume that $q_0 > q_1$. Let $\{f_j\}_{j=1}^\infty$ be a sequence in $\overset{*}{\mathcal{M}}{}^{\varphi}_{q}$ such that $f_j \to f$ in $\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$. For every $t \geq 0$, define

$$\Theta(t) := \begin{cases} 0, & 0 \leq t \leq \frac{a}{2} \text{ or } t > 2b\,, \\ 2t - a, & \frac{a}{2} \leq t \leq a\,, \\ a, & a \leq t \leq b\,, \\ -\frac{a}{b}t + 2a, & b < t \leq 2b\,. \end{cases}$$

Choose $\{g_j\}_{j=1}^{\infty} \subseteq \overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}$ and $\{h_j\}_{j=1}^{\infty} \subseteq \mathcal{M}^{\varphi_1}_{q_1}$ such that $f - f_j = g_j + h_j$, $\lim\limits_{j\to\infty} \|g_j\|_{\mathcal{M}^{\varphi_0}_{q_0}} = 0$, and $\lim\limits_{j\to\infty} \|h_j\|_{\mathcal{M}^{\varphi_1}_{q_1}} = 0$. Since $|\Theta(t) - \Theta(s)| \leq (2 + \frac{a}{b})|t - s|$ and $|\Theta(t) - \Theta(s)| \leq 2a$ for every $t, s \geq 0$, we have

$$\begin{aligned} |\chi_{\{a\leq|f|\leq b\}}\Theta(|f_j|) - \chi_{\{a\leq|f|\leq b\}}\Theta(|f|)| &\lesssim \chi_{\{a\leq|f|\leq b\}} \min(1, |f - f_j|) \\ &\leq \chi_{\{a\leq|f|\leq b\}}(\min(1, |g_j|) + \min(1, |h_j|))\,. \end{aligned} \tag{5.2}$$

By using the Hölder inequality, we have

$$\begin{aligned} \|\chi_{\{a\leq|f|\leq b\}} \min(1, |g_j|)\|_{\mathcal{M}^{\varphi}_{q}} &\leq \|\chi_{\{a\leq|f|\leq b\}} \min(1, |g_j|)\|^{1-\theta}_{\mathcal{M}^{\varphi_0}_{q_0}} \|\chi_{\{a\leq|f|\leq b\}} \min(1, |g_j|)\|^{\theta}_{\mathcal{M}^{\varphi_1}_{q_1}} \\ &\leq \frac{1}{a^{\frac{\theta q}{q_1}}} \|g_j\|^{1-\theta}_{\mathcal{M}^{\varphi_0}_{q_0}} \||f|^{q/q_1}\|^{\theta}_{\mathcal{M}^{\varphi_1}_{q_1}} \\ &= \frac{1}{a^{\frac{\theta q}{q_1}}} \|g_j\|^{1-\theta}_{\mathcal{M}^{\varphi_0}_{q_0}} \|f\|^{\theta q/q_1}_{\mathcal{M}^{\varphi}_{q}}\,. \end{aligned} \tag{5.3}$$

Meanwhile, by using $q > q_1$, we get

$$\|\min(1, |h_j|)\|_{\mathcal{M}^{\varphi}_{q}} \leq \||h_j|^{q_1/q}\|_{\mathcal{M}^{\varphi}_{q}} = \|h_j\|^{q_1/q}_{\mathcal{M}^{\varphi_1}_{q_1}}\,. \tag{5.4}$$

Combining (5.3) and (5.4), we get

$$\|\chi_{\{a\leq|f|\leq b\}}\Theta(|f_j|) - \chi_{\{a\leq|f|\leq b\}}\Theta(|f|)\|_{\mathcal{M}^{\varphi}_{q}} \lesssim \|g_j\|^{1-\theta}_{\mathcal{M}^{\varphi_0}_{q_0}} \|f\|^{\theta q/q_1}_{\mathcal{M}^{\varphi}_{q}} + \|h_j\|^{q_1/q}_{\mathcal{M}^{\varphi_1}_{q_1}}\,.$$

We combine the last inequality with $\lim\limits_{j\to\infty} g_j = 0$ in $\mathcal{M}^{\varphi_0}_{q_0}$ and $\lim\limits_{j\to\infty} h_j = 0$ in $\mathcal{M}^{\varphi_1}_{q_1}$ to have

$$\lim_{j\to\infty} \chi_{\{a\leq|f|\leq b\}}\Theta(|f_j|) = \chi_{\{a\leq|f|\leq b\}}\Theta(|f|) \tag{5.5}$$

in $\mathcal{M}^{\varphi}_{q}$. Since $\Theta(t) \leq t$, we have

$$\chi_{\{a\leq|f|\leq b\}}\Theta(|f_j|) \leq \chi_{\{a\leq|f|\leq b\}}|f_j| \leq |f_j|\,, \tag{5.6}$$

so $\chi_{\{a\leq|f|\leq b\}}\Theta(|f_j|) \in \overset{*}{\mathcal{M}}{}^{\varphi}_{q}$, and thus, $\chi_{\{a\leq|f|\leq b\}}\Theta(|f|) \in \overset{*}{\mathcal{M}}{}^{\varphi}_{q}$. Since $\chi_{\{a\leq|f|\leq b\}}|f| \leq \frac{b}{a}\chi_{\{a\leq|f|\leq b\}}\Theta(|f|)$, we have $\chi_{\{a\leq|f|\leq b\}} f \in \overset{*}{\mathcal{M}}{}^{\varphi}_{q}$. □

By combining the previous lemma, we are ready to prove Theorem 1.4.

PROOF OF THEOREM 1.4. Let $f \in \mathcal{M}^{\varphi}_q$ be such that $\chi_{\{a\le|f|\le b\}} f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q$. Since $\chi_{\{a\le|f|\le b\}} f \in L^\infty$ and $L^\infty \cap \overset{*}{\mathcal{M}}{}^{\varphi}_q \subseteq \widetilde{\mathcal{M}^{\varphi}_q}$, we have $\chi_{\{a\le|f|\le b\}} f \in \widetilde{\mathcal{M}^{\varphi}_q}$. By combining this with $[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}]^\theta \subseteq [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta$ and (1.16), we get the following inclusion

$$\bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}^{\varphi}_q : \chi_{\{a\le|f|\le b\}} f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q \right\} \subseteq \left[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}\right]^\theta . \tag{5.7}$$

Conversely, let $f \in [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta$. Since $[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta \subseteq [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta = \mathcal{M}^{\varphi}_q$, we have $f \in \mathcal{M}^{\varphi}_q$. Choose $G \in \mathcal{G}(\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$ such that $f = G'(\theta)$. For $z \in \overline{S}$ and $k \in \mathbb{N}$, define

$$h_k(z) := \frac{G(z + 2^{-k} i) - G(z)}{2^{-k} i} .$$

By virtue of Lemma 2.3 and $[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_0}_{q_0}]^\theta = \mathcal{M}^{\varphi}_q$, we have

$$\begin{aligned}
\|\chi_{\mathbb{R}^n\setminus B(0,R)} h_k(\theta)\|_{\mathcal{M}^{\varphi}_q} &\le \|\chi_{\mathbb{R}^n\setminus B(0,R)} h_k(\theta)\|_{[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_0}_{q_0}]_\theta} \\
&\le \left(\frac{1}{1-\theta} \int_{\mathbb{R}} \|\chi_{\mathbb{R}^n\setminus B(0,R)} h_k(it)\|_{\mathcal{M}^{\varphi_0}_{q_0}} P_0(\theta,t)\, dt \right)^{1-\theta} \\
&\quad \times \left(\frac{1}{\theta} \int_{\mathbb{R}} \|h_k(1+it)\|_{\mathcal{M}^{\varphi_1}_{q_1}} P_1(\theta,t)\, dt \right)^{\theta} .
\end{aligned}$$

Since $h_k(it) \in \overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}$, we see that $\lim\limits_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} h_k(it)\|_{\mathcal{M}^{\varphi_0}_{q_0}} = 0$. Hence, by the dominated convergence theorem, we have $\lim\limits_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} h_k(\theta)\|_{\mathcal{M}^{\varphi}_q} = 0$. Consequently, $h_k(\theta) \in \overset{*}{\mathcal{M}}{}^{\varphi}_q$. Since

$$\lim_{k\to\infty} \|f - h_k(\theta)\|_{\widetilde{\mathcal{M}^{\varphi_0}_{q_0}} + \mathcal{M}^{\varphi_1}_{q_1}} = \lim_{k\to\infty} \left\| G'(\theta) - \frac{G\left(\theta + 2^{-k} i\right) - G(\theta)}{2^{-k} i} \right\|_{\widetilde{\mathcal{M}^{\varphi_0}_{q_0}} + \mathcal{M}^{\varphi_1}_{q_1}} = 0 ,$$

we have $f \in \overline{\overset{*}{\mathcal{M}}{}^{\varphi}_q}^{\widetilde{\mathcal{M}^{\varphi_0}_{q_0}} + \mathcal{M}^{\varphi_1}_{q_1}}$. By virtue of Lemma 5.3, we have $\chi_{\{a\le|f|\le b\}} f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q$. From this fact and (5.7), we conclude that (1.13) holds.

Now, we move on to (1.14). We combine (1.13) and $[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta \subseteq [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta$ to obtain

$$\left[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}\right]^\theta \subseteq \bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}^{\varphi}_q : \chi_{\{a\le|f|\le b\}} f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q \right\} . \tag{5.8}$$

Conversely, let $f \in \mathcal{M}^{\varphi}_q$ be such that $\chi_{\{a\le|f|\le b\}} f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q$ for every $0 < a < b < \infty$. Since $\chi_{\{a\le|f|\le b\}} f \in L^\infty$ and $L^\infty \cap \overset{*}{\mathcal{M}}{}^{\varphi}_q \subseteq \widetilde{\mathcal{M}^{\varphi}_q}$, we have $\chi_{\{a\le|f|\le b\}} f \in \widetilde{\mathcal{M}^{\varphi}_q}$. As a consequence of (1.16) and $[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \widetilde{\mathcal{M}^{\varphi_1}_{q_1}}]^\theta \subseteq [\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta$, we have $f \in [\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta$. By combining this and (5.8), we arrive at (1.14).

Finally, we prove (1.15). Observe that the inclusions $[\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta \subseteq \mathcal{M}^{\varphi}_{q}$ and $[\mathcal{M}^{\varphi_0}_{q_0}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}]^\theta \subseteq \mathcal{M}^{\varphi}_{q}$ follow from Lemma 2.8 and $[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta = \mathcal{M}^{\varphi}_{q}$. Meanwhile, by combining (1.17) and $[\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}]^\theta \subseteq [\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta, [\mathcal{M}^{\varphi_0}_{q_0}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}]^\theta$, we get $\mathcal{M}^{\varphi}_{q} \subseteq [\overline{\mathcal{M}^{\varphi_0}_{q_0}}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta, [\mathcal{M}^{\varphi_0}_{q_0}, \overline{\mathcal{M}^{\varphi_1}_{q_1}}]^\theta$. Thus, we conclude that (1.15) holds. □

6. The description of the complex interpolation of L^∞ and closed subspaces of Morrey spaces

In this section, we give the proof of Theorem 1.6 and also give the description of other complex interpolation spaces between L^∞ and closed subspaces of $\mathcal{M}^{\varphi}_{q}$.

6.1. The proof of Theorem 1.6. In the proof of Theorem 1.6, we use the following lemma:

LEMMA 6.1. *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then, for every $f \in L^\infty \cap \widetilde{\mathcal{M}^{\varphi}_{q}}$, we have*

$$\|f\|_{[L^\infty, \widetilde{\mathcal{M}^{\varphi}_{q}}]^\theta} \simeq \|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}.$$

PROOF. Since $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_{q}}]^\theta \subseteq [L^\infty, \mathcal{M}^{\varphi}_{q}]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$, we have

$$\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \lesssim \|f\|_{[L^\infty, \widetilde{\mathcal{M}^{\varphi}_{q}}]^\theta}.$$

Assume that $\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 1$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}}, \ G(z) := \int_\theta^z F(u)\, du, \ \text{and } G_1(z) := \chi_{\{|f|>1\}} G(z).$$

For every $u \in \overline{S}$, we have

$$|\chi_{\{|f|>1\}} F(u)| = \chi_{\{|f|>1\}} |f|^{\frac{\mathrm{Re} u}{\theta}} \le |f|^{\frac{1}{\theta}} \le \|f\|_{L^\infty}^{\frac{1}{\theta}-1} |f|,$$

so $|G_1(z)| \le (1+|z|)\|f\|_{L^\infty}^{\frac{1-\theta}{\theta}} |f|$. Since $f \in \widetilde{\mathcal{M}^{\varphi}_{q}}$, we see that $G_1(z) \in \widetilde{\mathcal{M}^{\varphi}_{q}}$. Let $t_1, t_2 \in \mathbb{R}$. Since $f \in L^\infty \cap \widetilde{\mathcal{M}^{\varphi}_{q}}$ and

$$|G(1+it_2) - G(1+it_1)| = \left| i \int_{t_1}^{t_2} F(1+it)\, dt \right| \le |t_2 - t_1| |f|^{1/\theta} \le |t_2 - t_1| \|f\|_{L^\infty}^{\frac{1-\theta}{\theta}} |f|,$$

we have $G(1+it_2) - G(1+it_1) \in \widetilde{\mathcal{M}^{\varphi}_{q}}$. Combining $G_1(z) \in \widetilde{\mathcal{M}^{\varphi}_{q}}$, $G(1+it_2) - G(1+it_1) \in \widetilde{\mathcal{M}^{\varphi}_{q}}$, and $G \in \mathcal{G}(L^\infty, \mathcal{M}^{\varphi}_{q})$ we have $G \in \mathcal{G}(L^\infty, \widetilde{\mathcal{M}^{\varphi}_{q}})$. Moreover,

$$\|G\|_{\mathcal{G}(L^\infty, \widetilde{\mathcal{M}^{\varphi}_{q}})} = \max\left(\sup_{t<s} \left\| \frac{G(it) - G(is)}{t-s} \right\|_{L^\infty}, \sup_{t<s} \left\| \frac{G(1+it) - G(1+is)}{t-s} \right\|_{\widetilde{\mathcal{M}^{\varphi}_{q}}} \right)$$

$$= \max\left(\sup_{t<s}\left\|\frac{G(it)-G(is)}{t-s}\right\|_{L^\infty}, \sup_{t<s}\left\|\frac{G(1+it)-G(1+is)}{t-s}\right\|_{\mathcal{M}^\varphi_q}\right)$$
$$\le \max\big(1, \||f|^{1/\theta}\|_{\mathcal{M}^\varphi_q}\big)$$
$$= \max\Big(1, \|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}\Big) = 1 = \|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}.$$

Since $f = G'(\theta)$, we have

$$\|f\|_{[L^\infty,\mathcal{M}^\varphi_q]^\theta} \le \|G\|_{\mathcal{G}(L^\infty,\widetilde{\mathcal{M}^\varphi_q})} \le \|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}},$$

as desired. □

The proof of Theorem 1.6 is given as follows:

Proof of Theorem 1.6. For $f \in [L^\infty, \widetilde{\mathcal{M}^\varphi_q}]_\theta$, choose $F \in \mathcal{F}(L^\infty, \widetilde{\mathcal{M}^\varphi_q})$ such that $f = F(\theta)$. By combining Lemma 2.3 and Theorem 1.1, we have

$$\begin{aligned}\|\chi_{\mathbb{R}^n\setminus B(0,R)}f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} &\le \|\chi_{\mathbb{R}^n\setminus B(0,R)}F(\theta)\|_{[L^\infty,\mathcal{M}^\varphi_q]_\theta}\\ &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}}\|F(it)\|_{L^\infty}P_0(\theta,t)\,dt\right)^{1-\theta}\\ &\quad\times\left(\frac{1}{\theta}\int_{\mathbb{R}}\|\chi_{\mathbb{R}^n\setminus B(0,R)}F(1+it)\|_{\mathcal{M}^\varphi_q}P_1(\theta,t)\,dt\right)^{\theta}. \end{aligned} \tag{6.1}$$

From $F(1+it) \in \widetilde{\mathcal{M}^\varphi_q} \subseteq \overset{*}{\mathcal{M}}{}^\varphi_q$, we see that

$$\lim_{R\to\infty}\|\chi_{\mathbb{R}^n\setminus B(0,R)}F(1+it)\|_{\mathcal{M}^\varphi_q} = 0. \tag{6.2}$$

We combine (6.1), (6.2), and the dominated convergence theorem to obtain

$$\lim_{R\to\infty}\|\chi_{\mathbb{R}^n\setminus B(0,R)}f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0.$$

According to (1.11), we have $f \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. Since

$$\big[L^\infty, \widetilde{\mathcal{M}^\varphi_q}\big]_\theta \subseteq \big[L^\infty, \mathcal{M}^\varphi_q\big]_\theta = \overline{L^\infty \cap \mathcal{M}^\varphi_q}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}},$$

we see that $f \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta} \cap \overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$, as desired.

Now, let $f \in \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. We shall show that $f \in [L^\infty, \widetilde{\mathcal{M}^\varphi_q}]_\theta$. Since $L^\infty_c \subseteq \widetilde{\mathcal{M}^\varphi_q}$, we have $f \in \overline{L^\infty \cap \widetilde{\mathcal{M}^\varphi_q}}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Then, there exists a sequence $\{f_j\}_{j=1}^\infty \subseteq L^\infty \cap \widetilde{\mathcal{M}^\varphi_q}$ such that

$$\|f - f_j\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{j}. \tag{6.3}$$

Therefore, for every $j, k \in \mathbb{N}$ with $j > k$, we have

$$\|f_j - f_k\|_{[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta} \simeq \|f_j - f_k\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{j} + \frac{1}{k} < \frac{2}{k}\,,$$

so $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta$. By completeness of $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta$, there exists $g \in [L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta$ such that

$$\lim_{j\to\infty} \|f_j - g\|_{[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta} = 0\,. \tag{6.4}$$

Combining $\mathcal{M}^{\varphi^\theta}_{q/\theta} \subseteq L^\infty + \mathcal{M}^{\varphi}_q$, $[L^\infty, \mathcal{M}^{\varphi}_q]^\theta \subseteq L^\infty + \mathcal{M}^{\varphi}_q$, (6.3), and (6.4), we get $f = g \in \overline{L^\infty \cap \widetilde{\mathcal{M}^{\varphi}_q}}^{[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta}$. Finally, by using (2.3), we have $f \in [L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]_\theta$, as desired.

We shall show that $[L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Let $f \in [L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta$. By virtue of (1.7), we have $[L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta \subseteq [L^\infty, \mathcal{M}^{\varphi}_q]_\theta \subseteq \overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$, so $f \in \overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. By Lemma 2.2, for each $\varepsilon > 0$, there exists $g \in L^\infty \cap \overset{*}{\mathcal{M}}{}^{\varphi}_q$ such that

$$\|f - g\|_{[L^\infty, \mathcal{M}^{\varphi}_q]_\theta} < \varepsilon\,.$$

Since $L^\infty \cap \overset{*}{\mathcal{M}}{}^{\varphi}_q \subseteq \overline{\mathcal{M}^{\varphi}_q} \cap \overset{*}{\mathcal{M}}{}^{\varphi}_q = \widetilde{\mathcal{M}^{\varphi}_q}$, we have $g \in \widetilde{\mathcal{M}^{\varphi}_q}$. Therefore,

$$\|\chi_{\{|g|>R\}\cup(\mathbb{R}^n\setminus B(0,R))}g\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \|g\|_{L^\infty}^{1-\theta}\|\chi_{\{|g|>R\}\cup(\mathbb{R}^n\setminus B(0,R))}g\|_{\mathcal{M}^{\varphi}_q}^\theta \to 0$$

as $R \to \infty$. Consequently, $g \in \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Since $[L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta \subseteq [L^\infty, \mathcal{M}^{\varphi}_q]_\theta \subseteq \mathcal{M}^{\varphi^\theta}_{q/\theta}$, we have

$$\|f - g\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \lesssim \varepsilon\,.$$

This implies $f \in \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Thus, $[L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta \subseteq \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$.

Meanwhile, the inclusion $\widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq [L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta$ follows from $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]_\theta \subseteq [L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta$ and $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. □

6.2. The spaces $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]^\theta$, $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]_\theta$, $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta$, and $[L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]^\theta$. Next, we move on to the description of the spaces $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]^\theta$, $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]_\theta$, $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta$, and $[L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]^\theta$. First, we prove the following lemma:

LEMMA 6.2. *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

$$\overline{\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}}^{L^\infty + \mathcal{M}^{\varphi}_q} \cap \mathcal{M}^{\varphi^\theta}_{q/\theta} \subseteq \bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a \le |f| \le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta} \right\}.$$

PROOF. Let $\overline{\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}}^{L^\infty+\mathcal{M}^\varphi_q} \cap \mathcal{M}^{\varphi^\theta}_{q/\theta}$ and $0 < a < b < \infty$. For every $t \geq 0$, define

$$\psi_{a,b}(t) := \chi_{(\frac{a}{2},2b)}(t)(t-a/2)^2(t-2b)^2 .$$

Since

$$\chi_{\{a\le|f|\le b\}} \le \frac{1}{a}\chi_{\{a\le|f|\le b\}}|f| \le \frac{b}{a}\chi_{\{a\le|f|\le b\}} \le C_{a,b}\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f|) , \tag{6.5}$$

we only need to show that $\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f|) \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. Let $\{f_j\}_{j=1}^\infty$ be such that

$$\lim_{j\to\infty} \|f - f_j\|_{L^\infty+\mathcal{M}^\varphi_q} = 0 .$$

Choose $\{g_j\}_{j=1}^\infty \subseteq L^\infty$ and $\{h_j\}_{j=1}^\infty \subseteq \mathcal{M}^\varphi_q$ such that

$$f - f_j = g_j + h_j, \quad \lim_{j\to\infty} \|g_j\|_{L^\infty} = 0, \text{ and } \lim_{j\to\infty} \|h_j\|_{\mathcal{M}^\varphi_q} = 0 . \tag{6.6}$$

Since $\psi_{a,b} \in C^1(\mathbb{R})$ and $\psi_{a,b}, \psi'_{a,b} \in L^\infty(\mathbb{R})$, we have

$$\begin{aligned}|\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f|) - \chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f_j|)| &\lesssim \chi_{\{a\le|f|\le b\}} \min(1, |f-f_j|)\\ &\lesssim \chi_{\{a\le|f|\le b\}}(\min(1,|g_j|) + \min(1,|h_j|)) .\end{aligned}$$

Since $\min(1, |h_j|) \le |h_j|^\theta$, we have

$$\|\min(1,|h_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \||h_j|^\theta\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = \|h_j\|^\theta_{\mathcal{M}^\varphi_q} . \tag{6.7}$$

Meanwhile,

$$\|\chi_{\{a\le|f|\le b\}} \min(1,|g_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{a}\|g_j\|_{L^\infty}\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} . \tag{6.8}$$

By combining (6.7) and (6.8), we get

$$\|\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f|) - \chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{a}\|g_j\|_{L^\infty}\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} + \|h_j\|^\theta_{\mathcal{M}^\varphi_q} .$$

According to (6.6), we have $\lim_{j\to\infty} \|\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f|) - \chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0$. Since $\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f_j|) \lesssim |f_j|$, we have $\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f_j|) \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$, and hence, $\chi_{\{a\le|f|\le b\}}\psi_{a,b}(|f|) \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. As a consequence of (6.5), we conclude that $\chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. □

The description of the spaces $[L^\infty, \overline{\mathcal{M}^\varphi_q}]^\theta$, $[L^\infty, \overline{\mathcal{M}^\varphi_q}]_\theta$, $[L^\infty, \widetilde{\mathcal{M}^\varphi_q}]^\theta$, and $[L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q]^\theta$ is given as follows:

THEOREM 6.3. *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

(i) $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$,

(ii) $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]_\theta = \overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$,

(iii) $[L^\infty, \widetilde{\mathcal{M}^{\varphi}_q}]^\theta = [L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]^\theta = \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a \le |f| \le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\}$.

PROOF. Note that $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]^\theta \subseteq [L^\infty, \mathcal{M}^{\varphi}_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Now, let $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Define $F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}}$ and $G(z) := \int_\theta^z F(w)\, dw$. In the proof of (1.8), we know that $G \in \mathcal{G}(L^\infty, \mathcal{M}^{\varphi}_q)$, so it suffices to show that

1. $G_1(z) := \chi_{\{|f|>1\}} G(z) \in \overline{\mathcal{M}^{\varphi}_q}$ for every $z \in \overline{S}$;
2. $G(1+it) - G(1) \in \overline{\mathcal{M}^{\varphi}_q}$ for every $t \in \mathbb{R}$.

From the inequalities

$$|G_1(z)| = \left| \chi_{\{|f|>1\}} \frac{F(z) - F(\theta)}{\log |f|^{1/\theta}} \right| \lesssim \chi_{\{|f|>1\}} \frac{|f|^{1/\theta}}{\log |f|^{1/\theta}}$$

and $|G_1(z)| \le (1+|z|)|f|^{1/\theta}$, it follows that

$$\begin{aligned} \|\chi_{\{|G_1(z)|>R\}} G_1(z)\|_{\mathcal{M}^{\varphi}_q} &\lesssim \left\| \chi_{\{|f|^{1/\theta} > \frac{R}{1+|z|}\}} \frac{|f|^{1/\theta}}{\log |f|^{1/\theta}} \right\|_{\mathcal{M}^{\varphi}_q} \\ &\lesssim \frac{1}{\log(R/(1+|z|))} \|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0 \end{aligned}$$

as $R \to \infty$. Therefore, $G_1(z) \in \overline{\mathcal{M}^{\varphi}_q}$. Similarly, for every $t \in \mathbb{R}$, we have

$$\|\chi_{\{|G(1+it)-G(1)|>R\}}(G(1+it) - G(1))\|_{\mathcal{M}^{\varphi}_q} \lesssim \frac{\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}}{\log(R/(1+|t|))} \to 0\ (R \to \infty),$$

so $G(1+it) - G(1) \in \overline{\mathcal{M}^{\varphi}_q}$. Hence, $G \in \mathcal{G}(L^\infty, \overline{\mathcal{M}^{\varphi}_q})$ and $f = G'(\theta) \in [L^\infty, \overline{\mathcal{M}^{\varphi}_q}]^\theta$.

We now move on to the proof of (ii). Let $f \in [L^\infty, \overline{\mathcal{M}^{\varphi}_q}]_\theta$. By virtue of Lemma 2.2 and $[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]_\theta \subseteq \mathcal{M}^{\varphi^\theta}_{q/\theta}$, for each $\varepsilon > 0$, there exists $g \in L^\infty \cap \mathcal{M}^{\varphi}_q$ such that

$$\|f - g\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \lesssim \varepsilon. \tag{6.9}$$

By combining (6.9) and $\|g\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \|g\|^{1-\theta}_{L^\infty} \|g\|^\theta_{\mathcal{M}^{\varphi}_q} < \infty$, we see that $f \in \overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Meanwhile, by virtue of Theorem 6.3 (i) and (2.3), we have

$$\overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overline{L^\infty \cap \overline{\mathcal{M}^{\varphi}_q}}^{[L^\infty, \overline{\mathcal{M}^{\varphi}_q}]^\theta} = [L^\infty, \overline{\mathcal{M}^{\varphi}_q}]_\theta,$$

as desired.

Finally, let us prove (iii). Let $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$ be such that $\chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$ for every $0 < a < b < \infty$. Since

$$\|\chi_{\mathbb{R}^n\setminus B(0,R)}\chi_{\{a\le|f|\le b\}}\|_{\mathcal{M}^\varphi_q} = \|\chi_{\mathbb{R}^n\setminus B(0,R)}\chi_{\{a\le|f|\le b\}}\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $R \to \infty$, we have $\chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^\varphi_q$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}} \text{ and } G(z) := \int_\theta^z F(w)\,dw\,.$$

In the proof of Theorem 6.3 (i), we know that $G \in \mathcal{G}(L^\infty, \overline{\mathcal{M}^\varphi_q})$. Hence, in order to prove that $G \in \mathcal{G}(L^\infty, \widetilde{\mathcal{M}^\varphi_q})$, we only need to show that

$$G_1(z) := \chi_{\{|f|>1\}}G(z) \in \overset{*}{\mathcal{M}}{}^\varphi_q \text{ and } G(1+it) - G(1) \in \overset{*}{\mathcal{M}}{}^\varphi_q$$

for each $z \in \overline{S}$ and $t \in \mathbb{R}$. For every $R > 0$, we have

$$\left|\chi_{\{|f|\le R\}}G_1(z)\right| \le (1+|z|)R^{1/\theta}\chi_{\{1\le|f|\le R\}}\,,$$

so $\chi_{\{|f|\le R\}}G_1(z) \in \overset{*}{\mathcal{M}}{}^\varphi_q$. Since

$$\left\|G_1(z) - \chi_{\{|f|\le R\}}G_1(z)\right\|_{\mathcal{M}^\varphi_q} \lesssim \frac{1}{\log(R/(1+|z|))}\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $R \to \infty$, we have $G_1(z) \in \overset{*}{\mathcal{M}}{}^\varphi_q$. For every $t \in \mathbb{R}$ and $R > 1$, we have

$$|G(1+it) - G(1)|\chi_{\{\frac{1}{R}\le|f|\le R\}} \le (1+|t|)R^{1/\theta}\chi_{\{\frac{1}{R}\le|f|\le R\}}\,,$$

so $(G(1+it) - G(1))\chi_{\{\frac{1}{R}\le|f|\le R\}} \in \overset{*}{\mathcal{M}}{}^\varphi_q$. Meanwhile,

$$\left\|(G(1+it) - G(1))\chi_{\mathbb{R}^n\setminus\{\frac{1}{R}\le|f|\le R\}}\right\|_{\mathcal{M}^\varphi_q} \lesssim \frac{\theta}{\log R}\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $R \to \infty$, so $G(1+it) - G(1) \in \overset{*}{\mathcal{M}}{}^\varphi_q$. Since $G \in \mathcal{G}(L^\infty, \widetilde{\mathcal{M}^\varphi_q})$ and $f = G'(\theta)$, we conclude that $f \in [L^\infty, \widetilde{\mathcal{M}^\varphi_q}]^\theta$. Combining with $[L^\infty, \widetilde{\mathcal{M}^\varphi_q}]^\theta \subseteq [L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q]^\theta$, we have

$$\bigcap_{0<a<b<\infty}\left\{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\right\} \subseteq \left[L^\infty, \widetilde{\mathcal{M}^\varphi_q}\right]^\theta \subseteq \left[L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q\right]^\theta. \qquad (6.10)$$

Let $f \in [L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q]^\theta$. From $[L^\infty, \mathcal{M}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$, it follows that $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Choose $G \in \mathcal{G}(L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q)$ such that $f = G'(\theta)$. For each $z \in \overline{S}$, define

$$h_k(z) := \frac{G(z+2^{-k}i) - G(z)}{2^{-k}i}\,.$$

By Lemma 2.9 and Theorem 1.6, we have $h_k(\theta) \in [L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. Since $\lim_{k\to\infty} h_k(\theta) = f$ in $L^\infty + \mathcal{M}^\varphi_q$, we have $f \in \overline{\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}}^{L^\infty+\mathcal{M}^\varphi_q} \cap \mathcal{M}^{\varphi^\theta}_{q/\theta}$. By virtue of Lemma 6.2, we conclude that $\chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$, for every $0 < a < b < \infty$. Hence,

$$\left[L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q\right]^\theta \subseteq \bigcap_{0<a<b<\infty} \left\{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\right\}. \tag{6.11}$$

As a consequence of (6.10) and (6.11), we have Theorem 6.3 (iii). □

6.3. The spaces $[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]_\theta$, $[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$, $[\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]_\theta$, and $[\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta$. Finally, we also consider the complex interpolation between $\widetilde{L^\infty}$ and closed subspaces of Morrey spaces. Recall that $\widetilde{L^\infty}$ denotes the closure of $L^\infty_{\rm c}$ in L^∞.

THEOREM 6.4. *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

(i) $[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$,

(ii)

$$\bigcap_{0<a<b<\infty} \left\{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \widetilde{L^\infty}\right\} \subseteq \left[\widetilde{L^\infty}, \mathcal{M}^\varphi_q\right]^\theta \subseteq \bigcap_{0<a<b<\infty} \left\{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\right\}$$

(iii) *If* $\inf\varphi > 0$, *then*

$$\left[\widetilde{L^\infty}, \mathcal{M}^\varphi_q\right]^\theta = \bigcap_{0<a<b<\infty} \left\{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\right\}. \tag{6.12}$$

(iv) $[\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]_\theta = [\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$.

PROOF. Let $f \in \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Since $L^\infty_{\rm c} \subseteq \widetilde{L^\infty}$, we have $f \in \overline{\widetilde{L^\infty} \cap \mathcal{M}^\varphi_q}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Then, there exists a sequence $\{f_j\}_{j=1}^\infty \subseteq \widetilde{L^\infty} \cap \mathcal{M}^\varphi_q$ such that

$$\|f - f_j\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{j}. \tag{6.13}$$

By using a similar argument as in the proof of Lemma 6.1, we have

$$\|f_j\|_{[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta} \simeq \|f_j\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}.$$

Therefore, for every $j, k \in \mathbb{N}$ with $j > k$, we have

$$\|f_j - f_k\|_{[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta} \simeq \|f_j - f_k\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{j} + \frac{1}{k} < \frac{2}{k},$$

so $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $[\widetilde{L^\infty}, \mathcal{M}_q^\varphi]^\theta$. By completeness of $[\widetilde{L^\infty}, \mathcal{M}_q^\varphi]^\theta$, there exists $g \in [\widetilde{L^\infty}, \mathcal{M}_q^\varphi]^\theta$ such that

$$\lim_{j\to\infty} \|f_j - g\|_{[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta} = 0\,. \tag{6.14}$$

Combining $\mathcal{M}_{q/\theta}^{\varphi^\theta} \subseteq L^\infty + \mathcal{M}_q^\varphi$, $[L^\infty, \mathcal{M}_q^\varphi]^\theta \subseteq L^\infty + \mathcal{M}_q^\varphi$, (6.13), and (6.14), we get $f = g \in \overline{\widetilde{L^\infty} \cap \mathcal{M}_q^\varphi}^{[\widetilde{L^\infty}, \mathcal{M}_q^\varphi]^\theta}$. As a consequence of (2.3), we have $f \in [\widetilde{L^\infty}, \mathcal{M}_q^\varphi]_\theta$.

Conversely, let $f \in [\widetilde{L^\infty}, \mathcal{M}_q^\varphi]_\theta$ and choose $F \in \mathcal{F}(\widetilde{L^\infty}, \mathcal{M}_q^\varphi)$ such that $f = F(\theta)$. Since $F(it) \in \widetilde{L^\infty}$, we have

$$\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} F(it)\|_{L^\infty} = 0\,. \tag{6.15}$$

By Lemma 2.3, we have

$$\begin{aligned}\|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|\chi_{\mathbb{R}^n\setminus B(0,R)} F(it)\|_{L^\infty} P_0(\theta, t)\, dt\right)^{1-\theta} \\ &\quad \times \|F\|_{\mathcal{F}(\widetilde{L^\infty}, \mathcal{M}_q^\varphi)}\,. \end{aligned} \tag{6.16}$$

By virtue of the dominated convergence theorem, (6.15), and (6.16), we have

$$\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} = 0\,,$$

so $f \in \overset{*}{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta}$. Since $[\widetilde{L^\infty}, \mathcal{M}_q^\varphi]_\theta \subseteq [L^\infty, \mathcal{M}_q^\varphi]_\theta = \overline{L^\infty \cap \mathcal{M}_q^\varphi}^{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \subseteq \overline{L^\infty \cap \mathcal{M}_{q/\theta}^{\varphi^\theta}}^{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$, we have

$$f \in \overline{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \cap \overset{*}{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta} = \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}\,,$$

as desired.

The proof of (ii) goes as follows. Let $f \in \mathcal{M}_{q/\theta}^{\varphi^\theta}$ be such that $\chi_{\{a\le|f|\le b\}} \in \widetilde{L^\infty}$ for every $0 < a < b < \infty$. For each $z \in \overline{S}$, define

$$F(z) := \mathrm{sgn}(f)|f|^{z/\theta} \text{ and } G(z) := \int_\theta^z F(w)\, dw\,.$$

Since $G \in \mathcal{G}(L^\infty, \mathcal{M}_q^\varphi)$, we shall show that $G_0(z) := \chi_{\{|f|\le 1\}} G(z) \in \widetilde{L^\infty}$ for every $z \in \overline{S}$ and $G(it) - G(0) \in \widetilde{L^\infty}$ for every $t \in \mathbb{R}$. For each $N \in \mathbb{N}$, we have

$$|G_0(z)\chi_{\{|f|>\frac{1}{N}\}}| \le (1+|z|)\chi_{\{\frac{1}{N}<|f|<1\}}\,,$$

so $G_0(z)\chi_{\{|f|>\frac{1}{N}\}} \in \widetilde{L^\infty}$. Meanwhile,

$$\left\|G_0(z) - G_0(z)\chi_{\{|f|>1/N\}}\right\|_{L^\infty} = \left\|\theta \frac{\mathrm{sgn}(f)|f|^{z/\theta} - \mathrm{sgn}(f)|f|}{\log|f|} \chi_{\{1/N\le|f|\le1\}}\right\|_{L^\infty}$$

$$\leq \frac{2\theta}{\log N} \to 0$$

as $N \to \infty$. Therefore, $G_0(z) \in \widetilde{L^\infty}$.

Next, for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$|G(it) - G(0)|\chi_{\{1/N \leq |f| \leq N\}} \leq (1 + |t|)\chi_{\{1/N \leq |f| \leq N\}},$$

so $(G(it) - G(0))\chi_{\{1/N \leq |f| \leq N\}} \in \widetilde{L^\infty}$. Since $|F(it)| = 1$ for every $t \in \mathbb{R}$, we have

$$\begin{aligned}\left\|(G(it) - G(0))\chi_{\mathbb{R}^n \setminus \{1/N \leq |f| \leq N\}}\right\|_{L^\infty} &= \left\|\theta \frac{F(it) - F(0)}{\log |f|}\chi_{\{|f| < 1/N\} \cup \{|f| > N\}}\right\|_{L^\infty} \\ &\leq \frac{2\theta}{\log N} \to 0,\end{aligned}$$

as $N \to \infty$. Therefore, $G(it) - G(0) \in \widetilde{L^\infty}$. In total, $G \in \mathcal{G}(\widetilde{L^\infty}, \mathcal{M}^\varphi_q)$. Since $f = G'(\theta)$, we see that $f \in [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$.

Now, let $f \in [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$. Since $[L^\infty, \mathcal{M}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$, we have $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Let $G \in \mathcal{G}(\widetilde{L^\infty}, \mathcal{M}^\varphi_q)$ be such that such that $f = G'(\theta)$. For each $z \in \overline{S}$, define

$$h_k(z) := \frac{G(z + 2^{-k}i) - G(z)}{2^{-k}i}.$$

As a consequence of Lemma 2.9 and Theorem 6.4(i), we have $h_k(\theta) \in \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Since $\lim_{k\to\infty} h_k(\theta) = f$ in $L^\infty + \mathcal{M}^\varphi_q$, we have $f \in \overline{\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}}^{L^\infty + \mathcal{M}^\varphi_q} \cap \mathcal{M}^{\varphi^\theta}_{q/\theta}$. By virtue of Lemma 6.2, we conclude that $\chi_{\{a \leq |f| \leq b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$, as desired.

Finally, let us prove (iii) and (iv). Recall that, when $\inf \varphi > 0$, we have $\mathcal{M}^{\varphi^\theta}_{q/\theta} \subseteq L^\infty$; see [17, Proposition 3.3]. Therefore, $\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta} \subseteq \widetilde{L^\infty}$. Combining this fact with Theorem 6.4 (ii), we get (6.12). From Theorem 6.4 (i), it follows that $[\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]_\theta \subseteq \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. By the same argument as in the proof of Theorem 6.4 (i), we have

$$\widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overline{\widetilde{L^\infty} \cap \widetilde{\mathcal{M}^\varphi_q}}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overline{\widetilde{L^\infty} \cap \widetilde{\mathcal{M}^\varphi_q}}^{[\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]^\theta} = \left[\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}\right]_\theta.$$

By combining $[\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta \subseteq [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$ and $\widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = [\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]_\theta \subseteq [\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta$, we have $[\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$.

□

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