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From Colored Jones Invariants to Logarithmic Invariants

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Abstract. In this paper, we express the logarithmic invariant of knots in terms of derivatives of the colored Jones invariants. Logarithmic invariant is defined by using the Jacobson radicals of the restricted quantum group $\overline{\mathcal{U}}_\xi(sl_2)$ where ξ is a root of unity. We also propose a version of the volume conjecture stating a relation between the logarithmic invariants and the hyperbolic volumes of the cone manifolds along a knot, which is proved for the figure-eight knot.

Introduction

The logarithmic invariants of knots are introduced by Nagatomo and the author [12] by using the centers in the Jacobson radical of the restricted quantum group $\overline{\mathcal{U}}_q(sl_2)$ at root of unity. In this paper, we give a formula for the logarithmic invariant in terms of the colored Jones invariant. Let N be a positive integer greater than 1 and let ξ be the $2N$-th root of unity given by $\xi = \exp(\pi\sqrt{-1}/N)$. The center of $\overline{\mathcal{U}}_\xi(sl_2)$ is $3N-1$ dimensional, and its good basis

$$\{\hat{\boldsymbol{\rho}}_1, \hat{\boldsymbol{\rho}}_2, \dots, \hat{\boldsymbol{\rho}}_{N-1}, \hat{\boldsymbol{\varphi}}_1, \hat{\boldsymbol{\varphi}}_2, \dots, \hat{\boldsymbol{\varphi}}_{N-1}, \hat{\boldsymbol{\kappa}}_0, \hat{\boldsymbol{\kappa}}_1, \dots, \hat{\boldsymbol{\kappa}}_N\} \tag{1}$$

is given by [1, §5.2] which behaves well under certain action of $SL(2,\mathbb{Z})$. For a knot L, let $\gamma_s^{(N)}(L)$ be the logarithmic invariant corresponding to $\hat{\boldsymbol{\kappa}}_s$ of the above basis, and let $V_m(L)$ be the colored Jones invariant corresponding to the m dimensional representation of $\mathcal{U}_q(sl_2)$ at generic q. We get the following two formulas to explain the logarithmic invariant $\gamma_s^{(N)}(L)$ by using derivatives of the colored Jones invariant $V_m(L)$.

THEOREM (in Theorem 1). *The invariant $\gamma_s^{(N)}(L)$ $(1 \le s \le N)$ is given by*

$$\begin{aligned}\gamma_s^{(N)}(L) &= \frac{\xi}{2N}\,\frac{d}{dq}(q-q^{-1})\big(V_s(L)+V_{2N-s}(L)\big)\Big|_{q=\xi} \\ &= \frac{N}{\pi\sqrt{-1}}\,(\xi-\xi^{-1})\,\frac{d}{dm}V_m(L)\Big|_{\substack{m=s\\ q=\xi}}\ .\end{aligned} \tag{2}$$

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REMARK 1. The first formula in (2) is given by the derivative of $V_m(L)$ with respect to the parameter q. The second formula is given by the derivative of $V_m(L)$ with respect to the parameter m, which is an integer. However, we can differentiate $V_m(L)$ with respect to m by using the following universal expression of $V_m(L)$ given by Habiro [3, Theorem 3.1] (see also [6]);

$$V_m(L) = \sum_{i=0}^{\infty} a_i(L) \, \frac{\{m+i, 2i+1\}_q}{\{1\}_q} \,. \tag{3}$$

Here $\{n\}_q = q^n - q^{-n}$, $\{n, m\}_q = \prod_{j=0}^{m-1} \{n-j\}_q$ and the coefficient $a_i(L)$ is a Laurent polynomial in q which does not depend on m (see [3, Theorem 2.1]). For $\frac{d}{dm} V_m(L)$ in (2), $V_m(L)$ is given by (3) and is considered to be an infinite sum with the indeterminate m. The integer s is substituted to m after the differentiation, and the sum reduces to a finite sum when q is specialized to ξ.

The above theorem suggests some relation between the logarithmic invariant and the hyperbolic volume since relations between the colored Jones invariants and the hyperbolic volume are known for various cases by [4], [9], [10], [2], [8] and [11]. Let L be a hyperbolic knot. In [4], Kashaev found a relation between the hyperbolic volume of the knot complement and the series of invariants $\langle L \rangle_N$ he constructed. Kashaev's invariant turned out to be a specialization of the colored Jones invariant by [9], more precisely, $\langle L \rangle_N = V_N(L)|_{q=\xi}$. Then Kashaev's conjecture is generalized as follows.

CONJECTURE 1 (Complexified volume conjecture [10]). *Let L be a hyperbolic knot in S^3. Then*

$$\lim_{N\to\infty} \frac{2\pi \, \log \langle L \rangle_N}{N} = \mathrm{Vol}(\mathrm{S}^3 \setminus \mathrm{L}) + \sqrt{-1}\, \mathrm{CS}(\mathrm{S}^3 \setminus \mathrm{L}) \,, \tag{4}$$

where $\mathrm{Vol}(\mathrm{S}^3 \setminus \mathrm{L})$ *and* $\mathrm{CS}(\mathrm{S}^3 \setminus \mathrm{L})$ *are the hyperbolic volume and the Chern-Simons invariant of $S^3 \setminus L$ respectively.*

There are several generalizations of this conjecture. For example, if we deform ξ to $\xi^\alpha = \exp(\pi\sqrt{-1}\alpha/N)$ by a complex number α near 1, a conjecture for the relation between $V_N(L)$ at $q = \xi^\alpha$ and the complex volume of certain deformation of the hyperbolic structure of $S^3 \setminus K$ is proposed by [2] and [8]. For the figure-eight knot, this conjecture is proved partially by Murakami-Yokota [11].

Our invariant $\gamma_s^{(N)}(L)$ can be considered as a deformation of $\langle L \rangle_N$ since $\langle L \rangle_N$ is equal to $\gamma_N^{(N)}(L)$. Changing the parameter N to s can be considered as a deformation (not continuous but discrete) of the weight parameter λ instead of the deformation of the parameter q. Comparing with the deformations in [2], [8], [11], we propose the following conjecture.

CONJECTURE 2 (Volume conjecture for the logarithmic invariant). *Let L be a hyperbolic knot and let M_α be the cone manifold along the singularity set L with the cone angle α*

with $0 \le \alpha \le \pi$. *Let* s_N^α *be a sequence of integers such that* $\lim_{N\to\infty} s_N^\alpha/N = 1 - \alpha/2\pi$. *If* M_α *is a hyperbolic manifold, then*

$$\lim_{N\to\infty} \frac{2\pi \log \gamma^{(N)}_{s_N^\alpha}(L)}{N} = \mathrm{Vol}(\mathrm{M}_\alpha) + \sqrt{-1}\,\mathrm{CS}(\mathrm{M}_\alpha)\,. \tag{5}$$

For the figure-eight knot, we prove this conjecture for α satisfying $0 \le \alpha < \pi/3$, and check numerically for all α.

This paper is organized as follows. In Sect. 2, we recall the construction of the colored Jones invariant. In Sect. 3, we recall the restricted quantum groups, their representations and their centers. These materials are explained in [1]. In Sect. 4, we discuss about the logarithmic invariant of knots. For a knot L, there is a tangle T_L corresponding to L, and by passing through the universal invariant by Lawrence [5] and Ohtsuki [13], we get a center $z(T_L)$ of $\overline{\mathcal{U}}_\xi(sl_2)$, which is an invariant of L. We introduce a representation of $\mathcal{U}_q(sl_2)$ for generic q, which coincides with a projective representation of the restricted quantum group $\overline{\mathcal{U}}_\xi(sl_2)$ when q is specialized to ξ. Then, by applying this specialization to the colored Jones invariant $V_m(L)$, we get a formula to explain the logarithmic invariant in terms of $V_m(L)$. Moreover, since the invariant $z(T_L)$ is a linear combination of the basis in (1), these coefficients are again invariants of L, and they are expressed in terms of $V_m(L)$. In Sect. 5, we investigate the relation between the logarithmic invariant of the figure-eight knot K_{4_1} and the hyperbolic volume of a cone manifold along K_{4_1}.

1. Colored Jones invariant

1.1. Notations. Let q be a parameter, $\xi = \exp(\pi\sqrt{-1}/N)$ be the primitive $2N$-th root of unity, and we use the following notations.

$$\{n\}_q = q^n - q^{-n}\,, \quad \{n, m\}_q = \prod_{k=0}^{m-1} \{n-k\}_q\,, \quad \{n\}_q! = \{n, n\}_q\,,$$

$$\{n\} = \{n\}_\xi\,, \quad \{n\}! = \{n\}_\xi!\,, \quad [n] = [n]_\xi\,, \quad [n]! = [n]_\xi\,, \quad \{n\}_+ = \xi^n + \xi^{-n},$$

$$[n]_q = \frac{\{n\}_q}{\{1\}_q}\,, \quad [n]_q! = \prod_{k=1}^{n} [k]_q\,, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}\,, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_\xi\,.$$

1.2. Quantum group $\mathcal{U}_q(sl_2)$. Let $\mathcal{U}_q(sl_2)$ be the quantum group defined by

$$\mathcal{U}_q(sl_2) = \Big\langle K,\ E,\ F \ \mid\ K\,E\,K^{-1} = q^2\,E,\ \ K\,F\,K^{-1} = q^{-2}\,F\,,$$
$$E\,F - F\,E = \frac{K - K^{-1}}{q - q^{-1}} \Big\rangle.$$

The Hopf algebra structure of $\mathcal{U}_q(sl_2)$ is given by

$$\Delta(K) = K \otimes K\,, \quad \Delta(E) = 1 \otimes E + E \otimes K\,, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1\,,$$
$$\varepsilon(K) = 1\,, \quad \varepsilon(E) = \varepsilon(F) = 0\,,$$
$$S(K) = K^{-1}\,, \quad S(E) = -E\,K^{-1}\,, \quad S(F) = -K\,F\,,$$

where Δ is the coproduct, ε is the count and S is the antipode. The universal R-matrix of $\mathcal{U}_q(sl_2)$ is given by

$$R = q^{\frac{1}{2}H\otimes H} \sum_{n=0}^{\infty} \frac{\{1\}_q^{2n}}{\{n\}_q!} q^{\frac{n(n-1)}{2}} \,(E^n \otimes F^n)\,, \tag{6}$$

where H is an element such that $q^H = K$.

1.3. Irreducible representations of $\mathcal{U}_q(sl_2)$. Let W_m be the highest weight representation of the quantum group $\mathcal{U}_q(sl_2)$ given by the following basis and actions. Let $\boldsymbol{f}_0$, $\boldsymbol{f}_1$, $\dots$, $\boldsymbol{f}_{m-1}$ be the weight basis of W_m and the actions of E, F, K are given by

$$E\,\boldsymbol{f}_i = [i]_q\,\boldsymbol{f}_{i-1}\,, \quad F\,\boldsymbol{f}_i = [m-1-i]_q\,\boldsymbol{f}_{i+1}\,, \quad K\,\boldsymbol{f}_i = q^{m-1-2i}\,\boldsymbol{f}_i\,.$$

Then W_m is irreducible if q is generic. Let $\rho_m : \mathcal{U}_q(sl_2) \to \mathrm{End}(W_m)$ be the algebra homomorphism defined by the above actions.

1.4. Colored Jones invariants. Here we explain the colored Jones invariants briefly. For detail, see [14]. For a knot L, let b_L be a braid whose closure is isotopic to L as a framed knot. Let n be the number of strings of b_L. By assigning universal R matrix at each clossing of a braid, a represntation of the braid group B_n is defined on $W_m^{\otimes n}$, i.e., we have a homomorphism $\rho_m^{(n)} : B_n \to \mathrm{End}(W_m^{\otimes n})$. The colored Jones invariant $V_m(L)$ of L is given by the quantum trace of $\rho_m^{(n)}(b_L)$. More precisely, $V_m(L) = \mathrm{tr}\left(\rho^m(K)^{\otimes n}\,\rho_m^{(n)}(b_L)\right)$. Later, we use the *normalized colored Jones invariant* $\widetilde{V}_m(L)$ which is defined by $\widetilde{V}_m(L) = V_m(L)/[m]$.

1.5. Tangle invariant. The normalized colored Jones invariant $\widetilde{V}_m(L)$ can be interpreted as an invariant of a $(1,1)$-tangle T_L whose closure is the knot L.

Let V be a d dimensional representation of $\mathcal{U}_\xi(sl_2)$ with basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{d-1}\}$ and $\rho_V : \mathcal{U}_\xi(sl_2) \to \mathrm{End}(V)$ be the corresponding algebra homomorphism. Let $b_L \in B_n$ as before, then there is a homomorphism $\rho_V^{(n)} : B_n \to \mathrm{End}(V^{\otimes n})$ denifed by the universal R matrix. Let T_L be a (1,1)-tangle obtained from b_L by taking the closure of the right $(n-1)$ strings, then the closure of T_L is L. On the other hand, by taking the partial trace of $\rho_V^{(n)}(b_L)$ corresponding to the right $(n-1)$ components of $V^{\otimes n}$, we get a operator in $\mathrm{End}(V)$. Here the partial trace $\widetilde{\mathrm{tr}}$ is given as follows.

$$\widetilde{\mathrm{tr}}\big(\rho_V^{(n)}(b_L)\big)_{i_1}^{j_1} = \sum_{i_2, i_3, \dots, i_n = 0}^{d-1} \Big((id \otimes \rho_V(K)^{\otimes(n-1)})\,\rho_V^{(n)}(b_L)\Big)_{i_1, i_2, \dots, i_n}^{j_1, i_2, \dots, i_n}\,.$$

The operator $\widetilde{\mathrm{tr}}(\rho_V^{(n)}(b_L))$ is an isotopy invariant of the knot L, and if $V = W_m$, $\widetilde{\mathrm{tr}}(\rho_m^{(n)}(b_L))$ is a scalar matrix since W_m is irreducible. This scalar is equal to $\widetilde{V}_m(L)$.

2. Restricted quantum group $\overline{\mathcal{U}}_\xi(sl_2)$

We introduce the restricted quantum group and its representations.

2.1. Restricted quantum group $\overline{\mathcal{U}}_\xi(sl_2)$

DEFINITION 1. The *restricted quantum group* $\overline{\mathcal{U}}_\xi(sl_2)$ is given by

$$\overline{\mathcal{U}}_\xi(sl_2) = \mathcal{U}_\xi(sl_2)/(E^N,\ F^N,\ K^{2N}-1)\,,$$

i.e. $\overline{\mathcal{U}}_\xi(sl_2)$ is defined from $\mathcal{U}_\xi(sl_2)$ by adding new relations $E^N = F^N = 0$ and $K^{2N} = 1$.

The R matrix of $\overline{\mathcal{U}}_\xi(sl_2)$ is given by

$$R = (\xi^{\frac{1}{2}})^{H\otimes H} \sum_{n=0}^{N-1} \frac{\{1\}^{2n}}{\{n\}!} \xi^{\frac{n(n-1)}{2}} (E^n \otimes F^n)\,. \tag{7}$$

Here $\xi^{\frac{1}{2}} = \exp\left(\pi\sqrt{-1}/2N\right)$, H is given by $\xi^H = K$ and satisfies

$$H\,E - E\,H = 2\,E\,, \quad H\,F - F\,H = -2\,F\,. \tag{8}$$

Moreover, for every $\overline{\mathcal{U}}_\xi(sl_2)$-module V, if $K\,v = \xi^\alpha\,v$ for $v \in V$, then we assume $H\,v = \alpha\,v$. Here α is determined up to modulo $2N$, and we choose α so that H satisfies the relation (8). With the above assumptions, the representation of the R matrix on the tensor representation of two projective modules explained in the next subsection is uniquely determined, and coincides with the representation of the universal R-matrix of $\overline{\mathcal{U}}_\xi(sl_2)$ given by Drinfeld's quantum double constriction in [1].

2.2. Projective modules of $\overline{\mathcal{U}}_\xi(sl_2)$. We first explain irreducible representations of $\overline{\mathcal{U}}_\xi(sl_2)$. Let $U_s^\pm$ be the s-dimensional irreducible representations of $\overline{\mathcal{U}}_\xi(sl_2)$ labeled by $1 \le s \le N$. The module $U_s^\pm$ is spanned by elements $\boldsymbol{u}_n^\pm$ for $0 \le n \le s-1$, where the action of $\overline{\mathcal{U}}_\xi(sl_2)$ is given by

$$\begin{aligned}
&K\,\boldsymbol{u}_n^\pm = \pm\xi^{s-1-2n}\,\boldsymbol{u}_n^\pm\,, && 0 \le n \le s-1\,, && \\
&E\,\boldsymbol{u}_n^\pm = \pm[n][s-n]\,\boldsymbol{u}_{n-1}^\pm\,, && 1 \le n \le s-1\,, && E\,\boldsymbol{u}_0^\pm = 0\,, \\
&F\,\boldsymbol{u}_n^\pm = \boldsymbol{u}_{n+1}^\pm\,, && 0 \le n \le s-2\,, && F\,\boldsymbol{u}_{s-1}^\pm = 0\,.
\end{aligned}$$

Especially, U_1^+ is the trivial module for which K acts by 1 and E, F act by 0. The weights (eigenvalues of K) occurring in U_s^+ are $\xi^{s-1}, \xi^{s-3}, \dots, \xi^{-s+1}$, and the weights occurring in U_{N-s}^- are $-\xi^{N-s-1}, -\xi^{N-s-3}, \dots, -\xi^{-N+s+1}$.

Let $V_s^{\pm}$ ($1 \le s \le N$) be the N dimensional representation with highest-weight ξ^{s-1} spanned by elements $\boldsymbol{v}_n^{\pm}$ for $0 \le n \le N-1$, where the action of $\overline{\mathcal{U}}_\xi(sl_2)$ is given by

$$\begin{aligned}
K\boldsymbol{v}_n^{\pm} &= \pm\xi^{s-1-2n}\,\boldsymbol{v}_n^{\pm}, && 0 \le n \le N-1, && \\
E\boldsymbol{v}_n^{\pm} &= \pm[n][s-n]\,\boldsymbol{v}_{n-1}^{\pm}, && 1 \le n \le N-1, && E\boldsymbol{v}_0^{\pm} = 0, \\
F\boldsymbol{v}_n^{\pm} &= \boldsymbol{v}_{n+1}^{\pm}, && 0 \le n \le N-2, && F\boldsymbol{v}_{N-1}^{\pm} = 0.
\end{aligned}$$

Note that $V_N^{\pm} = U_N^{\pm}$. For $1 \le s \le N-1$, $V_s^{\pm}$ satisfies the exact sequence

$$0 \longrightarrow U_{N-s}^{\mp} \longrightarrow V_s^{\pm} \longrightarrow U_s^{\pm} \longrightarrow 0,$$

and there are projective modules $P_s^{\pm}$ satisfying the following exact sequence.

$$0 \longrightarrow V_{N-s}^{\mp} \longrightarrow P_s^{\pm} \longrightarrow V_s^{\pm} \longrightarrow 0.$$

The module $\mathcal{P}_s^{+}$ has a basis $\{\boldsymbol{x}_j^{+},\ \boldsymbol{y}_j^{+}\}_{0 \le j \le N-s-1} \cup \{\boldsymbol{a}_n^{+},\ \boldsymbol{b}_n^{+}\}_{0 \le n \le s-1}$, and the actions of $\overline{\mathcal{U}}_\xi(sl_2)$ are determined by

$$\begin{aligned}
&K\boldsymbol{x}_j^{+} = \xi^{2N-s-1-2j}\,\boldsymbol{x}_j^{+}, \qquad K\boldsymbol{y}_j^{+} = \xi^{-s-1-2j}\,\boldsymbol{y}_j^{+}, \quad 0 \le j \le N-s-1, \\
&K\boldsymbol{a}_n^{+} = \xi^{s-1-2n}\,\boldsymbol{a}_n^{+}, \qquad K\boldsymbol{b}_n^{+} = \xi^{s-1-2n}\,\boldsymbol{b}_n^{+}, \quad 0 \le n \le s-1, \\
&E\boldsymbol{x}_j^{+} = -[j][N-s-j]\,\boldsymbol{x}_{j-1}^{+}, \ 0 \le j \le N-s-1, \qquad E\boldsymbol{y}_j^{+} = \begin{cases} -[j][N-s-j]\,\boldsymbol{y}_{j-1}^{+}, & 1 \le k \le N-s-1, \\ \boldsymbol{a}_{s-1}^{+}, & j = 0, \end{cases} \\
&E\,\boldsymbol{a}_n^{+} = [n][s-n]\boldsymbol{a}_{n-1}^{+}, \ 0 \le n \le s-1, \qquad E\boldsymbol{b}_n^{+} = \begin{cases} [n][s-n]\boldsymbol{b}_{n-1}^{+} + \boldsymbol{a}_{n-1}^{+}, & 1 \le n \le s-1, \\ \boldsymbol{x}_{N-s-1}^{+}, & n = 0, \end{cases} \qquad (9) \\
&F\boldsymbol{x}_j^{+} = \begin{cases} \boldsymbol{x}_{j+1}^{+}, & 0 \le j \le N-s-2, \\ \boldsymbol{a}_0^{+}, & j = N-s-1, \end{cases} \qquad F\boldsymbol{y}_j^{+} = \boldsymbol{y}_{j+1}^{+}, \quad 0 \le j \le N-s-2, \\
&F\boldsymbol{a}_n^{+} = \boldsymbol{a}_{n+1}^{+}, \quad 0 \le n \le s-1, \qquad F\boldsymbol{b}_n^{+} = \begin{cases} \boldsymbol{b}_{n+1}^{+}, & 0 \le n \le s-2, \\ \boldsymbol{y}_0^{+}, & n = s-1. \end{cases}
\end{aligned}$$

Here we assume that $\boldsymbol{x}_{-1}^{+} = \boldsymbol{a}_{-1}^{+} = \boldsymbol{y}_{N-1}^{+} = \boldsymbol{a}_s^{+} = 0$.

The module $\mathcal{P}_{N-s}^{-}$ has a basis $\{\boldsymbol{x}_j^{-},\ \boldsymbol{y}_j^{-}\}_{0 \le j \le N-s-1} \cup \{\boldsymbol{a}_n^{-},\ \boldsymbol{b}_n^{-}\}_{0 \le n \le s-1}$, and the action of $\overline{\mathcal{U}}_\xi(sl_2)$ is determined by

$$
\begin{aligned}
&K\boldsymbol{x}_j^- = \xi^{-s-1-2j}\,\boldsymbol{x}_j^- , \qquad K\boldsymbol{y}_j^- = \xi^{-s-1-2j}\,\boldsymbol{y}_j^- , \quad 0\le j\le N-s-1 ,\\
&K\boldsymbol{a}_n^- = \xi^{s-1-2n}\,\boldsymbol{a}_n^- , \qquad K\boldsymbol{b}_n^- = \xi^{-2N+s-1-2n}\,\boldsymbol{b}_n^- , \quad 0\le n\le s-1 ,\\
&E\boldsymbol{x}_j^- = -[j][N-s-j]\,\boldsymbol{x}_{j-1}^- , \; 0\le k\le N-s-1 , \qquad E\boldsymbol{y}_j^- = \begin{cases} -[j][N-s-j]\,\boldsymbol{y}_{j-1}^- + \boldsymbol{x}_{j-1}^- , & 1\le j\le N-s-1 ,\\ \boldsymbol{a}_{s-1}^- , & j=0 ,\end{cases}\\
&E\boldsymbol{a}_n^- = [n][s-n]\,\boldsymbol{a}_{n-1}^- , \; 0\le n\le s-1 , \qquad E\boldsymbol{b}_n^- = \begin{cases} [n][s-n]\,\boldsymbol{b}_{n-1}^- , & 1\le n\le s-1 ,\\ \boldsymbol{x}_{N-s-1}^- , & n=0 ,\end{cases}\\
&F\boldsymbol{x}_j^- = \boldsymbol{x}_{j+1}^- , \; 0\le j\le N-s-2 , \qquad F\boldsymbol{y}_j^- = \begin{cases} \boldsymbol{y}_{j+1}^- , & 0\le j\le N-s-2 ,\\ \boldsymbol{b}_0^- , & j=N-s-1 ,\end{cases}\\
&F\boldsymbol{a}_n^- = \begin{cases} \boldsymbol{a}_{n+1}^- , & 0\le n\le s-2 ,\\ \boldsymbol{x}_0^- , & n=s-1 .\end{cases} \qquad F\boldsymbol{b}_n^- = \boldsymbol{b}_{n+1}^- , \quad 0\le n\le s-1 .
\end{aligned} \tag{10}
$$

Here we assume that $\boldsymbol{x}_{-1}^- = \boldsymbol{a}_{-1}^- = \boldsymbol{x}_{N-s}^- = \boldsymbol{b}_s^- = 0$.

2.3. Centers of $\overline{\mathcal{U}}_\xi(sl_2)$. The center of $\overline{\mathcal{U}}_\xi(sl_2)$ is investigated in [1].

PROPOSITION 1 ([1], Proposition 4.4.4). *The dimension of the center Z of $\overline{\mathcal{U}}_\xi(sl_2)$ is $3N-1$. Its commutative algebra structure is described as follows. There are two special central idempotents $\boldsymbol{e}_0$ and $\boldsymbol{e}_N$, other central idempotents $\boldsymbol{e}_s$ $(1\le s\le N-1)$, centers in the radical $\boldsymbol{w}_s^{\pm}$ $(1\le s\le N-1)$, and they satisfy the following commutation relation.*

$$
\begin{aligned}
&\boldsymbol{e}_s\,\boldsymbol{e}_t = \delta_{s,t}\,\boldsymbol{e}_s , && 0\le s,t\le N ,\\
&\boldsymbol{e}_s\,\boldsymbol{w}_t^{\pm} = \delta_{s,t}\,\boldsymbol{w}_t^{\pm} , && 0\le s\le N , \quad 1\le t\le N-1 ,\\
&\boldsymbol{w}_s^{\pm}\,\boldsymbol{w}_t^{\pm} = \boldsymbol{w}_s^{\pm}\,\boldsymbol{w}_t^{\mp} = 0 , && 1\le s , \quad t\le N-1 .
\end{aligned}
$$

The center $\boldsymbol{e}_N$ acts as identity on U_N^+ and as 0 on the other modules. $\boldsymbol{e}_0$ acts as identity on U_N^- and as 0 on the other modules. $\boldsymbol{e}_s$ acts as identity on P_s^+, P_{N-s}^- and as 0 on the other modules. The center $\boldsymbol{w}_s^+$ acts on $\mathcal{P}_s^+$ by $\boldsymbol{w}_s^+\,\boldsymbol{b}_n^+ = \boldsymbol{a}_n^+$, $\boldsymbol{w}_s^+\,\boldsymbol{a}_n^+ = \boldsymbol{w}_s^+\,\boldsymbol{x}_k^+ = \boldsymbol{w}_s^+\,\boldsymbol{y}_k^+ = 0$, and acts on the other modules as 0. Similarly, $\boldsymbol{w}_s^-$ acts on $\mathcal{P}_{N-s}^-$ by $\boldsymbol{w}_s^-\,\boldsymbol{y}_k^- = \boldsymbol{x}_k^-$, $\boldsymbol{w}_s^-\,\boldsymbol{x}_k^- = \boldsymbol{w}_s^-\,\boldsymbol{a}_n^- = \boldsymbol{w}_s^-\,\boldsymbol{b}_n^- = 0$, and acts on the other modules as 0.

According to [1], the basis (1) is expressed by $\boldsymbol{e}_s$ and $\boldsymbol{w}_s^{\pm}$ as follows.

$$
\hat{\boldsymbol{\rho}}_s = (-1)^{N+s}\,\frac{1}{N\,(q^s-q^{-s})}\left(\boldsymbol{e}_s - \frac{q^s+q^{-s}}{[s]^2}\,(\boldsymbol{w}_s^+ + \boldsymbol{w}_s^-)\right) \quad (1\le s\le N-1) ,
$$

$$\hat{\boldsymbol{\varphi}}_s = \frac{1}{[s]^2}\left(\frac{N-s}{N}\,\boldsymbol{w}_s^+ - \frac{s}{N}\,\boldsymbol{w}_s^-\right) \qquad (1 \le s \le N-1)\,.$$

$$\hat{\boldsymbol{\kappa}}_0 = \boldsymbol{e}_0\,, \quad \hat{\boldsymbol{\kappa}}_s = \frac{1}{[s]^2}\left(\boldsymbol{w}_s^+ + \boldsymbol{w}_s^-\right) \quad (1 \le s \le N-1)\,, \quad \hat{\boldsymbol{\kappa}}_N = -\boldsymbol{e}_N\,.$$

Let z be a center of $\overline{\mathcal{U}}_q(sl_2)$ given by

$$z = a_0\,\boldsymbol{e}_0 + a_N\,\boldsymbol{e}_N + \sum_{s=1}^{N-1}(a_s\,\boldsymbol{e}_s + b_s^+\,\boldsymbol{w}_s^+ + b_s^-\,\boldsymbol{w}_s^-)\,. \tag{11}$$

Then z can be also expressed by the good basis $\hat{\boldsymbol{\kappa}}_s$, $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\varphi}}$ by

$$z = \sum_{s=1}^{N-1}\alpha_s^{(N)}\,\hat{\boldsymbol{\rho}}_s + \sum_{s=1}^{N-1}\beta_s^{(N)}\,\hat{\boldsymbol{\varphi}}_s + \sum_{s=0}^{N}\gamma_s^{(N)}\,\hat{\boldsymbol{\kappa}}_s\,, \tag{12}$$

where

$$\begin{aligned} &\alpha_s^{(N)} = (-1)^{N+s}\,(q^s - q^{-s})\,N\,a_s\,, \quad \beta_s^{(N)} = [s]^2\,(b_s^+ - b_s^-)\,,\\ &\gamma_s^{(N)} = [s]^2\left(\frac{s}{N}b_s^+ + \frac{N-s}{N}\,b_s^-\right) + (q^s + q^{-s})\,a_s\,, \quad (1 \le s \le N-1)\\ &\gamma_0^{(N)} = a_0\,, \quad \gamma_N^{(N)} = -a_N. \end{aligned} \tag{13}$$

3. Logarithmic invariants of knots

3.1. Logarithmic invariants. Let L be a knot with framing 0, T_L be a tangle obtained from L and $z(T_L)$ be the center corresponding to the universal invariant constructed by Lawrence and Ohtsuki, where we assign the R matrix given by (6) and $K^{\pm 1}$ to the maximal and the minimal points as in Figure 1. In [12], $K^{N\pm 1}$ is assigned instead of $K^{\pm 1}$, and so the invariant defined here and that in [12] is different by the sign $(-1)^{(m-1)f}$ where f is the framing of the knot. So these invariants coincide for an unframed knot.

From (11) and (12), we define $a_k(L)$, $b_k^{\pm}(L)$, $\alpha_k^{(N)}(L)$, $\beta_k^{(N)}(L)$, $\gamma_k^{(N)}(L)$ as follows.

$$\begin{aligned} z(T_L) &= a_0(L)\,\boldsymbol{e}_0 + a_N(L)\,\boldsymbol{e}_N + \sum_{s=1}^{N-1}\left(a_s(L)\,\boldsymbol{e}_s + b_s^+(L)\,\boldsymbol{w}_s^+ + b_s^-(L)\,\boldsymbol{w}_s^-\right)\\ &= \sum_{s=1}^{N-1}\alpha_s^{(N)}(L)\,\hat{\boldsymbol{\rho}}_s + \sum_{s=1}^{N-1}\beta_s^{(N)}(L)\,\hat{\boldsymbol{\varphi}}_s + \sum_{s=0}^{N}\gamma_s^{(N)}(L)\,\hat{\boldsymbol{\kappa}}_s\,. \end{aligned}$$

The purpose of this section is to express the above coefficients in terms of the colored Jones invariant. We first consider $b_s^{\pm}(L)$.

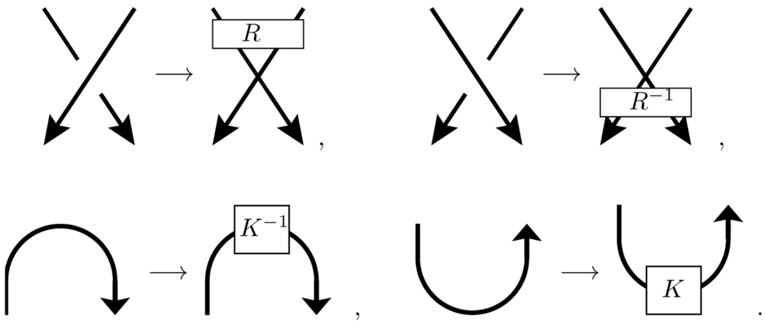

FIGURE 1. Assignment of the R matrix and $K^{\pm 1}$.

PROPOSITION 2. *Let L be a knot. Then we have*

$$\begin{aligned} b_s^+(L) &= \frac{\xi}{2\,N\,[s]}\ \frac{d}{dq}\{1\}_q \left(\frac{V_s(L)}{[s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q}\right)\bigg|_{q=\xi}, \\ b_s^-(L) &= \frac{\xi}{4\,N\,[s]}\ \frac{d}{dq}\{1\}_q \left(\frac{V_{2N+s}(L)}{[2N+s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q}\right)\bigg|_{q=\xi}. \end{aligned} \tag{14}$$

The proof of this proposition is given in Sect. 3.5.

3.2. Modified representations of $\mathcal{U}_q(sl_2)$. Let W_m be the highest weight representation of the quantum group $\mathcal{U}_q(sl_2)$ given in §1.3. For an integer m ($1 \le m \le N-1$), we introduce a $2N$ dimensional representation $\mathcal{Y}_m^+$ which is isomorphic to $W_{2N-m} \oplus W_m$. The basis of $\mathcal{Y}_m^+$ is $\boldsymbol{\alpha}_0^+, \boldsymbol{\alpha}_1^+, \dots, \boldsymbol{\alpha}_{2N-m-1}^+, \boldsymbol{\beta}_0^+, \boldsymbol{\beta}_1^+, \dots, \boldsymbol{\beta}_{m-1}^+$, and the actions of E, F, $K \in \mathcal{U}_q(sl_2)$ are given by

$$E\boldsymbol{\alpha}_i^+ = \begin{cases} [i]_q\,\boldsymbol{\alpha}_{i-1}^+ & \text{if } i \le N-m \text{ or } i \ge N+1\,, \\ [i]_q\,\boldsymbol{\alpha}_{i-1}^+ + \begin{bmatrix} 2N-m-i-1 \\ N-i \end{bmatrix}_q \boldsymbol{\beta}_{m-N+i-1}^+ & \text{if } N-m+1 \le i \le N\,, \end{cases}$$

$$E\boldsymbol{\beta}_i^+ = [i]_q\,\boldsymbol{\beta}_{i-1}^+\,,$$

$$F\boldsymbol{\alpha}_i^+ = \begin{cases} [2N-m-i-1]_q\,\boldsymbol{\alpha}_{i+1}^+ & \text{if } i \ne N-m-1\,, \\ [N]_q\,\boldsymbol{\alpha}_{i+1}^+ + \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^+ & \text{if } i = N-m-1\,, \end{cases}$$

$$F\boldsymbol{\beta}_i^+ = [m-i-1]_q\,\boldsymbol{\beta}_{i+1}^+\,,$$

$$K\boldsymbol{\alpha}_i^+ = q^{2N-m-1-2i}\,\boldsymbol{\alpha}_i^+\,, \qquad K\boldsymbol{\beta}_i^+ = q^{m-1-2i}\,\boldsymbol{\beta}_i^+\,.$$

PROPOSITION 1. *If q is specialized to ξ, then $\mathcal{Y}_m^+$ is isomorphic to the projective mod-*

ule $\mathcal{P}_m^+$ given by (9).

PROOF. We compare the actions of $\overline{\mathcal{U}}_\xi(sl_2)$ on $\mathcal{Y}_m^+$ and $\mathcal{P}_m^+$. Let f be a linear map defined by

$$f(\boldsymbol{x}_k^+) = \frac{(-1)^{N-m-1-k}\,[m]}{[N-m-1-k]!}\,\boldsymbol{\alpha}_k^+\,, \quad f(\boldsymbol{y}_k^+) = \frac{(-1)^k\,[N-1]!}{[N-m-1-k]!}\,\boldsymbol{\alpha}_{N+k}^+ \qquad \text{for } 0 \le k \le N-m-1\,, \tag{15}$$

$$f(\boldsymbol{a}_k^+) = \frac{[m]!}{[m-1-k]!}\,\boldsymbol{\beta}_k^+, \quad f(\boldsymbol{b}_k^+) = [k]!\,\boldsymbol{\alpha}_{k+N-m}^+ \quad \text{for } 0 \le k \le m-1\,.$$

Then a simple computation shows that the actions of K, E, F on $\mathcal{Y}_m^+$ and $\mathcal{P}_m^+$ are commute with f. Therefore, the specialization of $\mathcal{Y}_m^+$ at $q = \xi$ is isomorphic to $\mathcal{P}_m^+$ as an $\overline{\mathcal{U}}_\xi(sl_2)$ module. □

For an integer m $(1 \le m \le N-1)$, we introduce a $4N$ dimensional representation $\mathcal{Y}_m^-$ which is isomorphic to $W_{2N+m} \oplus W_{2N-m}$. The basis of $\mathcal{Y}_m^-$ is $\boldsymbol{\alpha}_0^-, \boldsymbol{\alpha}_1^-, \dots, \boldsymbol{\alpha}_{2N+m-1}^-, \boldsymbol{\beta}_0^-, \boldsymbol{\beta}_1^-, \dots, \boldsymbol{\beta}_{2N-m-1}^-$, and the actions of E, F, $K \in \mathcal{U}_q(sl_2)$ are given by

$$\begin{aligned}
E\,\boldsymbol{\alpha}_i^- &= \begin{cases} [i]_q \boldsymbol{\alpha}_{i-1}^- & \text{if } i \le m \text{ or } i \ge 2N+1\,,\\ [i]_q \boldsymbol{\alpha}_{i-1}^- + \begin{bmatrix} 2N+m-1-i \\ 2N-i \end{bmatrix}_q \boldsymbol{\beta}_{i-m-1}^- & \text{if } m+1 \le i \le 2N\,, \end{cases}\\
E\boldsymbol{\beta}_i^- &= [i]_q\,\boldsymbol{\beta}_{i-1}\,,\\
F\boldsymbol{\alpha}_i^- &= \begin{cases} [2N+m-1-i]_q\,\boldsymbol{\alpha}_{i+1}^- & \text{if } i \ne m-1\,,\\ [2N]_q\,\boldsymbol{\alpha}_{i+1}^- + \begin{bmatrix} 2N-1 \\ 2N-m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^- & \text{if } i = m-1\,, \end{cases}\\
F\boldsymbol{\beta}_i^- &= [2N-m-1-i]_q\,\boldsymbol{\beta}_{i+1}^-\,,\\
K\boldsymbol{\alpha}_i^- &= q^{2N+m-1-2i}\,\boldsymbol{\alpha}_i^-\,, \qquad K\boldsymbol{\beta}_i^- = q^{2N-m-1-2i}\,\boldsymbol{\beta}_i^-\,.
\end{aligned}$$

As for $\mathcal{Y}_m^+$, we get the following.

PROPOSITION 2. *If q is specialized to ξ, then $\mathcal{Y}_m^-$ is isomorphic to the direct sum $\mathcal{P}_{N-m}^- \oplus \mathcal{P}_{N-m}^-$ of the projective module $\mathcal{P}_{N-m}^-$ given by* (10).

PROOF. Let Y_1 be the subspace of $\mathcal{Y}_m^-$ spanned by $\boldsymbol{\alpha}_0^-, \boldsymbol{\alpha}_1^-, \dots, \boldsymbol{\alpha}_{N+m-1}^-, \boldsymbol{\beta}_0^-, \boldsymbol{\beta}_1^-, \dots, \boldsymbol{\beta}_{N-m-1}^-$, and let Y_2 be the subspace spanned by the remaining basis $\boldsymbol{\alpha}_{N+m}^-, \boldsymbol{\alpha}_{N+m+1}^-, \dots, \boldsymbol{\alpha}_{2N+m-1}^-, \boldsymbol{\beta}_{N-m}^-, \boldsymbol{\beta}_{N-m+1}^-, \dots, \boldsymbol{\beta}_{2N-m-1}^-$. Then Y_1 is invariant under the action of $\overline{\mathcal{U}}_\xi(sl_2)$. We prove that Y_1 and $\mathcal{Y}_m^-/Y_1$ are both isomorphic to $\mathcal{P}_{N-m}^-$. Let g be a linear map from $\mathcal{P}_{N-m}^-$

to Y_1 defined by

$$
\begin{aligned}
&g(\boldsymbol{x}_k^-) = \frac{(-1)^{m+k}[N-m]!}{[N-m-1-k]!}\,\boldsymbol{\beta}_k^- , \quad g(\boldsymbol{y}_k^-) = (-1)^k\,[k]!\,\boldsymbol{\alpha}_{m+k}^- \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for } 0 \le k \le N-m-1, \\
&g(\boldsymbol{a}_k^-) = \frac{[m]}{[m-1-k]!}\,\boldsymbol{\alpha}_k^- , \quad g(\boldsymbol{b}_k^-) = \frac{(-1)^{N+m+k}\,[N-1]!}{[m-1-k]!}\,\boldsymbol{\alpha}_{N+k}^- \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for } 0 \le k \le m-1\,.
\end{aligned}
\tag{16}
$$

Then, by checking the actions of K, E, F, we see that g gives an isomorphism from $\mathcal{P}_{N-m}^-$ to Y_1 as $\overline{\mathcal{U}}_\xi(sl_2)$ modules.

Next, we define a linear map h from $\mathcal{P}_{N-m}^-$ to Y_2 to show that $\mathcal{P}_{N-m}^-$ are isomorphic to $\mathcal{Y}_m^-/Y_1$.

$$
\begin{aligned}
&h(\boldsymbol{x}_k^-) = \frac{[N-m]!}{[N-m-1-k]!}\,\boldsymbol{\beta}_{N+k}^- , \quad h(\boldsymbol{y}_k^-) = [k]!\,\boldsymbol{\alpha}_{N+m+k}^- \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for } 0 \le k \le N-m-1\,, \\
&h(\boldsymbol{a}_k^-) = \frac{[k]!}{[m-1]!}\,\boldsymbol{\beta}_{N-m+k}^- , \quad h(\boldsymbol{b}_k^-) = \frac{(-1)^k\,[N-1]!}{[m-1-k]!}\,\boldsymbol{\alpha}_{2N+k}^- \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for } 0 \le k \le m-1\,.
\end{aligned}
\tag{17}
$$

Then h defines an isomorphism from $\mathcal{P}_{N-m}^-$ to $\mathcal{Y}_m^-/Y_1$. This isomorphism induces an inclusion from $\mathcal{P}_{N-m}^-$ to $\mathcal{Y}_m^-$ since $\mathcal{P}_{N-m}^-$ is a projective module. Hence $\mathcal{Y}_m^-$ at $q=\xi$ is isomorphic to $\mathcal{P}_{N-m}^- \oplus \mathcal{P}_{N-m}^-$. □

3.3. Specialization of the R matrix at $q=\xi$. We cannot specialize the universal R matrix (6) at $q=\xi$ since it has a pole at $q=\xi$. However, we can sepcialize its action on the representation spaces we are considering.

PROPOSITION 3. *Let W_m $(m=1,2,\dots)$ and $\mathcal{Y}_m^\pm$ $(1 \le m \le N-1)$ be the representations of $\mathcal{U}_q(sl_2)$ introduced in* §3.2, *and let V_1 and V_2 be two of these representations. If q is generic, the universal R matrix of $\mathcal{U}_q(sl_2)$ given by* (6) *acts on $V_1 \otimes V_2$, and this action can be specialized at $q=\xi$. The resulting action is the same as the action of the R matrix of $\overline{\mathcal{U}}_\xi(sl_2)$ given by* (7).

PROOF. For a large n, E^n and F^n vanish on V_1 and V_2. Hence it is enough to show that every term of the universal R matrix can be specialized at $q=\xi$. We investigate the degrees of zero at $q=\xi$ for the matrix elements of the representations W_m and $\mathcal{Y}_m^\pm$. By counting the factors of the form $[kN]$ in the matrix elements of representations of E^n and F^n constructed in §3.2, we see that the zero degrees of them at $q=\xi$ are at least ℓ if $n \ge \ell N$. (Sometimes, they act by 0, whose degree is considered to be ∞ for any factor.) Hence, for

$\ell N \le n < (\ell+1)\,N$, the zero degree of the term $\frac{1}{\{n\}_q!}\,E^n \otimes F^n$ at $q=\xi$ is a least ℓ since its numerator has degree at least 2ℓ and its denominator has degree ℓ. Hence, the action of the term $\frac{1}{\{n\}_q!}\,E^n \otimes F^n$ at $q=\xi$ is well-defined even if $n \ge N$, and the terms for $n \ge N$ are all specialized to 0 on $V_1 \otimes V_2$ at $q=\xi$. □

3.4. Specialization of the representation on $\mathcal{Y}_m^{\pm}$ at $q=\xi$. Let W_m be the irreducible representation of $\mathcal{U}_q(sl_2)$ introduced in §1.3, and let ρ_m, $\eta_m^{\pm}$ be the homomorphisms from $\mathcal{U}_q(sl_2)$ to W_m and $\mathcal{Y}_m^{\pm}$ respectively. Moreover, for a knot L, we define $\rho_m(L)$ and $\eta_m^{\pm}(L)$ by

$$\rho_m(L) = \widetilde{\mathrm{tr}}\big(\rho_{W_m}(b_L)\big)\,, \quad \eta_m^{\pm}(L) = \widetilde{\mathrm{tr}}\big(\rho_{\mathcal{Y}_m^{\pm}}(b_L)\big)\,,$$

where $\rho_{\mathcal{Y}_m^{\pm}}(b_L)$ is the representation of b_L on $\mathcal{Y}_m^{\pm} \otimes \cdots \otimes \mathcal{Y}_m^{\pm}$ given in §1.5. Then $\rho_m(L)$ and $\eta_m^{\pm}(L)$ are elements of $\mathrm{End}(W_m)$ and $\mathrm{End}(\mathcal{Y}_m^{+})$ respectively. Moreover, $\rho_m(L)$ is a scalar matrix such that the corresponding scalar is the normalized colored Jones invariant $\widetilde{V}_m(L)$. The matrix $\eta_m^{\pm}(L)$ may not be scalar matrix since $\mathcal{Y}_m^{\pm}$ is not irreducible. However, for any $g \in \mathcal{U}_q(sl_2)$, $\eta_m^{\pm}(L)$ satisfies $\eta_m^{\pm}(g)\,\eta_m^{\pm}(L) = \eta_m^{\pm}(L)\,\eta_m^{\pm}(g)$, and $\eta_m^{\pm}(L)$ is given by the following with some scalars $x_k^{\pm}$.

$$\eta_m^{+}(L)\,\boldsymbol{\alpha}_k^{+} = \begin{cases} \widetilde{V}_{2N-m}(L)\,\boldsymbol{\alpha}_k^{+}\,, & (0 \le k \le N-m-1) \\ \widetilde{V}_{2N-m}(L)\,\boldsymbol{\alpha}_k^{+} + x_{k-N+m}^{+}\,\boldsymbol{\beta}_{k-N+m}^{+}\,, & (N-m \le k \le N-1) \\ \widetilde{V}_{2N-m}(L)\,\boldsymbol{\alpha}_k^{+}\,, & (N \le k \le 2N-m-1) \end{cases}$$

$$\eta_m^{+}(L)\,\boldsymbol{\beta}_k^{+} = \widetilde{V}_m(L)\,\boldsymbol{\beta}_k^{+}\,, \qquad (0 \le k \le m-1)$$

$$\eta_m^{-}(L)\,\boldsymbol{\alpha}_k^{-} = \begin{cases} \widetilde{V}_{2N+m}(L)\,\boldsymbol{\alpha}_k^{-}\,, & (0 \le k \le m-1) \\ \widetilde{V}_{2N+m}(L)\,\boldsymbol{\alpha}_k^{-} + x_{k-m}^{-}\,\boldsymbol{\beta}_{k-m}^{-}\,, & (m \le k \le N-1) \\ \widetilde{V}_{2N+m}(L)\,\boldsymbol{\alpha}_k^{-}, & (N \le k \le 2N+m-1) \end{cases}$$

$$\eta_m^{-}(L)\,\boldsymbol{\beta}_k^{-} = \widetilde{V}_{2N-m}(L)\,\boldsymbol{\beta}_k^{-}\,. \qquad (0 \le k \le 2N-m-1)$$

Now, we obtain $x_0^{\pm}$. We have

$$\begin{aligned} \eta_m^{+}(F)\,\eta_m^{+}(L)\,\boldsymbol{\alpha}_{N-m-1}^{+} &= \widetilde{V}_{2N-m}(L)\,\eta_m^{+}(F)\,\boldsymbol{\alpha}_{N-m-1}^{+} \\ &= \widetilde{V}_{2N-m}(L)\left([N]_q\,\boldsymbol{\alpha}_{N-m}^{+} + \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^{+}\right), \end{aligned}$$

and

$$\begin{aligned} \eta_m^{+}(L)\,\eta_m^{+}(F)\,\boldsymbol{\alpha}_{N-m-1}^{+} &= \eta_m^{+}(L)\left([N]_q\,\boldsymbol{\alpha}_{N-m}^{+} + \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^{+}\right) \\ &= [N]_q\,\widetilde{V}_{2N-m}(L)\,\boldsymbol{\alpha}_{N-m}^{+} + [N]_q x_0^{+}\boldsymbol{\beta}_0^{+} + \widetilde{V}_m(L)\begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^{+}\,. \end{aligned}$$

Hence we get

$$x_0^+ = \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q \frac{\widetilde{V}_{2N-m}(L) - \widetilde{V}_m(L)}{[N]_q} . \tag{18}$$

We also get x_0^- similarly as follows.

$$\begin{aligned} \eta_m^-(F)\,\eta_m^-(L)\,\boldsymbol{\alpha}_{m-1}^- &= \widetilde{V}_{2N+m}(L)\,\eta_m^-(F)\,\boldsymbol{\alpha}_{m-1}^- \\ &= \widetilde{V}_{2N+m}(L)\Bigl([2N]_q\,\boldsymbol{\alpha}_m^- + \begin{bmatrix} 2N-1 \\ 2N-m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^-\Bigr), \end{aligned}$$

which is equal to

$$\begin{aligned} \eta_m^-(L)\,\eta_m^-(F)\,\boldsymbol{\alpha}_{m-1}^- &= \eta_m^-(L)\left([2N]_q\,\boldsymbol{\alpha}_m^- + \begin{bmatrix} 2N-1 \\ 2N-m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^-\right) \\ &= \widetilde{V}_{2N+m}(L)[2N]_q\,\boldsymbol{\alpha}_m^- + [2N]_q x_0^- \boldsymbol{\beta}_0^- + \widetilde{V}_{2N-m}(L)\begin{bmatrix} 2N-1 \\ 2N-m-1 \end{bmatrix}_q \boldsymbol{\beta}_0^- . \end{aligned}$$

Hence we get

$$x_0^- = \begin{bmatrix} 2N-1 \\ m \end{bmatrix}_q \frac{\widetilde{V}_{2N+m}(L) - \widetilde{V}_{2N-m}(L)}{[2N]_q} . \tag{19}$$

By specializing q to ξ at (18) and (19), we get

$$\begin{aligned} \lim_{q\to\xi} x_0^+ &= -\frac{\xi}{2N}\,\frac{d}{dq}\{1\}_q\bigl(\widetilde{V}_{2N-m}(L) - \widetilde{V}_m(L)\bigr)\Big|_{q=\xi} , \\ \lim_{q\to\xi} x_0^- &= \frac{(-1)^m\,\xi}{4N}\,\frac{d}{dq}\{1\}_q\bigl(\widetilde{V}_{2N+m}(L) - \widetilde{V}_{2N-m}(L)\bigr)\Big|_{q=\xi} , \end{aligned}$$

by using l'Hopital's rule. Now we are ready to prove Proposition 2.

3.5. Proof of Proposition 2. We compare $x_0^\pm$ with the coefficients $b_m^\pm(L)$ of $\boldsymbol{w}_s^\pm$ introduced in [12]. Let $\widetilde{\eta}_m^+$, $\widetilde{\eta}_m^-$ be the representations on $\mathcal{P}_m^+$ and $\mathcal{P}_{N-m}^-$ in [12], and $\widetilde{\eta}_m^+(L)$, $\widetilde{\eta}_m^-(L)$ be the elements of $\mathrm{End}(\mathcal{P}_m^+)$ and $\mathrm{End}(\mathcal{P}_{N-m}^-)$ defined as $\eta_m^\pm(L)$. Then we have

$$\begin{aligned} \widetilde{\eta}_m^+(L)\,\boldsymbol{b}_0^+ &= \widetilde{V}_{2N-m}(L)\big|_{q=\xi}\,\boldsymbol{b}_0^+ + b_m^+(L)\,\boldsymbol{a}_0^+ , \\ \widetilde{\eta}_m^-(L)\,\boldsymbol{y}_0^- &= \widetilde{V}_{2N+m}(L)\big|_{q=\xi}\,\boldsymbol{y}_0^- + b_m^-(L)\,\boldsymbol{x}_0^- . \end{aligned}$$

From Proposition 3, $\widetilde{\eta}_m^\pm(L)$ are essentially the same as the specialization of $\eta_m^\pm(L)$ at $q=\xi$. Hence the matrices of $\widetilde{\eta}_m^\pm(L)$ and $\eta_m^\pm(L)$ should be related by the isomorphisms f, g in (15), (16) as follows.

$$\widetilde{\eta}_m^+(L) = f^{-1}\circ\eta_m^+(L)\big|_{q=\xi}\circ f , \quad \widetilde{\eta}_m^-(L) = g^{-1}\circ\eta_m^-(L)\big|_{q=\xi}\circ g .$$

Therefore, by using

$$f(\boldsymbol{b}_0^+) = \boldsymbol{\alpha}_{N-m}^+ , \quad f(\boldsymbol{a}_0^+) = [m]\,\boldsymbol{\beta}_0^+ , \quad g(\boldsymbol{y}_0^-) = \boldsymbol{\alpha}_m^- , \quad g(\boldsymbol{x}_0^-) = (-1)^m\,[m]\,\boldsymbol{\beta}_0^- ,$$

we have

$$\begin{aligned} b_m^+(L) &= \frac{\xi}{2\,N\,[m]}\,\frac{d}{dq}\{1\}_q\big(\widetilde{V}_m(L) - \widetilde{V}_{2N-m}(L)\big)\Big|_{q=\xi} , \\ b_m^-(L) &= \frac{\xi}{4\,N\,[m]}\,\frac{d}{dq}\{1\}_q\big(\widetilde{V}_{2N+m}(L) - \widetilde{V}_{2N-m}(L)\big)\Big|_{q=\xi} . \end{aligned} \tag{20}$$

□

3.6. Habiro's formula. Let s be an integer satisfying $1 \le s \le N-1$ and put

$$\underline{s} = \min(s, N-s) , \quad \overline{s} = \max(s, N-s) .$$

By using Habiro's universal formula (3), $b_s^{\pm}(L)$ is expressed in terms of $a_i(L)$ as follows. We put $a_i(L)_\xi = a_i(L)|_{q=\xi}$.

PROPOSITION 4. *For a knot L, we have*

$$b_s^+(L) = b_s^-(L) =$$

$$\frac{\{1\}^2}{\{s\}}\left(\sum_{i=0}^{\underline{s}-1}\frac{a_i(L)_\xi\{s+i\}!}{[s]\{s-i-1\}!}\sum_{\substack{s-i\le k\le s+i\\ k\ne s}}\frac{\{k\}_+}{\{k\}} + 2\sum_{i=\underline{s}}^{\overline{s}-1} a_i(L)_\xi\,\widetilde{\{s+i, i\}}\,\widetilde{\{s-1, i\}}\right), \tag{21}$$

where $\widetilde{\{n, j\}}$ is given by the following.

$$\widetilde{\{n, j\}} = \prod_{\substack{0\le k\le j-1\\ n-k=Nt}} (-1)^t \prod_{\substack{0\le k\le j-1\\ n-k\notin N\mathbf{Z}}} \{n-k\}_q . \tag{22}$$

COROLLARY 1. *Let L^f be the framed knot with framing f which is isotopic to L as a non-framed knot. The colored Jones invariant V_m is generalized to a framed knot by $V_m(L^f) = q^{\frac{m^2-1}{2}f}\,V_m(L)$, and the invariants $b_s^+(L^f)$ and $b_s^-(L^f)$ are generalized as follows.*

$$\begin{aligned} b_s^+(L^f) &= q^{\frac{s^2-1}{2}f}\,b_s^+(L) + \frac{(-N+s)\,f\,\{1\}}{[s]^2}\,V_s(L)|_{q=\xi} , \\ b_s^-(L^f) &= q^{\frac{s^2-1}{2}f}\,b_s^-(L) + \frac{s\,f\,\{1\}}{[s]^2}\,V_s(L)|_{q=\xi} . \end{aligned} \tag{23}$$

PROOF OF PROPOSITION 4. Let $\tilde{a}_i(L) = \{1\}_q\,a_i(L)$ for $a_i(L)$ in (3) and $\tilde{a}_i(L)_\xi = \tilde{a}_i(L)\big|_{q=\xi}$, then

$$\frac{d}{dq}\,\frac{\{1\}_q V_s(L)}{[s]_q}\bigg|_{q=\xi} = \sum_{i=0}^{s-1}\frac{\tilde{a}_i(L)_\xi\,\{s+i\}!}{\{s\}\,\{s-i-1\}!}\left(\frac{\frac{d}{dq}\,\tilde{a}_i(L)}{\tilde{a}_i(L)}\bigg|_{q=\xi} + \sum_{\substack{s-i\le k\le s+i\\ k\ne s}}\frac{k\,\{k\}_+}{\xi\,\{k\}}\right),$$

where $\{k\}_+ = \xi^k + \xi^{-k}$. Now we compute (14) by using the relation $\{2N-k\} = -\{k\}$ and $\frac{d}{dq}F(q)\Big|_{q=\xi} = -\frac{2N}{\xi}\lim_{q\to\xi}\frac{F(q)}{\{N\}}$ for a function $F(q)$ of q.

$$
\begin{aligned}
&\frac{d}{dq}\{1\}_q\left(\frac{V_s(L)}{[s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q}\right)\Bigg|_{q=\xi} \\
&\quad = \sum_{i=0}^{\underline{s}-1}\frac{\tilde{a}_i(L)_\xi\,\{s+i\}!}{\{s\}\,\{s-i-1\}!}\left(\frac{\frac{d}{dq}\tilde{a}_i(L)}{\tilde{a}_i(L)}\Bigg|_{q=\xi} + \sum_{\substack{s-i\le k\le s+i\\ k\ne s}}\frac{k\,\{k\}_+}{\xi\,\{k\}}\right) \\
&\qquad - \sum_{i=\underline{s}}^{s-1}\frac{2N}{\xi}\lim_{q\to\xi}\frac{\tilde{a}_i(L)\,\{s+i,i\}_q\,\{s-1,i\}_q}{\{N\}_q} \\
&\qquad - \sum_{i=0}^{\underline{s}-1}\frac{\tilde{a}_i(L)_\xi\,\{2N-s+i\}!}{\{2N-s\}\,\{2N-s-i-1\}!}\left(\frac{\frac{d}{dq}\tilde{a}_i(L)}{\tilde{a}_i(L)}\Bigg|_{q=\xi} + \sum_{\substack{2N-s-i\le k\le 2N-s+i\\ k\ne 2N-s}}\frac{k\,\{k\}_+}{\xi\,\{k\}}\right) \\
&\qquad + \sum_{i=\underline{s}}^{\overline{s}-1}\frac{2N}{\xi}\lim_{q\to\xi}\frac{\tilde{a}_i(L)\,\{2N-s+i,i\}_q\,\{2N-s-1,i\}_q}{\{N\}_q}.
\end{aligned}
$$

For $\underline{s} \le i \le s-1$, we have

$$
\lim_{q\to\xi}\frac{\{s+i,i\}_q\,\{s-1,i\}_q}{\{N\}_q} = \begin{cases} 0 & \text{if } s \le \frac{N}{2}, \\ -\widetilde{\{s+i,i\}}\,\{s-1,i\} & \text{if } s > \frac{N}{2}, \end{cases} \tag{24}
$$

and for $\underline{s} \le i \le \overline{s}-1$,

$$
\lim_{q\to\xi}\frac{\{2N-s+i,i\}_q\,\{2N-s-1,i\}_q}{\{N\}_q} = \begin{cases} 2\{s+i,i\}\widetilde{\{s-1,i\}} & \text{if } s \le \frac{N}{2}, \\ \widetilde{\{s+i,i\}}\,\{s-1,i\} & \text{if } s > \frac{N}{2}. \end{cases} \tag{25}
$$

We also know that

$$
\{2N-k\} = -\{k\}, \quad \frac{\{2N-s+i\}!}{\{2N-s\}\,\{2N-s-i-1\}!} = \frac{\{s+i\}!}{\{s\}\,\{s-i-1\}!}. \tag{26}
$$

By using (24), (25), (26), we get the following. If $s \le N/2$, then

$$
\begin{aligned}
&\frac{d}{dq}\{1\}_q \left(\frac{V_s(L)}{[s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q}\right)\Bigg|_{q=\xi} \\
&\quad = \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \, \{s+i\}!}{\{s\}\,\{s-i-1\}!} \left(\frac{\frac{d}{dq}\tilde{a}_i(L)}{\tilde{a}_i(L)}\Bigg|_{q=\xi} + \sum_{\substack{s-i \le k \le s+i \\ k \ne s}} \frac{k\,\{k\}_+}{\xi\,\{k\}} \right) \\
&\qquad - \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \, \{s+i\}!}{\{s\}\,\{s-i-1\}!} \left(\frac{\frac{d}{dq}\tilde{a}_i(L)}{\tilde{a}_i(L)}\Bigg|_{q=\xi} - \sum_{\substack{s-i \le k \le s+i \\ k \ne s}} \frac{(2N-k)\,\{k\}_+}{\xi\,\{k\}} \right) \\
&\qquad + \frac{4\,N}{\xi} \sum_{i=\underline{s}}^{\overline{s}-1} \tilde{a}_i(L)_\xi \, \{s+i, i\}\,\{\widetilde{s-1}, i\} \\
&\quad = \frac{2\,N}{\xi} \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \{s+i\}!}{\{s\}\,\{s-i-1\}!} \sum_{\substack{s-i \le k \le s+i \\ k \ne s}} \frac{\{k\}_+}{\{k\}} + \frac{4\,N}{\xi} \sum_{i=\underline{s}}^{\overline{s}-1} \tilde{a}_i(L)_\xi \{\widetilde{s+i}, i\}\{\widetilde{s-1}, i\}\,.
\end{aligned}
$$

If $s > N/2$, then

$$
\begin{aligned}
&\frac{d}{dq}\{1\}_q \left(\frac{V_s(L)}{[s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q}\right)\Bigg|_{q=\xi} \\
&\quad = \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \, \{s+i\}!}{\{s\}\,\{s-i-1\}!} \left(\frac{\frac{d}{dq}\tilde{a}_i(L)}{\tilde{a}_i(L)}\Bigg|_{q=\xi} + \sum_{\substack{s-i \le k \le s+i \\ k \ne s}} \frac{k\,\{k\}_+}{\xi\,\{k\}} \right) \\
&\qquad + \frac{2\,N}{\xi} \sum_{i=\underline{s}}^{s-1} \tilde{a}_i(L)_\xi \, \{\widetilde{s+i}, i\}\,\{s-1, i\} \\
&\qquad - \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \, \{s+i\}!}{\{s\}\,\{s-i-1\}!} \left(\frac{\frac{d}{dq}\tilde{a}_i(L)}{\tilde{a}_i(L)}\Bigg|_{q=\xi} - \sum_{\substack{s-i \le k \le s+i \\ k \ne s}} \frac{(2N-k)\,\{k\}_+}{\xi\,\{k\}} \right) \\
&\qquad + \frac{2\,N}{\xi} \sum_{i=\underline{s}}^{\overline{s}-1} \tilde{a}_i(L)_\xi \, \{\widetilde{s+i}, i\}\,\{s-1, i\} \\
&\quad = \frac{2\,N}{\xi} \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \{s+i\}!}{\{s\}\,\{s-i-1\}!} \sum_{\substack{s-i \le k \le s+i \\ k \ne s}} \frac{\{k\}_+}{\{k\}} + \frac{4\,N}{\xi} \sum_{i=\underline{s}}^{\overline{s}-1} \tilde{a}_i(L)_\xi \{\widetilde{s+i}, i\}\{\widetilde{s-1}, i\}\,.
\end{aligned}
$$

Therefore, for all s with $1 \le s \le N-1$, we have

$$\frac{d}{dq}\,\{1\}_q \left(\frac{V_s(L)}{[s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q} \right)\Bigg|_{q=\xi}$$

$$= \frac{2\,N}{\xi} \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \{s+i\}!}{\{s\}\,\{s-i-1\}!} \sum_{\substack{s-i\le k\le s+i\\ k\ne s}} \frac{\{k\}_+}{\{k\}} + \frac{4\,N}{\xi} \sum_{i=\underline{s}}^{\overline{s}-1} \tilde{a}_i(L)_\xi \{\widetilde{s+i,i}\}\{\widetilde{s-1,i}\}\,. \quad (27)$$

Similarly we get

$$\frac{d}{dq}\{1\}_q \left(\frac{V_{2N+s}(K)}{[2N+s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q} \right)\Bigg|_{q=\xi}$$

$$= \frac{4\,N}{\xi} \sum_{i=0}^{\underline{s}-1} \frac{\tilde{a}_i(L)_\xi \{s+i\}!}{\{s\}\{s-i-1\}!} \sum_{\substack{s-i\le k\le s+i\\ k\ne s}} \frac{\{k\}_+}{\{k\}} + \frac{8\,N}{\xi} \sum_{i=\underline{s}}^{\overline{s}-1} \tilde{a}_i(L)_\xi \{\widetilde{s+i,i}\}\{\widetilde{s-1,i}\} \quad (28)$$

by using (25), (26),

$$\lim_{q\to\xi} \frac{\{2N+s+i,i\}_q\,\{2N+s-1,i\}_q}{\{N\}_q} = \begin{cases} 2\{s+i,i\}\{\widetilde{s-1,i}\} & \text{if } s \le \frac{N}{2}\,, \\ 3\{\widetilde{s+i,i}\}\{s-1,i\} & \text{if } s > \frac{N}{2}\,, \end{cases} \quad (29)$$

and

$$\frac{\{2N+s+i\}!}{\{2N+s\}\,\{2N+s-i-1\}!} = \frac{\{s+i\}!}{\{s\}\,\{s-i-1\}!}\,. \quad (30)$$

Combining (20) with (27) and (28), we get

$$b_s^+(L) = b_s^-(L)$$

$$= \frac{\{1\}^2}{\{s\}} \left(\sum_{i=0}^{\underline{s}-1} \frac{a_i(L)_\xi \{s+i\}!}{\{s\}\{s-1-i\}!} \sum_{\substack{s-i\le k\le s+i\\ k\ne s}} \frac{\{k\}_+}{\{k\}} + 2 \sum_{i=\underline{s}}^{\overline{s}-1} a_i(L)_\xi\, \{\widetilde{s+i,i}\}\,\{\widetilde{s-1,i}\} \right). \qquad \square$$

PROOF OF COROLLARY 1. Since $V_s(L^f) = q^{\frac{s^2-1}{2}f}\,V_s(L)$, $V_s(L^f)\big|_{q=\xi} = V_{2N-s}(L^f)\big|_{q=\xi} = V_{2N+s}(L^f)\big|_{q=\xi}$, and $b_s^+(L^f)$, $b_s^-(L^f)$ are given by (20), we have

$$b_s^+(L^f) = \xi^{\frac{s^2-1}{2}f}\, b_s^+(L) + \frac{1}{2N}\left(\frac{s^2-1}{2} - \frac{(2N-s)^2-1}{2} \right) \frac{f\,\{1\}}{[s]^2}\; V_s(L)|_{q=\xi}$$

$$= \xi^{\frac{s^2-1}{2}f}\, b_s^+(L) - f\,\frac{(N-s)\,\{1\}}{[s]^2}\; V_s(L)|_{q=\xi}\,,$$

$$b_s^-(L^f) = \xi^{\frac{s^2-1}{2}f}\, b_s^-(L) + f\,\frac{s\,\{1\}}{[s]^2}\; V_s(L)|_{q=\xi}\,. \qquad \square$$

3.7. Coefficients of centers. The center of the restricted quantum group $\overline{\mathcal{U}}_\xi(sl_2)$ is spanned by $\boldsymbol{e}_0, \dots, \boldsymbol{e}_N$, $\boldsymbol{w}_1^\pm, \dots, \boldsymbol{w}_{N-1}^\pm$, where $\boldsymbol{e}_s$ is the central idempotent and $\boldsymbol{w}_s^\pm$ is the center in the radical of $\mathcal{P}_s^\pm$. For a framed knot L^f, let $z = z(L^f)$ be the center of $\overline{\mathcal{U}}_\xi(sl_2)$ determined from L^f by using a tangle T_{L^f} obtained from L^f. Then z is expressed as a linear combination of the good basis (1).

$$z(L^f) = \sum_{s=1}^{N-1} \alpha_s^{(N)}(L^f)\,\hat{\boldsymbol{\rho}}_s + \sum_{s=1}^{N-1} \beta_s^{(N)}(L^f)\,\hat{\boldsymbol{\varphi}}_s + \sum_{s=0}^{N} \gamma_s^{(N)}(L^f)\,\hat{\boldsymbol{\kappa}}_s\,.$$

In the following, L represents the non-framed knot which is istopic to L^f. By using (13), we have

$$\begin{aligned}
\alpha_s^{(N)}(L^f) &= (-1)^{N+s}\, N\,\{1\}\ \ V_s(L^f)\Big|_{q=\xi}\,, && 1 \le s \le N-1\,,\\
\beta_s^{(N)}(L^f) &= [s]^2\,\Big(b_s^+(L^f) - b_s^-(L^f)\Big) = -N\,f\,\{1\}\ \ V_s(L^f)\Big|_{q=\xi}, && 1 \le s \le N-1\,,\\
\gamma_0^{(N)}(L^f) &= \frac{V_{2N}(L^f)}{[2N]}\bigg|_{q=\xi} = \frac{\xi^{1-\frac{f}{2}}}{4\,N}\ \frac{d}{dq}\{1\}_q V_{2N}(L)\bigg|_{q=\xi}\\
&= \frac{N\,\xi^{-\frac{f}{2}}\,\{1\}}{2\,\pi\,\sqrt{-1}}\ \frac{d}{dm} V_m(L)\bigg|_{\substack{m=2N\\ q=\xi}}\,,\\
\gamma_s^{(N)}(L^f) &= [s]^2\,\left(\frac{s}{N}\,b_s^+(L^f) + \frac{N-s}{N}\,b_s^-(L^f)\right) + \frac{\{s\}_+}{[s]}\ \ V_s(L^f)\Big|_{q=\xi}\\
&= \xi^{\frac{s^2-1}{2}f}\left(\sum_{i=0}^{\underline{s}-1} a_i(L)_\xi\{s+i, 2i+1\}\sum_{k=s-i}^{s+i}\frac{\{k\}_+}{\{k\}} + 2\sum_{i=\underline{s}}^{\overline{s}-1} a_i(L)_\xi\,\widetilde{\{s+i, 2i+1\}}\right)\\
&= \frac{\xi}{2N}\,\xi^{\frac{s^2-1}{2}f}\ \frac{d}{dq}\{1\}_q\big(V_s(L) + V_{2N-s}(L)\big)\bigg|_{q=\xi}\\
&= \frac{N\,\{1\}}{\pi\,\sqrt{-1}}\,\xi^{\frac{s^2-1}{2}f}\ \frac{d}{dm} V_m(L)\bigg|_{\substack{m=s\\ q=\xi}}\,, \quad 1 \le s \le N-1\,,\\
\gamma_N^{(N)}(L^f) &= -\,\frac{V_N(L^f)}{[N]}\bigg|_{q=\xi} = \frac{\xi^{1+\frac{N^2-1}{2}f}}{2\,N}\ \frac{d}{dq}\{1\}_q V_N(L)\bigg|_{q=\xi}\\
&= \frac{N\,\xi^{\frac{N^2-1}{2}f}\,\{1\}}{2\,\pi\,\sqrt{-1}}\ \frac{d}{dm} V_m(L)\bigg|_{\substack{m=N\\ q=\xi}}\,.
\end{aligned}$$

In $\frac{d}{dm}\,V_m(L)$, the colored Jones invariant $V_m(L)$ is expressed by Habiro's universal formula (3) and considered to be an infinite sum with the variable m. The integer s is substituted after obtaining the derivative. The sum reduces to a finite sum when q is specialized to ξ. Hence we get the following.

THEOREM 1. *For a framed knot L^f with framing f and let L be the same knot without framing. Then we have*

$$\alpha_s^{(N)}(L^f) = (-1)^{N+s}\, N\,\{1\}\ V_s(L^f)\Big|_{q=\xi}\,, \qquad 1 \le s \le N-1\,,$$

$$\beta_s^{(N)}(L^f) = -N\,f\,\{1\}\ V_s(L^f)\Big|_{q=\xi}\,, \qquad 1 \le s \le N-1\,,$$

$$\gamma_0^{(N)}(L^f) = \frac{V_{2N}(L^f)}{[2N]}\Big|_{q=\xi} = \frac{\xi^{1-\frac{f}{2}}}{4N}\,\frac{d}{dq}\{1\}_q V_{2N}(L)\Big|_{q=\xi} = \frac{N\,\xi^{-\frac{f}{2}}\,\{1\}}{2\pi\sqrt{-1}}\,\frac{d}{dm}V_m(L)\Big|_{\substack{m=2N\\ q=\xi}}\,,$$

$$\gamma_s^{(N)}(L^f) = \frac{\xi^{1+\frac{s^2-1}{2}f}}{2N}\ \frac{d}{dq}\{1\}\big(V_s(L)+V_{2N-s}(L)\big)\Big|_{q=\xi} = \frac{N\,\{1\}}{\pi\sqrt{-1}}\,\xi^{\frac{s^2-1}{2}f}\,\frac{d}{dm}V_m(L)\Big|_{\substack{m=s\\ q=\xi}}\,,$$
$$1 \le s \le N-1\,,$$

$$\gamma_N^{(N)}(L^f) = -\,\frac{V_N(L^f)}{[N]}\Big|_{q=\xi} = \frac{\xi^{1+\frac{N^2-1}{2}f}}{2N}\ \frac{d}{dq}\{1\}_q V_N(L)\Big|_{q=\xi}$$
$$= \frac{N\,\xi^{\frac{N^2-1}{2}f}\,\{1\}}{2\pi\sqrt{-1}}\,\frac{d}{dm}V_m(L)\Big|_{\substack{m=N\\ q=\xi}}\,.$$

Especially, if the framing $f = 0$,

$$\alpha_s^{(N)}(L) = (-1)^{N+s}\, N\,\{1\}\ V_s(L)|_{q=\xi}\,, \qquad 1 \le s \le N-1\,,$$

$$\beta_s^{(N)}(L) = 0\,, \qquad 1 \le s \le N-1\,,$$

$$\gamma_0^{(N)}(L) = \frac{V_{2N}(L)}{[2N]}\Big|_{q=\xi} = \frac{\xi}{4N}\,\frac{d}{dq}\{1\}_q V_{2N}(L)\Big|_{q=\xi} = \frac{N\,\{1\}}{2\pi\sqrt{-1}}\,\frac{d}{dm}V_m(L)\Big|_{\substack{m=2N\\ q=\xi}}\,,$$

$$\gamma_s^{(N)}(L) = \frac{\xi}{2N}\,\frac{d}{dq}\{1\}\big(V_s(L)+V_{2N-s}(L)\big)\Big|_{q=\xi} = \frac{N\,\{1\}}{\pi\sqrt{-1}}\,\frac{d}{dm}V_m(L)\Big|_{\substack{m=s\\ q=\xi}}\,,$$
$$1 \le s \le N-1\,,$$

$$\gamma_N^{(N)}(L) = -\,\frac{V_N(L)}{[N]}\Big|_{q=\xi} = \frac{\xi}{2N}\,\frac{d}{dq}\{1\}_q V_N(L)\Big|_{q=\xi} = \frac{N\,\{1\}}{2\pi\sqrt{-1}}\,\frac{d}{dm}V_m(L)\Big|_{\substack{m=N\\ q=\xi}}\,.$$

4. Relation to the hyperbolic volume

In this section, we check the volume conjecture (5) of the logarithmic invariant $\gamma_s(K_{4_1})$ for the figure-eight knot K_{4_1}.

4.1. Logarithmic invariant of figure-eight knot. The normalized colored Jones invariant $V_\lambda(K_{4_1})$ is expressed as follows.

$$V_s(K_{4_1}) = \sum_{i=0}^{\infty} \frac{\{s+i, 2i+1\}}{\{1\}}.$$

This means that the coefficients $a_i(K_{4_1})$ are all equal to 1 in Habiro's formula (3), and $\gamma_s^{(N)}(K_{4_1})$ is given by

$$\gamma_s^{(N)}(K_{4_1}) = \sum_{i=0}^{\underline{s}-1} \{s+i, 2i+1\} \sum_{k=s-i}^{s+i} \frac{\{k\}_+}{\{k\}} + 2 \sum_{i=\underline{s}}^{\overline{s}-1} \{s+\widetilde{i, 2i}+1\}, \tag{31}$$

where $\underline{s} = \min(s,\ N-s)$, $\overline{s} = \max(s,\ N-s)$ as before.

4.2. Limit of the logarithmic invariant. For $\gamma_s^{(N)}(K_{4_1})$, the following theorem holds.

THEOREM 2. *Let α be a real number with $0 \leq \alpha < \pi/3$ and let $s_N^\alpha = \lfloor N\alpha/2\pi \rfloor$ where $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \leq x$. Then*

$$\lim_{N\to\infty} \frac{2\pi \left|\gamma_{s_N^\alpha}^{(N)}(K_{4_1})\right|}{N} = \lim_{N\to\infty} \frac{2\pi \left|\gamma_{N-s_N^\alpha}^{(N)}(K_{4_1})\right|}{N} = \mathrm{Vol}\,(\mathrm{M}_\alpha) \tag{32}$$

where M_α is the cone manifold along singular set K_{4_1} with cone angle α.

REMARK 2. Numerical computation suggests that

$$\lim_{N\to\infty} \frac{2\pi \log\left|\gamma_{s_N^\alpha}^{(N)}(K_{4_1})\right|}{N} = \begin{cases} \mathrm{Vol}(\mathrm{M}_\alpha) & \text{for } 0 \leq \alpha < 2\pi/3, \\ 0 & \text{for } 2\pi/3 \leq \alpha \leq 4\pi/3, \\ \mathrm{Vol}(\mathrm{M}_{2\pi-\alpha}) & \text{for } 4\pi/3 < \alpha \leq 2\pi. \end{cases}$$

The values of $2\pi \log\left|\gamma_s^{(N)}(K_{4_1})\right|/N$ for $N = 200$ and $N = 400$ are shown by graphs in Figure 2.

PROOF OF THEOREM 2. We first prove $\lim_{N\to\infty} 2\pi \left|\gamma_{s_N^\alpha}^{(N)}(K_{4_1})\right|/N = \mathrm{Vol}\,(\mathrm{M}_\alpha)$ for $0 \leq \alpha < \pi/3$. To do this, we start by estimating the sum

$$\sum_{i=0}^{s_N^\alpha-1} \{s_N^\alpha+i, 2i+1\} \sum_{k=s_N^\alpha-i}^{s_N^\alpha+i} \frac{\{k\}_+}{\{k\}}.$$

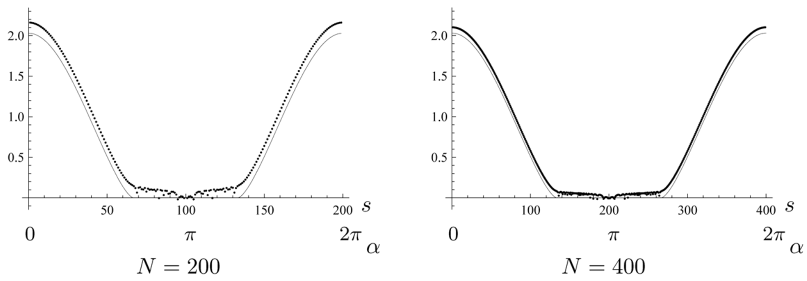

FIGURE 2. Numerical computation for $2\pi \log \left|\gamma_s^{(N)}(K_{4_1})\right|/N$ (dots) and the hyperbolic volume of the cone manifold M_α ($\alpha \leq \pi$), $M_{2\pi-\alpha}$ ($\alpha > \pi$) along K_{4_1} (thin line)

We know $|\{k\}_+/\{k\}| = |\cot k\pi/N| \leq 2\,|\cot \pi/N| \leq N$ since $1 \leq k \leq N-1$, and so $\left|\sum_{k=s_N^\alpha-i}^{s_N^\alpha+i} \{k\}_+/\{k\}\right| \leq N^2$. We also know $\{s_N^\alpha + i\}\{s_N^\alpha - i\} \leq 1$ since $0 \leq s_N^\alpha \leq N/6$ and $0 \leq i \leq s_N^\alpha$. Therefore, we have

$$\left|\sum_{i=0}^{s_N^\alpha-1} \{s_N^\alpha + i, 2i+1\} \sum_{k=s_N^\alpha-i}^{s_N^\alpha+i} \frac{\{k\}_+}{\{k\}}\right| \leq N^3\,.$$

Next we estimate $\sum_{i=s_N^\alpha}^{N-s_N^\alpha-1} \{s_N^\alpha \widetilde{+\, i, 2i} + 1\}$. Let $a_i = (-1)^{s_N^\alpha}\{s_N^\alpha \widetilde{+\, i, 2i} + 1\}$. Then $a_i \geq 0$ and we have

$$a_{i_{\max}^{(N)}} \leq \sum_{i=s_N^\alpha+1}^{N-s_N^\alpha} a_k \leq N\, a_{i_{\max}^{(N)}}$$

where $a_{i_{\max}^{(N)}} = \max_{s_N^\alpha \leq i \leq N-s_N^\alpha-1}(a_i)$. Therefore,

$$-N^3 + a_{i_{\max}^{(N)}} \leq \left|\gamma_{s_N^\alpha}(K_{4_1})\right| \leq N^3 + N\, a_{i_{\max}^{(N)}}\,. \tag{33}$$

The index $i_{\max}^{(N)}$ for the maximum a_i must be equal to $i_1^{(N)} = N - s_N^\alpha - 1$ or $i_2^{(N)}$ satisfying $\{s_N^\alpha + i_2^{(N)}\}\{s_N^\alpha - i_2^{(N)}\} \geq 1$ and $\{s_N^\alpha + i_2^{(N)} + 1\}\{s_N^\alpha - i_2^{(N)} - 1\} \leq 1$ since $i_1^{(N)}$ and $i_2^{(N)}$ correspond to the local maximal of a_i. The index $i_2^{(N)}$ satisfies

$$\cos\frac{2\pi\,(i_2^{(N)}+1)}{N} \leq \cos\frac{2\pi\, s_{N,\alpha}}{N} - \frac{1}{2} \leq \cos\frac{2\pi\, i_2^{(N)}}{N}\,. \tag{34}$$

If N is not small, such $i_2^{(N)}$ exists uniquely between $N/2 + s_N^\alpha$ and $N - s_N^\alpha - 1$ because of $\{N/2 + 2\,s_N^\alpha\}\{-N/2\} > 1$, $\{2\,s_N^\alpha + 1 - N\}\{N-1\} < 1$ and the shape of the graph of the cosine function. The log of a_i is given by

$$\log a_i = \sum_{k=s_N^\alpha-i}^{-1} \log|\{k\}| + \sum_{k=1}^{s_N^\alpha+i} \log|\{k\}|$$

and is estimated as

$$N\int_{\frac{s_N^\alpha-i}{N}}^{0} \log|2\sin\pi t|\,dt + N\int_0^{\frac{s_N^\alpha+i}{N}} \log|2\sin\pi t|\,dt < \log a_i$$

$$< N\int_{\frac{s_N^\alpha-i+1}{N}}^{-\frac{1}{N}} \log|2\sin\pi t|\,dt + N\int_{\frac{1}{N}}^{\frac{s_N^\alpha+i+1}{N}} \log|2\sin\pi t|\,dt\,.$$

Therefore,

$$\lim_{N\to\infty}\frac{2\pi\log a_{i_1^{(N)}}}{N} = -2\,\Lambda(\alpha)\,, \quad \lim_{N\to\infty}\frac{2\pi\log a_{i_2^{(N)}}}{N} = -2\left(\Lambda\left(\frac{\alpha+\theta}{2}\right)-\Lambda\left(\frac{\alpha-\theta}{2}\right)\right),$$

where $\theta = \lim_{N\to\infty} 2\pi i_2^{(N)}/N$ and $\Lambda(x) = -\int_0^x \log|2\,\sin t|\,dt$ is the Lobachevski function. Then $\theta > \pi$ since $N/2 + s_N^\alpha < i_2^{(N)} < N - s_N^\alpha - 1$, and this implies that $\Lambda\big((\alpha-\theta)/2\big) > \Lambda\big((\alpha+\theta)/2\big)$. We also know that $\Lambda(2\,\alpha) > 0$. Therefore, $\lim_{N\to\infty} a_{i_1^{(N)}} < 1$, $\lim_{N\to\infty} a_{i_2^{(N)}} > 1$, and we have $i_{\max}^{(N)} = i_2^{(N)}$ for sufficient large N. By using (33) and the fact that $\lim_{N\to\infty}\log N/N = 0$, we get

$$\lim_{N\to\infty}\frac{2\,\pi\,\log\left|\gamma_{s_N^\alpha}^{(N)}(K_{4_1})\right|}{N} = -2\left(\Lambda\left(\frac{\alpha+\theta}{2}\right)-\Lambda\left(\frac{\alpha-\theta}{2}\right)\right), \tag{35}$$

where θ satisfies $\cos\theta = \cos\alpha - 1/2$ by (34). The volume of M_α is given by Mednykh [7] as follows.

$$\mathrm{Vol}(\mathrm{M}_\alpha) = \int_\alpha^{\frac{2\pi}{3}} \mathrm{arccosh}(1+\cos\mathrm{t}-\cos 2\mathrm{t})\mathrm{dt}\,.$$

The right hand side of (35) is equal to $\mathrm{Vol}(\mathrm{M}_\alpha)$ since

$$\frac{d}{d\alpha}\left(-2\left(\Lambda\left(\frac{\alpha+\theta}{2}\right)-\Lambda\left(\frac{\alpha-\theta}{2}\right)\right)\right)=\log\left|t-\sqrt{t^2-1}\right| = -\,\mathrm{arccosh}\,t = \frac{d}{d\alpha}\mathrm{Vol}(\mathrm{M}_\alpha)$$

where $t = 1+\cos\alpha-\cos 2\alpha$, and, if $\alpha = 0$, then $\theta = \pi/3$ and $-2\big(\Lambda(\pi/6)-\Lambda(-\pi/6)\big) = \mathrm{Vol}(\mathrm{S}^3\setminus \mathrm{K}_{4_1}) = \mathrm{Vol}(\mathrm{M}_0)$. □

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