

# On periodic self-homeomorphisms of closed orientable surfaces determined by their orders

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**Abstract** The fundamentals for the topological classification of periodic orientation-preserving self-homeomorphisms of a closed orientable topological surface  $X$  of genus  $g \geq 2$  have been established, by Nielsen, in the thirties of the last century. Here we consider two concepts related to this classification; rigidity and weak rigidity. A cyclic action  $G$  of order  $N$  on  $X$  is said to be topologically rigid if any other cyclic action of order  $N$  on  $X$  is topologically conjugate to it. If this assertion holds for arbitrary other action but having, in addition, the same orbit genus and the same structure of singular orbits, then  $G$  is said to be weakly topologically rigid. We give a precise description of rigid and weakly rigid cyclic quasi-platonic actions which mean actions having three singular orbits and for which  $X/G$  is a sphere.

**Keywords** Periodic self-homeomorphisms of closed orientable surfaces ·  
Their classification · Riemann surfaces · Their automorphisms · (Real) algebraic curves

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## 1 Introduction

The fundamentals for the topological classification of periodic orientation-preserving self-homeomorphisms of a closed topological surface  $X$  of genus  $g \geq 2$  have been established, by Nielsen, in the thirties of the last century [8]. Certain classifications have been given also by Yokayama [11] and Nakagawa [7] who proved that a closed Riemann surface of genus  $g \neq 4, 6, 9, 10, 12$  having an automorphism of order  $N \geq 3g$  is determined (up to birational equivalence) by  $N$ . More recently, these results have been reproved independently by Hirose [5] using less explicit methods which, however, are closer to our approach in this paper. Actually in this case, there are very few possibilities for such orders;  $N = 4g + 2, 4g, 3g$  or  $3g + 3$  and for the first three cases such configuration actually exists for arbitrary  $g$  while the last one exists if and only if  $g \not\equiv 2 \pmod{3}$ . The order  $N$  of a cyclic action of  $G = \mathbb{Z}_N$  on  $X$  is said to be topologically rigid if for any other cyclic action  $G'$  of order  $N$  on  $X$ ,  $G$  and  $G'$  are conjugate by a certain orientation-preserving self-homeomorphism of  $X$  which does not need to be periodic.

It seems that the above rigidity property, for mentioned orders of cyclic actions, is a rather rare phenomenon and here, apart of it, we consider and study another related property of periodic cyclic actions called *weak rigidity*. We say that the cyclic action  $G$  on  $X$  is *weakly rigid* if any other cyclic action  $G'$  on  $X$  of the same order, with the same orbit genus and lengths structure of singular orbits, is topologically conjugate to it. It is clear that rigidity implies weak rigidity. Further if any action of order  $N$  on  $X$  is weakly rigid, then  $N$  is said to be *weakly rigid order*. In Theorem 3.9, we use combinatorial techniques to characterize the weakly rigid cyclic actions having three singular orbits and the orbit space  $X/G$  of genus zero. These actions are examples of so called quasi-platonic actions which get a great deal of attention in the literature. Observe first that they can be realized as birational actions on complex algebraic curves, due to the geometrization theorem of Nielsen and the Riemann uniformization. Due to the famous theorem of Belyi [1], these curves can be defined over the algebraic numbers. Their equations were found by Wootton in [10] for prime order  $N$ . They are also building blocks for the construction of Beauville surfaces which are important since they form a natural domain for the action of the absolute Galois group,  $\text{Aut}(\bar{\mathbb{Q}}/\mathbb{Q})$ . By another result of Singerman [9], Riemann surfaces considered here are symmetric and so they have some real forms. Furthermore, by Köck and Singerman [6] these forms can be defined over the real algebraic numbers. In [3], Bujalance et al. have found the number of such forms and their topological type. This gives us the number of connected components and separability of the smooth projective models for them. Finally in a recent paper [2], Benim and Wootton gave nice theoretical formulae for the number of topologically distinct actions of cyclic groups on quasi-platonic Riemann surfaces. Our result does not seem, however, possible to get from [2], at least in an easy direct way.

## 2 Preliminaries

We shall use the following ingredients. Throughout this paper,  $X$  will denote a closed orientable surface of genus  $g \geq 2$ .

### 2.1 Nielsen’s geometrization

Let  $\varphi$  be an orientation-preserving self-homeomorphism of  $X$  of finite order  $N$ . There exists a structure of a Riemann surface on  $X$  (we still denote the resulting Riemann surface also by  $X$  by abuse of language) so that  $\varphi$  is a conformal automorphism. In the rest of the paper, we assume  $X$  to have such a Riemann surface structure when necessary.

### 2.2 Riemann uniformization theorem and elementary covering theory

The Riemann uniformization theorem asserts that a closed orientable Riemann surface  $X$  of genus  $g \geq 2$  is isomorphic to the orbit space  $\mathcal{H}/\Gamma$  of the hyperbolic upper half plane  $\mathcal{H}$  with the holomorphic structure inherited from  $\mathcal{H}$ , where  $\Gamma$  is a Fuchsian group with signature  $(g; -)$ . Such a group  $\Gamma$  is algebraically isomorphic to the fundamental group of  $X$  and so it has the presentation

$$\langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] \rangle, \tag{1}$$

where  $[a, b]$  denotes the commutator  $aba^{-1}b^{-1}$ . Furthermore, elementary covering theory shows that, having a group  $G$  of conformal automorphisms of  $X$  so represented, we have an isomorphism  $G \cong \Lambda/\Gamma$  for some Fuchsian group  $\Lambda$ , say with signature  $(h; m_1, \dots, m_r)$ , which means that it has the presentation

$$\langle x_1, \dots, x_r, a'_1, b'_1, \dots, a'_h, b'_h \mid x_1^{m_1}, \dots, x_r^{m_r}, x_1 \dots x_r [a'_1, b'_1] \dots [a'_h, b'_h] \rangle. \tag{2}$$

So each group  $G$  of conformal automorphisms of a closed Riemann surface  $X$  leads to a Fuchsian group  $\Lambda$  and an epimorphism  $\theta : \Lambda \rightarrow G$  with a torsion-free kernel (which in such case is isomorphic to the fundamental group of  $X$  without any extra conditions). Conversely an epimorphism  $\theta : \Lambda \rightarrow G$  with a torsion-free kernel  $\Gamma$  gives rise to an action of  $G$  on  $X = \mathcal{H}/\Gamma$  through conformal automorphisms defined by

$$gx = \pi(\lambda(h))$$

if  $g = \theta(\lambda)$  and  $x = \pi(h)$  for the canonical projection  $\pi : \mathcal{H} \rightarrow X$ . Such an epimorphism will be called a *smooth* or *surface-kernel epimorphism* throughout the whole paper.

In this paper, we will mainly be interested in triangle groups, being Fuchsian groups with signatures  $(0; m_1, m_2, m_3)$  which will also be abbreviated as  $(m_1, m_2, m_3)$ . They have the presentations

$$\langle x_1, x_2, x_3 \mid x_1^{m_1}, x_2^{m_2}, x_3^{m_3}, x_1x_2x_3 \rangle. \tag{3}$$

Furthermore, the Riemann–Hurwitz formula says in this special situation that

$$2(g - 1) = |G| \left( 1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} \right). \tag{4}$$

### 2.3 Harvey criterion

Let  $\Lambda$  be a Fuchsian group with signature  $(h; m_1, \dots, m_r)$  and let  $M = \text{lcm}(m_1, \dots, m_r)$ . Then there exists a smooth epimorphism from  $\Lambda$  onto  $\langle x \mid x^N \rangle$  if and only if

- (a)  $M = \text{lcm}(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_r)$  for all  $i$ ;
  - (b)  $M$  divides  $N$  and if  $h = 0$  then  $M = N$ ;
  - (c)  $r \neq 1$  and if  $h = 0$  then  $r \geq 3$ ;
  - (d) if  $N$  is even then the number of periods  $m_i$  with  $N/m_i$  odd is even.
- (5)

### 2.4 Maclachlan decomposition

In the case that  $h = 0, r = 3$ , condition (a) in (5) above for the signature  $(m_1, m_2, m_3)$  is equivalent to the following canonical decomposition:

$$\begin{cases} m_1 = AA_2A_3, \\ m_2 = AA_1A_3, \\ m_3 = AA_1A_2, \end{cases} \tag{6}$$

where  $A = \gcd(m_1, m_2, m_3)$  and  $A_k = \gcd(m_i, m_j)/A$ , for  $k \neq i, j$ . Note that the integers  $A_i$  are pairwise relatively prime and, by (b) in (5), that  $N = AA_1A_2A_3$ . Condition (d) in (5), states that  $N = AA_1A_2A_3$  even is equivalent to having just one of the  $A_i$  even. This decomposition has been discovered in [4]. The collection  $A, A_1, A_2, A_3$  will be called the *Maclachlan decomposition* of  $(m_1, m_2, m_3)$ . A signature  $(m_1, m_2, m_3)$  satisfying (5) for  $h = 0, r = 3$  will be called *admissible* or *N-admissible* and a quadruple  $(A, A_1, A_2, A_3)$  uniquely determined by (6) we will be called *admissible* as well.

### 2.5 Few words on topological conjugacy

Two orientation-preserving self-homeomorphisms  $\varphi_1, \varphi_2$  of a surface  $X$  are *topologically equivalent* if they are conjugate by an orientation-preserving self-homeomorphism  $f$  of  $X$ . We shall use conformal models for these actions described in Sects. 2.1 and 2.2. In the conformal setting, the general definition is that two conformal actions  $G_1$  and  $G_2$  on Riemann surfaces  $X_1$  and  $X_2$  given by smooth epimorphisms  $\theta_1 : \Lambda_1 \rightarrow G_1$  and  $\theta_2 : \Lambda_2 \rightarrow G_2$  are topologically equivalent if and only if the diagram

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\Phi} & \Lambda_2 \\ \theta_1 \downarrow & & \downarrow \theta_2 \\ G_1 & \xrightarrow{\Psi} & G_2 \end{array} \tag{7}$$

commutes for some isomorphisms  $\Phi : \Lambda_1 \rightarrow \Lambda_2, \Psi : G_1 \rightarrow G_2$ . The Nielsen isomorphism theorem asserts that  $\Phi$  can be chosen to be the conjugation by a self-homeomorphism  $f$  of  $\mathcal{H}$  and, throughout the paper, we understand that  $f$  preserves orientation.

Furthermore, the signature  $(m_1, m_2, m_3)$  is said to be *rigid* if for any group  $\Lambda$  with such signature, any two smooth epimorphism  $\theta_1, \theta_2 : \Lambda \rightarrow \mathbb{Z}_N$  are equivalent in the above sense. If an  $N$ -admissible signature is unique, up to ordering of  $m_1, m_2, m_3$ , and it is rigid, then  $N$  is said to be *rigid* order. If there is more than one  $N$ -admissible signature but all of them are rigid then the order is said to be *weakly rigid*. Finally if none of these holds, then the order is simply said to be *non-rigid*.

### 3 Rigid and weakly rigid orders

Given a positive integer  $N$  and a triangle Fuchsian group  $\Lambda$  with an  $N$ -admissible signature  $(m_1, m_2, m_3)$ , let

$$\mathcal{K} = \{k < m_2 \mid \gcd(k, m_2) = 1\}.$$

Fix a generator  $a$  of  $\mathbb{Z}_N$  and let  $\mathcal{L}$  be the subset of  $\mathcal{K}$  consisting of those  $k$  for which

$$a^{N/m_1+kN/m_2} \tag{8}$$

has order  $m_3$ . Observe that  $K = \varphi(m_2)$ , where  $\varphi$  is the Euler’s totient function, is the cardinality of  $\mathcal{K}$  and denote by  $L$  be cardinality of  $\mathcal{L}$ . Then we have the following lemma by which the number of topological equivalency classes of cyclic quasiplatonic actions of type  $(m_1, m_2, m_3)$  is bounded by  $L$ .

**Lemma 3.1** *An arbitrary smooth epimorphism  $\theta : \Lambda \rightarrow \mathbb{Z}_N = \langle a \rangle$  is topologically equivalent to  $\theta_k$  for some  $k \in \mathcal{L}$ , where  $\theta_k(x_1) = a^{N/m_1}$  and  $\theta_k(x_2) = a^{kN/m_2}$ .*

*Proof* Obviously we have

$$\theta(x_1) = a^{mN/m_1}, \quad \theta(x_2) = a^{k'N/m_2}, \quad \theta(x_3) = a^{l'N/m_3},$$

where  $\gcd(m, m_1) = \gcd(k', m_2) = \gcd(l', m_3) = 1$ , for  $\theta$  to preserve the orders of  $x_1, x_2$ , and  $x_3$ , and

$$m \frac{N}{m_1} + k' \frac{N}{m_2} + l' \frac{N}{m_3} \equiv 0 \pmod{N},$$

to guarantee  $\theta(x_1x_2x_3) = 1$ . Now let  $m'$  be the inversion of  $m$  modulo  $m_1$ . Let also  $A'_1$  be the maximal divisor of  $A_1 = N/m_1$  coprime to  $m_1$ . Then by the Chinese Remainder Theorem there exists  $\alpha$  such that  $\alpha \equiv m' \pmod{m_1}$  and  $\alpha \equiv 1 \pmod{A'_1}$ . Then  $\gcd(\alpha, N) = 1$  and

$$(a^{mN/m_1})^\alpha = (a^{N/m_1})^{m\alpha} = a^{N/m_1}.$$

So  $k = k'\alpha \in \mathcal{L}$  and for  $\Phi = \text{id}_\Lambda$  and  $\Psi(a) = a^\alpha$  in (7), we obtain that  $\theta$  is topologically equivalent to  $\theta_k$  defined by

$$\theta_k(x_1) = a^{N/m_1}, \quad \theta_k(x_2) = a^{kN/m_2}, \quad \theta_k(x_3) = a^{lN/m_3}, \tag{9}$$

where  $l = l'\alpha$ . □

The next lemma will be used later, to decide when two smooth epimorphisms  $\theta_{k_1}$  and  $\theta_{k_2}$  from the Lemma 3.1 define topologically equivalent actions.

**Lemma 3.2** *Let  $S = \text{Stab}_{\mathbb{Z}_N^*}(a^{N/m_1})$  be the stabilizer of  $a^{N/m_1}$  in the group  $\text{Aut}(\langle a \rangle)$  and let  $S$  be its cardinality. Then  $|S| = \varphi(N)/\varphi(m_1)$ .*

*Proof* Indeed

$$\begin{aligned} \varphi(m_1) &= |\text{Orb}_{\mathbb{Z}_N^*}(a^{N/m_1})| \\ &= [\mathbb{Z}_N^* : \text{Stab}_{\mathbb{Z}_N^*}(a^{N/m_1})] \\ &= \frac{\varphi(N)}{|\text{Stab}_{\mathbb{Z}_N^*}(a^{N/m_1})|} \end{aligned}$$

and so the assertion. □

**Lemma 3.3** *Each element of  $S$  acts on  $\mathcal{K}$  without fixed points. That is, the group  $S$  acts faithfully on  $\mathcal{K}$ .*

*Proof* Let us take  $k \in \mathcal{K}$  and  $s, s' \in S$ . Recall that these mean  $\gcd(k, m_2) = 1$  and  $\gcd(s', N) = \gcd(s, N) = 1$ . Furthermore, from the definition of  $S$  we have  $(a^{N/m_1})^s = a^{N/m_1}$  which in turn gives  $N(s - 1)/m_1 \equiv 0 \pmod{N}$ . This means in particular that  $m_1$

divides  $s - 1$ . Similarly we can show that  $m_1$  divides  $s' - 1$ . Consequently  $m_1$  divides  $s - s'$ . Suppose that

$$(a^{Nk/m_2})^s = (a^{Nk/m_2})^{s'}$$

Then

$$N \frac{k(s - s')}{m_2} \equiv 0 \pmod{N}.$$

We see that  $m_2$  divides  $k(s - s')$  and therefore  $m_2$  divides  $s - s'$  which give  $s \equiv s' \pmod{N}$ . □

Now we shall look for  $L$ . The decomposition from (6) will be crucial here. We have

$$N = \text{lcm}(m_1, m_2, m_3) = \text{lcm}(m_1, m_2) = \text{lcm}(m_1, m_3) = \text{lcm}(m_2, m_3),$$

where  $N = AA_1A_2A_3$ ,  $N/m_i = A_i$ . Furthermore we know by (5) that if  $N$  is even, then the number of those  $A_i$  which are odd is even and therefore, since all  $A_i$  can not be even, only one of them is even. With these notations, the elements of the form (8) can be written as  $a^{A_1+kA_2}$  and so the set  $\mathcal{L}$  can be seen as

$$\mathcal{L} = \{k < AA_1A_3 \mid \text{gcd}(A_1 + kA_2, N) = A_3, \text{gcd}(k, AA_1A_3) = 1\}. \tag{10}$$

To state the next theorem, we define  $\varphi'(1) = 1$  and for given a prime decomposition  $n = p_1^{\alpha_1} \dots p_r^{\alpha_r} > 1$  we put

$$\varphi'(n) = \prod_{i=1}^r (p_i - 2) p_i^{\alpha_i - 1}. \tag{11}$$

*Remark 3.4* Observe the analogy of the function  $\varphi'$  defined by (11) with the classical Euler's totient function  $\varphi$ , both in this what concern its algebraic properties and the explicit formula. We see that both of them can be seen as particular cases of function

$$\varphi_k(n) = \prod_{i=1}^r (p_i - k) p_i^{\alpha_i - 1}$$

which can be defined for arbitrary  $k$ . Namely we have  $\varphi = \varphi_1$  and  $\varphi' = \varphi_2$ .

**Lemma 3.5** *Let  $D$  be the largest divisor of  $A$  coprime with  $A_1A_2A_3$  and let  $B = A/D$ . Then*

$$L = \varphi(A_1B)\varphi'(D). \tag{12}$$

*Proof* Let  $D = p_1^{\delta_1} \dots p_d^{\delta_d}$  and  $N = q_{11}^{\alpha_1} \dots q_{1a}^{\alpha_a} q_{21}^{\beta_1} \dots q_{2b}^{\beta_b} q_{31}^{\gamma_1} \dots q_{3c}^{\gamma_c} D$ , for different primes  $p_i$  and  $q_{ij}$ . Then

$$A_1 = q_{11}^{\alpha'_1} \dots q_{1a}^{\alpha'_a}, \quad A_2 = q_{21}^{\beta'_1} \dots q_{2b}^{\beta'_b}, \quad A_3 = q_{31}^{\gamma'_1} \dots q_{3c}^{\gamma'_c}$$

for some  $\alpha'_i, \beta'_i, \gamma'_i$  in ranges  $0 < \alpha'_i \leq \alpha_i, 0 < \beta'_i \leq \beta_i, 0 < \gamma'_i \leq \gamma_i$ . Let

$$B_1 = q_{11}^{\alpha_1 - \alpha'_1} \dots q_{1a}^{\alpha_a - \alpha'_a}, \quad B_2 = q_{21}^{\beta_1 - \beta'_1} \dots q_{2b}^{\beta_b - \beta'_b}, \quad B_3 = q_{31}^{\gamma_1 - \gamma'_1} \dots q_{3c}^{\gamma_c - \gamma'_c}.$$

Then  $B = B_1B_2B_3$  and  $D$  together with

$$A_1B_1 = q_{11}^{\alpha_1} \dots q_{1a}^{\alpha_a}, \quad A_2B_2 = q_{21}^{\beta_1} \dots q_{2b}^{\beta_b}, \quad A_3B_3 = q_{31}^{\gamma_1} \dots q_{3c}^{\gamma_c}$$

are all pairwise coprime. Therefore

$$\mathbb{Z}_N \cong \mathbb{Z}_{A_1B_1} \oplus \mathbb{Z}_{A_2B_2} \oplus \mathbb{Z}_{A_3B_3} \oplus \mathbb{Z}_{p_1^{\delta_1}} \oplus \dots \oplus \mathbb{Z}_{p_r^{\delta_d}}$$

and

$$\mathbb{Z}_{AA_1A_3} \cong \mathbb{Z}_{A_1B_1} \oplus \mathbb{Z}_{B_2} \oplus \mathbb{Z}_{A_3B_3} \oplus \mathbb{Z}_{p_1^{\delta_1}} \oplus \dots \oplus \mathbb{Z}_{p_r^{\delta_d}}$$

Hence every element  $x \in \mathbb{Z}_{AA_1A_3}$  can be represented as a sequence

$$(x_1, x_2, x_3, x'_1, \dots, x'_r)$$

with  $x_1 \in \mathbb{Z}_{A_1B_1}$ ,  $x_2 \in \mathbb{Z}_{B_2}$ ,  $x_3 \in \mathbb{Z}_{A_3B_3}$  and  $x'_j \in \mathbb{Z}_{p_j^{\delta_j}}$ . Similarly the elements of  $\mathbb{Z}_N$  are represented. Moreover if  $x$  is invertible in  $\mathbb{Z}_{AA_1A_3}^*$  then all components of the sequence corresponding to  $x$  are invertible in suitable rings.

Now, for  $x \in \mathbb{Z}_{AA_1A_3}$ , the correspondence  $x \rightarrow A_1 + xA_2 \pmod N$  is an injection from  $\mathbb{Z}_{AA_1A_3}$  to  $\mathbb{Z}_N$ . In fact, if  $A_1 + xA_2 \equiv A_1 + x'A_2 \pmod N$ , then  $N$  divides  $A_2(x - x')$  which in turn means that  $AA_1A_3$  divides  $x - x'$ .

Let  $(A_1 + x_1A_2, A_1 + x_2A_2, A_1 + x_3A_2, A_1 + x'_1A_2, \dots, A_1 + x'_rA_2)$  be the sequence corresponding to  $A_1 + xA_2$  with entries taken modulo a suitable number. If we assume that  $x \in \mathbb{Z}_{AA_1A_3}^*$ , we need to know how many images of the function satisfy the condition  $\gcd(A_1 + xA_2, N) = A_3$ . That is how many corresponding sequences of  $A_1 + xA_2$  for invertible  $x \in \mathbb{Z}_{AA_1A_3}$  have all components invertible beside the third one, which has to be of the form  $A_1 + x_3A_2 = tA_3$ , where  $\gcd(t, B_3) = 1$ . (Note that if  $\gcd(t, B_3) = t_1 > 1$ , then  $\gcd(A_1 + xA_2, N) = t_1A_3 > A_3$ .)

For counting all possible values of  $x$  satisfying the above, we determine the numbers of possible choices of  $x_i$ 's corresponding to  $x$ . Note first that for  $x \in \mathbb{Z}_{AA_1A_3}^*$  all possible values of  $t$  satisfying  $A_1 + x_3A_2 = tA_3$  are achievable. Actually, the condition  $A_1 + x_3A_2 \equiv tA_3 \pmod{A_3B_3}$  means  $A_2x_3 \equiv (tA_3 - A_1) \pmod{A_3B_3}$ . Since  $tA_3 - A_1$  and  $A_2$  are relatively prime to  $A_3B_3$ , there exist such invertible  $x_3$  in  $\mathbb{Z}_{A_3B_3}$ . That is, there are  $\varphi(B_3)$  choices for  $t$  and therefore  $\varphi(B_3)$  choices for  $x_3$ .

Next, for  $x \in \mathbb{Z}_{AA_1A_3}^*$  we have  $\gcd(A_1 + xA_2, A_1B_1A_2B_2) = 1$ , so for a fixed  $t$  satisfying  $A_1 + xA_2 = tA_3$ ,  $\gcd(t, B_3) = 1$ , we have  $\varphi(A_1B_1)\varphi(B_2)$  possibilities for the first two entries of the sequence.

Finally, let us consider  $A_1 + x'_jA_2$ . Write  $x'_j = y_j + p_jz_j$ , where  $1 \leq y_j \leq p_j - 1$  and  $0 \leq z_j \leq p_j^{\delta_j - 1} - 1$ . Then  $A_1 + x'_jA_2 = (A_1 + y_jA_2) + p_jz_jA_2$  and for exactly one value of  $y_j$  we have  $A_1 + y_jA_2 \equiv 0 \pmod{p_j}$ . Therefore we have  $(p_j - 2)p_j^{\delta_j - 1}$  values of  $x'_j$  modulo  $p_j^{\delta_j}$  such that  $A_1 + x'_jA_2$  is coprime to  $p_j$ .

Summarizing we have

$$\begin{aligned} L &= \varphi(B_3)\varphi(A_1B_1)\varphi(B_2) \prod_{i=1}^r (p_i - 2)p_i^{\delta_i - 1} \\ &= \varphi(A_1B_1B_2B_3)\varphi'(D) \\ &= \varphi(A_1B)\varphi'(D). \end{aligned}$$

□

We obtain the following as a corollary of the previous result.

**Lemma 3.6** *Let  $D$  be the largest divisor of  $A$  coprime with  $A_1A_2A_3$  and let  $B = A/D$ . Then  $L = S$  if and only if  $B \in \{1, 2\}$  and  $D \in \{1, 3\}$ .*

*Proof* Let  $B = B_1B_2B_3$ , where  $B_i$  are defined in the proof of Lemma 3.5. Then

$$\begin{aligned} S &= \frac{\varphi(N)}{\varphi(m_1)} \\ &= \frac{\varphi(A_1B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(D)}{\varphi(AA_2A_3)} \\ &= \frac{\varphi(A_1B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(D)}{\varphi(B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(D)} \\ &= \frac{\varphi(A_1B_1)}{\varphi(B_1)}. \end{aligned}$$

Hence  $L = S$  if and only if  $\varphi(A_1B)\varphi(B_1)\varphi'(D) = \varphi(A_1B_1)$ . Since, in general,  $\varphi(A_1B) \geq \varphi(A_1B_1)$ , we obtain  $\varphi(A_1B) = \varphi(A_1B_1)$  and  $\varphi(B_1) = \varphi'(D)$ . This can only happen when  $D \in \{1, 3\}$  and  $B_1 \in \{1, 2\}$ . If  $B_1 = 2$  then for  $B > B_1$  we get a contradiction as in this case  $\varphi(A_1B) > \varphi(A_1B_1)$ . So  $B = B_1 = 2$ . If  $B_1 = 1$  then we get  $\varphi(A_1B) = \varphi(A_1)$  which means that either  $B = 1$  or  $B = 2$  and  $A_1$  is odd.

Conversely, suppose  $D \in \{1, 3\}$  and  $B_1 \in \{1, 2\}$ . Then  $L = \varphi(A_1B)\varphi(B_1)$  and  $S = \varphi(A_1B_1)/\varphi(B_1)$ . If  $B = 1$  then also  $B_1 = 1$  and obviously  $L = S$ . If  $B = 2$ , then either  $B_1 = 1$  or  $B_1 = 2 = B$ . In the first case  $A_1$  is odd and then  $L = \varphi(A_1) = S$ . In the second case,  $L = \varphi(2A_1) = S$ . The lemma is proved. □

**Lemma 3.7** *Let  $A, A_1, A_2, A_3$  be the Maclachlan decomposition 2.4 of an admissible signature  $(m_1, m_2, m_3)$  for  $N$  with  $m_1 = m_2$  and let  $S = 1$ . Then  $1 < L \leq 6$  if and only if it is given in Table 1 or in Table 2.*

*Proof* Observe that  $S = 1$  if and only if  $m_1 = N$ , as by Sect. 2.3,  $N = \text{lcm}(m_1, m_2) = m_1 = m_2$ . The list of all cases for which  $L \leq 6$  can be easily derived from the formula (12). First of all, it follows from it that  $D \in \{1, 3, 5, 7, 9, 15, 21\}$  (note that  $D$  cannot be even). After fixing  $D$  we take the condition  $\varphi'(D)\varphi(A_1B) \leq 6$  to be satisfied.

Let first  $m_3 = N$  and let us list all cases giving a signature  $(N, N, N)$  with  $1 < L \leq 6$ . Here we have  $A = N, A_1 = A_2 = A_3 = 1, D = N, B = 1$  and so  $1 < L \leq 6$  if and only if it is given in Table 1. Observe here that  $L = 1$  implies  $N = 3$ .

Now let  $m = m_3 < N$ . In this case we have  $A_1 = A_2 = 1$  and so  $B_1 = B_2 = 1$ . The only condition on  $A_3$  is that it must be divisible by primes dividing  $B = B_3$  if  $B > 1$ , and coprime to  $D$ . So we have all cases listed in Table 2. □

**Table 1**  $L \leq 6$  for  $m_1 = m_2 = m_3 = N$

$A_1$	$A_2$	$A_3$	$A$	$D$	$B$	$L$	$N$
1	1	1	5	5	1	3	5
1	1	1	7	7	1	5	7
1	1	1	9	9	1	3	9
1	1	1	15	15	1	3	15
1	1	1	21	21	1	5	21

**Table 2**  $L \leq 6$  for  $m_1 = m_2 = N, m_3 < N$

$A_1$	$A_2$	$A_3$ where $\gcd(A_3, D) = 1$	$A$	$D$	$B$	$L$	$N$ divisible by
1	1	$N/3, N/4, N/6$	3, 4, 6	1	3, 4, 6	2	9, 8, 36
		$N/5, N/8, N/10, N/12$	5, 8, 10, 12		5, 8, 10, 12	4	25, 16, 100, 72
		$N/7, N/9, N/18$	7, 9, 18		7, 9, 18	6	49, 27, 108
		$N/12$	12	3	4	2	24
		$N/15, N/24, N/30$	15, 24, 30		5, 8, 10	4	45, 48, 300
		$N/21$	21		7	6	147
		$N/5$	5	5	1	3	5
		$N/15, N/20, N/30$	15, 20, 30		3, 4, 6	6	45, 40, 180
		$N/7$	7	7	1	5	7
		$N/9, N/18$	9, 18	9	1, 2	3	9, 36
		$N/36$	36		4	6	72
		$N/15, N/30$	15, 30	15	1, 2	3	15, 60
		$N/60$	60		4	6	120
		$N/21, N/42$	21, 42	21	1, 2	5	21, 84

**Proposition 3.8** *If  $S = L$  for an admissible signature  $(m_1, m_2, m_3)$  then this signature is rigid. The converse holds for all admissible signatures except for  $(5, 5, 5), (9, 9, 9), (15, 15, 15)$ , which are rigid and  $(S, L) = (1, 3)$ , and signatures:*

- $(N, N, 3), \quad \text{for } N = 9t, \quad t \in \mathbb{N},$
- $(N, N, 4), \quad \text{for } N = 16t, \quad t \in \mathbb{N},$
- $(N, N, 6), \quad \text{for } N = 36t, \quad t \in \mathbb{N},$
- $(N, N, 12), \quad \text{for } N = 48t, \quad t \in \mathbb{N}, 3 \nmid t$

which are rigid and  $(S, L) = (1, 2)$ .

*Proof* For  $L = S$ , the rigidity follows from the faithfulness of the action of  $S$  on  $\mathcal{L}$ , which we proved in the Lemma 3.3. The converse is also true if the  $m_i$  are pairwise distinct or  $L > 6$ . So assume that  $L \leq 6$  and  $m_i$  are not pairwise distinct. Then  $m_1$  and  $m_2$  are equal to  $N$ , and in particular  $S = 1$  and the list of all configurations of  $N, m_1, m_2, m_3$  are given in Lemma 3.7. The case  $m_3 = N$  is easy; here one can show that  $(5, 5, 5), (9, 9, 9)$  and  $(15, 15, 15)$  are rigid signatures, while each of the signatures  $(7, 7, 7)$  and  $(21, 21, 21)$  is not rigid since it allows two nonequivalent actions corresponding to  $(a, a, a^5), (a, a^2, a^4)$  and  $(a, a, a^{19}), (a, a^4, a^{16})$ , respectively.

The case  $m_3 < N$  described in Table 2 is a bit more involved. First, note that there is no rigid action for  $L > 2$  and so we can assume that  $(m_1, m_2, m_3) = (N, N, 3), (N, N, 4), (N, N, 6)$  or  $(N, N, 12)$ .

It is easy to show that the signatures  $(N, N, 3)$  and  $(N, N, 6)$  are rigid. For  $(N, N, 3)$ , we know  $N = 9t$  and we have to consider the actions  $(a, a^{6t-1}, a^{3t})$  and  $(a, a^{3t-1}, a^{6t})$  as  $a^{3t}$  and  $a^{6t}$  are the only elements of order 3 in the group  $\langle a \mid a^{9t} \rangle$ . But the function determined by the correspondence  $a \mapsto a^{3t-1}$  moves  $a^{6t-1}$  to  $a^{(3t-1)(6t-1)} = a$  and  $a^{3t}$  moves to  $a^{(3t-1)3t} = a^{-3t} = a^{6t}$ . Hence these two actions are equivalent. Similarly, for the signature  $(N, N, 6)$  we have  $N = 36t$  and in the group  $\langle a \mid a^{36t} \rangle$  we have only two elements of order 6, namely  $a^{6t}$  and  $a^{-6t}$ . So we have to consider the actions  $(a, a^{30t-1}, a^{6t})$  and  $(a, a^{6t-1}, a^{-6t})$  only, which are equivalent by the analogous arguments as for the previous case.

Let us consider the signature  $(N, N, 4)$ ,  $N = 8t$ . Similar to the previous cases, we have two actions:

$$(a, a^{2t-1}, a^{6t}), (a, a^{6t-1}, a^{2t}).$$

Now the function induced by the correspondence  $a \mapsto a^{6t-1}$  moves  $a^{2t-1}$  to  $a^{(2t-1)(6t-1)} = a^{4t^2+1}$ . If  $t$  is even then this element is equal to  $a$  and we have equivalence of both actions. If  $t$  is odd then  $t^2 \equiv 1 \pmod{8}$  hence  $4t^2 + 1 \equiv 5 \pmod{8}$ , which means that  $a^{4t^2+1} \neq a$  and so the actions are not equivalent.

Finally take the signature  $(N, N, 12)$ , where  $N = 24t$ . Note that in this case, 3 does not divide  $t$  as by Table 2,  $A_3 = N/m_3$  is coprime to  $D = 3$ . Since, a group of order  $24t$  generated by an element  $a$  has exactly four elements of order 12,  $a^{2t}$ ,  $a^{10t}$ ,  $a^{14t}$  and  $a^{22t}$ , we have four possibilities for actions:

$$(a, a^{2t-1}, a^{22t}), (a, a^{10t-1}, a^{14t}), (a, a^{14t-1}, a^{10t}), (a, a^{22t-1}, a^{2t}).$$

But only two these are of the type  $(N, N, 12)$ . In fact, if  $t \equiv 1 \pmod{3}$ , then  $10t - 1, 22t - 1 \equiv 0 \pmod{3}$  and if  $t \equiv 2 \pmod{3}$ , then  $2t - 1, 14t - 1 \equiv 0 \pmod{3}$ . Let us consider the first case. The function induced by the correspondence  $a \mapsto a^{14t-1}$  moves  $a^{2t-1}$  to  $a^{(2t-1)(14t-1)} = a^{4t^2+8t+1}$ . Since  $4t^2 + 8t + 1 = 4t(t + 2) + 1$ , we see that only for even  $t$  we have  $4t^2 + 8t + 1 \equiv 1 \pmod{24t}$ . Hence, only in this case both actions are equivalent. Let  $t \equiv 2 \pmod{3}$  and take the function induced by the correspondence  $a \mapsto a^{22t-1}$ . It moves  $a^{10t-1}$  to  $a^{(22t-1)(10t-1)} = a^{4t^2+16t+1}$ . Again, since  $4t^2 + 16t + 1 = 4t(t + 4) + 1 \equiv 1 \pmod{24t}$  only for even  $t$ , the actions  $(a, a^{10t-1}, a^{14t})$  and  $(a, a^{22t-1}, a^{2t})$  are equivalent just for  $t$  even which completes the proof.  $\square$

All of these allow us to deduce our principal result in this paper.

**Theorem 3.9** *Let  $\mathbb{Z}_N$  be a cyclic action with signature  $(m_1, m_2, m_3)$  and let  $A, A_1, A_2, A_3$  be the Maclachlan decomposition (6) of  $(m_1, m_2, m_3)$ . Then the action is weakly rigid if and only if one the following happen*

- (1)  $A = 1, 2$ ,
- (2)  $A = 3, 6$  and  $\gcd(A_1 A_2 A_3, 3) = 1$ ,
- (3)  $A = 5, 9, 15$  and  $A_1 = A_2 = A_3 = 1$ ,
- (4)  $A = 3, A_1 = A_2 = 1$ , and  $A_3 \equiv 0 \pmod{3}$ ,
- (5)  $A = 4, A_1 = A_2 = 1$ , and  $A_3 \equiv 0 \pmod{4}$ ,
- (6)  $A = 6, A_1 = A_2 = 1$ , and  $A_3 \equiv 0 \pmod{6}$ ,
- (7)  $A = 12, A_1 = A_2 = 1, A_3 \equiv 0 \pmod{4}$ , and  $\gcd(A_3/4, 3) = 1$ .

*In particular the signature  $(m_1, m_2, m_3)$  is not rigid for  $A \notin \{1, 2, 3, 4, 5, 6, 12\}$ .*  $\square$

### 4 Examples, equations, remarks, and open problems

The signature  $(m_1, m_2, m_3)$  is said to be  $(g, N)$ -admissible if there exists a self-homeomorphism  $\varphi$  of order  $N$  acting on a closed orientable surface  $X$  of genus  $g$  so that  $X/\varphi$  is the sphere ramified over three points with ramification indices  $m_1, m_2, m_3$  or equivalently having three singular orbits of the lengths  $N/m_1, N/m_2, N/m_3$ .

### 4.1 Rigid signature vs rigid order

Let  $X$  be a fixed, closed orientable surface of genus  $g \geq 2$ . The notion of rigid cyclic action on  $X$  leads us to define the concept of rigid order  $N$  for  $g \geq 2$  and, similarly, the notion of weakly rigid action gives rise to the concept of rigid signature for  $g$ . This, in turn, allows to define the concept of weakly rigid order which means that all admissible signatures are rigid. In this subsection we shall consider cyclic actions with the orbit genus zero and having three singular orbits of the lengths  $m_1, m_2, m_3$ , calling such actions *quasi-platonic*. Let  $N_1, \dots, N_k$  be all possible orders of all such actions on  $X$ . Now  $N = N_i$  can fail to be rigid order for two reasons. The first is that there may exist few distinct admissible signatures and, in principle, some of them may be rigid and the others not. The second, more subtle reason for non-rigidity of  $N$  for genus  $g$ , is that although there may exist just one admissible signature, this signature is not rigid. The previously mentioned results of Nakagawa and Hirose imply that the cyclic actions of orders  $4g + 1, 4g, 3g + 3, 3g$  on closed orientable surfaces of genus  $g$ , where  $g \geq 12$  are rigid. Additionally, if  $g \not\equiv 2 \pmod{3}$ , then  $N$  is also rigid. All of these phenomena, which show that rigidity of cyclic actions is essentially coarser than weak rigidity, are well illustrated in Tables 3 and 4.

### 4.2 Cyclic actions with fixed-points free self-homeomorphisms

The pair  $(g, N) = (11, 30)$  in Table 4 is an interesting case because these are the smallest  $g$  and  $N$  values for which there exists an action with fixed-point free acting self-homeomorphism. There are many more cyclic actions containing fixed point free acting self-homeomorphisms. The other two examples are  $(g, N) = (16, 42)$  or  $(25, 60)$  with signatures  $(21, 14, 6)$  and  $(20, 15, 12)$ , respectively.

Another interesting case is  $N = 210$  in which  $\mathbb{Z}_N$  can act on surfaces of three consecutive genera,  $g = 95, 96,$  and  $97$ . The corresponding signatures are  $(70, 42, 15), (70, 30, 21),$  and  $(42, 35, 30)$ , respectively, all of which are rigid since  $A = 1$  in each case. Additionally, there are no other 210-admissible signatures for  $g = 95$ , so 210 is a rigid order for  $g = 95$ . For  $g = 96$ , we have one more 210-admissible signature,  $(210, 42, 15)$ , which gives rigid action. In fact, let  $G$  be a cyclic group of order 210 and let us represent elements of  $G$  as 4-tuples of elements of the direct product  $G = \langle a_2 \rangle \times \langle a_3 \rangle \times \langle a_5 \rangle \times \langle a_7 \rangle$ , where  $a_i$  has order  $i$ . Now for a fixed element  $x$  of order 210, say  $x = (a_2, a_3, a_5, a_7)$  we have exactly one element  $y$  of order 42 such that  $xy$  has order 15, namely  $y = (a_2, a_3, 1, a_7^{-1})$ . So we see that  $N = 210$  is a weakly rigid order for  $g = 96$ . Finally for  $g = 97$ , we have three more 210-admissible signatures  $(210, 30, 21), (210, 70, 15),$  and  $(210, 105, 14)$  whose Maclachlan decomposition are respectively  $(3, 1, 7, 10), (5, 1, 3, 14),$  and  $(7, 1, 2, 15)$ . The first case defines rigid action, as for  $x = (a_2, a_3, a_5, a_7)$  again we have exactly one element  $y = (a_2, a_3, a_5^{-1}, 1)$  of order 30 such that  $xy$  has order 21. The last two cases are not rigid. In the case  $(210, 70, 15)$ , for  $x = (a_2, a_3, a_5, a_7)$ , we have exactly three elements  $y$  of order 70 such that  $xy$  has order 15, namely  $y = (a_2, 1, a_5^k, a_7^{-1}), k = 1, 2, 3$ . In the case  $(210, 105, 14)$ , for  $x = (a_2, a_3, a_5, a_7)$ , we have exactly five elements  $y$  of order 105 such that  $xy$  has order 14, namely  $y = (1, a_3^{-1}, a_5^{-1}, a_7^k), k = 1, 2, 3, 4, 5$ . So for  $g = 97, N = 210$  is not a weakly rigid order. So, all three phenomena: rigidity, weak rigidity and not weak rigidity can occur for cyclic action, allowing fixed point free self-homeomorphisms for the same order, on surfaces of three consecutive genera.

**Table 3** Rigidity of quasi-platonic cyclic actions on surfaces of genera  $2 \leq g \leq 7$ 

Genus	Order	Signature	Rigidity of order	Rigidity of signature
2	10	(10, 5, 2)	Rigid	Rigid
	8	(8, 8, 2)	Rigid	Rigid
	6	(6, 6, 3)	Rigid	Rigid
	5	(5, 5, 5)	Rigid	Rigid
3	14	(14, 7, 2)	Rigid	Rigid
	12	(12, 4, 3)	Weakly-rigid	Rigid
		(12, 12, 2)		Rigid
	9	(9, 9, 3)	Rigid	Rigid
	8	(8, 8, 4)	Non-rigid	Non-rigid
	7	(7, 7, 7)	Non-rigid	Non-rigid
4	18	(18, 9, 2)	Rigid	Rigid
	16	(16, 16, 2)	Rigid	Rigid
	15	(15, 5, 3)	Rigid	Rigid
	12	(12, 6, 4)	Weakly-rigid	Rigid
		(12, 12, 3)		Rigid
	10	(10, 10, 5)	Non-rigid	Non-rigid
	9	(9, 9, 9)	Rigid	Rigid
5	22	(22, 11, 2)	Rigid	Rigid
	20	(20, 20, 2)	Rigid	Rigid
	15	(15, 15, 3)	Rigid	Rigid
	12	(12, 12, 6)	Rigid	Rigid
	11	(11, 11, 11)	Non-rigid	Non-rigid
6	26	(26, 13, 2)	Rigid	Rigid
	24	(24, 24, 2)	Rigid	Rigid
	21	(21, 7, 3)	Rigid	Rigid
	20	(20, 5, 4)	Rigid	Rigid
	18	(18, 18, 3)	Rigid	Rigid
	16	(16, 16, 4)	Rigid	Rigid
	15	(15, 15, 5)	Non-rigid	Non-rigid
	14	(14, 14, 7)	Non-rigid	Non-rigid
	13	(13, 13, 13)	Non-rigid	Non-rigid
7	30	(30, 15, 2)	Rigid	Rigid
	28	(28, 28, 2)	Rigid	Rigid
	24	(24, 8, 3)	Rigid	Rigid
	21	(21, 21, 3)	Rigid	Rigid
	20	(20, 10, 4)	Rigid	Rigid
	18	(18, 9, 6)	Non-rigid	Non-rigid
	16	(16, 16, 8)	Non-rigid	Non-rigid
	15	(15, 15, 15)	Rigid	Rigid

**Table 4** Rigidity of quasi-platonic cyclic actions on surfaces of genera  $8 \leq g \leq 11$

Genus	Order	Signature	Rigidity of order	Rigidity of signature
8	34	(34, 17, 2)	Rigid	Rigid
	32	(32, 32, 2)	Rigid	Rigid
	24	(24, 24, 3)	Rigid	Rigid
	20	(20, 20, 5)	Non-rigid	Non-rigid
	18	(18, 18, 9)	Non-rigid	Non-rigid
	17	(17, 17, 17)	Non-rigid	Non-rigid
9	38	(38, 19, 2)	Rigid	Rigid
	36	(36, 36, 2)	Rigid	Rigid
	30	(30, 10, 3)	Rigid	Rigid
	28	(28, 7, 4)	Rigid	Rigid
	27	(27, 27, 3)	Rigid	Rigid
	24	(24, 8, 6)	Non-rigid	Rigid
		(24, 24, 4)		Non-rigid
	21	(21, 21, 7)	Non-rigid	Non-rigid
	20	(20, 20, 10)	Non-rigid	Non-rigid
	19	(19, 19, 19)	Non-rigid	Non-rigid
10	42	(42, 21, 2)	Rigid	Rigid
	40	(40, 40, 2)	Rigid	Rigid
	33	(33, 11, 3)	Rigid	Rigid
	30	(30, 6, 5)	Weakly-rigid	Rigid
		(30, 30, 3)		Rigid
	28	(28, 14, 4)	Rigid	Rigid
	25	(25, 25, 5)	Non-rigid	Non-rigid
	24	(24, 24, 6)	Non-rigid	Rigid
		(24, 12, 8)		Non-rigid
	22	(22, 22, 11)	Non-rigid	Non-rigid
11	46	(46, 23, 2)	Rigid	Rigid
	44	(44, 44, 2)	Rigid	Rigid
	33	(33, 33, 3)	Rigid	Rigid
	30	(15, 10, 6)	Rigid	Rigid
	24	(24, 24, 12)	Non-rigid	Non-rigid
	23	(23, 23, 23)	Non-rigid	Non-rigid

### 4.3 Infinite series of non-rigidity examples

*Example 1* Let  $p$  be an odd prime and let  $\mathbb{Z}_p^n$  be a quasi-platonic action on a closed orientable surface  $X = X_g$  corresponding to a signature  $(m_1, m_2, m_3)$ , where  $m_1 \geq m_2 \geq m_3$ . By Theorem 3.9, this action is not rigid for  $p > 7$  or  $p = 5$  with  $n > 1$ . By (5),  $m_1 = m_2 = p^n$  and  $m_3 = p^m$  for some  $m \leq n$ . We can find explicit examples of non-conjugated actions. Let us first assume  $m = n$ . Then

$$(a, a, a^{-2}) \text{ and } (a, a^2, a^{-3})$$

are non-equivalent under the action of  $\mathbb{Z}_{p^n}^* \rtimes S_3$ . These two signatures correspond to the algebraic curves

$$y^{p^n} = x(x - 1) \text{ and } y^{p^n} = x(x - 1)^2.$$

Now let  $m < n$ . Then for arbitrary  $q < p$ ,  $(a, a^{qp^{n-1}}, a^{qp^n})$  is a defining signature. For  $q = 1$  and  $q$  not satisfying the congruence

$$(p^{n-m} - 1)(qp^{n-m} = 1) \equiv 1 \pmod{p^n},$$

the corresponding curves given by

$$y^{p^n} = x(x - 1)^{qp^{n-1}}$$

are not isomorphic.

*Example 2* Let  $p, q$  be distinct odd primes with  $p < q$ , and let  $\mathbb{Z}_{pq}$  act on a surface  $X$  defined by an admissible signature  $(m_1, m_2, m_3)$ . Then for  $(m_1, m_2, m_3) = (p, q, pq)$  the action is rigid while for  $(m_1, m_2, m_3) = (pq, pq, pq)$  we have two generating signatures of elements from  $\mathbb{Z}_{pq}$ ,

$$(a, a, a^{-2}) \text{ and } (a, a^2, a^{-3}),$$

which define non-equivalent actions for  $p \geq 5$ . By direct calculations we obtain non-rigidity for  $p = 3$  except for  $q = 5$  and  $q = 7$ . These two signatures correspond to the algebraic curves

$$C_1 := y^{pq} = x(x - 1) \text{ and } C_2 := y^{pq} = x(x - 1)^2.$$

### 4.4 Conformal rigidity of weakly rigid actions

Observe that a triangle in the hyperbolic plane is determined, up to isometry, by its angles and so a Fuchsian group with signature  $(0; k, l, m)$  is unique up to conjugation by an isometry. This means that a rigid signature determines uniquely the structure described in Sect. 2.1. In particular, for a rigid order  $N$ , there is just one conformal structure on  $X_g$  for which  $\mathbb{Z}_N$  is the group of its conformal automorphisms. So the topological rigidity implies conformal rigidity.

### 4.5 Real forms

Observe that all quasi-platonic surfaces with cyclic actions are symmetric, due to the result of Singerman [9], since the map  $a \mapsto a^{-1}, b \mapsto b^{-1}$  induces an automorphism of any abelian group generated by  $a$  and  $b$ . The topological types of the corresponding real forms have been found in [3]. If all periods corresponding to the action are distinct then  $\mathbb{Z}_N$  is the full group of automorphisms of the surface  $X$  and it has one or two distinct real forms, according to  $N$ , being odd or even, respectively. In addition these forms are both connected and nonseparable. If some of periods are equal, say  $a$  and  $b$  have the same order, then the situation is more involved, because there is another automorphism of  $\mathbb{Z}_N$  induced by the map  $a \mapsto b^{-1}, b \mapsto a^{-1}$ . This implies that  $\mathbb{Z}_N$  is no longer the full group of automorphisms of  $X$ , so  $X$  may have 2, 3, or 4 distinct real forms. They can be both separable or not and connected or not, or even purely imaginary.

## 4.6 Open problems

- (1) In [3], the authors give equations of the corresponding surfaces  $X$ , and explicit formulas for the generators of  $\text{Aut}^{\pm}(X)$ . It could be interesting to find defining equations for the corresponding real forms, at least for the rigid signatures described in this paper.
- (2) Find sharp bounds for the number  $R(g)$  of rigid and  $WR(g)$  of weakly rigid actions on a closed orientable surface of genus  $g \geq 2$ .
- (3) Investigate asymptotic behavior of the ratios  $A(g)/R(g)$ ,  $A(g)/WR(g)$  and  $R(g)/A(g)$ ,  $WR(g)/A(g)$ , where  $A(g)$  denotes the number of all quasi-platonic actions.

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