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A taxonomy of best-reply multifunctions in 2 × 2 × 2 trimatrix games

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Abstract This paper provides an overview of the various shapes the best-reply multifunctions can take in $2 \times 2 \times 2$ trimatrix games. It is shown that, unlike in 2×2 bimatrix games, the best replies to the opponents' pure strategies do not completely determine the structure of the Nash equilibrium set.

Keywords Non-cooperative games · Best-reply multifunction · Nash equilibrium

Mathematics Subject Classification (2000) 91A06 · 91A05

1 Introduction

Vorobev (1958) introduced an easy graphical way of representing best-reply multifunctions for 2×2 bimatrix games and used this method to find all the Nash equilibria. These best-reply multifunction can take only a few forms. Moreover, the best-replies to the pure strategies of one's opponent completely determine the structure of the set of Nash equilibria. Borm (1987) enumerated all these different shapes.

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In $2 \times 2 \times 2$ trimatrix games, the picture is a bit more complicated. Not only are there simply more possibilities, but there is an added complication in that in three dimensions, it is not sufficient any more to know what happens in the extreme points. In this short paper we provide a taxonomy of all the possible shapes of the best-reply multifunctions and give an indication of what the set of Nash equilibria can look like.

2 Best-reply functions of 2 × 2 games

In this section we briefly review the study of best-reply multifunctions in 2×2 bimatrix games in Borm (1987).

Let (N, S, P) be a 2-person game in strategic form where $S = S_1 \times S_2$ is the space of strategies where each set S_i of pure strategies of Player i has only two elements denoted by 1 and 0. Functions $P_i : S \to \mathbb{R}$ are the payoffs of Player i $(i = 1, 2)$. Without loss of generality we develop our exposition for Player 2.

We note by a_{ij} the payoff to Player 2 when Players 1 and 2 use their pure strategies i and j, respectively. Mixed strategies of each player are determined by a value in the interval [0, 1] giving the probability of using the first pure strategy. So, the pure strategies 1 and 0 can be identified with probability values of 1 and 0, respectively.

Given a mixed strategy profile $(p, q) \in [0, 1]^2$, the payoff function of Player 2 can be expressed as

$$P_2(p,q) = (p, 1-p) \begin{bmatrix} a_{11} & a_{10} \\ a_{01} & a_{00} \end{bmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}.$$

Denoting by U the interval [0, 1], the general best-reply multifunction of Player 2 against a mixed strategy $p \in U$ is given by

$$B_2(p) = \begin{cases} 0, & \text{if } pa_{11} + (1-p)a_{01} < pa_{10} + (1-p)a_{00}, \\ U, & \text{if } pa_{11} + (1-p)a_{01} = pa_{10} + (1-p)a_{00}, \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

So, we can say that the shape of B_2 depends on the sign of the linear function

$$p(a_{11} - a_{10}) + (1-p)(a_{01} - a_{00}). \tag{2}$$

Where this function is positive, the best-reply is the pure strategy 1, whereas 0 is the best response where it is negative; where (2) is null any strategy is a best-reply. By linearity of expression (2) the shape of B_2 is given by the signs of $a_{11} - a_{10}$ and $a_{01} - a_{00}$ which indicate the best response in the extreme points of the interval [0, 1], as can be checked from (1). Since each of these amounts can be positive, negative or zero, we have nine possible shapes for the best-reply functions as Fig. 1 shows. Note that in almost all the cases the shape is completely determined. Only in the 0-1 and 1-0 cases a certain point of change must be calculated in which (2) is null.

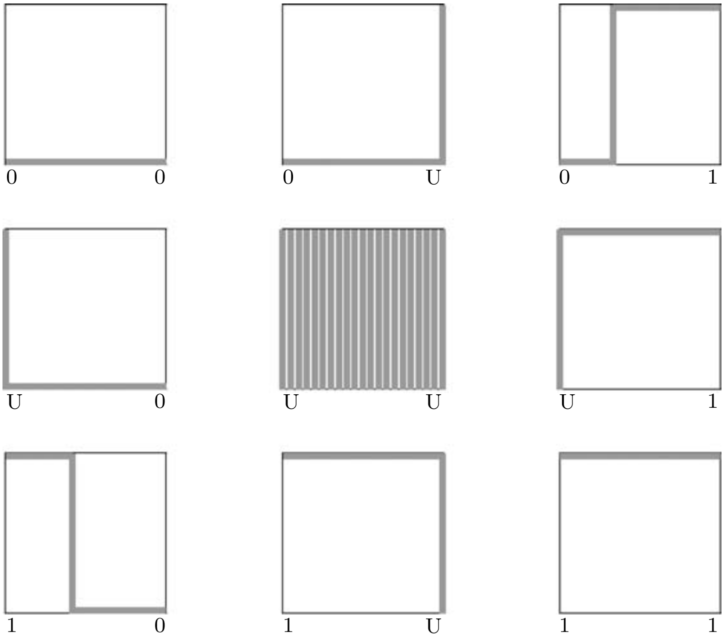


Fig. 1 The nine different shapes of best-reply multifunction for the second player in 2×2 games, $B_2(p)$. The best response against pure strategies of Player 1 is indicated on the *bottom side of each unit square*: on the *left* against 0, on the *right* against 1

3 Best-replies in 2 × 2 × 2 games

Now, let (N, S, P) be a 3-person game in strategic form, and as above, $S_i = \{0, 1\}$ $(i = 1, 2, 3)$. To simplify the discussion, we make our exposition for Player 3. We note by

$$A_1 = \begin{bmatrix} a_{111} & a_{101} \\ a_{011} & a_{001} \end{bmatrix} \quad \text{and} \quad A_0 = \begin{bmatrix} a_{110} & a_{100} \\ a_{010} & a_{000} \end{bmatrix}$$

Player 3's payoffs matrix from pure strategies: A_1 for Player 3's pure strategy 1, and A_0 for the pure strategy 0. The subscripts indicate the first, second and third player's pure strategy, respectively.

Given a mixed strategy profile $(p, q, r) \in [0, 1]^3$, the payoff function of Player 3 is

$$\begin{aligned} P_3(p, q, r) &= (p, 1-p)\big[rA_1 + (1-r)A_0\big]\begin{pmatrix} q \\ 1-q \end{pmatrix} \\ &= r(p, 1-p)A_1(q, 1-q)^t + (1-r)(p, 1-p)A_0(q, 1-q)^t. \quad (3) \end{aligned}$$

Expression (3) gives us an easy way to obtain the best-reply multifunction for Player 3, given the Player 1 and 2's strategies $p, q \in [0, 1]$:

$$B_3(p,q) = \begin{cases} 0, & \text{if } (p, 1-p)(A_1 - A_0)(q, 1-q)^t < 0, \\ U, & \text{if } (p, 1-p)(A_1 - A_0)(q, 1-q)^t = 0, \\ 1, & \text{otherwise.} \end{cases} \quad (4)$$

The unit square $[0, 1]^2$ in which the pair (p, q) moves can be divided into two regions in which the best reply is 1 and 0, respectively. The border between these regions is given by the points (p, q) satisfying

$$(p, 1-p)(A_1 - A_0) \begin{pmatrix} q \\ 1-q \end{pmatrix} = 0. \quad (5)$$

On these points the best reply is any point in the interval [0, 1].

Equation $(p, 1-p)(A_1 - A_0)(q, 1-q)^t = r$ gives a hyperbolic paraboloid in the space (p, q, r). It is one of the three doubly ruled quadrics: through every point on the surface there are two distinct lines that lie on the surface. In our case, for each point on the surface, the projections on the pq plane of these two lines are parallel to the p and q axes, respectively (see Fig. 2). As a degenerate case, the surface can be a plane (the other ruled surface together with the hyperboloid of one sheet).

In order to study the best-reply multifunctions we have to determine the sign that the left side of (5) takes for each point (p, q) in the unit square. For points with positive sign, the best reply is the pure strategy 1. If the sign is negative, the best-reply of Player 3 is the second pure strategy, 0. The expression is zero on the indifference points: both pure strategies are equally good responses and indeed, each mixed strategy $(r, 1-r)$ is a best-reply strategy.

The level curves of a hyperbolic paraboloid are hyperbolas. Furthermore, in this case its axes are parallel to the coordinate axes (and the axes are indeed level curves for some level)—see bottom part of the two boxes in Fig. 2. In the degenerate case in

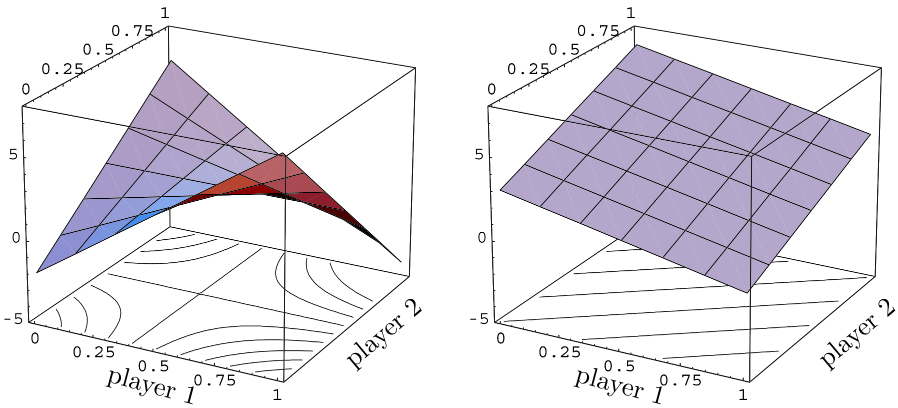


Fig. 2 Player 3's payoff function is a section of an hyperbolic paraboloid (*left*) and or a plane (*right*). On the *bottom*, some level curves are shown

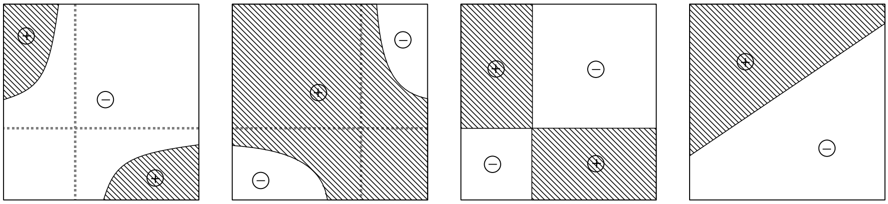


Fig. 3 Different possibilities of sign (best replies) regions on the unit square. Plus and minus signs can be interchanged to obtain other four possibilities. Of course, a trivial case with no change of sign could be added

which the surface is a plane, the level curves are straight lines. Figure 3 shows different possibilities of regions. The different situations of the unit square with respect to the zero-level curves determine the different forms the best-reply multifunction can adopt. However, the trimatrix case is more complicated than the previous 2-person games.

As a starting point, we could have a look at the signs of the four elements of the matrix $A_1 - A_0$, as they give the best response to the four possible combinations of pure strategies for Player 1 and 2: $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. From the three possible signs of these differences (positive, negative, or unsigned) we have $3^4 = 81$ cases to consider. Each corner of the strategy square can be labeled with 0, 1, or U by the best reply against it. Since many of these cases can be obtained from others by relabeling the strategies of the players, we group them into thirteen classes in Table 1.

Although we classify the best-reply multifunctions by the best response to the four combinations of pure strategies (the four corners of the unit square), the exact shape of that function can vary. Figure 4 shows an example of three different shapes belonging to the same best-reply pattern against combinations of pure strategies. These three (sometimes four, as in class 6) possibilities are indicated in Table 1 by gray lines.

4 Nash equilibria

An interesting aspect of studying the best-reply multifunctions is to obtain the Nash equilibria. We have a Nash equilibrium at each point at which the best replies of all the players coincide. In trimatrix games a point $(p, q, r) \in [0, 1]^3$ constitutes a Nash equilibrium if the best-reply multifunctions satisfy

$$p \in B_1(q, r), \qquad q \in B_2(p, r), \quad \text{and} \quad r \in B_3(p, q).$$

Given the three best-reply multifunctions of the players of a game, we are able to obtain the Nash equilibrium set. For example, suppose that the three best-reply multifunctions belong to the first case of class 7, which we represent by 0U0U. Figure 5 shows consecutively the best replies of player 1, 2, and 3 and the intersection with the previous player. The last one is the whole Nash equilibrium set: four isolated corner points.

Table 1 Classification of best-reply multifunctions of $2 \times 2 \times 2$ games. The 4-tuples represent best reply to the $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$ pure strategies combinations of the other players. Indications of how the indifference curves could go are given in the square in the same way as in Fig. 4. The 81 possible shapes of the best-reply curves are grouped into 13 classes on the basis of symmetry (pure strategies labels could be interchanged)

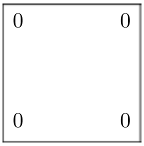
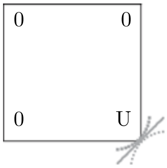
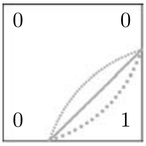
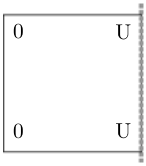
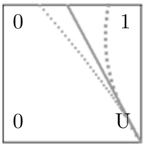
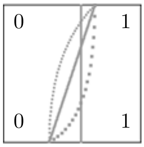
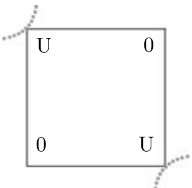
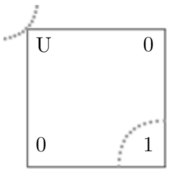
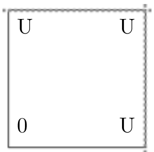
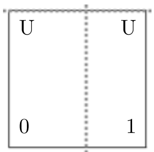
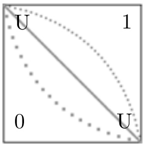
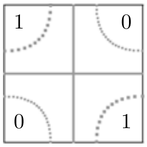
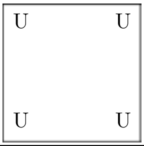
Class	Best-reply pattern			Class	Best-reply pattern		
1		0000	1111	2		000U U000 0U00 00U0	111U U111 1U11 11U1
3		0001 1000 0100 0010	1110 0111 1011 1101	4		00UU U00U UU00 0UU0	11UU U11U UU11 1UU1
5		00U1 100U U100 0U10 001U U001 1U00 01U0	11U0 011U U011 1U01 110U U110 0U11 10U1	6		0011 1001	1100 0110
7		0U0U U0U0	1U1U U1U1	8		0U01 10U0 010U U010	1U10 01U1 101U U101
9		0UUU U0UU UU0U UUU0	1UUU U1UU UU1U UUU1	10		0UU1 10UU U10U UU10	1UU0 01UU U01U UU01

Figure 6 shows an example of how the Nash equilibria can adopt different combinations of isolated points, lines, and surfaces.

As the interaction among the best-reply functions gives the Nash equilibrium set of the game, one could think that the best-reply functions can be used to classify games by their equilibria. This argument works perfectly for 2×2 games, in which

Table 1 (*Continued*)

Class	Best-reply pattern			Class	Best-reply pattern		
11		0U1U U0U1	1U0U U1U0	12		0101	1010
13		UUUU					

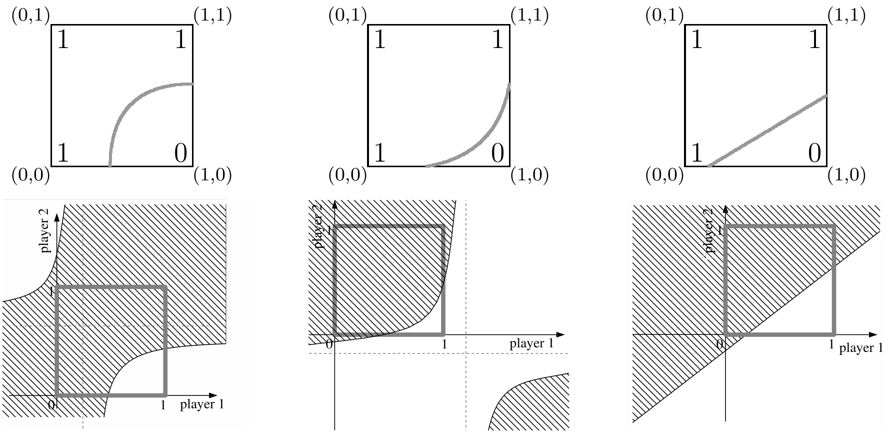


Fig. 4 Three different shapes of best-reply function with the same best responses against the pure strategies. It is the case 1110, from class 3

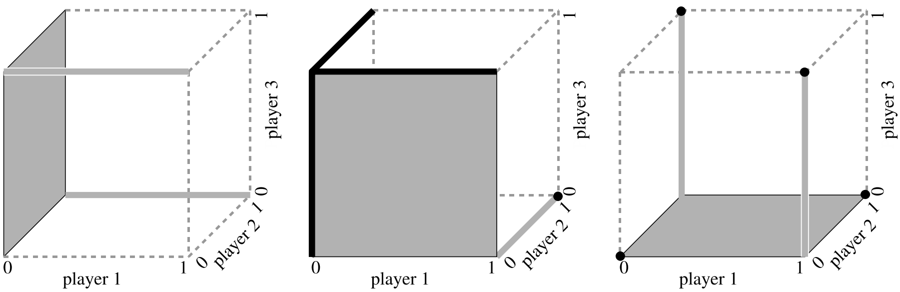


Fig. 5 *Left to right*: best-reply multifunctions of Players 1, 2, and 3 showing a 0U0U pattern. The *dark zones* in the second graph are the intersection of Player 1 and 2's best-replies; in the third one the Nash equilibrium set is given

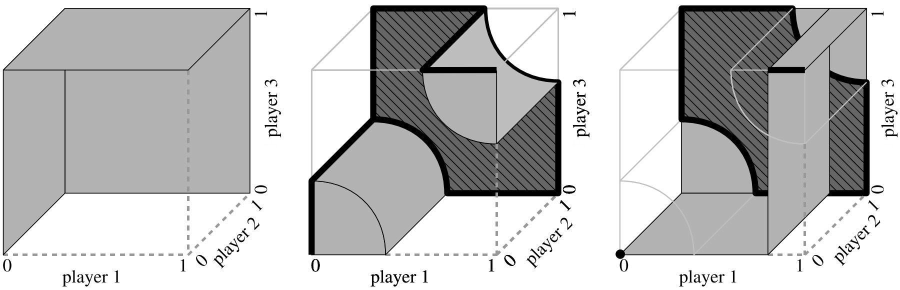


Fig. 6 *Left to right*: best reply multifunctions of Players 1 (0UUU), 2 (0101), and 3 (0UU1). Now, in the third graph it can be seen that the Nash equilibrium set can be a composition of points, straight segments, and surfaces

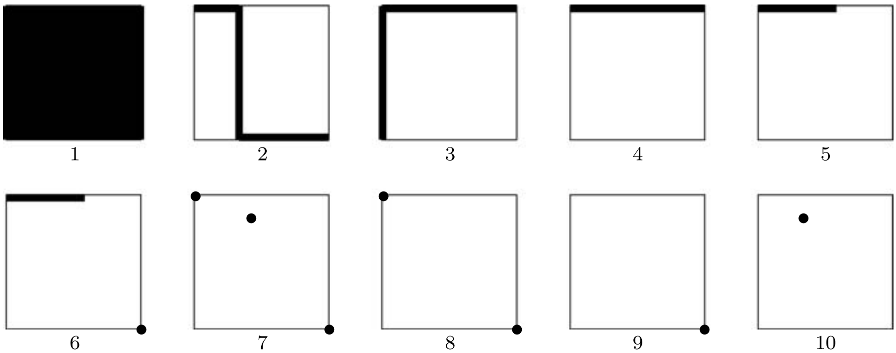


Fig. 7 The ten different shapes of Nash equilibria in 2×2 games produced by the 81 combinations of the nine best-reply multifunctions of Fig. 1. Each drawing represents the whole group of its symmetries

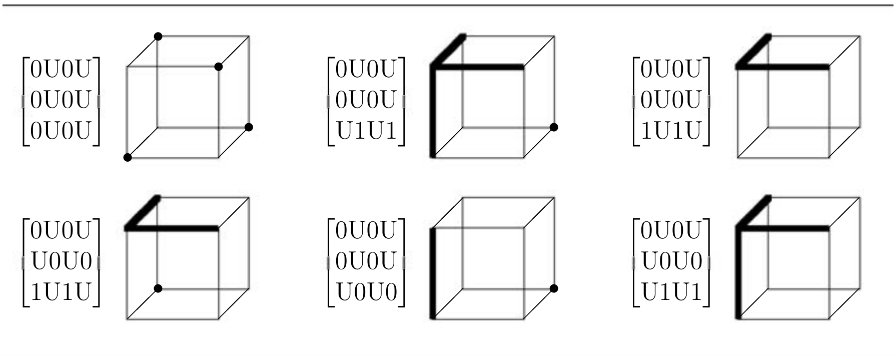


Fig. 8 Examples of shapes of the Nash equilibrium sets from the interaction of different best replies from class 7

the shape of the interaction between the same class of best-reply functions of the two players presents the same structure. For instance, when the players have both a best-reply multifunction of type 1-0, the equilibrium set has always three elements (two

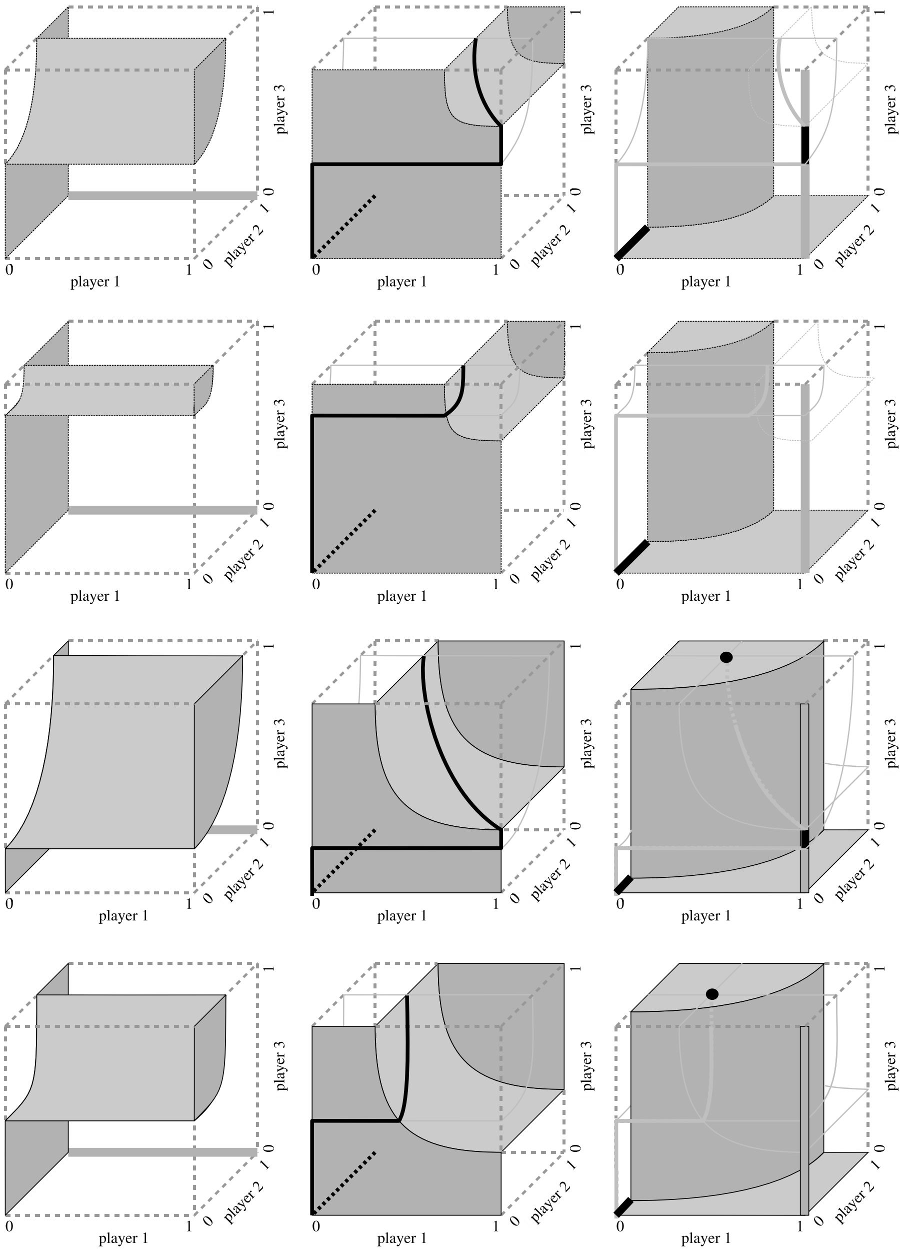


Fig. 9 Different Nash equilibrium sets (*dark region in the right-hand graphs*) from the same best-reply functions combination: a 010U pattern for all players

pure strategy equilibria (1, 0) and (0, 1), plus a mixed equilibrium) no matter where the multifunctions change from 1 to 0. From nine classes of best-reply function, the 81 interactions can be classified into ten shapes shown in Fig. 7.

However, in the $2 \times 2 \times 2$ case, although the classes group best-reply multifunctions showing similar shapes, the interaction among functions in the same class can lead to different equilibria sets. Figure 8 shows examples of shapes of the Nash equilibrium sets from the interaction of different best replies patterns all belonging to class 7.

Moreover, we can obtain different equilibria set configurations from the interaction of best-reply functions with the same pattern, as Fig. 9 shows. The three players have a best-reply multifunction with pattern 010U and indifference curves with similar curvatures, but for slightly different parameters the Nash equilibrium set varies.

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