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Hidden Variable A-Fractal Functions and Their Monotonicity Aspects

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Abstract

Fractal interpolation that possesses the ability to produce smooth and nonsmooth interpolants is a novice to the subject of interpolation. Apart from appropriate degree of smoothness, a good interpolant should reflect shape properties, for instance monotonicity, inherent in a prescribed data set. Despite the flexibility offered by these shape preserving fractal interpolants developed recently in the literature are well-suited only for the representation of self-referential functions. In this article we present hidden variable $\mathbf{A}$-fractal interpolation function as a tool to associate an entire family of $\mathbb{R}^2$-valued continuous functions $\mathbf{f}[\mathbf{A}]$ parameterized by a suitable block matrix $\mathbf{A}$ with a prescribed function $\mathbf{f} \in \mathcal{C}(I, \mathbb{R}^2)$. Depending on the choice of parameters, the members of the family may be self-referential, or non-self-referential, and preserve some properties of original function $\mathbf{f}$, thus yielding more diversity and flexibility in the process of approximation. As an application of the developed theory, we introduce a new class of monotone $\mathcal{C}^1$-cubic interpolants by taking full advantage of flexibility offered by the hidden variable $\mathbf{A}$-fractal interpolation functions (HFIFs). This theory invoked to the $\mathcal{C}^1$-cubic spline HFIF, which can be viewed as a fractal perturbation of the traditional $\mathcal{C}^1$-cubic spline, culminates with the desired monotonicity preserving $\mathcal{C}^1$-cubic HFIF. The monotonicity preserving interpolation scheme developed herein generalizes and enriches its traditional nonrecursive counterpart and its fractal extension.

Keywords. Fractal Operator, Fractal Cubic Spline, Hidden Variable Fractal Interpolation Function, Monotonicity, Parameter Identification

MSC 2000 Classifications. 28A80, 41A05, 41A10, 65D07, 26C15

1 Introduction and Motivation

Fractals are known to construct extremely complicated and impressive shapes, which many times resemble objects of the physical world. Fractal interpolation functions (FIFs), introduced

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by Barnsley [2, 3] not only open a new research field in the approximation theory of functions but also draw considerable attention of the researchers in various scientific areas applied in natural sciences [21, 22], engineering applications [7], image compression and processing [17] and economics [19]. This technique is followed in earnest by a host of researchers (see, for instance [8,9,12,13,23,24,26,31]). FIFs provide a basis for the constructive approximation theory of nondifferentiable functions. Further, differentiable FIFs constitute an alternative to the traditional nonrecursive interpolation and approximation methods (see, for instance, [4, 8, 12, 26]). In this way, the fractal methodology provides more flexibility and versatility on the shape of an interpolant. Consequently, this function class can be useful for mathematical and engineering problems where the classical spline interpolation approach does not work satisfactorily. FIFs are generally self-referential in the sense that the graph of the function is a union of transformed copies of itself.

To approximate non-self-similar objects found in nature, Barnsley et al. [5] extended the idea of FIFs to produce more flexible univariate interpolation functions, namely, hidden variable FIFs (HFIFs). Bouboulis and Dalla [6] have constructed hidden variable vector valued FIFs on random grids in $\mathbb{R}^2$. As the values of HFIF continuously depend on the parameters which define it because of that it is useful in adjusting the shape of interpolation data. HFIF is more diverse and appealing than a FIF. In some practical applications, the interpolation data might be generated simultaneously from self-referential and non-self-referential functions. To study such data, Chand and Kapoor [9] introduced the notion of coalescence hidden variable FIF (CHFIF).

In practice, it is very desirable for the shape of the approximant/interpolant to be compatible with the given data. For this, scientists and engineers usually demand that approximation/interpolation methods accurately represent the physical reality. The problem of searching a sufficiently smooth function that preserves the qualitative shape property inherent in the data is generally referred to as shape preserving interpolation/approximation, which is important in practical ground, and received considerable attention in the literature (see, for instance, [15, 16, 18, 20] and references quoted therein). The shape properties are mathematically expressed in terms of conditions such as positivity, monotony and convexity. As a submissive contribution to this goal, Chand and collaborators have initiated the study on shape preserving fractal interpolation and approximation using various families of polynomial and rational iterated function system (see, for instance, [10,11,29,30]). These shape preserving fractal interpolation schemes possess the novelty that the interpolants inherit the shape property in question and at the same time the suitable derivatives of these interpolants own irregularity in finite or dense subsets of the interpolation interval. This attribute of shape preserving FIFs finds potential applications in various nonlinear phenomena.

The primary intent of this article is to employ $\mathcal{C}^1$-cubic spline hidden variable FIF for monotonicity preserving interpolation, thereby giving an entire class of monotonic interpolants that include traditional monotonic $\mathcal{C}^1$-cubic spline and their fractal extensions studied recently as special cases. In this regard, the traditional monotone cubic spline interpolation has been extensively studied. Necessary and sufficient condition for a cubic spline to be monotone in an interval is studied by Fritsch and Carlson [16] and they used it to a develop a two-pass algorithm for constructing a monotone cubic interpolant to a given set of monotone data. The algorithm discussed by Fritsch and Butland [15] provides an improvement to Fritsch-Carlson (FC) algorithm by providing a flexibility of one pass. Both these methods are simple and easy to implement. Subsequently, many variants and improvements to FC algorithm were proposed, see for instance, [14, 32].

By using suitable IFS, Barnsley and Navascués have provided a method to perturb a continuous function $f \in \mathcal{C}(I)$ so as to yield a class of continuous functions $f^\alpha \in \mathcal{C}(I)$, where α is a free parameter, called the scale vector. For suitable values of the scale vector α, the fractal functions f^α simultaneously interpolate and approximate f. By this method one can define fractal analogues of any continuous function. Navascués [25] introduced an operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ defined through $f \mapsto f^\alpha$ and developed properties of this operator. This enriched the fractal approximation theory and facilitated the theory of fractal interpolation to interact with the fields such as functional analysis and operator theory (see, for instance, [26–28]).

We apply hidden variable FIF as a tool to fraternize a family of $\mathbb{R}^2$-valued continuous fractal functions with a prescribed continuous function $\mathbf{f} : I \to \mathbb{R}^2$ defined on a real compact interval I, where $\mathbb{R}^2$ being endowed with the l^1-norm. In detail, we obtain a family of continuous fractal functions $\mathbf{f}[\mathbf{A}]$ for a given continuous function $\mathbf{f} : I \to \mathbb{R}^2$, parameterized by a block matrix $\mathbf{A} = [A_n]_{n=1}^M$, where each A_n is a suitable matrix in $M^{2\times 2}(\mathbb{R})$, the space of all 2×2 matrices having real entries. This is a natural extension of the "fractal perturbation" process applied in the case of real-valued functions to obtain α-fractal function f^α corresponding to f studied in detail by Navascués (see, for instance, [26, 28]). The advantage gained is that the function whose graph is the orthogonal projection of graph $\mathbf{f}[\mathbf{A}]$ provides non-self-referential fractal function corresponding to a given real valued continuous function in contrast to the self-referential fractal generalizations obtained by α-fractal technique. Further, by the proper selection of parameters of the hidden variable FIF, the projection can be made self-referential as well, thus providing more flexibility and diversity in the process of approximation. We may refer $\mathbf{f}[\mathbf{A}]$ as $\mathbf{A}$-fractal function corresponding to $\mathbf{f}$ or fractal perturbation of $\mathbf{f}$. The presence of the block matrix parameter $\mathbf{A}$ in the constructed function undoubtedly provides more flexibility, which may be exploited in various approximation and engineering problems.

To invite the class of approximants $\mathbf{f}[\mathbf{A}]$ parameterized by $\mathbf{A}$ to the area of shape preserving approximation, we identify suitable parameters so that $\mathbf{f}[\mathbf{A}]$ preserves $\mathcal{C}^r$-continuity and monotonicity inherent in the function $\mathbf{f}$ being perturbed. This monotonicity preserving hidden variable fractal perturbation scheme is applied to construct $\mathcal{C}^1$-cubic hidden variable FIFs corresponding to a given monotonic Hermite data. These conditions are then used to develop an algorithm which constructs a monotonic $\mathcal{C}^1$-cubic spline HFIF to monotone data. The curve produced contains no extraneous "bumps" or "wiggles" which makes it more readily acceptable to scientists and engineers. Compared to most other shape preserving methods, the method proposed in this paper is characterized by its efficiency, in terms of time required to determine the interpolant, storage to required to represent it, and/or time required to evaluate it. Examples are included which compare this algorithm with other piecewise cubic interpolation methods. Consequently, the paper also provides an entire class of monotonic interpolants that include the traditional monotonic $\mathcal{C}^1$-cubic spline and their fractal analogue as special cases.

The organization of the paper is as follows: In Section 2, we recall some of the required basic tools. In Section 3, we construct $\mathbf{A}$-fractal function corresponding to a $\mathbb{R}^2$-valued continuous function $\mathbf{f}$. We consider the operator $\mathcal{F}[\mathbf{A}] : \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$ which assigns $\mathbf{f}[\mathbf{A}]$ to the function $\mathbf{f}$, establish some properties and some results on Schauder basis in Section 4. In Section 5, we identify suitable parameters in the IFS so that $\mathbf{f}[\mathbf{A}]$, which are regarded as the fractal perturbation of a given function $\mathbf{f}$, preserves the properties (for instance, regularity and monotonicity) inherent in $\mathbf{f}$. We develop a monotonicity preserving cubic spline interpolation scheme that extends the methods described in the reference [16] of Fritsch and Carlson in Section 5 which is followed by numerical illustration.

2 Background and Preliminaries

In this section we briefly recall requisite general material for our study like the notions of Cubic Interpolation, Iterated function system (IFS), FIF, CHFIF, and establish some of their basic properties. For a detailed study, reader may refer to [2,5,16,24] for any additional information.

2.1 *Piecewise cubic interpolation*

For $r \in \mathbb{N}$, let $\mathbb{N}_r$ denote the subset $\{1, 2, \dots, r\}$ of $\mathbb{N}$. Let a set of data points $D = \{(x_n, y_n) \in I \times \mathbb{R} : n \in \mathbb{N}_N\}$ satisfying $x_1 < x_2 < \dots < x_N$, where $I = [x_1, x_N]$ be given. The local mesh spacing is $h_n = x_{n+1} - x_n$, and the secant slope of the linear interpolant between the data points is $\Delta_n = \frac{y_{n+1} - y_n}{h_n}$. A piecewise cubic function $s \in \mathcal{C}^1(I)$ is uniquely determined by y_n and d_n, where $s(x_n) = y_n$, $s^{(1)}(x_n) = d_n$, $n \in \mathbb{N}_N$. The traditional $\mathcal{C}^1$-cubic interpolant s defined over the subinterval $I_n = [x_n, x_{n+1}]$ is defined as follows:

$$s_n(x) = \frac{d_n + d_{n+1} - 2\Delta_n}{h_n^2}(x - x_n)^3 + \frac{-2d_n - d_{n+1} + 3\Delta_n}{h_n}(x - x_n)^2 + d_n(x - x_n) + y_n. \quad (2.1)$$

2.2 *Monotone piecewise cubic interpolation with Fritsch-Carlson algorithm*

For our convenience, we consider the interpolated data to be monotonic increasing throughout the remainder of the paper. For a monotonic increasing data (i.e., $y_n \le y_{n+1}$, for all $n \in \mathbb{N}_{N-1}$), we invoke here the well-known Fritsch-Carlson algorithm which ensures that the corresponding cubic interpolant s is monotone. The basis of this algorithm is to check whether a cubic polynomial s defined on $[x_n, x_{n+1}]$ is monotone on that interval, and it is given in the following proposition.

Proposition 2.1 (Fritsch and Carlson [16])**.** *For the data set* $\{(x_n, y_n, d_n) : n \in \mathbb{N}_N\}$*, consider the traditional* $\mathcal{C}^1$*-cubic spline* s *defined as (2.1). Let* $\Delta_n \neq 0$*, let* $\eta_n = \dfrac{d_n}{\Delta_n}, \xi_n = \dfrac{d_{n+1}}{\Delta_n}$*. Then* s *is monotone on* $[x_n, x_{n+1}]$ *if and only if: (i)* $d_n = d_{n+1} = 0$ *if* $\Delta_n = 0$*, or (ii)* $(\eta_n, \xi_n) \in \mathcal{M}$ *if* $\Delta_n \neq 0$*,* $\mathcal{M}$ *is the closed region bounded by the axes and the "upper half" of the ellipse* $x^2 + y^2 + xy - 6x - 6y + 9 = 0$ *shown in the following Figure 1.*

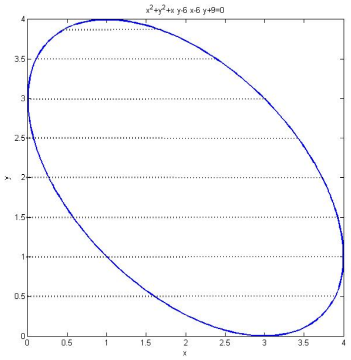


Figure 1: Fritsch-Carlson monotone region.

Algorithm (Fritsch-Carlson)

Step 1: Initialize derivatives $d_n, n \in \mathbb{N}_N$ such that $sgn(d_n) = sgn(d_{n+1}) = sgn(\Delta_n)$. If Δ_n = 0, set $d_n = d_{n+1} = 0$.

Step 2: For each interval $I_n = [x_n, x_{n+1}]$ in which $(d_n, d_{n+1}) \notin \mathcal{M}_n$, modify d_n and d_{n+1} to $d_n^\diamond$ and $d_{n+1}^\diamond$, such that $(d_n^\diamond, d_{n+1}^\diamond) \in \mathcal{M}_n$, where the closed region $\mathcal{M}_n = \mathcal{M}.\Delta_n = \{(x\Delta_n, y\Delta_n) : (x, y) \in \mathcal{M}\}$.

Fritsch and Carlson observed that decreasing the magnitude of d_n in moving (d_n, d_{n+1}) into $\mathcal{M}_n$ may force (d_{n-1}, d_n) out of $\mathcal{M}_{n-1}$ and vice versa. Due to this reason, they suggested to work with a subregion ρ of $\mathcal{M}$ of enjoying the property that if $(x, y) \in \rho$ and $0 \leq \widehat{x} \leq x$, $0 \leq \widehat{y} \leq y$, then $(\widehat{x}, \widehat{y}) \in \rho$. The recommended regions are (see Fig. 2):
(i) Ω_1: the largest subset of $\mathcal{M}$ bounded by the four lines $\eta = 0, 3$, and $\xi = 0, 3$,
(ii) Ω_2: region bounded by $\eta = 0$, $\xi = 0$, and the circle $\eta^2 + \xi^2 = 3^2$,
(iii) Ω_3: the subset of $\mathcal{M}$ determined by the lines $\eta = 0$, $\xi = 0$, and $\eta + \xi - 3 = 0$,
(iv) Ω_4: the subset of $\mathcal{M}$ bounded by $\eta = 0$, $\xi = 0$, $2\eta + \xi - 3 = 0$, and $\eta + 2\xi - 3 = 0$.
Fritsch and Carlson also observed that the choice of Ω_1 produces the least change in the derivatives and the graph more closely resembles the graph obtained using the standard three point difference formula. The choice of Ω_4 produces the greatest change in the derivatives and the

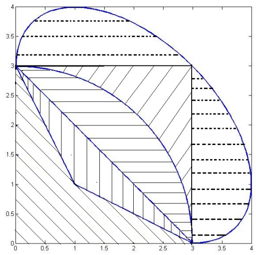


Figure 2: Fritsch-Carlson subregions. Diagonal hatching (- slope): Ω_4; Vertical hatching: $\Omega_3 - \Omega_4$; Horizontal hatching: $\Omega_2 - \Omega_3$; Diagonal hatching (+ slope): $\Omega_1 - \Omega_2$; Dotted: $\mathcal{M} - \Omega_1$.

graph more closely resembles a piecewise linear function. The choice of Ω_2 and Ω_3 lies somewhere in between. So it is highly recommended here to take the choice Ω_2 for most pleasing results. This kind of algorithm is known as fit and modify type algorithm.

2.3 *Iterated function system*

Let (X, d_X) be a complete metric space with metric d_X. If $w_n : X \to X$, $n \in \mathbb{N}_N$ are continuous mappings, then $\mathcal{I} = \{X; w_n : n \in \mathbb{N}_N\}$ is called an IFS. If, in addition, each $w_n, n \in \mathbb{N}_N$ is a contraction map, then the IFS $\mathcal{I}$ is referred to as a hyperbolic IFS or contractive IFS. The attractor associated with the IFS $\mathcal{I}$ is the unique fixed point of the Hutchinson map $W : H(X) \to H(X)$ defined by $W(B) = \bigcup_{n=1}^{N} w_n(B)$, where $H(X)$ is the set of all nonempty compact subsets of X endowed with the Hausdorff metric h. The Hausdorff metric h completes $H(X)$. When $\mathcal{I}$ is a hyperbolic IFS with contractivity c, the IFS $\mathcal{I}$ has an attractor, an attractor being a fixed point of the collage map W. It is well known that W is a contraction on the complete metric space $(H(X), h)$ with the same contractivity c. A self-referential object is a set or measure that can be defined in terms of a finite collection of geometric transformations applied to it. Since $A = \bigcup_{n=1}^{N} w_n(B)$, the attractor A of the IFS $\mathcal{I}$ is self-referential e.g. the middle third Cantor set is the union of two shrunken version of the whole set.

2.4 *Fractal interpolation function*

For the construction of FIF, a suitable IFS whose attractor is the graph of the desired interpolant be defined as follows. Let a set of data points $D = \{(x_n, y_n) \in \mathbb{R}^2 : n \in \mathbb{N}_N\}$ satisfying $x_1 < x_2 < \cdots < x_N$, $N > 2$, be given. Set $I = [x_1, x_N]$, $I_n = [x_n, x_{n+1}]$ for $n \in \mathbb{N}_{N-1}$. Suppose $L_n : I \to I_n \subset I, n \in \mathbb{N}_{N-1}$ be contraction homeomorphisms such that $L_n(x_1) = x_n$, $L_n(x_N) = x_{n+1}$, and mappings $F_n : I \times \mathbb{R} \to \mathbb{R}$ that are contraction in second argument fulfilling $F_n(x_1, y_1) = y_n$, $F_n(x_N, y_N) = y_{n+1}$, $n \in \mathbb{N}_{N-1}$. Let $X := I \times \mathbb{R}$ and consider the IFS $\mathcal{I} = \{X; w_n = (L_n, F_n) : n \in \mathbb{N}_{N-1}\}$. According to the IFS theory [3], such an IFS has a unique attractor $G = G(f)$ is the graph of a continuous function $f : I \to \mathbb{R}$ interpolating the given data set and satisfying $f(x_n) = y_n$, for $n \in \mathbb{N}_N$. The aforementioned function f is referred to as a FIF corresponding to D and is obtained as the fixed point of the Read-Bajraktarević (RB) operator T on $(\mathcal{G}, \rho)$ as $(Tg)(x) = F_n\big(L_n^{-1}(x), g \circ L_n^{-1}(x)\big)$, for $x \in I_n = [x_n, x_{n+1}]$, $n \in \mathbb{N}_{N-1}$, where $\mathcal{G}$ be the set of continuous functions $h : I \to \mathbb{R}$ such that $h(x_1) = y_1, h(x_N) = y_N$ equipped with the metric $\rho(h, h^*) = \max\{|h(x) - h^*(x)| : x \in I\}$ for $h, h^* \in \mathcal{G}$. Since graph of f, $G(f)$ is a union of transformed copies of itself, specifically $G(f) = \bigcup_{n \in \mathbb{N}_{N-1}} w_n\big(G(f)\big)$, the map f is a *self-referential function*.

2.5 *Hidden variable fractal interpolation function*

To approximate non-self-affine patterns, hidden variable FIFs are constructed by projecting vector valued FIF corresponding to a generalized interpolation data, which we shall succinctly review in the following.

For constructing an interpolation function $g_1 : I = [x_1, x_N] \to \mathbb{R}$ such that $g_1(x_n) = y_n$ for all $n \in \mathbb{N}_N$, consider a generalized set of data of D, $\widehat{D} = \{(x_n, y_n, z_n) \in I \times \mathbb{R}^2 : n \in \mathbb{N}_N\}$. Here $\{z_n : n \in \mathbb{N}_N\}$ are real parameters, whose selection is highly subjective. The idea is to construct a fractal interpolation function for $\widehat{D}$, and project its graph into $I \times \mathbb{R}$ such a way that the projection is the graph of a function that interpolates D. For $n \in \mathbb{N}_{N-1}$, let the contraction homeomorphisms $L_n : I \to I_n \subset I$ be defined so as to satisfy

$$L_n(x) = a_n x + b_n,\ L_n(x_1) = x_n \text{ and } L_n(x_N) = x_{n+1}. \tag{2.2}$$

Let $\mathbb{R}^2$ be endowed with the Manhattan metric $d_M\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$. Here we note that an element in $\mathbb{R}^2$ may be regarded as an ordered pair (a_1, a_2) or as a column matrix $(a_1, a_2)^t$ which will be clear from the context. Let $F_n : I \times \mathbb{R}^2 \to \mathbb{R}^2$:

$$F_n(x, \mathbf{y}) = F_n(x, y, z) = \big(F_n^1(x, y, z), F_n^2(x, z)\big)^t := A_n(y, z)^t + \big(p_n(x), q_n(x)\big)^t, \tag{2.3}$$

where t denotes the transpose, A_n are upper-triangular matrices $\begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix}$, and p_n, q_n are suitable real valued continuous functions so that the following conditions are satisfied for all $n \in \mathbb{N}_{N-1}$:

(i) $d_M(F_n(x, y, z), F_n(x^*, y, z)) \le c_1|x - x^*|$ for some constant $c_1 > 0$,

(ii) $d_M\big(F_n(x, y, z), F_n(x, y^*, z^*)\big) \le s\, d_M\big((y, z)(y^*, z^*)\big)$ for $0 \le s < 1$,

(iii) join-up conditions: $F_n(x_1, y_1, z_1) = (y_n, z_n)$ and $F_n(x_N, y_N, z_N) = (y_{n+1}, z_{n+1})$.

The variables α_n, β_n, and γ_n are chosen such that $\|A_n\|_1 < 1$ for all $n \in \mathbb{N}_{N-1}$. Consider $w_n : I \times \mathbb{R}^2 \to I \times \mathbb{R}^2$ defined by $w_n(x, y, z) = \big(L_n(x), F_n(x, y, z)\big)$. It follows from the conditions on maps L_n and F_n that w_n are contraction maps with respect to the metric d_M^* defined on $I \times \mathbb{R}^2$ by $d_M^*\big((x, y, z), (x^*, y^*, z^*)\big) = |x - x^*| + \theta d_M\big((y, z), (y^*, z^*)\big)$ for some $\theta > 0$. Consequently, the generalized IFS $\{I \times \mathbb{R}^2; w_n : n \in \mathbb{N}_{N-1}\}$ admits an attractor $A \in H(I \times \mathbb{R}^2)$. It follows from the generalized IFS theory that A is the graph of a continuous function $\mathrm{g} : I \to \mathbb{R}^2$

such that $\mathbf{g}(x_n) = (y_n, z_n)$ for all $n \in \mathbb{N}_{N-1}$. Letting $\mathbf{g} = (g_1, g_2)$ it follows that $g_1 : I \to \mathbb{R}$ is a continuous function interpolating D. The aforementioned function $g_1 : I \to \mathbb{R}$ is called (coalescence) *hidden variable fractal interpolation function* associated with the set of data D (see, for instance, [9]).

Let $\mathcal{G}^*$ be the set of continuous functions $\mathbf{h} : I \to \mathbb{R}^2$ such that $\mathbf{h}(x_1) = (y_1, z_1), \mathbf{h}(x_N) = (y_N, z_N)$ equipped with the metric $d(\mathbf{h}, \mathbf{h}^*) = \max\{d_M\big(\mathbf{h}(x), \mathbf{h}^*(x)\big) : x \in I\}$. To obtain a functional equation for $\mathbf{g}$, we recall that $\mathbf{g}$ is the fixed point of the operator $\mathrm{T}^* : \mathcal{G}^* \to \mathcal{G}^*$ defined by

$$(\mathrm{T}^*\mathbf{h})(x) = F_n\big(L_n^{-1}(x), \mathbf{h}\big(L_n^{-1}(x)\big)\big), \text{ for } x \in I_n,\ n \in \mathbb{N}_{N-1}.$$

Whence, the vector-valued function $\mathbf{g}$ satisfies the functional equation

$$\mathbf{g}\big(L_n(x)\big) = A_n\mathbf{g}(x) + \big(p_n(x), q_n(x)\big)^t,\ x \in I,$$

and the component functions g_1 and g_2 obey the following coupled functional equations.

$$\begin{aligned} g_1\big(L_n(x)\big) &= \alpha_n g_1(x) + \beta_n g_2(x) + p_n(x), \\ g_2\big(L_n(x)\big) &= \gamma_n g_2(x) + q_n(x),\ x \in I. \end{aligned} \tag{2.4}$$

Throughout the remainder of the paper, we use the block matrix $\mathbf{A} = [A_1\ A_2\ \dots\ A_{N-1}] = [A_n]_{n\in\mathbb{N}_{N-1}}$ to collectively represent the parameters involved in the definition of HFIFs.

3 Construction of A-fractal Function Corresponding to a $\mathbb{R}^2$-valued Continuous Function f

In this section we deal with the construction of an $\mathbf{A}$-fractal function corresponding to a $\mathbb{R}^2$-valued continuous function $\mathbf{f}$ with the help of theory reported in the previous section. We enunciate that a continuous function $\mathbf{f} : I \to \mathbb{R}^2$ gives rise to an entire family of fractal functions $\mathbf{f}[\mathbf{A}]$ parameterized by a certain block matrix $\mathbf{A} = [A_n]_{n\in\mathbb{N}_{N-1}}$ with $A_n = \begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix}$, where $\mathbf{f}[\mathbf{0}] = \mathbf{f}$. Let $\mathbb{R}^2$ be endowed with the Manhattan metric and $\mathbf{f} = (f_1, f_2) \in \mathcal{C}(I, \mathbb{R}^2)$, the space of all continuous $\mathbb{R}^2$-valued functions on $I = [x_1, x_N]$. Choose a partition $\{x_1, x_2, \dots, x_N\}$ satisfying $x_1 < x_2 < \cdots < x_N$ in I and consider the data set $D = \{\big(x_n, f_1(x_n), f_2(x_n)\big) : n \in \mathbb{N}_N\}$. In the IFS $\{I \times \mathbb{R}^2; \big(L_n, F_n\big) : n \in \mathbb{N}_{N-1}\}$ defined through (2.1)-(2.2), we consider the special case

$$p_n(x) = f_1 \circ L_n(x) - \alpha_n b_1(x) - \beta_n b_2(x), \quad q_n(x) = f_2 \circ L_n(x) - \gamma_n b_2(x),$$

where $\mathbf{b} = (b_1, b_2) \in \mathcal{C}(I, \mathbb{R}^2)$ satisfies $\mathbf{b}(x_1) = \mathbf{f}(x_1)$ and $\mathbf{b}(x_N) = \mathbf{f}(x_N)$. In this case, the IFS $\{I \times \mathbb{R}^2; \big(L_n, F_n\big) : n \in \mathbb{N}_{N-1}\}$ provides a fixed point that is the graph of a continuous function denoted here as $\mathbf{f}[\mathbf{A}] = (f_1[\mathbf{A}], f_2[\mathbf{A}])$. Following (2.3), we stipulate that $\mathbf{f}[\mathbf{A}]$ satisfies:

$$\mathbf{f}[\mathbf{A}](x) = \mathbf{f}(x) + A_n(\mathbf{f}[\mathbf{A}] - \mathbf{b})\big(L_n^{-1}(x)\big),\ x \in I_n,\ n \in \mathbb{N}_{N-1}. \tag{3.5}$$

The fixed point $\mathbf{f}[\mathbf{A}]$ also depends on the choice of $\mathbf{b} \in \mathcal{C}(I, \mathbb{R}^2)$. We call this function $\mathbf{f}[\mathbf{A}]$ as $\mathbf{A}$-fractal function of $\mathbf{f}$ with respect to the partition $x_1 < x_2 < \cdots < x_N$ and the function $\mathbf{b}$. Let $\mathbf{f}$ be a continuous classical interpolant for the data set $D = \{(x_n, y_n, z_n) : n \in \mathbb{N}_N\}$. Since $\mathbf{f}[\mathbf{A}](x_n) = \mathbf{f}(x_n)$ for all $n \in \mathbb{N}_N$, for any choice of $\mathbf{A} = [A_n]_{n\in\mathbb{N}_{N-1}}$ and any choice of $\mathbf{b}$ fulfilling the conditions specified earlier, the fractal function $\mathbf{f}[\mathbf{A}]$ can be regarded as "fractal generalization" of the function $\mathbf{f}$. For a given continuous function $f : I \to \mathbb{R}$, we can select $\mathbf{f} = (f, f)$ and $\mathbf{b} = (b, b)$ satisfying $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$, $\alpha_n + \beta_n = \gamma_n$ to construct $\mathbf{A}$-fractal function for $\mathbf{f}$. In this case we obtain $\mathbf{f}[\mathbf{A}] = \big(f[\mathbf{A}], f[\mathbf{A}]\big)$, where $f[\mathbf{A}]$ coincides with the standard γ-fractal function f^γ corresponding to f with $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{N-1})$.

Remark 3.1. The projection $G(g_1)$ of the attractor $G(\mathbf{g})$ is not always the union of transformed copies of itself. Hence, g_1 is, in general, non-self-referential. It can be observed that $G(g_2) = \bigcup_{n\in\mathbb{N}_{N-1}} w_n^2\big(G(g_2)\big)$, where $w_n^2(x, z) := \big(L_n(x), F_n^2(x, z)\big) = \big(a_n x + b_n, \gamma_n z + q_n(x)\big)$ for all $n \in \mathbb{N}_{N-1}$. Thus, g_2 is a self-referential fractal function interpolating $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. In particular, $f_2[\mathbf{A}]$ is self-referential as (α-fractal function according to the definition given in the reference [26]).

Remark 3.2. With $|\gamma|_\infty := \max\{|\gamma_n| : n \in \mathbb{N}_{N-1}\}$, it is easy to see that $\|f_2[\mathbf{A}] - f_2\|_\infty \leq \dfrac{|\gamma|_\infty}{1 - |\gamma|_\infty}\|f_2 - b_2\|_\infty$. Thus, for proper choices of γ_n, $n \in \mathbb{N}_{N-1}$, the self-referential function $f_2[\mathbf{A}]$ simultaneously interpolates and approximates f_2. A similar remark holds for $f_1[\mathbf{A}]$.

Remark 3.3. If the elements of the hidden variable FIF are chosen such that $z_n = y_n$ for all $n \in \mathbb{N}_N$, and $\alpha_n + \beta_n = \gamma_n$, $p_n = q_n$ for all $n \in \mathbb{N}_{N-1}$, then $f_1[\mathbf{A}]$ coincides with $f_2[\mathbf{A}]$, and hence in this case one obtains a self-referential hidden variable FIF.

Remark 3.4. If $\beta_n = 0$ for all $n \in \mathbb{N}_{N-1}$, then $G(g_1) = \bigcup\limits_{n\in\mathbb{N}_{N-1}} w_n^1\big(G(g_1)\big)$, where

$$w_n^1(x, y, z) : \ = \big(L_n(x), F_n^1(x, y, z)\big) = \big(a_n x + b_n, \alpha_n y + p_n(x)\big) \text{ for all } n \in \mathbb{N}_{N-1}.$$

Therefore, we infer that g_1, and hence, in particular, $f_1[\mathbf{A}]$ is self-referential in this case as well as (α-fractal function according to the definition given in the reference [26]).

4 Approximation results

In this section, we consider certain properties of the corresponding map $\mathbf{f} \mapsto \mathbf{f}[\mathbf{A}]$ for a fixed $\mathbf{A}$ such as $\mathbf{f}[\mathbf{0}] = \mathbf{f}$. There may be many choices to select an appropriate $\mathbf{b}$. Among them to fulfill our desire, we assume that $\mathbf{b}$ depends linearly on $\mathbf{f}$, that is to say, $\mathbf{b}_{\lambda\mathbf{f}+\mathbf{g}} = \lambda\mathbf{b}_\mathbf{f} + \mathbf{b}_\mathbf{g}$ or $\mathbf{b} = L\mathbf{f}$, where $L : \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$ is a linear operator which is bounded with respect to the norm $\|\mathbf{f}\|_\infty := \sup\big\{\|\mathbf{f}(x)\|_{l^1} : x \in I\big\} = \sup\big\{|f_1(x)| + |f_2(x)| : x \in I\big\}$. Let $\|L\|$ denote the operator norm of L with respect to $\|.\|_\infty$ in $\mathcal{C}(I, \mathbb{R}^2)$. For a fixed partition D, parameter matrix $\mathbf{A}$, and continuous function $\mathbf{b}$, let us consider the operator $\mathcal{F}[\mathbf{A}] : \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$

which assigns $\mathbf{f}[\mathbf{A}]$ to the function $\mathbf{f}$. The block matrix $\mathbf{A}$ can be viewed as an element in $M^{2\times 2(N-1)}(\mathbb{R})$ and $\|\mathbf{A}\|_1 = \max\limits_{n\in\mathbb{N}_{N-1}} \{|\alpha_n|, |\beta_n| + |\gamma_n|\} < 1$.
The researches of Navascués (see, for instance, [26, 28]) bring influence for our work in this section. While the results in this section share a natural kinship with the corresponding results in the case of real valued fractal function, the reader will also discern a considerable degree of disparity due to the vector valuedness considered herein.

Theorem 4.1. *The following holds:*

I. $\mathbf{f}[\mathbf{0}] = \mathbf{f}$. *Consequently, if* $\mathbf{A} = \mathbf{0}$, *then the fractal operator* $\mathcal{F}[\mathbf{A}]$ *is the identity operator on* $\mathcal{C}(I, \mathbb{R}^2)$.

II. If $\mathbf{b} = \mathbf{f}$, *then* $\mathbf{f}[\mathbf{A}] = \mathbf{f}$.

III. The fractal function $\mathbf{f}[\mathbf{A}]$ *corresponding to* $\mathbf{f}$ *satisfies the inequality*

$$\big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty \le \frac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1}\|\mathbf{f} - \mathbf{b}\|_\infty.$$

IV. For suitable choices of parameters, the fractal function $\mathbf{f}[\mathbf{A}]$ *simultaneously interpolates and approximates* $\mathbf{f}$.

V. If the vector-valued function $\mathbf{b}$ *depends linearly on* $\mathbf{f}$, *then* $\mathcal{F}[\mathbf{A}] : \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$, $\mathbf{f} \mapsto \mathbf{f}[\mathbf{A}]$ *is linear.*

VI. If $\mathbf{b} = L\mathbf{f}$, *where* $L : \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$ *is a bounded linear map with respect to the uniform norm then the fractal operator* $\mathcal{F}[\mathbf{A}] : \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$, $\mathbf{f} \mapsto \mathbf{f}[\mathbf{A}]$ *is bounded.*

VII. If $\|\mathbf{A}\|_1 < \|L\|^{-1}$, *then* $\mathcal{F}[\mathbf{A}]$ *is injective and its range (*$Rg(\mathcal{F}[\mathbf{A}])$*) is closed,* $\mathcal{F}[\mathbf{A}]^{-1}$ *is bounded on (*$Rg(\mathcal{F}[\mathbf{A}])$*) and* $\mathcal{F}[\mathbf{A}]$, $\mathcal{F}[\mathbf{A}]^{-1}$ *are both closed operators.*

Proof. I. Follows directly from the functional equation for $\mathbf{f}[\mathbf{A}]$ (cf. (3.5)).

II. Let $\mathbf{b} = \mathbf{f}$. In this case, the functional equation (3.5) for $\mathbf{f}[\mathbf{A}]$ reads

$$\mathbf{f}[\mathbf{A}](x) = \mathbf{f}(x) + A_n\big(\mathbf{f}[\mathbf{A}] - \mathbf{f}\big)\big(L_n^{-1}(x)\big) \text{ on } I_n,\ n \in \mathbb{N}_{N-1},$$

which is obviously satisfied by $\mathbf{f}[\mathbf{A}] = \mathbf{f}$. Since $\mathbf{f}[\mathbf{A}]$ is obtained as a fixed point of the map T, from the uniqueness of the fixed point it follows that $\mathbf{f}[\mathbf{A}] = \mathbf{f}$.

III. By definition

$$\begin{aligned}\big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty &= \sup\Big\{\big\|\mathbf{f}[\mathbf{A}](x) - \mathbf{f}(x)\big\|_{l^1} : x \in I\Big\},\\ &= \max_{n\in\mathbb{N}_{N-1}} \sup\Big\{\big\|\big(\mathbf{f}[\mathbf{A}] - \mathbf{f}\big)(x)\big\|_{l^1} : x \in I_n\Big\},\\ &= \max_{n\in\mathbb{N}_{N-1}} \sup\Big\{\big\|A_n\big(\mathbf{f}[\mathbf{A}] - \mathbf{b}\big)\big(L_n^{-1}(x)\big)\big\|_{l^1} : x \in I_n\Big\}.\end{aligned}$$

Letting $\mathbf{f}[\mathbf{A}] = \big(f_1[\mathbf{A}], f_2[\mathbf{A}]\big)$ and performing the matrix multiplication, through a series of self-explanatory steps we obtain:

$$
\begin{aligned}
\big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty &= \max_{n\in\mathbb{N}_{N-1}} \sup \Big\{ \big|\alpha_n\big(f_1[\mathbf{A}] - b_1\big)\big(L_n^{-1}(x)\big) + \beta_n\big(f_2[\mathbf{A}] \\
&\quad - b_2\big)\big(L_n^{-1}(x)\big)\big| + \big|\gamma_n\big(f_2[\mathbf{A}] - b_2\big)\circ L_n^{-1}(x)\big| : x \in I_n \Big\}, \\
&\le \max_{n\in\mathbb{N}_{N-1}} \sup \Big\{ |\alpha_n|\big|\big(f_1[\mathbf{A}] - b_1\big)\big(L_n^{-1}(x)\big)\big| + (|\beta_n| + |\gamma_n|). \\
&\quad \big|\big(f_2[\mathbf{A}] - b_2\big)\big(L_n^{-1}(x)\big)\big| : x \in I_n \Big\}, \\
&\le \max_{n\in\mathbb{N}_{N-1}} \|A_n\|_1 \sup \Big\{ \big|\big(f_1[\mathbf{A}] - b_1\big)\big(L_n^{-1}(x)\big)\big| + \big|\big(f_2[\mathbf{A}] \\
&\quad - b_2\big)\big(L_n^{-1}(x)\big)\big| : x \in I_n \Big\}, \\
&= \|\mathbf{A}\|_1 \big\|\mathbf{f}[\mathbf{A}] - \mathbf{b}\big\|_\infty, \\
&\le \|\mathbf{A}\|_1 \big(\big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty + \big\|\mathbf{f} - \mathbf{b}\big\|_\infty\big),
\end{aligned}
$$

from which the desired estimate can be deduced.

IV. For an arbitrary selection of the partition, free and constrained parameters, and function $\mathbf{b}$, the interpolation property of $\mathbf{f}[\mathbf{A}]$, i.e., $\mathbf{f}[\mathbf{A}](x_n) = \mathbf{f}(x_n)$ is evident and it is in fact a content of the construction.
Let $\epsilon > 0$. To show $\big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty < \epsilon$, it suffices to show, thanks to Part III, that $\dfrac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1}\|\mathbf{f} - \mathbf{b}\|_\infty < \epsilon$. Choose the parameters α_n, β_n, and γ_n such that

$$\|\mathbf{A}\|_1 < \frac{\epsilon}{\epsilon + \|\mathbf{f} - \mathbf{b}\|_\infty} < 1.$$

With this selection, it is a matter of direct verification that $\dfrac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1}\|\mathbf{f} - \mathbf{b}\|_\infty < \epsilon$, whence the stated result follows.

V. Let $\mathbf{f}$, $\mathbf{g}$ be in $\mathcal{C}(I, \mathbb{R}^2)$ and $\lambda, \mu \in \mathbb{R}$. We have $(\mathcal{F}[A])(\mathbf{f}) = \mathbf{f}[\mathbf{A}]$ and $(\mathcal{F}[A])(\mathbf{g}) = \mathbf{g}[\mathbf{A}]$. To prove that $(\mathcal{F}[A])(\lambda\mathbf{f} + \mu\mathbf{g}) = \lambda\mathbf{f}[\mathbf{A}] + \mu\mathbf{g}[\mathbf{A}]$. Recall that

$$
\begin{aligned}
&\mathbf{f}[\mathbf{A}](x) = \mathbf{f}(x) + A_n\big(\mathbf{f}[\mathbf{A}] - \mathbf{b}_\mathbf{f}\big)\big(L_n^{-1}(x)\big), \\
&\mathbf{g}[\mathbf{A}](x) = \mathbf{g}(x) + A_n\big(\mathbf{g}[\mathbf{A}] - \mathbf{b}_\mathbf{g}\big)\big(L_n^{-1}(x)\big),\ \forall\, x \in I_n,
\end{aligned}
$$

and then

$$\big(\lambda\mathbf{f}[\mathbf{A}] + \mu\mathbf{g}[\mathbf{A}]\big)(x) = \big(\lambda\mathbf{f} + \mu\mathbf{g}\big)(x) + A_n\big(\lambda\mathbf{f}[\mathbf{A}] + \mu\mathbf{g}[\mathbf{A}] - \mathbf{b}_{\lambda\mathbf{f}+\mu\mathbf{g}}\big)\big(L_n^{-1}(x)\big).$$

Therefore, $\lambda\mathbf{f}[\mathbf{A}] + \mu\mathbf{g}[\mathbf{A}]$ is the fixed point of the operator

$$(T\mathbf{h})(x) = F_n\big(L_n^{-1}(x), \mathbf{h}\big(L_n^{-1}(x)\big)\big) = (\lambda\mathbf{f} + \mu\mathbf{g})(x) + A_n\big(\mathbf{h} - \mathbf{b}_{\lambda\mathbf{f}+\mu\mathbf{g}}\big)\big(L_n^{-1}(x)\big).$$

From the uniqueness of the fixed point we gather that

$$(\lambda \mathbf{f} + \mu \mathbf{g})[\mathbf{A}] = \lambda \mathbf{f}[\mathbf{A}] + \mu \mathbf{g}[\mathbf{A}],$$

demonstrating the linearity of $\mathcal{F}[\mathbf{A}]$.

VI. From Part III, we have

$$\begin{aligned}
\big\|(\mathcal{F}[\mathbf{A}])(\mathbf{f})\big\|_\infty &= \big\|(\mathcal{F}[\mathbf{A}])(\mathbf{f}) - \mathbf{f} + \mathbf{f}\big\|_\infty, \\
&\leq \big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty + \|\mathbf{f}\|_\infty, \\
&\leq \Big(\frac{\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}\|\mathbf{f} - \mathbf{b}\|_\infty + \|\mathbf{f}\|_\infty\Big), \\
&\leq \Big(\frac{\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}\|I_d - L\| + 1\Big)\|\mathbf{f}\|_\infty,
\end{aligned}$$

where I_d is the identity operator on $\mathcal{C}(I, \mathbb{R}^2)$. This shows that the linear map $\mathcal{F}[\mathbf{A}]$ is bounded, and the operator norm satisfies $\big\|\mathcal{F}[\mathbf{A}]\big\| \leq \dfrac{\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}\|I_d - L\| + 1.$

VII. It is patent from a moment's reflection on the proof of Part III that

$$\big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty \leq \|\mathbf{A}\|_1 \big\|\mathbf{f}[\mathbf{A}] - \mathbf{b}\big\|_\infty = \|\mathbf{A}\|_1 \big\|\mathbf{f}[\mathbf{A}] - L\mathbf{f}\big\|_\infty.$$

Assume $(\mathcal{F}[\mathbf{A}])(\mathbf{f}) = \mathbf{f}[\mathbf{A}] = \mathbf{0}$. Then, we have

$$\|\mathbf{f}\|_\infty \leq \|\mathbf{A}\|_1 \|L\| \|\mathbf{f}\|_\infty.$$

Since $\|\mathbf{A}\|_1 < \|L\|^{-1}$, it is manifest that $\|\mathbf{f}\|_\infty = 0$. That is $\mathbf{f} = \mathbf{0}$, yielding injectivity of the operator $\mathcal{F}[\mathbf{A}]$. Again using Part III,

$$\|\mathbf{f}\|_\infty \leq \big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty + \big\|\mathbf{f}[\mathbf{A}]\big\|_\infty \leq \|\mathbf{A}\|_1 \big\|\mathbf{f}[\mathbf{A}] - L\mathbf{f}\big\|_\infty + \big\|\mathbf{f}[\mathbf{A}]\big\|_\infty.$$

By some simple manipulations on the above inequality we obtain

$$\|\mathbf{f}\|_\infty \leq \frac{1 + \|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1 \|L\|} \big\|\mathbf{f}[\mathbf{A}]\big\|_\infty,$$

from which we see that the inverse operator $\mathcal{F}[\mathbf{A}]^{-1}$ is bounded on $(Rg(\mathcal{F}[\mathbf{A}]))$. To prove the range of $\mathcal{F}[\mathbf{A}]$ is closed one can follow the arguments provided in Theorem 3.5 of [26]. The operators $\mathcal{F}[\mathbf{A}]$ and $\mathcal{F}[\mathbf{A}]^{-1}$ are continuous in this case and thus closed.

□

Definition 4.1. A sequence (x_m) of a Banach space X is a Schauder basis, if for all $x \in X$ there exists a unique representation of x as

$$x = \sum c_m x_m,$$

where (c_m) is a sequence of scalars.

Definition 4.2. A sequence (x_m) of a Banach space is a Schauder sequence, if it is a Schauder basis for $[x_m] = \overline{span}(x_m)$.

Note: The set $span(x_m)$ is the family of finite linear combinations of the elements x_m and $[x_m]$ is the topological closure of $span(x_m)$.

Theorem 4.2. *If* $(\boldsymbol{f}_m)$ *is a Schauder basis of* $\mathcal{C}(I, \mathbb{R}^2)$ *and* $\|\mathbf{A}\|_1 < \|L\|^{-1}$, *then* $(\boldsymbol{f}_m[\boldsymbol{A}])$ *is a Schauder sequence.*

Proof. Although the norm and the space of functions used are different, the arguments are similar to those provided in the proof of Theorem 2.12 of the reference [27]. The operator $\mathcal{F}[\mathbf{A}]$ is in this case a topological isomorphism onto $Rg(\mathcal{F}[\mathbf{A}]) = [\mathbf{f}_m[\mathbf{A}]]$, and these types of transformations preserve the bases. $\square$

On lines similar to Theorem 4.1, elementary property of the operator $\mathcal{F}[\mathbf{A}]$ can be established, for instance, we have the following.

Theorem 4.3. *For the variables* α_n, β_n, *and* γ_n, $n \in \mathbb{N}_{N-1}$ *selected so that* $\|\mathbf{A}\|_1 < (1 + \|I_d - L\|)^{-1}$, *the corresponding fractal operator* $\mathcal{F}[\mathbf{A}] : \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$ *is a topological isomorphism.*

Proof. According to Part III of Theorem 4.1, $\big\|(\mathcal{F}[\mathbf{A}])(\mathbf{f}) - \mathbf{f}\big\|_\infty \leq \dfrac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1}\|\mathbf{f} - L\mathbf{f}\|_\infty$ and consequently $\big\|I_d - \mathcal{F}[\mathbf{A}]\big\| \leq \dfrac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1}\|I_d - L\|$. The hypothesis $\|\mathbf{A}\|_1 < (1 + \|I_d - L\|)^{-1}$ now yields $\big\|I_d - \mathcal{F}[\mathbf{A}]\big\| < 1$. That the operator $\mathcal{F}[\mathbf{A}] = I_d - \big(I_d - \mathcal{F}[\mathbf{A}]\big)$ has bounded inverse follows from the standard theorem which reads: If T is a bounded linear operator from a Banach space into itself such that $\|T\| < 1$, then $I - T$ has bounded inverse and the Neumann series $\sum\limits_{k=0}^{\infty} T^k$ converges in operator norm to $(I - T)^{-1}$ [1]. This completes the proof. $\square$

Theorem 4.4. *If* $(\boldsymbol{f}_m)$ *is a Schauder basis of* $\mathcal{C}(I, \mathbb{R}^2)$ *and* $\|\mathbf{A}\|_1 < (1 + \|I_d - L\|)^{-1}$, *then* $(\boldsymbol{f}_m[\boldsymbol{A}])$ *is a Schauder basis as well.*

Proof. The operator $\mathcal{F}[\mathbf{A}]$ is in this case a topological isomorphism according to the previous Theorem. Consequently $(\mathbf{f}_m[\mathbf{A}])$ is a Schauder basis of $\mathcal{C}(I, \mathbb{R}^2)$. $\square$

5 Fractal Function f[A] Preserving Some Properties of Original Function f

In this section, we identify suitable parameters in the IFS so that $\mathbf{A}$-fractal function $\mathbf{f}[\mathbf{A}]$ preserves the properties (for instance, regularity and monotonicity) inherent in $\mathbf{f}$.

Theorem 5.1. *Let* $\mathbf{f} \in \mathcal{C}^r(I, \mathbb{R}^2)$. *Suppose* $D = \{x_1, x_2, \dots, x_N\}$ *be an arbitrary partition on* I *satisfying* $x_1 < x_2 < \dots < x_N$, *block matrix* $\boldsymbol{A} = [A_n]_{n \in \mathbb{N}_{N-1}}$, $A_n = \begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix}$ *of* *parameters satisfy* $|\alpha_n| < a_n^r$, $|\beta_n| + |\gamma_n| < a_n^r$ *for all* $n \in \mathbb{N}_{N-1}$. *Further suppose that* $\mathbf{b} = (b_1, b_2) \in \mathcal{C}^r(I, \mathbb{R}^2)$ *fulfills* $\mathbf{b}^{(j)}(x_1) = \mathbf{f}^{(j)}(x_1)$, $\mathbf{b}^{(j)}(x_N) = \mathbf{f}^{(j)}(x_N)$ *for* $j = 0, 1, \dots, r$. *Then the corresponding* $\mathbb{R}^2$*-valued fractal function* $\mathbf{f}[\mathbf{A}]$ *is* r*-smooth, and* $\mathbf{f}[\mathbf{A}]^{(j)}(x_n) = \mathbf{f}^{(j)}(x_n)$ *for* $n \in \mathbb{N}_N$ *and* $j = 0, 1, \dots, r$.

Proof. Let $\Re := \{\mathbf{h} \in \mathcal{C}^r(I, \mathbb{R}^2),\ \mathbf{h}^{(j)}(x_1) = \mathbf{f}^{(j)}(x_1)$ and $\mathbf{h}^{(j)}(x_N) = \mathbf{f}^{(j)}(x_N), j = 0, 1, \dots, r\}$. Here $\Re$ is closed subset of the complete metric space $(\mathcal{C}^r(I, \mathbb{R}^2), \|.\|_{\mathcal{C}^r(I,\mathbb{R}^2)})$, where

$$\|\mathbf{f}\|_{\mathcal{C}^r(I,\mathbb{R}^2)} := \max\{\|\mathbf{f}^{(j)}\|_\infty : j = 0, 1, \dots, r\}, \quad \|\mathbf{f}^{(j)}\|_\infty := \sup\left\{\|\mathbf{f}^{(j)}(x)\|_{l^1} : x \in I\right\}$$

and hence the space $\Re$ is complete. Suppose $\mathrm{T}_\mathbf{A} : \Re \to \Re$ is defined as follows.

$$\begin{aligned}(\mathrm{T}_\mathbf{A}\mathbf{h})(x) &= F_n\Big(L_n^{-1}(x), \mathbf{h}\big(L_n^{-1}(x)\big)\Big),\\ &= \mathbf{f}(x) + A_n(\mathbf{h} - \mathbf{b})\big(L_n^{-1}(x)\big)\ x \in I_n,\ n \in \mathbb{N}_{N-1}.\end{aligned} \tag{5.6}$$

It follows upon the assumptions on $\mathbf{f}, \mathbf{b}$ and the parameters involved in block matrix $\mathbf{A} = [A_n]_{n\in\mathbb{N}_{N-1}}$, that $\mathrm{T}_\mathbf{A}\mathbf{h}$ is r-times differentiable with a continuous r-th derivative on each interval (x_n, x_{n+1}). We prove that $\mathrm{T}_\mathbf{A}$ maps $\Re$ into $\Re$. With the choice of $\mathbf{b}$, we have $(\mathrm{T}_\mathbf{A}\mathbf{h})(x_1) = \mathbf{f}(x_1)$ and $(\mathrm{T}_\mathbf{A}\mathbf{h})(x_N) = \mathbf{f}(x_N)$. Next we verify

$$(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_n^+) = (\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_n^-)\ \forall\ n \in \mathbb{N}_{N-1}.$$

For $j = 1, 2, \dots, r$, from (5.6) we obtain

$$(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(L_n(x)) = \mathbf{f}^{(j)}(L_n(x)) + \frac{A_n(\mathbf{h} - \mathbf{b})^{(j)}(x)}{a_n^j}. \tag{5.7}$$

Therefore,

$$\begin{aligned}(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_n^+) &= \mathbf{f}^{(j)}(x_n) + \frac{A_n(\mathbf{h} - \mathbf{b})^{(j)}(x_1)}{a_n^j},\\ (\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_n^-) &= \mathbf{f}^{(j)}(x_n) + \frac{A_n(\mathbf{h} - \mathbf{b})^{(j)}(x_N)}{a_{n-1}^j}.\end{aligned} \tag{5.8}$$

Since $\mathbf{h} \in \Re$, it is apparent that $\mathbf{h}^{(j)}(x_1) = \mathbf{f}^{(j)}(x_1)$ *and* $\mathbf{h}^{(j)}(x_N) = \mathbf{f}^{(j)}(x_N), j = 0, 1, \dots, r$. Using the conditions on $\mathbf{b}$, namely, its contact of order r at the extremes of the interval I with the function $\mathbf{f}$, we find from the above equations that $(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_n^+) = (\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_n^-)$ for $j = 0, 1, \dots, r$ and $n = 2, 3, \dots, N-1$. Furthermore, from above equations it can be deduced that $(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_1) = \mathbf{f}^{(j)}(x_1)$, $(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x_N) = \mathbf{f}^{(j)}(x_N); j \in \{0, 1, \dots, r\}$. Thus, $\mathrm{T}_\mathbf{A}\mathbf{h} \in \Re$ and $\mathrm{T}_\mathbf{A}$ is well defined. Adhering to above equations again, we see that

$$\begin{aligned}\big\|(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)} - (\mathrm{T}_\mathbf{A}\mathbf{h}^*)^{(j)}\big\|_\infty &= \sup\Big\{\big\|(\mathrm{T}_\mathbf{A}\mathbf{h})^{(j)}(x) - (\mathrm{T}_\mathbf{A}\mathbf{h}^*)^{(j)}(x)\big\|_{l^1} : x \in I\Big\},\\ &= \max_{n\in\mathbb{N}_{N-1}} \sup\Big\{\Big\|\frac{A_n(\mathbf{h} - \mathbf{h}^*)^{(j)}(L_n^{-1}(x))}{a_n^j}\Big\|_{l^1} : x \in I_n\Big\},\\ &\le \max_{n\in\mathbb{N}_{N-1}} \frac{\|A_n\|_1}{a_n^j} \sup\Big\{\big\|(\mathbf{h} - \mathbf{h}^*)^{(j)}(L_n^{-1}(x))\big\|_{l^1} : x \in I_n\Big\},\\ &\le \max_{n\in\mathbb{N}_{N-1}} \frac{\|A_n\|_1}{a_n^r} \big\|(\mathbf{h} - \mathbf{h}^*)^{(j)}\big\|_\infty.\end{aligned}$$

Thus, for $j \in \{0, 1, \dots, r\}$ one obtains

$$\begin{aligned}\left\|\mathrm{T}_{\mathbf{A}}\mathbf{h} - \mathrm{T}_{\mathbf{A}}\mathbf{h}^*\right\|_{\mathcal{C}^r(I,\mathbb{R}^2)} &= \max\{\|(\mathrm{T}_{\mathbf{A}}\mathbf{h})^{(j)} - (\mathrm{T}_{\mathbf{A}}\mathbf{h}^*)^{(j)}\|_\infty : j = 0, 1, \dots, r\}\\ &\le \max\{\max_{n\in\mathbb{N}_{N-1}} \frac{\|A_n\|_1}{a_n^r}\left\|(\mathbf{h} - \mathbf{h}^*)^{(j)}\right\|_\infty : j = 0, 1, \dots, r\}\\ &\le \max_{n\in\mathbb{N}_{N-1}} \frac{\|A_n\|_1}{a_n^r}\left\|\mathbf{h} - \mathbf{h}^*\right\|_{\mathcal{C}^r(I,\mathbb{R}^2)}.\end{aligned}$$

Since $|\alpha_n| < a_n^r$, $|\beta_n| + |\gamma_n| < a_n^r$ for all $n \in \mathbb{N}_{N-1}$, then it follows that $\|A_n\|_1 < a_n^r$. Consequently, $\max\limits_{n\in\mathbb{N}_{N-1}} \dfrac{\|A_n\|_1}{a_n^r} < 1$ and $\mathrm{T}_{\mathbf{A}}$ is a contraction. The fixed point $\mathbf{f}[\mathbf{A}]$ of $\mathrm{T}_{\mathbf{A}}$ is the r-smooth fractal function corresponding to $\mathbf{f}$. It is clear from the above discussion that the derivatives of $\mathbf{f}$ and $\mathbf{f}[\mathbf{A}]$ agree at the endpoints of the interpolation interval i.e. $\mathbf{f}[\mathbf{A}]^{(j)}(x_n) = \mathbf{f}^{(j)}(x_n)$ for $n \in \mathbb{N}_N$ and $j = 0, 1, \dots, r$, completing the proof. □

The fractal function $\mathbf{f}[\mathbf{A}]$ established in the previous section may not preserve the monotonicity property hidden in a given data set. As indicated at the start of this section, the next theorem points to the conditions on the elements of the (hidden variable) IFS so that the $\mathbf{A}$-fractal function $\mathbf{f}[\mathbf{A}]$ retains the monotonicity of $\mathbf{f}$ inherent in the prescribed data set. We need the following notation to best describe it. Let

$$M_i = \max_{x\in I} b_i^{(1)}(x),\ m_{i,n} = \min_{x\in I_n} f_i^{(1)}(x) \text{ for } i = 1, 2;\ n \in \mathbb{N}_{N-1}.$$

Note that the existence of these parameters follows from the continuity of functions involved in their definition and the compactness of the domain.

Theorem 5.2. *Let $\mathbf{f} \in \mathcal{C}^1(I, \mathbb{R}^2)$ be a monotonic increasing function. Consider an arbitrary partition $D = \{x_1, x_2, \dots, x_N\}$ on I satisfying $x_1 < x_2 < \cdots < x_N$ and a function $\mathbf{b} = (b_1, b_2) \in \mathcal{C}^1(I, \mathbb{R}^2)$ satisfying $\mathbf{b}(x_1) = \mathbf{f}(x_1)$, $\mathbf{b}(x_N) = \mathbf{f}(x_N)$, $\mathbf{b}^{(1)}(x_1) = \mathbf{f}^{(1)}(x_1)$, $\mathbf{b}^{(1)}(x_N) = \mathbf{f}^{(1)}(x_N)$. Further, let the block matrix $\mathbf{A} = [A_n]_{n\in\mathbb{N}_{N-1}}$ with $A_n = \begin{bmatrix}\alpha_n & \beta_n\\ 0 & \gamma_n\end{bmatrix}$ be selected such that the entries $\alpha_n, \beta_n, \gamma_n$ that lie in $[0, 1)$ satisfy*

$$\alpha_n < a_n,\ \gamma_n \le \frac{a_n m_{2,n}}{M_2},\ \beta_n + \gamma_n < a_n,\ \alpha_n M_1 + \beta_n M_2 \le a_n m_{1,n}.$$

Then the corresponding $\mathbf{A}$-fractal function $\mathbf{f}[\mathbf{A}]$ preserves the monotonicity of $\mathbf{f}$.

Proof. Note that the $\mathbf{A}$-fractal function $\mathbf{f}[\mathbf{A}] = \big(f_1[A], f_2[A]\big)$ is constructed iteratively using the functional equations

$$\begin{aligned}f_1[\mathbf{A}]\big(L_n(x)\big) &= F_n^1\big(x, f_1[\mathbf{A}](x), f_2[\mathbf{A}](x)\big),\\ &= \alpha_n f_1[\mathbf{A}](x) + \beta_n f_2[\mathbf{A}](x) + f_1\big(L_n(x)\big) - \alpha_n b_1(x) - \beta_n b_2(x),\\ f_2[\mathbf{A}]\big(L_n(x)\big) &= F_n^2\big(x, f_2[\mathbf{A}](x)\big),\\ &= \gamma_n f_2[\mathbf{A}](x) + f_2\big(L_n(x)\big) - \gamma_n b_2(x).\end{aligned}$$

Differentiating the expression for $f_1[\mathbf{A}]\big(L_n(x)\big)$ and $f_2[\mathbf{A}]\big(L_n(x)\big)$, we obtain

$$a_n f_1^{(1)}[\mathbf{A}]\big(L_n(x)\big) = \alpha_n f_1^{(1)}[\mathbf{A}](x) + \beta_n f_2^{(1)}[\mathbf{A}](x) + a_n f_1^{(1)}\big(L_n(x)\big) - \alpha_n b_1^{(1)}(x) - \beta_n b_2^{(1)}(x),$$
$$a_n f_2^{(1)}[\mathbf{A}]\big(L_n(x)\big) = \gamma_n f_2^{(1)}[\mathbf{A}](x) + a_n f_2^{(1)}\big(L_n(x)\big) - \gamma_n b_2^{(1)}(x).$$

To prove that the $\mathbf{A}$-fractal function $\mathbf{f}[\mathbf{A}]$ preserves the monotonicity of the function $\mathbf{f}$, we will see $f_j^{(1)}[\mathbf{A}](x) \geq 0$ for all $x \in I$, for $j = 1, 2$. For $j = 1, 2$, it is enough (by induction) to prove that $f_j^{(1)}[\mathbf{A}] \geq 0$ holds good at points on I obtained at $(i+1)$-th iteration whenever $f_j^{(1)}[\mathbf{A}] \geq 0$ is satisfied for distinct points on I at i-th iteration. First we consider $f_2^{(1)}[\mathbf{A}]\big(L_n(x)\big) \geq 0$ for all $n \in \mathbb{N}_{N-1}$. This is equivalent to prove that $(F_n^2)^{(1)}(x,z) = \dfrac{\gamma_n z + a_n f_2^{(1)}\big(L_n(x)\big) - \gamma_n b_2^{(1)}(x)}{a_n} \geq 0$ for all $n \in \mathbb{N}_{N-1}$ whenever $(x,z) \in I \times \mathbb{R}$ and $z \geq 0$. Again for $\gamma_n \geq 0$, the conditions $(F_n^2)^{(1)}(x,z) \geq 0$ for all $(x,z) \in I \times \mathbb{R}$ and $z \geq 0$ are met if $a_n f_2^{(1)}\big(L_n(x)\big) - \gamma_n b_2^{(1)}(x) \geq 0$. By the definition of $m_{2,n}$ and M_2, we have $a_n f_2^{(1)}\big(L_n(x)\big) - \gamma_n b_2^{(1)}(x) \geq a_n m_{2n} - \gamma_n M_2$. With the aforementioned points one can deduce that $(F_n^2)^{(1)}(x,z) \geq 0$ for all $x \in I$ is satisfied if $\gamma_n \in [0,1)$ is selected so that $\gamma_n \leq \dfrac{a_n m_{2,n}}{M_2}$ for all $n \in \mathbb{N}_{N-1}$. Note also that if $M_2 = 0$, then no additional constraint on γ_n needs to be imposed.

Having selected γ_n, $n \in \mathbb{N}_{N-1}$ according to the aforementioned prescription, by similar arguments it can be seen that $(F_n^1)^{(1)}(x,y,z) \geq 0$ for all $x \in I$ is fulfilled, if $y \geq 0$, $z \geq 0$, $\alpha_n \geq 0$, $\beta_n \geq 0$ and $a_n f_1^{(1)}\big(L_n(x)\big) - \alpha_n b_1^{(1)}(x) - \beta_n b_2^{(1)}(x) \geq 0$ for all $x \in I$ and $n \in \mathbb{N}_{N-1}$. Note that $a_n f_1^{(1)}\big(L_n(x)\big) - \alpha_n b_1^{(1)}(x) - \beta_n b_2^{(1)}(x) \geq a_n m_{1,n} - \alpha_n M_1 - \beta_n M_2$. Consequently, the desired condition turns out to be true if $\alpha_n M_1 + \beta_n M_2 \leq a_n m_{1,n}$. This completes the proof. □

Remark 5.1. With a slight modification of the arguments as in the foregoing theorem, analogous result may be obtained for a nonincreasing function $\mathbf{f} \in \mathcal{C}^1(I, \mathbb{R}^2)$.

Remark 5.2. The aforementioned fractal scheme can be modified and extended to produce piecewise defined $\mathbf{A}$-fractal function which is comonotone with the given $\mathbf{f} \in \mathcal{C}^1(I, \mathbb{R}^2)$. For this, the interval I has to be subdivided into subintervals, say $I_j, j = 1, 2, \dots, r$ in such a way that the function $\mathbf{f}|I_j = \mathbf{f}_j$ is monotonic increasing or decreasing throughout the subinterval I_j. In each of these subintervals I_j, we take base function $\mathbf{b}_j$, and the variables α_n^j, β_n^j and γ_n^j in the block matrix $\mathbf{A}_j = [A_n^j]_{n \in \mathbb{N}_{N-1}}$ so as to meet the specification in Theorem 5.2 and in Remark 5.1. Consequently we can get the fractal functions $\mathbf{f}[\mathbf{A}_j]$ that retain the monotonicity of the functions $\mathbf{f}_j = \mathbf{f}|I_j, j = 1, 2, \dots, r$.

6 Monotonicity of $\mathcal{C}^1$-Cubic Spline Hidden Variable FIF

In this section we illustrate the fractal perturbation process, its monotonicity aspect enunciated in the previous section by taking cubic spline as an example and to develop a monotonicity preserving cubic spline hidden variable interpolation scheme that extends the methods described in the reference [16]. Note that Fritch and Carlson [16] have established the condition on the derivative parameters so that the $\mathcal{C}^1$-cubic spline reflects the monotonicity property inherent in a prescribed data set. The desired $\mathcal{C}^1$-cubic spline hidden variable FIF can be employed to represent self-referential as well as non-self-referential monotonic function Ψ with derivative $\Psi^{(1)}$ having irregularity in a finite or dense subset of the interpolation interval.

Consider a set of Hermite data $D = \big\{(x_n, y_n, d_n) : n \in \mathbb{N}_N\big\}$, where y_n denote the function value and d_n denote the derivative value of an unknown function Ψ_1 at the knot point x_n. To obtain a $\mathcal{C}^1$-cubic spline hidden variable interpolant corresponding to D, we extend it to a generalized data set $\widehat{D} = \big\{(x_n, y_n, d_n, y_n^*, d_n^*) : n \in \mathbb{N}_N\big\}$, where y_n^* and d_n^* are real parameters that are assumed to be the function values and the derivative values of a function Ψ_2 at the knot point x_n. By making use of the general theory given in Section 2 coupled with conditions of differentiability and by taking p_n and q_n as cubic polynomials, we construct the $\mathcal{C}^1$-cubic spline hidden variable FIF corresponding to D. With $h_n = x_{n+1} - x_n$ and $t := \frac{x-x_1}{x_N-x_1}$, the traditional nonrecursive $\mathcal{C}^1$-cubic splines f_1 and f_2 corresponding to D and $\widehat{D}$, respectively can be represented as

$$\begin{aligned} f_1\big(L_n(x)\big) = \quad & \big\{h_n(d_n + d_{n+1}) - 2(y_{n+1} - y_n)\big\}t^3 + \big\{-h_n(2d_n + d_{n+1}) \\ & \qquad\qquad +3(y_{n+1} - y_n)\big\}t^2 + h_n d_n t + y_n, \\ f_2\big(L_n(x)\big) = \quad & \big\{h_n(d_n^* + d_{n+1}^*) - 2(y_{n+1}^* - y_n^*)\big\}t^3 + \big\{-h_n(2d_n^* + d_{n+1}^*) \\ & \qquad\qquad +3(y_{n+1}^* - y_n^*)\big\}t^2 + h_n d_n^* t + y_n^*. \end{aligned} \tag{6.9}$$

According to the descriptions in Theorem 5.2, we have to select the parameter matrix $\mathbf{A}$ and function $\mathbf{b} = (b_1, b_2)$ in such a way to obtain a continuously differentiable fractal perturbation for $\mathbf{f} = (f_1, f_2) \in \mathcal{C}^1(I, \mathbb{R}^2)$. A natural choice of $\mathbf{b} = (b_1, b_2)$ is the one in which b_1 and b_2 are the two-point Hermite interpolants (with knots at x_1 and x_N) corresponding to f_1 and f_2 respectively. That is,

$$\begin{aligned} b_1(x) = \quad & \big[(x_N - x_1)(d_1 + d_N) - 2(y_N - y_1)\big]t^3 + \big[-(x_N - x_1)(2d_1 + d_N) \\ & \qquad\qquad +3(y_N - y_1)\big]t^2 + d_1(x - x_1) + y_1, \\ b_2(x) = \quad & \big[(x_N - x_1)(d_1^* + d_N^*) - 2(y_N^* - y_1^*)\big]t^3 + \big[-(x_N - x_1)(2d_1^* + d_N^*) \\ & \qquad\qquad +3(y_N^* - y_1^*)\big]t^2 + d_1^*(x - x_1) + y_1^*, \end{aligned} \tag{6.10}$$

With these choices of component functions and with $\mathbf{A} = [A_n]_{n \in \mathbb{N}_{N-1}}$, where $A_n = \begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix}$, $n \in \mathbb{N}_{N-1}$, satisfy $|\alpha_n| < a_n$, $|\beta_n| + |\gamma_n| < a_n$, we obtain $\mathbf{A}$-fractal function $\mathbf{f}[\mathbf{A}] = \big(f_1[\mathbf{A}], f_2[\mathbf{A}]\big) \in \mathcal{C}^1(I, \mathbb{R}^2)$ corresponding to $\mathbf{f} = (f_1, f_2) \in \mathcal{C}^1(I, \mathbb{R}^2)$ defined as:

$$\begin{aligned} f_1[\mathbf{A}]\big(L_n(x)\big) = \quad & f_1\big(L_n(x)\big) + \alpha_n\big(f_1[\mathbf{A}] - b_1\big)(x) + \beta_n\big(f_2[\mathbf{A}] - b_2\big)(x), \\ f_2[\mathbf{A}]\big(L_n(x)\big) = \quad & f_2\big(L_n(x)\big) + \gamma_n\big(f_2[\mathbf{A}] - b_2\big)(x),\ x \in I,\ n \in \mathbb{N}_{N-1}. \end{aligned} \tag{6.11}$$

The function $f_1[\mathbf{A}] : I \to \mathbb{R}$ enjoying the Hermite interpolation conditions $f_1[\mathbf{A}](x_n) = y_n$ and $f_1[\mathbf{A}]^{(1)}(x_n) = d_n$ is the desired $\mathcal{C}^1$*-cubic spline hidden variable FIF* corresponding to D.

Remark 6.1. If we choose the "hidden variables" y_n^* and d_n^* such that $y_n^* = y_n$ and $d_n^* = d_n$ for all $n \in \mathbb{N}_N$, and the parameters according to the relation $\alpha_n + \beta_n = \gamma_n$ for all $n \in \mathbb{N}_{N-1}$, then the cubic hidden variable FIF $f_1[\mathbf{A}]$ coincides with $f_2[\mathbf{A}]$, representing a self-referential $\mathcal{C}^1$-cubic FIF approached constructively by Chand and Viswanathan [12]. For other choices of the hidden variables and parameters, $f_1[\mathbf{A}]$ is, in general, non-self-referential. Thus, the method is suitable for representing both self-referential and non-self-referential function, hence referred to as cubic spline coalescence hidden variable FIF.

Suppose a monotonic interpolation data set $D = \{(x_n, y_n) : n \in \mathbb{N}_N\}$ wherein $y_1 \leq y_2 \leq \cdots \leq y_N$. Extend it to $\widehat{D} = \{(x_n, y_n, y_n^*) : n \in \mathbb{N}_N\}$ by augmenting real parameters (hidden variables) y_n^* such that $y_1^* \leq y_2^* \leq \cdots \leq y_N^*$.

An Algorithm for monotonic $\mathcal{C}^1$-cubic spline hidden variable FIF

Step 1: Compute the approximate derivative values d_n, d_n^* for $n \in \mathbb{N}_N$. Ensure that each $d_n \geq 0$, $d_n^* \geq 0$. If $\Delta_n = 0$, $\Delta_n^* = 0$, let $d_n = d_{n+1} = 0$, $d_n^* = d_{n+1}^* = 0$ respectively.

Step 2: For each interval I_n for which $(\eta_n, \xi_n) = (\frac{d_n}{\Delta_n}, \frac{d_{n+1}}{\Delta_n}) \notin \rho$, modify d_n, d_{n+1} to $d_n^\diamond, d_{n+1}^\diamond$ such that $(\eta_n^\diamond, \xi_n^\diamond) = (\frac{d_n^\diamond}{\Delta_n}, \frac{d_{n+1}^\diamond}{\Delta_n}) \in \rho$. Similarly, do for d_n^*, d_{n+1}^*.
Further, construct the corresponding monotonic cubic splines f_i, $i = 1, 2$, (cf. (6.9)) and the functions b_i, $i = 1, 2$, (cf. (6.10)).

Step 3: Denoting the derivative values obtained at the end of Step 2 by d_n, d_n^* for $n \in \mathbb{N}_N$. For f_i and b_i, $i = 1, 2$, as obtained at the end of the previous step, compute the constants $M_i = \max_{x \in I} b_i^{(1)}(x)$, $m_{i,n} = \min_{x \in I_n} f_i^{(1)}(x)$ for $i = 1, 2$. Choose variables satisfying the following constraints:

$$0 \leq \alpha_n < a_n,\ \beta_n \geq 0,\ \gamma_n \in \big[0, \frac{a_n m_{2n}}{M_2}\big], \beta_n + \gamma_n < a_n,\ \text{and } \alpha_n M_1 + \beta_n M_2 \leq a_n m_{1n}.$$

Step 4: Input the derivative values chosen in Step 2 and parameters as prescribed by Step 3 in the functional equations represented by (6.11) whereupon the points of the graph of $f_1[\mathbf{A}]$ and $f_2[\mathbf{A}]$ are computed. The parameters of this cubic FIF $\mathbf{f}[\mathbf{A}]$ satisfy sufficient conditions in Theorem 5.2, and hence $\mathbf{f}[\mathbf{A}]$ is monotone.
Note: We can apply any classical method available in literature, for instance, [14, 15, 32] up to Step 2 first, then we can obtain fractal analogue of that particular method or algorithm with combination of Step 3 and Step 4.
We prove the following theorem which provides convergence order of monotonic $\mathcal{C}^1$-cubic spline hidden variable FIF.

Theorem 6.1. *Suppose that* $\mathbf{\Phi} \in \mathcal{C}^3(I, \mathbb{R}^2)$ *is monotonic increasing. Let the approximate derivative values* d_n, d_n^* *satisfy* $|\Phi_1^{(1)}(x_n) - d_n| \leq k_1 h^2$, $|\Phi_2^{(1)}(x_n) - d_n^*| \leq k_2 h^2$ *for all* $n \in \mathbb{N}_{N-1}$ *and for some constants* k_1, k_2, *where* $h = \max\{h_n : n \in \mathbb{N}_{N-1}\}$. *Further, let the closed triangle with vertices* $(0,0), (2,0), (0,2)$ *is contained in the subregion* ρ, *and the projection of* (η_n, ξ_n) *onto* ρ *satisfies* $\eta_n^\diamond + \xi_n^\diamond \geq 2$, *and the variables in the block matrix* $\mathbf{A} = [A_n]_{n \in \mathbb{N}_{N-1}}$ *are such that* $|\alpha_n| < a_n^3$, $|\beta_n| + |\gamma_n| < a_n^3$ *for all* $n \in \mathbb{N}_{N-1}$. *Then the associated monotone cubic spline hidden variable FIF* $\mathbf{f}[\mathbf{A}]$ *is a third order approximation to* $\mathbf{\Phi}$.

Proof. Let $\mathbf{\Phi}$ be the original function and $\mathbf{f}$ be a traditional cubic non-recursive approximant for $\mathbf{\Phi}$. We begin with the triangle inequality

$$\|\mathbf{\Phi} - \mathbf{f}[\mathbf{A}]\|_\infty \leq \|\mathbf{\Phi} - \mathbf{f}\|_\infty + \|\mathbf{f} - \mathbf{f}[\mathbf{A}]\|_\infty \tag{6.12}$$

We know [14] that Fritsch and Carlson algorithm is third order accurate under the given hypothesis. The first term in the above inequality:

$$\|\mathbf{\Phi} - \mathbf{f}\|_\infty = O(h^3). \tag{6.13}$$

Similarly, the second term in the right hand side in (6.12) can be bounded by using Theorem 4.1, Part III as follows:

$$\big\|\mathbf{f}[\mathbf{A}] - \mathbf{f}\big\|_\infty \leq \frac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1} \|\mathbf{f} - \mathbf{b}\|_\infty. \tag{6.14}$$

Now

$$\begin{aligned}
\|\mathbf{\Phi} - \mathbf{f}[\mathbf{A}]\|_\infty &\leq \|\mathbf{\Phi} - \mathbf{f}\|_\infty + \frac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1} \|\mathbf{f} - \mathbf{b}\|_\infty, \\
&\leq \|\mathbf{\Phi} - \mathbf{f}\|_\infty + \frac{\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1} (\|\mathbf{f}\|_\infty + \|\mathbf{b}\|_\infty), \\
&\leq \|\mathbf{\Phi} - \mathbf{f}\|_\infty + \frac{2\|\mathbf{A}\|_1}{1 - \|\mathbf{A}\|_1} \|\mathbf{f}\|_\infty.
\end{aligned}$$

Under the hypothesis, the associated monotone cubic spline hidden variable FIF $\mathbf{f}[\mathbf{A}]$ is third order approximation to $\mathbf{\Phi}$. This completes the proof. □

7 Numerical Illustration

In this section, we illustrate the monotonicity preserving $\mathcal{C}^1$-cubic spline hidden variable FIF scheme with some computationally generating examples. For this purpose, let us take the following subset of the Akima data: $D = \{(8,10), (9,10.5), (11,15), (12,50), (14,60), (15,85)\}$. We have written a simple computer program in `MatLab` for plotting the graphs of $\mathcal{C}^1$-cubic spline hidden variable FIFs. One inputs the data points, derivative values, hidden variables, and scaling parameters, whereupon points on the graph are recursively generated. Theoretically, to

obtain the actual fractal interpolant, one needs to continue the iterations indefinitely. However, in practice, computation is very fast (note that for a data set with N points exactly $(N-1)^{r+1}+1$ points with distinct x-coordinate are obtained at the r-th iteration) and a good view of the whole function is quickly obtained and may be printed with a graphics printer.
Extend the given monotonic data set to

$$\widehat{D}=\{(x_n,y_n,y_n^*)=(8,10,20),(9,10.5,23),(11,15,25),(12,50,30),(14,60,36),(15,85,40)\}.$$

Note that for the implementation of the $\mathcal{C}^1$-cubic spline hidden variable FIF scheme one requires in input the values of the derivatives at the knot points. Therefore, in the absence of other conditions/information, estimates of derivatives are necessary. The derivative values at data points are estimated using the arithmetic mean method: At the end points x_1 and x_N, set

$$d_1=\begin{cases}0 & \text{if } \Delta_1=0 \text{ or } sgn(d_{1_*})\neq sgn(\Delta_1),\\ d_{1_*}=\Delta_1+\frac{(\Delta_1-\Delta_2)h_1}{h_1+h_2}, & \text{otherwise}\end{cases}$$

$$d_N=\begin{cases}0 & \text{if } \Delta_{N-1}=0 \text{ or } sgn(d_{N_*})\neq sgn(\Delta_{N-1}),\\ d_{N_*}=\Delta_{N-1}+\frac{(\Delta_{N-1}-\Delta_{N-2})h_{N-1}}{h_{N-1}+h_{N-2}}, & \text{otherwise}\end{cases}$$

and at the interior points x_n; $n=2,\dots,N-1$, set

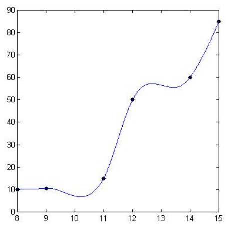
$$d_n=\begin{cases}0 & \text{if } \Delta_n=0 \text{ or } \Delta_{n-1}=0,\\ \frac{h_n\Delta_{n-1}+h_{n-1}\Delta_n}{h_n+h_{n-1}}, & \text{otherwise}\end{cases}$$

For the present data set, we have $d_1=0, d_2=1.0833, d_3=24.0833, d_4=25, d_5=18.3333, d_6=31.6667$ and $d_1^*=3.6667, d_2^*=2.3333, d_3^*=3.6667, d_4^*=4.3333, d_5^*=3.6667, d_6^*=4.3333$ calculated by above arithmetic mean method. Taking $\alpha_n=\beta_n=\gamma_n=0$ for all $n=1,2,\dots,5$, Fig. 3(a) shows the traditional piecewise cubic interpolant corresponding to this initial choice of derivative values. Note that this cubic interpolant is not monotone and the non-monotonic $\mathcal{C}^1$-cubic spline hidden variable FIF $f_1[\mathbf{A}]$ displayed in Fig. 3(a) is obtained by iterating the functional equations given in (6.11) via (6.9)-(6.10). To obtain a monotone cubic interpolant $f_1[\mathbf{A}]$, we apply the FC-algorithm (Section 2) with monotonicity region ρ_2 (the disc $\eta^2+\xi^2=3^2$). One procedure for modifying the derivative values in Step 2 (FC algorithm) is to construct the line joining the origin to the point (η,ξ). Let $(\eta^\diamond,\xi^\diamond)$ be the intersection of this line with the boundary of ρ_2. For ρ_2, $\eta^\diamond=\tau\eta, \xi^\diamond=\tau\xi$, where $\tau=3(\eta^2+\xi^2)^{-1/2}$. This procedure modifies the initial derivative values to $d_1=0, d_2=0.3033, d_3=6.7432, d_4=10.6065, d_5=7.7800, d_6=31.6667$ and $d_1^*=3.6667, d_2^*=1.6106, d_3^*=2.5310, d_4^*=4.3333, d_5^*=3.6667, d_6^*=4.3333$. Taking $\alpha_n,\beta_n,\gamma_n$ for all $n=1,2,\dots,5$ as in Fig. 3(a) and the derivative parameters as recommended by Fritsch and Carlson (see Step 2 of our algorithm). We obtain a monotonic self-referential $\mathcal{C}^1$-cubic spline in Fig. 3(b). Selecting $\alpha=(-0.2,0.001,0.01,0.01,0.01)$, $\beta=(0,0.002,0.01,0.03,0.01,0.01)$,

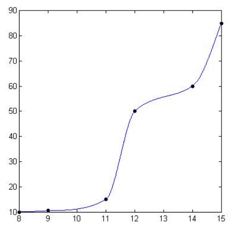
and $\gamma = (-0.4, 0.008, 0.011, 0.04, 0.017)$ arbitrarily and $\mathbf{b}$ as the two-point cubic Hermite interpolant corresponding to s and the derivative parameters as in Fig. 3(b), we obtain a non-monotonic non-self-referential cubic spline HFIF in Fig. 3(c). This illustrates the importance of the monotonicity preserving $\mathcal{C}^1$-cubic spline hidden variable FIF algorithm developed in the previous section. Next to obtain a monotone cubic HFIF we apply Steps 3 and 4 of our monotone cubic HFIF algorithm. Observe that the end derivatives $d_1 = 0, d_6 = 31.6667, d_1^* = 3.6667, d_6^* = 4.3333$ obtain through the FC-algorithm satisfy condition prescribed in Step 3, and hence there is no need of any modification, where we take ρ_2 to be the disc specified earlier. Taking $\alpha = (0, 0.001, 0.01, 0.01, 0.01)$, $\beta = (0, 0.002, 0.01, 0.03, 0.01, 0.01)$ and $\gamma = (0.007, 0.008, 0.011, 0.04, 0.017)$ arbitrarily from the range of permissible values given in Step 3 and the derivative parameters as in Fig. 3(b), the corresponding non-self-referential $\mathcal{C}^1$-cubic spline HFIF retains the monotonicity is plotted in Fig. 3(d).
The derivatives of the traditional monotone cubic interpolant s, and monotone $\mathcal{C}^1$-cubic spline HFIF $f_1[\mathbf{A}]$ are given in Fig. 3(e)-(f). Here the Fig. 3(b) and Fig. 3(d) look alike, but the derivative of both figures are different in nature. The function $s^{(1)}$ is smooth except possibly at the knots whereas $f_1^{(1)}[\mathbf{A}]$ shows irregularity in finite number of points or on dense subsets of the interpolation interval. Further, the irregularity can be quantified using the notion of fractal dimension. In CAGD and geometric modeling, in addition to have methods for monotone interpolation, it is desirable to have one or more parameters that can influence the shape of the interpolant and/or its derivative. So, the proposed scheme can be exploited to construct an interpolant satisfying chosen properties such as monotonicity and fractality in the derivative.

8 Concluding Remarks

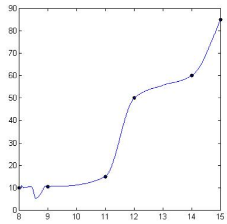
In the present work, we have applied hidden variable fractal interpolation as a tool to associate an entire class of $\mathbb{R}^2$-valued continuous fractal functions $\mathbf{f}[\mathbf{A}]$ with a prescribed continuous function $\mathbf{f}$. Suitable values of the parameters in the block matrix $\mathbf{A}$ are identified so that the fractal functions retain regularity and monotonicity of the germ function $\mathbf{f}$. We have derived estimate for the approximation of function $\mathbf{f}$ by their fractal analogue $\mathbf{f}[\mathbf{A}]$. As an application of the developed theory, we obtain monotonic $\mathcal{C}^1$-cubic spline hidden variable fractal interpolation functions corresponding to a prescribed set of monotonic data, thus initiating a new approach to shape preserving approximation via hidden variable fractal function. The monotonicity preserving interpolation scheme developed herein generalizes and enriches its traditional nonrecursive counterpart and its fractal extension. In practice, there are many instances where we desire a monotonic interpolant with its derivative receiving varying irregularity, and introduction of monotonicity of cubic HFIFs $\mathbf{f}[\mathbf{A}]$ accomplishes this. Thus, in conclusion, the hidden variable fractal methodology can be exploited in the field of shape preserving interpolation/approximation for providing more diverse and flexible shape preserving curves.



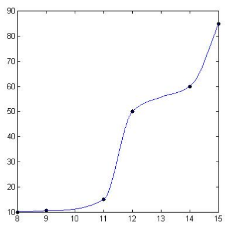
(a): Traditional cubic spline



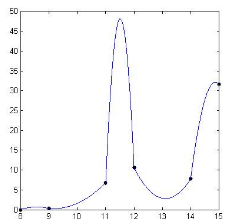
(b): Monotone self-referential cubic spline



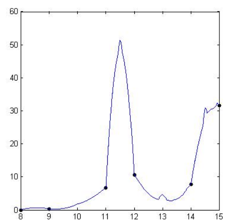
(c): Non monotone non-self-referential cubic spline HFIF



(d): Monotone non-self-referential cubic spline HFIF



(e): Derivative of monotone cubic spline in Fig. (b)



(f): Derivative of monotone cubic HFIF in Fig. (d)

Figure 3: Cubic spline hidden variable fractal interpolation functions (HFIFs) (the interpolating data points are given by the circles and the relevant hidden variable fractal interpolants by the solid lines).

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