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# On the connectedness of the branch locus of the moduli space of Riemann surfaces

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**Abstract** The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g$ has the structure of an orbifold and the set of singular points of such orbifold is the *branch locus* $\mathcal{B}_g$. In this article we present some results related with the topology of $\mathcal{B}_g$. We study the connectedness of $\mathcal{B}_g$ for $g \leq 8$, the existence of isolated equisymmetric strata in the branch loci and finally we stablish the connectedness of the branch locus of the moduli space of Riemann surfaces considered as Klein surfaces. We just sketch the proof of some of the results; complete proofs will be published elsewhere.

### Sobre la conexión del conjunto singular del espacio de moduli de las surperficies de Riemann

**Resumen.** El espacio de moduli $\mathcal{M}_g$ de las superficies de Riemann de género $g$ tiene estructura de orbifold y el conjunto de puntos singulares de tal orbifold es el conjunto singular $\mathcal{B}_g$. En este artículo presentamos algunos resultados acerca de la topología de $\mathcal{B}_g$. Concretamente se estudia la conexión de $\mathcal{B}_g$ para $g \leq 8$, la existencia de estratos equisimétricos aislados de ciertas dimensiones en $\mathcal{B}_g$ y finalmente se establece la conexión del conjunto singular del espacio de moduli de las superficies de Riemann consideradas como superficies de Klein.

## 1 Introduction

The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g$ being the quotient of the Teichmüller space by the discontinuous action of the mapping class group, has the structure of a complex orbifold, whose set of singular points is called the *branch locus* $\mathcal{B}_g$. In this article we present some new contributions in the understanding of the topology of the branch locus. More precisely we shall study the connectedness of $\mathcal{B}_g$, for low $g$, the existence of isolated equisymmetric strata of a given dimension in $\mathcal{B}_g$ and finally the connectedness of the branch locus of the moduli space of Riemann surfaces considered as Klein surfaces.

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In order to study Riemann surfaces our main tool will be the uniformization by Fuchsian groups. Given a Riemann surface $X$ of genus $g > 1$, we consider the universal covering $\mathcal{H} \xrightarrow{\pi_1(X)} X$, where $\mathcal{H}$ is the complex upperplane. Hence there is a representation $r\colon \pi_1(X) \to \mathrm{Isom}^+(\mathcal{H}) = \mathrm{PSL}(2,\mathbb{R})$ such that $X = \mathcal{H}/r(\pi_1(X))$ and $r(\pi_1(X))$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ (i.e. a Fuchsian group).

If there is $\gamma \in \mathrm{PSL}(2,\mathbb{R})$, such that $r_1(\pi_1(X)) = \gamma(r_2(\pi_1(X)))\gamma^{-1}$, clearly the Fuchsian groups $r_1(\pi_1(X))$ and $r_2(\pi_1(X))$ uniformize the same Riemann surface. The space:

$$\{\, r\colon \pi_1(X) \longrightarrow \mathrm{PSL}(2,\mathbb{R}) : \mathcal{H}/r(\pi_1(X)) \text{ is a genus } g \text{ surface} \,\}/\text{conjugation in } \mathrm{PSL}(2,\mathbb{R})$$

is the Teichmüller space $\mathbf{T}_g$. The Teichmüller space $\mathbf{T}_g$ has complex structure of dimension $3g-3$ and it is simply connected.

The group $\mathrm{Aut}^+(\pi_1(X))/\,\mathrm{Inn}(\pi_1(X)) = \mathrm{Mod}_g$ is the modular group or mapping class group, acting by composition on $\mathbf{T}_g$. Now we define the *moduli space* by $\mathcal{M}_g = \mathbf{T}_g/\,\mathrm{Mod}_g$.

The projection $\mathbf{T}_g \to \mathcal{M}_g = \mathbf{T}_g/\,\mathrm{Mod}_g$ is a regular branched covering with *branch locus* $\mathcal{B}_g$, in other words, $\mathcal{M}_g$ is an orbifold with singular locus $\mathcal{B}_g$. The branch locus $\mathcal{B}_g$ consists of the Riemann surfaces with symmetry, i.e. Riemann surfaces with non-trivial automorphism group (except when $g = 2$, where $\mathcal{B}_2$ consists of the surfaces with automorphisms different from the hyperelliptic involution and the identity). Our goal is the study of the topology of $\mathcal{B}_g$.

As an example, let us describe $\mathcal{B}_1$. Each elliptic surface is uniformized by a lattice $\{\, z_1, z_2 :$ with $\mathrm{Im}(z_1/z_2) < 0\,\}$, that can be normalized and parametrized by a complex number, the modulus of the basis $\{z_1, z_2\}$: $z_1/z_2 = \tau \in \{\, z \in \mathbb{C} : \mathrm{Im}\, z > 0\,\}$. Then the Teichmüller space is: $\mathbf{T}_1 = \mathcal{H} = \{\, \tau \in \mathbb{C} : \mathrm{Im}\, \tau > 0\,\}$.

The modular group $\mathrm{Mod}_1$ is $\mathrm{PSL}(2,\mathbb{Z})$ and the orbifold $\mathcal{M}_1$ is the Riemann sphere with a cusp and two conic points: $[i]$ with isotropy group of order 2 and $[\mathrm{e}^{2\pi i/3}]$ with isotropy group of order 3: $\mathcal{M}_1 = \widehat{\mathbb{C}}_{2,3,\infty}$. Then $\mathcal{B}_1 = \{[i], [\mathrm{e}^{2\pi i/3}]\}$.

It is known that $\mathcal{B}_2$ is not connected, since R. Kulkarni (see [11]) showed that the curve $w^2 = z^5 - 1$ is isolated in $\mathcal{B}_2$, i.e. this single surface is a connected component of $\mathcal{B}_2$. It is easy to prove that $\mathcal{B}_2$ has exactly two connected components (see Section 3). In Section 3 we present the following results: the branch loci $\mathcal{B}_i$, $i = 3, 4, 7$ are connected, $\mathcal{B}_5$, $\mathcal{B}_6$ are connected with the exception of one isolated point, $\mathcal{B}_8$ is connected with the exception of two isolated points.

There is a natural stratification of $\mathcal{B}_g$ by equisymmetric strata: $\mathcal{M}_g = \bigcup \overline{\mathcal{M}}^{G,a}$, see Section 2. In such stratification the isolated points of $\mathcal{B}_g$ are the isolated strata of smaller dimension. Given the results on $\mathcal{B}_g$, with $g \leq 8$, it is natural to ask if the only obstruction to the connectedness of the branch locus is the existence of isolated points. In Section 4 we give a negative answer to the above question studying the possible isolated one-dimension equisymmetric strata. We obtain that such strata exist in $\mathcal{B}_{p-1}$, with $p$ a prime $\geq 11$. Furthermore we show that for $g$ large enough we can find isolated equisymmetric strata of $\mathcal{B}_g$ of dimension as large as wanted.

We can find in [3] a geometrical interpretation of the existence of isolated points in $\mathcal{B}_g$ using the moduli space $\mathcal{M}_g^K$ of Riemann surfaces considered as Klein surfaces, i.e. the space of classes of Riemann surfaces of a given genus considering in the same class the surfaces that are conformal or anti-conformally equivalent. In such a context $\mathcal{M}_g^K$ has also an orbifold structure with branch loci $\mathcal{B}_g^K$ and there is a two fold covering $c\colon \mathcal{M}_g \to \mathcal{M}_g^K$. The isolated points in $\mathcal{B}_g$ are the preimage by $c$ of some intersection of strata of $\mathcal{B}_g^K$ (see [3]). In Section 5 we announce that $\mathcal{B}_g^K$ is connected for every $g$.

We just sketch the proof of some of the results; complete proofs will be published elsewhere.

## 2 Symmetric Riemann surfaces

Let $X$ be a Riemann surface and assume that $\mathrm{Aut}(X) \neq \{1\}$. Hence $X/\,\mathrm{Aut}(X)$ is an orbifold and there is a Fuchsian group $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$, such that:

$$\mathcal{H} \longrightarrow X = \mathcal{H}/\pi_1(X) \longrightarrow X/\,\mathrm{Aut}(X) = \mathcal{H}/\Gamma$$

The algebraic structure of $\Gamma$ is given by the signature $s(\Gamma) = (h; m_1, \dots, m_r)$, where $h$ is the genus of $\mathcal{H}/\Gamma$ and $m_1, \dots, m_r$ are the orders of the conic points of the orbifold $\mathcal{H}/\Gamma$.

For $\mathcal{G}$ an abstract group isomorphic to all the Fuchsian groups of signature $s = (h; m_1, \dots, m_r)$, the Teichmüller space of Fuchsian groups of signature $s$ is:

$$\{r\colon \mathcal{G} \longrightarrow \mathrm{PSL}(2,\mathbb{R}),\ \text{such that } s(r(\mathcal{G})) = s\}/\text{conjugation in } \mathrm{PSL}(2,\mathbb{R}) = \mathbf{T}_s.$$

The Teichmüller space $\mathbf{T}_s$ is a complex ball of dimension $2g - 3 + r$.

If $X/\operatorname{Aut}(X) = \mathcal{H}/\Gamma$ and $\operatorname{genus}(X) = g$, there is a natural inclusion $i\colon \mathbf{T}_s \subset \mathbf{T}_g$:

$$r\colon \mathcal{G} \longrightarrow \mathrm{PSL}(2,\mathbb{R}), \pi_1(X) \subset \mathcal{G}, r' = r\,|_{\pi_1(X)}\colon \pi_1(X) \longrightarrow \mathrm{PSL}(2,\mathbb{R}).$$

If we have $\pi_1(X) \lhd \mathcal{G}$, then there is a topological action of a finite group $G = \mathcal{G}/\pi_1(X)$ on surfaces of genus $g$. The inclusion $a\colon \pi_1(X) \to \mathcal{G}$ produces $i_a(\mathbf{T}_s) \subset \mathbf{T}_g$. The image of $i_a(\mathbf{T}_s)$ by $\mathbf{T}_g \to \mathcal{M}_g$ produces $\overline{\mathcal{M}}^{G,a}$, where $\mathcal{M}_g^{G,a}$ is the set of Riemann surfaces with automorphisms group containing a subgroup acting in a topologically equivalent way to the action of $G$ on $X$ given by the inclusion $a$.

Furthermore $\mathcal{M}_g = \bigcup \overline{\mathcal{M}}^{G,a}$ and $\mathcal{B}_g = \bigcup_{G\neq\{1\}} \overline{\mathcal{M}}^{G,a}$, such covers are called the equisymmetric stratifications [2].

Since each non-trivial group $G$ contains subgroups of prime order, we have the following remark that will be very useful in the sequel:

**Remark 1**

$$\mathcal{B}_g \subset \bigcup_{p \text{ prime}} \overline{\mathcal{M}}^{C_p,a}$$

*where $\overline{\mathcal{M}}^{C_p,a}$ is the set of Riemann surfaces of genus $g$ with an automorphism group containing $C_p$, the cyclic group of $p$ elements, acting on surfaces of genus $g$ in a fixed way given by $a$.*

# 3 The connectedness of $\mathcal{B}_g$ for $g \leq 8$

For $g = 2$, we have that: $\mathcal{B}_2 = \{\text{surfaces with } \operatorname{Aut}(X) \gneqq \{id, \text{hyperelliptic involution}\}\}$.

**Theorem 1** *$\mathcal{B}_2$ has two connected components.*

PROOF. It is easy to show that the prime order groups that can act on a surface $X$ of genus 2 are: $C_2$, $C_3$ and $C_5$. There are two possible actions of groups of order two: the action topologically equivalent to the hyperelliptic involution and the action with exactly two fixed points (giving as orbit space a surface of genus 1). For $C_3$ and $C_5$ there is only a topological type of actions. Hence we have:

$$\mathcal{B}_2 \subset \overline{\mathcal{M}}^{C_2} \cup \overline{\mathcal{M}}^{C_3} \cup \overline{\mathcal{M}}^{C_5}$$

To complete the proof of the theorem it is sufficient to show the existence of a surface in $\overline{\mathcal{M}}^{C_2} \cap \overline{\mathcal{M}}^{C_3}$. For that, we observe that we can construct a finite group of homeomorphisms acting on a topological surface of genus 2 having a homeomorphism of order three and an homeomorphism of order two with exactly two fixed points. Now, consider the unit sphere $S^2$ in $\mathbb{R}^3$ and in $S^2$ consider the graph $\delta$ consisting in three geodesic arcs from the north to the south pole, making on the poles three angles $2\pi/3$. Let $U_\varepsilon(\delta)$ be the set of points at distance $\leq \varepsilon$ of $\delta$ and $X = \partial U_\varepsilon(\delta)$. On $X$ acts a group isomorphic to $D_3$ which is the restriction of a group of rotations of $\mathbb{R}^3$. Such a group contains homeomorphisms of order three and homeomorphisms of order two with two fixed points. Finally, $\overline{\mathcal{M}}^{C_5}$ consists exactly of one surface: the isolated Kulkarni curve, with automorphism group $\mathbb{Z}_{10}$. ■

**Theorem 2** *$\mathcal{B}_3$ is connected.*

PROOF. The prime orders of cyclic actions on surfaces of genus three are $2$, $3$ and $7$. There are several actions of $C_2$ and $C_3$, each topological action being determined for the genus $h$ of the orbit surface of the action. For $C_2$ there are three actions where $h = 0$, $1$, $2$ and for order $3$ two actions with $h = 0$, $1$. For order $7$ there are two different actions, the two producing as quotient a sphere. Hence:

$$\mathcal{B}_3 \subset \bigcup_{h=0}^{2} \overline{\mathcal{M}}^{2,h} \bigcup_{h=0}^{1} \overline{\mathcal{M}}^{3,h} \bigcup_{i=1}^{2} \overline{\mathcal{M}}^{7,a_i}$$

Considering finite groups of rotations in the space $\mathbb{R}^3$ and surfaces embedded in $\mathbb{R}^3$ invariant by such rotation groups it is possible to show that $\overline{\mathcal{M}}^{2,0} \cap \overline{\mathcal{M}}^{2,1} \cap \overline{\mathcal{M}}^{2,2} \neq \varnothing$ and $\overline{\mathcal{M}}^{2,1} \cap \overline{\mathcal{M}}^{3,1} \neq \varnothing$.

A family of surfaces in $\left(\bigcup_{h=0}^{2} \overline{\mathcal{M}}^{2,h}\right) \cap \overline{\mathcal{M}}^{3,0}$ is uniformized by the kernel of $\theta : \Delta \to C_6 = \langle a : a^6 = 1 \rangle$, where $\Delta$ is a Fuchsian group with signature $(0; 2, 3, 3, 6)$ and $\theta(x_1) = a^3$, $\theta(x_1) = a^4$, $\theta(x_1) = a^4$, $\theta(x_1) = a$.

Finally there are exactly two surfaces of genus $3$ having automorphisms of order $7$. One is the Klein quartic $K$, where $\mathrm{Aut}(K) = \mathrm{PSL}(2, \mathbb{Z}_7)$, hence with order two and three automorphisms, and the other one is a surface with automorphism group $\mathbb{Z}_{14}$, hence admitting an involution. ■

**Theorem 3 ([6])** *$\mathcal{B}_4$ is connected.*

SKETCH OF THE PROOF. Note that the number of equisymmetric strata for genus $4$ is $41$ (see [8]).

The prime integers $p$ such that $C_p$ acts on a surface of genus $4$ are: $2$, $3$ and $5$. Let $X$ be a Riemann surface of genus $4$ such that $C_p \leq \mathrm{Aut}(X)$; we denote by $h$ the genus of $X/C_p$. There are three possible actions of order two classified by $h$ and giving the equisymmetric strata: $\overline{\mathcal{M}}^{C_2,h}$, where $h = 0$, $1$, $2$. Two actions of order three determined by the genus of the orbit space, producing the strata $\overline{\mathcal{M}}^{3,h}$, $h = 1$, $2$ and two classes of actions of order three with $h = 0$: $\overline{\mathcal{M}}^{3,0,i}$, $i = 1$, $2$. Finally there are three actions of order $5$ groups, all of them with $h = 0$, $\overline{\mathcal{M}}^{5,0,i}$, $i = 1$, $2$, $3$. Hence:

$$\mathcal{B}_4 \subset \bigcup_{h=0}^{2} \overline{\mathcal{M}}^{2,h} \bigcup_{i=1}^{2} \overline{\mathcal{M}}^{3,0,i} \bigcup_{h=1}^{2} \overline{\mathcal{M}}^{3,h} \bigcup_{i=1}^{3} \overline{\mathcal{M}}^{5,0,i}$$

Using the Singerman list of non-maximal signatures of Fuchsian groups it is possible to establish the following inclusions: $\overline{\mathcal{M}}^{3,2} \subset \overline{\mathcal{M}}^{2,1}$, $\overline{\mathcal{M}}^{5,0,2} \subset \overline{\mathcal{M}}^{2,2}$, $\overline{\mathcal{M}}^{5,0,3} \subset \overline{\mathcal{M}}^{2,2}$. Thus

$$\mathcal{B}_4 \subset \overline{\mathcal{M}}^{2,0} \cup \overline{\mathcal{M}}^{2,1} \cup \overline{\mathcal{M}}^{2,2} \cup \overline{\mathcal{M}}^{3,0,1} \cup \overline{\mathcal{M}}^{3,0,2} \cup \overline{\mathcal{M}}^{3,1} \cup \overline{\mathcal{M}}^{5,0,1}$$

We denote by $\mathcal{F}^G$ a family of Riemann surfaces with group of automorphisms isomorphic to $G$. Now the connectedness is a consequence of the following facts:

1. $\overline{\mathcal{M}}^{5,0,1} \subset \overline{\mathcal{M}}^{2,0} \cap \overline{\mathcal{M}}^{2,2}$
2. The existence of $\mathcal{F}^{C_6 \times C_2}$ such that $\mathcal{F}^{C_6 \times C_2} \subset \overline{\mathcal{M}}^{2,1} \cap \overline{\mathcal{M}}^{2,2} \cap \overline{\mathcal{M}}^{3,0,2}$.
3. The existence of $\mathcal{F}^{D_6}$ such that $\mathcal{F}^{D_6} \subset \overline{\mathcal{M}}^{2,2} \cap \overline{\mathcal{M}}^{3,0,1}$.
4. The existence of $\mathcal{F}^{D_3 \times C_3}$ such that $\mathcal{F}^{D_3 \times C_3} \subset \overline{\mathcal{M}}^{3,0,2} \cap \overline{\mathcal{M}}^{3,1}$.
5. The existence of $\mathcal{F}^{D_3 \times D_3}$ such that $\mathcal{F}^{D_3 \times D_3} \subset \overline{\mathcal{M}}^{3,0,1} \cap \overline{\mathcal{M}}^{3,2}$. ■

Recently, further results were obtained:

**Theorem 4 ([1])** *$\mathcal{B}_5$, $\mathcal{B}_6$ are connected with the exception of one isolated point, $\mathcal{B}_7$ is connected and $\mathcal{B}_8$ is connected with the exception of two isolated points.*

One of the main results in [1] is that the set $\bigcup_{G=C_2,C_3} \overline{\mathcal{M}}^{G,a} \subset \mathcal{B}_g$ is connected for every $g$.

# 4 Isolated strata of dimension $> 0$

The isolated strata of the equisymmetric stratification are defined as follows: $\mathcal{M}_g^H$ is an isolated stratum if and only if $\overline{\mathcal{M}}_g^{H,a_H} \cap \overline{\mathcal{M}}_g^{G,a_G} = \varnothing$, for $(G, a_G) \neq (H, a_H)$. Note that for $\overline{\mathcal{M}}_g^{H,a_H}$ to be an isolated stratum, the group $H$ must be isomorphic to $C_p$ with $p$ a prime. The isolated strata of smaller dimension, i.e. the isolated points have been studied by R. Kulkarni in 1991. Kulkarni showed that isolated points appear in $\mathcal{B}_g$ when $2g+1$ is an odd prime distinct from 7 ([11]).

Given the results in Section 3 is natural to ask if $\mathcal{B}_g$ is connected with the exception of isolated points. But the answer is shown to be negative by constructing isolated strata of dimension 1. We present the following complete result:

**Theorem 5 ([7])** *The branch locus $\mathcal{B}_g$ of the moduli space of Riemann surfaces of genus $g$ contains isolated connected components of (complex) dimension $1$ if and only if $g = p-1$, with $p$ a prime $\geq 11$.*

Sketch of the proof. Let $s$ be the signature $(0; p, p, p, p)$, note that $\dim \mathbf{T}_s = 1$. Let $\Delta$ be a group with signature $s$ and $\theta\colon \Delta \to C_p = \langle \gamma : \gamma^p = 1 \rangle$ defined by $\theta(x_1) = \gamma, \theta(x_2) = \gamma^i,\ \theta(x_3) = \gamma^j, \theta(x_4) = \gamma^{p-1-i-j}$, where $i, j$ are integers such that $\#\{\pm 1 \bmod p, \pm i \bmod p, \pm j \bmod p, \pm(i+j+1) \bmod p\} = 8$. Then $\ker\theta$ is a surface group of genus $p-1$, i.e. isomorphic to the fundamental group of a surface of genus $p-1$. The inclusion $i_\theta\colon \ker\theta \to \Delta$ produces $i_\theta(\mathbf{T}_s) \subset \mathbf{T}_g$ and the isolated one-dimensional strata $\mathcal{M}^\theta$ are the image of $i_\theta(\mathbf{T}_s)$ by $\mathbf{T}_g \to \mathcal{M}_g$. By the way of construction of $\mathcal{M}^\theta$ implies that these are the only possible one dimensional isolated strata.

The following result shows that there exists isolated strata of large dimension.

**Theorem 6** *Let $p$ be a prime and $d$ be an integer such that $p > (d+2)^2$, then there are isolated equisymmetric strata of dimension $d$ in $\mathcal{M}_{(d+1)(p-1)/2}$.*

Proof. Let $g = (d+1)(p-1)/2$. Let $s$ be the signature $(0; p, \overset{d+3}{\ldots}, p)$. Let $\Delta$ be a group with signature $s$ and $\theta\colon \Delta \to C_p = \langle \gamma : \gamma^p = 1 \rangle$ defined by $\theta(x_i) = \gamma^i$, $i = 1, \ldots, d+2$ and $\theta(x_{d+3}) = \gamma^{p-(d+2)(d+3)/2}$. Then $\ker\theta$ is a surface group of genus $g$. The inclusion $i_\theta\colon \ker\theta \to \Delta$ produces $i_\theta(\mathbf{T}_s) \subset \mathbf{T}_g$, and we assert that the image of $i_\theta(\mathbf{T}_s)$ by $\mathbf{T}_g \to \mathcal{M}_g$, $\mathcal{M}^{C_p,\theta}$ is isolated. In fact, if $\mathcal{M}^{C_p,\theta}$ is not isolated, then there is a surface $X$ in $\mathcal{M}^{C_p,\theta}$ admitting an automorphism group $G \supsetneqq C_p$. Since, [9], $C_p$ is normal in $G$, there is an action of $G/C_p$ on $X/C_p$ producing a finite automorphism $\alpha$ of $\pi_1 O(X/C_p)$ such that $\theta \circ \alpha = \beta \circ \theta$, where $\beta$ is an automorphism of $C_p$, i.e. $\theta \circ \alpha(x) = (\theta(x))^j$, where $j \in \{1, \ldots, p-1\}$. The way of construction of $\theta$ and the condition that $p > (d+2)^2$ imply that such an automorphism does not exist. ■

# 5 On the connectedness of the branch locus of the moduli space of Riemann surfaces considered as Klein surfaces.

Let $X$ be a surface of genus $g > 1$ and $r_i\colon \pi_1(X) \to \mathrm{Isom}^+(\mathcal{H}) = \mathrm{PSL}(2,\mathbb{R})$, $i = 1, 2$ be two representations, with $r_i(\pi_1(X))$ discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathcal{H}/r_i(\pi_1(X))$ homeomorphic to $X$. If there is $\gamma \in \mathrm{Isom}^\pm(\mathcal{H})$ such that $r_1(\pi_1(X)) = \gamma(r_2(\pi_1(X)))\gamma^{-1}$: then the Fuchsian groups $r_1(\pi_1(X))$ and $r_2(\pi_1(X))$ uniformize equivalent Klein surfaces. The space of classes of representations $r\colon \pi_1(X) \to \mathrm{PSL}(2,\mathbb{R})$, such that $\mathcal{H}/r(\pi_1(X)) \simeq X$, by conjugation in $\mathrm{Isom}^\pm(\mathcal{H})$, is the Teichmüller space $\mathbf{T}_g^K$.

The group $\mathrm{Aut}^\pm(\pi_1(X))/\mathrm{Inn}(\pi_1(X)) = \mathrm{Mod}_g^\pm$ acts by composition on $\mathbf{T}_g^k$. We define the Moduli space of Riemann surfaces considered as Klein surfaces to be the quotient $\mathcal{M}_g^K = \mathbf{T}_g^K / \mathrm{Mod}_g^\pm$.

The projection $\mathbf{T}_g^K \to \mathcal{M}_g^K$ is a regular branched covering with *branch locus* $\mathcal{B}_g^K$, that is, $\mathcal{M}_g^K$ is an orbifold with singular locus $\mathcal{B}_g^K$. Note that there is a two fold branched covering $\mathcal{M}_g \xrightarrow{2:1} \mathcal{M}_g^K$.

**Theorem 7** *$\mathcal{B}_g^K$ is connected for every $g \geq 2$.*

SKETCH OF THE PROOF. The branch locus admits a cover $\mathcal{B}_g^K = \bigcup \overline{\mathcal{M}}^{C_p,a}$. In each $\overline{\mathcal{M}}^{C_p,a}$ there are Riemann surfaces $X$ with $\mathrm{Aut}^{\pm}(X) \cong D_p$ and such that $X/\mathrm{Aut}^{\pm}(X)$ is a surface with boundary, then the surfaces $X$ have reflections. Now the theorem follows from the fact that the locus of Riemann surfaces with reflections (real locus) is connected. ([4], [5], [12]). ■

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