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P-spaces and an unconditional closed graph theorem

Marek Wójtowicz and Waldemar Sieg

Abstract Let X be a completely regular (Tychonoff) space, and let $C(X)$, $U(X)$, and $B_1(X)$ denote the sets of all real-valued functions on X that are continuous, have a closed graph, and of the first Baire class, respectively.

We prove that $U(X) = C(X)$ if and only if X is a P-space (i.e., every G_δ-subset of X is open) if and only if $B_1(X) = U(X)$. This extends a list of equivalences obtained earlier by Gillman and Henriksen, Onuchic, and Iséki. The first equivalence can be regarded as an *unconditional* closed graph theorem; it implies that if X is perfectly normal or first countable (e.g., metrizable), or locally compact, then there exist *discontinuous* functions on X with a closed graph. This complements earlier results by Doboš and Baggs on discontinuity of closed graph functions.

P-espacios y un teorema incondicional de gráfica cerrada

Resumen. Sea X un espacio completamente regular (Tychonoff). Por $C(X)$, $U(X)$ y $B_1(X)$ se denotan los conjuntos de funciones reales definidas en X que son continuas, que tienen gráfica cerrada y que son de primera clase de Baire, respectivamente. Se prueba que $U(X) = C(X)$ si y sólo si X es un P espacio (es decir que cada subconjunot G_δ de X es abierto) o si y sólo si $B_1(X) = U(X)$. Estos resultados extienden una relación de equivalencias obtenidas por Gillman y Henriksen, Onuchic e Iséki. La primera equivalencia es un teorema incondicional de gráfica cerrada; implica que si X es perfectamente normal o cumple el primer axioma de numerabilidad (por ejemplo si es metrizable), o es localmente compacto, entonces existen funciones discontinuas en X con gráfica cerrada. Así se complementan resultados obtenidos por Dobos y Bags sobre discontinuidad de funciones con gráfica cerrada.

1 Introduction

Throughout this paper X, Y denote Hausdorff spaces, $C(X)$, $U(X)$, and $B_1(X)$ have the same meanings as in the Abstract, and $\mathbb{R}$ denotes the set of real numbers endowed with the natural topology. If f is a function on X, then $D(f)$ denotes the set of discontinuous points of f; hence $C(f) := X \setminus D(f)$ is the set of continuity points of f.

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The study of *discontinuous* closed graph functions was initiated in 1974 by Ivan Baggs [1]. He showed that if $X = \mathbb{R}$ then, for every closed and nowhere dense subset F of X, there is a function $f \in U(X)$ such that $D(f) = F$. In 1985 Doboš [7, Theorems 7 and C] generalized Baggs's result to two cases: for X a perfectly normal space and F of first category in X, and for X a metric Baire (e.g., complete) space and F closed and nowhere dense in X. In 1989 Baggs considered a similar problem under a weaker assumption on X: he showed in [3, Theorem 4.3] that if X is completely regular then, given F a compact G_δ and of first category subset of X, the equation $D(f) = F$ has a solution $f \in U(X)$. The above-cited results led obviously to the natural question:

> *Whether there exists a space X such that every closed graph function on X is continuous, i.e., if $U(X) = C(X)$?*

In 1989 Baggs [2, Example 4.2] showed this question has a positive answer: he has constructed a regular (yet non-completely regular) space X on which every real-valued function with a closed graph is constant; hence

$$U(X) = C(X) = B_1(X). \tag{1}$$

It is worth to add that Baggs [2, Example 5.2] has also given an example of a regular and non-completely regular space Y such that $C(Y) = B_1(Y)$, yet $U(Y) \neq C(Y)$.

The purpose of this paper is to give a full characterization of equations (1) for X a completely regular space (Theorem 1 below). In particular, the characterization implies that the phenomenon described in the latter example by Baggs does not hold whenever X is merely completely regular. We recall [12, pp. 62–63] that such a space X is said to be a P-space if every G_δ-subset [F_σ-subset] of X is open [closed]; equivalently, every co-zero subset of X is closed (the examples of such spaces are included in the monograph by Gillman and Jerison [12, Examples 4JKL, 4N], and in the papers [8, 18, 23, 26, 28]; P-spaces illustrate also the essentiality of the notion of *Dedekind σ-completeness* of $C(X)$ [20, Theorem 43.8]).

Our main result reads as follows (its proof is given in Section 3).

Theorem 1 *Let X be a completely regular space. Then the following four conditions are equivalent:*

(i) $U(X) = C(X)$*;*

(ii) $B_1(X) = C(X)$*;*

(iii) $B_1(X) = U(X)$*;*

(iv) *X is a P-space.*

Theorem 1 extends the lists of equivalent conditions for X to be a P-space obtained by Gillman and Henriksen [11, Theorem 5.3] (cf. [12, 4J]), and Tucker [30, Theorem 5].

The above equivalence (i) $\iff$ (iv) may be regarded as an *unconditional* closed graph theorem (CGT) (here a *conditional CGT* is understood as: *a function f is continuous on X iff f has a closed graph + an extra condition on f*). Conditional CGTs (in a larger setting, where the targeted space $\mathbb{R}$ is replaced by a topological space Y) were examined by a number of authors. The most known such a result is the linear Banach's CGT (see [4, Chapter 6] for a survey of its generalizations, cf. [34]), and its versions for topological groups were studied by Grant [13], Husain [14], and Wilhelm [33]. Purely topological conditional CGTs were obtianed by Fuller [10], Piotrowski and Szymański [27], Wilhelm [32], and Moors [22].

The equivalence (ii) $\iff$ (iv) in Theorem 1 is a characterization of those $C(X)$-spaces, for X completely regular, for which the Baire order is zero (cf. [21, p. 418 and Corollary 4.1]). This equivalence was obtained independently in 1957 by Onuchic [25], and in 1958 by Iséki [15]; it is also included implicitly in Theorem 4 by Ohno [24], and appears (without any reference) on p. 562 of Tucker's paper [30].

Remark 1 *The equivalence* (i) $\iff$ (ii) *and theorems on extensions of Baire-one functions* [16, 19, 29] *suggest we can deduce the existence of discontonuous closed graph functions on X from the relation*

$B_1(A) \neq C(A)$ *for a proper subset* A *of* X *(see, e.g.,* [21, pp. 431-441]*). However, this method (i.e., an appeal to extension theorems) of verification of the relation* $U(X) \neq C(X)$ *seems to be improper, because from the equivalence* (i) $\iff$ (iv) *of Theorem* 1 *and property* [12, 4K(4)] *(that every subspace of a* P*-space is a* P*-space again) it follows it is enough only to check if the subspace* A *is not a* P*-space (and in a few typical cases even if* A *is non-discrete: see Corollaries* 1 *and* 2 *below).*

Since every singleton of a perfectly normal or first countable space X is a G_δ-set, every such a P-space is discrete. Moreover, every locally compact P-space is discrete too (this follows easily from [12, 4K(3)]). Hence Theorem 1 has the following immediate consequence.

Corollary 1 *Let* X *be a perfectly normal or first countable space (e.g., metrizable), or a locally compact space. Then the following four conditions are equivalent:*

(i) $U(X) \neq C(X)$;

(ii) $B_1(X) \neq C(X)$;

(iii) $B_1(X) \neq U(X)$;

(iv) X *is non-discrete.*

The next corollary follows from the fact that every P-space is basically disconnected [12, 4K(7)] (the latter means that every co-zero subset of X has an open closure).

Corollary 2 *Let* X *be a completely regular space. If, additionally,* X *is connected, then there exist discontinuous closed graph functions on* X.

The last corollary deals with two spaces of functions of the first Baire class. A function $f\colon X \to \mathbf{R}$ is said to be

- *piecewise continuous* if there is a sequence (X_n) of closed subsets of X such that $X = \bigcup_{n=1}^{\infty} X_n$ and the restrictions $F_{|X_n}$ are continuous for all n's;
- *Baire-one-star function* if for every closed subset F of X the set $C(f_{|F})$ has nonempty interior (in the induced topology on F).

The linear spaces of piecewise continuous and Baire-one-star functions on X are denoted by $\mathcal{P}(X)$ and $B_1^*(X)$, respectively. These two spaces were studied by Borsik [5], Borsik, Doboš and Repický [6], and Kirchheim [17], among others.

It is known that $U(X) \subset \mathcal{P}(X)$ for X arbitrary [5, p. 119], that $\mathcal{P}(X) = U(X) + U(X)$ for X is perfectly normal [5, Theorem 1], and that $\mathcal{P}(X) = B_1^*(X)$ for X a complete metric space [17, Theorem 2.3]. It is easy to check that if X is a P-space, then every piecewise continuous function f on X is continuous (this follows from the equality $f^{-1}(F) = \bigcup_{n=1}^{\infty} (f_{|X_n})^{-1}(F)$, where $X = \bigcup_{n=1}^{\infty} X_n$). Moreover, $U(X) \subset \mathcal{P}(X)$ (because every $f \in U(X)$ is continuous on the closed set $X_n := f^{-1}[-n, n]$). Hence, by Theorem 1 (i), we obtain yet another characterization of P-spaces:

Corollary 3 *Let* X *be a completely regular space. Then* X *is a* P*-space if and only if* $\mathcal{P}(X) = C(X)$*. In particular, if* X *is a complete metric space, then* $B_1^*(X) \neq C(X)$ *if and only if* X *is non-discrete.*

Remark 2 *Looking at Theorem* 1 *we can conjecture that the equality* $\mathcal{P}(X) = B_1(X)$ *should imply* X *to be a* P*-space, but this is not the case. Indeed, let* X *denote the set of all rational numbers endowed with the natural (metric) topology. Since* X *is a metric space, from the Tietze theorem we get* $\mathcal{P}(X) \subset B_1(X)$*. It is also easy to check that* $\mathcal{P}(X) = \mathbb{R}^X$*, whence* $\mathcal{P}(X) = B_1(X)$*. On the other hand, by* [12, 4K(1)] *(that* every countable P-space is discrete*),* X *is not a* P*-space.*

2 Notations

For the basic facts concerning topology and continuous functions we refer the reader to the monographs [9, 12]. We recall that a subset A of X is said to be a *zero-set* if there is a continuous function $f\colon X \to [0,1]$ such that $A = [f = 0] := f^{-1}(0)$; then the set $[f > 0] := X \setminus A$ is called *co-zero*. The symbol $\mathrm{Fr}(A)$ denotes the frontier of A, i.e., $\mathrm{Fr}(A) := \overline{A} \cap \overline{X \setminus A} = \overline{A} \setminus \mathrm{Int}(A)$, where $\overline{A}$ is the closure of A, and $\mathrm{Int}(A)$ denotes the interior of A.

3 The proof of Theorem 1

The proof of Theorem 1 is based on a few properties of continuous and closed graph functions collected in Lemmas 1 and 2 below.

We start with a result generalizing the constructions of Doboš [7] and Baggs [3] of discontinuous closed graph functions. The reader should note that in our Lemma 1 the (nonempty) zero-set $[f = 0]$ is arbitrary, while the above-mentioned authors assume it to be nowhere dense [7, Theorem 5], or compact [3, Theorem 3.2, Theorem 4.3].

Lemma 1 *Let $f\colon X \to [0,1]$ be a continuous function, and let the set $[f = 0]$ be not empty. Consider the function*

$$f^*(x) = \begin{cases} \dfrac{1}{f(x)} & \text{if } x \in [f > 0], \\ 0 & \text{if } x \in [f = 0]. \end{cases}$$

Then f^* has a closed graph, and $D(f^*) = \mathrm{Fr}([f = 0])$.

PROOF. A proof that $f^* \in U(X)$ is actually due to Doboš [7, Proof of Theorem 5]; Baggs [1, Proof of Theorem 4.3] applies elementary arguments: define $\phi\colon \mathbb{R} \to \mathbb{R}$ as $\phi(0) = 0$ and $\phi(x) = 1/x$ otherwise, notice that $\phi \in U(\mathbb{R})$, and that the composition $\phi \circ f$ equals f^*, whence $f^* \in U(X)$.

Now we prove that $D(f^*) = \mathrm{Fr}([f = 0])$; equivalently,

$$C(f^*) = [f > 0] \cup \mathrm{Int}([f = 0]). \tag{2}$$

Since the both sets on the right side of (2) are open and f^* is continuous on each of them, the inclusion $\supseteq$ in (2) is obvious.

On the other hand,

$$x_0 \notin [f > 0] \cup \mathrm{Int}([f = 0])$$

iff $x_0 \in \mathrm{Fr}([f = 0])$; hence

$$f^*(x_0) = 0 = f(x_0). \tag{3}$$

Let us fix a basis $\mathcal{V}(x_0)$ of neighbourhoods of such an x_0. Since

$$x_0 \in \overline{[f > 0]} \supset \mathrm{Fr}([f = 0]),$$

for every $V \in \mathcal{V}(x_0)$ there is $y_V \in V$ such that

$$f(y_V) > 0. \tag{4}$$

From (4) we obtain

$$f^*(y_V) = 1/f(y_V) \geq 1.$$

Hence, by (3) and (4), for every $V \in \mathcal{V}(x_0)$ there is $y \in V$ such that

$$|f^*(y) - f^*(x_0)| \geq 1.$$

Therefore $x_0 \in D(f^*)$, i.e., $x_0 \notin C(f^*)$. We thus have proved the inclusion $\subseteq$ in (2) is also true. The proof of Lemma 1 is complete. ∎

In the proof of Theorem 1 we shall apply the following three properties.

Lemma 2 *Let X be a Hausdorff space.*

(a) *If X is a P-space and $\xi \in X$, then every function $f \in C(X)$ is constant on a neighborhood of ξ* [12, 4J(2)].

(b) *Let $f \in U(X)$. Then (see* [31, p. 196 - Facts (iii) and (iv)]*)*

(*) for every closed interval $[a, b]$ the set $f^{-1}[a, b]$ is closed;

(**) if f is bounded, it is continuous.

PROOF OF THEOREM 1.

Non-(iv) implies non-(i). Since X is not a P-space, there is a continuous function f on X such that the zero set $[f = 0]$ is non-open. The latter implies that the set $\mathrm{Fr}([f = 0])$ is not empty. By Lemma 1, the function f^* has a closed graph and $D(f^*) \neq \emptyset$, i.e., f^* is discontinuous. Hence $U(X) \neq C(X)$.

Both (ii) and (iii) implies (iv). Set $U = [f > 0]$, where f is an arbitrary fixed element of $C(X)$. It is known (see [16, Proof of Proposition 2]) that the characteristic function χ_U of U belongs to $B_1(X)$. In case (ii) we obtain that U is closed, whence (as f were arbitrary) X is a P-space. In case (iii), by Lemma 2(b)(**), χ_U is continuous, and so, as previously, X is a P-space.

(iv) implies (ii). This is a simple consequence of Lemma 2(a): the pointwise limit f of a sequence of functions $(f_n) \subset C(X)$, each constant on a neighborhood V_n of $\xi \in X$, is constant on their intersection V, which is open by the definition of a P-space. Hence f is continuous at ξ.

(iv) implies (i). Let $f \in U(X)$. We claim that, for every open interval (a, b), the set $f^{-1}(a, b)$ is G_δ, hence open by the definition of a P-space. By Lemma 2(b)(*), every set $A_n = f^{-1}[b, b + n]$, $n = 1, 2, \ldots$ is closed, whence the set $f^{-1}[b, \infty) = \bigcup_{n=1}^{\infty} A_n$ is F_σ. Similarly, the set $f^{-1}(-\infty, a]$ is F_σ too. Consequently, $f^{-1}(a, b)$ is G_δ, as claimed.

(iv) implies (iii). Since, as we have already showed, (iv) implies both (i) and (ii), we obtain that condition (iv) implies further that $U(X) = C(X) = B_1(X)$.

The proof of Theorem 1 is complete. ■

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Marek Wójtowicz
Instytut Matematyki,
Uniwersytet Kazimierza Wielkiego,
Pl. Weyssenhoffa 11,
85-072 Bydgoszcz,
Poland
mwojt@ukw.edu.pl

Waldemar Sieg
Instytut Matematyki,
Uniwersytet Kazimierza Wielkiego,
Pl. Weyssenhoffa 11,
85-072 Bydgoszcz,
Poland
waldeks@ukw.edu.pl