

# Nonparametric estimation of the expected accumulated reward for semi-Markov chains

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## Abstract

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In this paper a nonparametric estimator of the expected value of a discounted semi-Markov reward chain is proposed. Its asymptotic properties are established and as a consequence of the asymptotic normality the confidence sets are obtained. An application in quality of life modelling is described.

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## 1 Introduction

Homogeneous semi-Markov chains (HSMC) have been recognized as a flexible and efficient tool in the modelling of stochastic systems. Recent results and applications are retrievable in Barbu, Boussemart and Limnios (2004) and Janssen and Manca (2007).

The idea to link rewards to the occupancy of a semi-Markov state led to the construction of semi-Markov reward processes. These processes have been analyzed and applied by many authors; see Howard (1971), De Dominicis and Manca (1986), Limnios and Oprişan (2001), Khorshidian and Soltani (2002), Janssen and Manca (2006), Stenberg, Manca and Silvestrov (2006, 2007) and Janssen and Manca (2007).

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The inferential problems related to reward processes are seldom considered. Gardiner, Luo, Bradley, Sirbu and Given (2006) considered an estimator of the expected accumulated reward for non-homogeneous Markov reward processes with deterministic reward functions. D'Amico (2009) proposed Markov reward processes, with stochastic rate and impulse rewards, to study accumulated measure of the quality of life. In that paper the asymptotic properties of the nonparametric estimator of the higher order moments of the reward process have been established.

In this paper we face the nonparametric inference problems related to a discrete time semi-Markov reward process. We define an estimator of the expected accumulated reward and we prove that it is uniformly strongly consistent, and if properly centralized and normalized that it converges in distribution to a normal random variable. The goal is achieved developing the techniques of estimation for HSMC presented in Barbu and Linnios (2006).

The paper is divided in this way: first, the semi-Markov reward model is briefly depicted and the definition of the functional to which we are interested is given. Next, the asymptotic properties of the nonparametric estimator of the expected accumulated reward process are assessed. Finally, the practical usefulness of the results is shown by exposing a possible application to measure the quality of life.

## 2 The semi-Markov reward model

Homogeneous semi-Markov chains are a generalization of discrete time Markov chains allowing the times between transitions to occur at random times distributed according to any kind of distribution function which may depend on the current and the next state.

Let us consider a finite set of states  $E = \{1, 2, \dots, S\}$  in which the system can be into and a complete probability space  $(\Omega, F, P)$  on which we define the following random variables:

$$X_n : \Omega \rightarrow E, T_n : \Omega \rightarrow \mathbb{N}. \quad (2.1)$$

They denote the state occupied at the  $n$ -th transition and the time of the  $n$ -th transition respectively.

Suppose that the process  $(X_n, T_n)_{n \in \mathbb{N}}$  is a discrete time homogeneous Markov renewal process of kernel  $\mathbf{q} = (q_{ij}(t))$ ; see Barbu *et al.* (2004). Elements of the kernel represent the following probabilities

$$q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n = t | X_n = i]. \quad (2.2)$$

From these quantities it is possible to define

$$Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i] = \sum_{\tau=1}^t q_{ij}(\tau), \quad (2.3)$$

the probability to join, with next transition, state  $j$  within time  $t$  given the starting, at time zero, from the state  $i$ .

The process  $\{X_n\}$  is a Markov chain with state space  $E$  and transition probability matrix  $\mathbf{P} = \mathbf{Q}(\infty)$ . We shall refer to it as the embedded Markov chain.

The unconditional waiting time distribution function in state  $i$  is

$$H_i(t) = P[T_{n+1} - T_n \leq t | X_n = i] = \sum_{j \in E} Q_{ij}(t). \quad (2.4)$$

Now it is possible to define the conditional cumulative distribution functions of the waiting time in each state, given the state subsequently occupied:

$$G_{ij}(t) = P\{T_{n+1} - T_n \leq t | X_n = i, X_{n+1} = j\} = \frac{1}{p_{ij}} \sum_{s=1}^t q_{ij}(s) \cdot 1_{\{p_{ij} \neq 0\}} + 1_{\{p_{ij} = 0\}} \quad (2.5)$$

Define  $\{N(t)\}$  by  $N(t) = \sup\{n : T_n \leq t\} \forall t \in \mathbb{N}$ . The discrete time process  $Z = (Z(t), t \in \mathbb{N})$  defined by  $Z(t) = X_{N(t)}$  is a semi-Markov process of kernel  $\mathbf{q}$ . It represents, for each waiting time, the state occupied by the process  $X_n$ .

We define,  $\forall i, j \in E$ , and  $t \in \mathbb{N}$ , the semi-Markov transition probabilities:

$$\phi_{ij}(t) = P[X_{N(t)} = j | X_0 = i]. \quad (2.6)$$

They are obtained by solving the system of equations:

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_{k \in E} \sum_{\tau=1}^t q_{ik}(\tau) \phi_{kj}(t - \tau). \quad (2.7)$$

Algorithms to solve equations (2.7) are well known, see for example Janssen and Manca (2007).

To introduce a reward structure, we consider the score function  $g : E \rightarrow \mathbb{R}$ . This function assigns a reward (score)  $g(j)$  when the process visits state  $j \in E$ . Define  $\{Y(t)\}$  by  $Y(t) = \sum_{s=1}^t d^s g(Z(s))$ . It represents the discounted accumulated semi-Markov reward process. The quantity  $d \in [0, 1]$  is a discount factor introduced to compare present scores with future scores. The process  $Y(t)$  is of interest, for example, in insurance mathematics see e.g. Stenberg *et al.* (2006, 2007) as well as in quality of life measurement – see D'Amico (2009) in which  $Z(t)$  is considered to be a finite and ergodic Markov chain and  $g$  a stochastic score function.

The expected value of  $Y(t)$  is of interest to synthesize the process behaviour. Let us denote  $M_i(t) = E[Y(t)|X_0 = i]$ . Then it results that

$$\begin{aligned} M_i(t) &= E\left[\sum_{s=1}^t d^s g(Z(s))|X_0 = i\right] \\ &= \sum_{s=1}^t d^s E[g(Z(s))|X_0 = i] = \sum_{s=1}^t d^s \sum_{j \in E} g(j) \phi_{ij}(s) \\ &= \sum_{s=1}^t \sum_{j \in E} d^s g(j) \phi_{ij}(s). \end{aligned} \quad (2.8)$$

$M_i(t)$  represents the functional we wish to estimate.

### 3 The estimation of the expected accumulated reward

Let us suppose now that we have a right-censored history of the HSMC until the observation time  $L$ :

$$H(L) = \{X_0, T_1, X_1, T_2, X_2, \dots, T_{N(L)}, X_{N(L)}, u_L\} \quad (3.1)$$

where  $N(L) = \max\{n \in \mathbb{N} | T_n \leq L\}$  and  $u_L = L - T_{N(L)}$ .

Following the line of research in Barbu and Limnios (2006, 2008), to estimate the semi-Markov kernel, we use the empirical estimator:

$$\hat{q}_{ij}(k, L) = \frac{\sum_{n=1}^{N(L)} 1_{\{X_{n-1}=i, X_n=j, T_n-T_{n-1}=k\}}}{\sum_{n=1}^{N(L)} 1_{\{X_n=i\}}} \quad (3.2)$$

To estimate the functional (2.8) we propose the estimator

$$\hat{M}_i(t; L) = \sum_{s=1}^t \sum_{j \in E} d^s g(j) \hat{\phi}_{ij}(s; L). \quad (3.3)$$

Estimator  $\hat{\phi}_{ij}(s; L)$  is the  $(i, j)$ -th element of the transition probability matrix  $\hat{\Phi}$  which satisfies the matrix equation  $\hat{\Phi}(t) = \mathbf{I} - \hat{\mathbf{H}}(t) + \hat{\mathbf{q}} * \hat{\Phi}(t)$ , where  $*$  denotes the matrix convolution product – see Barbu and Limnios (2006) for more details.

The following asymptotic property holds true:

**Proposition 3.1** For all  $i \in E$  and  $\theta \in \mathbb{N}$  the estimator  $\hat{M}_i(\theta, L)$  is uniformly strongly consistent, that is

$$\max_{i \in E} \max_{0 \leq \theta \leq L} |\hat{M}_i(\theta, L) - M_i(\theta)| \xrightarrow{a.s.} 0 \text{ as } L \rightarrow \infty. \quad (3.4)$$

*Proof.* We use the following inequalities:

$$\begin{aligned} \max_{i \in E} \max_{0 \leq \theta \leq L} |\hat{M}_i(\theta, L) - M_i(\theta)| &= \max_{i \in E} \max_{0 \leq \theta \leq L} \left| \sum_{s=1}^{\theta} \sum_{j \in E} d^s g(j) (\hat{\phi}_{ij}(s, L) - \phi_{ij}(s)) \right| \\ &\leq \max_{i \in E} \max_{0 \leq \theta \leq L} \sum_{s=1}^{\theta} \sum_{j \in E} d^s g(j) |\hat{\phi}_{ij}(s, L) - \phi_{ij}(s)| \\ &\leq \sum_{s=1}^{\theta} \sum_{j \in E} d^s g(j) \max_{i \in E} \max_{0 \leq \theta \leq L} |\hat{\phi}_{ij}(s, L) - \phi_{ij}(s)| \end{aligned} \quad (3.5)$$

and this last quantity goes to zero almost surely as a consequence of the uniform strongly consistency of the estimators  $\hat{\phi}_{ij}(s, L)$  given in Barbu and Limnios (2006). ■

To prove the asymptotic normality of estimator  $\hat{M}_i(t; L)$  we need to introduce the following variables:

$$q_{ij}^{(n)}(t) = P[X_n = j, T_n = t | X_0 = i], \quad (3.6)$$

$$\psi_{ij}(t) \doteq \sum_{n=0}^t q_{ij}^{(n)}(t), \quad (3.7)$$

$$\Psi_{ij}(t) \doteq E_i[N_j(t)] = \sum_{n=0}^t Q_{ij}^{(n)}(t). \quad (3.8)$$

Finally, with  $\mu_{ii}$  and  $\mu_{ii}^*$  we shall denote the mean recurrence time of state  $i$  for the Markov renewal process  $(X_n, T_n)_{n \in \mathbb{N}}$  and the mean recurrence time of state  $i$  for the embedded Markov chain  $(X_n)_{n \in \mathbb{N}}$ , respectively.

The following theorem describes the asymptotic normality of the estimator  $\hat{M}_i(t; L)$ .

**Theorem 3.2** For any fixed time  $h \in \mathbb{N}$  and state  $i \in E$ , it results that

$$\sqrt{L}(\hat{M}_i(h, L) - M_i(h)) \xrightarrow{d} N(0, \sigma_{M_i}^2(h)) \text{ as } L \rightarrow \infty \quad (3.9)$$

where

$$\sigma_{M_i}^2(h) = \frac{\mu_{ii}^*}{\mu_{ii}} \sum_{m \in E} \frac{\mu_{mm}^2}{\mu_{mm}^*} \left\{ \sum_{r \in E, s=1}^h d^{2s} [C_{imr} - g(m)\Psi_{im}]^2 * q_{mr}(s) - \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 \right\} \quad (3.10)$$

and

$$C_{imr} = \sum_{j \in E} g(j)[1 - H_j] * \psi_{im} * \psi_{rj} \quad (3.11)$$

$$D_{im}(s) = \sum_{l \in E} (C_{ml} * q_{ml}(s) - g(m)\psi_{im} * Q_{ml}(s)) \quad (3.12)$$

*Proof.*

$$\sqrt{L}(\hat{M}_i(h, L) - M_i(h)) = \sqrt{L} \left( \sum_{s=1}^h \sum_{j \in E} d^s g(j) (\hat{\phi}_{ij}(s, L) - \phi_{ij}(s)) \right) \quad (3.13)$$

In Barbu and Limnios (2006) it was proved that  $\sqrt{L}(\hat{\phi}_{ij}(k, L) - \phi_{ij}(k))$  has the same asymptotic behaviour as

$$\sqrt{L} \left\{ \sum_{n=1}^m \sum_{u=1}^m [(1 - H_j) * \psi_{in} * \psi_{uj} * \Delta q_{nu}](k) - \sum_{u=1}^m \psi_{ij} * \Delta Q_{ju}(k) \right\}, \quad (3.14)$$

where  $\Delta q_{ij}(k) \doteq \hat{q}_{ij}(k, L) - q_{ij}(k)$  and  $\Delta Q_{ij}(k) \doteq \hat{Q}_{ij}(k, L) - Q_{ij}(k)$ .

Applying this result to our functional we obtain that  $\sqrt{L}(\hat{M}_i(h, L) - M_i(h))$  has the same asymptotic distribution as

$$\begin{aligned} & \sqrt{L} \sum_{s=1}^h \sum_{v \in E} \sum_{l \in E} \left[ \sum_{j \in E} d^s (g(j)(1 - H_j) * \psi_{iv} * \psi_{lj}) * \Delta q_{vl} \right](s) \\ & - \sqrt{L} \sum_{s=1}^h \sum_{l \in E} \sum_{j \in E} d^s g(j) \psi_{ij} * \Delta Q_{jl}(\theta) \end{aligned} \quad (3.15)$$

Let us denote  $C_{ivl} = \sum_{j \in E} g(j)[1 - H_j] * \psi_{iv} * \psi_{lj}$ , then by substitution of the kernel estimator (3.2) in formula (3.15) we get

$$\begin{aligned} & = \frac{1}{\sqrt{L}} \sum_{n=1}^{N(L)} \left( \sum_{v \in E} \sum_{l \in E} \frac{L}{N_n(L)} \left\{ \sum_{s=1}^h [d^s C_{vl} * (1_{\{X_{n-1}=v, X_n=l, T_n-T_{n-1}=\cdot\}} \right. \right. \\ & - q_{vl}(\cdot) 1_{\{X_{n-1}=v\}})(s) - d^s g(v) \psi_{iv} * (1_{\{X_{n-1}=v, X_n=l, T_n-T_{n-1} \leq \cdot\}} \\ & \left. \left. - Q_{vl}(\cdot) 1_{\{X_{n-1}=v\}})(s) \right\} \right) = \frac{1}{\sqrt{L}} \sum_{n=1}^{N(L)} f(X_{n-1}, X_n, T_n - T_{n-1}) \end{aligned} \quad (3.16)$$

where the function  $f : E \times E \times \mathbb{N} \longrightarrow \mathbb{R}$  is defined as follows:

$$\begin{aligned}
f(m, r, z) &\equiv \sum_{v \in E} \sum_{l \in E} \frac{L}{N_n(L)} \left\{ \sum_{s=1}^h [d^s C_{ivl} * (1_{\{X_{n-1}=v, X_n=l, T_n-T_{n-1}=\cdot\}} - q_{vl}(\cdot) \right. \\
&\quad \cdot 1_{\{X_{n-1}=v\}})(s) - d^s g(v) \psi_{iv} * (1_{\{X_{n-1}=v, X_n=l, T_n-T_{n-1} \leq \cdot\}} - Q_{vl}(\cdot) 1_{\{X_{n-1}=v\}})(s)] \left. \right\} \\
&= \frac{L}{N_n(L)} \left\{ \sum_{s=1}^h [d^s C_{imr} * (1_{\{z=\cdot\}})(s) - \sum_{l \in E} d^s C_{ml} * q_{ml}(s)] \right. \\
&\quad \left. - d^s g(m) \psi_{im} * 1_{\{z \leq \cdot\}}(s) \sum_{l=1}^s d^s g(m) \psi_{im} * Q_{ml}(s)] \right\} \tag{3.17} \\
&= \frac{L}{N_n(L)} \left\{ \sum_{s=1}^h d^s [C_{imr} * 1_{\{z=\cdot\}} - g(m) \psi_{im} * 1_{\{z \leq \cdot\}}](s) \right. \\
&\quad \left. - \sum_{l \in E} (C_{iml} * q_{ml}(s) - g(m) \psi_{im} * Q_{ml}(s)) \right\}
\end{aligned}$$

Pyke and Schaufele (1964) provide a central limit theorem for expressions of the type (3.16). Then, its application, as suggested by Barbu and Limnios (2006) for reliability indicators, will give us the asymptotic variance of  $\hat{M}_i(t)$ . The application of this theorem requires the computation of several quantities marked below in bold.

Let

$$\begin{aligned}
A_{imr} &\doteq \sum_{z=1}^{\infty} f(m, r, z) q_{mr}(z) \\
&= \left( \frac{L}{N_n(L)} \right) \left\{ \sum_{s=1}^h d^s [C_{imr} * q_{mr}(s) - g(m) \psi_{im} * Q_{mr}(s)] \right. \\
&\quad \left. - \sum_{l \in E} (C_{iml} * q_{ml}(s) - g(m) \psi_{im} * Q_{ml}(s)) p_{mr} \right\} \tag{3.18}
\end{aligned}$$

consequently we have

$$A_{im} \doteq \sum_{r \in E} A_{imr} = 0 \tag{3.19}$$

Let

$$\begin{aligned}
B_{imr} &\doteq \sum_{z=1}^{\infty} f^2(m, r, z) q_{mr}(z) \\
&= \sum_{z=1}^{\infty} \left( \frac{L}{N_m(L)} \right)^2 \left\{ \sum_{s=1}^h d^s [C_{imr} * 1_{\{z=\cdot\}}(s) - g(m) \psi_{im} * 1_{\{z \leq \cdot\}}(s)] \right. \\
&\quad \left. - \sum_{l \in E} (C_{iml} * q_{ml}(s) - g(m) \psi_{im} * Q_{ml}(s)) \right\}^2 q_{mr}(z) \tag{3.20}
\end{aligned}$$

Denoting  $D_{im}(s) = \sum_{l \in E} [C_{iml} * q_{ml}(s) - g(m)\psi_{im} * Q_{ml}(s)]$  and developing the square we get

$$\begin{aligned} B_{imr} &= \left(\frac{L}{N_m(L)}\right)^2 \left\{ \sum_{z=1}^{\infty} \left[ \left( \sum_{s=1}^h d^s [C_{imr} * 1_{\{z=\cdot\}}(s) - g(m)\psi_{im} * 1_{\{z\leq\cdot\}}(s)] \right)^2 \right. \right. \\ &+ \left. \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 - 2 \left( \sum_{a=1}^h d^a D_{im}(a) \right) \left( \sum_{s=1}^h d^s [C_{imr} * 1_{\{z=\cdot\}}(s) \right. \right. \\ &\left. \left. - g(m)\psi_{im} * 1_{\{z\leq\cdot\}}(s)] \right) \right] q_{mr}(z) \left. \right\} \end{aligned} \quad (3.21)$$

Then

$$\begin{aligned} B_{imr} &= \left(\frac{L}{N_m(L)}\right)^2 \left\{ \sum_{s=1}^h d^s [C_{imr}^2 * q_{mr}(s) + g(m)\Psi_{im}^2 * q_{mr}(s) \right. \\ &- 2g(m)C_{imr}\Psi_{im} * q_{mr}(s)] + \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 p_{mr} \\ &\left. - 2 \left( \sum_{a=1}^h d^a D_{im}(a) \right) \left( \sum_{s=1}^h d^s [C_{imr} * q_{mr}(s) - g(m)\psi_{im} * Q_{ml}(s)] \right) \right\} \\ &= \left(\frac{L}{N_m(L)}\right)^2 \left( \sum_{s=1}^h d^{2s} [C_{imr} - g(m)\Psi_{im}]^2 * q_{mr}(s) + \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 p_{mr} \right. \\ &\left. - 2 \left( \sum_{a=1}^h d^a D_{im}(a) \right) \left( \sum_{s=1}^h d^s [C_{imr} * q_{mr}(s) - g(m)\psi_{im} * Q_{ml}(s)] \right) \right) \end{aligned} \quad (3.23)$$

Now let us compute

$$\begin{aligned} B_{im} &\doteq \sum_{r \in E} B_{imr} = \left(\frac{L}{N_m(L)}\right)^2 \left\{ \left( \sum_{r \in E} \sum_{s=1}^h d^{2s} [C_{imr} - g(m)\Psi_{im}]^2 * q_{mr}(s) \right) \right. \\ &+ \left. \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 - 2 \left( \sum_{a=1}^h d^a D_{im}(a) \right) \left( \sum_{s=1}^h d^s D_{im}(s) \right) \right\} \\ &= \left(\frac{L}{N_m(L)}\right)^2 \left\{ \left( \sum_{r \in E} \sum_{s=1}^h d^{2s} [C_{imr} - g(m)\Psi_{im}]^2 * q_{mr}(s) \right) - \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 \right\} \end{aligned} \quad (3.24)$$

Since  $A_{ij} = 0$ ,  $m_i \doteq \sum_{j=1}^s A_{ij} \frac{\mu_{ii}^*}{\mu_{jj}^*} = 0$  and then  $m_f \doteq \frac{m_i}{\mu_{ii}} = 0$ . Consequently in the Pyke-Schaufele's central limit theorem

$$\begin{aligned}
\sigma_i^2 &\doteq \sum_{m \in E} B_{im} \frac{\mu_{ii}^*}{\mu_{mm}^*} \\
&= \mu_{ii}^* \sum_{m \in E} \left( \frac{L}{N_m(L)} \right)^2 \frac{1}{\mu_{mm}^*} \left\{ \left( \sum_{r \in E} \sum_{s=1}^h d^{2s} [C_{imr} - g(m)\Psi_{im}]^2 * q_{mr}(s) \right) \right. \\
&\quad \left. - \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 \right\} \quad (3.25)
\end{aligned}$$

Moreover  $B_f \doteq \frac{\sigma_i^2}{\mu_{ii}^*}$  and since  $\frac{N_m(L)}{L} \xrightarrow{a.s.} \frac{1}{\mu_{mm}^*}$  as  $L \rightarrow \infty$ , then we get

$$\begin{aligned}
\sigma_{(M)}^2(h) &= B_f \\
&= \frac{\mu_{ii}^*}{\mu_{ii}^*} \sum_{m \in E} \frac{\mu_{mm}^2}{\mu_{mm}^*} \left\{ \sum_{r \in E} \sum_{s=1}^h d^{2s} [C_{imr} - g(m)\Psi_{im}]^2 * q_{mr}(s) - \left( \sum_{s=1}^h d^s D_{im}(s) \right)^2 \right\} \quad (3.26)
\end{aligned}$$

■

Note that at this time it is an easy task to construct the confidence intervals for this estimate, in fact at first we have to estimate the variance  $\sigma_M^2(h)$  by replacing in expression (3.26) each element with its corresponding estimator then, since  $\hat{\sigma}_M^2(h)$  is a consistent estimator, the resulting confidence interval for  $M_i(t)$  still has asymptotic level  $100(1 - \alpha)\%$  and is given by:

$$\hat{M}_i(t) - z_{\frac{\alpha}{2}} \times \frac{\hat{\sigma}_M(t)}{\sqrt{L}} \leq M_i(t) \leq \hat{M}_i(t) + z_{\frac{\alpha}{2}} \times \frac{\hat{\sigma}_M(t)}{\sqrt{L}} \quad (3.27)$$

#### 4 Application in quality of life estimation

The results can be applied in order to solve many real life problems which require semi-Markov processes, such as disability insurance models, see D'Amico, Guillén and Manca (2009). Here, we discuss a possible application in the modelling and estimation of the quality of life evolution of a person.

One of the most recent approaches in the quality of life modelling and estimation is to assume that the observed quality of life of a person is, at any time, a discrete variable which can be assessed through a self-rated questionnaire or by an interviewer, see Limnios, Mesbah and Sadek (2004).

The use of Markov chains to describe the longitudinal process of the quality of life of a person has been suggested by Chen and Sen (2001, 2004), Limnios *et al.* (2004)

and more recently by D'Amico (2009). In particular in D'Amico (2009), Markov reward processes have been proposed to study accumulated measure of the quality of life of a person.

As already stated, semi-Markov chains have sojourn time distributions (2.5) of any type, this is why they are more appropriate to applications than the Markov chains. For this reason we suppose that, at any time, the quality of life of a person is described by a discrete variable (state) and that its evolution in time is described by a HSMC.

Following D'Amico (2009), we give the following definition:

**Definition 4.1** *The accumulated quality of life index at time  $t$  is*

$$AIQL_i(t) = Y_i(t) \quad (4.1)$$

where  $Y_i(t)$  is the discounted accumulated semi-Markov reward process given that  $X_0 = i$ .

The functional (2.8) represents the expected value of the accumulated quality of life index and is an important indicator for comparing different quality of life policies.

To illustrate the results obtained in the previous section we adopt a simulation strategy. In general we do not know the true form of the semi-Markov kernel which should be estimated via historical data. Unfortunately we are not in possession of real data, so we assume that data are generated by the unknown kernel  $\tilde{Q}$  identified by the following embedded Markov chain:

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0.70 & 0.30 & 0.00 \\ 0.50 & 0.00 & 0.50 \\ 0.00 & 0.35 & 0.65 \end{pmatrix} \end{matrix}$$

and the following conditional waiting time distribution functions:

$$\begin{aligned} G_{1,1}(\cdot) &= cdf(\text{Lognormal})(4, 2); G_{1,2}(\cdot) = cdf(\text{Lognormal})(2, 1) \\ G_{2,1}(\cdot) &= cdf(\text{Exponential})(3); G_{2,3}(\cdot) = cdf(\text{Exponential})(5.8) \\ G_{3,2}(\cdot) &= cdf(\text{Lognormal})(4, 0.5); G_{3,3}(\cdot) = cdf(\text{Exponential})(2) \end{aligned}$$

Thus, when the process is in state  $i = 1$ , the next state is sampled from the probability distribution (0.70, 0.30, 0.00). If, for example, the state  $j = 2$  is selected then a waiting time in state  $i = 1$  has to be sampled from the distribution  $G_{1,1}(\cdot)$  which is a Lognormal with parameters (4, 2). At this time a new state is sampled from the distribution (0.50, 0.00, 0.50) and so on. We construct a trajectory of length  $L = 2000$  of the semi-Markov process generated by the assumed kernel  $\tilde{Q}$ . From this trajectory we estimate the quantity of interest by using the proposed estimators.

In Figure 1 we show the estimation of the transition probabilities (dashed lines) with starting state  $i = 1$  and we compare them with those obtained assuming as true the kernel  $\tilde{Q}$  (continuous lines). The lower right hand plot shows the estimation of the expected value of the accumulated quality of life (dashed line) and the true value calculated by using kernel  $\tilde{Q}$  (continuous line) assumed to generate the data.

Finally, notice that it could be possible and interesting to construct estimators of the higher order moments of a semi-Markov chain with rewards. To this end, we shall estimate the renewal type equations established by Stenberg *et al.* (2006, 2007) since no explicit expression, as simple and manageable as formula (2.8), exists for higher order moments.

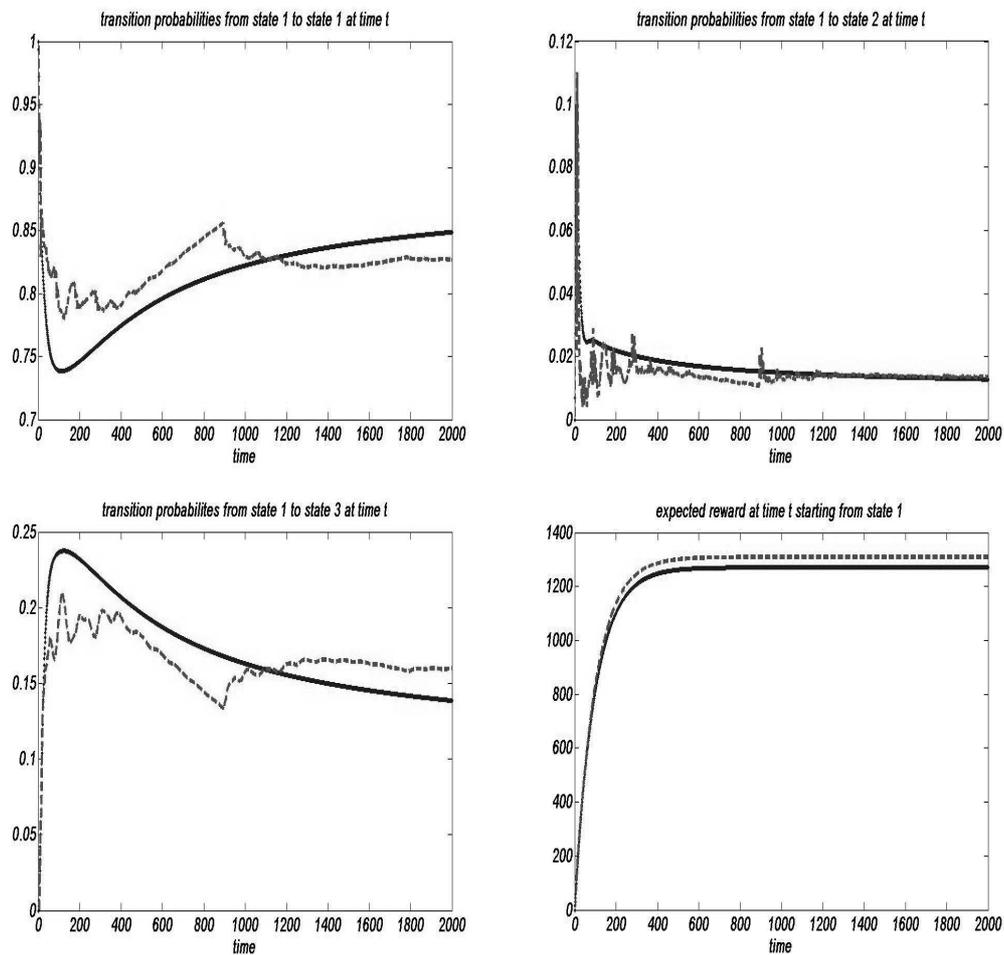


Figure 1: Comparison between true and estimated values.

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