PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 12, December 2009, Pages 4007–4017
S 0002-9939(09)09994-8
Article electronically published on July 30, 2009

JORDAN GRADINGS ON EXCEPTIONAL SIMPLE LIE ALGEBRAS

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(Communicated by Gail R. Letzter)

ABSTRACT. Models of all the gradings on the exceptional simple Lie algebras induced by Jordan subgroups of their groups of automorphisms are provided.

1. Introduction

Given a simple Lie algebra $\mathfrak{g}$ and a complex Lie group G with $\mathrm{Int}(\mathfrak{g}) \leq G \leq \mathrm{Aut}(\mathfrak{g})$ (here $\mathrm{Int}(\mathfrak{g})$ denotes the group of inner automorphisms and $\mathrm{Aut}(\mathfrak{g})$ the group of all the automorphisms of $\mathfrak{g}$), an abelian subgroup A of G is a *Jordan subgroup* if [Ale74]:

(i) its normalizer $N_G(A)$ is finite,

(ii) A is a minimal normal subgroup of its normalizer, and

(iii) its normalizer is maximal among the normalizers of those abelian subgroups satisfying (i) and (ii).

The Jordan subgroups are shown in [Ale74] to be elementary (that is, isomorphic to $\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ for some prime number p), and they induce gradings, called *Jordan gradings*, on the Lie algebra $\mathfrak{g}$.

The classification of Jordan subgroups is given in [Ale74] in two tables. Table 1 deals with the classical Lie algebras. Detailed models of the corresponding Jordan gradings are given in [OV91, Chapter 3, §3.12]. On the other hand, Table 2 in [Ale74] gives the classification of the Jordan subgroups for the exceptional Lie algebras (see also [OV91, Chapter 3, §3.13]). Table 1 below summarizes some properties of these Jordan subgroups and of the corresponding Jordan gradings. In all of them, the zero homogeneous subspace is trivial.

It turns out that the Jordan gradings on the exceptional simple complex Lie algebras, with the exception of a $\mathbb{Z}_5^3$-grading on E_8, look like the gradings recently obtained (see [Eld08, Remark 5.30]) from gradings on an octonion algebra and from gradings on a different type of (nonunital) composition algebra: the Okubo algebras.

Received by the editors October 15, 2008, and, in revised form, April 5, 2009.
2000 *Mathematics Subject Classification.* Primary 17B25.
Key words and phrases. Jordan grading, simple, exceptional, Lie algebra, orthogonal decomposition.
The author was supported by the Spanish Ministerio de Educación y Ciencia and FEDER (MTM 2007-67884-C04-02) and by the Diputación General de Aragón (Grupo de Investigación de Álgebra).

$\mathfrak{g}$	A	$\dim \mathfrak{g}_\alpha$ $(\alpha \neq 0)$
G_2	$\mathbb{Z}_2^3$	2
F_4	$\mathbb{Z}_3^3$	2
E_8	$\mathbb{Z}_5^3$	2
D_4	$\mathbb{Z}_2^3$	4
E_8	$\mathbb{Z}_2^5$	8
E_6	$\mathbb{Z}_3^3$	3

TABLE 1. The exceptional Jordan gradings

The first purpose of this paper is to check that those gradings induced by octonion and Okubo algebras are indeed Jordan gradings.

However, checking that a given subgroup of the automorphism group of a simple Lie algebra is a Jordan subgroup is not an easy task, so a different approach will be followed which consists of proving the next result, which is of independent interest:

Main Theorem. *Let $\mathbb{F}$ be an algebraically closed ground field of characteristic 0. Then, up to equivalence:*

(i) *There is a unique $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type G_2 over $\mathbb{F}$ such that* $\dim \mathfrak{g}_\alpha = 2$ *for any* $0 \neq \alpha \in \mathbb{Z}_2^3$.

(ii) *There is a unique $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type F_4 over $\mathbb{F}$ such that* $\dim \mathfrak{g}_\alpha = 2$ *for any* $0 \neq \alpha \in \mathbb{Z}_3^3$.

(iii) *There is a unique $\mathbb{Z}_5^3$-grading on the simple Lie algebra of type E_8 over $\mathbb{F}$ such that* $\dim \mathfrak{g}_\alpha = 2$ *for any* $0 \neq \alpha \in \mathbb{Z}_5^3$.

(iv) *There is a unique $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type D_4 over $\mathbb{F}$ such that* $\dim \mathfrak{g}_\alpha = 4$ *for any* $0 \neq \alpha \in \mathbb{Z}_2^3$.

(v) *There is a unique $\mathbb{Z}_2^5$-grading on the simple Lie algebra of type E_8 over $\mathbb{F}$ such that* $\dim \mathfrak{g}_\alpha = 8$ *for any* $0 \neq \alpha \in \mathbb{Z}_2^5$.

(vi) *There is a unique $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type E_6 over $\mathbb{F}$ such that* $\dim \mathfrak{g}_\alpha = 3$ *for any* $0 \neq \alpha \in \mathbb{Z}_3^3$.

Recall that two gradings $\mathfrak{g} = \bigoplus_{g\in G} \mathfrak{g}_g$ and $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ are said to be *equivalent* if there is an automorphism φ of $\mathfrak{g}$ such that for any $g \in G$ with $\mathfrak{g}_g \neq 0$, there is a $\gamma \in \Gamma$ with $\varphi(\mathfrak{g}_g) = \mathfrak{g}_\gamma$.

As mentioned above, in the recent paper [Eld08], some natural gradings on either octonion algebras or Okubo algebras over fields of characteristic $\neq 2, 3$ have been used to construct some nice gradings on the exceptional simple Lie algebras. Okubo algebras constitute a class of eight dimensional nonunital composition algebras. They are then endowed with a nondegenerate quadratic multiplicative form n (so that $n(x * y) = n(x)n(y)$ for any x, y), and this form is such that the associated polar form $n(x, y) = n(x + y) - n(x) - n(y)$ is associative: $n(x * y, z) = n(x, y * z)$ for any x, y, z. These algebras were introduced by S. Okubo [Oku78] and have some remarkable features (see for instance [KMRT98, Chapter 8]). Actually, over fields of characteristic $\neq 2, 3$, Okubo algebras are precisely the forms of the so-called *pseudo-octonion algebra*, which is the algebra defined over the subspace of traceless 3×3 matrices over a field containing the cubic roots of 1 with multiplication and norm given by

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\operatorname{trace}(xy)1, \qquad n(x) = -\frac{1}{2}\operatorname{trace}(x^2)$$

for any x, y, where $\omega \neq 1 = \omega^3$. It is an interesting fact that in this way a nonassociative composition algebra appears inside the associative algebra of 3×3 matrices.

More precisely, the following gradings on exceptional simple Lie algebras were obtained in [Eld08]:

(1) A $\mathbb{Z}_2^3$-grading on any octonion algebra $\mathbb{O}$ induces a $\mathbb{Z}_2^3$-grading on the simple Lie algebras $\mathfrak{g}$ of derivations of $\mathbb{O}$ (of type G_2) and also a $\mathbb{Z}_2^3$-grading on the orthogonal simple Lie algebra $\hat{\mathfrak{g}}$ of the skew-symmetric maps relative to the norm of $\mathbb{O}$ (of type D_4), with $\mathfrak{g}_0 = 0 = \hat{\mathfrak{g}}_0$ and such that $\mathfrak{g}_\alpha$ (respectively $\hat{\mathfrak{g}}_\alpha$) is a Cartan subalgebra of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) for any $0 \neq \alpha \in \mathbb{Z}_2^3$. (See [Eld08, Subsection 5.2], and note that in many respects D_4 is exceptional.)

(2) A $\mathbb{Z}_3^2$-grading on any Okubo algebra $\mathcal{O}$ induces a $\mathbb{Z}_3^3$-grading on some attached simple Lie algebras $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ of types F_4 and E_6, with $\mathfrak{g}_0 = 0 = \hat{\mathfrak{g}}_0$ and such that $\mathfrak{g}_\alpha$ (respectively $\hat{\mathfrak{g}}_\alpha$) is a two dimensional subalgebra of $\mathfrak{g}$ (respectively a three dimensional subalgebra of $\hat{\mathfrak{g}}$) with $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ (respectively $\hat{\mathfrak{g}}_\alpha \oplus \hat{\mathfrak{g}}_{-\alpha}$) being a Cartan subalgebra of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) for any $0 \neq \alpha \in \mathbb{Z}_3^3$. (See [Eld08, Subsection 5.3].)

(3) A $\mathbb{Z}_2^3$-grading on each of two octonion algebras induces a $\mathbb{Z}_2^5$-grading on some attached simple Lie algebra $\mathfrak{g}$ of type E_8, with $\mathfrak{g}_0 = 0$ and such that $\mathfrak{g}_\alpha$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_2^5$. (See [Eld08, Subsection 5.4].)

An immediate corollary of the Main Theorem is that indeed the gradings above obtained from gradings on octonion or Okubo algebras are Jordan gradings.

Actually, parts (i), (iv) and (v) of the Main Theorem follow from results by Hesselink. In fact, by [Hes82, Proposition 3.6], any grading of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with the properties of the gradings in the Main Theorem satisfies the property that for any $0 \neq \alpha$ in the grading group (which is $\mathbb{Z}_p^r$ for $p = 2$, 3 or 5 and $r = 3$ or 5) the subspace

$$\mathfrak{g}_{[\alpha]} = \bigoplus_{i=0}^{p-1} \mathfrak{g}_{i\alpha} \tag{1.1}$$

is always a Cartan subalgebra of $\mathfrak{g}$. Now, by [Hes82, Theorem 6.2], these gradings are unique (up to equivalence) in cases (i), (iv) and (v) of the Main Theorem, where $\mathfrak{g}_\alpha$ is a Cartan subalgebra for any $\alpha \neq 0$. For E_8 this was proved earlier in [Tho76], where it was shown that there is a unique (up to conjugation by automorphisms) *Dempwolff decomposition* of E_8.

Moreover, the gradings in parts (i), (iv) and (v) are all obtained from the natural $\mathbb{Z}_2^3$-grading on the algebra of octonions $\mathbb{O}$ (see [Eld98] and [Eld08]). The $\mathbb{Z}_2^3$-gradings on $G_2 = \mathfrak{Der}\,\mathbb{O}$ and on $D_4 = \mathfrak{so}(\mathbb{O})$ are just the gradings induced from the one in $\mathbb{O}$, while the $\mathbb{Z}_2^5$-grading on E_8 is obtained from the model of E_8 as a direct sum of two copies of the triality Lie algebra of the octonions (which is isomorphic to

$\mathfrak{so}(\mathbb{O})$) and three copies of the tensor product of two copies of the octonions:

$$E_8 = \big(\mathfrak{tri}(\mathbb{O}) \oplus \mathfrak{tri}(\mathbb{O})\big) \oplus \iota_0(\mathbb{O}\otimes\mathbb{O}) \oplus \iota_1(\mathbb{O}\otimes\mathbb{O}) \oplus \iota_2(\mathbb{O}\otimes\mathbb{O})$$

(see [Eld08] and the references therein). This model of E_8 is naturally $\mathbb{Z}_2^2$-graded, with the zero homogeneous part given by the direct sum of the two copies of the triality Lie algebra and the nonzero homogeneous parts given by the three copies of the tensor product of two copies of $\mathbb{O}$. And this $\mathbb{Z}_2^2$-grading is now refined by means of the $\mathbb{Z}_2^3$-grading of $\mathbb{O}$ to get a $\mathbb{Z}_2^5$-grading of E_8 with the required properties (see [Eld08, §5.4] for the details).

Therefore, the rest of this paper will be devoted to proving parts (ii), (iii), and (vi) of the Main Theorem. In the process, very concrete models of the corresponding Jordan gradings will emerge.

As a consequence, detailed models of all the Jordan gradings in Table 2 of [Ale74] (the exceptional Jordan gradings) are obtained.

To finish this introduction, note that all these gradings are related to the so-called *orthogonal decompositions* introduced in [KKU81] (see [KP94] and the references therein). For any of these gradings, if we denote by $\mathbb{P}(\mathbb{Z}_p^r)$ the projective space of dimension $r-1$ over the finite field $\mathbb{Z}_p$ and if for $0\neq\alpha\in\mathbb{Z}_p^r$, $[\alpha]$ denotes the corresponding point in $\mathbb{P}(\mathbb{Z}_p^r)$, then the subalgebras $\mathfrak{g}_{[\alpha]}$ in (1.1) are Cartan subalgebras of $\mathfrak{g}$, and the decomposition

$$\mathfrak{g} = \bigoplus_{[\alpha]\in\mathbb{P}(\mathbb{Z}_p^r)} \mathfrak{g}_{[\alpha]} \tag{1.2}$$

is a decomposition of $\mathfrak{g}$ into a direct sum of Cartan subalgebras which are orthogonal relative to the Killing form (as the homogeneous subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are always orthogonal unless $\beta=-\alpha$). That is, the decomposition in (1.2) is an orthogonal decomposition of $\mathfrak{g}$.

The next section will be devoted to proving the Main Theorem for the $\mathbb{Z}_5^3$-gradings on E_8 (part (iii)), and then Section 3 will deal with parts (vi) and (ii).

2. $\mathbb{Z}_5^3$-grading on E_8

The purpose of this section is to prove part (iii) of the Main Theorem, that is:

Theorem 2.1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic* 0 *and let $\mathfrak{g}$ be the simple Lie algebra of type E_8 over $\mathbb{F}$. Then up to equivalence there is a unique $\mathbb{Z}_5^3$-grading of $\mathfrak{g}$ such that* $\dim\mathfrak{g}_\alpha = 2$ *for any* $0\neq\alpha\in\mathbb{Z}_5^3$.

Proof. First note that by a dimension count $\mathfrak{g}_0=0$ holds. The proof will follow several steps.

Step 1. The construction of a suitable model of the simple Lie algebra of type E_8.

Let V_1 and V_2 be two vector spaces over $\mathbb{F}$ of dimension 5, and consider the $\mathbb{Z}_5$-graded vector space

$$\mathfrak{g} = \bigoplus_{i=0}^{4} \mathfrak{g}_{\bar{i}}, \tag{2.2}$$

where

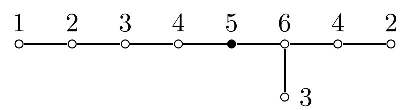
$$
\begin{aligned}
\mathfrak{g}_{\bar 0} &= \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2),\\
\mathfrak{g}_{\bar 1} &= V_1 \otimes \textstyle\bigwedge^2 V_2,\\
\mathfrak{g}_{\bar 2} &= \textstyle\bigwedge^2 V_1 \otimes \bigwedge^4 V_2,\\
\mathfrak{g}_{\bar 3} &= \textstyle\bigwedge^3 V_1 \otimes V_2,\\
\mathfrak{g}_{\bar 4} &= \textstyle\bigwedge^4 V_1 \otimes \bigwedge^3 V_2.
\end{aligned}
\tag{2.3}
$$

(All the tensor products are considered over the ground field $\mathbb{F}$.) This is a $\mathbb{Z}_5$-graded Lie algebra, with the natural action of the semisimple algebra $\mathfrak{g}_{\bar 0}$ on each of the other homogeneous components and where the brackets between elements in different components are given by suitable scalar multiples of the only $\mathfrak{g}_{\bar 0}$-invariant possibilities. In this way, $\mathfrak{g}$ is the exceptional simple Lie algebra of type E_8. The details of the Lie multiplication have been computed in [Dra05]. This decomposition has received some attention lately [Kos08].

Step 2. Up to conjugation in $\operatorname{Aut}\mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type E_8 such that the dimension of the subalgebra of fixed elements is 48.

Actually, as shown in [Kac90, §8.6], up to conjugation, the finite order automorphisms of E_8 are in one-to-one correspondence with subsets of nodes of the affine Dynkin diagram $E_8^{(1)}$:

(2.4)



such that the sum of the integers that label the nodes in the subset is exactly 5. Given such a subset of, say, r nodes, the fixed subalgebra is the direct sum of the semisimple Lie algebra whose Dynkin diagram is obtained by removing from (2.4) the nodes in the subset and a center of dimension $r-1$. Now it is easy to see that the only possibility is the automorphism σ obtained when considering the subset that consists exactly of the node with label 5. In this case, one gets a $\mathbb{Z}_5$-grading of $\mathfrak{g}$ where $\mathfrak{g}_{\bar 0}$ is a direct sum of two copies of the simple Lie algebra of type A_4. The uniqueness shows us that, up to conjugation, σ is the automorphism of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\bar\imath}$ (with notation as in Step 1) is ξ^i times the identity, where ξ is a fixed primitive fifth root of unity.

Step 3. Assume that $\mathfrak{g} = \bigoplus_{0\neq\alpha\in\mathbb{Z}_5^3} \mathfrak{g}_\alpha$ is a $\mathbb{Z}_5^3$-graded simple Lie algebra of type E_8 with $\dim \mathfrak{g}_\alpha = 2$ for any $0 \neq \alpha \in \mathbb{Z}_5^3$. The homogeneous spaces are given by the common eigenspaces of three commuting order 5 automorphisms σ_1, σ_2, and σ_3 of $\mathfrak{g}$ which generate a subgroup of $\operatorname{Aut}\mathfrak{g}$ isomorphic to $\mathbb{Z}_5^3$.

3.1 (σ_1): Step 2 shows us that, without loss of generality, we may assume that σ_1 is the automorphism such that $\sigma_1(x) = \xi^i x$ for any $x \in \mathfrak{g}_{\bar\imath}$ (notation as in Step 1).
3.2 (σ_2): Consider now the order 5 automorphism σ_2. As it commutes with σ_1, the restriction $\sigma_2|_{\mathfrak{g}_{\bar 0}}$ is an automorphism of $\mathfrak{g}_{\bar 0}$. Its order is then either 1 or 5. Given a subset of automorphisms of $\mathfrak{g}$, let us denote by $\operatorname{Fix}(S)$ the subset of elements that

are fixed by all the elements in S. Note that

$$\mathrm{Fix}(\sigma_2|_{\mathfrak{g}_{\bar 0}}) = \mathrm{Fix}(\sigma_1,\sigma_2) = \bigoplus_{i=0}^{4} \mathfrak{g}_{i\alpha}$$

for some $0 \neq \alpha \in \mathbb{Z}_5^3$ and that this subspace has dimension 8. We conclude that $\sigma_2|_{\mathfrak{g}_{\bar 0}}$ has order 5. Since $\mathfrak{sl}(V_1)$ and $\mathfrak{sl}(V_2)$ are the only ideals of $\mathfrak{g}_{\bar 0}$ and σ_2 induces a permutation of these two ideals of order 1 or 5, it follows that both $\mathfrak{sl}(V_1)$ and $\mathfrak{sl}(V_2)$ are invariant under the action of σ_2; and since $\dim \mathrm{Fix}(\sigma_1,\sigma_2)$ is 8, it turns out that the restriction of σ_2 to $\mathfrak{sl}(V_i)$ has order 5 ($i = 1, 2$).

Recall (see [Jac62, Chapter IX]) that $\mathrm{Int}(\mathfrak{sl}(V_i)$ is the group generated by the set $\{\exp \mathrm{ad}_a : a \in \mathfrak{sl}(V_i),\ a \text{ nilpotent}\}$ and that the quotient $\mathrm{Aut}(\mathfrak{sl}(V_i))/\,\mathrm{Int}(\mathfrak{sl}(V_i))$ is a cyclic group of order 2. Since the order of the restriction $\sigma_2|_{\mathfrak{sl}(V_i)}$ is 5, this restriction belongs to $\mathrm{Int}(\mathfrak{sl}(V_i))$, $i = 1, 2$.

Therefore, there are nilpotent endomorphisms $a_{ij} \in \mathfrak{sl}(V_i)$, $j = 1, \dots, m_i$, $i = 1, 2$, such that

$$\sigma_2|_{\mathfrak{sl}(V_i)} = \exp \mathrm{ad}_{a_{i1}} \cdots \exp \mathrm{ad}_{a_{im_i}}\,.$$

Hence, the restriction $\sigma_2|_{\mathfrak{g}_{\bar 0}}$ extends to the automorphism $\hat\sigma_2$ of $\mathfrak{g}$ given by the formula

$$\hat\sigma_2 = \exp \mathrm{ad}_{a_{11}} \cdots \exp \mathrm{ad}_{a_{1m_1}} \exp \mathrm{ad}_{a_{21}} \cdots \exp \mathrm{ad}_{a_{2m_2}}\,.$$

Note that $\hat\sigma_2$ leaves invariant the subspaces $\mathfrak{g}_{\bar i}$, for $0 \le i \le 4$. Thus the automorphism $\hat\sigma_2^{-1}\sigma_2$ leaves invariant all the subspaces $\mathfrak{g}_{\bar i}$ and its restriction to $\mathfrak{g}_{\bar 0}$ is the identity. But each $\mathfrak{g}_{\bar 1}$ is an irreducible module for $\mathfrak{g}_{\bar 0}$, so Schur's Lemma shows that there is a nonzero scalar $\lambda \in \mathbb{F}$ such that

$$\hat\sigma_2^{-1}\sigma_2|_{\mathfrak{g}_{\bar 1}} = \lambda 1;$$

and, as $\mathfrak{g}_{\bar 1}$ generates $\mathfrak{g}$ as a Lie algebra, it follows that the restriction of $\hat\sigma_2^{-1}\sigma_2$ to $\mathfrak{g}_{\bar i}$ is λ^i times the identity map, where $\lambda^5 = 1$. Also note that given any endomorphism $a \in \mathfrak{sl}(V_i)$, $\exp \mathrm{ad}_a = \mathrm{Ad}_{\exp a}$ on $\mathfrak{sl}(V_i)$ ($\mathrm{Ad}_g(x) = gxg^{-1}$ for any $g \in GL(V_i)$ and $a \in \mathfrak{sl}(V_i)$), while ad_a acts on each $\bigwedge^j V_i$ in the natural way, so that $\exp \mathrm{ad}_a$ acts on $\bigwedge^j V_i$ as $\wedge^j \exp a$ (where $(\wedge^j f)(w_1 \wedge \cdots \wedge w_j) = f(w_1) \wedge \cdots \wedge f(w_j)$).

Consider the elements $b_{ij} = \exp a_{ij} \in SL(V_i)$, and let $b_i = b_{i1} \cdots b_{im_i}$. Then the restrictions of $\hat\sigma_2$ to $\mathfrak{sl}(V_i)$ ($i = 1, 2$) and $\mathfrak{g}_{\bar 1} = V_1 \otimes \bigwedge^2 V_2$ are, respectively, the automorphism Ad_{b_i} and the linear isomorphism $b_1 \otimes \wedge^2 b_2$. If b_1 is changed to λb_1, then we get a new automorphism $\tilde\sigma_2$ such that $\tilde\sigma_2|_{\mathfrak{g}_{\bar 0}} = \hat\sigma_2|_{\mathfrak{g}_{\bar 0}} = \sigma_2|_{\mathfrak{g}_{\bar 0}}$ and $\tilde\sigma_2|_{\mathfrak{g}_{\bar 1}} = \lambda\hat\sigma_2|_{\mathfrak{g}_{\bar 1}} = \sigma_2|_{\mathfrak{g}_{\bar 1}}$. It follows that $\tilde\sigma_2 = \sigma_2$ (recall that $\mathfrak{g}_{\bar 1}$ generates $\mathfrak{g}$).

Summarizing the previous arguments, it has been proven that there are elements $b_i \in SL(V_i)$, $i = 1, 2$, such that

$$\sigma_2|_{\mathfrak{sl}(V_i)} = \mathrm{Ad}_{b_i} \quad (i = 1, 2), \qquad \sigma_2|_{\mathfrak{g}_{\bar 1}} = b_1 \otimes \wedge^2 b_2. \tag{2.5}$$

Moreover, the order of σ_2 is 5, so $(\sigma_2|_{\mathfrak{g}_{\bar 0}})^5 = 1$, which implies that $b_i^5 = \lambda_i 1_{V_i}$ for some $0 \neq \lambda_i \in \mathbb{F}$, $i = 1, 2$. But also $(\sigma_2|_{\mathfrak{g}_{\bar 1}})^5 = 1$, whence $\lambda_1\lambda_2^2 = 1$. Since $\mathbb{F}$ is algebraically closed, we can take scalars $\mu_1, \mu_2 \in \mathbb{F}$ such that $\mu_1^5 = \lambda_1^{-1}$, $\mu_2^5 = \lambda_2^{-1}$ and $\mu_1\mu_2^2 = 1$. We may replace b_i by $\mu_i b_i$, $i = 1, 2$, in (2.5) and hence assume that $b_i^5 = 1_{V_i}$, $i = 1, 2$.

Besides, for $i = 1, 2$, since $b_i^5 = 1$, b_i is a diagonalizable endomorphism of V_i whose eigenvalues are fifth roots of unity. Note that the subspace $\{x \in \mathfrak{sl}(V_i) : b_i x b_i^{-1} = x\}$ has dimension at least 4, because the endomorphisms which act diagonally on a basis of eigenvectors of b_i commute with b_i. But if an eigenvalue of b_i has multiplicity ≥ 2, then the dimension above is strictly greater than 4, and this contradicts the dimension of $\mathrm{Fix}(\sigma_1,\sigma_2)=\mathrm{Fix}(\sigma_2|_{\mathfrak{g}_{\bar 0}})$ being exactly 8. Therefore, all the eigenvalues of b_i have multiplicity 1, and therefore a basis $\{v_{i1},\dots,v_{i5}\}$ of V_i can be taken with $b_i(v_{ij})=\xi^j v_{ij}$ ($i=1,2,\ j=1,2,3,4,5$). That is, the matrix of b_i in this basis is precisely

$$\begin{pmatrix}1&0&0&0&0\\0&\xi&0&0&0\\0&0&\xi^2&0&0\\0&0&0&\xi^3&0\\0&0&0&0&\xi^4\end{pmatrix}.$$

3.3 (σ_3): Finally, let us consider the automorphism σ_3. Using the same arguments as in **3.2**, elements $c_i\in SL(V_i)$ ($i=1,2$) can be found, with $c_i^5=1$ and no repeated eigenvalues, such that

$$\sigma_3|_{\mathfrak{sl}(V_i)}=\mathrm{Ad}_{c_i},\qquad \sigma_3|_{\mathfrak{g}_{\bar 1}}=c_1\otimes\wedge^2 c_2.$$

As σ_2 and σ_3 commute, it follows in particular that $\mathrm{Ad}_{b_i}\mathrm{Ad}_{c_i}=\mathrm{Ad}_{c_i}\mathrm{Ad}_{b_i}$ in $\mathfrak{sl}(V_i)$ or that $b_ic_i=\mu_ic_ib_i$ for some $0\neq\mu_i\in\mathbb{F}$. Since $b_i^5=1$, we have $\mu_i^5=1$, $i=1,2$.

But if μ_i were equal to 1, then c_i would belong to $\{x\in\mathfrak{gl}(V_i): xb_i=b_ix\}=\mathrm{span}\left\{b_i^j: j=0,\dots,4\right\}$ and so the subspace $\{x\in\mathfrak{sl}(V_i):\sigma_2(x)=\sigma_3(x)=x\}=\{x\in\mathfrak{sl}(V_i): xb_i=b_ix\}$ would have dimension 4, while we have

$$\{x\in\mathfrak{sl}(V_i):\sigma_2(x)=\sigma_3(x)=x\}\subseteq\mathrm{Fix}(\sigma_1,\sigma_2,\sigma_3)=\mathfrak{g}_0=0,$$

a contradiction. Therefore, $\mu_i\neq 1$, $i=1,2$.

We may change σ_3 to σ_3^j for $1\leq j\leq 4$, which implies changing c_i to the corresponding power, and in this way we may assume that $\mu_1=\xi$, the fixed primitive fifth root of unity we have been using so far. (Note that the grading induced by $\sigma_1,\sigma_2,\sigma_3$ is induced also by σ_1,σ_2 and σ_3^j.)

Moreover, the commutativity of σ_2 and σ_3 on $\mathfrak{g}_{\bar 1}$ gives

$$b_1c_1\otimes\wedge^2(b_2c_2)=c_1b_1\otimes\wedge^2(c_2b_2)=\mu_1\mu_2^2 b_1c_1\otimes\wedge^2(b_2c_2)$$

so that $\mu_1\mu_2^2=1$, and thus we may assume that $\mu_1=\xi$ and $\mu_2=\xi^2$.

Since $b_1c_1=\xi c_1b_1$, we have $b_1c_1(v_{1j})=\xi^jc_1(v_{1j})$ ($j=1,\dots,5$), and hence we may scale the basic vectors v_{1j} so that $c_1(v_{1j})=v_{1(j+1)}$ for $j=1,2,3,4$. In other words, a basis can be taken in V_1 such that the coordinate matrices of b_1 and c_1 are

$$b_1\leftrightarrow\begin{pmatrix}1&0&0&0&0\\0&\xi&0&0&0\\0&0&\xi^2&0&0\\0&0&0&\xi^3&0\\0&0&0&0&\xi^4\end{pmatrix},\qquad c_1\leftrightarrow\begin{pmatrix}0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}. \tag{2.6}$$

In the same vein, since $b_2c_2=\xi^2c_2b_2$, by permuting and scaling the previous basic vectors on V_2 a new basis can be taken in V_2 such that the coordinate matrices of

b_2 and c_2 are

$$b_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 \\ 0 & 0 & \xi^4 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & \xi^3 \end{pmatrix}, \qquad c_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.7}$$

In conclusion, up to equivalence, the only $\mathbb{Z}_5^3$-grading of $\mathfrak{g}$ such that $\dim \mathfrak{g}_\alpha = 2$ for any $0 \neq \alpha \in \mathbb{Z}_5^3$ is given by the automorphisms $\sigma_1, \sigma_2, \sigma_3$ such that

$$\begin{aligned} &\sigma_1(x) = \xi^i x \quad \text{for any } x \in \mathfrak{g}_{\bar{\imath}} \text{ and } 0 \leq i \leq 4, \\ &\sigma_2|_{\mathfrak{g}_{\bar{1}}} = b_1 \otimes \wedge^2 b_2, \\ &\sigma_3|_{\mathfrak{g}_{\bar{1}}} = c_1 \otimes \wedge^2 c_2, \end{aligned} \tag{2.8}$$

where the $\mathfrak{g}_{\bar{\imath}}$'s are the homogeneous components in (2.3) and, on fixed bases of V_1 and V_2, b_1 and c_1 (respectively b_2 and c_2) are the endomorphisms of V_1 (respectively V_2) in (2.6) (respectively (2.7)). □

Remark 2.9. The proof of the previous theorem gives a precise model for the $\mathbb{Z}_5^3$ Jordan grading of E_8.

3. $\mathbb{Z}_3^3$-gradings on E_6 and F_4

In this section parts (ii) and (vi) of the Main Theorem will be proved. Many arguments are quite similar to the ones used for E_8, so they will just be sketched.

We start with E_6:

Theorem 3.1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic 0 and let $\mathfrak{g}$ be the simple Lie algebra of type E_6 over $\mathbb{F}$. Then up to equivalence there is a unique $\mathbb{Z}_3^3$-grading of $\mathfrak{g}$ such that $\dim \mathfrak{g}_\alpha = 3$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.*

Proof. The same steps as for E_8 will be followed.

Step 1. The construction of a suitable model of the simple Lie algebra of type E_6.

Here the model appears in [Ada96, Chapter 13] (see also [Dra08, §3]). Let V_1, V_2 and V_3 be three vector spaces of dimension 3 over $\mathbb{F}$ and consider the $\mathbb{Z}_3$-graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}}, \tag{3.2}$$

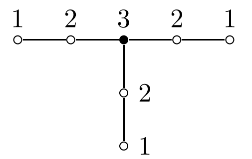
where

$$\begin{aligned} \mathfrak{g}_{\bar{0}} &= \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \oplus \mathfrak{sl}(V_3), \\ \mathfrak{g}_{\bar{1}} &= V_1 \otimes V_2 \otimes V_3, \\ \mathfrak{g}_{\bar{2}} &= V_1^* \otimes V_2^* \otimes V_3^*. \end{aligned} \tag{3.3}$$

This is a $\mathbb{Z}_3$-graded Lie algebra, with the natural action of the semisimple algebra $\mathfrak{g}_{\bar{0}}$ on each of the other homogeneous components and where the brackets between elements in different components are given by suitable scalar multiples of the only $\mathfrak{g}_{\bar{0}}$-invariant possibilities. In this way, $\mathfrak{g}$ is the exceptional simple Lie algebra of type E_6.

Step 2. Up to conjugation in $\operatorname{Aut} \mathfrak{g}$, there is a unique order 3 automorphism of the simple Lie algebra $\mathfrak{g}$ of type E_6 such that the dimension of the subalgebra of fixed elements is 24.

Actually, this automorphism σ is the one that corresponds to the only node labeled by 3 in the affine Dynkin diagram $E_6^{(1)}$:



The uniqueness shows us that, up to conjugation, σ is the automorphism of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\bar{\imath}}$ (with notation as in Step 1) is ω^i times the identity, where ω is a fixed primitive third root of unity.

Step 3. Assume that $\mathfrak{g} = \bigoplus_{0\neq\alpha\in\mathbb{Z}_3^3} \mathfrak{g}_\alpha$ is a $\mathbb{Z}_3^3$-graded simple Lie algebra of type E_6 with $\dim \mathfrak{g}_\alpha = 3$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$. The homogeneous spaces are given by the common eigenspaces of three commuting order 3 automorphisms σ_1, σ_2, and σ_3 of $\mathfrak{g}$ which generate a subgroup of $\operatorname{Aut}\mathfrak{g}$ isomorphic to $\mathbb{Z}_3^3$.

3.1 (σ_1): Step 2 shows us that, without loss of generality, we may assume that σ_1 is the automorphism such that $\sigma_1(x) = \omega^i x$ for any $x \in \mathfrak{g}_{\bar{\imath}}$ (notation as in Step 1).
3.2 (σ_2): As for E_8, the restriction $\sigma_2|_{\mathfrak{g}_{\bar{0}}}$ is an order 3 automorphism (otherwise the dimension of $\operatorname{Fix}(\sigma_1, \sigma_2)$ would be > 6). Now, σ_2 induces a permutation of the three simple ideals of $\mathfrak{g}_{\bar{0}}$ of order 1 or 3, but if the order were 3, then the eight dimensional subspace $\{x+\sigma_2(x)+\sigma_2^2(x) : x \in \mathfrak{sl}(V_1)\}$ would be contained in the six dimensional subspace $\operatorname{Fix}(\sigma_1, \sigma_2)$, a contradiction. Therefore, σ_2 leaves invariant $\mathfrak{sl}(V_i)$ for all i.

Now the same arguments as for E_8 show that one may find elements $b_i \in SL(V_i)$, $i = 1, 2, 3$, such that $b_i^3 = 1$ and

$$\sigma_2|_{\mathfrak{sl}(V_i)} = \operatorname{Ad}_{b_i} \quad (i = 1, 2, 3), \qquad \sigma_2|_{\mathfrak{g}_{\bar{1}}} = b_1 \otimes b_2 \otimes b_3.$$

Moreover, the minimal polynomial of $b_i \in SL(V_i)$ is exactly $X^3 - 1$ (its eigenvalues have multiplicity 1).
3.3 (σ_3): In the same vein, there are endomorphisms $c_i \in SL(V_i)$, $i = 1, 2, 3$, with minimal polynomial $X^3 - 1$ such that

$$\sigma_3|_{\mathfrak{gsl}(V_i)} = \operatorname{Ad}_{c_i}, \qquad \sigma_3|_{\mathfrak{g}_{\bar{1}}} = c_1 \otimes c_2 \otimes c_3.$$

As for E_8, the commutation of σ_2 and σ_3 and a dimension count show that $b_ic_i = \mu_ic_ib_i$, with $1 \neq \mu_i \in \mathbb{F}$ and $\mu_i^3 = 1$ $(i = 1, 2, 3)$. Hence $\mu_i \in \{\omega, \omega^2\}$. Replacing σ_3 by σ_3^2 if necessary, it can be assumed that $\mu_1 = \omega$.

Moreover, the commutativity of σ_2 and σ_3 on $\mathfrak{g}_{\bar{1}}$ forces the equality $\mu_1\mu_2\mu_3 = 1$ or $\mu_2\mu_3 = \omega^2$. We conclude that $\mu_1 = \mu_2 = \mu_3 = \omega$. Hence, a basis can be chosen on each V_i such that the coordinate matrices of b_i and c_i are

$$b_i \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad c_i \leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.4}$$

In conclusion, up to equivalence, the only $\mathbb{Z}_3^3$-grading of $\mathfrak{g}$ such that $\dim \mathfrak{g}_\alpha = 3$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$ is given by the automorphisms $\sigma_1, \sigma_2, \sigma_3$ such that

$$\begin{aligned} &\sigma_1(x) = \omega^i x \quad \text{for any } x \in \mathfrak{g}_{\bar{i}} \text{ and } i = 0, 1, 2,\\ &\sigma_2|_{\mathfrak{g}_{\bar{1}}} = b_1 \otimes b_2 \otimes b_3,\\ &\sigma_3|_{\mathfrak{g}_{\bar{1}}} = c_1 \otimes c_2 \otimes c_3, \end{aligned}$$

where the $\mathfrak{g}_{\bar{i}}$'s are the homogeneous components in (3.3) and, on fixed bases of V_1, V_2 and V_3, b_i and c_i are the endomorphisms of V_i in (3.4). □

The corresponding result for F_4 is the following:

Theorem 3.5. *Let $\mathbb{F}$ be an algebraically closed field of characteristic* 0 *and let $\mathfrak{g}$ be the simple Lie algebra of type F_4 over $\mathbb{F}$. Then up to equivalence there is a unique $\mathbb{Z}_3^3$-grading of $\mathfrak{g}$ such that* $\dim \mathfrak{g}_\alpha = 2$ *for any* $0 \neq \alpha \in \mathbb{Z}_3^3$.

Proof. Here we will be even more sketchy, since the situation is simpler.

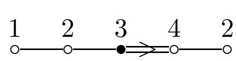
Just consider the model of E_6 obtained above, and consider the order 2 automorphism τ which permutes V_2 and V_3. The subalgebra $\mathfrak{g}$ of elements fixed by τ is a simple Lie algebra of type F_4 (see [Dra08, §3]). Therefore, $\mathfrak{g}$ appears as the $\mathbb{Z}_3$-graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}}, \tag{3.6}$$

$$\begin{aligned} \mathfrak{g}_{\bar{0}} &= \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2),\\ \mathfrak{g}_{\bar{1}} &= V_1 \otimes S^2(V_2),\\ \mathfrak{g}_{\bar{2}} &= V_1^* \otimes S^2(V_2^*). \end{aligned} \tag{3.7}$$

Here $S^2(V)$ denotes the subspace of symmetric tensors in $V \otimes V$.

Up to conjugation in $\operatorname{Aut} \mathfrak{g}$, there is a unique order 3 automorphism of the simple Lie algebra $\mathfrak{g}$ of type F_4 such that the dimension of the subalgebra of fixed elements is 16. Actually, this automorphism σ is the one that corresponds to the only node labeled by 3 in the affine Dynkin diagram $F_4^{(1)}$:



Now, the same types of arguments as for E_6 give the result. □

Again, the proofs of Theorems 3.1 and 3.5 give precise models of the corresponding Jordan gradings. In [DM07, §7] it was shown that for F_4 this grading is fine. These models are different from those obtained in [Eld08, §5.3], which were based on a $\mathbb{Z}_3^2$-grading of the Okubo algebra over $\mathbb{F}$, complemented by an extra order three automorphism induced by the triality automorphism associated to the Okubo algebra. For E_6, this unique $\mathbb{Z}_3^3$-grading is not fine, as the construction of E_6 in terms of an Okubo algebra requires the use of another two dimensional symmetric composition algebra, which in turn can be graded over $\mathbb{Z}_3$ and used to get a $\mathbb{Z}_3^4$-grading on E_6 (see [Eld08] for details).

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