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Weighted composition operators between weighted Bergman spaces

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Abstract. We study the boundedness of weighted composition operators acting between weighted Bergman spaces.

Operadores de composición ponderados entre espacios de Bergman con pesos

Resumen. Se estudia la acotación de los operadores de composición ponderados entre espacios de Bergman con pesos.

1 Introduction

We consider strictly positive bounded continuous functions (*weights*) v and w on the open unit disk D in the complex plane. Moreover let $H(D)$ denote the space of all holomorphic functions on D and let ϕ be an analytic self map of D as well as $\psi\colon D \to \mathbb{C}$ be analytic. Such maps induce a linear weighted composition operator $\psi C_\phi(f) = \psi(f \circ \phi)$. We are interested in weighted composition operators acting on weighted Bergman spaces

$$A_w^p := \left\{ f \in H(D); \quad \|f\|_{w,p} = \left(\int_D |f(z)|^p \, w(z) \, \mathrm{d}A(z) \right)^{\frac{1}{p}} < \infty \right\}, \qquad 1 \leq p < \infty,$$

where $\mathrm{d}A(z)$ is the area measure on D normalized so that area of D is 1. Thus A_1^2 denotes the usual Bergman space. An introduction to the concept of Bergman spaces is given in [7] and [8]. Composition operators and weighted composition operators have been studied on various spaces of holomorphic functions, see e.g. [10, 9, 1, 2, 3, 4, 12]. For more general information on composition operators we refer to the monographs [5] and [11]. In this article we want to charaterize boundedness of composition operators acting between weighted Bergman spaces.

2 Preliminaries

For $a, z \in D$ let $\sigma_a(z)$ be the Möbius transformation of D which interchanges 0 and a, that is

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

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Furthermore we use the fact that

$$-\sigma_a'(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2}, \qquad z \in D.$$

Moreover let $K_a(z) = \frac{1}{(1-\overline{a}z)^2}$ denote the Bergman kernel and $k_a(z) = -\sigma_a'(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2} = (1-|a|^2)\, K_a(z)$ the normalized Bergman kernel in A_1^2 so that $\|k_a\|_{1,2} = 1$. For an analytic self map ϕ of D and weights v, w on D we define the weighted (ϕ, v)-Berezin transform of w as follows

$$[B_{\phi,v}(w)](a) = \int_D |\sigma_a'(\phi(z))|^2 \frac{w(z)}{v(\phi(z))}\, \mathrm{d}A(z).$$

In order to find results on composition operators acting on weighted Bergman spaces we need the Carleson measure. To use this we collect some facts. Let μ be a positive Borel measure on D. Then μ is called a Carleson measure on the Bergman space if there is a constant $C > 0$ such that, for any $f \in A_1^2$

$$\int_D |f(z)|^2\, \mathrm{d}\mu(z) \leq C\|f\|_{1,2}^2.$$

For an arc I in the unit circle ∂D let $S(I)$ be the Carleson square defined by

$$S(I) = \left\{ z \in D; \quad 1 - |I| \leq |z| < 1, \quad \frac{z}{|z|} \in I \right\}.$$

The following result is well-known. In its present form it is taken from [6] (see there Theorem A).

Theorem 1 ([6, Theorem A]) *Let μ be a positive Borel measure on D. Then the following statements are equivalent.*

(i) *There is a constant $C_1 > 0$ such that for any $f \in A_1^2$*

$$\int_D |f(z)|^2\, \mathrm{d}\mu(z) \leq C_1\|f\|_{1,2}^2.$$

(ii) *There is a constant $C_2 > 0$ such that, for any arc $I \in \partial D$,*

$$\mu(S(I)) \leq C_2|I|^2.$$

(iii) *There is a constant $C_3 > 0$ such that, for every $a \in D$,*

$$\int_D |\sigma_a'(z)|^2\, \mathrm{d}\mu(z) \leq C_3.$$

In the sequel we consider the following weights. Let ν be a holomorphic function on D, non-vanishing, strictly positive on $[0, 1[$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight v as follows $v(z) = \nu(|z|^2)$ for every $z \in D$.
Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1-z)^\alpha$, $\alpha \geq 1$. Then the corresponding weight is the so-called standard weight $v(z) = (1-|z|^2)^\alpha$.

(ii) Select $\nu(z) = \mathrm{e}^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$. Then we obtain the weight $v(z) = \mathrm{e}^{-\frac{1}{(1-|z|^2)^\alpha}}$.

(iii) Choose $\nu(z) = \sin(1-z)$ and the corresponding weight is given by $v(z) = \sin(1-|z|^2)$.

For a fixed point $a \in D$ we introduce a function $v_a(z) := \nu(\overline{a}z)$ for every $z \in D$. Since ν is holomorphic on D, so is the function v_a.

3 Boundedness

We first need the following auxiliary result. The following lemma is well-known for standard weights (see [7] or [8]) but to the best of our knowledge not known for the weights described above.

Lemma 1 *Let v be a radial weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that $\sup_{a\in D} \sup_{z\in D} \frac{v(z)|v_a(\sigma_a(z))|}{v(\sigma_a(z))} \leq C < \infty$. Then*

$$|f(z)| \leq \frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\|f\|_{v,p}$$

for all $z \in D$, $f \in A_{v,p}$.

PROOF. Let $\alpha \in D$ be an arbitrary point. Consider the map

$$T_\alpha : A_v^p \to A_v^p, \qquad T_\alpha(f(z)) = f(\sigma_\alpha(z))\sigma'_\alpha(z)^{\frac{2}{p}}v_\alpha(\sigma_\alpha(z))^{\frac{1}{p}}.$$

Then a change of variables yields

$$\begin{aligned}
\|T_\alpha f\|_{v,p}^p &= \int_D v(z)\,|f(\sigma_\alpha(z))|^p\,|\sigma'_\alpha(z)|^2\,|v_\alpha(\sigma_\alpha(z))|\,\mathrm{d}A(z)\\
&= \int_D \frac{v(z)|v_\alpha(\sigma_\alpha(z))|}{v(\sigma_\alpha(z))}\,|f(\sigma_\alpha(z))|^p\,|\sigma'_\alpha(z)|^2\,v(\sigma_\alpha(z))\,\mathrm{d}A(z)\\
&\leq \sup_{z\in D}\frac{v(z)|v_\alpha(\sigma_\alpha(z))|}{v(\sigma_\alpha(z))}\int_D |f(\sigma_\alpha(z))|^p\,|\sigma'_\alpha(z)|^2\,v(\sigma_\alpha(z))\,\mathrm{d}A(z)\\
&\leq C\int_D v(t)\,|f(t)|^p\,\mathrm{d}A(t) = C\|f\|_{v,p}^p.
\end{aligned}$$

Now put $g(z) = T_\alpha(f(z))$. By the mean-value property we obtain

$$v(0)\,|g(0)|^p \leq \int_D v(z)\,|g(z)|^p\,\mathrm{d}A(z) = \|g\|_{v,p}^p \leq C\|f\|_{v,p}^p.$$

Hence

$$v(0)\,|g(0)|^p = v(0)\,|f(\alpha)|^p\,(1-|\alpha|^2)^2\,v(\alpha) \leq C\|f\|_{v,p}^p.$$

Thus $|f(\alpha)| \leq C^{\frac{1}{p}}\frac{\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|\alpha|^2)^{\frac{2}{p}}v(\alpha)^{\frac{1}{p}}}$. Since α was arbitrary, the claim follows. ■

Thus, we can give the following sufficient condition for the boundedness of an operator $\psi C_\phi : A_v^p \to A_w^p$.

Proposition 1 *Let w be a weight and v be a weight as in Lemma 1. If*

$$\sup_{z\in D}\frac{|\psi(z)|w(z)^{\frac{1}{p}}}{(1-|\phi(z)|^2)^{\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}} < \infty,$$

then the operator $\psi C_\phi\colon A_v^p \to A_w^p$ is bounded.

PROOF. Applying Lemma 1 we get for every $f \in A_v^2$

$$\begin{aligned}
\|\psi C_\phi f\|_{w,p}^p &= \int_D |\psi(z)|^p\,|f(\phi(z))|^p\,w(z)\,\mathrm{d}A(z)\\
&\leq \int_D \frac{|\psi(z)|^p\,C}{v(0)\,(1-|\phi(z)|^2)^2\,v(\phi(z))}w(z)\,\|f\|_{v,p}^p\,\mathrm{d}A(z)\\
&\leq \sup_{z\in D}\frac{|\psi(z)|^p\,C}{v(0)\,(1-|\phi(z)|^2)^2\,v(\phi(z))}w(z)\,\|f\|_{v,p}^p,
\end{aligned}$$

and the claim follows. ■

Next, we turn our attention to weights v of the form $v = |u|$, where u is a holomorphic function on D without any zeros on D. The proof of the following theorem was inspired by the proof of [6, Theorem 1].

Theorem 2 *Let u be an analytic function on D without any zeros on D. Put $v(z) = |u(z)|$, $z \in D$. Moreover let w be an arbitrary weight on D and ϕ be an analytic self-map of D. Furthermore let ψ be analytic on D. Then the weighted composition operator*

$$\psi C_\phi \colon A_v^2 \to A_w^2, \quad f \to \psi(f \circ \phi)$$

is bounded if and only if the weighted Berezin transform $B_{\phi,v}(|\psi|^2 w) \in L^\infty(D)$.

PROOF. Our proof uses a reformulation of the Carleson measure condition. By definition, $\psi C_\phi \colon A_v^2 \to A_w^2$ is bounded if and only if there is $C > 0$ such that for every $f \in A_v^2$:

$$\int_D |f(\phi(z))|^2 \, |\psi(z)|^2 \, w(z) \, \mathrm{d}A(z) \le C \int_D |f(z)|^2 \, v(z) \, \mathrm{d}A(z) \tag{1}$$

Since $f \in A_v^2$ if and only if $g = u^{\frac{1}{2}} f \in A_1^2$ (which means $f = \frac{g}{u^{1/2}}$), (1) is equivalent to the following condition: There is a constant $C > 0$ such that for every $g \in A_1^2$

$$\int_D \frac{|g(\phi(z))|^2}{v(\phi(z))} |\psi(z)|^2 \, w(z) \, \mathrm{d}A(z) \le C \int_D |g(z)|^2 \, \mathrm{d}A(z). \tag{2}$$

Let $\mathrm{d}\nu_{v,w,\psi}(z) = |\psi(z)|^2 \frac{w(z)}{v(\phi(z))} \, \mathrm{d}A(z)$ and let $\mu_{v,w,\psi} = \nu_{v,w,\psi} \circ \phi^{-1}$ be the pull-back measure induced by ϕ. If we change variable $s = \phi(z)$, then we get

$$\int_D |g(\phi(z))|^2 \, |\psi(z)|^2 \frac{w(z)}{v(\phi(z))} \, \mathrm{d}A(z) = \int_D |g(\phi(z))|^2 \, \mathrm{d}\nu_{v,w,\psi}(z) = \int_D |g(s)|^2 \, \mathrm{d}\mu_{v,w,\psi}(s).$$

Thus, (1) is equivalent to $\int_D |g(s)|^2 \, \mathrm{d}\mu_{v,w,\psi}(s) \le C \int_D |g(s)|^2 \, \mathrm{d}A(s)$. By Theorem 1 this holds if and only if

$$\sup_{a \in D} \int_D |\sigma_a'(s)|^2 \, \mathrm{d}\mu_{v,w,\psi}(s) < \infty.$$

Changing the variable back to z, we get

$$\sup_{a \in D} \int_D |\sigma_a'(\phi(z))|^2 \, |\psi(z)|^2 \frac{w(z)}{v(\phi(z))} \, \mathrm{d}A(z) < \infty,$$

and the claim follows. ■

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