A MULTIOBJECTIVE APPROACH USING CONSISTENT RATE CURVES TO THE CALIBRATION OF A GAUSSIAN HEATH–JARROW–MORTON MODEL*

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ABSTRACT

In this paper we propose an alternate calibration algorithm, by using a consistent family of yield curves, that fits a Gaussian Heath–Jarrow– Morton model jointly to the implied volatilities of caps and zero-coupon bond prices. The algorithm is capable for finding several Pareto optimal points as is expected for a general nonlinear multicriteria optimization problem. The calibration approach is evaluated in terms of in-sample data fitting as well as stability of parameter estimates. Furthermore, the efficiency is tested against a non-consistent traditional method by using simulated and US market data.

Keywords: HJM models, consistent forward rate curves, multiobjective calibration

JEL Classification: E43, C1

1 Introduction

Any acceptable model which prices interest rate derivatives must fit the observed term structure. This idea pioneered by Ho and Lee [18], has been explored in the past by many other researchers like Black and Karasinski [11] and Hull and White [19].

The contemporary models are more complex because they consider the evolution of the whole forward curve as an infinite system of stochastic differential equations (Heath, Jarrow and Morton [17]). In particular, they use as initial input, a continuous forward rate curve. In reality, we just observe a discrete set composed either by bond prices or swap rates. So, in practice, the usual approach is to interpolate the forward curve by using splines or other parametrized families of functions.

A very plausible question arises at this point: Choose a specific parametric family, $\mathcal{G}$, of functions that represent the forward curve, and also an arbitrage free interest rate model $\mathcal{M}$. Assume that we use an initial curve that lay within as input for model $\mathcal{M}$. Will this interest rate model evolve through forward curves that lay within the family? Motivated by this question, Björk and Christensen [8] define the so–called consistent pairs ($\mathcal{M}$, $\mathcal{G}$) as ones whose answer to the above question is positive. In particular, they studied the problem of consistency the family of curves proposed by Nelson and Siegel [24] and any HJM interest rate model with deterministic volatility, obtaining that there is no such interest model consistent with it.

We remark that the Nelson and Siegel interpolating scheme is an important example of a parametric family of forward curves, because it is widely adopted by central banks (see for instance BIS [3]). Its forward curve shape, $G_{NS}(z, \cdot)$ is given by the expression

$$G_{NS}(z, x) = z_1 + z_2 e^{-z_4 x} + z_3 x e^{-z_4 x},$$

where x denotes time to maturity and z the parameter vector

$$z = (z_1,\ z_2,\ \dots).$$

Despite all the positive empirical features and general acceptance by the financial community, Filipović [16] has shown that there is no Itô process that is consistent with the Nelson-Siegel family. In a recent study De Rossi [14] applies consistency results to propose a consistent exponential dynamic model, and estimates it using data on LIBOR and UK swap rates. On the other hand, Buraschi and Corielli [12] add results to theoretical framework indicating that the use of inconsistent parametric families to obtain smooth interest rate curves, violates the standard self financing arguments of replicating strategies, with direct consequences in risk management procedures.

In order to illustrate this situation, we describe a very common fixed-income market procedure. In the real world, the practitioners usually re-estimate yield curve and HJM model parameters on a dayly basis. This procedure consists of two steps:

- They fit the initial yield curve from discrete market data (bond prices, swap rates, short-term zero rates), then
- They obtain an estimation of the parameters of the HJM model, minimizing the pricing error of some actively traded (plain vanilla) interest rate derivatives (commonly swap options or caps)

Perhaps, this fact is not relevant for mark to market, but it could have practical consequences on the hedging portfolios associated with these financial instruments. Recall that such dynamic strategies depend on the model assumptions. Thus, re-calibration is conceivable because the practitioners are aware of *model risk*. A particular HJM model is not a perfect description of reality, and they are forced to re-estimate day to day model parameters in order to include new information that arrives from the market. On the other hand, unstable estimates may be caused by reasons that are more theoretical, because the above mentioned setup does not take into account that HJM model parameters are linked, in general, to the the initial yield curve fit parameters. If a practitioner uses an interpolation scheme which is not consistent with the model, then the parameters will be artificially forced to change. Thus, it seems there are a plethora of motivations for the study of the empirical evidence and the practical implications that are predicted by a consistent HJM build model.

The consistency hypothesis stated by Björk, implies that the zero coupon bond curve has to be determined at the same time as the parameters of the model. In [1] and [2], Angelini and Herzel, propose the use of a optimization program related to the mentioned dayly calibrations, which is compatible with this joint estimation. The milestone of this methodology is the use of an objective function based on an error measure for just the caps portfolio. Then, the theoretical prices for the caps along with the minimization of this measure can be calculated at the same time that yield-curve is fitted. This is an efficient method because consistent families of yield-curves have a good behaviour in a Gaussian framework.

The purpose of this work is to extend the above strategy to a more general framework. It modifies the objective function mentioned, by taking into account the error measure for the discount bonds estimation. To this scope, we construct the objective function using a convex combination of the cap and the bond error measures, by means of a fixed parameter. As a matter of fact, this rigorous approach is richer in possible outcomes.

To this end, we restrict ourselves to a particular humped volatility HJM model, proposed by Mercurio and Moraleda [22] and Ritchen and Chuang [25] independently. We will discuss this formalization and give some theoretical results relevant to our analysis. We chose this model because it is quite popular and analytically treatable. In particular, it provides closed formulas for europeans caps. Moreover, it is the one-factor Gaussian model that seems more capable for reproducing the humped shape of the implied volatility term structure for caps, that the normal market scenarios usually depicts. Moreover, it is also the most flexible in other market conditions. We perform our study by calibrating this model, first by using simulated data and second by a US–market data set composed by the US discount factors and the cap at-the-money flat volatilities quotes in two different periods, see Figure 4. The first one depicts a normal fixed-income market scenario, the term structure of implied volatilities for caps (hereafter TSV) have humped shape and the term structure of interest rates (hereafter TSIR) is decreasing in the short term with a local minimum, and then it increases to mid-long term maturities –spoon-shaped. Second period lays within the Post-2001 era. Its may be considered an excited period with a TSV monotonically decreasing, and with a higher overall implied volatility and a TSIR monotonically increasing without local minimum. To our knowledge is the first attempt to search for empirical evidence focusing on US–market data.

This paper is organized as follows. In Section 2 we give a brief overview of the model and present in this context the option valuation and the construction of the consistent families with the model. In Section 3 the calibration procedure is described. Section 4 is devoted to empirical results, first comparing the consistent calibration algorithm to the non-consistent approaches with simulated data, then presenting the results of the fitting of the different models with US–market data. In the last section we give some final conclusions and remarks.

2 The Model

Let W be a one dimensional Wiener stochastic process defined in a complete probability space $(\Omega, \mathcal{F}, P)$.

Single factor Heath-Jarrow-Morton [17] framework is based on the dynamics of the entire forward rate curve, $\{r_t(x),\ x > 0\}$. Thus, under Musiela's [23] parameterization it follows that the infinite dimensional diffusion process given by

$$\left\{ \begin{array}{rcl} dr_t(x) & = & \beta(r_t, x)dt + \sigma(r_t, x)dW_t \\ r_0(x) & = & r^*(x), \end{array} \right. \tag{1}$$

where $\{r^*(x),\ x \geq 0\}$, can be interpreted as the *observed* forward rate curve. The standard drift condition derived in Heath, Jarrow and Morton [17] can easily transferred to the Musiela parametrization (see, for instance, Musiela [23]),

$$\beta(r_t,x)=\frac{\partial}{\partial x}r_t(x)+\sigma(r_t,x)\int_0^x \sigma(r_t,s)ds.$$

Thus, a particular model is constructed by the choice of an explicit volatility function $\sigma(r_t,x)$.

Our work is devoted to a Gaussian humped volatility model where

$$\sigma(r_t,x)=\sigma(x)=(\alpha+\beta x)e^{-ax},$$

i.e. σ is a deterministic function depending only on time to maturity, and then $r_t(x)$ is a Gaussian process.

Finite Dimensional Realizations of Gaussian Models

It should be also noted that $\sigma(x)$ is a one dimension quasi-exponential function (QE for short), because is of the form

$$f(x)=\sum_i e^{\lambda_i x}+\sum_i e^{\alpha_i x}[p_i(x)\cos(\omega_i x)+q_i(x)\sin(\omega_i x)],$$

with $\lambda_i,\alpha_i,\omega_i$ being real numbers and p_i,q_i are real polynomials.

It is well-known that if $f(x)$ is a m-dimensional QE function, then it admits the following matrix representation

$$f(x)=ce^{Ax}B,$$

where A is a $(n\times n)$-matrix, B is a $(n\times m)$-matrix and c is a n–dimensional row vector (see Lemma 2.1 in Björk [5]). Thus, $\sigma(x)$ can be written as

$$\begin{aligned}\sigma(x) &= ce^{Ax}b, \text{ where} &&(2)\\ c &= [\alpha \quad \beta-a\alpha],\\ A &= \begin{bmatrix} 0 & -a^2\\ 1 & -2a\end{bmatrix},\\ b &= \begin{bmatrix}1\\0\end{bmatrix}.\end{aligned}$$

By means of Proposition 2.1 in Björk [6] we can write the forward rate equation (1) as:

$$\begin{aligned} dq_t(x) &= \mathbf{F}q_t(x)\,dt+\sigma(x)\,dW_t, \quad q_0(x)=0 &&(3)\\ r_t(x) &= q_t(x)+\delta_t(x), &&(4)\end{aligned}$$

here $\mathbf{F}$ is a linear operator that is defined by

$$\mathbf{F} = \frac{\partial}{\partial x},$$

and $\delta_t(x)$ is the deterministic process given by

$$\delta_t(x) = r^*(x+t) + \int_0^t \Sigma(x+t-s)\,ds,$$

with $\Sigma(\cdot) = \sigma(\cdot)\int_0^{\cdot}\sigma(s)\,ds$. Moreover, $q_t(x)$ has the concrete *finite dimensional* realization

$$\begin{aligned} dZ_t &= AZ_t\,dt + b\,dW_t, \quad Z_0 = 0, &(5)\\ q_t(x) &= C(x)Z_t, &(6)\end{aligned}$$

because σ is a QE function (see, for instance, Proposition 2.3 in Björk [5]) with A, b as in (2) and $C(x) = ce^{Ax}$. Thus, (5) is a linear SDE in the narrow sense (see Kloeden and Platen [21] for details) with explicit solution

$$Z_t = \Phi_t \int_0^t \Phi_s^{-1} b\,dW_s, \tag{7}$$

where

$$\Phi_t = e^{At} = e^{-at}\begin{bmatrix} 1+at & -a^2 t \\ t & 1-at \end{bmatrix}.$$

Now, with the definition of $S(x) = \int_0^x \sigma(u)du$, it is easy to obtain that

$$\int_0^t \Sigma(t+x-s)\,ds = \frac{1}{2}\left[S^2(t+x) - S^2(x)\right],$$

and, therefore, combining these explicit results with decomposition (4) we arrive to the operative expression

$$r_t(x) = r^*(x+t) + \frac{1}{2}\left[S^2(t+x) - S^2(x)\right] + C(x)Z_t. \tag{8}$$

The last expression allows to perform the Monte Carlo simulation of future forward curves produced by this HJM particular model. On the other hand, as we will show later, equation (8) can also be used to build the initial forward rate curve $r^*(x)$. Recall that it is consistent with the dynamics of the model.

Interest Rate Option Pricing

To calibrate the model by means of real data, we actually need to determine the vector parameter $\theta = (\alpha,\ \beta,\ a)$. In order to estimate the forward rate volatility, the statistical analysis of past data can be a possible approach, but the practitioners usually prefer implied volatility, laying within some derivative market prices, based techniques. This way involves a minimization problem where the loss function can be taken as

$$l(\theta) = \sum_{i=1}^{n} (C_i^* - C_i(\theta, T_i))^2,$$

where $C_i(\theta)$ are the i–th theoretical derivative price and $C_i^* \equiv C^*(T_i)$ is the i–th market price one. As it is well known, see Proposition 24.15 and pages 364–366 in Björk [4], the price, at $t = 0$, of the cap is given by

$$C(T) = (1 + \tau K) \left(\sum_{j=0}^{n-1} \kappa D(x_j) N(-d_+) - D(x_{j+1}) N(-d_-) \right), \qquad (9)$$

where

$$d_\pm = \frac{\ln \frac{D(0,x_j)}{\kappa D(0,x_{j+1})} \pm \frac{1}{2}\vartheta^2(x_j)}{\vartheta(x_j)}. \qquad (10)$$

the interval $[0, T]$ is subdivided with equidistant points, i.e.,

$$x_j = (j + 1)\tau \qquad j = 0,\ 1,\ \ldots,\ n; \qquad (11)$$

$D(\cdot)$ is the initial discount function; and κ equals to $(1 + \tau K)^{-1}$ with K denoting the *cap rate*.

The variable ϑ in (10) is intimately related with the concrete multifactor Gaussian HJM model realization via the particular $[A,\ B,\ c]$ forward rate TSV selection:

$$\vartheta^2(x_j) = M(x_j) F(x_j) M'(x_j),$$

where $M(x_j)$ is the matrix

$$M(x_j) = cA^{-1} \left(e^{A(x_j+\tau)} - e^{Ax_j} \right),$$

and $F(\cdot)$ satisfies

$$F(\cdot) = \int_0^{\cdot} e^{-As} BB' e^{-A's} ds.$$

Although the inversion of the matrix A, the series expansion of e^{Ax}, reveals that M is not a singular matrix even for small values of parameter a. This result is also true for another Gaussian HJM models built from QE forward TSV families, because the matrix elements of A are, fortunately, polynomial functions of the model parameters. However, due to numerical instability of the calibration process, when $a \to 0$, an asymptotically equivalent expression for ϑ must be used.

The equations (9) and (10), also expres the effective influence of *ab initio* yield curve estimation on cap pricing[2].

[2] See Appendix

3 Calibration to Market Data Approaches

The calibration procedures can be described formally as follows. Let θ be the vector (α, β, a) of parameter values for the model under consideration. Assume that we have time series observations of the implicit volatilities, σ_i^B, of N caps, with different ATM *strikes*, K_i, and maturities T_i with $i = 1, \ldots, N$, here $N = 7$. Suppose, that at time $t = 0$ we are also equipped with the discount function estimation, $D(x)$, and that the market participants translate volatility quotes to cash quotes adopting *Black* framework. In doing so, they adopt the convention that K_i quantities must match forward swap rates of the interest rate swaps (IRS) with same reset periods that the i-th cap (these IRS starts its cashflows at $t = x_0 + \tau$ as the corresponding cap and have no cash value at $t = 0$):

$$K_i = \frac{D(\tau) - D(T_i)}{\tau \sum_{j=1}^{n} D(x_j)}, \tag{12}$$

where τ is the length of the underlying caplets, and $x_1 = 2\tau, \ldots, x_n = T_i$. The derivation of the formula (12) can be found, for example, in Björk [4] (Proposition 20.7 on page 313). Now, by inspection, it is clear that this market convention makes that K_i depends on the yield-curve estimation. It allows to us to denote market prices of caps with $C^*(T_i, D(x), K_i(D(x)), \sigma_i^B)$. This last expression emphasizes explicit and implicit dependence (through ATM *strikes*) on discount function estimation even for market prices. Let $C(T_i, D(x), K_i(D(x)), \theta)$ be the corresponding theoretical price under our particular model.

The Two-Step Traditional Method

First, we choose a non-consistent parametrized family of forward rate curves $G(z, x)$. Let $D(z, x)$ be the zero-coupon bond prices reported by $G(z, x)$. Let D_k^* be the corresponding discount factor observations on maturities x_k with $k = 1, \ldots, M = 11$. For each zero-coupon bond denoted with subscript k, the logarithmic pricing error[3] is written as follows

$$\epsilon_k(z) = \log D_k^* - \log D(z, x_k).$$

Then, we have chosen in this work the sum of squared logarithmic pricing errors, l_D, as the objective loss function to minimize:

$$l_D = \min_z \| \log D^* - \log D(z, x) \|^2 = \min_z \sum_{k=1}^{M} \epsilon_k^2(z). \tag{13}$$

[3]Recall that, for small ϵ_k, it is also the relative pricing error $\frac{D_k^* - D(z, x_k)}{D(z, x_k)}$.

Now, via the least squares estimators $\hat{z}$, an entire discount factor estimation allows the pricing of caps using market practice or a HJM model. Following a similar scheme for the derivatives fitting than the used at the bond side we have

$$\eta_i(\theta) = \log C_i^* - \log C(\theta, T_i).$$

and

$$l_C = \min_\theta \| \log C^* - \log C(\theta, T)\|^2 = \min_\theta \sum_{i=1}^{N} \eta_i^2(\theta), \tag{14}$$

where we have summarized dependencies for simplicity. Note that yield-curve estimation is external to the model in the sense that there is no need to know first any of the model parameters θ for solving non-linear program (13).

The Joint Calibration to Cap and Bond Prices

Let us now describe in detail the joint cap-bond calibration procedure which has sense in a consistent family framework. We note that in this situation the parameters of the model are determined together with the initial forward rate curve. A simplified derivation for the consistent families used can be found in the Appendix available at the end of this work.

This is different from the traditional fitting of HJM models, where the two steps are separate, as we discussed before. From the expression (21) in the Appendix, we notice the dependency of the family from the parameter a. Let $G(z, x, a)$ be a family consistent with our gaussian model (for instance, G_m and G_{ANS}) and define least-squares estimators, $\hat{z}(a)$

$$\hat{z}(a) = \arg\min_z \sum_{k=1}^{M} (\log D_k^* - \log D(z, x_k, a))^2. \tag{15}$$

From the expression

$$\log D(z, x_k, a) = -\int_0^{x_k} G(z, s, a)\, ds = \sum_{j=1}^{n_p} M_{kj}(a) z_j, \tag{16}$$

we note that, for consistent families and for a fixed a the problem (15) is linear in z-parameters (for the G_m family $n_p = 5$, and for the G_{ANS} family $n_p = 6$). Thus, $\hat{z}$ is an explicit and continuous function of $a \equiv \theta_3$. Strictly speaking, joint calibration of interest rate cap derivatives and bonds based upon a consistent family must be formalized as a multicriteria optimization problem (MOP):

$$\min_{\theta \in S} \boldsymbol{l}(\theta) = \begin{bmatrix} l_C(\theta) \\ l_D(\theta_3) \end{bmatrix}, \qquad \boldsymbol{l} : \mathcal{R}^3 \to \mathcal{R}^2$$

where

$$S = \{\theta : h(\theta) = 0, a \leq \theta \leq b\}$$

with

$$h(\theta) = h(\theta_3) = z - R(\theta_3)Q^{-1}(\theta_3)\log D^* \qquad h : \mathcal{R} \to \mathcal{R}^5$$

being Q, R matrices of the reduced QR decomposition of $M(\theta_3)$ which is defined by the relation (16) $a \in (\mathcal{R} \cup \{-\infty\})^3$ and $b \in (\mathcal{R} \cup \{\infty\})^3$, and *partial loss functions*, $l_i(\theta)$, defined as

$$\begin{aligned} l_C(\theta) &= \|\log C^*\left(D(z,\theta)\right) - \log\left(D(z,\theta),\theta\right)\|^2 \\ l_D(\theta) &= \|\log D^* - \log D(z,\theta)\|^2 \end{aligned}$$

Since no single θ^* would generally minimize every l_i simultaneously, we are dealing with Pareto optimality. A popular and rigorous method for finding the Pareto optimal points consists on minimizing a convex combination of the objectives, see for instance [15]:

$$\min_{\theta \in S} \boldsymbol{\lambda}^T \boldsymbol{l}(\theta) \tag{17}$$

with $\boldsymbol{\lambda} \in (\mathcal{R}_+ \cup \{0\})^2$ and $\lambda_C + \lambda_D = 1$.

This algorithm provides a whole collection of Pareto optimal points representative of the entire spectrum of efficient solutions by carrying out the optimization for different choices of $\boldsymbol{\lambda}$ as noted in [13]. Thus ideally, consistent calibration based upon consistent families involves the entire Pareto optimal set, in contrast to the uniqueness for the solution that appears in the two-step scalar problem.

At this point, note that the program used by Angelini and Herzel [1, 2] in their works, uses a different goal attainment

$$l = \min_{\theta} l_C(\theta) \tag{18}$$

where $l_C(\theta)$, and $\hat{z}(a)$ are defined trough the identities (14) and (15). As a consequence, the program used by these authors is a degenerate case of (17) with λ fixed equal to 1.

4 Empirical Results

We compare three different estimations of initial yield curve based on Nelson-Siegel family (henceforth NS), MC and ANS.

Our first objective is to test the stability of the implicit estimation of the model parameters θ. We consider mean, standard deviation and coefficient of variation of parameter estimates time series. In this context the main goal is to analyze the impact that an alternative interpolation scheme has on the fitting capabilities of the model. To this end, we use as a measure, the mean of the dayly sum of squared errors of derivatives log prices, hereafter l_C. The same measure is used for the zero-coupon bond prices (we denote it with l_D) and it is included in the analysis with the market data.

It consists of 150 dayly observations divided in two periods: first period covers from 1/1/2001 to 13/4/2001 (75 trading dates) and the second one starts in 15/3/2002 and finish on 27/6/2002 (75 trading dates). The data set is composed of US discount factors for thirteen maturities (3, 6 and 9 months and from 1 to 10 years) and of implied volatilities of at-the-money interest rate caps with maturities 1,2,3,4,5,7,10 years. This database is provided by Datastream Financial Service. The simulated data was obtained from 360 extractions from the model of bond and cap prices under identical contractual features.

Simulations

We simulate, departing from alternative initial conditions $r^*(x)$, the forward curve until the time t attainable by this model. We accomplished this by working out the expression (22), and writing the explicit formula for the stochastic as well as the deterministic coefficients which are actually variable in time evolution: the aforementioned $g_i(t)$, Z_t^i and the extra ones coming from initial curve translation, $r^*(x+t)$. Now, it is possible to compute the prices of a set of zero-coupon bonds using exact integration of $r_t(x)$ over cross-sectional variable x at a fixed time t, and then, the prices of the seven caps with formula (9).

The fixed model parameters, $\theta = (0.002, 0.007, 0.35)$, have been taken. This particular choice has similar order of magnitude as typical empirical estimations for the model reported by Angelini and Herzel [2]. As alternative initial curves, we choose MC, ANS and NS fitted to the zero coupon bond prices shown in Figure 1.

Starting from the initial fitted curves, which may be denoted with $r_m^*(x)$, $r_{ANS}^*(x)$ and $r_{NS}^*(x)$, and according to (8), the corresponding three different model evolutions are calibrated to MC, ANS and NS. In order to make calibration results more comparable, Monte Carlo simulations are built in from identical random sequence $(Z_t^1,\ Z_t^2)$ in all three cases. In addition, the results reported are restricted to one particular Pareto point, with weights $(1,0)$[4]. Following the expression (8), it is easy to observe that there are two consistent families, G_m and G_{ANS}, for the first simulation E1, just one, G_{ANS}, for the second simulation E2, and no one for the last simulation E3. Figure 2 shows main consequences of the theory when

[4]Numerical conclusions reported by the three experiments do not vary due to the effective scale of the Pareto frontier.

the model is the *truth* model. Notice that perfect calibration just occurs, although model parameters are fixed *a priory*, when the used family to perform calibrations is consistent with all the future forward curves generated from initial curve $r^*(x)$. This fact explains, for instance, bad performance for the NS family even on E3 experiment. Moreover, parameter instability and imprecision that produces an incorrect yield-curve selection can also be checked in Figure 3.

Real Data

The objective of this section is to compare the performance of the two different calibration approaches on two different periods of real trading dates. Thus, from now on we will only consider the calibration results obtained with the US market data. Calibration with consistent families, as it have been explained before, must be properly carried out by setting several combinations of the weight vector $\boldsymbol{\lambda}$ on a date-by-date basis. Figure 5 shows the comparative results in the objective space (l_C, l_D) for two particular and representative dates on both subsamples mentioned at the beginning of this section. We have chosen in doing this the set of vectors with second component $\lambda_D = 0,\ 0.05,\ \ldots,\ 0.95$.

The table on Figure 6 exhibits the sample mean of the dayly error fitting measures, namely l_C and l_D, and the mean and the coefficient of variation of parameter estimates. On the other hand, Figure 7 shows in-sample fitting time series. For the shake of simplicity we restrict the results shown to the Pareto point $(0.05, 0.95)$ because this particular choice performs the best results for the consistent calibrations on the bond side with a reasonable trade-off within the derivative side.

The two consistent families under study report better in-sample fitting results when dealing with bond data. However, on the derivatives side calibration, only the ANS family performs similarly to the NS one in the two periods. This fact may be motivated by the extra factor, z_1, common for the families G_{ANS} and G_{NS}, which is independent of zero-coupon bond maturities and responsible that these families fit better observed short-term discount factors than G_m family (note that this is not incompatible with the better summary l_D reported in this sample by the minimal family when compared against the Nelson-Siegel ones).

Euro-market results reported in Angelini and Herzel [2] for the same model, shows a slightly better performance for the minimal consistent family when compared against Nelson-Siegel family, even on cap prices[5].

This empirical fact can be explained by the existing differences within the short-term TSIR between Euro and US-market on both excited periods

[5] It have to be remarked that the program used by these authors does not take into account vector nature of this optimization problem.

taken into consideration.

5 Conclusions

When calibrating a HJM model, a TSIR curve choice to fit a few market data observations is needed. In particular it seems to be natural to use families of curves which do not modify their structure under the future evolution of the model, the so-called consistent families.

In this work, we choose three families of curves (two consistent families and the popular Nelson-Siegel family) and we conclude that this choice have an effective impact on the quality of in-sample fitting as well as parameter estimates on both simulated and US-market data.

When using simulated data it is very clear that the consistent families for the E1 ans E2 experiments performs much better than the non-consistent ones. Moreover, Nelson-Siegel family does not work even if it is chosen as the starting yield-curve (recall E3 experiment). These empirical facts constitute a nice demonstration of the theory, in the sense that even on absence of *model risk* only when consistent families are used, perfect calibration may occur.

Translation of these consequences to real data is less clear, due to *model risk* and quality of data, but we can infer similar encouraging concluding remarks. In this case, the introduction of a sufficiently rich *consistent families*, MC as well as ANS, well motivated theoretically by Björk et al., improves in-sample fitting capabilities on bonds. However, consistent families leads to somewhat stable parameter estimates and worse in-sample derivatives fitting results than the NS family, this could be an insight that consistent families may exhibit undesired asymptotic features in different markets. In this sense, this work complements the empiral findings shown in Angelini and Herzel [1, 2] analyzing another data sets. On the other hand, note that the vector extension to the consistent calibration presents more general features. Such extension is structured to allow more numerical outcomes. According to our experience with other databases, this leads, in general, to better results also in derivatives calibration as compared to non-extended consistent calibration and non-consistent methodologies. Thus, comparative studies between the fitting of short-term zero-coupon bond capabilities and its consequences on cap pricing performance for several consistent families with a particular model and on different market basis (for instance, using Euro-market inputs as well as US-market data) should be analyzed deeply in the future. Moreover, we restrict our studies to a flexible one-factor Gaussian HJM model. Then, future empirical research on the matter should include multi-factor models for capturing more appropriately the TSIR and TSV observed in the market.

Another technical point regards the adaptation of the *Normal Boundary Intersection* (NBI) method to use it in the calibration problems that usually appear in the private and public financial institutions. As is mentioned by Das and Dennis in [13], NBI method surpass in flexibility as well as efficiency the popular method of minimizing weighted combinations of objective functions.

Figure 1: Discrete data for initial yield-curve estimation.

Maturity, x	0.25	1	2	3	4	
Discount Factor, $D^*(x)$	0.9886	0.9538	0.9069	0.8602	0.8142	
Maturity, x	5	6	7	8	9	10
Discount Factor, $D^*(x)$	0.7693	0.7260	0.6843	0.6445	0.6066	0.5706

Figure 2: Summary statistics for calibration results with simulated data.

		MC	ANS	NS
E1: $r_0(x) = r^*_m(x)$	$\varepsilon_r(\alpha)$	0	0	0.23
	$\varepsilon_r(\beta)$	0	0	0.13
	$\varepsilon_r(a)$	0	0	$8.7\ 10^{-2}$
	$C_v(\alpha)$	0	0	0.18
	$C_v(\beta)$	0	0	0.14
	$C_v(a)$	0	0	$9.7\ 10^{-2}$
	l_C	0	0	$1.9\ 10^{-3}$
E2: $r_0(x) = r^*_{ANS}(x)$	$\varepsilon_r(\alpha)$	0.25	0	0.28
	$\varepsilon_r(\beta)$	0.16	0	0.16
	$\varepsilon_r(a)$	0.12	0	$9.5\ 10^{-2}$
	$C_v(\alpha)$	$3.8\ 10^{-2}$	0	0.117
	$C_v(\beta)$	$3.9\ 10^{-2}$	0	$9.1\ 10^{-2}$
	$C_v(a)$	$3.2\ 10^{-2}$	0	$4.8\ 10^{-2}$
	l_C	$2.6\ 10^{-4}$	0	$6.7\ 10^{-4}$
E3: $r_0(x) = r^*_{NS}(x)$	$\varepsilon_r(\alpha)$	0.313	$2.7\ 10^{-4}$	0.18
	$\varepsilon_r(\beta)$	0.20	$2.10\ 10^{-4}$	0.10
	$\varepsilon_r(a)$	0.16	$1.6\ 10^{-5}$	$6.7\ 10^{-2}$
	$C_v(\alpha)$	$2.3\ 10^{-2}$	$1.4\ 10^{-4}$	0.17
	$C_v(\beta)$	$2.6\ 10^{-2}$	$1.0\ 10^{-4}$	0.111
	$C_v(a)$	$2.2\ 10^{-2}$	$8.3\ 10^{-5}$	$6.3\ 10^{-2}$
	l_C	$3.8\ 10^{-4}$	$3.9\ 10^{-9}$	$3.5\ 10^{-4}$

Sample statistics of the calibration on simulated data. Relative errors of the parameters estimates are expressed in absolute value. We set to 0 table entries with value $< 10^3 \cdot$`eps` (variable `eps` $\sim 10^{-16}$ measures MATLAB internal accuracy).

Figure 3: a and α estimates from Monte Carlo run E2, where $r_0(x) = r^*_{ANS}(x)$.

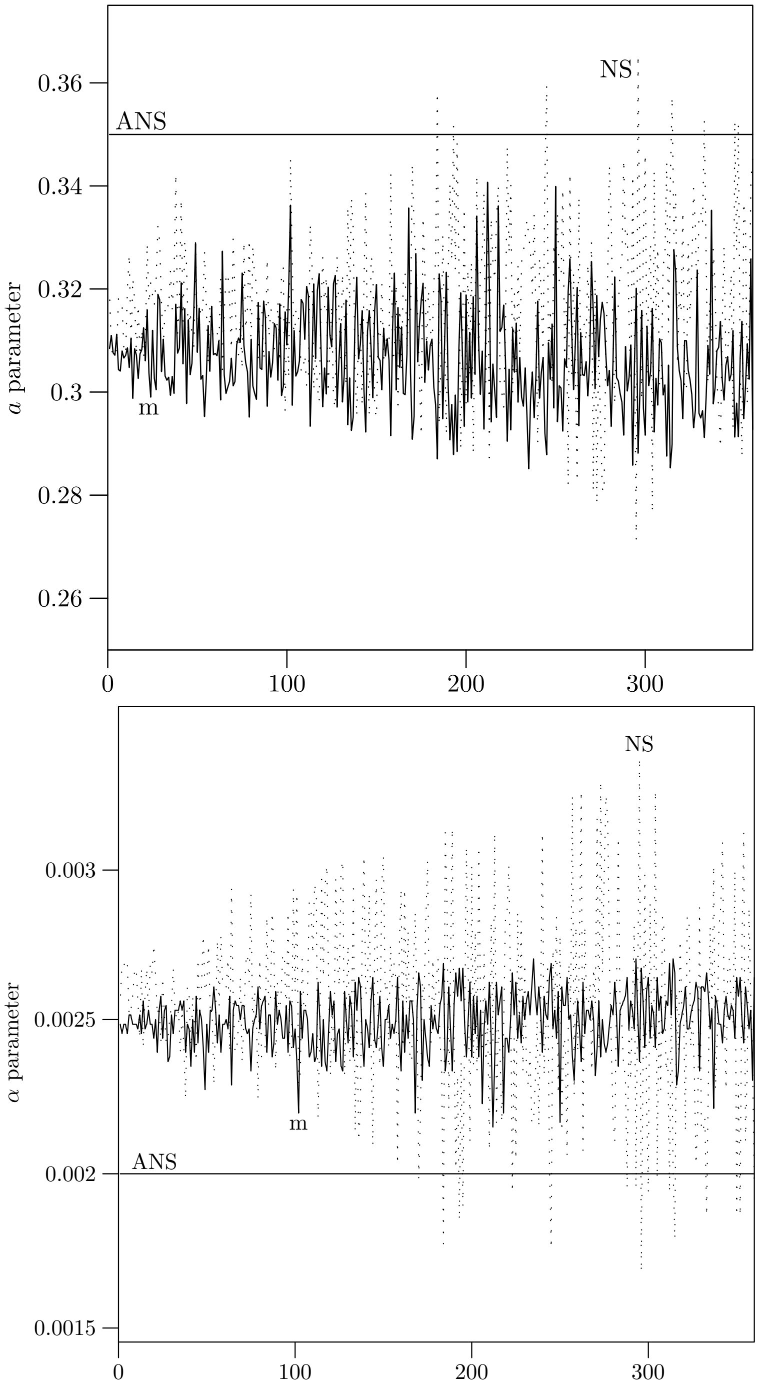


Figure 4: Market TSIR and TSV data in the two different market scenarios.

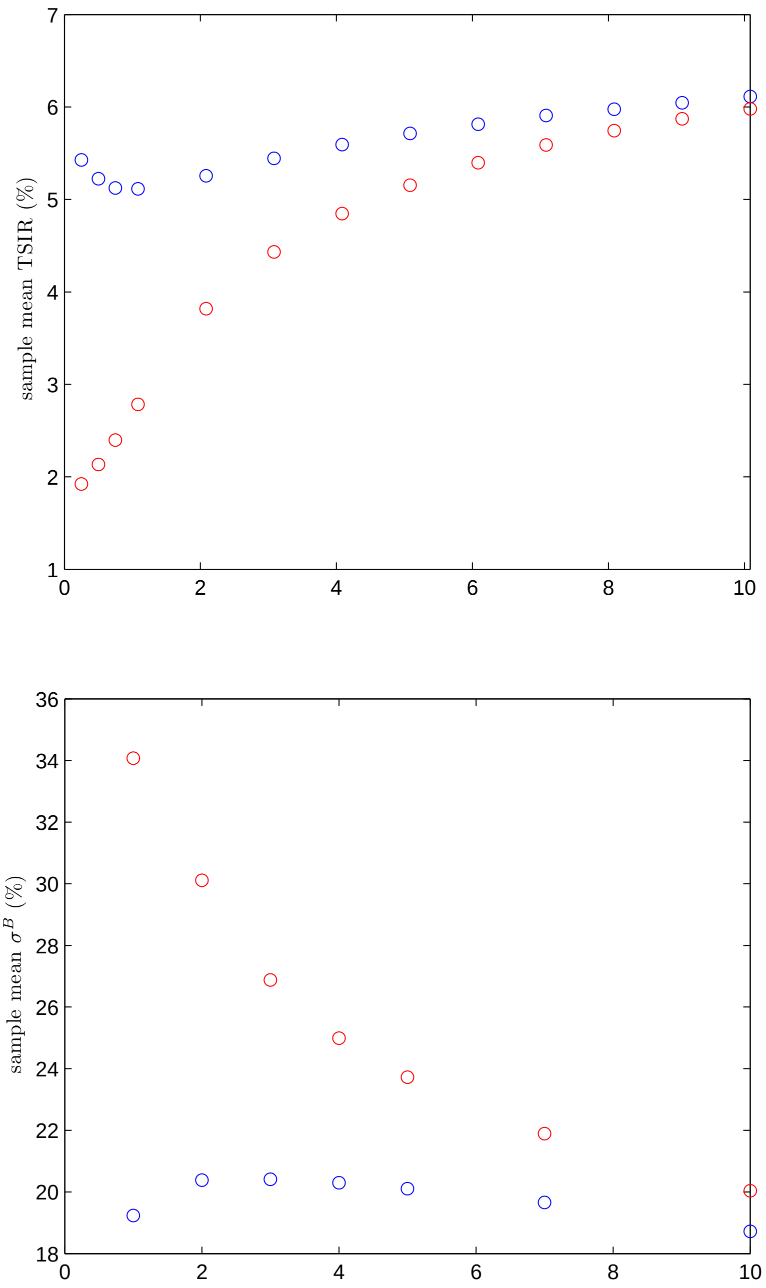


Figure 5: Calibration results for the minimal consistent (crosses), augmented Nelson-Siegel (circles) and Nelson-Siegel (squares) families in the Day 1 (top) and the Day 2 (bottom). On the right column, a zoom of the the highlighted regions reveals the scale of the Pareto boundaries.

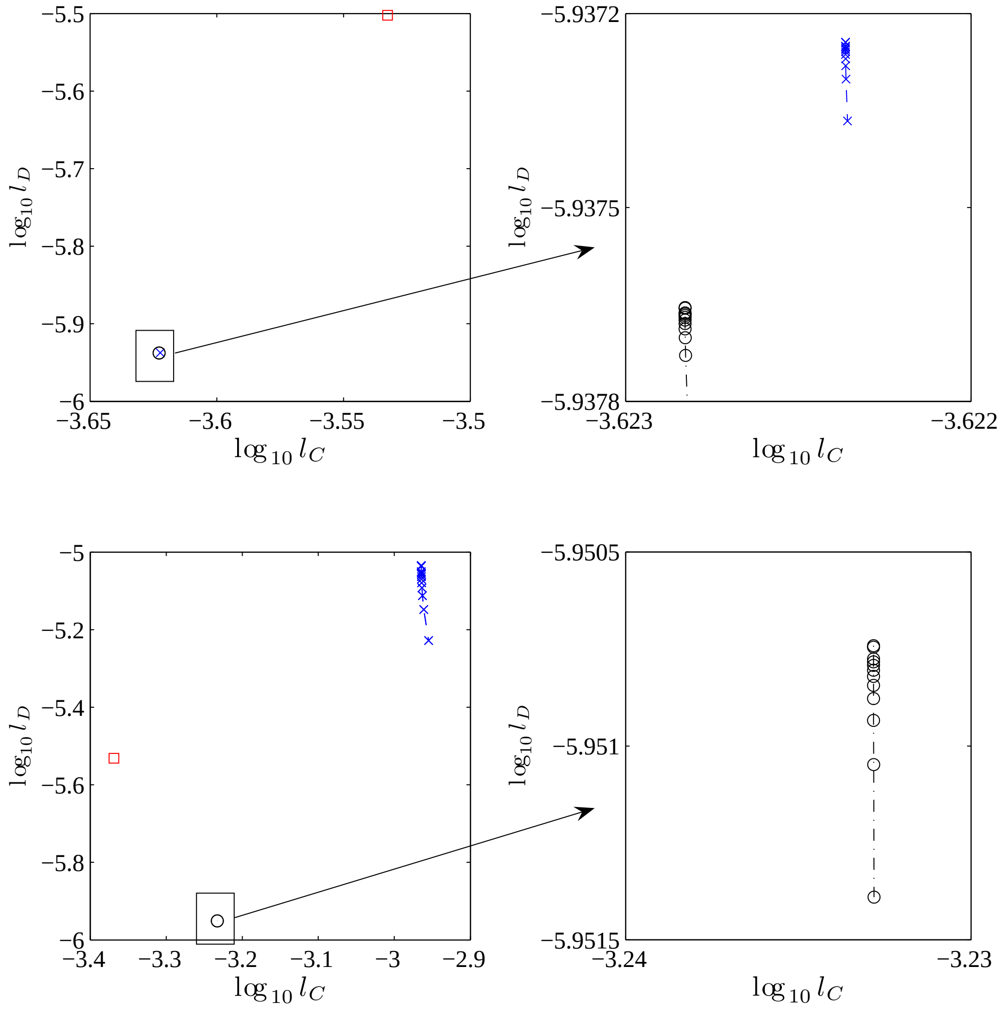
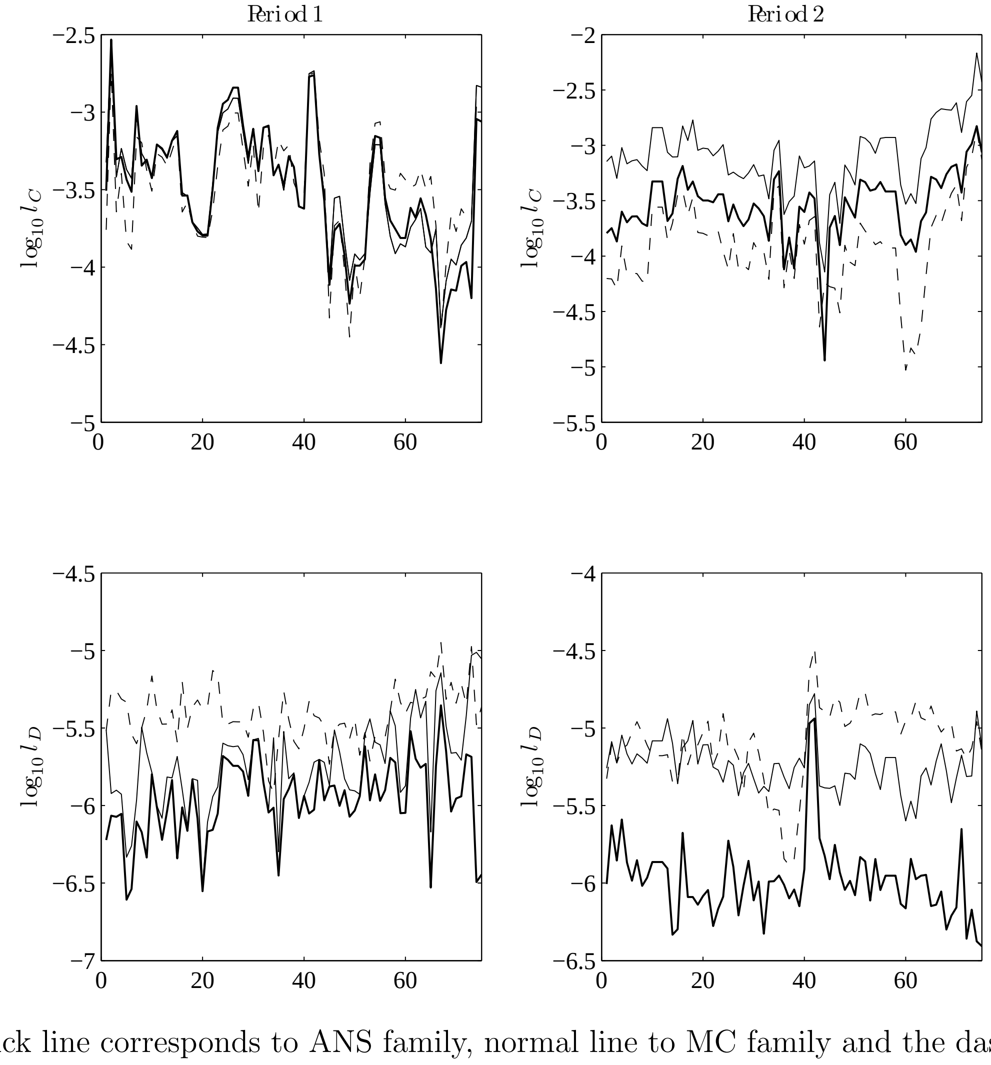


Figure 6: Summary statistics for the calibration results with real data on both periods. In-sample descriptive statistics are carried out using the particular Pareto point $\boldsymbol{\lambda} = (0.05, 0.95)$.

		MC	ANS	NS
PERIOD 1	α	0.0084	0.0085	0.0087
	β	0.0055	0.0055	0.0053
	a	0.23	0.23	0.23
	$C_v(\alpha)$	0.12	0.114	0.11
	$C_v(\beta)$	0.31	0.29	0.29
	$C_v(a)$	0.24	0.23	0.23
	l_C	$4.9\ 10^{-4}$	$4.8\ 10^{-4}$	$4.5\ 10^{-4}$
	l_D	$2.4\ 10^{-6}$	$1.3\ 10^{-6}$	$4.17\ 10^{-6}$
PERIOD 2	α	0.0076	0.0073	0.0079
	β	0.0092	0.011	0.0095
	a	0.31	0.36	0.33
	$C_v(\alpha)$	0.10	0.14	0.09
	$C_v(\beta)$	0.11	0.20	0.17
	$C_v(a)$	0.04	0.11	0.10
	l_C	0.001	$3.5\ 10^{-4}$	$1.9\ 10^{-4}$
	l_D	$6.15\ 10^{-6}$	$1.3\ 10^{-6}$	$9.3\ 10^{-6}$

Figure 7: In-sample fitting time series for the first period (left) and the second period (right) in logarithmic terms.



Thick line corresponds to ANS family, normal line to MC family and the dashed one to the Nelson-Siegel family. Pareto point with $\lambda_D = 0.95$ is used for drawing the consistent families lines.

Appendix

Consistent Curves with Gaussian Models

If we want to measure the actual impact that alternative choices to the Nelson-Siegel yield curve interpolating approach produces on derivatives pricing and hedging, we need to determine consistent families for this particular model. The fundamental results can be found in Björk and Christensen [8] in more detail. We adapt some of them to our Gaussian case study without further technical discussion for the general case.

Definition 1 *Consider the space $\mathcal{H}$ is defined as the space of all $\mathcal{C}^{\infty}$-functions,*

$$r : \mathcal{R}_{+} \rightarrow \mathcal{R}$$

satisfying the norm condition:

$$||r||^2 = \sum_{n=0}^{\infty} 2^{-n} \int_0^{\infty} \left(\frac{d^n r}{dx^n}(x) \right)^2 e^{-\gamma x} \, dx < \infty$$

where γ is a fixed positive real number.

As proved by Björk and Landen [9], this space $\mathcal{H}$ is a Hilbert space.

Theorem 1 *Consider as given the mapping*

$$G : \mathcal{Z} \rightarrow \mathcal{H}$$

where the parameter space $\mathcal{Z}$ is an open connected subset of R^d, $\mathcal{H}$ a Hilbert space and the forward curve manifold $\mathcal{G} \subseteq H$ is defined as $\mathcal{G} = Im(G)$. The family $\mathcal{G}$ is consistent with the one-factor model $\mathcal{M}$ with deterministic volatility function $\sigma(\cdot)$, if and only if

$$\partial_x G(z,x) + \sigma(x) \int_0^x \sigma(s) ds \quad \in \quad \mathrm{Im}\left[\partial_z G(z,x)\right], \tag{19}$$

$$\sigma(x) \quad \in \quad \mathrm{Im}\left[\partial_z G(z,x)\right], \tag{20}$$

for all $z \in Z$.

The statements 12 and 13 are called, respectively, *the consistent drift* and *the consistent volatility* conditions. These are easy to apply in concrete cases as shown Björk and Christensen [8] or De Rossi [14], among others. For the particular one-factor model we consider along this work, Proposition 7.2 and 7.3 in Björk and Christensen [8] may be directly applied to get the useful result:

Proposition 1 *The family*

$$G_m(z,x) = (z_1 + z_2 x)e^{-ax} + (z_3 + z_4 x + z_5 x^2)e^{-2ax}, \tag{21}$$

is the minimal dimension consistent family with the model characterized by $\sigma(x) = (\alpha + \beta x)e^{-ax}$.

Moreover, it should be also noted that *augmented* families related from the (21) can be constructed by adding to G_m an arbitrary function ϕ, that is, the map

$$G(z,x) = G_m(z,x) + \phi(z,x),$$

is also consistent with this model.

There is an alternative way to justify (21) focusing on forward rate evolution deduced at (8), and to get an insight on how the Monte-Carlo procedure is implemented, we describe it next. By the definition of $S(x)$, we have that $S'(x) = \sigma(x)$. Then it is easy to derive that deterministic term $\frac{1}{2}\left[S^2(t+x) - S^2(x)\right]$ is of the form

$$g_1(t)e^{-2ax} + g_2(t)xe^{-2ax} + g_3(t)x^2e^{-2ax} + h_1(t)e^{-ax} + h_2(t)xe^{-2ax}.$$

On the other hand, the explicit expansion of stochastic term $C(x)Z_t$

$$ce^{Ax}\begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix} = e^{-ax}\left[\alpha \ \beta - a\alpha\right]\begin{bmatrix} 1+ax & -a^2x \\ x & 1-ax \end{bmatrix}\begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix}$$

$$= e^{-ax}\left(\alpha Z_t^1 - a\alpha Z_t^2 + \beta Z_t^2\right) + xe^{-ax}\left(\beta Z_t^1 - a\beta Z_t^2\right),$$

and the forward rate evolution becomes

$$\begin{aligned} r_t(x) = r^*(x+t) + g_1(t)e^{-2ax} + g_2(t)xe^{-2ax} + g_3(t)x^2e^{-2ax} + \\ \left(h_1(t) + \alpha Z_t^1 - a\alpha Z_t^2 + \beta Z_t^2\right)e^{-ax} + \left(h_2(t) + \beta Z_t^1 - a\beta Z_t^2\right)xe^{-ax}. \end{aligned} \tag{22}$$

From (22) we see that a family which is invariant under time translation is consistent with the model if and only if it contains the linear space $\{e^{-ax}, xe^{-ax}, e^{-2ax}, xe^{-2ax}, x^2e^{-2ax}\}$. Consequently, to make a consistent version of a translation invariant family $\phi(z,x)$ it is enough to add $G_m(z,x)$.

The following concluding remarks about the families used along this work should now be clear:

- The Nelson-Siegel family (henceforth NS)

  $$G_{NS}(z,x) = z_1 + z_2 e^{-z_4 x} + z_3 x e^{-z_4 x},$$

  is not consistent with the model.

- The family

$$G_m(z,x) = (z_1 + z_2 x)e^{-ax} + (z_3 + z_4 x + z_5 x^2)e^{-2ax},$$

  it is the lowest dimension family consistent with the model (hereafter MC).

- The family

$$G_{ANS}(z,x) = z_1 + z_2 e^{-ax} + z_3 x e^{-ax} + (z_4 + z_5 x + z_6 x^2)e^{-2ax},$$

  is the simplest adjustment based on restricted NS family that allows model consistency (hereafter ANS).

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