

Some versions of the regular polygon problem of $n + 1$ bodies

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Abstract

During the last decade special attention has been paid to a problem that is known as the regular polygon problem of $(N + 1)$ bodies or, simply, the ring problem and describes the motion of a small body under the action of N much bigger bodies that are called primaries. $(N - 1)$ of these primaries are arranged in equal distances at the vertices of a regular polygon and have equal masses m . Another primary with different mass m_0 is located at the center of mass of the configuration. Here we present a survey of the research work that has been done on some versions of the problem by summarizing the main results.

Key words: ring problem, regular polygon configurations, particle motion, periodic orbits, stationary solutions, gyrostats, rigid body dynamics, effect of radiation pressure.

1 Introduction

The problem deals with the motion of a small body S (natural or artificial) in the force field of a system that consists of many bodies P_i , $i = 1, 2, \dots, \nu$, ($\nu = N - 1$) located at the vertices of a regular ν -gon and a central body P_0 that rests on the center of mass of the system (Scheeres, [11]; Kalvouridis, [5], [6]; Arribas and Elipe, [1]; Pinotsis, [9])(Fig.1). We assume that the primaries rotate around their center of mass with constant angular velocity. In the pure gravitational case, two parameters determine the dynamic behavior of the system: the mass parameter $\beta = m_0/m$ and the number ν of the peripheral primaries. The model has application to satellite

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motion in the neighborhood of a planetary ring or near a planetary system that consists of a central "sun" and some co-orbital planets. The importance of the above problem is also reinforced by the fact that it is reducible to a number of known problems of Celestial Mechanics by a simple alteration of the values of its two parameters. Apart from the original model, some other versions of the problem have recently appeared, aiming to improve it and to make it more realistic without destroying its simplicity.

2 Basic features of the gravitational version

2.1 Equations of motion and equilibrium positions

The motion of the particle is described in a synodic coordinate system $Oxyz$ by means of the dimensionless equations,

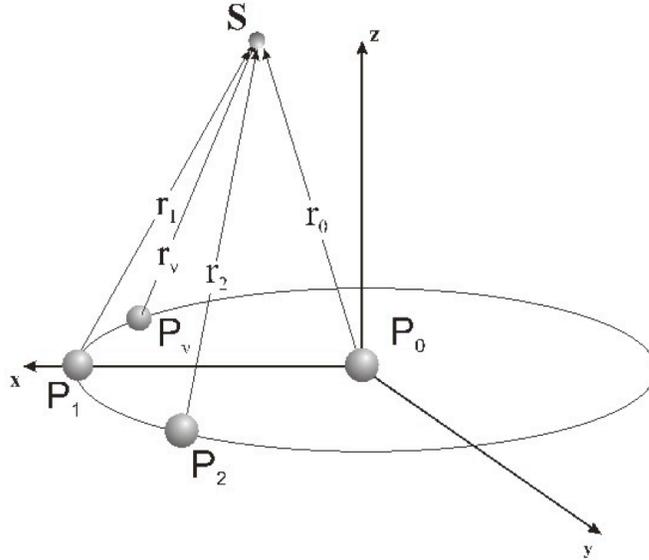


Figure 1: The three-dimensional ring problem of $n + 1$ bodies.

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}, \quad \ddot{z} = \frac{\partial U}{\partial z}$$

$$U = \frac{1}{2}(x^2 + y^2) + \frac{1}{\Delta} \left[\frac{\beta}{r_0} + \sum_{i=1}^{\nu} \frac{1}{r_i} \right]$$

$$r_0 = \sqrt{x^2 + y^2 + z^2}, \quad r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + z^2}$$

$$x_i = \frac{\cos[2(i-1)\theta]}{2 \sin \theta}, \quad y_i = \frac{\sin[2(i-1)\theta]}{2 \sin \theta}$$

$$\Delta = M(\Lambda + \beta M^2), \quad \Lambda = \sum_{i=2}^{\nu} \frac{\sin^2 \theta}{\sin(1-i)\theta}, \quad M = 2 \sin \theta, \quad \theta = \frac{\pi}{\nu}$$

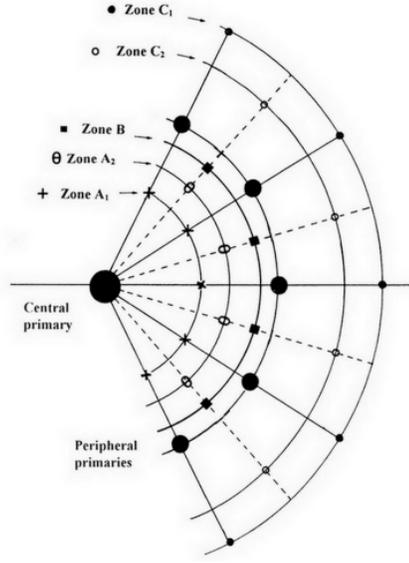


Figure 2: Equilibrium zones in the ring configuration.

In general the equilibrium positions are arranged in five imaginary coplanar circles that are centered at the origin of the synodic system. They are named equilibrium zones and are symbolized by A_1 , A_2 , B , C_2 , and C_1 as they appear from the center outwards. Their number depends on β :

- If $0 < \beta < l_\nu$, where l_ν is a bifurcation point, different for each ν , all the equilibrium zones exist. The number of the equilibria is 5ν in this case.
- When $\beta > l_\nu$, only $1, C_2$ and C_1 exist. The number of the equilibria is 3ν .

These zones are pushed toward the imaginary circle of the primaries as parameter β increases.

2.2 A property of the zero-velocity curves

All the zvc curves drawn in the (x_0, C) diagram for a particular ν and for any value of the mass parameter β , pass through two different focal points, which are symmetric with respect to the C axis when ν is even, and non-symmetric when ν is odd (Kalvouridis, [8]).

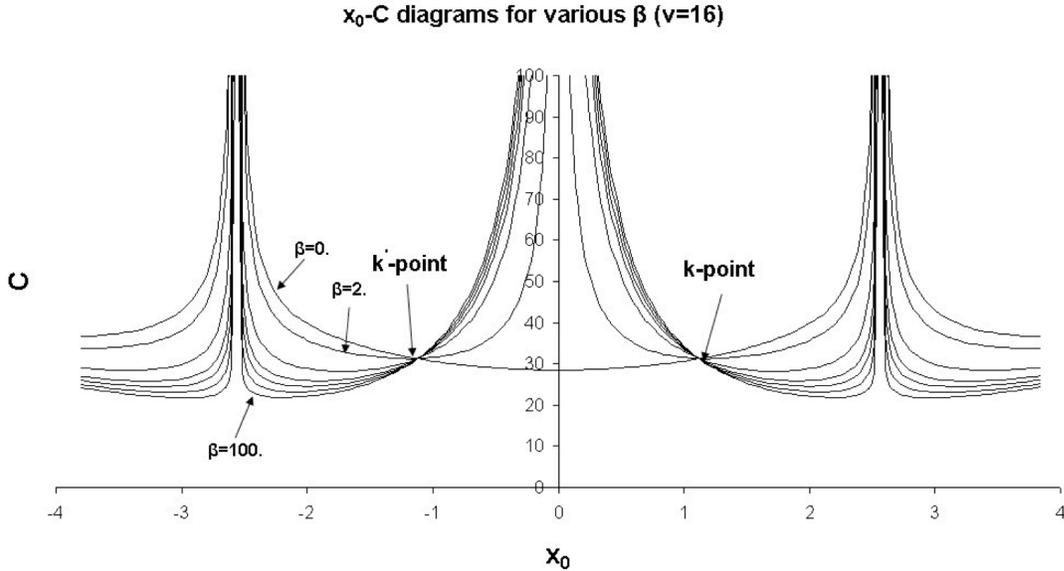


Figure 3: Common intersection points of the zero-velocity curves in the diagram (x_0, C) in a ring system with $\nu = 16$ and $\beta = 2$.

Propositions

- All the 3-D surfaces $C = C(x, y; \beta)$ intersect along a unique curve,

$$\sum_{i=1}^{\nu} \frac{r_0}{r_i} = \frac{\Lambda}{M^2}$$

regardless the value of the parameter β .

- In all cases, the common curve is on the central funnel.

2.3 Planar Periodic orbits. Effect of the mass parameter

Periodic orbits form families according to their multiplicity and their particular characteristics. The planar symmetric ones constitute curves in the x_0 - C diagram,

that are usually called characteristic curves. Mass parameter plays a very important role in the evolution of the families, the particular characteristics of the orbits and their stability (Psaros and Kalvouridis, [10]). Figures 4 and 5 depict this effect and the main conclusions are summarized in the comments that follow.

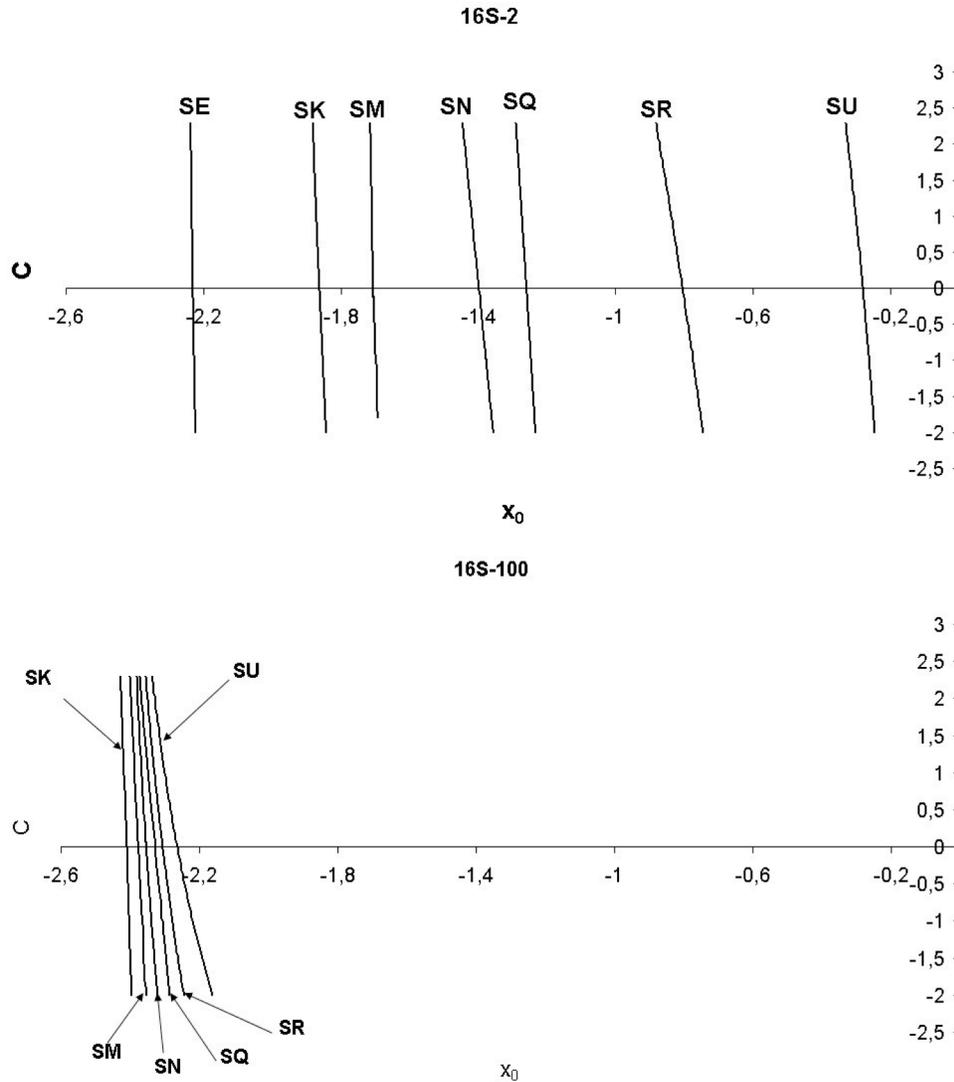


Figure 4: Evolution of families of simple periodic orbits for $\beta = 2$ and $\beta = 100$.

Comments

- The characteristic curves shift toward the boundary circle of the primaries as parameter β increases and at the same time, they crowd in such a way that they form a dense bunch. When $\beta > 100.0$, the distances among them, become very small.

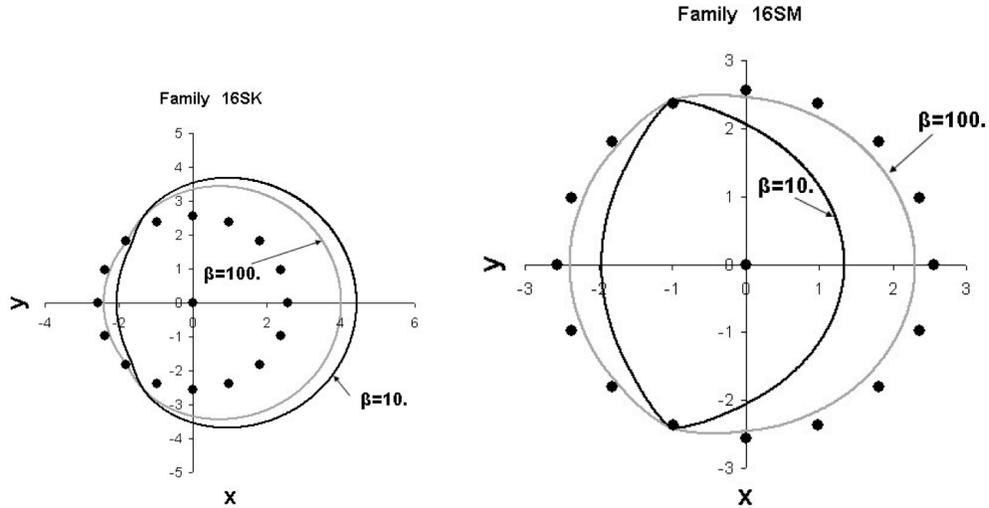


Figure 5: Orbits of families $S16K$ and $S16M$ having the same Jacobian constant and drawn for $\beta = 10$ (black) and $\beta = 100$ (gray).

- A slight bending appears in curves for large values of β (Fig.4). This bending is particularly obvious in curves that lie near the central primary.
- As β increases, the part of the orbits described outside the imaginary ring of the peripheral primaries shrinks while the one described inside the ring extends. As a consequence of this, simple periodic orbits tend to become circular. The same tendency is observed for both loops of double periodic orbits (Fig.5).
- The periods of the orbits of the same Jacobian constant increase while the absolute values of the velocities at $t = T/2$ decrease.
- The state of stability of the orbits does not change in the considered range of values of the mass parameter.

2.4 Three-dimensional periodic orbits

2.4.1 VERTICAL CRITICAL POINTS ON THE CHARACTERISTIC CURVES OF THE PLANAR SYMMETRIC PERIODIC FAMILIES

Families of three-dimensional symmetric periodic orbits are created from bifurcation points of families of planar symmetric periodic orbits. Starting from these points, a numerical investigation was carried out in the three-dimensional space and several three-dimensional families of symmetric periodic orbits were located (Had-

jifotinou and Kalvouridis, [2]; Hadjifotinou, et al., [3]). The above study was done for $\nu = 7$ and taking $\beta = 2$, that is, considering a central primary with mass double of the mass of a peripheral primary.

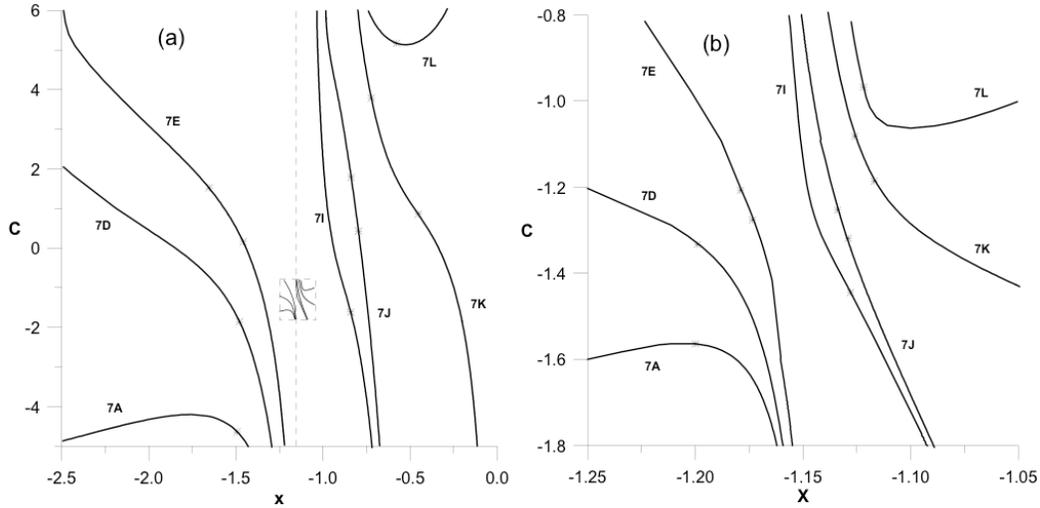


Figure 6: Families of periodic orbits of the planar ring problem for $\nu = 7$. Stars: bifurcation points from the planar to the three-dimensional families of periodic orbits. (a) $\beta = 2$. The dotted line shows the discontinuity at $x = -1.15$. In the small dotted square, we have included the plot (b), in order to show the difference in scale. (b) $\beta = 1000$

3 The photo-gravitational version

The existence of strong radiating sources in the universe led many scientists, to consider the effect of radiation pressure as well as gravitational forces. Radiation plays an important role in particle dynamics and in several cases may produce significant deviations from purely gravitational behavior. Here we present some of the results obtained by investigating the consequences of radiation on the particle dynamics in a regular polygon model of N bodies (Kalvouridis, [7]; Kazazakis and Kalvouridis, [4]).

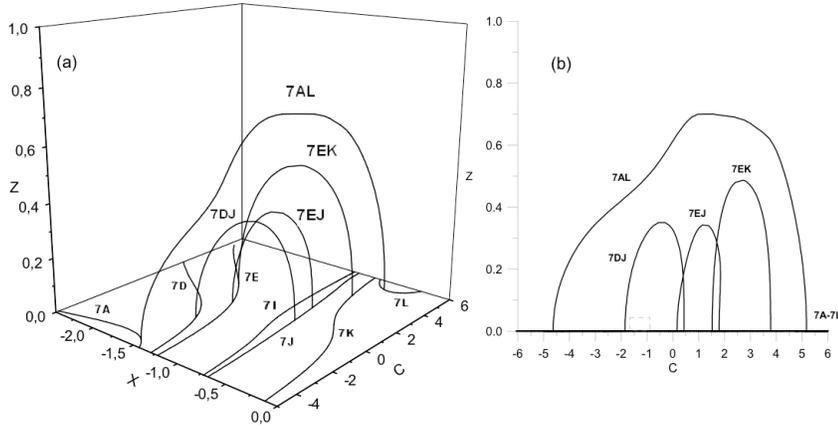


Figure 7: (a) The families $7AL, 7DJ, 7EJ$ and $7EK$ of the three-dimensional ring problem for $\nu = 7, \beta = 2$ and their bifurcation from the families $7A - 7L$ of the planar problem. (b) The same figure in the Cz -projection. The little dotted square shows the dimensions of the same figure for $\beta = 1000$.

3.1 Potential function

The potential function when some or all primaries are radiation sources, takes the form,

$$U(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{\Delta} \left[\frac{\beta q_0}{r_0} + \sum_{i=1}^n \nu \frac{q_i}{r_i} \right]$$

where $q_i = 1 - b_i$ and $b = \frac{F_r}{F_g}$

are the radiation coefficients (i.e ratios of the radiation force to the gravitational one), which depend on the luminosity and the mass of the emitting sources as well as on the physical, geometrical and chemical properties of the grains.

3.2 Effect of radiation on the stationary solutions

Remarks

- As the number of radiation sources in the system increases, the equilibrium points belonging to various sets are generally reduced.
- In all the examined cases and for the values of the parameters used, the stability of the equilibrium points does not change.

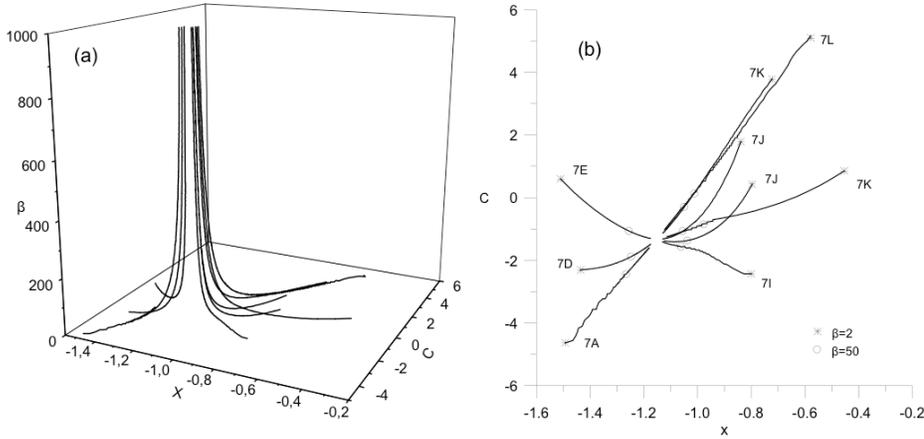


Figure 8: β -parametric families of bifurcation points from two-dimensional to three-dimensional symmetric periodic orbits for $\nu = 7$. The bifurcation points accumulate at the center of the drawing (around $x = -1.15$) as β increases beyond $\beta = 50$.

- The value for the Jacobian constant of a particular equilibrium point reduces as new radiation sources are added to the system. Its distance from the center of the system is also reduced.

3.3 Effect of radiation on the particle periodic motions

Radiation pressure which acts on small particles i.e. interplanetary or interstellar dust, should not be generally ignored. The consequences of this action may be significant when the bodies are strong radiation emitters ($b, b_0 > 0.1$). In this case, small changes are observed on the evolution of the particle orbits and their measures. For a wide range of values of the above coefficients, radiation doesn't really affect the state of stability found in the purely gravitational case.

4 Version where the small body is a gyrostat

As it is known, gyrostat is a mechanical system that consists of a platform and a number of rotors that are rigidly attached to the carrier. Each rotor is spinning independently about an axis fixed on the platform and its motion does not modify the mass distribution of the gyrostat. The platform may rotate about an inertial reference frame, so that a gyrostat is generally characterized by $n+1$ angular velocities, the n of which are the angular velocities of the spinning rotors relative to the platform. Sometimes, in the relevant literature, a gyrostat is also referred to as

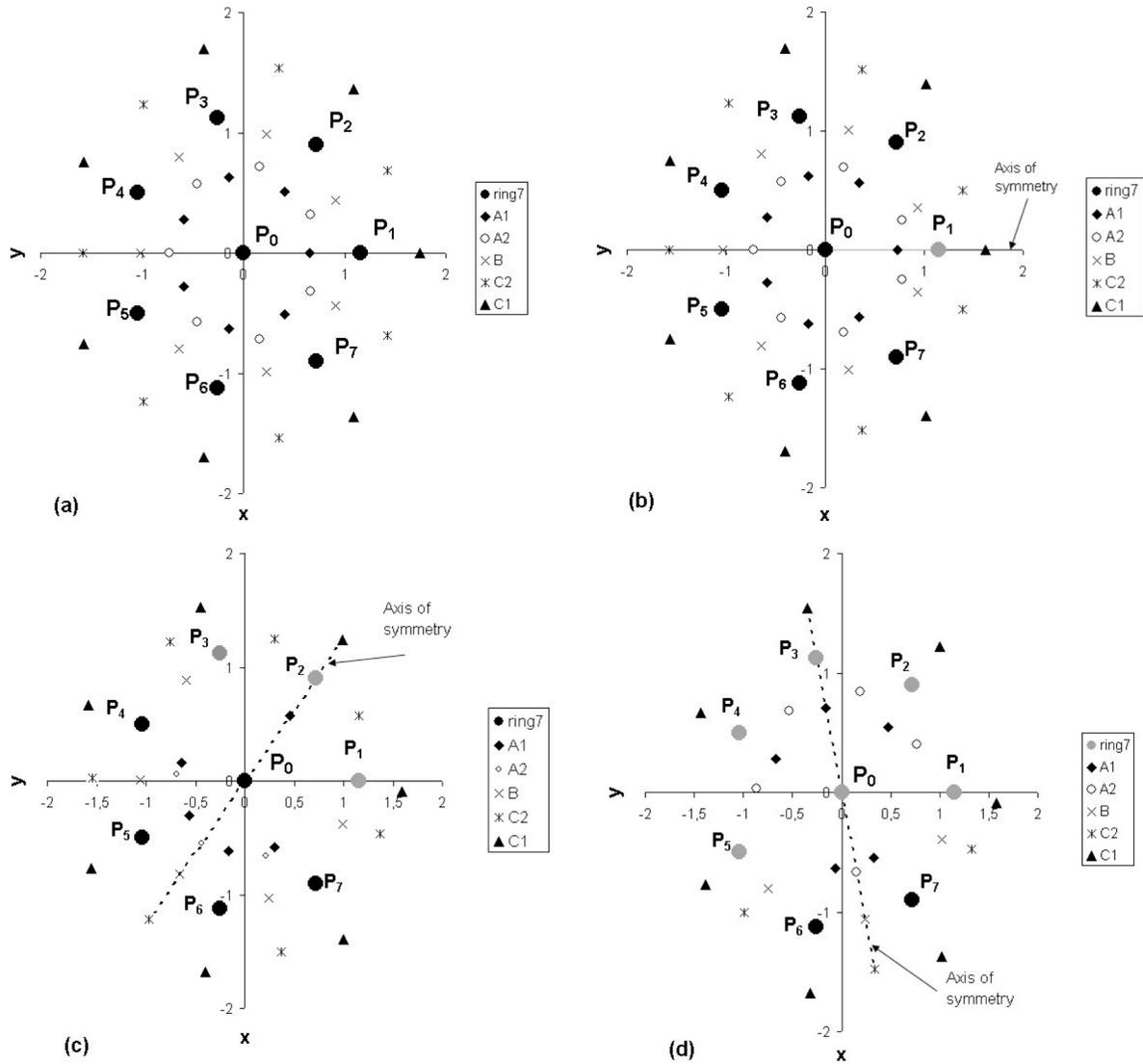


Figure 9: Distribution of equilibrium points in various cases of the photogravitational version when $\nu = 7$ and $\beta = 2$. (a)Gravitational case. (b)Primary P_1 radiates. (c)Primaries P_1, P_2 and P_3 radiate. (d)Primaries P_1, P_2, P_3, P_4 and P_5 radiate.

a 'dual-spin' body. Here we present a version of the ring problem where the small body S is a gyrostat (Tsogas, Kalvouridis and Mavraganis, [12]).

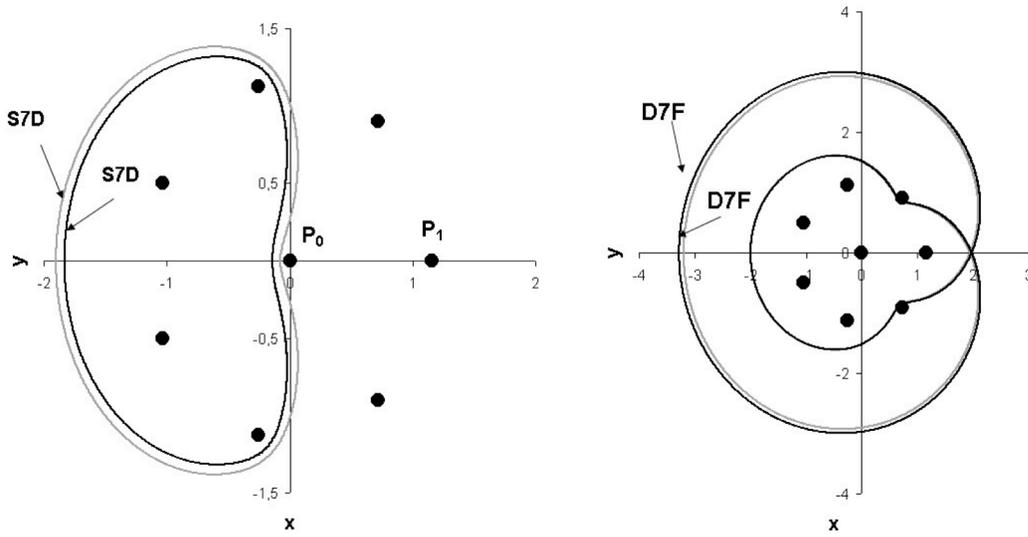


Figure 10: Simple (left) and double (right) periodic orbits in the case where the central body radiates. The unperturbed (black) and the perturbed (gray) orbits have the same Jacobian constant.

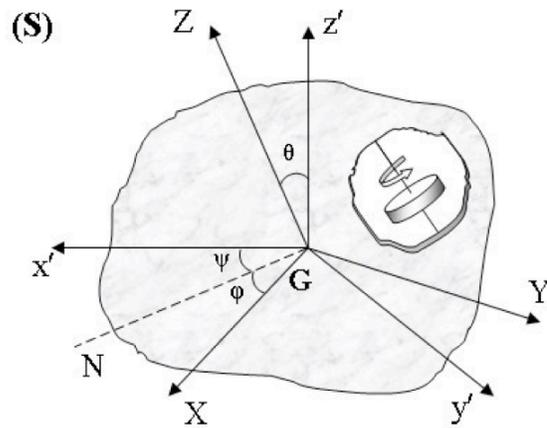


Figure 11: The small body S is a gyrostat consisting of a platform and a rotor.

4.1 Kinetic and potential energy of the gyrostat

The kinetic energy of the translational-rotational motion of the gyrostat is given by the relation,

$$\begin{aligned}
T = & \frac{1}{2}m_g [(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2] + \frac{1}{2}(I_1 - I_2) \sin \theta \sin 2\phi \dot{\theta}(\dot{\psi} + \omega) \\
& + \frac{1}{2} [(I_1 \sin^2 \phi + I_2 \cos^2 \phi) \sin^2 \theta + I_3 \cos^2 \theta] (\dot{\psi} + \omega)^2 + \frac{1}{2}(I_1 \cos^2 \phi + \\
& + I_2 \sin^2 \phi) \dot{\theta}^2 + \frac{1}{2}I_3 \dot{\phi}^2 + I_3 \cos \theta \dot{\phi}(\dot{\psi} + \omega) + (a \cos \phi - b \sin \phi) \dot{\theta} \\
& + c \dot{\phi} + [(a \sin \phi + b \cos \phi) \sin \theta + c \cos \theta] (\dot{\psi} + \omega) + T_r
\end{aligned}$$

where m_g is the mass of the gyrostat, I_i , $i = 1, 2, 3$ are its principal central moments of inertia, a, b, c are the components of the rotor's internal moment of momentum and T_r is its kinetic energy relative to the platform.

We assume that the rotor rotates with relation to the carrier with constant angular velocity, therefore the quantity T_r is constant and does not appear in the final form of the equations of motion. The potential energy is expressed with the help of MacCullagh's formula,

$$\begin{aligned}
V = & -Gmm_g \left[\sum_{i=1}^{\nu} \frac{1}{r_i} + \frac{\beta}{r_C} \right] + Gm \left\{ \sum_{i=1}^{\nu} \frac{1}{r_i^5} [(x - x_i)^2 f + (y - y_i)^2 g + z^2 p + (x - x_i)(y - y_i)h \right. \\
& \left. + (x - x_i)zs + (y - y_i)zq] + \frac{\beta}{r_C^5} [x^2 f + y^2 g + z^2 p + xyh + xzs + yzq] \right\}
\end{aligned}$$

where

$$\begin{aligned}
r_0 &= (x^2 + y^2 + z^2)^{1/2} \\
r_i &= [(x - x_i)^2 + (y - y_i)^2 + z^2]^{1/2}, \quad i = 1, 2, \dots, \nu.
\end{aligned}$$

are the distances of the gyrostat from the primaries and

$$\begin{aligned}
f &= -\frac{3}{2}\varepsilon \sin^2 \psi \sin^2 \theta + \frac{3}{4}\zeta \cos^2 \psi \cos 2\phi - \frac{3}{4}\zeta \sin^2 \psi \cos^2 \theta \cos 2\phi - \\
& - \frac{3}{4}\zeta \sin 2\psi \cos \theta \sin 2\phi + \frac{\varepsilon}{2} \\
g &= -\frac{3}{2}\varepsilon \cos^2 \psi \sin^2 \theta + \frac{3}{4}\zeta \sin^2 \psi \cos 2\phi - \frac{3}{4}\zeta \cos^2 \psi \cos^2 \theta \cos 2\phi + \\
& + \frac{3}{4}\zeta \sin 2\psi \cos \theta \sin 2\phi + \frac{\varepsilon}{2}
\end{aligned}$$

$$\begin{aligned}
h &= \frac{3}{2}\varepsilon \sin 2\psi \sin^2 \theta + \frac{3}{4}\zeta \sin 2\psi \cos 2\phi + \frac{3}{2}\zeta \cos 2\psi \cos \theta \sin 2\phi + \\
&\quad + \frac{3}{4}\zeta \sin 2\psi \cos^2 \theta \cos 2\phi \\
p &= -\frac{3}{2}\varepsilon \cos^2 \theta - \frac{3}{4}\zeta \sin^2 \theta \cos 2\phi + \frac{\varepsilon}{2} \\
q &= \frac{3}{2}\varepsilon \cos \psi \sin 2\theta + \frac{3}{2}\zeta \sin \psi \sin \theta \sin 2\phi - \frac{3}{4}\zeta \cos \psi \sin 2\theta \cos 2\phi \\
s &= -\frac{3}{2}\varepsilon \sin \psi \sin 2\theta + \frac{3}{2}\zeta \cos \psi \sin \theta \sin 2\phi + \frac{3}{4}\zeta \sin \psi \sin 2\theta \cos 2\phi
\end{aligned}$$

$$\text{with } \varepsilon = \frac{I_1 + I_2}{2} - I_3 \quad \text{and} \quad \zeta = I_1 - I_2$$

4.2 Equations of gyrostat's translational motion

The equations of the translational motion of the gyrostat are,

$$\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= \omega^2 x - Gm \left[\sum_{i=1}^{\nu} \frac{(x - x_i)}{r_i^3} + \frac{\beta x}{r_0^3} \right] - \\
&\quad - G \frac{m}{m_g} \left\{ \sum_{i=1}^{\nu} \frac{\partial}{\partial x} \left[\frac{1}{r_i^5} [(x - x_i)^2 f + (y - y_i)^2 g + z^2 p + (x - x_i)(y - y_i)h + (x - x_i)zs + (y - y_i)zq] \right] \right. \\
&\quad \left. + \beta \frac{\partial}{\partial x} \left[\frac{1}{r_0^5} (x^2 f + y^2 g + z^2 p + xyh + xzs + yzq) \right] \right\} \tag{1}
\end{aligned}$$

$$\begin{aligned}
\ddot{y} + 2\omega\dot{x} &= \omega^2 y - Gm \left[\sum_{i=1}^{\nu} \frac{(y - y_i)}{r_i^3} + \frac{\beta y}{r_0^3} \right] - \\
&\quad - G \frac{m}{m_g} \left\{ \sum_{i=1}^{\nu} \frac{\partial}{\partial y} \left[\frac{1}{r_i^5} [(x - x_i)^2 f + (y - y_i)^2 g + z^2 p + (x - x_i)(y - y_i)h + (x - x_i)zs + (y - y_i)zq] \right] \right. \\
&\quad \left. + \beta \frac{\partial}{\partial y} \left[\frac{1}{r_0^5} (x^2 f + y^2 g + z^2 p + xyh + xzs + yzq) \right] \right\} \tag{2}
\end{aligned}$$

$$\begin{aligned}
\ddot{z} &= -Gm \left[\sum_{i=1}^{\nu} \frac{z}{r_i^3} + \frac{\beta z}{r_0^3} \right] - \\
&\quad - G \frac{m}{m_g} \left\{ \sum_{i=1}^{\nu} \frac{\partial}{\partial z} \left[\frac{1}{r_i^5} [(x - x_i)^2 f + (y - y_i)^2 g + z^2 p + (x - x_i)(y - y_i)h + (x - x_i)zs + (y - y_i)zq] \right] \right. \\
&\quad \left. + \beta \frac{\partial}{\partial z} \left[\frac{1}{r_0^5} (x^2 f + y^2 g + z^2 p + xyh + xzs + yzq) \right] \right\} \tag{3}
\end{aligned}$$

4.3 Equations of gyrostat's rotational motion

The equations of the rotational motion of the gyrostat are,

$$\begin{aligned}
& [(I_1 \sin^2 \phi + I_2 \cos^2 \phi) \sin^2 \theta + I_3 \cos^2 \theta] \ddot{\psi} + \frac{1}{2}(I_1 - I_2) \sin 2\phi \sin \theta \ddot{\theta} + I_3 \cos \theta \ddot{\phi} + \\
& + \left[(I_1 - I_2) \sin 2\phi \sin^2 \theta \dot{\phi} + (I_1 \sin^2 \phi + I_2 \cos^2 \phi) \sin 2\theta \dot{\theta} - I_3 \sin 2\theta \dot{\theta} \right] (\dot{\psi} + \omega) + \\
& + (I_1 - I_2) \cos 2\phi \sin \theta \dot{\theta} \dot{\phi} + \frac{1}{2}(I_1 - I_2) \sin 2\phi \cos \theta \dot{\theta}^2 - I_3 \sin \theta \dot{\theta} \dot{\phi} + \\
& + [(a \sin \phi + b \cos \phi) \cos \theta - c \sin \theta] \dot{\theta} + (a \cos \phi - b \sin \phi) \sin \theta \dot{\phi} = \\
& = -Gm \left\{ \sum_{i=1}^{\nu} \frac{1}{r_i^5} \left[(x - x_i)^2 \frac{\partial f}{\partial \psi} + (y - y_i)^2 \frac{\partial g}{\partial \psi} + z^2 \frac{\partial p}{\partial \psi} + (x - x_i)(y - y_i) \frac{\partial h}{\partial \psi} + (x - x_i)z \frac{\partial s}{\partial \psi} \right. \right. \\
& \left. \left. + (y - y_i)z \frac{\partial q}{\partial \psi} \right] + \frac{\beta}{r_0^5} \left[x^2 \frac{\partial f}{\partial \psi} + y^2 \frac{\partial g}{\partial \psi} + z^2 \frac{\partial p}{\partial \psi} + xy \frac{\partial h}{\partial \psi} + xz \frac{\partial s}{\partial \psi} + yz \frac{\partial q}{\partial \psi} \right] \right\} \quad (4)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}(I_1 - I_2) \sin 2\phi \sin \theta \ddot{\psi} + (I_1 \cos^2 \phi + I_2 \sin^2 \phi) \ddot{\theta} - (I_1 - I_2) \sin 2\phi \dot{\theta} \dot{\phi} + \\
& + (I_1 - I_2) \cos 2\phi \sin \theta \dot{\phi} (\dot{\psi} + \omega) - \frac{1}{2} [(I_1 \sin^2 \phi + I_2 \cos^2 \phi) - I_3] \sin 2\theta (\dot{\psi} + \omega)^2 + \\
& + I_3 \sin \theta \dot{\phi} (\dot{\psi} + \omega) - (a \sin \phi + b \cos \phi) \dot{\phi} - [(a \sin \phi + b \cos \phi) \cos \theta - c \sin \theta] (\dot{\psi} + \omega) = \\
& = -Gm \left\{ \sum_{i=1}^{\nu} \frac{1}{r_i^5} \left[(x - x_i)^2 \frac{\partial f}{\partial \theta} + (y - y_i)^2 \frac{\partial g}{\partial \theta} + z^2 \frac{\partial p}{\partial \theta} + (x - x_i)(y - y_i) \frac{\partial h}{\partial \theta} + (x - x_i)z \frac{\partial s}{\partial \theta} \right. \right. \\
& \left. \left. + (y - y_i)z \frac{\partial q}{\partial \theta} \right] + \frac{\beta}{r_0^5} \left[x^2 \frac{\partial f}{\partial \theta} + y^2 \frac{\partial g}{\partial \theta} + z^2 \frac{\partial p}{\partial \theta} + xy \frac{\partial h}{\partial \theta} + xz \frac{\partial s}{\partial \theta} + yz \frac{\partial q}{\partial \theta} \right] \right\} \quad (5)
\end{aligned}$$

$$\begin{aligned}
& I_3 \cos \theta \ddot{\psi} + I_3 \ddot{\phi} - I_3 \sin \theta \dot{\theta} (\dot{\psi} + \omega) - \frac{1}{2}(I_1 - I_2) \sin 2\phi \sin^2 \theta (\dot{\psi} + \omega)^2 + \\
& + \frac{1}{2}(I_1 - I_2) \sin 2\phi \dot{\theta}^2 - (I_1 - I_2) \cos 2\phi \sin \theta \dot{\theta} (\dot{\psi} + \omega) - \\
& - (a \cos \phi + b \sin \phi) \sin \theta (\dot{\psi} + \omega) + (a \sin \phi + b \cos \phi) \dot{\theta} = \\
& = -Gm \left\{ \sum_{i=1}^{\nu} \frac{1}{r_i^5} \left[(x - x_i)^2 \frac{\partial f}{\partial \phi} + (y - y_i)^2 \frac{\partial g}{\partial \phi} + z^2 \frac{\partial p}{\partial \phi} + (x - x_i)(y - y_i) \frac{\partial h}{\partial \phi} + (x - x_i)z \frac{\partial s}{\partial \phi} \right. \right. \\
& \left. \left. + (y - y_i)z \frac{\partial q}{\partial \phi} \right] + \frac{\beta}{r_0^5} \left[x^2 \frac{\partial f}{\partial \phi} + y^2 \frac{\partial g}{\partial \phi} + z^2 \frac{\partial p}{\partial \phi} + xy \frac{\partial h}{\partial \phi} + xz \frac{\partial s}{\partial \phi} + yz \frac{\partial q}{\partial \phi} \right] \right\} \quad (6)
\end{aligned}$$

4.4 Equilibrium states of the gyrostat

The equilibrium states of the gyrostat are determined by the conditions

$\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = \dot{\theta} = \ddot{\theta} = \dot{\phi} = \ddot{\phi} = \dot{\psi} = \ddot{\psi} = 0$ as being applied to

equations (1) to (6). There are a lot of equilibrium solutions, but here we shall confine to those for which

$$\begin{aligned}\psi &= \lambda_\psi \frac{\pi}{2} \quad \text{with} \quad \lambda_\psi = 0, 1, 2, 3 \\ \theta &= \lambda_\theta \frac{\pi}{2} \quad \text{with} \quad \lambda_\theta = 0, 1, 2 \quad \text{and} \\ \phi &= \lambda_\phi \frac{\pi}{2} \quad \text{with} \quad \lambda_\phi = 0, 1, 2, 3\end{aligned}$$

For these values it holds that

$$h = s = q = 0 \quad \text{and} \quad \frac{\partial f}{\partial \psi} = \frac{\partial g}{\partial \psi} = \frac{\partial f}{\partial \theta} = \frac{\partial g}{\partial \theta} = \frac{\partial h}{\partial \theta} = \frac{\partial f}{\partial \phi} = \frac{\partial g}{\partial \phi} = 0$$

These conditions result in various combinations of the above values of angles for which equilibrium states exist. Some of them are exposed in Table I.

Table I: Some cases of equilibrium states

$I_1 = I_2$			$I_1 = I_3$			$I_2 = I_3$		
$\mathbf{a} = \mathbf{0}$	$\mathbf{b} \neq \mathbf{0}$	$\mathbf{c} \neq \mathbf{0}$	$\mathbf{a} = \mathbf{0}$	$\mathbf{b} \neq \mathbf{0}$	$\mathbf{c} = \mathbf{0}$	$\mathbf{a} \neq \mathbf{0}$	$\mathbf{b} = \mathbf{0}$	$\mathbf{c} = \mathbf{0}$
ψ	θ	ϕ	ψ	θ	ϕ	ψ	θ	ϕ
0	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$
0	0	$\frac{3\pi}{2}$	0	$\frac{\pi}{2}$	π	0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$
π	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
π	0	$\frac{3\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$

We observe that each condition $I_1 = I_2$, $I_1 = I_3$ or $I_2 = I_3$ implies $f = g$. The stability analysis showed that all the equilibrium states of the gyrostat are unsteady.

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