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A sequence space representation of the space of bounded ultradistributions of Beurling type

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Abstract. Sequence space representations of the spaces $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ and of its dual $\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N)$, the space of bounded ultradistributions of Beurling type, are presented, in case the weight ω is a strong weight.

Representación del espacio de las ultradistribuciones acotadas de tipo Beurling como espacio de sucesiones

Resumen. Representamos $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ y su dual $\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N)$, el espacio de las ultradistribuciones acotadas de tipo Beurling, como espacio de sucesiones cuando la función peso ω es un peso fuerte

1. Introduction and notation

The representation of several spaces of (ultra)-differentiable functions and (ultra)-distributions has been intensively investigated by many authors, like Meise and Taylor [6], Valdivia [11] and Vogt [13] among others. There are several reasons for the investigation of such representation, for example it gives the information about the linear topological structure of the spaces under consideration.

The classical space $\mathcal{D}_{L_1}$ was introduced by Schwartz [10] as the Fréchet space of all C^∞ functions f such that f and all its derivatives are in L_1. Its dual consists of the bounded distributions and contains as a subspace the almost periodic distributions. Cioranescu [2] obtained the characterization of bounded and almost periodic ultradistributions of Beurling type. The same topic was considered in [4], where some of the Cioranescu's results were extended to study bounded and almost periodic ultradistributions of Beurling or Roumieu type.

In this paper a representation of the space $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ as a Köthe sequence space is given. As a corollary we will obtain that such a space is always quasinormable. In particular, the space of bounded ultradistributions of Beurling type is an (LB)-space.

We introduce the spaces of functions and ultradistributions and most of the notation that will be used in the sequel.

Definition 1 ([1]) *A continuous increasing function $\omega : [0,\infty[\longrightarrow [0,\infty[$ is called a (non-quasianalytic) weight if it satisfies the following conditions:*
(α) *there exists $L \geq 0$ with $\omega(\mathrm{e}\, t) \leq L(\omega(t)+1)$ for all $t \geq 0$,*

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(β) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$,
(γ) $\log(t) = o(\omega(t))$ *as t tends to* ∞,
(δ) $\varphi : t \to \omega(e^t)$ *is convex.*

For a weight function ω *we define* $\widetilde{\omega} : \mathbb{C}^N \to [0, \infty[$ *by* $\widetilde{\omega}(z) := \omega(|z|)$, *and we call this function again* ω *by abuse of notation.*

A strong weight is a weight ω *satisfying the following aditional condition*

(ϵ) *there exists* $C \geq 1$ *with* $\int_1^\infty \frac{\omega(yt)}{t^2} dt \leq C\omega(y) + C$ *for all* $y > 0$.

The *Young conjugate* $\varphi^* : [0, \infty[\longrightarrow \mathbb{R}$ of φ is defined by

$$\varphi^*(s) := \sup\{st - \varphi(t) : t \geq 0\}.$$

There is no loss of generality to assume that ω vanishes on $[0, 1]$. Then φ^* has only non-negative values and $\varphi^{**} = \varphi$.

Definition 2 ([1]) *Let* ω *be a weight function. We define*

$$\mathcal{E}_{(\omega)}(\mathbb{R}^N) := \{f \in C^\infty(\mathbb{R}^N) : \ \forall K \subset\subset \mathbb{R}^N \ \forall m \in \mathbb{N} : \| f \|_{K,m} < \infty\}$$

where

$$\| f \|_{K,\lambda} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\big(- \lambda \varphi^* \left(\frac{|\alpha|}{\lambda} \right) \big) \quad \forall \lambda > 0.$$

We endow this space with its natural locally convex topology. The elements of $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ *are called* ω-*ultradifferentiable functions of Beurling type.*

For a compact set K in $\mathbb{R}^N$ we put

$$\mathcal{D}_{(\omega)}(K) := \{f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N) \ : \ \mathrm{supp}(f) \subset K\}$$

endowed with the induced topology. For a fundamental sequence $(K_j)_{j \in \mathbb{N}}$ of compact subsets of $\mathbb{R}^N$ we let

$$\mathcal{D}_{(\omega)}(\mathbb{R}^N) := \underset{j \to}{\mathrm{ind}} \ \mathcal{D}_{(\omega)}(K_j).$$

The elements of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ are called ω-*ultradistributions of Beurling type*.

Definition 3 ([4]) *For a weight function* ω, *we denote*

$$\mathcal{D}_{L^1,\omega,\lambda} := \{f \in \mathcal{D}_{L^1}(\mathbb{R}^N) : \ |f|_\lambda := \sup_{\alpha \in \mathbb{N}_0^N} \| f^{(\alpha)} \|_1 \exp\big(- \lambda \varphi^* \big(\frac{|\alpha|}{\lambda} \big) \big) < \infty \ \}.$$

We define $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N) := \underset{\leftarrow \lambda}{\mathrm{proj}} \ \mathcal{D}_{L^1,\omega,\lambda}$. *Then* $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ *is a Fréchet space.*

Remark 1 *The inclusions* $\mathcal{D}_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N) \subset \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ *are continuous and have dense range.*

The elements of $\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N)$ are called *bounded* $\omega-$*ultradistributions of Beurling* type. In what follows we will always consider $\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N)$ endowed with the strong topology $\beta(\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N), \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N))$.

For every $\lambda > 0$ and $f \in \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ we put

$$\mathrm{q}_\lambda(f) := \sum_{\alpha \in \mathbb{N}_0^N} \| f^{(\alpha)} \|_1 \exp\big(- \lambda \ \varphi^* \big(\frac{|\alpha|}{\lambda} \big) \big).$$

Proposition 1 *The topology of $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ is given by the sequence of seminorms $(\mathrm{q}_m)_{m\in\mathbb{N}}$.*

PROOF. We consider $f \in \mathcal{D}_{L^1,(\omega)}(K)$. It is clear that $|f|_m \le q_m(f)$ for every $m \in \mathbb{N}$. Now we take $L > 0$ such that $\omega(et) \le L(1+\omega(t))$ for all $t \ge 0$. It follows from [1, 1.4] that

$$\begin{aligned} \| f^{(\alpha)} \|_1 \exp\left(- m\varphi^* \left(\frac{|\alpha|}{m}\right) \right) & \le \| f^{(\alpha)} \|_1 \exp\left(- mL\varphi^* \left(\frac{|\alpha|}{mL}\right) \right) \mathrm{e}^{-|\alpha|+mL} \\ & \le |f|_{mL} \mathrm{e}^{-|\alpha|+mL} \end{aligned}$$

for every $m \in \mathbb{N}$, and then

$$\mathrm{q}_m(f) \le \mathrm{e}^{mL}\, |f|_{mL} \sum_{\alpha\in\mathbb{N}^N} \mathrm{e}^{-|\alpha|}.$$

Since $\sum_{\alpha\in\mathbb{N}^N} \mathrm{e}^{-|\alpha|}$ is convergent the conclusion follows. ■

According to Remark 1 and Proposition 1, for a compact set $K \subset \mathbb{R}^N$, the topology of $\mathcal{D}_{(\omega)}(K)$ is given by the sequence of seminorms $(q_m)_{m\in\mathbb{N}}$.

For a function $f \in \mathcal{C}^\infty(\mathbb{R}^N)$ and $h \in \mathbb{R}^N$ we denote τ_h the translation operator defined by $(\tau_h f)(x) = f(x-h), \quad x \in \mathbb{R}^N$.

2. The representation of $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ and $\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N)$

Our aim is to prove that if ω is a strong weight, then $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ is isomorphic to $\ell_1(\Lambda_\infty(\alpha(\omega,N)))$ where $\alpha(\omega,N) = (\omega(j^{\frac{1}{N}}))_{j\in\mathbb{N}}$ and to derive some consequences about the topological structure of $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$.

We recall that, for a locally convex space E, $\ell_1(E)$ denotes the linear subspace of $E^{\mathbb{Z}^N}$ which consists of all sequences $(x_\alpha)_{\alpha\in\mathbb{Z}^N}$ in E such that $\sum_{\alpha\in\mathbb{Z}^N} p(x_\alpha)$ is convergent for every continuous seminorm p in E.

Let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ be an increasing and unbounded sequence of positive real numbers. The *power series space of infinite type* $\Lambda_\infty(\alpha)$ is defined by

$$\Lambda_\infty(\alpha) := \{x \in \mathbb{C}^{\mathbb{N}} \ : \| x \|_r := \sum_{j=1}^{\infty} |x_j| r^{\alpha_j} < \infty \text{ for all } r > 0\}.$$

Obviously, $\Lambda_\infty(\alpha)$ is a Fréchet space under the locally convex topology induced by the norms $(\| \,.\, \|_r)_{r>0}$ [8].

Our first objective is to prove that $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ is isomorphic to $\ell_1(\mathcal{D}_{(\omega)}(K))$ for every compact cube $K \subset\subset \mathbb{R}^N$. To do this we will use the following modification of the Pelczynski's method due to Vogt [13].

Lemma 1 ([13, 1.1]) *Let E, F be locally convex spaces. If E is isomorphic to a complemented subspace of $\ell_1(F)$ and $\ell_1(F)$ is isomorphic to a complemented subspace of E then E and $\ell_1(F)$ are isomorphic.*

Proposition 2 *For every compact cube K in $\mathbb{R}^N$, $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ can be embedded as a complemented subspace into $\ell_1(\mathcal{D}_{(\omega)}(K))$.*

PROOF. We can assume $K = [-2,2]^N$. Choose $\phi_0 \in \mathcal{D}_{(\omega)}(K)$ such that, for $\phi_\nu(x) := \phi_0(x-\nu)$, we have $\sum_{\nu\in\mathbb{Z}^N} \phi_\nu(x) = 1$ for all $x \in \mathbb{R}^N$. We define the maps

$$\begin{array}{rcl} \Phi : \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N) & \longrightarrow & \ell_1(\mathcal{D}_{(\omega)}(K)) \\ f & \mapsto & (\tau_{-\nu}(f\phi_\nu))_{\nu\in\mathbb{Z}^N} \end{array}$$

and

$$\begin{array}{rcl} \Psi : \ell_1(\mathcal{D}_{(\omega)}(K)) & \longrightarrow & \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N) \\ (f_\nu)_{\nu\in\mathbb{Z}^N} & \mapsto & \sum_{\nu\in\mathbb{Z}^N} \tau_\nu(f_\nu) \end{array}$$

We have to show that Φ and Ψ are well defined continuous linear maps and $\Psi\circ\Phi = \mathrm{id}$.

(i) Φ is well defined and continuous: We consider $f \in \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$, $m \in \mathbb{N}_0$ and $L > 1$ such that $\omega(\mathrm{e}t) \le L(\omega(t)+1)$. For every $\nu \in \mathbb{Z}^N$,

$$\tau_{-\nu}(f\phi_\nu) = (\tau_{-\nu}f)\phi_0 \in \mathcal{D}_{(\omega)}(K).$$

For every $x \in K$ we have, after applying Leibniz's rule and the convexity of φ^*,

$$\begin{aligned}
&|\,(\tau_{-\nu}(f\phi_\nu))^{(\alpha)}\,(x)|\,\exp\big(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\big)\\
&\le \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\,\mathrm{e}^{-|\alpha|+Lm}|f^{(\gamma)}(x+\nu)|\,|\phi_0^{(\alpha-\gamma)}(x)|\,\mathrm{e}^{-Lm\varphi^*\left(\frac{|\alpha-\gamma|}{Lm}\right)}\,\mathrm{e}^{-Lm\varphi^*\left(\frac{|\gamma|}{Lm}\right)}\\
&\le \|\,\phi_0\,\|_{K,Lm}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\,\mathrm{e}^{-|\alpha|+Lm}|f^{(\gamma)}(x+\nu)|\,\mathrm{e}^{-Lm\varphi^*\left(\frac{|\gamma|}{Lm}\right)},
\end{aligned}$$

and consequently

$$\begin{aligned}
&\sum_{\nu\in\mathbb{Z}^N} q_m\,(\tau_{-\nu}(f\phi_\nu))\\
&\le \mathrm{e}^{Lm}\,\|\,\phi_0\,\|_{K,Lm}\sum_{\alpha\in\mathbb{N}_0^N}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\mathrm{e}^{-|\alpha|}\sum_{\nu\in\mathbb{Z}^N}\int_K|f^{(\gamma)}(x+\nu)|\,dx\,\mathrm{e}^{-Lm\varphi^*\left(\frac{|\gamma|}{Lm}\right)}\\
&\le 4^N\mathrm{e}^{Lm}\,\|\,\phi_0\,\|_{K,Lm}\sum_{\alpha\in\mathbb{N}_0^N}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\mathrm{e}^{-|\alpha|}\,\|\,f^{(\gamma)}\,\|_1\,\mathrm{e}^{-Lm\varphi^*\left(\frac{|\gamma|}{Lm}\right)}\\
&\le 4^N\mathrm{e}^{Lm}\,\|\,\phi_0\,\|_{K,Lm}\,|f|_{Lm}\sum_{\alpha\in\mathbb{N}_0^N}\left(\frac{2}{\mathrm{e}}\right)^{|\alpha|}.
\end{aligned}$$

(ii) Ψ is a continuous map: We fix $(f_\nu) \in \ell_1(\mathcal{D}_{(\omega)}(K))$. Since $\sum_{\nu\in\mathbb{Z}^N}\tau_\nu f_\nu$ is a locally finite sum we obtain

$$\int_{\mathbb{R}^N}|\left(\sum_{\nu\in\mathbb{Z}^N}\tau_\nu f_\nu\right)^{(\alpha)}(x)|dx \le \sum_{\nu\in\mathbb{Z}^N}\int_{\mathbb{R}^N}|f_\nu^{(\alpha)}(x-\nu)|dx = \sum_{\nu\in\mathbb{Z}^N}\|\,f_\nu^{(\alpha)}\,\|_1\,.$$

Consequently, for every $m \in \mathbb{N}_0$,

$$|\sum_{\nu\in\mathbb{Z}^N}\tau_\nu f_\nu|_m \le \sup_{\alpha\in\mathbb{N}_0^N}\sum_{\nu\in\mathbb{Z}^N}\|\,f_\nu^{(\alpha)}\,\|_1\,\exp^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)} \le \sum_{\nu\in\mathbb{Z}^N}|f_\nu|_m,$$

from where it follows that Ψ is a continuous map.

(iii) Since for every $f \in \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ we have

$$(\Psi\circ\Phi)(f) = \Psi\big((\tau_{-\nu}(f\phi_\nu))_{\nu\in\mathbb{Z}^N}\big) = \sum_{\nu\in\mathbb{Z}^N} f\phi_\nu = f,$$

the proof is complete. ■

Definition 4 ([7]) *Let ω be a weight function. For a compact set K in $\mathbb{R}^N$ with $\overline{\mathring{K}} = K$ we define the ω-Whitney jets of Beurling type on K by*

$$\begin{aligned}
\mathcal{E}_{(\omega)}(K) \;:=\; &\{f = (f_\alpha)_{\alpha\in\mathbb{N}_0^N} \in \mathcal{C}(K)^{\mathbb{N}_0^N}\;:\; f_{(0)}|\,\mathring{K} \in \mathcal{C}^\infty(\mathring{K}),\\
&(f_{(0)}|\,\mathring{K})^{(\alpha)} = f_\alpha|\,\mathring{K}\ \textit{for each}\ \alpha\in\mathbb{N}_0^N\ \textit{and for each}\ m\in\mathbb{N}\\
&|||f|||_m := \sup_{\alpha\in\mathbb{N}_0^N}\sup_{x\in K}|f_\alpha(x)|\exp(-m\varphi^*\left(\frac{|\alpha|}{m}\right)) < \infty\}.
\end{aligned}$$

and we endow $\mathcal{E}_{(\omega)}(K)$ with the Fréchet space topology induced by the norm system $(|||.|||_m)_{m\in\mathbb{N}}$. The restriction map

$$\rho_K : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \longrightarrow \mathcal{E}_{(\omega)}(K) \qquad \text{by } \rho_K(f) := (f^{(\alpha)}|K)_{\alpha\in\mathbb{N}_0^N}$$

is continuous and linear.

The existence of a continuous linear extension operator has been studied by Meise and Taylor [7]. They obtained the following result, which is essential in the proof of Proposition 3.

Theorem 1 ([7, 3.1]) *Let ω be a strong weight function and let K be a compact set in $\mathbb{R}^N$ with $\overset{\circ}{K}\neq\emptyset$ which is of the form $K = \prod_{j=1}^m \overline{G_j}$, where $G_j \subset \mathbb{R}^{N_j}$, $1 \le N_j \le N$, is a bounded open set with real-analytic boundary for $1 \le j \le m$. Then there exists a continuous linear extension operator $E_K : \mathcal{E}_{(\omega)}(K) \longrightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ with $\rho_K \circ E_K = \mathrm{id}_{\mathcal{E}_{(\omega)}(K)}$.*

Proposition 3 *Let ω be a strong weight. For every compact cube K in $\mathbb{R}^N$, $\ell_1(\mathcal{E}_{(\omega)}(K))$ can be embedded as a complemented subspace into $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$.*

PROOF. We can assume $K := [-\frac{1}{4}, \frac{1}{4}]^N$. It follows from Theorem 1 that there is a continuous linear operator $E_K : \mathcal{E}_{(\omega)}(K) \longrightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ with $\rho_K \circ E_K = \mathrm{id}_{\mathcal{E}_{(\omega)}(K)}$. We can assume, by multiplying with a test function, that $E_K : \mathcal{E}_{(\omega)}(K) \longrightarrow \mathcal{D}_{(\omega)}([-\frac{1}{2}, \frac{1}{2}]^N)$. Now we define

$$\begin{array}{rcl}\Phi : \ell_1(\mathcal{E}_{(\omega)}(K)) & \longrightarrow & \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N) \\ (f_\nu)_{\nu\in\mathbb{Z}^N} & \mapsto & \sum_{\nu\in\mathbb{Z}^N} \tau_\nu(E_K f_\nu)\end{array}$$

and

$$\begin{array}{rcl}\Psi : \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N) & \longrightarrow & \ell_1(\mathcal{E}_{(\omega)}(K)) \\ f & \mapsto & (\rho_K(\tau_{-\nu} f))_{\nu\in\mathbb{Z}^N}\end{array}$$

We claim that Φ and Ψ are well defined, linear and continuous maps and $\Psi \circ \Phi = id$.

(i) Ψ is a well defined and continuous map:

For every $f \in \mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ we have that $\rho_K(\tau_{-\nu} f) \in \mathcal{E}_{(\omega)}(K)$ for all $\nu \in \mathbb{Z}^N$. Now, it follows from Sobolev's Lemma [9, 3.5.12] and the convexity of φ^* that

$$\begin{aligned}
&\sum_\nu |||\rho_K(\tau_{-\nu} f)|||_m \\
&= \sum_\nu \| (\tau_{-\nu} f) \|_{K,m} \\
&= \sum_\nu \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x+\nu)| \mathrm{e}^{-m\varphi^*(\frac{|\alpha|}{m})} \\
&\le C \sum_\nu \sup_{\alpha\in\mathbb{N}_0^N} \sup_{|\beta|\le N+1} \int_{[-\frac{1}{2},\frac{1}{2}]^N} |f^{(\alpha+\beta)}(x+\nu)|\, \mathrm{e}^{-2m\varphi^*(\frac{|\alpha+\beta|}{2m}) + m\varphi^*(\frac{|\beta|}{m})} dx \\
&\le C \sum_{\nu\in\mathbb{Z}^N} \mathrm{e}^{m\varphi^*(\frac{N+1}{m})} \sum_{\gamma\in\mathbb{N}_0^N} \int_{[-\frac{1}{2},\frac{1}{2}]^N} |f^{(\gamma)}(x+\nu)|\, \mathrm{e}^{-2m\varphi^*(\frac{|\gamma|}{2m})} dx \\
&\le C\mathrm{e}^{m\varphi^*(\frac{N+1}{m})} \sum_{\gamma\in\mathbb{N}_0^N} \sum_{\nu\in\mathbb{Z}^N} \left(\int_{[-\frac{1}{2},\frac{1}{2}]^N} |f^{(\gamma)}(x+\nu)| dx \right) \mathrm{e}^{-2m\varphi^*(\frac{|\gamma|}{2m})} \\
&\le C\mathrm{e}^{m\varphi^*(\frac{N+1}{m})} \sum_{\gamma\in\mathbb{N}_0^N} \| f^{(\gamma)} \|_1 \, \mathrm{e}^{-2m\varphi^*(\frac{|\gamma|}{2m})}.
\end{aligned}$$

for some positive constant C. This shows that Ψ is a well defined and continuous map.

(ii) Φ is well defined and continuous: Let $m \in \mathbb{N}_0^N$, $(f_\nu)_{\nu\in\mathbb{Z}^N} \in \ell_1(\mathcal{E}_{(\omega)}(K))$. Since $\sum_\nu \tau_\nu(E_K f_\nu)(x)$ is a locally finite sum we obtain

$$\begin{aligned}
&|\sum_\nu \tau_\nu(E_K f_\nu)|_m \\
&= \sup_{\alpha\in\mathbb{N}_0^N} \int_{\mathbb{R}^N} |\left(\sum_\nu \tau_\nu(E_K f_\nu)\right)^{(\alpha)}(x)| e^{-m\varphi^*(\frac{|\alpha|}{m})} dx \\
&\le \sum_{\alpha\in\mathbb{N}_0^N} \int_{\mathbb{R}^N} \sum_\nu |(E_K f_\nu)^{(\alpha)}(x-\nu)| e^{-m\varphi^*(\frac{|\alpha|}{m})} dx \\
&\le \sum_\nu \sum_{\alpha\in\mathbb{N}_0^N} \int_{\mathbb{R}^N} |(E_K f_\nu)^{(\alpha)}(x-\nu)| e^{-m\varphi^*(\frac{|\alpha|}{m})} dx \\
&\le \sum_\nu \sum_{\alpha\in\mathbb{N}_0^N} \| (E_K f_\nu)^{(\alpha)} \|_1 \, e^{-m\varphi^*(\frac{|\alpha|}{m})} \\
&\le \sum_\nu q_m(E_K f_\nu).
\end{aligned}$$

Since E_K is continuous, there exist a positive constant $C > 0$ and $n \in \mathbb{N}$ such that

$$|q_m(E_K f_\nu)| \le C|||f_\nu|||_n \quad \text{for all } \nu.$$

Then

$$|\sum_\nu \tau_\nu(E_K f_\nu)|_m \le C \sum_\nu |||f_\nu|||_n.$$

(iii) $\Psi \circ \Phi = id$: Let $(f_\nu)_{\nu\in\mathbb{Z}^N} \in \ell_1(\mathcal{E}_{(\omega)}(K))$, $f_\nu = (f_{\nu,\alpha})_{\alpha\in\mathbb{N}_0^N}$,

$$(\Psi\circ\Phi)\Big((f_\nu)_{\nu\in\mathbb{Z}^N}\Big) = \Psi\left(\sum_\nu \tau_\nu(E_K f_\nu)\right) = \left(\rho_K\big(\tau_{-\gamma}\big(\sum_\nu \tau_\nu(E_K f_\nu)\big)\big)\right)_{\gamma\in\mathbb{Z}^N}.$$

If $\gamma \in \mathbb{Z}^N$ and $x \in K$,

$$\left(\tau_{-\gamma}\left(\sum_\nu \tau_\nu(E_K f_\nu)\right)\right)(x) = \sum_\nu \tau_\nu(E_K f_\nu)(x+\gamma) = \sum_\nu (E_K f_\nu)(x+\gamma-\nu).$$

If $\gamma \neq \nu$ implies $x+\gamma-\nu \notin [-\frac{1}{2}, \frac{1}{2}]^N$ and then $E_K f_\nu(x+\gamma-\nu) = 0$. Hence $\tau_{-\gamma}(\sum_\nu \tau_\nu(E_K f_\nu)) = E_k f_\gamma$ on K and $(\psi\circ\phi)((f_\nu)) = (\rho_K(E_K f_\gamma))_\gamma = (f_\gamma)_\gamma$. ■

Now, we can prove the announced representation of $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ as a sequence space.

Theorem 2 *Let ω be a strong weight. For every compact cube K in $\mathbb{R}^N$ $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ is isomorphic to $\ell_1(\mathcal{D}_{(\omega)}(K))$. Consequently $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ is isomorphic to $\ell_1(\Lambda_\infty(\alpha(\omega, N)))$, where $\alpha(\omega, N) = (\omega(j^{\frac{1}{N}}))_{j\in\mathbb{N}}$.*

PROOF. Since for every compact cube K in $\mathbb{R}^N$, both spaces $\mathcal{E}_{(\omega)}(K)$ and $\mathcal{D}_{(\omega)}(K)$ are isomorphic to $\Lambda_\infty(\alpha(\omega, N))$ where $\alpha(\omega, N) = (\omega(j^{\frac{1}{N}}))_{j\in\mathbb{N}}$ [6, 5.9],[7, 3.2], it follows from the Lemma 1 and the Propositions 2 and 3 that $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ is isomorphic to $\ell_1(\mathcal{D}_{(\omega)}(K))$. The conclusion follows from [7, 3.2]. ■

As a consequence of the previous theorem we obtain some topological properties of $\mathcal{D}_{L^1,(\omega)}(\mathbb{R}^N)$ and its dual.

Corollary 1 *If ω is a strong weight, then the Fréchet space $\mathcal{D}_{L^1,(\omega)}(\mathbb{R})^N$ is a quasinormable space. In particular it is distinguished.*

Corollary 2 *If ω is a strong weight then $\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N) \simeq \ell_\infty(\Lambda_\infty(\alpha(\omega, N))'_b)$, where $\alpha(\omega, N) = (\omega(j^{\frac{1}{N}}))_{j\in\mathbb{N}}$. Moreover $\mathcal{D}'_{L^1,(\omega)}(\mathbb{R}^N)$ is a strongly boundedly retractive (LB)-space.*

PROOF. It follows from [3, 5.18]. ■

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