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Equivariant tori which are critical points of the conformal total tension functional

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Abstract. We give a new method to obtain Willmore tori over principal circle bundles. This method can be viewed as a reduction of variables criterion for the Willmore variational problem in conformal structures associated with metrics, on principal circle bundles, which are obtained via the generalized inverse Kaluza-Klein mechanism. The problem of finding critical points for the conformal total tension functional in those conformal structures is transfered to the search of critical points for certain elastic energy functionals acting on spaces of curves in the base. This technique is applied to construct wide families of equivariant tori which are critical points for the conformal total tension functional in an ample class of conformal structures.

Toros equivariantes que son puntos críticos de la tensión total conforme

Resumen. Se obtiene un nuevo método para obtener toros de Willmore en estructuras conformes de Kaluza-Klein sobre fibrados principales con fibra la circunferencia. Diversas aplicaciones de esta técnica son consideradas.

1. Introduction

Let $\mathcal{N}$ be the space of immersions of a compact, smooth surface N in a Riemannian manifold $(L, ds^2 =<,>)$. The tension field of $\varphi \in \mathcal{N}$ is the Euler-Lagrange operator associated with the energy of φ, [9]. It is known from the time of Laplace that the tension field of φ is precisely its mean curvature vector field H. We can measure the tension globally and then wonder for the minimal amount of total tension that a surface receives from the surrounding space $(L, ds^2 =<,>)$. More generally, we ask for the critical points of the functional $\mathcal{E} : \mathcal{N} \to \mathbb{R}$ defined by

$$\mathcal{E}(\varphi) = \int_N < H, H > dv,$$

where dv is the volume element of the induced metric $\varphi^*(<,>)$ on N.

To obtain a functional invariant under conformal transformations of the ambient space, we need to modify the integrand of $\mathcal{E}$ by attaching the extrinsic Gaussian curvature as a potential. To be precise, the Willmore functional $\mathcal{W} : \mathcal{N} \to \mathbb{R}$ is defined to be

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$$\mathcal{W}(\varphi) = \int_N (< H, H > +K) dv,$$

where K is the sectional curvature of $(L, ds^2 =<,>)$ restricted to $\varphi_*(TN)$ (notice that both functionals coincide if the ambient space is flat). This action is also known as the *conformal total mean curvature (or conformal total tension) functional*, because its invariance by conformal transformations of $(L, ds^2 =<,>)$. The critical points of $\mathcal{W}$ are called *Willmore surfaces* and its associated variational problem is actually stated in $(L, [ds^2])$, where $[ds^2]$ standard for the conformal structure defined by ds^2. The importance of this variational problem partially comes from the Willmore conjecture. In 1965, T.J.Willmore [22, 23] conjectured that if N has genus one and $[ds^2]$ is the standard conformal class on the sphere (i.e. $(L, ds^2 =<,>)$ is a round sphere) then $\mathcal{W} \geq 2\pi^2$ and the equality is attained for any conformal image of the Clifford torus in the unit round 3-sphere. The conjecture is still an open problem.

In this paper we are going to exhibit a method to obtain $\mathbb{S}^1$-invariant, Willmore tori in Kaluza-Klein conformal structures over principal $\mathbb{S}^1$-bundles. The core of this process is the principle of symmetric criticality, [15], which combined with the above mentioned extrinsic conformal invariance allows one to reduce the search of Willmore tori in those conformal structures to that for critical points of certain elastic energy functionals defined on spaces of curves in the base of the principal $\mathbb{S}^1$-bundle.

It is well known that the Clifford torus is the only Willmore torus with constant mean curvature (actually it is minimal) one can find in the round 3-sphere. This is not true if we remove the conformal structure on the 3-sphere. A torus immersed in a Riemannian manifold is said to be conformal constant mean curvature if there exists a conformal transformation of that Riemannian manifold which carries the torus to one with constant mean curvature. Among the Willmore tori, those with constant mean curvature (respectively conformal total mean curvature) have great interest because they naturally appear as critical points of a certain functional associated with another classical variational problem (respectively up to conformal transformation). As an illustration, we will give many examples where our procedure is applied to obtain Willmore tori in different conformal structures. Also, by applying the algorithm, we will obtain conformal total mean curvature Willmore tori in a wide class of conformal structures.

2. The inverse Kaluza-Klein mechanism

The inverse Kaluza-Klein method of interpreting gravity plus a $\mathbb{S}^1 = U(1)$ gauge field as pure gravity in one higher dimension can be explained in the following general framework. Let $p : P \rightarrow M$ be a principal fibre $\mathbb{S}^1$-bundle endowed with principal connection and denote by ω its connection 1-form. For any Riemannian metric h on M and any positive smooth function u in M, we define the generalized Kaluza-Klein metric $\bar{h}_u$ on P by

$$\bar{h} = p^*(h) + (u \circ p)^2 \omega^*(dt^2),$$

where dt^2 denotes the standard metric on the unit circle.

These metrics are like local warped product metrics. In particular, if u is chosen to be constant, then it works as a global scalling factor on the fibres, which is usually called constant *squashing* parameter. In these cases, the metrics $\bar{h}_u$ are simply called Kaluza-Klein or *bundle like* metrics. It is not difficult to see that the projection map $p : (P, \bar{h}_u) \rightarrow (M, h)$ has the following properties

1. This is a Riemannian submersion whose leaves are the fibres. Furthermore it has geodesic leaves if and only if u is constant, that is $\bar{h}_u$ is bundle like.
2. The natural $\mathbb{S}^1$-action on P is carried out by isometries of $(P, \bar{h}_u)$.

This paper concerns the Willmore variational problem in $(P, [\bar{h}_u])$. It should be noticed that in any Kaluza-Klein conformal class, $[\bar{h}_u]$, we can find a unique bundle like metric. In fact, just put $\tilde{h}_u = \frac{1}{(u \circ p)^2} \bar{h}_u$ and then

$$\tilde{h}_u = p^*\left(\frac{1}{u^2}.h\right) + \omega^*(dt^2).$$

With this choice in $[\bar{h}_u]$, we obtain a Riemannian submersion, $p : (P, \tilde{h}_u) \to \left(M, \frac{1}{u^2}.h\right)$, with geodesic fibres. Actually $\tilde{h}_u$ is the only Riemannian metric in P such that $p : (P, \tilde{h}_u) \to \left(M, \frac{1}{u^2}.h\right)$ is a Riemannian submersion with geodesic fibres isometric to the unit circle, [18]. This conformal change will be very useful in the next section.

3. The principle of symmetric criticality

In many areas, including mathematics and physics, it has proved extremely useful to look for symmetries and to exploit them, if they exist, in problem solving. The sucess of this procedure is based on the principle of symmetric criticality. This has been used in many applications of the calculus of variations, in particular in theoretical physics, without being particulary noticed. A typical example of this implicit use can be found in the H.Weyl derivation of the Schwarschild solution of the Einstein field equations, [21]. A suggestive formulation of this principle, although it is not valid in this general form, is: *Any critical symmetric point is a symmetric critical point*. The precise formulation of the principle is due to R.S.Palais [15]. In this paper we will discuss a simplified version of the Palais formulation, which will be enough for our purposes.

The starting point is a smooth manifold $\mathcal{M}$ on which a group G acts through diffeomorphisms. One also has a G-invariant functional $\mathcal{B} : \mathcal{M} \to \mathbb{R}$, e.g. $\mathcal{B}(a.\varphi) = \mathcal{B}(\varphi)$, for all $a \in G$ and $\varphi \in \mathcal{M}$. The set of symmetric points is defined to be $\mathcal{M}_G = \{\varphi \in \mathcal{M} \,:\, a.\varphi = \phi \,,\, \forall a \in G\}$. Let Σ be the set of critical points of $\mathcal{B}$ and denote by Σ_G the set of critical points of $\mathcal{B}$ when it is restricted to $\mathcal{M}_G$. Naturally this setting forces $\mathcal{M}_G$ to be a differentiable manifold and this is assured if $\mathcal{M}_G$ is a smooth submanifold of $\mathcal{M}$. A sufficient condition to guarantee this is to assume that G is a compact Lie group and then the principle of symmetric criticality simply states that

$$\Sigma \cap \mathcal{M}_G = \Sigma_G.$$

4. An algorithm to reduce variables in the Willmore variational problem

We consider the Willmore functional $\mathcal{W} : \mathcal{N} \to \mathbb{R}$, acting on the smooth manifold $\mathcal{N}$ of immersions from a genus one, compact surface N in $(P, \bar{h}_u)$. The set of critical points, Σ, of this functional is nothing but the set of Willmore tori in $(P, [\bar{h}_u])$.

On the other hand, for any curve γ immersed in M, its complete lift, $N_\gamma = p^{-1}(\gamma)$, is a $\mathbb{S}^1$-invariant surface immersed in P. It is not difficult to see that the converse also holds, indeed for any $\mathbb{S}^1$-invariant surface N in P, one can integrate the distribution $\bar{h}_u$-orthogonal to the fibres because it has dimension one. Hence, one finds a curve γ immersed in M such that $N = p^{-1}(\gamma)$. We also observe that N_γ is embedded if and only if γ is simple. To obtain N_γ, we begin from a horizontal lift, $\bar{\gamma}$, of γ and then $N_\gamma = \{a.\bar{\gamma}(s) \,:\, a \in \mathbb{S}^1\}$, thus we can obtain a natural parametrization of N_γ where coordinate curves are fibres and horizontal lifts of γ, respectively. In general the horizontal lifts of a closed curve are not closed, because the holonomy of ω could be non trivial. However, if γ is closed, then N_γ is compact. As a consequence, the set of symmetric points, $\mathcal{N}_{\mathbb{S}^1}$, is identified with

$$\mathcal{N}_{\mathbb{S}^1} = \{N_\gamma = p^{-1}(\gamma) \,:\, \gamma \,\text{is a closed curve immersed in}\, M\}.$$

Since the Willmore functional is $\mathbb{S}^1$-invariant, because the $\mathbb{S}^1$-action on P is made up through isometries, we have all the ingredients to apply the above stated formulation of the principle of symmetric criticality.

Therefore, to obtain Willmore tori in $(P, [\bar{h}_u])$ which do not break the $\mathbb{S}^1$-symmetry of the problem, we only need to compute $\mathcal{W}$ over $\mathcal{N}_{\mathbb{S}^1}$ and then to proceed in due course.

To calculate $\mathcal{W}(N_\gamma)$ we will use $\tilde{h}_u$ and recall that $p : (P, \tilde{h}_u) \to (M, \frac{1}{u^2}.h)$ is a Riemannian submersion with geodesic fibres. In [2], the author obtained the following relationship between the mean curvature function α of N_γ in $(P, \tilde{h}_u)$ and the curvature function κ of γ in $(M, \frac{1}{u^2}.h)$

$$\alpha^2 = \frac{1}{4}(\kappa^2 \circ p), \tag{1}$$

this formula holds for any harmonic Riemannian submersion.

The computation of the second term, K, appearing in the integrand of the Willmore functional involves several concepts from the theory of Riemannian submersions. In this framework, a pair of geometric invariants appear, they are usually called the O'Neill invariants and denoted by A and T, respectively [7]. The later is defined using the second fundamental form of the fibres, in particular it vanishes identically when those are totally geodesic. The former invariant measures the obstruction to integrability of the horizontal distribution, in particular it vanishes identically when ω is flat. In terms of these invariants, one can compute the fundamental relationships between the curvatures of the Riemannian manifolds involved in the Riemannian submersion. On the other hand, the tangent plane of N_γ is a *mixed* (also called *vertizontal*, [20]) section in $(P, \tilde{h}_u)$ anywhere. Since in our case T vanishes identically, then K is given [7]

$$K = \tilde{h}_u(A_{\bar{X}}V, A_{\bar{X}}V), \tag{2}$$

where $\{\bar{X}, V\}$ is a $\tilde{h}_u$-orthonormal basis in the above mentioned mixed section made up by the horizontal lift $\bar{X} = \bar{\gamma}'$ of the unit tangent vector field $X = \gamma'$ (assuming that γ is arclength parametrized in $(M, \frac{1}{u^2}.h)$) and V is nothing but the fundamental vector field associated with the standard unit vector field in the Lie algebra of $\mathbb{S}^1 = U(1)$ (actually V defines the unit global vector field in P generating the leaves flow). Next, we denote by r and $\tilde{r}$ the Ricci curvatures of $(M, \frac{1}{u^2}.h)$ and $(P, \tilde{h}_u)$, respectively. Again we use that the Riemannian submersion has geodesic fibres to see [7]

$$\tilde{h}_u(A_{\bar{X}}V, A_{\bar{X}}V) = \frac{1}{2}\left(r(X,X) \circ p - \tilde{r}(\bar{X}, \bar{X})\right). \tag{3}$$

Now, we combine (2,3) to obtain

$$K = \frac{1}{2}\left(r(X,X) \circ p - \tilde{r}(\bar{X}, \bar{X})\right). \tag{4}$$

Let UM be the unit tangent bundle of $(M, \frac{1}{u^2}.h)$, we define $\phi : UM \to \mathbb{R}$, such that

$$\phi(\gamma') \circ p = 2\left(r(\gamma', \gamma') \circ p - \tilde{r}(\bar{\gamma}', \bar{\gamma}')\right). \tag{5}$$

We put (1) and (4) in the integrand of the Willmore functional to obtain

$$\mathcal{W}(N_\gamma) = \int_N \left(\frac{1}{4}(\kappa^2 \circ p) + \frac{1}{2}\left(r(\gamma', \gamma') \circ p - \tilde{r}(\bar{\gamma}', \bar{\gamma}')\right)\right) dv,$$

and so

$$\mathcal{W}(N_\gamma) = \frac{\pi}{2}\int_N \left(\kappa^2 + \phi(\gamma')\right) ds.$$

The last formula suggests to define an *elastic energy density*, with *potential* ϕ, for closed curves in $(M, \frac{1}{u^2}.h)$, by

$$\psi(\gamma) = \kappa^2 + \phi(\gamma'),$$

then, we consider the following elastic energy functional acting on closed curves in $(M, \frac{1}{u^2}.h)$

$$\mathcal{F}(\gamma) = \int_\gamma \psi(\gamma) ds.$$

As a consequence of all this information, we have

$$\mathcal{W}(N_\gamma) = \frac{\pi}{2}\mathcal{F}(\gamma).$$

That can be summed up in the following theorem, which is regarded as a criterion to reduce variables for Willmore surfaces.

Theorem 1 *Let $p : P \to M$ be a $\mathbb{S}^1$-bundle endowed with a principal connection. Let $\bar{h}_u$ be a generalized Kaluza-Klein metric on P and $[\bar{h}_u]$ its conformal class. Given an immersed closed curve γ in M, then $N_\gamma = p^{-1}(\gamma)$ is a Willmore surface in $(P, [\bar{h}_u]$ if and only if γ is a critical point of the elastic energy action $\mathcal{F}$ acting on closed curves in $(M, \frac{1}{u^2}.h)$.*

Corollary 1 *Let ω be the connection 1-form of a principal connection on the principal $\mathbb{S}^1$-bundle $p : P \to M$. For any Riemannian metric h on M, let $\bar{h} = p^*(h) + \omega^*(dt^2)$ be the only metric on P which makes $p : (P, \bar{h}) \to (M, h)$ to be a Riemannian submersion with geodesic fibres isometric to the unit circle. Given an immersed closed curve γ in M, then $N_\gamma = p^{-1}(\gamma)$ is a Willmore surface in $(P, [\bar{h}]$ if and only if γ is a critical point of $\mathcal{F}$ in (M, h).*

5. Early applications

Most of the important applications of this result occur when the potential ϕ is constant. In this case, we will refer the critical points of $\mathcal{F}$ as *elasticae* (or elastic curves), [12], and ϕ works as a Lagrange multiplier for the total squared curvature functional. A sufficient condition to guarantee the constancy of ϕ is obtained when we assume that both (M, h) and $(P, \bar{h})$ are Einstein. In this case λ and $\bar{\lambda}$ will denote the corresponding Einstein constants and then, [7]

$$\phi = 2(\lambda - \bar{\lambda}). \tag{6}$$

However, the previous sufficient condition is not necessary. To do clear this claim, we consider a Riemannian submersion $p : (P, \bar{h}) \to (M, h)$ with geodesic fibres isometric to the unit circle and assume that both (M, h) an $(P, \bar{h})$ are Einstein, so we get a constant potential ϕ. Now, we deformate the metric $\bar{h}$ of P by changing the relative scales of the base and the fibre. To be precise, for any positive real number t, we consider $\bar{h}_t$ to be the unique Riemannian metric on P which makes $p : (P, \bar{h}_t) \to (M, h)$ a Riemannian submersion with geodesic fibres isometric to the radius $\sqrt{t}$ circle and horizontal distribution defined via the same ω. In this way, we get a one-parameter family of Riemannian submersions, with $\bar{h}_1 = \bar{h}$, which constitutes the so called *canonical variation* of the starting Riemannian submersion. Since we are assuming that $\bar{h}$ is Einstein, then there is at most one more Einstein metric in $\{\bar{h}_t : t > 0\}$, (see [7] for details). On the other hand, if ϕ_t denotes the potential associated with $\bar{h}_t$, it is not difficult to see that $\phi_t = t.\phi$ and so this is constant for any t.

5.1. Example 1

Let $(\mathbb{S}^3, \bar{h})$ and $(\mathbb{S}^2, h)$ be the round spheres of radii 1 and $1/2$, respectively. The usual Hopf map $p : (\mathbb{S}^3, \bar{h}) \to (\mathbb{S}^2, h)$ is a Riemannian submersion with fibres being geodesics. Since both metrics are Einstein, we apply (8) to obtain $\phi = 4$. Corollary 1 gives the following result due to U.Pinkall, [17]

Corollary 2 *Let γ be a closed curve, immersed in $\mathbb{S}^2$. Then $p^{-1}(\gamma)$ is a Willmore torus in $(\mathbb{S}^3, [\bar{h}])$ if and only if γ is an elastica in $(\mathbb{S}^2, h)$* (with Lagrange multiplier 4).

In [12], J. Langer and D.A. Singer have shown the existence of infinitely many elasticae ($\phi = 4$) in $(\mathbb{S}^2, h)$. Even one can get an infinite series of simple elasticae. Hence, we have the following result, [17]

Corollary 3 *There exist infinitely many Willmore tori in $(\mathbb{S}^3, [\bar{h}])$ which are obtained as Hopf tori over closed elasticae ($\phi = 4$) in $(\mathbb{S}^2, h)$. Furthermore, the only such a torus with constant mean curvature in $(\mathbb{S}^3, \bar{h})$ is the Clifford torus which is shaped on a geodesic of $(\mathbb{S}^2, h)$ and so it is minimal.*

5.2. Example 2

Let $\{p : (\mathbb{S}^3, \bar{h}_t) \to (\mathbb{S}^2, h) \, : \, t > 0\}$ be the canonical variation of the previous Hopf Riemannian submersion. It is obvious that $\bar{h}_1 = \bar{h}$ is the unique Einstein metric in this one-parameter family of metrics on $\mathbb{S}^3$. However, $\bar{h}_t$ has constant scalar curvature for any $t > 0$. Actually these three-spheres can be considered as geodesic spheres in a complex projective plane. The potential ϕ_t is computed to be $4t$ and so we can use once more the results of [12] to obtain the following result, [2]

Corollary 4 *There exist infinitely many Willmore tori in $(\mathbb{S}^3, [\bar{h}_t])$ which are obtained as Hopf tori over closed elasticae ($\phi = 4t$) in $(\mathbb{S}^2, h)$. Furthermore, for $0 < t < 1$, there exist Willmore tori $p^{-1}(\gamma)$ in this series such that*

$$\mathcal{W}(p^{-1}(\gamma)) < 2\pi^2.$$

The last claim in the above statement, strongly contrasts with the well known Willmore conjecture, which ensures that $\mathcal{W}(T) \geq 2\pi^2$ for any torus T immersed in $(\mathbb{S}^3, \bar{h})$.

5.3. Example 3

A particular case in Corollary 1 was obtained in [4]. It corresponds with the case where ω is a flat connection. In this case the O'Neill invariant A, of $p : (P, \bar{h}) \to (M, h)$ vanishes identically. Thus, we combine (3) with (5) to see that ϕ vanishes identically too.

Let $\pi_1(M)$ be the Poincare group of the smooth manifold M. It is well known that G-bundles on M admitting a flat connection are classified by the class of monomorphisms from $\pi_1(M)/H$ (H being a normal subgroup of $\pi_1(M)$) into G. Our next example can be regarded in this context. We consider $M = \mathbb{R} \times \mathbb{S}^1$, its fundamental group is isomorphic to $(\mathbb{Z}, +)$. The universal covering $\mathbb{R}^2$ of M is a principal $\mathbb{Z}$-bundle which admits an obvious trivial flat connection ω_o. Let l be a real number such that l/π is not rational, the map $f_l : \mathbb{Z} \to \mathbb{S}^1$ defined by $f_l(a) = e^{ial}$ is a monomorphism from $(\mathbb{Z}, +)$ in $\mathbb{S}^1 \subset \mathbb{C}$ regarded as a multiplicative group. The transition functions of $\mathbb{R}^2(M, \mathbb{Z})$ can be extended, via f_l, to be valued in $\mathbb{S}^1$ and then considered as transition functions to define a principal $\mathbb{S}^1$-bundle, say $P_l(M, \mathbb{S}^1)$, over M. Now, f_l can also be extended to a monomorphism, $\bar{f}_l$, from $\mathbb{R}^2(M, \mathbb{Z})$ to $P_l(M, \mathbb{S}^1)$ which maps ω_o in a flat connection, also called ω_o, on $P_l(M, \mathbb{S}^1)$. It should be noticed that the holonomy subbundle of this connection is isomorphic to $\mathbb{R}^2(M, \mathbb{Z})$. Corollary 1 can be translated to this case giving

Corollary 5 *Let h be a Riemannian metric on $M = \mathbb{R} \times \mathbb{S}^1$ and $\bar{h} = p^*(h) + \omega_o^*(dt^2)$ the unique Riemannian metric on P_l such that $p : (P_l, \bar{h}) \to (M, h)$ is a Riemannian submersion with fibres being geodesics isometric to the unit circle $(\mathbb{S}^1, dt^2)$. Let γ be any closed curve immersed in M, then $p^{-1}(\gamma)$ is a Willmore torus in $(P_l, [\bar{h}])$ if and only if γ is a critical point of the total squared curvature functional acting on closed curves in (M, g). Moreover, if γ has constant curvature in (M, h), then $p^{-1}(\gamma)$ has constant mean curvature in $(P_l, \bar{h})$.*

The critical points of the elastic energy action $\mathcal{F}(\gamma) = \int_\gamma \kappa^2 ds$ are called *free* elasticae because any constraint on the length of curves is required. As an illustration, we consider the following particular case. Choose h on M such that (M, h) is a catenoid. In other words, we look M as the surface of revolution in $\mathbb{R}^3$ which is obtained when rotate a catenary and then h is the induced metric on M for the Euclidean one in

$\mathbb{R}^3$. It was shown in [5] that (M, h) has exactly two non-geodesic parallels, γ_1 and γ_2, being free elastica. They are placed symmetrically with respect to the unique geodesic parallel, γ_o, and so they are congruent in $(M; h)$. Now, $p^{-1}(\gamma_o)$ and $p^{-1}(\gamma_1)$ are non-congruent Willmore tori in $(P_l, [\bar{h}])$. The former is minimal in $(P_l, \bar{h})$ while the later has non-zero constant mean curvature in $(P_l, \bar{h})$. This result nicely contrasts with the uniqeness for the Clifford torus in $(\mathbb{S}^3, \bar{h})$ previously stated in Corollary 3. It should be finally observed that (compact) minimal surfaces in, for example, a three space are Willmore surfaces, and so considered as trivial, only if this three space has constant curvature.

5.4. Example 4

The next example uses the extrinsic conformal invariance of the Willmore variational problem as an additional ingredient, [1] (see also [13] for another conformal model). Let M be the open hemisphere in the unit round two-sphere defined in $\mathbb{R}^4$ by $x_1 > 0$ and $x_2 = 0$. Denote by h its standard metric of constant curvature one. Then, $P = M \times \mathbb{S}^1$ is the three-sphere where one geodesic was removed. It is evident that it admits a flat connection ω as a principal $\mathbb{S}^1$-bundle over M. The metric of constant curvature one on P is given by $\bar{h} = h + f^2 dt^2$, with the obvious meaning and f being the positive smooth function on M defined as the x_1-projection. In other words, $(P, \bar{h})$ is the warped product $M \times_f \mathbb{S}^1$ (see [7] for details about warped products). It is noteworthy that $p : (P, \bar{h}) \to (M, h)$ is a Riemannian submersion althougth the fibres are not geodesics. In spite of this, we take advantage of the above mentioned conformal invariance to give the following argument. First, we make the conformal change in $(P, \bar{h})$ by considering $\tilde{h} = \frac{1}{f^2}.\bar{h}$. Now, $p : (P, \tilde{h}) \to (M, \frac{1}{f^2}.h)$is a Riemannian submersion with geodesic fibres. Moreover, it is not difficult to see that $(M, \frac{1}{f^2}.h)$ is the standard hyperbolic two-plane with constant curvature -1. Since the complete classification of free elasticae in this surface was provided in [12], we can use it to obtain Willmore tori in the three-sphere endowed with its standard conformal structure. Of course the family of Willmore tori so obtained is different to that obtained by U. Pinkall, [17] and which was reported in Corollary 3.

6. Further applications

We recall that $\mathbb{S}^1$-bundles on a compact manifold M are classified by the cohomology group $H^2(M, \mathbb{Z})$. Given $\beta \in H^2(M, \mathbb{Z})$, we denote by $\beta^{\mathbb{R}} \in H^2(M, \mathbb{R})$ its image under the universal change of coefficients morphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. Let $p : P \to M$ be a principal $\mathbb{S}^1$-bundle associated with β and ω a principal connection with curvature 2-form Ω. Then, $\Omega = p^*(\Theta)$ for a closed 2-form Θ on M. Moreover, the cohomology class $[\Theta]$ satisfies $[\Theta] = 2\pi\beta^{\mathbb{R}}$. The converse also holds, i.e. for any closed 2-form Θ with $[\Theta] = 2\pi\beta^{\mathbb{R}}$, one can find a principal connection ω on the $\mathbb{S}^1$-bundle associated with β, whose curvature is $\Omega = p^*(\Theta)$.

Let (M, h) be a compact Kaehler-Einstein manifold with Kaehler 2-form F. Suppose it has positive scalar curvature. We denote by $c_1(M)$ the first Chern class of M and take β as a rational multiple of $c_1(M)$. Since $[F] = 2\pi_1(M)^{\mathbb{R}}$, then $[F]$ is a multiple of $\beta^{\mathbb{R}}$. Now, a classical result of S. Kobayashi, [11], guarantees the existence of a unique $\mathbb{S}^1$-invariant Einstein Riemannian metric, $\bar{h}$, on P such that $p : (P, \bar{h}) \to (M, h)$ is a Riemannian submersion with totally geodesic fibres. In particular, $\bar{h} = p^*(h) + \omega^*(dt^2)$.

6.1. Example 5

We choose $M = \mathbb{CP}^n$, the complex projective space, endowed with its canonical (Fubini-Study) Kaehler-Einstein metric h and take β as the positive generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$. In this setting, $P = \mathbb{S}^{2n+1}$ with its standard metric and the usual Hopf fibration $p : S^{2n+1} \to \mathbb{CP}^n$ gives a $\mathbb{S}^1$-bundle associated with β. A natural principal connection ω can be defined in this bundle. By choosing h with constant holomorphic sectional curvature 4 and $\bar{h}$ with constant sectional curvature one, then $p : (S^{2n+1}, \bar{h}) \to (\mathbb{CP}^n, h)$ becames a Riemannian submersion between Einstein spaces with geodesic fibres isometric to the unit circle. Then we have

Corollary 6 *Let γ be a closed curve, immersed in $\mathbb{CP}^n$. The $N_\gamma = p^{-1}(\gamma)$ is a Willmore torus in $(S^{2n+1}, [\bar{h})]$ if and only if γ is an elastica in $(\mathbb{CP}^n, h)$* (with potential $\phi = 4$).

A curve γ immersed in $(\mathbb{CP}^n, h)$ is said to have constant slant if the angle between the complex tangent plane and the osculating plane of γ is constant along γ. Curves with osculating plane either holomorphic or Lagrangian obviously have constant slant 0 or $\frac{\pi}{2}$, respectively. In [6], the author joint O.J.Garay and D.A.Singer have obtained the complete classification of elasticae with constant slant in $(\mathbb{CP}^2, h)$. This essentially consists in three families of elasticae. Two of them are torsion free elasticae, they lie fully in totally geodesic surfaces of $(\mathbb{CP}^2, h)$ and their slants reach the extremal values 0 and $\frac{\pi}{2}$ according the surface is holomorphic or Lagrangian, respectively. The third family is a real two-parameter class of helices lying fully in $(\mathbb{CP}^2, h)$. This contains a rational one-parameter subfamily of closed helices which are elasticae in $(\mathbb{CP}^2, h)$ for an arbitrary given potential. Combining this classification with Corollary 1, we have [6]

Corollary 7 *There exist infinitely many Willmore tori in $(\mathbb{S}^5, [\bar{h}])$. This class includes the following three subfamilies:*

1. $\Sigma_1 = \{p^{-1}(\gamma) \ : \ \gamma \,\text{is a closed elastica in}\, \mathbb{S}^2(1/2)\}$, *where $S^2(1/2)$ is a holomorphic and totally geodesic surface in $\mathbb{CP}^n$. This subfamily essentially coincides with that studied by Pinkall which was reported in Example 1.*
2. $\Sigma_2 = \{p^{-1}(\gamma) \ : \ \gamma \,\text{is a closed elastica in}\, \mathbb{RP}^2\}$, *where $\mathbb{RP}^2$ is a Lagrangian and totally geodesic surface in $\mathbb{CP}^2$. The tori of this subfamily lie fully in $(\mathbb{S}^5, \bar{h})$ and contains to the Ejiri torus, [10], which is the only constant mean curvature torus of this family.*
3. $\Sigma_3 = \{p^{-1}(\gamma) \ : \ \gamma \,\text{is a closed elastic helix in}\, (\mathbb{CP}^2, h)\}$. *This subfamily is a rational one-parameter class of constant mean curvature tori in $(\mathbb{S}^5, \bar{h})$.*

In all the cases, the potential is $\phi = 4$.

6.2. Example 6

Let $M = Q_{n-1}$ be the Grassmannian of oriented two planes in $\mathbb{R}^{n+1}$, it can be viewed as the complex quadric in $\mathbb{CP}^n$. It is well known that Q_{n-1} is the only non totally geodesic, Einstein, complex hypersurface in $\mathbb{CP}^n$ and so it admits a natural symmetric Kaehler-Einstein structure, h. We take $\beta = \frac{1}{n} c_1(Q_{n-1})$. Now, the associated $\mathbb{S}^1$-bundle, P, is nothing but the unit tangent bundle of $\mathbb{S}^n$ and it admits a homogeneous Einstein metric $\bar{h}$ (the Stiefel manifold). Therefore, we have a Riemannian submersion $p : (T_1\mathbb{S}^n, \bar{h}) \to (Q_{n-1}, h)$ between Einstein spaces, which has geodesic fibres isometric to the unit circle.

Corollary 8 *Let γ be a closed curve, immersed in (Q_{n-1}, h). Then, $N_\gamma = p^{-1}(\gamma)$ is a Willmore torus in $(T_1\mathbb{S}^n, [\bar{h}])$ if and only if γ is an elastica in (Q_{n-1}, h).*

Now, we are going to construct elastic helices in the complex quadric of complex dimension two. Therefore, we start by standing the following question. Let γ_1 and γ_2 be a couple of closed curves in $\mathbb{S}^2$, we consider the flat torus $\mathcal{T} = \gamma_1 \times \gamma_2$ which is immersed as a product in the complex quadric $Q_2 = \mathbb{S}^2 \times \mathbb{S}^2$. Given a closed geodesic, γ In $\mathcal{T}$, the question is: When is γ an elastica in Q_2? In [3], the author answered this question by showing

Proposition 1 *The following statements hod:*

1. *If a geodesic of $\mathcal{T} = \gamma_1 \times \gamma_2$ is an elastica in Q_2, then both γ_1 and γ_2 have constant curvature.*

2. *In that case and for any constant potential, there exists a rational one-parameter family of closed geodescis in $\mathcal{T} = \gamma_1 \times \gamma_2$ which are elastic helices in Q_2.*

As a consequence, we have

Corollary 9 *There exists a rational one-parameter class of conformal constant mean curvature Willmore tori in $(T_1\mathbb{S}^n, [\bar{h}])$.*

6.3. Example 7

The above mentioned construction of Kobayashi can be extended as follows, [19]. Let (M_i, h_i), $1 \leq i \leq m$, be compact Kaehler-Einstein manifolds with positive scalar curvature. Choose $\beta_i \in H^2(M_i, \mathbb{Z})$ and $a_i \in \mathbb{Z}$ with $c_1(M_i) = a_i\beta_i$, $1 \leq i \leq m$. If $M = M_1 \times \cdots \times M_m$ and Π_i denotes the projection of M onto M_i, then we consider the $\mathbb{S}^1$-bundle $\Pi : P_{b_1,\ldots,b_m} \to M$ with Euler class $\Sigma_{i=1}^m b_i \Pi_i^*(\beta)$, $b_i \in \mathbb{Z}$. The following existence argument is due to M.Wang and W.Ziller, [19]: *For every choice of integers $b_1, \ldots, b_m$ which do not vanish simultaneously, the $\mathbb{S}^1$-bundle $P_{b_1,\ldots,b_m}$ admits an Einstein metric, $\bar{h}$, with positive scalar curvature. These Einstein metrics are obtained in a similar fashion as in the above mentioned Kobayashi's result. That is, Π is made into a Riemannian submersion with geodesic fibres. Moreover, the Yang-Mills connection ω has the harmonic representative of the Euler form as curvature form*.

The examples one gets from this construction in dimension seven are $\mathbb{S}^1$-bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ and over $\mathbb{CP}^2 \times \mathbb{CP}^1$ and the corresponding Einstein metrics were independently discovered in [8, 14, 16].

Let consider the spaces $Q^{abc} = \frac{SU(2)\times SU(2)\times SU(2)}{U(1)\times U(1)}$, where the integers a, b, c determine the embedding of $U(1) \times U(1)$ into $SU(2) \times SU(2) \times SU(2)$. These spaces can equivalently be observed as the above mentioned $\mathbb{S}^1$-bundles over $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ and the integers characterize the twisting degree of the $\mathbb{S}^1$ fibres over $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. The inverse Kaluza-Klein method allows to obtain the above mentioned Einstein metric, say $\bar{h}$ on Q^{abc}. We combine, once more the Corollary 1 with the Proposition 1, to deduce

Corollary 10 *For any choice of integers a, b, c, there exists a rational one-parameter family of Willmore tori in $(Q^{abc}, [\bar{h}])$, which have constant mean curvature in $(Q^{abc}, \bar{h})$.*

The second posibility gives the spaces of E.Witten, [24], named M^{abc}, which can be regarded as $\mathbb{S}^1$-bundles over $\mathbb{CP}^2 \times \mathbb{CP}^1$. If $\bar{h}$ denotes the above obtained Einstein metric, then we have

Corollary 11 *For any choice of $a, b, c \in \mathbb{Z}$, there exists a rational one-parameter class of Willmore tori in $(M^{abc}, [\bar{h}])$ which have constant mean curvature in $(M^{abc}, \bar{h})$. These tori are obtained as liftings of a corresponding family of closed elastic helices in a complex quadric totally geodesic in $\mathbb{CP}^2 \times \mathbb{CP}^1$.*

we also have

Corollary 12 *For any choice of $a, b, c \in \mathbb{Z}$, there exists a rational one-parameter class of Willmore tori in $(M^{abc}, [\bar{h}])$ which have constant mean curvature in $(M^{abc}, \bar{h})$. They are obtained by lifting the subfamily Σ_3 of closed elastic helices in a complex projective plane totally geodesic in $\mathbb{CP}^2 \times \mathbb{CP}^1$.*

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