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Nilpotent complex structures

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Abstract. This paper is a survey on some recent results about nilpotent complex structures J on compact nilmanifolds. We deal with the classification problem of compact nilmanifolds admitting such a J, the study of a Dolbeault minimal model and its formality, and the construction of nilpotent complex structures for which the Frölicher spectral sequence does not collapse at the second term.

Estructuras complejas nilpotentes

Resumen. Este artículo presenta un panorama de algunos resultados recientes sobre estructuras complejas nilpotentes J definidas sobre nilvariedades compactas. Tratamos el problema de clasificación de nilvariedades compactas que admiten una tal J, el estudio de un modelo minimal de Dolbeault y su formalidad, y la construcción de estructuras complejas nilpotentes para las cuales la sucesión espectral de Frölicher no colapsa en el segundo término.

1. Introduction

In the last few years the methods and techniques of *rational homotopy theory* have been applied successfully to some geometric problems, as for example the construction of symplectic manifolds with no Kähler structure (see [23] for a survey on this subject). In the context of complex manifolds, Neisendorfer and Taylor [19] built a *Dolbeault rational homotopy theory* which, roughly speaking, can be seen as a version of Sullivan's theory for certain bigraded algebras. In contrast with rational homotopy theory, the theory of [19] has been not much developed in many cases. For example, it is well known a minimal model for the de Rham complex of a (compact) homogeneous space $\Gamma\backslash G$, G being a simply-connected *nilpotent* Lie group and Γ a lattice in G of maximal rank; however, not much has been studied about Dolbeault (minimal) models of complex structures on such spaces $\Gamma\backslash G$. From now on we shall refer to $\Gamma\backslash G$ as a *compact nilmanifold*, and we shall restrict our attention to this special class of manifolds.

In the context of nilmanifolds some aspects of the Dolbeault homotopy theory have been studied by Sakane [21], who proved that the computation of the Dolbeault cohomology of $\Gamma\backslash G$, in the case of G being a *complex* Lie group, can be reduced to a calculation at the Lie algebra level. Such a $\Gamma\backslash G$ is *complex parallelizable* in the sense of Wang [24]. But there is a more general class of compact nilmanifolds having complex structures which are not complex parallelizable, although all of them are real parallelizable. In fact, any left invariant complex structure J on the Lie group G descends to the quotient, and therefore $\Gamma\backslash G$ becomes a complex manifold.

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In this paper we deal with *nilpotent complex structures* J, that is, left invariant complex structures for which there exists an ordered basis $\{\omega_1, \ldots, \omega_n\}$ of complex 1-forms of type $(1,0)$ such that the differential of a (complex) 1-form is expressed in terms of the preceding (complex) forms and their conjugates. Notice that since G is nilpotent, this property always holds for the (real) structure equations of G, but it may not be satisfied for its complex equations.

Recently, the authors of this paper have studied some aspects of the Dolbeault homotopy theory for compact nilmanifolds endowed with a nilpotent complex structure (see [3, 8, 9, 10]). Our main purpose here is to give a brief survey on some results on this type of complex structures (a detailed and complete development can be found in [6, 7, 8, 9, 10]).

In Section 2 we show that there are strong necessary conditions for the existence of a nilpotent complex structure on a compact nilmanifold $\Gamma\backslash G$. In particular, we show that the first Betti number $b_1(\Gamma\backslash G)$ of $\Gamma\backslash G$ must be at least 3. Moreover, we classify (up to dimension 6) the compact nilmanifolds admitting such complex structures. In Section 3 we get a minimal model for the Dolbeault complex of $\Gamma\backslash G$, and we show that such a model is formal if and only if $\Gamma\backslash G$ is a complex torus. Finally, Section 4 is devoted to the study of the Frölicher spectral sequence $\{E_r\}_{r\geq 1}$ of compact nilmanifolds with nilpotent complex structure. We exhibit a family of such nilmanifolds M_{ABC} (where A, B, C are rational parameters), which can be described as the total space of a holomorphic torus bundle over the well known Kodaira-Thurston manifold. Many complex manifolds M_{ABC} in this family satisfy $E_2 \ncong E_\infty$. Since $\dim_{\mathbb{C}} M_{ABC} = 3$, they have the lowest possible dimension for which one can obtain $E_2 \ncong E_\infty$.

2. Classification of six-dimensional compact nilmanifolds with nilpotent complex structures

In this section we show necessary conditions for the existence of a nilpotent complex structure on a compact nilmanifold and classify, up to real dimension 6, the compact nilmanifolds having such structures.

First, let us start by analysing the simply-connected nilpotent Lie groups G, of real dimension $2n$, endowed with a left invariant complex structure J. If G is s-step nilpotent and $\mathfrak{g}$ denotes its Lie algebra, then the ascending central series $\{\mathfrak{g}_l\}_{l\geq 0}$ of $\mathfrak{g}$ satisfies

$$\mathfrak{g}_0 = \{0\} \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{s-1} \subset \mathfrak{g}_s = \mathfrak{g}, \tag{1}$$

where each $\mathfrak{g}_l$ is an ideal in $\mathfrak{g}$ which is defined inductively by $\mathfrak{g}_l = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}\}$ for $l \geq 1$. It is well known that the sequence (1) increases strictly until $\mathfrak{g}_s$ and, in particular, the center $\mathfrak{g}_1$ of the Lie algebra is nonzero.

Since the spaces $\mathfrak{g}_l$ are not in general J-invariant, the sequence $\{\mathfrak{g}_l\}$ is not suitable to work with J. We introduce a new sequence $\{\mathfrak{a}_l(J)\}$ having the property of J-invariance. The *ascending series* $\{\mathfrak{a}_l(J)\}_{l\geq 0}$ of the Lie algebra $\mathfrak{g}$, *compatible* with the left invariant complex structure J on G, is defined inductively as follows:

$$\begin{cases} \mathfrak{a}_0(J) &= \{0\}\,, \\ \mathfrak{a}_l(J) &= \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1}(J) \ \text{ and } \ [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1}(J)\}\,, \quad l \geq 1. \end{cases} \tag{2}$$

It is clear that each $\mathfrak{a}_l(J)$ is a J-invariant ideal in $\mathfrak{g}$ and $\mathfrak{a}_l(J) \subseteq \mathfrak{g}_l$, for $l \geq 0$. It must be remarked that the terms $\mathfrak{a}_l(J)$ depend on the complex structure J considered on G (see [9] for explicit examples). Moreover, this ascending series, in spite of $\mathfrak{g}$ being nilpotent, can stop without reaching the Lie algebra $\mathfrak{g}$, that is, it may happen that $\mathfrak{a}_l(J) = \mathfrak{a}_t(J) \neq \mathfrak{g}$ for all $l \geq t$. The following definition is motivated by this fact.

Definition 1 *A left invariant complex structure J on G is called nilpotent if the series $\{\mathfrak{a}_l(J)\}_{l\geq 0}$ given by (2) satisfies $\mathfrak{a}_t(J) = \mathfrak{g}$ for some integer $t > 0$.*

Let us denote by $\{\mathfrak{g}^k\}_{k\geq 0}$ the descending central series of $\mathfrak{g}$, which is defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$, for $k \geq 1$. Recall that $\mathfrak{g}$ is s-step nilpotent if and only if $\mathfrak{g}^s = 0$ and $\mathfrak{g}^{s-1} \neq 0$. In the following result we show sufficient conditions on a nilpotent Lie group G which guarantee the nilpotency of a left invariant complex structure J on G.

Proposition 1 [9] *Let J be a left invariant complex structure on G.*

(i) *If all the terms $\mathfrak{g}_l$ are invariant under J, then $\mathfrak{a}_l(J) = \mathfrak{g}_l$ for all $l \geq 0$. In particular, J is nilpotent.*

(ii) *If all the terms $\mathfrak{g}^k$ are J-invariant, then $\mathfrak{a}_s(J) = \mathfrak{g}_s = \mathfrak{g}$, that is, J is nilpotent.*

However, there are nilpotent complex structures J on Lie groups G for which some of the terms $\mathfrak{g}_l$ or $\mathfrak{g}^k$ are not invariant under J.

Definition 1 includes the complex structures of *complex* nilpotent Lie groups as a particular case. In fact, if J is the canonical complex structure of such a Lie group then $J(\mathfrak{g}_l) = \mathfrak{g}_l$ for $l \geq 0$, because $\mathfrak{g}$ is a *complex* Lie algebra and any ideal of $\mathfrak{g}$ is J-invariant. From Proposition 1 we get:

Corollary 1 *The canonical complex structure of a complex nilpotent Lie group is nilpotent.*

Next we show a characterization of the nilpotency of J in terms of the (complex) structure equations of the Lie group.

Theorem 1 [9] *Let G be a simply-connected nilpotent Lie group of real dimension $2n$. A left invariant complex structure J on G is nilpotent if and only if there exists an ordered basis $\{\omega_i, \overline{\omega}_i; 1 \leq i \leq n\}$ of left invariant complex 1-forms on G satisfying*

$$d\omega_i = \sum_{j<k<i} A_{ijk}\, \omega_j \wedge \omega_k + \sum_{j,k<i} B_{ijk}\, \omega_j \wedge \overline{\omega}_k\,, \quad 1 \leq i \leq n\,, \tag{3}$$

where each ω_i is of type $(1,0)$, and $\overline{\omega}_i$ of type $(0,1)$, with respect to J.

Using this characterization it is very easy to construct many nilpotent complex structures which do not come from complex Lie groups. In fact:

Corollary 2 *The structure equations* (3) *with the coefficients chosen so that $d^2 = 0$ define a nilpotent Lie group G with a nilpotent left invariant complex structure.*

Notice that the nilpotent complex structures of Corollary 1 are precisely those for which all the coefficients B_{ijk} in (3) vanish. In fact, if $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{g}_{1,0}$ (resp. $\mathfrak{g}_{0,1}$) denotes the subalgebra of elements of type (1,0) (resp. (0,1)) then $[\mathfrak{g}_{1,0}, \mathfrak{g}_{0,1}] = 0$, which in view of equations (3), means that $B_{ijk} = 0$ for all i, j, k.

Next, we give some necessary conditions for the existence of nilpotent complex structures on a nilpotent Lie group G (see [9] for a proof).

Proposition 2 *Let G be a simply-connected s-step nilpotent Lie group of* (*real*) *dimension $2n$, with Lie algebra $\mathfrak{g}$. Suppose that G carries a nilpotent complex structure J. Then:*

(i) $\dim[\mathfrak{g}, \mathfrak{g}] \leq 2n - 3$.

(ii) $\dim \mathfrak{g}_l \geq \dim \mathfrak{a}_l(J) \geq 2l$, *for* $0 \leq l \leq s$.

(iii) *The first integer t for which $\mathfrak{a}_t(J) = \mathfrak{g}$ satisfies $s \leq t \leq n$.*

Let us denote by $H^1(\mathfrak{g})$ the first Chevalley-Eilenberg cohomology group of the Lie algebra $\mathfrak{g}$. If G is in the conditions of Proposition 2 then $\dim H^1(\mathfrak{g}) = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \geq 2n - (2n - 3) = 3$. Therefore:

Corollary 3 *Under the conditions of Proposition* 2 *we have that* $s \le n$, $\dim \mathfrak{g}_l \ge 2l$ *for* $0 \le l \le s$, *and* $\dim H^1(\mathfrak{g}) \ge 3$.

Next we classify the nilpotent Lie groups G of even real dimension ≤ 6 admitting nilpotent complex structures in terms of their Lie algebras.

It is obvious that any left invariant complex structure on the Abelian Lie group $\mathbb{R}^{2n}$ is nilpotent. Moreover, in [12] it is proved that if G is a non Abelian nilpotent Lie group of dimension ≤ 4 then, G has left invariant complex structures *if and only if* its Lie algebra $\mathfrak{g}$ is 2-step nilpotent or, equivalently, $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{K} = \langle X_1, X_2, X_3, X_4 \mid [X_1, X_2] = X_3 \rangle$. Denote by K the nilpotent Lie group defined by $\mathfrak{K}$. It is easy to check that the almost complex structure J given by $JX_1 = X_2$ and $JX_3 = X_4$, is integrable on K. Since $\mathfrak{a}_1(J) = \{X_3, X_4\}$ and $\mathfrak{a}_2(J) = \mathfrak{K}$, the structure J is nilpotent on K.

The moduli space of left invariant complex structures J on K has been computed explicitly in [1]: moreover, it is proved that for any such J the (complex) structure equations of K can be written as $d\omega_1 = 0$, $d\omega_2 = \omega_1 \wedge \overline{\omega}_1$. Therefore, from Theorem 1 we have:

Theorem 2 *Let* G *be a* s*-step nilpotent Lie group of dimension* ≤ 4. *Then,* G *has nilpotent complex structures if and only if* $s \le 2$. *In this case, any left invariant complex structure on G is nilpotent.*

This theorem shows that the necessary condition $s \le n$ given in Corollary 3 is also sufficient for $n \le 2$. However, for $n = 3$ the situation is more complicated. First, it must be remarked that there are 6-dimensional Lie groups with nonnilpotent left invariant complex structures (see [9]). Secondly, there is a nilpotent Lie group satisfying all the conditions of Corollary 3 but not admitting nilpotent complex structures. In fact, let G be the nilpotent Lie group defined by the nilpotent Lie algebra $\mathfrak{g} = \langle X_1, X_2, X_3, X_4 \mid [X_1, X_2] = X_3, [X_1, X_3] = X_4 \rangle \times \mathfrak{a}^2$, $\mathfrak{a}$ being the Abelian Lie algebra of dimension k. It is easy to check that $s = n = 3$, $\dim \mathfrak{g}_1 = 3$, $\dim \mathfrak{g}_2 = 4$, and $\dim H^1(\mathfrak{g}) = 4$. However, in [7] it is proved that G has no nilpotent complex structure. Therefore, the necessary conditions of Corollary 3 are not sufficient in dimension > 4.

Furthermore, in dimension 6 we have the following general result:

Theorem 3 [7] *Let* G *be a simply-connected* s*-step nilpotent Lie group of dimension* 6, *with Lie algebra* $\mathfrak{g}$.

(i) *If* $s \le 2$, *then* G *admits nilpotent complex structures.*

(ii) *If* $s = 3$ *then* G *admits nilpotent complex structures, except when* $\dim \mathfrak{g}_1$ *is odd and not equal to* $\dim [\mathfrak{g}, \mathfrak{g}]$.

(iii) *If* $s \ge 4$, *then* G *has no nilpotent complex structure.*

There are sixteen nonisomorphic classes of 6-dimensional nilpotent Lie algebras for which their corresponding Lie groups have nilpotent complex structures. They are given in the following table:

Algebra	**Defining bracket relations**	**dim** $H^1(\mathfrak{g})$	**dim** $\mathfrak{g}_l$
$\mathfrak{h}_1 = \mathfrak{a}^6$	$[\,,\,] \equiv 0$	6	(6)
$\mathfrak{h}_2$	$[X_1, X_2] = X_5, [X_3, X_4] = X_6$	4	$(2, 6)$
$\mathfrak{h}_3$	$[X_1, X_2] = X_6, [X_3, X_4] = X_6$	5	$(2, 6)$
$\mathfrak{h}_4$	$[X_1, X_2] = X_5, [X_1, X_3] = X_6, [X_2, X_4] = X_6$	4	$(2, 6)$
$\mathfrak{h}_5$	$[X_1, X_3] = X_5, [X_1, X_4] = X_6,$ $[X_2, X_4] = X_5, [X_2, X_3] = -X_6$	4	$(2, 6)$
$\mathfrak{h}_6$	$[X_1, X_2] = X_5, [X_1, X_3] = X_6$	4	$(3, 6)$
$\mathfrak{h}_7$	$[X_1, X_2] = X_6, [X_1, X_3] = X_4, [X_2, X_3] = X_5$	3	$(3, 6)$
$\mathfrak{h}_8 = \mathfrak{K} \times \mathfrak{a}^2$	$[X_1, X_2] = X_3$	5	$(4, 6)$
$\mathfrak{h}_9$	$[X_1, X_2] = X_3, [X_1, X_3] = X_6, [X_2, X_4] = X_6$	4	$(2, 4, 6)$
$\mathfrak{h}_{10}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_1, X_4] = X_6$	3	$(2, 4, 6)$
$\mathfrak{h}_{11}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5,$ $[X_1, X_4] = X_6, [X_2, X_4] = -X_5$	3	$(2, 4, 6)$
$\mathfrak{h}_{12}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_2, X_4] = -X_6$	3	$(2, 4, 6)$
$\mathfrak{h}_{13}$	$[X_1, X_2] = X_3 + X_4, [X_1, X_3] = X_5, [X_2, X_4] = X_6$	3	$(2, 4, 6)$
$\mathfrak{h}_{14}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5,$ $[X_1, X_4] = X_6, [X_2, X_3] = X_6$	3	$(2, 4, 6)$
$\mathfrak{h}_{15}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_1, X_4] = X_6,$ $[X_2, X_4] = X_5, [X_2, X_3] = -X_6$	3	$(2, 4, 6)$
$\mathfrak{h}_{16}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_2, X_3] = X_6$	3	$(3, 4, 6)$

Next we consider compact quotients of a nilpotent Lie group G. Let us first recall [17] that such a simply-connected G has compact quotients of the form $\Gamma\backslash G$, Γ being a lattice in G of maximal rank, provided there exists a basis of left invariant 1-forms on G for which the coefficients in the structure equations are rational numbers. Let us suppose next that G has such a lattice Γ.

Definition 2 *A compact nilmanifold with a nilpotent complex structure J is a complex manifold of the form $\Gamma\backslash G$ whose complex structure J is inherited from a nilpotent (left invariant) complex structure on G by passing to the quotient.*

If G is a *complex* Lie group then $\Gamma\backslash G$ is a *compact complex parallelizable nilmanifold*, that is, there are n holomorphic 1-forms on $\Gamma\backslash G$ which are linearly independent at each point. Corollary 1 implies that this class of manifolds is contained in the class of compact nilmanifolds with nilpotent complex structure. Moreover, from Theorem 1 and Mal'cev theorem [17] we have:

Corollary 4 *The structure equations* (3) *with rational coefficients chosen so that* $d^2 = 0$ *define a compact nilmanifold with nilpotent complex structure* $\Gamma\backslash G$. *Furthermore, if some* B_{ijk} *is nonzero then* $\Gamma\backslash G$ *is not complex parallelizable.*

As a consequence of Corollary 3 we get in particular the following necessary topological condition for $\Gamma\backslash G$ to have a nilpotent complex structure:

Corollary 5 *Let* $\Gamma\backslash G$ *be a compact nilmanifold with a nilpotent complex structure. Then* $b_1(\Gamma\backslash G) \geq 3$, *where* $b_1(\Gamma\backslash G)$ *denotes the first Betti number of* $\Gamma\backslash G$.

PROOF. From Nomizu's theorem [20] we know that $H^1(\Gamma\backslash G) \cong H^1(\mathfrak{g})$, where $H^1(\Gamma\backslash G)$ is the first de Rham cohomology group of $\Gamma\backslash G$, and $\mathfrak{g}$ denotes the Lie algebra of G. Therefore, the result follows from Corollary 3. ■

Next we restrict again our attention to dimension $2n$, with $n \leq 3$. It is obvious that the tori $\mathbb{T}^{2n} = \mathbb{Z}^{2n}\backslash\mathbb{R}^{2n}$ have nilpotent complex structures. Let Γ be a lattice in the 4-dimensional Lie group K given above. A compact nilmanifold $\Gamma\backslash K$ is known as a *Kodaira-Thurston manifold*. Thus, Theorem 2 implies:

Proposition 3 *Apart from a torus, the only* 4*-dimensional compact nilmanifold admitting nilpotent complex structures is a Kodaira-Thurston manifold* $\Gamma\backslash K$.

Remark 1 We recall that apart from $\mathfrak{a}^4$ and $\mathfrak{K}$, there is another 4-dimensional nilpotent Lie algebra. In [12] it is proved that no compact quotient of a Lie group defined by this algebra, can have a complex structure (homogeneous or otherwise).

Let H_r denote a simply-connected nilpotent Lie group with Lie algebra $\mathfrak{h}_r$ given in the above table ($1 \leq r \leq 16$). Since all the structure constants are rational numbers, Mal'cev's theorem [17] guarantees that each H_r has a discrete subgroup Γ_r such that $\Gamma_r\backslash H_r$ is compact. From Theorem 3 we have:

Proposition 4 *A* 6*-dimensional compact nilmanifold has nilpotent complex structures if and only if it is of the form* $\Gamma_r\backslash H_r$ *for some* $1 \leq r \leq 16$.

Explicit examples of nilpotent complex structures on each $\Gamma_r\backslash H_r$ can be found in [7].

3. Dolbeault minimal models and formality

In this section we construct a minimal model for the Dolbeault complex of a compact nilmanifold with nilpotent complex structure $\Gamma\backslash G$. Moreover, if $\Gamma\backslash G$ is not a torus then such a model is never formal.

First, we recall some basic definitions of the Dolbeault homotopy theory [19]. A *differential bigraded algebra*, DBA, $(\mathcal{A}^{*,*}, \bar\partial)$ is defined to be a bigraded commutative algebra $\mathcal{A}^{*,*}$ over $\mathbb{C}$ with a differential $\bar\partial$ of type (0,1), which is a derivation with respect to the total degree. We obtain the category of DBA's by requiring the maps to be bidegree preserving and commuting with $\bar\partial$.

A DBA $(\mathcal{A}^{*,*}, \bar\partial)$ is *minimal* if $\mathcal{A}^{*,*}$ is free as an algebra and the differential $\bar\partial$ is decomposable; it is said to be *formal* if there is a morphism $\psi\colon (\mathcal{A}^{*,*}, \bar\partial) \longrightarrow (H^{*,*}(\mathcal{A}), 0)$ inducing the identity on cohomology, where $H^{*,*}(\mathcal{A})$ denotes the cohomology of $(\mathcal{A}^{*,*}, \bar\partial)$. Nonzero Massey products for $H^{*,*}(\mathcal{A})$ are obstructions to the formality of $(\mathcal{A}^{*,*}, \bar\partial)$.

Recall that on a complex manifold M the exterior differential d decomposes as $d = \partial + \bar\partial$, with $\bar\partial$ of type (0,1). Therefore, the Dolbeault complex $(\Lambda^{*,*}(M), \bar\partial)$ of M, where $\Lambda^{*,*}(M)$ denotes the complex valued forms on M, belongs to the category of DBA's. By definition, a DBA $(\mathcal{A}^{*,*}, \bar\partial)$ is a *model* for the Dolbeault complex of M if there is a morphism $\varphi\colon (\mathcal{A}^{*,*}, \bar\partial) \longrightarrow (\Lambda^{*,*}(M), \bar\partial)$ inducing an isomorphism on cohomology. We shall say that M is *Dolbeault formal* if there exists a minimal model for $(\Lambda^{*,*}(M), \bar\partial)$ which is formal.

Let G be a nilpotent Lie group with a left invariant complex structure J. Denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of the Lie algebra $\mathfrak{g}$ of G, and by $\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*$ the bigraduation induced on $\Lambda^*(\mathfrak{g}^{\mathbb{C}})^*$ by J. The natural extension d of the Chevalley-Eilenberg differential to $\Lambda^*(\mathfrak{g}^{\mathbb{C}})^* = \bigoplus_{p,q\geq 0} \Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^*$ also admits a decomposition $d = \partial + \bar{\partial}$, where $\bar{\partial}\colon \Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^* \longrightarrow \Lambda^{p,q+1}(\mathfrak{g}^{\mathbb{C}})^*$ is of type (0,1). Then $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar{\partial})$ becomes a DBA.

Notice that each $\Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^*$ is canonically identified to the space of complex valued left invariant forms of type (p, q) on the Lie group G. Therefore, if $\Gamma\backslash G$ is a compact nilmanifold with nilpotent complex structure then there is a canonical morphism of DBA's

$$i\colon (\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar{\partial}) \longrightarrow (\Lambda^{*,*}(\Gamma\backslash G), \bar{\partial}).$$

In [9] it is proved that i induces an isomorphism on cohomology, that is:

Theorem 4 *Let $\Gamma\backslash G$ be a compact nilmanifold endowed with a nilpotent complex structure. Then, $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar{\partial})$ is a model for the Dolbeault complex of $\Gamma\backslash G$.*

Furthermore, $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar{\partial})$ is a *minimal* DBA. In fact, since it is an exterior algebra it is obviously free, and if $\dim_{\mathbb{R}} G = 2n$ then we can write

$$\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^* \equiv \Lambda^{*,*}(x^1_{1,0}, x^1_{0,1}, \ldots, x^n_{1,0}, x^n_{0,1}),$$

where the generators have total degree 1 and bidegree as indexed. From the nilpotency of J and Theorem 1 it follows that we can choose the generators as an ordered set $\{x^1_{1,0}, x^1_{0,1}, \ldots, x^n_{1,0}, x^n_{0,1}\}$ with respect to which $\bar{\partial}$ is given by

$$\bar{\partial} x^i_{1,0} = \sum_{j,k<i} B_{ijk}\, x^j_{1,0} \cdot x^k_{0,1}\,, \qquad \bar{\partial} x^i_{0,1} = \sum_{j<k<i} \bar{A}_{ijk}\, x^j_{0,1} \cdot x^k_{0,1}\,,$$

for $1 \leq i \leq n$. Therefore, the differential $\bar{\partial}$ is decomposable.

Corollary 6 *Under the conditions of the above theorem, $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar{\partial})$ is a minimal model for the Dolbeault complex of $\Gamma\backslash G$.*

Also, as a trivial consequence of Theorem 4, we have the following result which, in view of Corollary 1, extends Sakane's theorem [21] for compact complex parallelizable nilmanifolds.

Corollary 7 *Under the conditions of Theorem 4, there is a canonical isomorphism $H^{p,q}_{\bar{\partial}}(\Gamma\backslash G) \cong H^{p,q}_{\bar{\partial}}(\mathfrak{g}^{\mathbb{C}})$, for all p and q.*

The formality of the DBA $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar{\partial})$ has been studied in [10]: if the Lie algebra $\mathfrak{g}$ is nonAbelian then such a DBA is never formal. Therefore:

Theorem 5 *Let $\Gamma\backslash G$ be a compact nilmanifold with nilpotent complex structure. Then, $\Gamma\backslash G$ is Dolbeault formal if and only if $\Gamma\backslash G$ is a complex torus.*

There is a strong relation between Dolbeault formality and Kähler structures. In fact, according to Neisendorfer and Taylor [19], a minimal model for the Dolbeault complex of a compact Kähler manifold is always formal.

Corollary 8 *A compact nilmanifold with nilpotent complex structure does not admit Kähler structure unless it is a complex torus.*

4. Frölicher spectral sequence

In this section we show that there are nilpotent complex structures for which the Frölicher spectral sequence satisfies $E_2 \ncong E_\infty$. Moreover, such complex structures exist on compact nilmanifolds $\Gamma\backslash G$ of (real) dimension 6, which provides a solution, in the lowest possible dimension, to a problem posed by Griffiths and Harris [14].

For any complex manifold M, there is [13] a spectral sequence $\{E_r(M)\}_{r\geq 1}$ relating the Dolbeault cohomology $H^{*,*}_{\bar\partial}(M) \equiv E_1(M)$ of M to the de Rham cohomology $H^*(M) \equiv E_\infty(M)$ of the manifold. In fact, since $\partial\bar\partial = -\bar\partial\partial$ we have on M the double complex $(\Lambda^{*,*}(M), \partial, \bar\partial)$, and the first spectral sequence associated to it in a standard way is precisely $\{E_r(M)\}_{r\geq 1}$ (see [6] for a description of the terms $E_r^{p,q}(M)$).

Frölicher observed [13] that $E_1(M) \cong E_\infty(M)$ for any compact Kähler manifold M. Then, using Theorem 5, we have for compact nilmanifolds the following relation between Dolbeault formality and the nondegeneration of the sequence:

Corollary 9 *Let $\Gamma\backslash G$ be a compact nilmanifold with a nilpotent complex structure. If $E_1(\Gamma\backslash G) \ncong E_\infty(\Gamma\backslash G)$ then $\Gamma\backslash G$ is not Dolbeault formal.*

It is easy to construct compact complex parallelizable nilmanifolds $\Gamma\backslash G$ with $E_1(\Gamma\backslash G) \ncong E_2(\Gamma\backslash G)$. However, such manifolds always have $E_2 \cong E_\infty$ [5, 21]. Therefore, to construct nilpotent complex structures on a compact nilmanifold $\Gamma\backslash G$ for which $E_2 \ncong E_\infty$ it is necessary to consider a noncomplex Lie group G. The following result is an extension of Corollary 7 to all the terms in the sequence $\{E_r(\Gamma\backslash G)\}_{r\geq 1}$.

Theorem 6 [8] *Let $\Gamma\backslash G$ be a compact nilmanifold endowed with a nilpotent complex structure, and $\mathfrak{g}$ the Lie algebra of G. Denote by $\{E_r(\mathfrak{g}^{\mathbb{C}})\}_{r\geq 1}$ the first spectral sequence associated to the double complex $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \partial, \bar\partial)$. Then, for $r \geq 1$ and for any p, q, there is a canonical isomorphism $E_r^{p,q}(\Gamma\backslash G) \cong E_r^{p,q}(\mathfrak{g}^{\mathbb{C}})$.*

Let us consider the structure equations

$$\begin{cases} d\omega_1 = 0, \\ d\omega_2 = \omega_1 \wedge \overline{\omega}_1, \\ d\omega_3 = A\,\omega_1 \wedge \omega_2 + B\,\omega_1 \wedge \overline{\omega}_2 + C\,\omega_2 \wedge \overline{\omega}_1, \end{cases} \tag{4}$$

where $A, B, C \in \mathbb{Q}$. It follows from Corollary 2 that (4) defines a family of simply-connected nilpotent Lie groups G_{ABC}, endowed with a nilpotent (left invariant) complex structure. Moreover, since the coefficients A, B and C are rational numbers Mal'cev theorem [17] asserts the existence of a lattice Γ_{ABC} in G_{ABC} of maximal rank. Therefore:

Proposition 5 *The equations* (4) *define a* 3*-parametric family of compact nilmanifolds with nilpotent complex structure $M_{ABC} = \Gamma_{ABC}\backslash G_{ABC}$, of complex dimension* 3.

Since $d\omega_1 = 0, d\omega_2 = \omega_1 \wedge \overline{\omega}_1$ are precisely the structure equations of the Lie group K given in Section 2, it is easy to see that each M_{ABC} is the total space of a holomorphic principal bundle $\mathbb{T}^1 \hookrightarrow M_{ABC} \to \Gamma\backslash K$, where $\mathbb{T}^1$ is a complex torus and $\Gamma\backslash K$ the Kodaira-Thurston manifold. Since $\Gamma\backslash K$ has complex dimension 2, its associated Frölicher spectral sequence degenerates at E_1 [16]. However, as we shall see next, the sequence $\{E_r\}$ does not collapse even at the second term for many complex manifolds M_{ABC}.

The following table shows the behaviour of the sequence $\{E_r\}_{r\geq 1}$ for each manifold M_{ABC} in the family defined by (4). Observe that using Theorem 6 the computation of each $E_r^{p,q}(M_{ABC})$ is reduced to a calculation at the Lie algebra level (a detailed proof can be found in [6, 8]).

Parameters A, B, C		**dim** E_1	**dim** E_2	**dim** E_3	**Sequence** $\{E_r\}$
$A = B = C = 0$		48	48	48	$E_1 \cong E_\infty$
$A \neq 0,\ B = C = 0$		40	24	24	$E_1 \ncong E_2 \cong E_\infty$
$B \neq 0,\ A = C = 0$		40	24	24	$E_1 \ncong E_2 \cong E_\infty$
$C \neq 0,\ A = B = 0$		32	28	24	$E_1 \ncong E_2 \ncong E_3 \cong E_\infty$
$A, B \neq 0$ $C = 0$	$B = A$	36	36	36	$E_1 \cong E_\infty$
	$B \neq A$	36	24	24	$E_1 \ncong E_2 \cong E_\infty$
$A, C \neq 0$ $B = 0$	$C^2 = A^2$	28	28	28	$E_1 \cong E_\infty$
	$C^2 \neq A^2$	28	28	24	$E_1 \cong E_2 \ncong E_3 \cong E_\infty$
$B, C \neq 0$ $A = 0$	$C^2 = B^2$	32	32	32	$E_1 \cong E_\infty$
	$C^2 \neq B^2$	32	28	24	$E_1 \ncong E_2 \ncong E_3 \cong E_\infty$
$A, B, C \neq 0$	$C^2 = (B - A)^2$	28	28	28	$E_1 \cong E_\infty$
	$C^2 \neq (B - A)^2$	28	28	24	$E_1 \cong E_2 \ncong E_3 \cong E_\infty$

In the following theorem we sum up the information given above.

Theorem 7 *Let M_{ABC} be a compact complex manifold defined by* (4). *Then,*

$$E_2(M_{ABC}) \ncong E_\infty(M_{ABC}) \quad \textit{if and only if} \quad C \neq 0 \quad \textit{and} \quad C^2 \neq (B - A)^2.$$

Moreover, if $E_2(M_{ABC}) \ncong E_\infty(M_{ABC})$ then

$$E_1(M_{ABC}) \ncong E_2(M_{ABC}) \qquad \textit{if and only if} \qquad A = 0.$$

Finally, let us remark that the converse of Corollary 9 does not hold. In fact, from the table above and Theorem 5 we see that M_{000} (obtained for $A = B = C = 0$) is a non Dolbeault formal example with $E_1 \cong E_\infty$.

Remark 2 (added in Galley Proof) We would like to call the attention on the existence of some recent results about left invariant complex structures on nilpotent Lie groups, which are closed related to the topics of the present survey:
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