**RACSAM**
*Rev. R. Acad. Cien. Serie A. Mat.*
VOL. **100** (1-2), 2006, pp. 63–74
Análisis Matemático / Mathematical Analysis
Artículo panorámico / Survey

# On the range of the derivative of a smooth function and applications

**Robert Deville**

**Abstract.** We survey recent results on the structure of the range of the derivative of a smooth real valued function $f$ defined on a real Banach space $X$ and of a smooth mapping $F$ between two real Banach spaces $X$ and $Y$. We recall some necessary conditions and some sufficient conditions on a subset $A$ of $\mathcal{L}(X,Y)$ for the existence of a Fréchet-differentiable mapping $F$ from $X$ into $Y$ so that $F'(X) = A$. Whenever $F$ is only assumed Gâteaux-differentiable, new phenomena appear : we discuss the existence of a mapping $F$ from a Banach space $X$ into a Banach space $Y$, which is bounded, Lipschitz-continuous, and so that for all $x, y \in X$, if $x \neq y$, then $\|F'(x) - F'(y)\|_{\mathcal{L}(X,Y)} > 1$. Applications are given to existence and uniqueness of solutions of Hamilton-Jacobi equations.

**ESTRUCTURA DEL RANGO DE LA DERIVADA DE UNA FUNCIÓN**

**Resumen.** Recogemos recientes resultados sobre la estructura del rango de la derivada de una función real $f$ definida en un espacio de Banach real $X$ y de una aplicación diferenciable $F$ entre dos espacios de Banach reales $X$ e $Y$. Listamos algunas condiciones necesarias y otras suficientes acerca de un subconjunto $A$ de $\mathcal{L}(X,Y)$ para la existencia de una aplicación diferenciable Fréchet $F$ de $X$ en $Y$ de modo que $F'(X) = A$. Cuando se supone tan solo que $F$ es Gâteaux diferenciable, aparecen nuevos fenómenos: discutimos la existencia de una aplicación $F$ de un espacio de Banach $X$ en un espacio de Banach $Y$ acotada, Lipschitz-continua, de tal manera que, para todo $x, y \in X$, si $x \neq y$, entonces $\|F'(x) - F'(y)\|_{\mathcal{L}(X,Y)} > 1$. Se dan aplicaciones a la existencia de soluciones de ecuaciones de Hamilton-Jacobi.

## 1. Introduction

Let $X, Y$ be separable Banach spaces such that $dim(X) \geq 1$ and $F : X \to Y$ be a mapping differentiable at every point of $X$. We are interested by the structure of $F'(X) = \{F'(x); x \in X\} \subset \mathcal{L}(X,Y)$. Several notions of differentiability can be considered : we say that $F$ is Gâteaux-differentiable at $x \in X$ provided there exists $T \in \mathcal{L}(X,Y)$ such that for each $h \in X$,

$$\lim_{t \to 0} \frac{F(x+th) - F(x)}{t} = T(h)$$

$T$ is called the Gâteaux derivative of $F$ at $x$ and is denoted $T = F'(x)$. We say that $F$ is Frechet-differentiable at $x \in X$ provided there exists $T \in \mathcal{L}(X,Y)$ such that for each $h \in X$,

$$\lim_{\|h\| \to 0} \frac{F(x+h) - F(x) - T(h)}{\|h\|} = 0$$

Presentado por Vicente Montesinos Santalucía.
*Recibido*: 08/04/2006. *Aceptado*: 14/04/2006.
*Palabras clave / Keywords*: Fréchet derivative, Gâteaux derivative, Hamilton-Jacobi equations.
*Mathematics Subject Classifications*: 46G05, 58C20, 46B20, 35D05.

$T$ is called the Frechet derivative of $F$ at $x$ and is denoted $T = F'(x)$. We are interested in the following questions :

- What are the topological properties of the set $F'(X)$?
- For which sets $A \subset \mathcal{L}(X, Y)$ does there exist a smooth mapping $F : X \to Y$ such that $A = F'(X)$? Smoothness of $F$ may have several meanings : $\mathcal{C}^1$ mapping, everywhere Fréchet-differentiable mapping, everywhere Gâteaux-differentiable mapping, everywhere Fréchet-differentiable mapping with Lipschitz continuous derivative. How the notion of smoothness affects the above question?
- What is the interplay between the geometry of the Banach spaces $X$ and $Y$ and the structure of the set $F'(X)$?
- Whenever $f$ is a smooth mapping, what is the structure of $f'(X)$? Is this structure richer than the structure of the range of the derivative of a smooth mapping? What additional properties can be said when $f$ has a non empty bounded support (we then say that $F$ is a bump function)?

The first section investigates the structure of the range of the derivative of a smooth real valued function $f$ defined on a Banach space $X$ : we discuss the topological properties of the set $f'(X)$, with an emphasis whenever $f$ is a bump function. We then give sufficient conditions on a subset $A$ of a dual Banach space $X^*$ so that it can be expressed as the range of the derivative of a smooth real valued function on $X$. Finally, we give some examples showing that the the situation is far from being completly elucidated.

The second section investigates the structure of the range of the derivative of a smooth mapping $f$ between two Banach spaces $X$ and $Y$ : Some construction of examples of smooth functions with certain properties extend to the case of smooth mappings. It is also possible to construct some pathological examples in the framework of smooth mappings, although these constructions cannot be achieved in the framework of real valued functions. We shall in particular consider the following question : let $X, Y$ be two Banach spaces. Is it possible to construct a Lipschitz continuous mapping $f : X \to Y$, Gâteaux-differentiable at each point, and such that, for all $x, y \in X$, $x \neq y$, we have $\|f'(x) - f'(y)\| \geq 1$? We shall prove that this is not possible whenever $Y = I\!R$, but such a construction will be carried out for instance whenever $X = Y = \ell^2$.

The last section is about the interplay between the construction of smooth functions and the resolution of Hamilton-Jacobi equations. Some Hamilton-Jacobi equations fail to have classical solutions in finite dimensions, but they have classical solutions in infinite dimensions. For another Hamilton-Jacobi equation, there is uniqueness of a classical solution only in finite dimensions. Finally, we recall the construction of "almost classical" solutions of the Eikonal equation $\|u'\| = 1$ on a bounded open subset of a finite dimensional space, satisfying a Dirichlet boundary condition. It is well known that there is no classical solution of this equation.

# 2. Smooth real valued functions.

## 2.1. Necessary conditions.

In this section, we shall give some properties which are satisfied by the set $f'(X)$, whenever $f$ is a smooth real valued function defined on a Banach space $X$. We first examine the connectedness of the range of $f'$. Whenever $f : X \to I\!R$ is everywhere Frechet-differentiable, this question was answered by J. Maly [18] in 1996 :

**Theorem 1** *If $X$ is a Banach space and $f : X \to I\!R$ is Fréchet-differentiable at every point, then the set $f'(X)$ is connected in $(X^*, \|.\|)$.*

Whenever $f$ is assumed to be Gâteaux differentiable, the following result was obtained by R. Deville and P. Hajek [11] :

**Proposition 1** *Let $X$ be an infinite dimensional Banach space, and let $f$ be a real valued locally Lipschitz and Gâteaux-differentiable function on $X$. Then either $f$ is affine or $f'(X)$ has no $w^*$-isolated points.*

This result was improved by T. Matrai [20] :

**Theorem 2** *Let $X$ be a separable Banach space, and let $f$ be a real valued locally Lipschitz and Gâteaux-differentiable function on $X$. Then $f'(X)$ is connected in $(X^*, w^*)$.*

From now on, we say that a real valued function on a Banach space $X$ is a *bump* function if it has bounded non empty support. If $X$ is a Banach space, $x \in X$ and $r > 0$, we denote $B_X(x,r)$ (resp. $\overline{B}_X(x,r)$) the open ball (resp. closed ball) in $X$ of center $x$ and radius $r$.

**Proposition 2** *Let $X$ be a Banach space and $f$ be a $\mathcal{C}^1$ bump function on $X$.*

1. *If $X$ is finite dimensional then $f'(X)$ is compact, arc connected, and there exists $r > 0$ such that $f'(X) \supset B_{X^*}(0,r)$.*
2. *If $X$ is infinite dimensional and separable, then $f'(X)$ is analytic, arc connected, and the norm* closure of $f'(X)$ contains a ball $B_{X^*}(0,r)$ for some $r > 0$.

The last assertion of Proposition 2 follows from the following more general result :

**Proposition 3** *If $f$ is a continuous and Gâteaux-differentiable bump function on $X$, then the norm* closure of $f'(X)$ contains a ball $B_{X^*}(0,r)$ for some $r > 0$.

PROOF. The ranges of the derivative of $f$ and of the function $af\big((. - x_0)/a\big)$ are the same. So, there is no loss of generality if we assume that $f(0) \neq 0$ and the support of $f$ is contained in the unit ball. Furthermore, if the conclusion of the proposition holds for $f$, it also holds for $-f$. So we can assume that $f(0) < 0$. Let $g \in X^*$ be such that $\|g\| < -f(0)$, fix $\varepsilon$ arbitrary such that $0 < \varepsilon < -f(0) - \|g\|$, and denote $\varphi(x) = f(x) - g(x)$ if $x \leq 1$ and $\varphi(x) = +\infty$ otherwise. According to the Ekeland variational principle, there exists $x_0$ such that $\varphi(x) \geq \varphi(x_0) - \varepsilon\|x - x_0\|$. We have $\|x_0\| < 1$, otherwise $\|x_0\| = 1$ and $\varphi(0) = f(0) \geq \varphi(x_0) - \varepsilon\|x_0\| \geq \|g\| - \varepsilon$, contradiction with the choice of $\varepsilon$. Therefore $\|\varphi'(x_0)\| = \|f'(x_0) - g\| \leq \varepsilon$. This shows that $f'(X)$ contains the ball $B_{X^*}\big(0, -f(0)\big)$. ■

Necessary conditions on a subset $A$ of the dual space $X^*$ so that there exists a $\mathcal{C}^1$-smooth bump function on $X$ with Lipschitz derivative have been obtained by T. Gaspari [15], these conditions are very restrictive. Let us just mention here a result of L. Rifford.

**Theorem 3** *Let $f : \mathbb{R}^N \to \mathbb{R}$ be $\mathcal{C}^{N+1}$-smooth bump. Then $f'(\mathbb{R}^N)$ is the closure of its interior.*

## 2.2. Sufficient conditions.

In this section, we give sufficient conditions on a subset $A$ of a dual Banach space $X^*$ so that it is the range of the derivative of a smooth bump function on $X$. Results on this question have been obtained in [7], [8], [4], [13], [14] and [15].

Let us first consider the case of finite dimensional Banach spaces. We shall present here a sufficient condition so that a subset $A$ of $X^*$ is the range of the derivative of a $\mathcal{C}^1$-smooth bump due to D. Azagra, M. Fabian and M. Jimenez-Sevilla ([4]).

**Definition 1** *Let $A$ be an arc connected subset of $X^*$. We define:*

$$d(x,y) = \inf\big\{diam\big(\gamma([0,1])\big),\ \gamma : [0,1] \to A \text{ continuous},\ \gamma(0) = x \text{ and } \gamma(1) = y\big\}$$

*$d$ is clearly a distance in $A$. We then define, for each $n \in \mathbb{N}$ :*

$$R_n(A) = \sup_{(y_1,\dots,y_n)\in A^n} \big\{ \inf\{d(y_i,y_j),\ 1 \leq i < j \leq n\}\big\}$$

Observe that the condition $\lim_{n\to\infty} R_n(A)=0$ means that the metric space $(A,d)$ is totally bounded.

**Theorem 4** *Let $X$ be a finite dimensional Banach space. Let $A \subset X^*$ be the closure of a bounded, connected open subset $U$ of $X^*$ containing the origin.*

1. *If $\lim_{n\to\infty} R_n(U)=0$, then there exists a $\mathcal{C}^1$-smooth bump on $X$ such that $b'(X)=A$.*
2. *Conversely, if there exists a $\mathcal{C}^1$-smooth bump on $X$ such that $b'(X)=A$, then $\lim_{n\to\infty} R_n(A)=0$.*

**Problem 1** *Is there a characterization of subsets of the dual $X^*$ of a finite dimensional Banach space $X$ which can be written as ranges of the derivative a $\mathcal{C}^1$-smooth bump function on $X$?*

There are many more sets which can be written as the range of the derivative of a Frechet-differentiable bump function as shown by the following result of T. Gaspari ([14]).

**Theorem 5** *Let $X$ be a finite dimensional Banach space. Let $U \subset X^*$ be a bounded, connected open subset $U$ of $X^*$ containing the origin. Then there exists a Frechet-differentiable bump function $f$ on $X$ such that $f'(X)=U$.*

We now consider the case of infinite dimensional Banach spaces. The following sufficient condition so that a subset $A$ of $X^*$ is the range of the derivative of a $\mathcal{C}^1$-smooth bump is due to M. Fabian, O. Kalenda and J. Kolár [13].

**Theorem 6** *If $X$ is an infinite-dimensional Banach space, with separable dual, and $A \subset X^*$ is an analytic set such that any point $x^* \in A$ can be reached from $0$ by a continuous path contained (except for the point $x^*$) in the interior of $A$, then $A$ is the range of the derivative of a $C^1$-smooth function on $X$ with bounded nonempty support.*

The structure of the range of $f'$ whenever $f'$ satisfies a Lipschitz condition has been investigated in [15]. We shall restrict our attention to the case of starshaped subsets of the dual space.

**Definition 2** *Let $A$ be a subset of a Banach space $E$. We say that $A$ is starshaped (with respect to the origin $0$) if for every $x \in A$, the segment $[0,x]$ is included in $A$. We say that $A$ is uniformly starshaped if there exists $r>0$ such that for every $x \in A$, the drop $\mathrm{conv}\big(\overline{B}_E(0,r)\cup\{x\}\big)$ is included in $A$.*

**Theorem 7** *Let $X$ be a separable infinite dimensional Banach space such that there exists a bump function $b : X \to I\!R$ which is $\mathcal{C}^1$-smooth with Lipschitz derivative. If $A$ is a bounded uniformly starshaped open subset of $X^*$, then there exists a bump function $\beta : X \to I\!R$ which is $\mathcal{C}^1$-smooth with Lipschitz derivative such that $\beta'(X)=A$.*

The assumption on $X$ in Theorem 6 is for instance satisfied when $X = L^p([0,1])$ or $\ell^p(I\!N)$, with $2 \le p < +\infty$. The assumption "$A$ uniformly starshaped" cannot be replaced by the assumption "$A$ starshaped" as shown by a counterexample of T. Gaspari (see [15]).

## 2.3. Examples

If $A = X^*$, $A$ satisfies the assumptions of Theorem 4. Therefore, whenever $X$ is an infinite dimensional Banach space with separable dual, there exists a $\mathcal{C}^1$-smooth bump function such that $f'(X)=X^*$ : this is a result from [1]. In [3], it is proved that whenever $X$ is a separable infinite dimensional Banach space, there exists a Gâteaux-differentiable bump function $f$ such that $f'(X)=X^*$.

On the other hand, P. Hajek has proved in [16] that if $f$ is a function on $c_0$ with locally uniformly continuous derivative, then $f'(c_0)$ is included in a countable union of norm compact subsets of $\ell^1$ and therefore

$f'(c_0)$ has empty interior. There does exist bump functions on $c_0$ with locally uniformly continuous derivative : take for instance $f : c_0 \to I\!R$ defined by $f(x) = \varphi\Big(\sum_{n=1}^{+\infty} x_n^{2n}\Big)$ for $x = (x_n)_{n\geq 1} \in c_0$, where $\varphi : I\!R \to I\!R$ is a $\mathcal{C}^\infty$-smooth function such that $\varphi(0) = 1$ and $\varphi(x) = 0$ if $x \geq 1$ : $f$ is a $\mathcal{C}^\infty$-smooth bump function on $c_0$. Therefore, it seems extremely difficult to find a necessary and sufficient condition on a subset $A$ of a separable infinite dimensional Banach space $X^*$ so that it is the range of the derivative of a $\mathcal{C}^1$-smooth bump function on $X$.

Because of the lack of compactness, bump functions on infinite dimensional Banach spaces need not attain their maximum. Therefore, one may expect the failure of Rolles' theorem on infinite dimensional Banach spaces. This is indeed the case as shown by the following result due to D. Azagra and M. Jimenez-Sevilla [5].

**Theorem 8** *Let $X$ be an infinite dimensional separable Banach space such that $X^*$ is separable. Then, there exists a $\mathcal{C}^1$-smooth bump function $f$ on $X$ such that $f'(x) = 0$ implies $f(x) = 0$.*

If $f$ is a Gâteaux-differentiable bump function, in view of Proposition 3, a natural conjecture would be that the norm closure of $f'(X)$ is norm connected, or at least that $f'(X)$ does not contain an isolated point. This is not so as shown by the following construction from [11].

**Theorem 9** *Let $X$ be an infinite dimensional separable Banach space. Then, there exists a bump function $f$ on $X$ such that $f$ is Gâteaux-differentiable at every point, $f'$ is norm to weak$^*$ continuous and $\|f'(0) - f'(x)\| \geq 1$ whenever $x \neq 0$. If $X^*$ is separable, we can assume moreover that $f$ is $\mathcal{C}^1$ on $X\backslash\{0\}$.*

**Remark 1** *According to the above discussion, $0$ is not an isolated point of $f'(X)$, so necessarily $f'(0) \neq 0$.*

PROOF. The Theorem is proved using the following two lemmas.

**Lemma 1** *Let $X$ be a Banach space, $U$ be an open connected subset of $X^*$ such that $0 \in U$ and $x^* \in U$. Assume there exists on $X$ a Lipschitz continuous bump function which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point. Then there exists a Lipschitz continuous bump function $\beta$ on $X$ with support contained in the unit ball, which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point, such that $\beta'(X) \subset U$ and $\beta'(x) = x^*$ for all $x$ in a neighbourhood of $0$.*

**Lemma 2** *Let $X, Y$ be two Banach spaces, $a \in X$, $V$ be an open neighbourhood of $a$, and $f : V \to Y$ be continuous on $V$ and Gâteaux-differentiable at every point of $V\backslash\{a\}$. If $f'(x)$ has a limit $\ell$ in $\mathcal{L}(X,Y)$ endowed with the strong operator topology as $x$ tends to $a$, then $f$ is Gâteaux-differentiable at $a$ and $f'(a) = \ell$.*

**Hint of proof :** Let $a^* \in X^*$ such that $1 < \|a^*\| < 2$. It is possible to construct $(W_n)_{n\geq 0}$, a decreasing sequence of norm open, norm connected and weak$^*$ open subsets so that, if $y_n^* \in W_n$ and if $(y_n^*)$ is bounded, then $(y_n^*)$ converges to $a^*$ for the weak$^*$-topology, and so that for every $n$ and every $x \in W_n$, $\|x - a^*\| > 1$. Let $(x_n^*) \subset X^*$ be a sequence such that $x_1^* = 0$ and for every $n$, $x_n^* \in W_n$. Since $X$ is separable (resp. $X^*$ is separable) there exists on $X$ a Lipschitz continuous bump function which is Gâteaux-differentiable (resp. Frechet-differentiable) at each point. According to lemma 1, there exists a Lipschitz continuous bump $b_n$ which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point, such that $b_n'(X) \subset W_n - x_n^*$, with support in the unit ball and such that $b_n'(x) = x_{n+1}^* - x_n^*$ for all $x$ satisfying $\|x\| < \delta_n$. Denote $c_1 = 1$ and, for $n \geq 2$, $c_n = \prod_{i=1}^{n-1} \delta_n$.

$$b(x) = \sum_{n=1}^{+\infty} c_n b_n(x/c_n)$$

$b$ is Gâteaux-differentiable at each point of $X\backslash\{0\}$, and its support is contained in the unit ball. By construction, $b'(X\backslash\{0\}) \subset X^*\backslash B(a^*, 1)$, and $b'(x) \xrightarrow{w^*} a^*$ as $x \to 0$. Lemma 2 then shows that $b$ is Gâteaux-differentiable at 0 and that $b'(0) = a^*$. ■

# 3. Smooth mappings between Banach spaces.

## 3.1. First examples.

Some properties described for smooth functions extend to smooth mappings. We have seen the construction of smooth bump functions on a Banach space $X$ such that the range of its derivative is equal to the dual space of $X$. it was observed in [3] that the range of a smooth mapping $F$ beween separable Banach spaces $X$ and $Y$ such that $X$ is infinite dimensional can be as big as possible, even if $F$ is identically equal to 0 outside of a bounded set of $X$.

**Proposition 4** *If $X$ and $Y$ are separable Banach spaces and if $X$ is infinite dimensional, one can always find a Gâteaux-differentiable mapping $F : X \to Y$ such that $F$ has bounded support, $F'(X) = \mathcal{L}(X, Y)$ and $F(X) = Y$.*

**Proposition 5** *Let $X$ and $Y$ be Banach spaces so that $\mathcal{L}(X, Y)$ is separable and $X$ is infinite dimensional. Then, there exists a Frechet-differentiable mapping $F : X \to Y$ so that $F$ has bounded support, $F'(X) = \mathcal{L}(X, Y)$ and $F(X) = Y$.*

The structure of the range of the derivative of a smooth mapping $F$ can be very different from the structure of the range of the derivative of a smooth function $f$. Maly theorem states that if $f : X \to \mathbb{R}$ is everywhere differentiable, then $f'(X)$ is a connected subset of $X^*$. The following result shows that there is no analog of Maly's theorem for vector valued mappings.

**Example 1** *There exists a mapping $F$ from $\mathbb{R}^2$ into $\mathbb{R}^2$, Frechet-differentiable at each point, and so that the cardinal of $\{det(F'(x)); x \in \mathbb{R}^2\}$ is 2. Therefore $F'(\mathbb{R}^2)$ is not connected.*

Such an example was communicated to us by J. Saint-Raymond: take

$$F(x, y) = \left(x^2\sqrt{y}\cos 1/x^3, x^2\sqrt{y}\sin 1/x^3\right)$$

whenever $(x, y) \neq (0, 0)$ and $F(0, 0) = (0, 0)$. In this case, we have $\{det(F'(x)); x \in \mathbb{R}^2\} = \{0, 3/2\}$.

The aim of the end of this section is to investigate the following question : if $X, Y$ are Banach spaces, is it possible to construct a mapping $F : X \to Y$, everywhere differentiable, so that for every $x, y \in X$, if $x \neq y$, then $\|F'(x) - F'(y)\| \geq 1$? All results presented in the remainder of this section are due to R. Deville and P. Hajek [11] except the last one due to F. Bayart [6].

## 3.2. Necessary conditions.

We first notice that, under mild regularity assumptions, the answer to the above question is negative for real valued functions.

**Proposition 6** *Let $X$ be a Banach space and $f : X \to \mathbb{R}$ be a Lipschitz continuous (or merely locally uniformly continuous), everywhere Gâteaux-differentiable function. Then, for every $x \in X$ and every $\varepsilon > 0$, there exists $y, z \in B_X(x, \varepsilon)$ such that $\|f'(y) - f'(z)\| \leq \varepsilon$.*

PROOF. (sketch) Take any $h \in X$ such that $\|h\|$ small enough, and consider the lower semi-continuous function defined by $\varphi(y) = f(y+h) - f(y)$. The Ekeland variational principle then tells us the existence of $y \in X$, not far from $x$ such that $\|\varphi'(y)\| \le \varepsilon$. Therefore, if we denote $z = y + h$, $\|f'(y) - f'(z)\| \le \varepsilon$, and $y$ and $z$ are not far from $x$. ■

The answer to the above question will be also negative if $f$ is everywhere Fréchet differentiable.

**Proposition 7** *Let $X, Y$ be separable Banach spaces and $F : X \to Y$ be an everywhere Fréchet-differentiable locally uniformly continuous mapping. Then, for every $x \in X$ and every $\varepsilon > 0$, there exists $y, z \in B_X(x, \varepsilon)$, $y \neq z$, such that $\|F'(y) - F'(z)\| \le \varepsilon$.*

The answer to the above question will be also negative if $\mathcal{L}(X, Y)$ is separable.

**Proposition 8** *Let $X, Y$ be a Banach space and $F : X \to Y$ be an everywhere Gâteaux-differentiable function. If $\mathcal{L}(X, Y)$ is separable, then, for every $x \in X$ and every $\varepsilon > 0$, there exists $y, z \in B_X(x, \varepsilon)$ such that $\|F'(y) - F'(z)\| \le \varepsilon$.*

## 3.3. Examples.

In view of the above propositions, one could believe that whenever $X, Y$ are Banach spaces (or vector normed spaces) and $F : X \to Y$ is a mapping Gâteaux-differentiable at each point of $X$, then for every $\varepsilon > 0$, there exists $y, z \in X$ such that $\|F'(y) - F'(z)\| \le \varepsilon$. The next result from [11] proves that this is not so.

**Theorem 10** *There exists a Lipschitz mapping $F : \ell^1 \to I\!R^2$, Gâteaux-differentiable at each point of $\ell^1$, such that for every $x, y \in \ell^1$, $x \neq y$, we have :*

$$\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, I\!R^2)} \ge 1$$

We shall construct $F$ with the properties of Theorem 10 using series. We need an auxiliary construction.

**Lemma 3** *Given $\Delta = (a, b) \in I\!R^2$ such that $a < b$ and $\varepsilon > 0$, there exists a $\mathcal{C}^\infty$-function $\varphi = \varphi_{\Delta,\varepsilon} : I\!R^2 \to I\!R^2$ such that :*

*(i)* $|\varphi(x, y)| \le \varepsilon$ *for all* $(x, y) \in I\!R^2$,

*(ii)* $\varphi(x, y) = 0$ *whenever* $x < a$,

*(iii)* $\left\|\dfrac{\partial \varphi}{\partial x}(x, y)\right\| \le \varepsilon$ *for all* $(x, y) \in I\!R^2$,

*(iv)* $\left\|\dfrac{\partial \varphi}{\partial y}(x, y)\right\| = 1$ *whenever* $x \ge b$,

*(v)* $\left\|\dfrac{\partial \varphi}{\partial y}(x, y)\right\| \le 1$ *for all* $(x, y) \in I\!R^2$,

*(vi) If we denote* $\varphi(x, y) = \big(\varphi_1(x, y), \varphi_2(x, y)\big)$, *then* $\dfrac{\partial \varphi_1}{\partial y}(x, 0) = 1$ *whenever* $x \ge b$.

PROOF. Let $\beta : I\!R \to I\!R$ be a $\mathcal{C}^\infty$-smooth function such that $0 \le \beta(x) \le 1$ for all $x$, $\beta(x) = 0$ whenever $x < a$ and $\beta(x) = 1$ whenever $x \ge b$. If $n \ge 1$ is large enough, the function defined by $\varphi(x, y) = \dfrac{\beta(x)}{n}\big(\sin(ny), \cos(ny)\big)$ satisfies the desired properties. ■

We shall also use the following criterium of Gâteaux-differentiability of the sum of a series (the proof of this criterium is elementary and omitted here) :

**Lemma 4** ***(Gâteaux-differentiability criterium)*** *Let $X$ and $Y$ be Banach spaces and, for all $n$, let $f_n : X \to Y$ be Gâteaux-differentiable mappings. Assume that $\left(\sum f_n\right)$ converges pointwise on $X$, and that there exists a constant $K > 0$ so that for all $h$,*

$$\sum_{n\geq 1} \sup_{x\in X} \left\| \frac{\partial f_n}{\partial h}(x) \right\| \leq K\|h\| \tag{1}$$

*Then the mapping $F = \sum_{n\geq 1} f_n$ is $K$-Lipschitz, Gâteaux-differentiable on $X$, and for all $x$, $F'(x) = \sum_{n\geq 1} f'_n(x)$ (where the convergence of the series is in $\mathcal{L}(X,Y)$ for the strong operator topology).*

**Proof of theorem 10.** Fix an enumeration $\Delta_k = (a_k, b_k)$, $k \in N$, of all couples of dyadic numbers such that $a_k < b_k$. Select integers $m_k^n$ such that for each $n$, $n < m_k^n$ and $(m_k^n)_k$ is an increasing sequence, and satisfying

$$m_k^n = m_\ell^p \;\; \Rightarrow \;\; n = p \;\; \text{and} \;\; k = \ell \tag{2}$$

This condition is satisfied for instance whenever $m_k^n = 2^k.3^n$. Fix $\varepsilon > 0$ and let $\varepsilon_k^n$ be positive real numbers such that $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{\infty} \varepsilon_k^n = \varepsilon$. We shall notice $\varepsilon_k = \sum\limits_{n=1}^{\infty} \varepsilon_k^n$, so that $\sum\limits_{k=1}^{\infty} \varepsilon_k = \varepsilon$. Put $f_{n,k} : \ell^1 \to I\!R^2$ such that, if $x = (x_i) \in \ell^1$, then $f_{n,k}\big(x\big) = \varphi_{\Delta_k,\varepsilon_k^n}\big(x_n, x_{m_k^n}\big)$ : $f_{n,k}$ is a $\mathcal{C}^\infty$ function on $\ell^1$. The function $F : \ell^1 \to I\!R^2$ we are looking for is defined by :

$$F(x) = \sum_{n\in I\!N}\sum_{k\in I\!N} f_{n,k}(x)$$

It is easy to see that $F$ is well-defined. The Gâteaux-differentiability criterium shows that $F$ is Gâteaux-differentiable on $\ell^1$ and $F$ is $(1+\varepsilon)$-Lipschitz-continuous on $\ell^1$. If $x \neq y \in \ell^1$, let us prove that $\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, I\!R^2)} \geq 1 - 2\varepsilon$.

Indeed, let $n \in I\!N$ such that $x_n \neq y_n$. We may assume that $x_n < y_n$. Let $k$ such that $x_n < a_k < b_k < y_n$. According to $(ii)$ and $(iv)$ of Lemma 3,

$$\left\| \frac{\partial f_{n,k}}{\partial x_{m_k^n}}(x) \right\| = 1 \qquad \text{and} \qquad \frac{\partial f_{n,k}}{\partial x_{m_k^n}}(y) = 0$$

On the other hand, according to $(iii)$ of Lemma 3, for all $r$,

$$\left\| \frac{\partial f_{m_k^n,r}}{\partial x_{m_k^n}}(x) \right\| \leq \varepsilon_r \qquad \text{and} \qquad \left\| \frac{\partial f_{m_k^n,r}}{\partial x_{m_k^n}}(y) \right\| \leq \varepsilon_r$$

and, if $\ell \neq m_k^n$ and $(\ell, r) \neq (n,k)$, because of 2,

$$\frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(x) = 0 \qquad \text{and} \qquad \frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(y) = 0$$

Therefore,

$$\begin{aligned}
\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, I\!R^2)} &\geq \left\| \frac{\partial F}{\partial x_{m_k^n}}(x) - \frac{\partial F}{\partial x_{m_k^n}}(y) \right\| \\
&\geq 1 - \sum_{(\ell,r)\neq(n,k)} \left\| \frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(x) - \frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(y) \right\| \\
&\geq 1 - 2\varepsilon
\end{aligned}$$

■

**Remark 2** *The mapping $F$ constructed above has the property that for each $h \in \ell^1$, $x \to F'(x).h$ is continuous from $\ell^1$ into $I\!R^2$.*

**Corollary 1** *Let us denote $D$ the vector normed space of elements of $\ell^1$ with finite support. There exists a Lipschitz function $f : \ell^1 \to I\!R$, Gâteaux-differentiable at each point of $\ell^1$, such that for every $x, y \in D$, $x \neq y$, then $\|f'(x) - f'(y)\|_{\ell^\infty} \geq 1$.*

PROOF. Let $F : \ell^1 \to I\!R^2$ be the mapping constructed in Theorem 1 and denote $F = (f, g)$ : the function $f$ satisfies the corollary because of condition $(vi)$ of Lemma 3. ■

**Problem 2** *Let $X$ and $Y$ be two separable Banach spaces such that $dim(X) \geq 1$. By proposition 6 and proposition 8, if there exists a Lipschitz mapping $F : X \to Y$, Gâteaux-differentiable at each point of $X$, such that for every $x, y \in X$, if $x \neq y$, then*

$$\|F'(x) - F'(y)\|_{\mathcal{L}(X,Y)} \geq 1$$

*then $dim(Y) \geq 2$ and $\mathcal{L}(X, Y)$ is not separable. Is the converse true?*

Here are two partial answers. The following result is proved in [11].

**Theorem 11** *Let $X_p = \ell^p$ if $1 \leq p < +\infty$ and $X_\infty = c_0$. Let us fix $1 \leq p, q \leq +\infty$. The following assertions are equivalent :*

1. *There exists a Lipschitz continuous mapping $F : X_p \to X_q$, Gâteaux-differentiable at each point of $X_p$, such that $\|F'(x) - F'(y)\|_{\mathcal{L}(X_p, X_q)} \geq 1$ for every $x, y \in X_p$, $x \neq y$.*
2. *$p \leq q$.*
3. *$\mathcal{L}(X_p, X_q)$ is not separable.*

If $X$ and $Y$ are two real Banach spaces, we denote $Lip_b(X, Y)$ the space of Lipschitz bounded mappings from $X$ to $Y$. The space $Lip_b(X, Y)$, equipped with the norm $\|F\|_{Lip_b(X,Y)} = \sup\left\{\|F\|_\infty, \|F\|_{Lip}\right\}$ is a Banach space. The following result has been obtained by F. Bayart [6].

**Theorem 12** *Let $X$ be an infinite dimensional separable Banach space. Then there exists a closed infinite dimensional subspace $E$ of $Lip_b(X, c_0)$ and a constant $c > 0$ such that for every $F \in E$, $F$ is Gâteaux-differentiable at each point of $X$, and for every $x, y \in X$, $x \neq y$, then*

$$\|F'(x) - F'(y)\|_{\mathcal{L}(X,c_0)} \geq c\|F\|_{Lip_b(X,c_0)}$$

# 4. Construction of solutions of some Hamilton-Jacobi equations.

## 4.1. Existence and uniqueness of classical solutions.

Let $X$ be a separable Banach space, $\Omega$ be a non empty bounded open subset of $X$ and $f : \Omega \to (0, +\infty)$ be continuous. We denote $\partial\Omega$ the boundary of $\Omega$. We consider the following boundary value problem :

$$(BVP1) \qquad \begin{cases} \text{Find } u : \overline{\Omega} \to I\!R, \text{ continuous on } \overline{\Omega}, \text{ differentiable at every point of } \Omega, \\ \text{such that } \|u'(x)\| = f(x) \text{ for every } x \in \Omega \text{ and} \\ \text{satisfying the Dirichlet boundary condition : } u(x) = 0 \text{ for every } x \in \partial\Omega. \end{cases}$$

The equation $\|u'(x)\| = f(x)$ is sometimes called the Eikonal equation.

**Proposition 9** *Let $X$ be a separable Banach space. The following assertions are equivalent :*

1. *There exists a bounded open subset $\Omega$ of $X$ and there exists $f : \Omega \to (0, +\infty)$ continuous such that there exists a solution of $(BVP1)$.*
2. *$X$ is infinite dimensional and $X^*$ is separable.*

PROOF. Let $X$ be an infinite dimensional separable Banach space such that $X^*$ is separable. According to Theorem 8, there exists a $\mathcal{C}^1$-smooth bump function $u$ on $X$ such that $u'(x) = 0$ implies $u(x) = 0$. The set $\Omega = \{x \in X; u(x) \neq 0\}$ is a bounded open subset of $X$, and the function $f : \Omega \to I\!R$ given by $f(x) = \|u'(x)\|$ is continuous with values in $(0, +\infty)$. By construction $u$ is a solution of $(BVP1)$.

Conversely, let $\Omega$ be a bounded non empty subset of $X$ and $u : \overline{\Omega} \to I\!R$ be continuous on $\overline{\Omega}$ and differentiable at every point of $\Omega$. If $X$ is finite dimensional, by Rolles' Theorem, there exists $x_0 \in \Omega$ such that $u'(x_0) = 0$, so $\|u'(x_0)\| = 0 \neq f(x_0)$ and $u$ is not a solution of $(BVP1)$. On the other hand, if $X$ is separable and $X^*$ is non separable, $u$ is necessarily identically equal to 0 because there is no non trivial bump function on $X$, so $u$ is not a solution of $(BVP1)$. ■

Let us now turn to uniqueness of classical solutions of Hamilton-Jacobi equations.

Let $X$ be a separable Banach space, $\Omega$ be a bounded open subset of $X$ and $f : \Omega \to (0, +\infty)$ be continuous. We consider the following boundary value problem :

$$(BVP2) \qquad \begin{cases} \text{Find } u : \overline{\Omega} \to I\!R, \text{ continuous on } \overline{\Omega}, \text{ differentiable at every point of } \Omega, \\ \text{such that } u(x) + H(x)\|u'(x)\| = 0 \text{ for every } x \in \Omega \text{ and} \\ \text{satisfying the Dirichlet boundary condition : } u(x) = 0 \text{ for every } x \in \partial\Omega. \end{cases}$$

**Proposition 10** *Let $X$ be a separable Banach space. The following assertions are equivalent :*

1. *For every bounded open subset $\Omega$ of $X$ and for every $H : \Omega \to (0, +\infty)$ continuous, there exists at most one solution of $(BVP2)$.*
2. *$X$ is finite dimensional or $X^*$ is not separable.*

PROOF. Let $X$ be an infinite dimensional separable Banach space such that $X^*$ is separable. According to Theorem 8, there exists a $\mathcal{C}^1$-smooth bump function $u$ on $X$ such that $v'(x) = 0$ implies $v(x) = 0$. The set $\Omega = \{x \in X; v(x) \neq 0\}$ is a bounded open subset of $X$, and the function $f : \Omega \to I\!R$ given by $f(x) = \frac{v(x)}{\|v'(x)\|}$ is continuous with values in $(0, +\infty)$. By construction for all $\lambda \in I\!R$, $u = \lambda v$ is a solution of $(BVP2)$ which admits therefore infinitely many solutions.

Conversely, let $\Omega$ be a bounded non empty subset of $X$ and $u, v : \overline{\Omega} \to I\!R$ be two distinct solutions of $(BVP1)$. If $X$ is finite dimensional, by Rolles' Theorem applied to the function $u - v$, there exists $x_0 \in \Omega$ such that $u'(x_0) = v'(x_0)$ and $u(x_0) \neq v(x_0)$. Since $\|u'(x_0)\| = \|v'(x_0)\|$, the two equations $u(x_0) + H(x_0)\|u'(x_0)\| = 0$ and $v(x_0) + H(x_0)\|v'(x_0)\| = 0$ cannot be satisfied simultaneously. This proves the uniqueness of the solution of $(BVP2)$ in the case $X$ finite dimensional. On the other hand, if $X$ is separable and $X^*$ is non separable, $u$ is necessarily identically equal to 0 because there is no non trivial bump function on $X$, so 0 is the only solution of $(BVP2)$. ■

## 4.2. Almost classical solutions.

Let $\Omega$ be a bounded open subset of $I\!R^d$ and $F : I\!R^d \to I\!R$ be continuous.

**Definition 3** *A function $u : \overline{\Omega} \to I\!R$ is called an almost* classical solution of the following boundary value problem :

$$\text{(BVP3)} \qquad F(u') = 1 \qquad \text{and} \qquad u_{|\partial\Omega} = 0$$

if $u$ is continuous on $\overline{\Omega}$, differentiable at every point of $\Omega$, satisfies $F(u') = 1$ for *almost* every $x \in \Omega$ and $u(x) = 0$ for every $x \in \partial\Omega$.

We first state a theorem from [12] on the existence of almost classical solutions of $(BVP3)$.

**Theorem 13** *ssume that $F(0) \neq 0$ and that the connected component $\Omega$ of $\{F \neq 0\}$ containing $0$ is bounded. Then there exists an almost classical solution $u : \overline{\Omega} \rightarrow I\!R$ of $(BVP3)$, which is Lipschitz-continuous on $\overline{\Omega}$.*

Notice that since $\Omega$ is bounded, by Rolles theorem, if $u : \overline{\Omega} \rightarrow I\!R$ is continuous on $\overline{\Omega}$ and differentiable on $\Omega$, then there exists $x_0 \in \Omega$ such that $u'(x_0) = 0$. Therefore, the condition $F(u') = 0$ cannot be satisfied everywhere on $\Omega$ (so there is no classical solution of $(BVP3)$), and the function $u$ from the above corollary cannot be $\mathcal{C}^1$-smooth. Whenever $F(p) = \|p\| - 1$, we obtain the existence of almost classical solutions of the Eikonal equation:

**Corollary 2** *Let $\Omega$ be the unit ball of $I\!R^d$. Then there exists $u : \overline{\Omega} \rightarrow I\!R$, Lipschitz-continuous on $\overline{\Omega}$ with Lipschitz constant $1$, differentiable on $\Omega$, such that $u_{|\partial\Omega} = 0$ and $\|u'\| = 1$ almost everywhere in $\Omega$.*

The above results were inspired by the solution by Z. Buchzolich to the gradient problem of C. E. Weil (see [9]) : he proved that whenever $d \geq 2$, there exists a function $u : I\!R^d \rightarrow I\!R$, differentiable at every point, such that $u'(0) = 0$ and $\|u'(x)\| \geq 1$ for almost every $x \in I\!R^d$. Therefore, if $u : I\!R^d \rightarrow I\!R$ is differentiable at every point and if $U$ is an open subset of $I\!R^d$, it is possible that $\{x \in R^d; u(x) \in U\}$ is non empty of measure $0$, a phenomenum that never occurs whenever $d = 1$. The proofs of the above results are given in [12], following ideas from J. Malý and M. Zelený [19].

## 4.3. Viscosity solutions.

Let $X$ be a Banach space and $\Omega$ be a non empty bounded open subset of $X$. We consider the problem of finding a function $u : X \rightarrow I\!R$ solution of

$$(BVP4) \qquad \begin{cases} \|u'(x)\| = 1 & \text{for all } x \in \Omega \\ u(x) = 0 & \text{for all } x \in \partial\Omega \end{cases}$$

**Definition 4** *A function $u : X \longrightarrow I\!R$ is a classical solution of (HJ1) if $u$ is continuous on $\overline{\Omega}$, differentiable at every point of $\Omega$, and, for all $x \in \Omega$, $\|Du(x)\| = 1$.*

**Proposition 11** *Equation $(BVP4)$ has no classical solution.*

PROOF. Let us assume first that $dim(X) < +\infty$. Let $u : X \longrightarrow I\!R$ be a classical solution of $(BVP4)$. According to Rolle's theorem, There exists $x \in \Omega$ such that $u'(x) = 0$, which contradicts the fact that $\|u'(x)\| = 1$.

If $dim(X)$ is arbitrary and if $u : X \rightarrow I\!R$ is a classical solution of $(BVP4)$, then, by the mean value theorem, $u$ is Lipschitz continuous with Lipchitz constant 1. Since $\Omega$ is bounded, it follows that $u$ is bounded. Applying the approximate Rolle's theorem (see for instance [2]) : if we fix $0 < \varepsilon < 1$, there exists $x_0 \in \Omega$ such that $\|u'(x_0)\| < \varepsilon$. This contradicts the fact that $\|u'(x)\| = 1$ for every $x \in \Omega$. ■

We have seen in the previous section that $(BVP4)$ admits almost classical solutions. Actually, the function $u : \overline{\Omega} \rightarrow I\!R$ given by $u(x) = d(x, \partial\Omega)$ is the unique viscosity solution of $(BVP4)$ if there exists a $\mathcal{C}^1$-smooth bump function on $X$ which is Lipschitz-continuous (for instance whenever $X^*$ is separable), and uniqueness of the viscosity solution of $(BVP4)$ fails if $X$ is non Asplund (for instance whenever $X$ is separable and $X^*$ is non separable). We refer the reader to [10].

# References


[1] Azagra, D. and Deville, R. (2001). James' theorem fails for starlike bodies, *J. Funct. Anal.* **180**, 328–346.

[2] Azagra, D. and Deville, R. (1997). Subdifferential Rolle's and mean value inequality theorems, *Bull. Aust. Math. Soc.* **56**, 319–329.

[3] Azagra, D., Deville, R. and Jimenez-Sevilla, M. (2003). On the range of the derivatives of a smooth function between Banach spaces, *Math Proc. Camb. Philos. Soc.* **134**, 163–185.

[4] Azagra, D., Fabian, M. and Jimenez-Sevilla, M. (2005). Exact filling of figures with the derivatives of smooth mappings between Banach spaces, *Can. Math. Bull.* **48**, 481–499.

[5] Azagra, D. and Jimenez-Sevilla, M. (2001). The failure of Rolle's theorem in infinite-dimensional Banach spaces, *J. Funct. Anal.* **182**, 207–226.

[6] Bayart, F. (2005). Linearity of sets of strange functions, *Michigan Math. J.* **53**, 291–303.

[7] Borwein, J.M., Fabian, M., Kortezov, I. and Loewen, P.D. (2001). The range of the gradient of a continuously differentiable bump, *J. Nonlinear and Convex Anal.* **2**, 1–19.

[8] Borwein, J.M., Fabian, M. and Loewen, P.D. (2002). The range of the gradient of a Lipschitzian $\mathcal{C}^1$-smooth bump in infinite dimensions, *Isr. J. Math.* **132**, 239–251.

[9] Buczolich, Z. Solution to the gradient problem of C. E. Weil. *Revista. Mat. Iberoamericana* (To appear).

[10] Deville, R. (1998.) *Smooth variational principles and non-smooth analysis in Banach spaces*, F. H Clarke et al. (ed.), Nonlinear analysis, differential equations and control. Proceedings of the NATO Advanced Study Institute and séminaire de mathématiques supérieures, Montréal, Canada, July 27-August 7, 1998. Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. **528** (1999), 369–405.

[11] Deville, R. and Hajek, P. (2005). On the range of the derivative of Gâteaux-smooth functions on separable Banach spaces, *Isr. J. Math.* **145**, 257–269.

[12] Deville, R. and Matheron, E. Infinite games, Banach space geometry and the Eikonal equation, (to appear).

[13] Fabian, M., Kalenda O. and Kolár, J. (2005). Filling analytic sets by the derivatives of $C^1$-smooth bumps, *Proc. Amer. Math. Soc.* **133**, 295–303.

[14] Gaspari, T. (2002). On the range of the derivative of a real valued function with bounded support, *Studia Math.* **153**, 81–99.

[15] Gaspari, T. (2004). Bump functions with Holder derivatives, *Can. J. Math.* **56**, 699–715.

[16] Hájek, P. (1998). Smooth functions on $c_0$, *Israel J. Math.* **104**, 17–27.

[17] Rifford, L. (2003) Range of the gradient of a smooth bump function in finite dimensions, *Proc. Amer. Math. Soc.* **131**, 3063–3066.

[18] Maly, J. (1996). The Darboux property for gradients, *Real Anal. Exch.* **22**, (1), 167–173.

[19] Malý, J. and Zelený, M. A note on Buczolich's solution of the Weil gradient problem. (Preprint).

[20] Matrai, T. (2003-2004). Graphs of Gëteaux derivatives are $w^*$-connected, *Real Anal. Exch.* **29**, (1), 291–298.

[21] Saint-Raymond, J. (2002). Local Inversion for Differentiable Functions and the Darboux Property, *Mathematika,* **49**, 141–158.



**Robert Deville**
Institut de Mathématiques, Université de Bordeaux 1
351, Cours de la Libération
33405, Talence, France
*E-mail*: Robert.Deville@math.u-bordeaux1.fr