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Remarks on compact operators between interpolation spaces associated to polygons

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Abstract. This note deals with interpolation methods defined by means of polygons. We show necessary and sufficient conditions for compactness of operators acting from a J-space into a K-space.

Notas sobre operadores compactos entre espacios de interpolación asociados a polígonos

Resumen. Esta nota trata de métodos de interpolación definidos por medio de polígonos. Damos condiciones necesarias y suficientes para la compacidad de operadores actuando desde un J-espacio a un K-espacio.

1. Introduction

Interpolation theory has a number of important applications in geometry of Banach spaces, as can be seen, for example, in the monographs by Beauzamy [1], Lindenstraus and Tzafriri [18], or the papers by Kalton and Montgomery-Smith [17], and Pisier and Xu [19]. The more important interpolation methods are the real and complex methods, denoted by $(\cdot,\cdot)_{\theta,q}$ and $(\cdot,\cdot)_{[\theta]}$, respectively. Both of them work for Banach couples. In the special case of two Banach spaces A_0, A_1 with A_0 continuously embedded in A_1, they generate a "continuous scale" of spaces joining A_0 and A_1. It is useful to imagine A_0 and A_1 sitting on the endpoints of the segment $[0,1]$ and the space $(A_0,A_1)_{\theta,q}$ [respectively, $(A_0,A_1)_{[\theta]}$] located at the point θ. See Fig. 1.1.

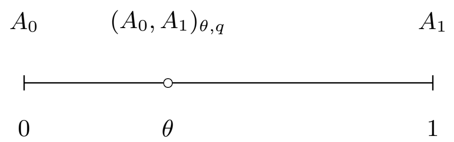

Figure 1.1

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Full details on interpolation methods can be found, for example, in the monographs by Bergh and Löfström [2], Triebel [21], Beauzamy [1] and Brudnyĭ and Krugljak [3]. As concern to applications in geometry of Banach spaces, the real method fits better than the complex method (see [18], [1] and [2]).

We deal here with interpolation methods similar to the real method, but working for finite families (N-tuples) of Banach spaces instead of couples and incorporating some geometrical elements which are essential in developing their theory. They were introduced by Peetre and one of the present authors in [11]. They are defined by using a convex polygon $\Pi = \overline{P_1 \cdots P_N}$ in the plane $\mathbb{R}^2$ with vertices P_j, an interior point (α, β) of Π, and two scalar parameters t, s. The Banach spaces $A_1, \ldots, A_N$ of the N-tuple should be thought of as sitting on the vertices of Π. The J- and K-methods associated to polygons give a unified point of view for the multidimensional methods which extend the real method to N-tuples. So, when the polygon Π is equal to the simplex, these methods give back (the first nontrivial case) of spaces studied by Sparr [20], and if Π coincides with the unit square we recover spaces studied by Fernandez [13], [14]. The geometrical point of view of Cobos and Peetre explains the restrictions on parameters in Fernandez' case.

A large part of the paper [11] is devoted to investigate the behaviour of compact operators under interpolation by the J- and K-methods. Later, Cobos, Kühn and Schonbek [9] studied the case when interpolated operators act from a J-space into a K-space. An estimate for the measure of non-compactness of operators acting from J- to K-spaces was established by Cobos, Fernández-Martínez and Martínez [7]. Other compactness results can be found in the papers by Cobos [4] and Cobos and Romero [12].

In this paper we compliment the results of [9] and [7] by characterizing compactness of interpolated operators in terms of weaker compactness conditions and an approximation stipulation involving the K- and the J-functionals.

In the case of the real method, the first result of this type was established by one of the present authors [15]. Later, the authors proved in [5], [6] similar results for the complex method and other classical methods for Banach couples. Very recently, two of the present authors [16] have investigated the multidimensional case, dealing with operators which act between two K-spaces or two J-spaces. For this aim, they imposed a certain geometrical condition on the polygon (to be *admissible*; see Section 2 below), already considered by Cobos and Peetre in [11] for their compactness theorems.

The result that we shall derive here do not require any extra assumption on the polygon. We achieve it thanks to the good estimate that holds for the norms of interpolated operators acting from a J- into a K-space. As in [16], our arguments are based on families of projections on vector-valued sequence spaces that come up when defining the K- and J-spaces but, in contrast to [16] where projections depend on the polygon, we work here with fixed sequences of projections.

The paper is organized as follows. In Section 2, we review the definitions of J- and K-methods associated to polygons and some of their basic properties. In Section 3, we establish the characterization of compact operators.

2. Interpolation methods defined by means of polygons

Let $\overline{A} = \{A_1, \ldots, A_N\}$ be a Banach N-tuple, that is, a family of N Banach spaces all of them continuously embedded in a common linear Hausdorff space. Let $\Sigma(\overline{A}) = A_1 + \cdots + A_N$ be their sum and $\Delta(\overline{A}) = A_1 \cap \cdots \cap A_N$ be their intersection. These two spaces become Banach spaces when endowed with the norms

$$\|a\|_{\Sigma(\overline{A})} = \inf \Big\{ \sum_{j=1}^{N} \|a_j\|_{A_j} : \ a = \sum_{j=1}^{N} a_j, \ a_j \in A_j \Big\}$$

and

$$\|a\|_{\Delta(\overline{A})} = \max \big\{ \|a\|_{A_j} : \ 1 \leq j \leq N \big\}$$

respectively.

Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$. We imagine each A_j as sitting on the vertex P_j. By means of the polygon Π, we define the following family of norms in $\Sigma(\overline{A})$

$$K(t,s;a) = \inf\Big\{\sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} \,:\, a = \sum_{j=1}^{N} a_j,\ a_j \in A_j\Big\}, t,s > 0.$$

Similarly, in $\Delta(\overline{A})$, we introduce the family of norms

$$J(t,s;a) = \max\big\{t^{x_j} s^{y_j} \|a\|_{A_j} \,:\, 1 \le j \le N\big\}, t,s > 0.$$

Given any interior point (α,β) of Π, $[(\alpha,\beta) \in Int\ \Pi]$ and $1 \le q \le \infty$, the K-space $\overline{A}_{(\alpha,\beta),q;K}$ consists of all elements a in $\Sigma(\overline{A})$ which have a finite norm

$$\|a\|_{\overline{A}_{(\alpha,\beta),q;K}} = \left(\sum_{(m,n)\in\mathbb{Z}^2} \big(2^{-\alpha m - \beta n} K(2^m, 2^n; a)\big)^q\right)^{1/q}$$

(the sum should be replaced by the supremum if $q = \infty$).

The J-space $\overline{A}_{(\alpha,\beta),q;J}$ is formed by all those a in $\Sigma(\overline{A})$ which can be represented as

$$a = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n} \quad (\text{convergence in } \Sigma(\overline{A})) \tag{2..1}$$

with $\{u_{m,n}\} \subseteq \Delta(\overline{A})$ and

$$\|\{u_{m,n}\}\|_{(\alpha,\beta),q;J} = \left(\sum_{(m,n)\in\mathbb{Z}^2} \big(2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n})\big)^q\right)^{1/q} < \infty. \tag{2..2}$$

The norm $\|\cdot\|_{\overline{A}_{(\alpha,\beta),q;J}}$ on $\overline{A}_{(\alpha,\beta),q;J}$ is given by the infimum of the values of the sums (2..2) over all such representations (2..1) of a.

In the special case when Π is equal to the simplex $\{(0,0),(1,0),(0,1)\}$, we recover (the first non-trivial case of) spaces introduced by Sparr [20], and in the case of the unit square $\{(0,0),(1,0),(1,1),(0,1)\}$, we obtain spaces studied by Fernandez [13], [14].

In general, K- and J-spaces do not coincide, but we have that $\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K}$ (see [11], Thm. 1.3). Here $\hookrightarrow$ means continuous inclusion.

In order to show a concrete example, consider the N-tuple of L_∞-spaces with weights

$$(L_\infty(w_1), \cdots, L_\infty(w_N)).$$

It is shown in [8], Thm. 2.3, that

$$(L_\infty(w_1), \cdots, L_\infty(w_N))_{(\alpha,\beta),\infty;K} = L_\infty(\breve{w}_{\alpha,\beta})$$

where the weight $\breve{w}_{\alpha,\beta}$ is defined by

$$\breve{w}_{\alpha,\beta}(x) = \min\Big\{w_i^{c_i}(x) w_k^{c_k}(x) w_r^{c_r}(x) : \{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}\Big\}.$$

Here $\mathcal{P}_{(\alpha,\beta)}$ is the set of all those triples $\{i,k,r\}$ such that (α,β) belongs to the triangle with vertices P_i, P_k, P_r (see Fig. 2.1) and (c_i, c_k, c_r) are the (unique) barycentric coordinates of (α,β) with respect to P_i, P_k, P_r.

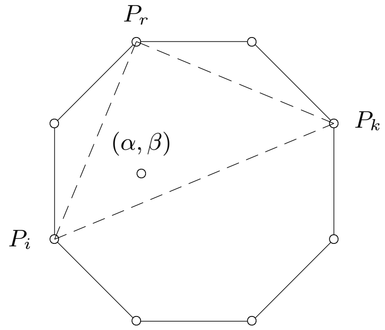

Figure 2.1

For N-tuples of weighted L_1-spaces, it is proved in [8], Thm. 2.5, that

$$(L_1(w_1), \cdots, L_1(w_N))_{(\alpha,\beta),1;J} = L_1(\hat{w}_{\alpha,\beta})$$

where

$$\hat{w}_{\alpha,\beta}(x) = \max\Big\{w_i^{c_i}(x) w_k^{c_k}(x) w_r^{c_r}(x) : \{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}\Big\}.$$

Other examples can be found in [11] or [10].

Let $\overline{B} = \{B_1, \ldots, B_N\}$ be another Banach N-tuple. We write $T \in \mathcal{L}(\overline{A}, \overline{B})$ to mean that T is a linear operator from $\Sigma(\overline{A})$ to $\Sigma(\overline{B})$ whose restriction to each A_j gives a bounded operator from A_j into B_j, $j = 1, \ldots, N$. It is easy to check that if $T \in \mathcal{L}(\overline{A}, \overline{B})$, then the restrictions

$$T : \overline{A}_{(\alpha,\beta),q;K} \to \overline{B}_{(\alpha,\beta),q;K} \quad , \quad T : \overline{A}_{(\alpha,\beta),q;J} \to \overline{B}_{(\alpha,\beta),q;J}$$

are bounded operators. The norms of these restrictions satisfy the following inequalities (see [8], Thm. 1.9)

$$\|T\|_{\overline{A}_{(\alpha,\beta),q;K}, \overline{B}_{(\alpha,\beta),q;K}} \leq C \max\Big\{\|T\|^{c_i}_{A_i,B_i} \|T\|^{c_k}_{A_k,B_k} \|T\|^{c_r}_{A_r,B_r} : \{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}\Big\},$$

where C is a constant depending only on Π and (α, β). A similar estimate holds for the restriction of T to the J-spaces.

If $T \in \mathcal{L}(\overline{A}, \overline{B})$, then it is clear that the restriction

$$T : \overline{A}_{(\alpha,\beta),q;J} \to \overline{B}_{(\alpha,\beta),q;K}$$

is bounded, as well. But in this case a more handy estimate holds. Let $\overline{\theta} = (\theta_1, \ldots, \theta_N)$ be any N-tuple of positive numbers with $\sum_{j=1}^N \theta_j = 1$ and $\sum_{j=1}^N \theta_j P_j = (\alpha, \beta)$. It was proved in [8], Thm. 3.2, that there is a constant $M > 0$, depending only on $\overline{\theta}$, such that for any Banach N-tuples $\overline{A}$, $\overline{B}$ and any $T \in \mathcal{L}(\overline{A}, \overline{B})$, it holds

$$\|T\|_{\overline{A}_{(\alpha,\beta),q;J}, \overline{B}_{(\alpha,\beta),q;K}} \leq M \prod_{j=1}^N \|T\|^{\theta_j}_{A_j,B_j}. \tag{2..3}$$

The following class of polygons was introduced in [11] to establish the compactness results for general N-tuples. If $\Pi = \overline{P_1 \cdots P_N}$, then for $j > N$ or $j < 1$, we put

$$P_j = P_{j_0} \quad \text{if} \quad j \equiv j_0 \ (\mathrm{mod}\ N), \ \ 1 \leq j_0 \leq N.$$

The convex polygon Π is said to be *admissible* if for each edge $\overline{P_j P_{j+1}}$, $j = 1, \ldots, N$, there is another $\overline{P_k P_{k+1}}$ satisfying the following two conditions:

(a) The extension of the segment $\overline{P_j P_{k+1}}$ in the direction of P_j meets the extension of $\overline{P_{j+1} P_{j+2}}$ in the direction of P_{j+1}; and

(b) the extension of the segment $\overline{P_{j+1} P_k}$ in the direction of P_{j+1} meets the extension of $\overline{P_{j-1} P_j}$ in the direction of P_j.

See Fig. 2.2.

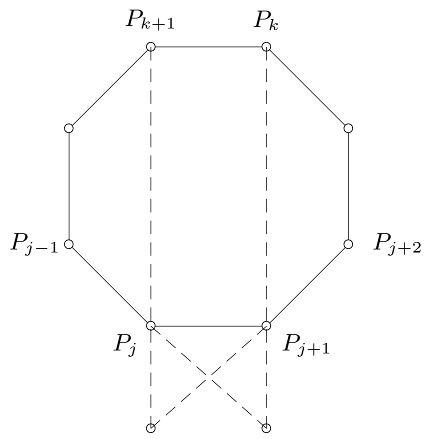

Figure 2.2

Characterizations for compact operators between J-spaces or K-spaces established by Fernández-Cabrera and Martínez require the polygon Π to be the simplex, the unit square, or any admissible polygon (see [16], Thms. 3.3 and 4.2). However, as we shall show in the next section, when operators act from a J-space into a K-space no geometrical condition on the polygon is needed.

3. Compact operators from J- to K-spaces

Let $\{W_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ be any sequence of Banach spaces and let $\{\lambda_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ be any sequence of non-negative numbers. We denote by $\ell_q(\lambda_{m,n} W_{m,n})$ the vector-valued ℓ_q-space formed by all sequences $w = \{w_{m,n}\}$ with $w_{m,n} \in W_{m,n}$ which have a finite norm

$$\|w\|_{\ell_q(\lambda_{m,n} W_{m,n})} = \Big(\sum_{(m,n)\in\mathbb{Z}^2} (\lambda_{m,n} \|w_{m,n}\|_{W_{m,n}})^q \Big)^{1/q}.$$

If $\lambda_{m,n} = 1$ for all $(m,n) \in \mathbb{Z}^2$, we write simply $\ell_q(W_{m,n})$.

Next we establish the characterization for compact operators.

Theorem 3..1 *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in $\mathbb{R}^2$ with vertices $P_j = (x_j, y_j)$, let $(\alpha,\beta) \in Int\, \Pi$ and $1 \le q \le \infty$. Assume that $\overline{A} = \{A_1, \dots, A_N\}$ and $\overline{B} = \{B_1, \dots, B_N\}$ are Banach N-tuples, and let $T \in \mathcal{L}(\overline{A}, \overline{B})$.*

If

(a) $T : \Delta(\overline{A}) \longrightarrow \overline{B}_{(\alpha,\beta),q;K}$ is compact,

(b) $T : \overline{A}_{(\alpha,\beta),q;J} \longrightarrow \Sigma(\overline{B})$ is compact, and

(c) $\sup\left\{\Big(\sum_{\substack{|m|>k\\|n|>k}}\big(2^{-\alpha m-\beta n}K(2^m,2^n;T\big(\sum_{\substack{|m|>k\\|n|>k}}u_{m,n}\big))\big)^q\Big)^{1/q}:\|\{u_{m,n}\}\|_{(\alpha,\beta),q;J}\leq 1\right\}\to 0$

as $k\to\infty$,

then

$$T:\overline{A}_{(\alpha,\beta),q;J}\longrightarrow\overline{B}_{(\alpha,\beta),q;K}\quad \textit{is compact}.$$

Conversely, if $q<\infty$*, then compactness of* $T:\overline{A}_{(\alpha,\beta),q;J}\longrightarrow\overline{B}_{(\alpha,\beta),q;K}$ *implies conditions (a), (b) and (c).*

PROOF. Making a change of variables if necessary (see [9], Remark 4.1), we may assume without loss of generality that Π has the form described in Fig. 3.1. That is

$$P_1=(0,0),\quad P_2=(1,0)\quad\text{and}\quad P_N=(0,1).$$

Vertices are numbered counterclockwise.

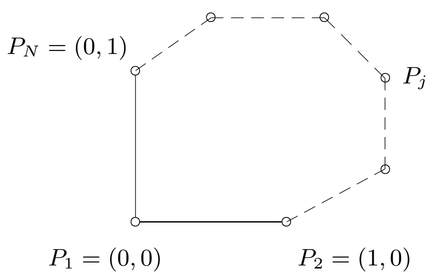

Figure 3.1

Let $F_{m,n}$ be the Banach space $\Sigma(\overline{B})$ provided with the norm $K(2^m,2^n;\cdot)$, and write

$$\ell_\infty(j)=\ell_\infty(2^{-x_jm-y_jn}F_{m,n})\ ,\ j=1,\dots,N,$$

and $\overline{\ell_\infty}=\{\ell_\infty(1),\dots,\ell_\infty(N)\}$. By [11], Thm. 3.1,

$$(\overline{\ell_\infty})_{(\alpha,\beta),q;K}=\ell_q(2^{-\alpha m-\beta n}F_{m,n})\quad(\text{equivalent norms}).\tag{3..1}$$

We denote by ι the operator assigning to each $b\in\Sigma(\overline{B})$ the constant sequence $\iota b=\{...,b,b,b,...\}$. Clearly, $\iota\in\mathcal{L}(B_j,\ell_\infty(j))$ for $j=1,\dots,N$, with norm ≤ 1. Furthermore, $\iota:\overline{B}_{(\alpha,\beta),q;K}\to\ell_q(2^{-\alpha m-\beta n}F_{m,n})$ is a metric embedding.

Following [9], for any $k\in\mathbb{N}$, we consider the partition of $\mathbb{Z}^2$ given by the sets

$$\begin{aligned}
\Gamma_k^{(0)}&=\big\{(m,n)\in\mathbb{Z}^2:|m|<k,|n|<k\big\}\,,\\
\Gamma_k^{(1)}&=\big\{(m,n)\in\mathbb{Z}^2:m\leq -k,|n|<k\big\}\,,\\
\Gamma_k^{(2)}&=\big\{(m,n)\in\mathbb{Z}^2:m\geq k,|n|<k\big\}\,,\\
\Gamma_k^{(3)}&=\big\{(m,n)\in\mathbb{Z}^2:n\leq -k\big\}\,,\\
\Gamma_k^{(4)}&=\big\{(m,n)\in\mathbb{Z}^2:n\geq k\big\}\,.
\end{aligned}$$

Let $\{S_k^{(r)}\}_{0\leq r\leq 4,\ k\in\mathbb{N}}$ be the projections on $\overline{\ell_\infty}$ defined by $S_k^{(r)}\{w_{m,n}\}=\{v_{m,n}\}$ where

$$v_{m,n}=\begin{cases} w_{m,n} & \text{if } (m,n)\in\Gamma_k^{(r)},\\ 0 & \text{otherwise .}\end{cases}$$

It is not hard to check that these operators satisfy the following conditions:

(I) The identity operator on $\Sigma(\overline{\ell_\infty})$ can be decomposed as

$$I=\sum_{r=0}^{4}S_k^{(r)} \quad , \quad k=1,2,...$$

(II) We have

$$\|S_k^{(r)}\|_{\ell_\infty(j),\ell_\infty(j)}=1 \text{ for any } k\in\mathbb{N}\ ,0\leq r\leq 4 \text{ and } 1\leq j\leq N.$$

(III) For each $k\in\mathbb{N}$, $S_k^{(0)}\in\mathcal{L}(\Sigma(\overline{\ell_\infty}),\Delta(\overline{\ell_\infty}))$.

(IV) For each $k\in\mathbb{N}$ we have

$$\begin{aligned}
S_k^{(1)}&:\ell_\infty(2^{-x_2m-y_2n}F_{m,n})\longrightarrow\ell_\infty(2^{-x_1m-y_1n}F_{m,n}),\\
S_k^{(2)}&:\ell_\infty(2^{-x_1m-y_1n}F_{m,n})\longrightarrow\ell_\infty(2^{-x_2m-y_2n}F_{m,n}),\\
S_k^{(3)}&:\ell_\infty(2^{-x_Nm-y_Nn}F_{m,n})\longrightarrow\ell_\infty(2^{-x_1m-y_1n}F_{m,n}),\\
S_k^{(4)}&:\ell_\infty(2^{-x_1m-y_1n}F_{m,n})\longrightarrow\ell_\infty(2^{-x_Nm-y_Nn}F_{m,n}),
\end{aligned}$$

and their norms are equal to 2^{-k}.

Let $G_{m,n}$ be the Banach space $\Delta(\overline{A})$ endowed with the norm $J(2^m,2^n;\cdot)$. We put

$$\ell_1(j)=\ell_1(2^{-x_jm-y_jn}G_{m,n})\ ,\ j=1,\dots,N,$$

and $\overline{\ell_1}=\{\ell_1(1),\dots,\ell_1(N)\}$. Relationship between these vector-valued sequence spaces and $\overline{A}$ is given by the operator $\pi\{u_{m,n}\}=\sum_{(m,n)\in\mathbb{Z}^2}u_{m,n}$. Clearly, $\pi\in\mathcal{L}(\ell_1(j),A_j)\ ,\ j=1,\dots,N$, and its norm is ≤ 1. Moreover, π acting from $\ell_q(2^{-\alpha m-\beta n}G_{m,n})$ into $\overline{A}_{(\alpha,\beta),q;J}$ is a metric surjection.

On the N-tuple $\overline{\ell_1}$ we can define analogous operators to $\{S_k^{(r)}\}$. We call them $\{D_k^{(r)}\}$. Note that they satisfy the corresponding versions of (I), (II), (III) and (IV).

In terms of projections $D_k^{(0)}$ and $S_k^{(0)}$ condition (c) can be equivalently stated as

$$\|(I-S_k^{(0)})\iota T\pi(I-D_k^{(0)})\|_{\ell_q(2^{-\alpha m-\beta n}G_{m,n}),\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\longrightarrow 0 \quad \text{as} \quad k\to\infty. \tag{c'}$$

Assume that (a), (b) and (c') holds. Using the diagrams of bounded operators

$$\ell_q\big(2^{-\alpha m-\beta n}G_{m,n}\big)\xrightarrow{D_k^{(0)}}\Delta(\overline{\ell_1})\xrightarrow{\pi}\Delta(\overline{A})\xrightarrow{T}\overline{B}_{(\alpha,\beta),q;K}\xrightarrow{\iota}\ell_q\big(2^{-\alpha m-\beta n}F_{m,n}\big)$$

$$\begin{array}{c}
\ell_q\big(2^{-\alpha m-\beta n}G_{m,n}\big)\xrightarrow{I-D_k^{(0)}}\ell_q\big(2^{-\alpha m-\beta n}G_{m,n}\big)\xrightarrow{\pi}\overline{A}_{(\alpha,\beta),q;J}\xrightarrow{T}\Sigma(\overline{B})\xrightarrow{\iota}\Sigma(\overline{\ell_\infty})\\
\hfill\downarrow S_k^{(0)}\\
\hfill\ell_q\big(2^{-\alpha m-\beta n}F_{m,n}\big)
\end{array}$$

and conditions (a) and (b), we derive that $\iota T\pi D_k^{(0)}$ and $S_k^{(0)}\iota T\pi(I-D_k^{(0)})$ act compactly from $\ell_q\big(2^{-\alpha m-\beta n}G_{m,n}\big)$ into $\ell_q\big(2^{-\alpha m-\beta n}F_{m,n}\big)$.

Since

$$\iota T\pi-\iota T\pi D_k^{(0)}-S_k^{(0)}\iota T\pi(I-D_k^{(0)})=(I-S_k^{(0)})\iota T\pi(I-D_k^{(0)}),$$

it follows from (c') that

$$\iota T\pi:\ell_q\big(2^{-\alpha m-\beta n}G_{m,n}\big)\longrightarrow\ell_q\big(2^{-\alpha m-\beta n}F_{m,n}\big)$$

is compact. Now, properties of π and ι yield that

$$T:\overline{A}_{(\alpha,\beta),q;J}\longrightarrow\overline{B}_{(\alpha,\beta),q;K}\quad\text{ is compact.}$$

Conversely, suppose that $q<\infty$ and that T acting from the J- into the K-space is compact. Then (a) and (b) follows easily using that $\Delta(\overline{A})\hookrightarrow\overline{A}_{(\alpha,\beta),q;J}$ and $\overline{B}_{(\alpha,\beta),q;K}\hookrightarrow\Sigma(\overline{B})$. In order to establish (c'), observe that for $w\in\ell_q\big(2^{-\alpha m-\beta n}G_{m,n}\big)$ we have

$$\begin{aligned}&\|(I-S_{k+1}^{(0)})\iota T\pi(I-D_{k+1}^{(0)})w\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\\&\qquad\le\|(I-S_k^{(0)})\iota T\pi(I-D_k^{(0)})\big((I-D_{k+1}^{(0)})w\big)\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}.\end{aligned}$$

Thus, the sequence

$$\Big\{\|(I-S_k^{(0)})\iota T\pi(I-D_k^{(0)})\|_{\ell_q(2^{-\alpha m-\beta n}G_{m,n}),\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\Big\}_{k\in\mathbb{N}}$$

is non-increasing. Let η be its limit. To complete the proof we must show that $\eta=0$.

Choose vectors $\{w_k\}$ in the unit ball of $\ell_q(2^{-\alpha m-\beta n}G_{m,n})$ such that

$$\eta=\lim_{k\to\infty}\|(I-S_k^{(0)})\iota T\pi(I-D_k^{(0)})w_k\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}.$$

Since $q<\infty$, we can select $\{w_k\}$ is such a way that each w_k has only finitely many non-zero co-ordinates. Therefore

$$\{w_k\}_{k\in\mathbb{N}}\subseteq\Delta(\overline{\ell_1}).\tag{3..2}$$

Using the compactness of $T:\overline{A}_{(\alpha,\beta),q;J}\longrightarrow\overline{B}_{(\alpha,\beta),q;K}$ and boundedness of the sequence

$$\{\pi(I-D_k^{(0)})w_k\},$$

we can find a subsequence $\{T\pi(I-D_{k'}^{(0)})w_{k'}\}$ converging to some b in $\overline{B}_{(\alpha,\beta),q;K}$. We claim that

$$\Big\{\|(I-S_{k'}^{(0)})\iota T\pi(I-D_{k'}^{(0)})w_{k'}\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\Big\}$$

is a null sequence.

Indeed, given any $\varepsilon>0$, we can find $\nu_1\in\mathbb{N}$ such that for all $k'\ge\nu_1$

$$\|b-T\pi(I-D_{k'}^{(0)})w_{k'}\|_{\overline{B}_{(\alpha,\beta),q;K}}\le\varepsilon/12.$$

Let $z=T\pi(I-D_{\nu_1}^{(0)})w_{\nu_1}$. By (3..2), we have $z\in\Delta(\overline{B})$. On the other hand, we know from (IV) that $\|S_k^{(1)}\|_{\ell_1(2),\ell_1(1)}\le 2^{-k}$. Hence, using equality (3..1), the norm estimate (2..3) and factorization

$$\begin{aligned}&\Delta(\overline{\ell_\infty})\hookrightarrow(\ell_\infty(2),\ell_\infty(2),\ell_\infty(3),\dots,\ell_\infty(N))_{(\alpha,\beta),q;J}\xrightarrow{S_k^{(1)}}\\&\qquad\longrightarrow(\ell_\infty(1),\ell_\infty(2),\ell_\infty(3),\dots,\ell_\infty(N))_{(\alpha,\beta),q;K}=\ell_q\big(2^{-\alpha m-\beta n}F_{m,n}\big)\end{aligned}$$

we obtain that

$$\|S_k^{(1)}\|_{\Delta(\overline{\ell_\infty}),\ell_q(2^{-\alpha m-\beta n}F_{m,n})} \longrightarrow 0 \quad \text{as} \quad k\to\infty.$$

A similar argument applies to $S_k^{(j)}$ for $j=2,3,4$. Let $\nu_2\in\mathbb{N}$ such that for all $k'\geq\nu_2$

$$\|S_{k'}^{(j)}\iota z\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})} < \varepsilon/12\,, \quad j=1,2,3,4.$$

For any $k'\geq\max\{\nu_1,\nu_2\}$ it follows that

$$\begin{aligned}
&\|(\sum_{j=1}^4 S_{k'}^{(j)})\iota T\pi(I-D_{k'}^{(0)})w_{k'}\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\\
&\qquad\leq \sum_{j=1}^4\Big[\|S_{k'}^{(j)}\iota(T\pi(I-D_{k'}^{(0)})w_{k'}-b)\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\\
&\qquad\quad+\|S_{k'}^{(j)}\iota(b-z)\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}+\|S_{k'}^{(j)}\iota z\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\Big]\\
&\leq\sum_{j=1}^4\Big[\|T\pi(I-D_{k'}^{(0)})w_{k'}-b\|_{\overline{B}_{(\alpha,\beta),q;K}}+\|b-z\|_{\overline{B}_{(\alpha,\beta),q;K}}+\|S_{k'}^{(j)}\iota z\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}\Big]\leq\varepsilon.
\end{aligned}$$

Consequently, $\eta=\lim_{k'\to\infty}\|(\sum_{j=1}^4 S_{k'}^{(j)})\iota T\pi(I-D_{k'}^{(0)})w_{k'}\|_{\ell_q(2^{-\alpha m-\beta n}F_{m,n})}=0$. This completes the proof. ■

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