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Existence and Stability of Periodic Solutions for a Nonlocal Evolution Population Problem

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Abstract. The theory of maximal monotone operators is applied to prove the existence of weak periodic solutions for a nonlinear nonlocal problem. The stability of these solutions is a consequence of the Lipschitz continuous assumption on the diffusivity matrix and the death rate.

Existencia y estabilidad de soluciones periódicas para un problema no local de evolución de la población

Resumen. La teoría de operadores monótonos maximales se aplica para demostrar la existencia de soluciones periódicas débiles de problemas no lineales y no locales. La estabilidad de estas soluciones es consecuencia de la suposición de la continuidad Lipschitz en la matriz de difusividad y de la tasa de defunción.

1 Introduction

In this paper we deal with the existence and stability of weak periodic solutions for a nonlinear parabolic problem modelling the evolution of a population of bacteria of density $u(x,t)$ at the localization (x,t), whose coefficients are depending on a weighted integral of the density u. In the bacteria diffusion process, the speed of diffusion is given by the Fourier law with a local dependence of the diffusion rate on the density u. With regard to these problems (see [3]), it is also natural to assume a nonlocal dependence of the diffusion rate on the entire population $\int_\Omega u(x,t)\,dx$. More generally, one can consider a weighted factor of the type $\int_\Omega g(x)u(x,t)\,dx$.

Let Ω be a regular bounded open set of $\mathbb{R}^n$, $n \geqslant 1$, with boundary $\partial\Omega$ and assume that Γ_D and $\Gamma_N := \partial\Omega \setminus \Gamma_D$ are two measurable subset of $\partial\Omega$ with positive measure. We would like to find $u = u(x,t)$ solution to

$$(P) \begin{cases} u_t - \sum_{i,j=1}^n \dfrac{\partial}{\partial x_j}\left(a_{ij}\big(l(u(t))\big)\dfrac{\partial u}{\partial x_i}\right) + a_0\big(l(u(t))\big)u = f & \text{in } Q := \Omega \times P, \\ u(x,t) = 0 & \text{on } \Gamma_D \times P, \\ \dfrac{\partial u}{\partial \nu}(x,t) = 0 & \text{on } \Gamma_N \times P, \\ u(x,t+\omega) = u(x,t) & \text{a.e. in } Q, \omega > 0, \end{cases}$$

where the diffusivity symmetric matrix $\{a_{ij}(\zeta)\}_{i\times j}$ in the diffusivity velocity term, depends on a nonlocal term and the nonlocal death rate occurs at rate $a_0(\zeta)$ proportional to the density of population. In the setting

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of the model, the $t-$periodic function $f(x,t)$ denotes the density of bacteria supplied from outside by means of births. A balance of population leads to consider problem (P). Dirichlet boundary condition describes a lethal crossing boundary Γ_D while Neumann boundary condition, excludes migration accross the boundary Γ_N. The period interval $[0,\omega]$ shall be denoted by $P := \frac{R}{\omega Z}$, thus for the functions defined on Q we are automatically imposing the time $\omega-$periodicity. Next, we summarize the structural assumptions that shall be done on the data.

For any $i,\, j \in \{1, 2, \ldots, n\}^2$, let us introduce an $n \times n$ symmetric matrix $a_{i,j}$ on $\mathbb{R}$, with bounded (i, j)-components such that

H$_{ij}$) $a_{ij} \in C(\mathbb{R})$ and there exist two positive constants λ, Λ such that

$$\lambda \mid \xi \mid^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(\zeta)\, \xi_i\, \xi_j \leqslant \Lambda \mid \xi \mid^2, \qquad \forall \xi \in \mathbb{R}^n,\, \forall \zeta \in \mathbb{R}.$$

On the death rate term a_0, we assume that

H$_0$) $a_0 \in C(\mathbb{R}), \quad 0 < \delta \leqslant a_0(\zeta) \leqslant \gamma,\, \forall \zeta \in \mathbb{R}.$

Moreover,

H$_l$) $l(u(t)) := \int_\Omega g(x)u(x,t)\,dx$, where the weight $g \in L^2(\Omega)$.

H$_f$) $f \in L^2(Q)$.

Since $\partial\Omega$ is regular, the unit outward normal to Γ_N is $\nu = (\nu_1, \ldots, \nu_n)$ and we let define

$$\frac{\partial u}{\partial \nu} := \sum_{i,j=1}^{n} a_{ij}(\zeta) \frac{\partial u}{\partial x_i} \nu_j.$$

We point out that the existence of weak periodic solutions to (P) shall be proven in the framework of known results on monotone operators. Concerning the stability of periodic solutions, this shall be obtained under an additional assumption on a_{ij} and a_0 i.e.

H$_0$) The functions a_{ij} and a_0 are Lipschitz continous with constant L.

For parabolic problems with coefficients depending on nonlocal term, the reader is referred to [3] where many references can be found with regard to existence, uniqueness and asymptotic behaviour of solutions. The periodic case seems no yet considered in literature. Our plan is the following: in section 3, in order to carry out our study we present the preliminary result on monotone operator that we use in the later section. Section 4, is devoted to the investigation of existence of weak periodic solutions utilizing a fixed point argument which allows to apply the Schauder theorem. Finally, in the last section we study the stability of weak periodic solutions.

2 Preliminaries and functional framework

The approach to weak periodicity of solutions, shall be done looking for them in a suitable t-periodic function space. Hence, to study our problem we introduce some useful functional spaces. Introduced the space

$$V_D(\Omega) := \{u \in W^{1,2}(\Omega) :\; u = 0 \text{ on } \Gamma_D\},$$

let $V_0 := L^2(P; V_D(\Omega))$ be a Hilbert space endowed with the norm

$$\|v\|_V := \left(\int_Q \mid v(x,t) \mid^2 \; dx\, dt + \int_Q \mid \nabla v(x,t) \mid^2 \; dx\, dt \right)^{1/2}. \tag{1}$$

The space V_0 is the closure of $C_0^\infty(Q)$, the space of the periodic functions vanishing on Γ_D, with respect to the norm (1). The topological dual space of V_0 is denoted by $V^* := L^2(P; V_D'(\Omega))$ and endowed by the norm $\|\cdot\|_*$. The duality inner product between V_0 and V^* shall be denoted by $\langle \cdot, \cdot \rangle$.

Fixed $w \in L^2(Q)$, consider the problem

$$(P_w) \begin{cases} u_t - \sum_{i,j=1}^n \dfrac{\partial}{\partial x_j}\left(a_{ij}\big(l(w(t)\big)\dfrac{\partial u}{\partial x_i}\right) + a_0\big(l(w(t))\big)u = f & \text{in } Q := \Omega \times P, \\ u(x,t) = 0 & \text{on } \Gamma_D \times P \\ \dfrac{\partial u}{\partial \nu}(x,t) = 0 & \text{on } \Gamma_N \times P, \\ u(x, t+\omega) = u(x,t) & \text{a.e. in } Q, \omega > 0. \end{cases}$$

Definition 1 *A weak periodic solution of (P_w), is a function $u \in V_0$ such that*

$$\begin{aligned} \int_Q u_t \xi \, dx\, dt + \sum_{i,j=1}^n \int_Q a_{ij}(l(w(t)) \frac{\partial u}{\partial x_i}\frac{\partial \xi}{\partial x_j}\, dx\, dt + \int_Q a_0(l(w(t)) u \xi \, dx\, dt \\ = \int_Q f(x,t)\xi \, dx\, dt. \forall \xi \in V_0. \end{aligned} \tag{2}$$

The argument of monotone operator utilized to show the existence of periodic solutions u for (2), is contained in the following result (see, [1, 2, 5]).

Theorem 1 ([1, 2, 5]) *Let L be a linear closed, densely defined operator from the reflexive Banach space V_0 to V^*, L maximal monotone and let B be a bounded, hemicontinuous monotone mapping from V_0 into V^*, then $L+B$ is maximal monotone in $V_0 \times V^*$. Moreover, if $L+B$ is coercive, then* $\mathrm{Range}(L+B) = V^*$.

3 Existence of periodic solutions

In order to apply theorem 1, we need to define operators L and B.

Let $L\colon D \to V^*$ be a closed skew-adjoint (i.e. $L = -L^*$) linear operator densely defined by

$$\langle L(u), \xi \rangle := \int_Q u_t \xi \, dx\, dt, \qquad \forall \xi \in V_0,$$

on the set

$$D := \{u \in V_0 := L^2(P; V_D(\Omega));\ u_t \in L^2(P; V_D'(\Omega))\},$$

thus L is a maximal monotone operator (see [5]).

Defined the operator $B\colon V_0 \to V^*$ by setting

$$\langle B(u), \xi \rangle := \sum_{i,j=1}^n \int_Q a_{ij}(l(w(t)) \frac{\partial u}{\partial x_i}\frac{\partial \xi}{\partial x_j}\, dx\, dt + \int_Q a_0(l(w(t)) u \xi \, dx\, dt, \qquad \forall \xi \in V_0,$$

we collect some properties of this operator

Proposition 1 *Assume* H_{ij})–H_f)*, then the operator B satisfies*

i) $B\colon V_0 \to V^*$ *is hemicontinuous;*

ii) B *is monotone;*

iii) B *is coercive;*

PROOF. Applying the Hölder inequality one has

$$\begin{aligned}
|\langle B(u),\xi\rangle| &\leqslant \sum_{i,j=1}^{n}\int_Q a_{ij}(l(w(t))\left|\frac{\partial u}{\partial x_i}\right|\left|\frac{\partial \xi}{\partial x_j}\right| dx\,dt + \int_Q a_0(l(w(t))|u||\xi|\,dx\,dt\\
&\leqslant \alpha\sum_{i,j=1}^{n}\left(\int_Q\left|\frac{\partial u}{\partial x_i}\right|^2 dx\,dt\right)^{1/2}\left(\int_Q\left|\frac{\partial \xi}{\partial x_j}\right|^2 dx\,dt\right)^{1/2}\\
&\quad+\gamma\left(\int_Q |u|^2\, dx\,dt\right)^{1/2}\left(\int_Q |\xi|^2\, dx\,dt\right)^{1/2}\\
&\leqslant (\alpha+\gamma)\,\|u\|_V\,\|\xi\|_V,
\end{aligned}$$

where $\alpha \geqslant a_{ij}(l(w(t))$, for any i, j, by which

$$\|B(u)\|_* \leqslant (\alpha+\gamma)\|u\|_V$$

and the implication of hemicontinuity follows from a result of [4].

ii)

$$\begin{aligned}
\langle B(u_1)-B(u_2),u_1-u_2\rangle &= \sum_{i,j=1}^{n}\int_Q a_{ij}(l(w(t))\frac{\partial(u_1-u_2)}{\partial x_i}\,\frac{\partial(u_1-u_2)}{\partial x_j}\,dx\,dt\\
&\quad+\int_Q a_0(l(w(t))(u_1-u_2)^2\,dx\,dt\\
&\geqslant \lambda\int_Q|\nabla(u_1-u_2)|^2 dx\,dt + \int_Q a_0(l(w(t))(u_1-u_2)^2\,dx\,dt \geqslant 0.
\end{aligned}$$

iii)

$$\begin{aligned}
\langle B(u),u\rangle &= \sum_{i,j=1}^{n}\int_Q a_{ij}(l(w(t))\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,dx\,dt + \int_Q a_0(l(w(t))u^2\,dx\,dt\\
&\geqslant \lambda\int_Q|\nabla u|^2\,dx\,dt + \delta\int_Q |u|^2\,dx\,dt \geqslant c\|u\|_V^2,
\end{aligned}$$

with $c := \min\{\lambda,\delta\}$.

Therefore, we get

$$\frac{\langle B(u),u\rangle}{\|u\|_V} \geqslant c\|u\|_V \to +\infty, \qquad \text{as } \|u\|_V \to +\infty. \quad \blacksquare$$

Finally, let $G \in V^*$ be the linear functional defined as follows

$$\langle G,\xi\rangle := \int_Q f(x,t)\xi(x,t)\,dx\,dt, \qquad \forall \xi \in V_0,$$

then (2) can be equivalently rewritten as

$$Lu + Bu = G. \tag{3}$$

Applying Theorem 1 to problem (3), remain showed the existence of weak periodic solutions u for problem (2). The uniqueness of the weak periodic solution corresponding to w, is a consequence of classical results.

Let $w_n \in L^2(Q)$ be such that $w_n \to w$ in $L^2(Q)$ as n goes to infinity, consider the problem

$$\int_Q u_{nt}\xi dx dt + \sum_{i,j=1}^{n} \int_Q a_{ij}(l(w_n(t)) \frac{\partial u_n}{\partial x_i} \frac{\partial \xi}{\partial x_j}\, dx\, dt + \int_Q a_0(l(w_n(t)) u_n \xi\, dx\, dt = \int_Q f(x,t)\xi\, dx\, dt, \qquad \forall \xi \in V_0, \tag{4}$$

choosing u_n as a test function in (4), by the Young inequality we obtain

$$\delta \int_Q |u_n|^2\, dx\, dt + \lambda \int_Q |\nabla u_n|^2\, dx\, dt \leqslant \frac{1}{2\delta} \int_Q f^2(x,t)\, dx\, dt + \frac{\delta}{2} \int_Q u_n^2(x,t)\, dx\, dt.$$

Thus,

$$\int_Q |\nabla u_n|^2\, dx\, dt \leqslant C \tag{5}$$

and the usual energy estimate follows

$$\int_Q u_n^2\, dx\, dt + \int_Q |\nabla u_n|^2\, dx\, dt \leqslant C, \tag{6}$$

(where the positive constant C is independent of n).

By virtue of (5), u_{nt} is bounded with respect to the norm of V^* hence u_n belongs to a bounded set of D i.e.

$$\|u_n\|_D \leqslant C.$$

Passing to subsequence, if necessary still denoted by u_n, one has

$$u_n \rightharpoonup u \qquad \text{in } D \text{ as } n \to +\infty.$$

By a result of [5], the sequence u_n is precompact in $L^2(Q)$ that is

$$u_n \to u \qquad \text{in } L^2(Q) \text{ and a.e. in } Q.$$

Therefore, we can collect the properties of solutions of (5) concerning convergence

$$\begin{array}{ll} \nabla u_n \rightharpoonup \nabla u & \text{in } L^2(P; L^2(\Omega)^n), \\ l(w_n(t)) \to l(w(t)) & \text{in } L^2(Q), \\ w_n \to w & \text{in } L^2(Q). \end{array}$$

4 A fixed point argument

The existence of weak periodic solutions to the problem (P), shall be showed by means of the Schauder fixed point theorem. We define the operator

$$\Theta\colon L^2(Q) \to L^2(Q)$$
$$\Theta(w) = u,$$

where u is the unique weak periodic solution to (2) corresponding to w.

Lemma 1 *The operator Θ is continuous.*

PROOF. The result follows from the above convergences, because $\Theta(w_n) = u_n$ converges strongly in $L^2(Q)$ to $\Theta(w) = u$. ■

Lemma 2 *There exists a constant $R > 0$ such that*

$$\|\Theta(w)\|_{L^2(Q)} \leqslant R.$$

PROOF. Passing to the limit in (6) one has the conclusion. ■

Since $\Theta(L^2(Q)) \subset D$ and the embedding of $D \hookrightarrow L^2(Q)$ is compact, the operator Θ is compact from $L^2(Q)$ into itself.

Next, we can state our main result

Theorem 2 *If* $\mathrm{H}_{ij})$–$\mathrm{H}_f)$ *are fulfilled, there exists at least a weak periodic solution to problem* (P).

PROOF. The assertion descends from the Schauder fixed point theorem applied to the operator Θ whose fixed points correspond to weak periodic solutions to (P) that is

$$\int_Q u_t \xi \, dx \, dt + \sum_{i,j=1}^n \int_Q a_{ij}(l(u(t)) \frac{\partial u}{\partial x_i} \frac{\partial \xi}{\partial x_j} \, dx \, dt + \int_Q a_0(l(u(t)) u \xi \, dx \, dt$$
$$= \int_Q f(x,t) \xi \, dx \, dt, \qquad \forall \xi \in V_0.$$

5 Local stability

This section is devoted to show the local stability of weak periodic solutions in the sense that the unique solution v of the initial-boundary problem

$$(P_{i,b}) \begin{cases} v_t - \sum_{i,j=1}^n \dfrac{\partial}{\partial x_j}(a_{ij}(l(v(t))) \dfrac{\partial v}{\partial x_i}) + a_0(l(v(t)))v = f & \text{in } Q := \Omega \times \mathbb{R}_+, \\ v(x,t) = 0 & \text{on } \Gamma_D \times \mathbb{R}_+, \\ \dfrac{\partial v}{\partial \nu}(x,t) = 0 & \text{on } \Gamma_N \times \mathbb{R}_+, \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

$v_0 \in L^2(\Omega)$ (see [3]), is such that

$$\|u(t) - v(t)\|_{L^2(\Omega)} < \varepsilon, \tag{7}$$

while

$$\|u(0) - v_0\|_{L^2(\Omega)} < \eta_\varepsilon.$$

This result shall be derived under the additional assumption $\mathrm{H}_0)$.

Theorem 3 *Assume* $\mathrm{H}_{ij})$–$\mathrm{H}_0)$*, then for any $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that for $\|u(0) - v_0\|_{L^2(\Omega)} < \eta_\varepsilon$,* (7) *holds.*

PROOF. If u is any weak periodic solution to (P) and v is the unique solution of $(P_{i,b})$, then

$$\frac{d}{dt}\int_\Omega (u(t)-v(t))\xi(t)\,dx + \sum_{i,j=1}^n a_{ij}(l(u(t)))\int_\Omega \frac{\partial(u(t)-v(t))}{\partial x_i}\frac{\partial \xi(t)}{\partial x_j}\,dx$$
$$- \sum_{i,j=1}^n (a_{ij}(l(v(t))) - a_{ij}(l(u(t))))\int_\Omega \frac{\partial(v(t))}{\partial x_i}\frac{\partial \xi(t)}{\partial x_j}\,dx$$
$$+ a_0(l(u(t)))\int_\Omega (u(t)-v(t))\xi(t)\,dx + (a_0(l(u(t))) - a_0(l(v(t))))\int_\Omega v(t)\xi(t)\,dx = 0.$$

Taking $\xi(x,t) = u(x,t) - v(x,t)$ as a test function, we infer that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t)-v(t))^2\,dx + \sum_{i,j=1}^n a_{ij}(l(u(t)))\int_\Omega \frac{\partial(u(t)-v(t))}{\partial x_i}\frac{\partial(u(t)-v(t))}{\partial x_j}\,dx$$
$$+ a_0(l(u(t)))\int_\Omega (u(t)-v(t))^2\,dx$$
$$= \sum_{i,j=1}^n (a_{ij}(l(v(t))) - a_{ij}(l(u(t))))\int_\Omega \frac{\partial v(t)}{\partial x_i}\frac{\partial(u(t)-v(t))}{\partial x_j}\,dx$$
$$+ L\,|l(v(t)) - l(u(t))|\,\|u(t)-v(t)\|_{L^2(\Omega)}\|v(t)\|_{L^2(\Omega)},$$

by which

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t)-v(t))^2 dx + \lambda\int_\Omega |\nabla(u(t)-v(t))|^2\,dx$$
$$\leqslant nL\,|l(v(t)) - l(u(t))|\,\|\nabla(u(t)-v(t))\|_{L^2(\Omega)}\|\nabla v(t)\|_{L^2(\Omega)}$$
$$+ L|l(v(t)) - l(u(t))|\|u(t)-v(t)\|_{L^2(\Omega)}\|v(t)\|_{L^2(\Omega)}.$$

Since
$$|l(v(t)) - l(u(t))| \leqslant \|g\|_{L^2(\Omega)}\|u(t)-v(t)\|_{L^2(\Omega)},$$
one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t)-v(t))^2\,dx + \lambda\int_\Omega |\nabla(u(t)-v(t))|^2\,dx$$
$$\leqslant nL\|g\|_{L^2(\Omega)}\|u(t)-v(t)\|_{L^2(\Omega)}\|\nabla v(t)\|_{L^2(\Omega)}\|\nabla(u(t)-v(t))\|_{L^2(\Omega)}$$
$$+ L\|g\|_{L^2(\Omega)}\|v(t)\|_{L^2(\Omega)}\|u(t)-v(t)\|^2_{L^2(\Omega)},$$

the Young inequality applied to the first term on the right hand side gives us

$$\frac{1}{2}\frac{d}{dt}\|u(t)-v(t)\|^2_{L^2(\Omega)} + \lambda\|\nabla(u(t)-v(t))\|^2$$
$$\leqslant \frac{\lambda}{2}\|\nabla(u(t)-v(t))\|^2_{L^2(\Omega)} + \Bigg(\frac{n^2L^2}{2\lambda}\|g\|^2_{L^2(\Omega)}\|\nabla v(t))\|^2_{L^2(\Omega)}$$
$$+ L\|g\|_{L^2(\Omega)}\|v(t)\|_{L^2(\Omega)}\Bigg)\|u(t)-v(t)\|^2_{L^2(\Omega)}.$$

Therefore,
$$\frac{d}{dt}\|u(t)-v(t)\|^2_{L^2(\Omega)} \leqslant c(t)\|u(t)-v(t)\|^2_{L^2(\Omega)},$$
where
$$c(t) := \frac{n^2L^2}{2\lambda}\|g\|^2_{L^2(\Omega)}\|\nabla v(t))\|^2_{L^2(\Omega)} + L\|g\|_{L^2(\Omega)}\|v(t)\|_{L^2(\Omega)} \in L^1(R_+).$$

From the Gronwall lemma one obtains
$$\|u(t)-v(t)\|^2_{L^2(\Omega)} \leqslant \|u(0)-v_0\|^2_{L^2(\Omega)} e^{\int_0^t c(s)\,ds}, \qquad \forall\, t>0$$
and the conclusion is achieved for $\|u(0)-v_0\|^2_{L^2(\Omega)} < \eta_\varepsilon := \dfrac{\varepsilon}{e^{\int_0^\infty c(s)\,ds}}$. ■

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