A Remark on Perturbed Elliptic Equations of Caffarelli-Kohn-Nirenberg Type

Boumediene Abdellaoui [1], Veronica Felli [2], and Ireneo Peral [1]

[1] Departamento de Matemáticas
Universidad Autónoma de Madrid
Cantoblanco 28049, Madrid — Spain.
boumediene.abdellaoui@uam.es
ireneo.peral@uam.es.

[2] Dipartimento di Matematica e Applicazioni
Università di Milano Bicocca
Via Bicocca degli Arcimboldi 8
20126 Milano — Italy.
felli@matapp.unimib.it.

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ABSTRACT

Using a perturbation argument based on a finite dimensional reduction, we find positive solutions to the following class of perturbed degenerate elliptic equations with critical growth

$$-\operatorname{div}(|x|^{-2a}(I+\varepsilon B(x))\nabla u)-\frac{\lambda}{|x|^{2(1+a)}}\,u=\frac{u^{p-1}}{|x|^{bp}},\qquad x\in\mathbb{R}^N$$

where $-\infty<a<\frac{N-2}{2}$, $-\infty<\lambda<\left(\frac{N-2a-2}{2}\right)^2$, $p=p(a,b)=\frac{2N}{N-2(1+a-b)}$, and $a\le b<a+1$.

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1. Introduction

We study the problem of existence of positive solutions to the following elliptic equation in $\mathbb{R}^N$ in dimension $N \geq 3$

$$-\operatorname{div}(|x|^{-2a}(I+\varepsilon B(x))\nabla u) - \frac{\lambda}{|x|^{2(1+a)}}\, u = \frac{u^{p-1}}{|x|^{bp}}, \qquad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \tag{1}$$

where

$$\begin{aligned} -\infty < a < \frac{N-2}{2}, \qquad -\infty < \lambda < \left(\frac{N-2a-2}{2}\right)^2, \\ p = p(a,b) = \frac{2N}{N-2(1+a-b)}, \quad a \leq b < a+1, \end{aligned} \tag{2}$$

and ε is a small real perturbation parameter. Concerning the perturbation $N \times N$ matrix $B(x) = (b_{ij}(x))_{ij}$, we assume

$$b_{ij} \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). \tag{3}$$

If (3) is satisfied, there exists a positive constant α such that, for any $x \in \mathbb{R}^N$, there holds $\|B(x)\|_{\mathcal{L}(\mathbb{R}^N)} \leq \alpha$ and hence

$$|B(x)\xi \cdot \xi| \leq \alpha|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^N.$$

For $\lambda = \varepsilon = 0$ equation (1) is related to the following class of inequalities established by Caffarelli, Kohn, and Nirenberg in [6],

$$\|u\|_{p,b}^2 := \left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p\, dx\right)^{2/p} \leq \mathcal{C}_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\, dx \qquad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{4}$$

which can be considered as a generalization of both Hardy and Sobolev inequalities. We refer to [7] for discussion on sharp constants and extremal functions associated to (4).

The new feature in this work is the perturbation of the principal part, which is singular or degenerate according with the sign of a. Previous results on this kind of problems are the following ones. In [5] a small perturbation of a regular problem is analyzed. In [8] problem (1) is studied in the case in which a perturbation appears not inside the divergence operator but in the coefficient of the nonlinear term. Related problems for $a = b = 0$ (hence $p = 2^*$) and $0 < \lambda < (N-2)^2/4$ are treated in [1] and [9].

The natural functional space to study (1) is $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_a := \left[\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\, dx\right]^{1/2}.$$

Since Catrina and Wang [7] proved that for $b = a + 1$ the best constant in (4) is

$$\mathcal{C}_{a,a+1}^{-1} = \inf_{\mathcal{D}_a^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2}{\int_{\mathbb{R}^N} |x|^{-2(1+a)} |u|^2} = \left(\frac{N-2-2a}{2}\right)^2,$$

we obtain that for $-\infty < \lambda < \left(\frac{N-2-2a}{2}\right)^2$ an equivalent norm is given by

$$\|u\| = \left[\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}} \, dx\right]^{1/2}.$$

We endow the Hilbert space $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ with the scalar product induced by $\|\cdot\|$

$$(u, v) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v \, dx - \lambda \int_{\mathbb{R}^N} \frac{u\, v}{|x|^{2(1+a)}} \, dx.$$

We will treat problem (1) via an abstract perturbative variational method discussed in [2]. The first step of this procedure is the study of the unperturbed problem, i.e., of equation (1) with $\varepsilon = 0$, for which it was proved in [8] the existence of a one dimensional manifold of radial solutions, which is non-degenerate in some sense we will precise later. Hence a one dimensional reduction of the perturbed variational problem in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ is possible and reduces the problem to the search for critical points of a function defined on the real line. Solutions of (1) can be found as critical points in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ of the perturbed functional

$$f_\varepsilon(u) := f_0(u) + \varepsilon G(u)$$

where

$$\begin{aligned} f_0(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}} \, dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{u_+^p}{|x|^{bp}} \, dx, \\ G(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} B(x) \nabla u \cdot \nabla u \, dx, \end{aligned}$$

and $u_+ := \max\{u, 0\}$. For $\varepsilon = 0$, it was shown in [8] that f_0 has a one dimensional manifold of critical points

$$Z_{a,b,\lambda} := \left\{ z_\mu^{a,b,\lambda} := \mu^{-\frac{N-2-2a}{2}} z_1^{a,b,\lambda}\left(\frac{x}{\mu}\right) \,\Big|\, \mu > 0 \right\},$$

where $z_1^{a,b,\lambda}$ is explicitly given by

$$\begin{aligned} z_1^{a,b,\lambda}(x) = \left[\frac{N(N-2-2a)\sqrt{(N-2-2a)^2 - 4\lambda}}{N - 2(1+a-b)}\right]^{\frac{N-2(1+a-b)}{4(1+a-b)}} \cdot \\ \cdot \left[|x|^{\left(1 - \frac{\sqrt{(N-2-2a)^2-4\lambda}}{N-2-2a}\right)\frac{(N-2-2a)(1+a-b)}{N-2(1+a-b)}} \left[1 + |x|^{\frac{2(1+a-b)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}}\right]\right]^{-\frac{N-2(1+a-b)}{2(1+a-b)}}. \end{aligned} \tag{5}$$

These radial solutions were computed in [7] for $\lambda = 0$, and in [10] in the case $a = b = 0$ and $-\infty < \lambda < (N-2)^2/4$. Moreover in [8] the following non-degeneracy theorem was proved. We will denote by $T_z Z_{a,b,\lambda}$ the tangent space to $Z_{a,b,\lambda}$ in Z.

Theorem 1.1 ([8])**.** *Suppose a, b, λ, p satisfy* (2)*. Then the critical manifold $Z_{a,b,\lambda}$ is non-degenerate, i.e.,*

$$T_z Z_{a,b,\lambda} = \ker D^2 f_0(z) \quad \forall\, z \in Z_{a,b,\lambda}, \tag{6}$$

if and only if

$$b \neq h_j(a,\lambda) := \frac{N}{2}\left[1 + \frac{4j(N+j-2)}{(N-2-2a)^2 - 4\lambda}\right]^{-1/2} - \frac{N-2-2a}{2} \qquad \forall\, j \in \mathbb{N}\setminus\{0\}. \tag{7}$$

For the values of the parameters a, b, λ, for which nondegeneracy condition holds, it is possible to follow the abstract scheme in [2] and construct a manifold

$$Z^{\varepsilon}_{a,b,\lambda} = \left\{\, z^{a,b,\lambda}_{\mu} + w(\varepsilon,\mu) \,\middle|\, \mu > 0 \,\right\},$$

such that any critical point of f_ε restricted to $Z^{\varepsilon}_{a,b,\lambda}$ is a solution to $(\mathcal{P}_{a,b,\lambda})$. Since the perturbed manifold we construct is globally diffeomorphic to the unperturbed one, we can estimate the difference $\|w(\varepsilon,\mu)\|$ when $\mu \to \infty$ or $\mu \to 0$ (see also [4, 8]). In particular we prove that $\|w(\varepsilon,\mu)\|$ vanishes as $\mu \to \infty$ or $\mu \to 0$ under the assumption that the entries of the matrix B vanish at 0 and at ∞.

We will prove the following existence results.

Theorem 1.2. *Suppose* (2), (3)*, and* (7) *hold. Then problem* (1) *has a solution for all $|\varepsilon|$ sufficiently small if*

$$b_{ij}(\infty) := \lim_{|x|\to\infty} b_{ij}(x) \text{ exists for any } i,j \quad \text{and} \quad b_{ij}(\infty) = b_{ij}(0) = 0. \tag{8}$$

Theorem 1.3. *Assume* (2), (3), (7)*, and*

$$b_{ij} \in C^2(\mathbb{R}^N), \quad |\nabla b_{ij}| \in L^\infty(\mathbb{R}^N), \quad \text{and} \quad |D^2 b_{ij}| \in L^\infty(\mathbb{R}^N). \tag{9}$$

Then (1) *is solvable for all small $|\varepsilon|$ under each of the following conditions*

$$\limsup_{|x|\to\infty} b_{ij}(x) \le b_{ij}(0) \text{ and } \Delta B(0) \text{ is positive definite}, \tag{10}$$

$$\liminf_{|x|\to\infty} b_{ij}(x) \ge b_{ij}(0) \text{ and } \Delta B(0) \text{ is negative definite}, \tag{11}$$

where $\Delta B(0)$ is the matrix $(\Delta b_{ij}(0))_{ij}$.

Theorem 1.4. *Suppose* (2), (3), (7) *hold, and that the functions b_{ij} are periodic, i.e., for any $i, j = 1, \dots, N$ there exists $T^{ij} = (T_1^{ij}, T_2^{ij}, \dots, T_N^{ij}) \in \mathbb{R}^N$ such that*

$$b_{ij}(x + T_k^{ij} e_k) = b_{ij}(x)$$

for any $k = 1, 2, \dots, N$, $x \in \mathbb{R}^N$, where e_k is the k-th vector of the canonical basis of $\mathbb{R}^N$. Let us denote by $\fint B$ the matrix

$$\begin{aligned}\fint B &= \left(\fint_{\{x=(x_1,\dots,x_N)\in\mathbb{R}^N\,:\,0\le x_j\le|T_j^{ij}|,\ \forall j=1,2,\dots,N\}} b_{ij}\right)_{ij}\\ &= \left(\frac{1}{|T_1^{ij}||T_2^{ij}|\cdots|T_N^{ij}|}\int_{\{x=(x_1,\dots,x_N)\in\mathbb{R}^N\,:\ 0\le x_j\le|T_j^{ij}|,\ \forall j=1,2,\dots,N\}} b_{ij}\right)_{ij}.\end{aligned}$$

Then problem (1) *has a solution for all $|\varepsilon|$ sufficiently small if either*

$$\Delta B(0) \quad \textit{is positive definite} \tag{12}$$

$$B(0) - \fint B \quad \textit{is positive definite} \tag{13}$$

or

$$\Delta B(0) \quad \textit{is negative definite} \tag{14}$$

$$B(0) - \fint B \quad \textit{is negative definite.} \tag{15}$$

2. The finite dimensional reduction

In this section we show that whenever the critical manifold is non-degenerated, i.e., if (7) holds, our problem can be reduced to a finite dimensional one through the perturbative method developed in [2]. For simplicity of notation, we write z_μ instead of $z_\mu^{a,b,\lambda}$ and Z instead of $Z_{a,b,\lambda}$ if there is no possibility of confusion. In the sequel, we will use the canonical identification of the Hilbert space $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ with its dual induced by the scalar product and denoted by $\mathcal{K}$, i.e.,

$$\mathcal{K} : \big(\mathcal{D}_a^{1,2}(\mathbb{R}^N)\big)' \to \mathcal{D}_a^{1,2}(\mathbb{R}^N), \quad (\mathcal{K}(\varphi), u) = \varphi(u),\\ \text{for any } \varphi \in \big(\mathcal{D}_a^{1,2}(\mathbb{R}^N)\big)',\ u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N).$$

Consequently we shall consider $f_\varepsilon'(u)$ as an element of $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ and $f_\varepsilon''(u)$ as one of $\mathcal{L}(\mathcal{D}_a^{1,2}(\mathbb{R}^N))$. We recall the following lemma from [8].

Lemma 2.1 ([8]**).** *Suppose a, b, λ, p satisfy* (2) *and v is a measurable function such that the integral $\int_{\mathbb{R}^n} |v|^{\frac{p}{p-2}} |x|^{-bp}$ is finite. Then the operator $J_v : \mathcal{D}_a^{1,2}(\mathbb{R}^N) \to \mathcal{D}_a^{1,2}(\mathbb{R}^N)$, defined by*

$$J_v(u) := \mathcal{K}\Big(\int_{\mathbb{R}^N} |x|^{-pb} v u \cdot\Big),$$

is compact.

Lemma 2.1 immediately leads to the following result.

Corollary 2.2. *For all $z \in Z$ the operator $f_0''(z) : \mathcal{D}_a^{1,2}(\mathbb{R}^N) \to \mathcal{D}_a^{1,2}(\mathbb{R}^N)$ may be written as $f_0''(z) = id - J_{|z|^{p-2}}$ and is consequently a self-adjoint Fredholm operator of index zero.*

Following [8], we introduce some notation. For all $\mu > 0$ we define the rescaling map

$$U_\mu : \mathcal{D}_a^{1,2}(\mathbb{R}^N) \to \mathcal{D}_a^{1,2}(\mathbb{R}^N), \qquad U_\mu(u) := \mu^{-\frac{N-2-2a}{2}} u\Big(\frac{x}{\mu}\Big).$$

Invariance by dilations ensures that U_μ conserves the norms $\|\cdot\|$ and $\|\cdot\|_{p,b}$ and

$$(U_\mu)^{-1} = (U_\mu)^t = U_{\mu^{-1}} \text{ and } f_0 = f_0 \circ U_\mu \quad \text{for every } \mu > 0 \tag{16}$$

where $(U_\mu)^t$ denotes the adjoint of U_μ. Differentiating twice the identity $f_0 = f_0 \circ U_\mu$ yields for all $h_1, h_2, v \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$

$$(f_0''(v)h_1, h_2) = (f_0''(U_\mu(v))U_\mu(h_1), U_\mu(h_2)),$$

namely

$$f_0''(v) = (U_\mu)^{-1} \circ f_0''(U_\mu(v)) \circ U_\mu \quad \text{for all } v \in \mathcal{D}_a^{1,2}(\mathbb{R}^N). \tag{17}$$

Differentiating (16) we get that $U(\mu, z) := U_\mu(z)$ maps $(0, \infty) \times Z$ into Z, hence

$$\frac{\partial U}{\partial z}(\mu, z) = U_\mu : \, T_z Z \to T_{U_\mu(z)} Z \quad \text{and} \quad U_\mu : \, (T_z Z)^\perp \to (T_{U_\mu(z)} Z)^\perp. \tag{18}$$

Moreover if Z is non-degenerated, the self-adjoint Fredholm operator $f_0''(z_1)$ maps the space $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ into $T_{z_1} Z^\perp$ and $f_0''(z_1) \in \mathcal{L}(T_{z_1} Z^\perp)$ is invertible. Consequently, using (17) and (18), we obtain in this case

$$\|(f_0''(z_1))^{-1}\|_{\mathcal{L}(T_{z_1} Z^\perp)} = \|(f_0''(z))^{-1}\|_{\mathcal{L}(T_z Z^\perp)} \quad \forall z \in Z. \tag{19}$$

Lemma 2.3. *Suppose a, b, p, λ satisfy* (2) *and* (3) *holds. Then there exists a constant $C_1 = C_1(\|B\|_{L^\infty(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N))}, a, \lambda) > 0$ such that for any $\mu > 0$ and for any $w \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$*

$$|G(z_\mu + w)| \le C_1 \bigg(\int_{\mathbb{R}^N} |y|^{-2a} B(\mu y) \nabla z_1(y) \cdot \nabla z_1(y) + \|w\| + \|w\|^2 \bigg), \tag{20}$$

$$\|G'(z_\mu + w)\| \le C_1 \bigg(\bigg(\int_{\mathbb{R}^N} |x|^{-2a} |(B + B^t)(\mu y) \nabla z_1(y)|^2 \bigg)^{1/2} + \|w\| \bigg), \tag{21}$$

$$\|G''(z_\mu + w)\| \le \|B\|_{L^\infty(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N))}, \tag{22}$$

where $B^t(x) = (b_{ji}(x))_{ij}$ *denotes the adjoint of* B*. Moreover, if* (8) *holds, then*

$$\begin{aligned} &\int_{\mathbb{R}^N} |x|^{-2a} |(B + B^t)(\mu y)\nabla z_1(y)|^2 \to 0, \\ &\int_{\mathbb{R}^N} |y|^{-2a} B(\mu y)\nabla z_1(y) \cdot \nabla z_1(y) \to 0, \end{aligned} \tag{23}$$

as $\mu \to \infty$ *or* $\mu \to 0$.

Proof. (20) is an easy consequence of the definition of G, Schwarz inequality and (3). To deduce (21) we observe that by Schwarz inequality

$$\begin{aligned} &\|G'(z_\mu + w)\| \\ &\quad = \sup_{\|v\|\le 1} \left| \int_{\mathbb{R}^N} |x|^{-2a} \Big(\frac{B + B^t}{2}\Big)(x)\nabla z_\mu \cdot \nabla v + \int_{\mathbb{R}^N} |x|^{-2a} \Big(\frac{B + B^t}{2}\Big)(x)\nabla w \cdot \nabla v \right| \\ &\quad \le \text{const}\bigg(\bigg(\int_{\mathbb{R}^N} |x|^{-2a} |(B + B^t)(x)\nabla z_\mu(x)|^2\bigg)^{1/2} + \|w\|\bigg) \end{aligned}$$

whereas (22) comes just from $(G''(z_\mu + w)v, u) = \frac{1}{2}\int |x|^{-2a}(B + B^t)(x)\nabla v \cdot \nabla u$. Under the additional assumption $b_{ij}(0) = b_{ij}(\infty) = 0$ estimate (23) follows by the dominated convergence theorem. □

Lemma 2.4. *Suppose* a, b, p, λ *satisfy* (2) *and* (3) *and* (6) *hold. Then there exist constants* $\varepsilon_0, C > 0$ *and a smooth function*

$$w = w(\mu, \varepsilon) : \quad (0, +\infty) \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathcal{D}^{1,2}_a(\mathbb{R}^N)$$

such that for any $\mu > 0$ *and* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$w(\mu, \varepsilon) \ \text{ is orthogonal to } \ T_{z_\mu} Z \tag{24}$$

$$f'_\varepsilon\big(z_\mu + w(\mu, \varepsilon)\big) \in T_{z_\mu} Z \tag{25}$$

$$\|w(\mu, \varepsilon)\| \le C\,|\varepsilon|. \tag{26}$$

Moreover, if (8) *holds then*

$$\|w(\mu, \varepsilon)\| \to 0 \ \text{ as } \mu \to 0 \ \text{ or } \mu \to \infty. \tag{27}$$

Proof. Let $H : (0, \infty) \times \mathcal{D}^{1,2}_a(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \to \mathcal{D}^{1,2}_a(\mathbb{R}^N) \times \mathbb{R}$ be defined by

$$H(\mu, w, \alpha, \varepsilon) := (f'_\varepsilon(z_\mu + w) - \alpha\dot{\xi}_\mu, (w, \dot{\xi}_\mu)),$$

where $\dot{\xi}_\mu$ denotes the normalized tangent vector $\frac{d}{d\mu} z_\mu$. If $H(\mu, w, \alpha, \varepsilon) = (0, 0)$ then w satisfies (24), (25) and $H(\mu, w, \alpha, \varepsilon) = (0, 0)$ if and only if $(w, \alpha) = F_{\mu,\varepsilon}(w, \alpha)$, where

$$F_{\mu,\varepsilon}(w, \alpha) := -\bigg(\frac{\partial H}{\partial (w, \alpha)}(\mu, 0, 0, 0)\bigg)^{-1} H(\mu, w, \alpha, \varepsilon) + (w, \alpha).$$

Note that

$$\left(\left(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\right)(w,\beta),(f_0''(z_\mu)w-\beta\dot{\xi}_\mu,(w,\dot{\xi}_\mu))\right)$$
$$=\|f_0''(z_\mu)w\|^2+\beta^2+|(w,\dot{\xi}_\mu)|^2, \quad (28)$$

where

$$\left(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\right)(w,\beta)=(f_0''(z_\mu)w-\beta\dot{\xi}_\mu,(w,\dot{\xi}_\mu)).$$

From Corollary 2.2 and (28) we infer that $\big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\big)$ is an injective Fredholm operator of index zero, hence invertible and by (19) and (28) we obtain

$$\begin{aligned}\left\|\left(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\right)^{-1}\right\| &\le \max\big(1,\|(f_0''(z_\mu))^{-1}\|\big)\\ &=\max\big(1,\|(f_0''(z_1))^{-1}\|\big)=:C_*.\end{aligned} \quad (29)$$

Suppose that $(w,\alpha)\in \bar{B}_\rho(0)=\big\{(x,\beta)\in \mathcal{D}_a^{1,2}(\mathbb{R}^N)\times\mathbb{R}\,\big|\,\|x\|+|\alpha|\le\rho\big\}$ with $\rho=\rho(\varepsilon)>0$ to be determined. From (17) and (29) we deduce

$$\begin{aligned}\|F_{\mu,\varepsilon}(w,\alpha)\| &\le C_*\left\|\left(H(\mu,w,\alpha,\varepsilon)-\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\Big)(w,\alpha)\right)\right\|\\ &\le C_*\|f_\varepsilon'(z_\mu+w)-f_0''(z_\mu)w\|\\ &\le C_*\int_0^1\|f_0''(z_\mu+tw)-f_0''(z_\mu)\|\,dt\|w\|+C_*|\varepsilon|\|G'(z_\mu+w)\|\\ &\le C_*\int_0^1\|f_0''(z_1+tU_{\mu^{-1}}(w))-f_0''(z_1)\|\,dt\|w\|+C_*|\varepsilon|\|G'(z_\mu+w)\|\\ &\le C_*\rho\sup_{\|w\|\le\rho}\|f_0''(z_1+w)-f_0''(z_1)\|+C_*|\varepsilon|\sup_{\|w\|\le\rho}\|G'(z_\mu+w)\|.\end{aligned} \quad (30)$$

Analogously we get for $(w_1,\alpha_1),(w_2,\alpha_2)\in B_\rho(0)$

$$\begin{aligned}\frac{\|F_{\mu,\varepsilon}(w_1,\alpha_1)-F_{\mu,\varepsilon}(w_2,\alpha_2)\|}{C_*\|w_1-w_2\|} &\le \frac{\|f_\varepsilon'(z_\mu+w_1)-f_\varepsilon'(z_\mu+w_2)-f_0''(z_\mu)(w_1-w_2)\|}{\|w_1-w_2\|}\\ &\le\int_0^1\|f_\varepsilon''(z_\mu+w_2+t(w_1-w_2))-f_0''(z_\mu)\|\,dt\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \|f_0''(z_\mu + w_2 + t(w_1 - w_2)) - f_0''(z_\mu)\| \, dt \\
&\quad + |\varepsilon| \int_0^1 \|G''(z_\mu + w_2 + t(w_1 - w_2))\| \, dt \\
&\leq \sup_{\|w\| \leq 3\rho} \|f_0''(z_1 + w) - f_0''(z_1)\| \\
&\quad + |\varepsilon| \sup_{\|w\| \leq 3\rho} \|G''(z_\mu + w)\|.
\end{aligned}
$$

We choose $\rho_0 > 0$ such that

$$
C_* \sup_{\|w\| \leq 3\rho_0} \|f_0''(z_1 + w) - f_0''(z_1)\| < \frac{1}{2}
$$

and $\varepsilon_0 > 0$ such that

$$
\varepsilon_0 < \min\left\{ \frac{1}{8C_1 C_* \|B\| \|z_1\|}, \frac{\rho_0}{8C_1 C_* \|B\| \|z_1\|}, \frac{1}{8C_1 C_*} \right\}
$$

and

$$
\begin{aligned}
3\varepsilon_0 &< \Big(\sup_{z \in Z, \|w\| \leq 3\rho_0} \|G''(z + w)\| \Big)^{-1} C_*^{-1}, \\
3\varepsilon_0 &< \Big(\sup_{z \in Z, \|w\| \leq \rho_0} \|G'(z + w)\| \Big)^{-1} C_*^{-1} \rho_0,
\end{aligned}
$$

where C_1 is given in Lemma 2.3. With these choices and the above estimates it is easy to see that for every $\mu < 0$ and $|\varepsilon| < \varepsilon_0$ the map $F_{\mu,\varepsilon}$ maps $\bar{B}_{\rho_0}(0)$ in itself and is a contraction there. Thus from the Contraction Mapping Theorem, $F_{\mu,\varepsilon}$ has a unique fixed point $(w(\mu, \varepsilon), \alpha(\mu, \varepsilon))$ in $B_{\rho_0}(0)$ and it is a consequence of the implicit function theorem that w and α are continuously differentiable.

From (30) we also infer that $F_{\mu,\varepsilon}$ maps $\bar{B}_\rho(0)$ into $\bar{B}_\rho(0)$, whenever $\rho \leq \rho_0$ and

$$
\rho > 2|\varepsilon| \big(\sup_{\|w\| \leq \rho} \|G'(z_\mu + w)\| \big) C_*.
$$

Consequently due to the uniqueness of the fixed-point we have

$$
\|(w(\mu, \varepsilon), \alpha(\mu, \varepsilon))\| \leq 3|\varepsilon| \big(\sup_{\|w\| \leq \rho_0} \|G'(z_\mu + w)\| \big) C_*,
$$

which gives (26) in view of (21). Let us now prove (27). Set

$$
\rho_\mu := 8\varepsilon_0 C_* C_1 \left(\int_{\mathbb{R}^N} |x|^{-2a} |(B + B^t)(\mu y) \nabla z_1(y)|^2 \right)^{1/2}.
$$

Note that $\rho_\mu < \min\{1, \rho_0\}$. In view of (21) we have that for any $|\varepsilon| < \varepsilon_0$ and $\mu > 0$

$$2|\varepsilon| C_* \sup_{\|w\| \le \rho_\mu} \|G'(z_\mu + w)\| \le \frac{1}{4}\rho_\mu + 2\varepsilon_0 C_* C_1 \rho_\mu < \frac{1}{2}\rho_\mu.$$

By the above argument, we can conclude that $F_{\mu,\varepsilon}$ maps $B_{\rho_\mu}(0)$ into $B_{\rho_\mu}(0)$. Consequently due to the uniqueness of the fixed-point we have

$$\|w(\mu, \varepsilon)\| \le \rho_\mu.$$

Since by (23) we have that $\rho_\mu \to 0$ for $\mu \to 0$ and for $\mu \to +\infty$, we get (27). □

Under the assumptions of Lemma 2.4 we may define for $|\varepsilon| < \varepsilon_0$

$$Z^\varepsilon_{a,b,\lambda} := \left\{ u \in \mathcal{D}^{1,2}_a(\mathbb{R}^N) \mid u = z^{a,b,\lambda}_\mu + w(\mu, \varepsilon),\ \mu \in (0, \infty) \right\}. \tag{31}$$

For simplicity of notation, we will write Z^ε for $Z^\varepsilon_{a,b,\lambda}$ if no confusion is possible. Note that Z^ε is a one dimensional manifold parameterized by the rescaling variable μ. Moreover arguing as in [8] we can prove that we may choose $\varepsilon_0 > 0$ such that for every $|\varepsilon| < \varepsilon_0$ the manifold Z^ε is a natural constraint for f_ε, i.e., every critical point of $f_\varepsilon|_{Z^\varepsilon}$ is a critical point of f_ε. Hence we end up facing a finite dimensional problem as it is enough to find critical points of the functional $\Phi_\varepsilon : (0, \infty) \to \mathbb{R}$ given by $f_\varepsilon|_{Z^\varepsilon}$.

3. Study of Φ_ε

In this section we assume that (7) holds, in such a way that the critical manifold is non-degenerate and the functional Φ_ε is defined. To find critical points of $\Phi_\varepsilon = f_\varepsilon|_{Z^\varepsilon}$ it is convenient to introduce the functional Γ given below.

Lemma 3.1. *Suppose a, b, p, λ satisfy* (2) *and* (3) *holds. Then*

$$\Phi_\varepsilon(\mu) = f_0(z_1) + \varepsilon\Gamma(\mu) + o(\varepsilon), \tag{32}$$

as $\varepsilon \to 0$ uniformly with respect to $\mu \in (0, \infty)$, where

$$\Gamma(\mu) = G(z_\mu) = \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a} B(x)\nabla z_\mu \cdot \nabla z_\mu = \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a} B(\mu x)\nabla z_1 \cdot \nabla z_1. \tag{33}$$

More precisely, there exists $C > 0$, independent of μ and ε, such that

$$|\Phi_\varepsilon(\mu) - f_0(z_1) - \varepsilon\Gamma(\mu)| \le C\big(\|w(\varepsilon, \mu)\|^2 + \|w(\varepsilon, \mu)\|^p + |\varepsilon|\|w(\varepsilon, \mu)\|\big). \tag{34}$$

Consequently, if there exist $0 < \mu_1 < \mu_2 < \mu_3 < \infty$ such that

$$\Gamma(\mu_2) > \max(\Gamma(\mu_1), \Gamma(\mu_3)) \ \textit{or}\ \Gamma(\mu_2) < \min(\Gamma(\mu_1), \Gamma(\mu_3)) \tag{35}$$

then Φ_ε will have a critical point, provided $|\varepsilon|$ is sufficiently small.

Proof. Invariance by dilations yields for all $\mu > 0$

$$f_0(z_\mu) = f_0(z_1). \tag{36}$$

Moreover since z_μ solves the unperturbed problem we have that

$$\|z_\mu\|^2 = \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} \qquad \text{and} \qquad \big(z_\mu, w(\varepsilon,\mu)\big) = \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} w(\varepsilon,\mu)}{|x|^{bp}}. \tag{37}$$

From (36) and (37) we deduce

$$\begin{aligned}\Phi_\varepsilon(\mu) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} + \frac{1}{2}\|w(\varepsilon,\mu)\|^2 \\ + \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} w(\varepsilon,\mu)}{|x|^{bp}} - \frac{1}{p}\int_{\mathbb{R}^N} \frac{\big(z_\mu + w(\varepsilon,\mu)\big)_+^p}{|x|^{bp}} \\ + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |x|^{-2a} B(x)\nabla(z_\mu + w(\varepsilon,\mu)) \cdot \nabla(z_\mu + w(\varepsilon,\mu))\end{aligned}$$

and

$$f_0(z_1) = f_0(z_\mu) = \frac{1}{2}\|z_\mu\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}}.$$

Hence

$$\Phi_\varepsilon(\mu) = f_0(z_1) + \varepsilon\Gamma(\mu) + \frac{1}{2}\|w(\varepsilon,\mu)\|^2 + R_\varepsilon(\mu),$$

where

$$\begin{aligned}R_\varepsilon(\mu) = -\frac{1}{p}\int_{\mathbb{R}^N} \frac{\big(z_\mu + w(\varepsilon,\mu)\big)_+^p - z_\mu^p - p\, z_\mu^{p-1} w(\varepsilon,\mu)}{|x|^{bp}} \\ + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |x|^{-2a} B(x)\nabla z_\mu \cdot \nabla w(\varepsilon,\mu) \\ + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |x|^{-2a} B(x)\nabla w(\varepsilon,\mu) \cdot \nabla(z_\mu + w(\varepsilon,\mu)).\end{aligned}$$

Using the inequality

$$|(a+b)_+^s - a_+^s - s a_+^{s-1} b| \le \begin{cases} C(a_+^{s-2}|b|^2 + |b|^s) & \text{if } s \ge 2, \\ C\ |b|^s & \text{if } 1 < s < 2, \end{cases}$$

where $C = C(s) > 0$, with $s = p$, and Hölder's inequality, we have for some $c_2, c_3 > 0$

$$\begin{aligned}
|R_\varepsilon(\mu)| &\le \frac{1}{p}\int_{\mathbb{R}^N} \frac{|(z_\mu + w(\varepsilon,\mu))_+^p - z_\mu^p - p\, z_\mu^{p-1} w(\varepsilon,\mu)|}{|x|^{bp}} \\
&\quad + \frac{|\varepsilon|}{2}\|B(x)\|_{\mathcal{L}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |x|^{-2a}\Big(|\nabla z_\mu||\nabla w(\varepsilon,\mu)| \\
&\quad + |\nabla w(\varepsilon,\mu)||\nabla(z_\mu + w(\varepsilon,\mu))|\Big) \\
&\le c_2\left[\int_{\mathbb{R}^N} \frac{z_\mu^{p-2} w^2(\varepsilon,\mu)}{|x|^{bp}} + \int_{\mathbb{R}^N} \frac{|w(\varepsilon,\mu)|^p}{|x|^{bp}} + |\varepsilon|(\|w(\varepsilon,\mu)\| + \|w(\varepsilon,\mu)\|^2)\right] \\
&\le c_3\left[\|w(\varepsilon,\mu)\|^2 + \|w(\varepsilon,\mu)\|^p + |\varepsilon|\|w(\varepsilon,\mu)\|\right]
\end{aligned}$$

and the claim follows. □

In view of expansion (32), we have that critical points of Γ which are stable under small uniform perturbations yield critical points of Φ_ε and hence of f_ε. On the other hand, although it is convenient to study only the reduced functional Γ instead of Φ_ε, it may lead in some cases to a loss of information, i.e., Γ may be constant even if B is a non-constant matrix. In this case we have to study the functional $\Phi_\varepsilon(\mu)$ directly.

Proof of Theorem 1.2. By (8), (23), (27), and (34),

$$\lim_{\mu\to 0^+} \Phi_\varepsilon(\mu) = \lim_{\mu\to+\infty} \Phi_\varepsilon(\mu) = f_0(z_1).$$

Hence, either the functional $\Phi_\varepsilon \equiv f_0(z_1)$, and we obtain infinitely many critical points, or $\Phi_\varepsilon \not\equiv f_0(z_1)$ and Φ_ε has at least a global maximum or minimum. In any case Φ_ε has a critical point that provides a solution of (1). The maximum principle applied in $\mathbb{R}^N \setminus \{0\}$ ensures the positivity of solutions in such a region. □

As in [3, Lemma 3.4], we can extend the C^2-functional Γ by continuity to $\mu = 0$.

Lemma 3.2. *Under the assumptions of Lemma 3.1,*

$$\Gamma(0) := \lim_{\mu\to 0} \Gamma(\mu) = \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a} B(0)\nabla z_1 \cdot \nabla z_1$$

and

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}\big(\liminf_{|y|\to\infty} B(y)\big)\nabla z_1\cdot\nabla z_1 &\le \liminf_{\mu\to\infty}\Gamma(\mu) \le \limsup_{\mu\to\infty}\Gamma(\mu) \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}\big(\limsup_{|y|\to\infty} B(y)\big)\nabla z_1\cdot\nabla z_1
\end{aligned}$$

where $\liminf_{|y|\to\infty} B(y)$ *is the matrix* $(\liminf_{|y|\to\infty} b_{ij}(y))_{ij}$ *and* $\limsup_{|y|\to\infty} B(y)$ *is the matrix* $(\limsup_{|y|\to\infty} b_{ij}(y))_{ij}$. *If, moreover,* (9) *holds we obtain*

$$\Gamma'(0) = 0 \text{ and } \Gamma''(0) = \frac{1}{2N}\int |x|^{2-2a}(\Delta B(0))\nabla z_1 \cdot \nabla z_1.$$

Proof of Theorem 1.3. We prove the theorem by showing that under assumptions (10) and (11) the function Γ has a critical point. Condition (10) and Lemma 3.2 imply that Γ has a global maximum strictly bigger than $\Gamma(0)$ and $\limsup_{\mu\to\infty}\Gamma(\mu)$. Consequently Φ_ε has a critical point in view of Lemma 3.1. The same argument yields a critical point under condition (11). □

Proof of Theorem 1.4. To see that assumptions (12) and (13) give rise to a critical point we use the functional Γ. Condition (12) and Lemma 3.2 imply that Γ is strictly convex at 0. From periodicity of b_{ij} and Riemann-Lebesgue Theorem, it follows that

$$\lim_{\mu\to\infty}\Gamma(\mu) = \frac{1}{2}\int |x|^{-2a}\Big(\fint B\Big)\nabla z_1 \cdot \nabla z_1.$$

Hence assumption (13) implies that $\Gamma(0) \geq \lim_{\mu\to\infty}\Gamma(\mu)$. Hence Γ must have a proper maximum point and consequently Φ_ε has a critical point in view of Lemma 3.1. The same reasoning yields a critical point under conditions (14), (15). □

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